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ASYMPTOTIC BEHAVIOR FOR A SEMILINEAR SECOND ORDER EVOLUTION EQUATION

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ABSTRACT. This paper is devoted to the qualitative analysis for a second order evolution equation $u_{tt}-\Delta u-\Delta u_t-\varepsilon\Delta u_{tt}+f(u)=g(x)$ ($\varepsilon\in[0,1]$) with critical nonlinearity. Some uniformly (w.r.t. $\varepsilon\in[0,1]$) asymptotic regularity about the solutions has been established for both $g(x)\in L^2(\Omega)$ and $g(x)\in H^{-1}$, which shows that the solutions are exponentially approaching a more regular fixed subset uniformly (w.r.t. $\varepsilon\in[0,1]$). As an application of this regularity result, a family $\{\mathcal{E}_\varepsilon\}_{\varepsilon\in[0,1]}$ of finite dimensional exponential attractors has been constructed. Moreover, to characterize the relation with a strongly damped wave equation ($\varepsilon=0$), the upper semicontinuity, at $\varepsilon=0$, of the global attractors has been proved.

1. Introduction

We study the long-time behavior of the following semilinear evolution equation of second order in time:

$$(E_\varepsilon)\qquad \begin{cases} u_{tt}-\Delta u-\Delta u_t-\varepsilon\Delta u_{tt}+f(u)=g(x) \text{ in } \Omega\times[0,\infty),\\ (u(x,0),\,u_t(x,0))=(u_0(x),\,v_0(x)),\\ u|_{\partial\Omega}=0,\end{cases}$$

where $\Omega\subset\mathbb{R}^N (N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\varepsilon\in[0,1]$, and the external forcing $g(x)$ is time-independent.

When $\varepsilon=0$, (E_0) is the usual strongly damped wave equation, and its asymptotic behavior has been studied extensively in terms of attractors; see [4, 5, 7, 13, 16, 23, 25, 32, 35, 36].

For each fixed $\varepsilon_0>0$, equation (E_{ε_0}) is a special form of the so-called improved Boussinesq equation (see [3, 19, 20, 31]) with damped term $-\Delta u_t$, which was used to describe ion-sound waves in plasma by Makhankov [20, 21] and also known to represent other sorts of 'propagation problems' of, for example, lengthways waves in nonlinear elastic rods and ion-sonic waves of space transformations by a weak nonlinear effect (see [3, 10]).

The main purpose of this paper is, based on the global well-posedness results given in [6] and motivated by the dynamical results in [9, 13, 23, 25, 28, 30, 36, 37],

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to give some uniform (w.r.t. the parameter $\varepsilon \in [0,1]$) qualitative analysis (or a priori estimates) for the solutions of (E_ε) and then provide some information about the relation between the solutions of (E_0) and those of (E_ε).

This paper is organized as follows. In §2, we introduce basic notation and state our main results. In §3, we recall some abstract results that we will use later. In §4, we present several dissipative estimates about the solution of (E_ε), which hold uniformly with respect to $\varepsilon \in [0,1]$. The main results are proved in §5 and §6 for $g(x) \in L^2(\Omega)$ and $g(x) \in H^{-1}$, respectively. Moreover, as an application, we construct a finite dimensional exponential attractor and prove upper semicontinuity of the global attractor in §5.5.

2. Main results

Before presenting our main results, we first state the basic mathematical assumptions for considering the long-time behaviors of second order evolution equations and then introduce some notation that we will use throughout this paper:

- $f \in \mathcal{C}^1(\mathbb{R})$ with $f(0)=0$ and satisfies the following conditions:

$$|f'(s)| \le C(1+|s|^{\frac{N+2}{N-2}-1}) \quad \text{for all } s \in \mathbb{R}, \tag{2.1}$$

  and

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{2.2}$$

  where λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$.
- Let $A=-\Delta$ with domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, and consider the family of Hilbert spaces $D(A^{s/2})$, $s \in \mathbb{R}$ with the standard inner products and norms, respectively, $\langle \cdot, \cdot\rangle_{D(A^{s/2})} = \langle A^{s/2}\cdot, A^{s/2}\cdot\rangle$ and $\|\cdot\|_{D(A^{s/2})} = \|A^{s/2}\cdot\|$. In particular, $\langle \cdot, \cdot\rangle$ and $\|\cdot\|$ mean the $L^2(\Omega)$ inner product and norm, respectively. We denote
  - $\mathcal{H}^s = D(A^{\frac{1+s}{2}}) \times D(A^{\frac{1+s}{2}})$, $s \in [0,1]$ with the usual norm

$$\|(u,v)\|^2_{\mathcal{H}^s} = \|A^{\frac{1+s}{2}}u\|^2 + \|A^{\frac{1+s}{2}}v\|^2.$$

    In particular, we denote $\mathcal{H}=\mathcal{H}^0 = H_0^1(\Omega)\times H_0^1(\Omega)$ and $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_{\mathcal{H}^0}$.
  - For each $(u,v)\in\mathcal{H}$, we define $\|\cdot\|_{\mathcal{H}^s_\varepsilon}$ $(\varepsilon, s \in [0,1])$ as

$$\|(u,v)\|^2_{\mathcal{H}^s_\varepsilon} = \|A^{\frac{1+s}{2}}u\|^2 + \|A^{\frac{s}{2}}v\|^2 + \varepsilon\|A^{\frac{1+s}{2}}v\|^2,$$

    and define $\mathcal{H}^s_\varepsilon$ as

$$\mathcal{H}^s_\varepsilon = cl_{\|\cdot\|_{\mathcal{H}^s_\varepsilon}}(\mathcal{H}^1).$$

    Then $(\mathcal{H}^s_\varepsilon, \|\cdot\|_{\mathcal{H}^s_\varepsilon})$ is a Banach space for every $\varepsilon, s \in [0,1]$.
- $\xi_u(t) = (u(t), u_t(t))$ for any $t \ge 0$.

For clarity, we would like to separate our results into two parts according to the external forcing $g(x)\in L^2(\Omega)$ and $g(x) \in H^{-1}$. For the well-posedness, there is no essential difference between the cases $g(x)\in L^2(\Omega)$ and $g(x)\in H^{-1}$ if we work in the weakly energy phase space $\mathcal{H}$. However, for the asymptotic regularity (and so the dynamics), there is a big difference: the stationary solutions of (E_ε) will belong to $H^2(\Omega)\cap H_0^1(\Omega)$ if $g(x) \in L^2(\Omega)$, and so one can expect the global attractor $\mathcal{A}_\varepsilon$ will be bounded in $\mathcal{H}^1$ for this case, but the stationary solutions of (E_ε) in general will only belong to $H_0^1(\Omega)$ if $g(x) \in H^{-1}$; consequently the global attractor $\mathcal{A}_\varepsilon$ now will only be bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$.

Part I: $g(x) \in L^2(\Omega)$.

We make the following assumption:

Assumption I. $\Omega \subset \mathbb{R}^N (N \geq 3)$ *is a bounded domain with smooth boundary,* $g \in L^2(\Omega)$ *and* f *satisfies* (2.1)-(2.2) *with* $f(0) = 0$.

The existence-uniqueness of solutions for (E_{ε_0}) has been proven in [27, 38] by the Faedo-Galerkin method, and then the long-time behavior of the solution of (E_{ε_0}) via proving the existence of a global attractor in $\mathcal{H}^0_{\varepsilon_0}$ under *Assumption I* has been discussed by Xie and Zhong in [33, 34]. Recently, Carvalho and Cholewa [6] presented systematic results including the existence-uniqueness and long-time behavior of (E_{ε_0}) by using the semigroup approach in $\mathcal{H}^0_{\varepsilon_0}$ (note that $(\mathcal{H}^0_{\varepsilon_0}, \|\cdot\|_{\mathcal{H}^0_{\varepsilon_0}}) \cong (\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ for each fixed $\varepsilon_0 > 0$). They showed that for each $\varepsilon > 0$, the solution of (E_ε) generates a C^0 semigroup $\{S_\varepsilon(t)\}_{t\geq 0}$ in $\mathcal{H}^0_\varepsilon$, and also obtain the asymptotic regularity of attractors for the subcritical case, i.e., require the exponent in (2.1) to be strictly less than $\frac{N+2}{N-2} - 1$.

The main result of this part is the following asymptotic regularity.

Theorem 2.1. *Under Assumption I, there exist a positive constant* ν*, a bounded (in* $\mathcal{H}^1$*) subset* $\mathbb{B} \subset \mathcal{H}^1$ *and a continuous increasing function* $Q(\cdot) : [0, \infty) \to [0, \infty)$ *such that, for any bounded (in* $\mathcal{H}$*) subset* $B \subset \mathcal{H}$*,*

$$\forall\, \varepsilon \in [0, 1],\ \mathrm{dist}_{\mathcal{H}}(S_\varepsilon(t)B, \mathbb{B}) \leq Q(\|B\|_{\mathcal{H}})e^{-\nu t} \quad \text{for all } t \geq 0,$$

where $\mathbb{B}$*,* ν *and* $Q(\cdot)$ *are all independent of* ε*, and* $\{S_\varepsilon(t)\}_{t\geq 0}$ *is the semigroup generated by* (E_ε) *in* $\mathcal{H}^0_\varepsilon$*.*

This result says that asymptotically, for each (E_ε), the solutions are exponentially approaching a more regular fixed subset $\mathbb{B}$ uniformly (w.r.t. $\varepsilon \in [0, 1]$). Moreover, it implies the following results:

(1) For each $\varepsilon \in [0, 1]$, $\{S_\varepsilon(t)\}_{t\geq 0}$ has a global attractor $\mathcal{A}_\varepsilon$ in $\mathcal{H}$, and
$$\bigcup_{\varepsilon\in[0,1]} \mathcal{A}_\varepsilon \subset cl_{\mathcal{H}^1}(\mathbb{B}).$$
(2) For the case $g \in L^\infty$ (e.g., $g(x) \equiv f(0)$) as considered in Carvalho and Cholewa [6], *Theorem* 2.1 means that we have proved [6, Lemma 3.4] for the critical nonlinear case. Then applying [6, Lemma 3.5 and Lemma 3.6] (which hold certainly for the critical case), we indeed have shown that [6, Theorem 1.3] holds for the critical case.
(3) Based on *Theorem* 2.1, applying the abstract result devised in [9, 14, 22], for each $\varepsilon \in [0, 1]$ we can prove the existence of a finite dimensional exponential attractor $\mathcal{E}_\varepsilon$ in $\mathcal{H}$. Moreover, our attraction is uniform (w.r.t. $\varepsilon \in [0, 1]$) under the $\mathcal{H}$-norm (not only with the $\mathcal{H}^0_\varepsilon$-norm); see *Lemma* 5.10.
(4) Since the global attractor $\mathcal{A}_\varepsilon \subset \mathcal{E}_\varepsilon$, it also implies that the fractal dimension of the global attractor $\mathcal{A}_\varepsilon$ is finite. Moreover, based on *Theorem* 2.1, we show the upper semicontinuity of $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$; see *Lemma* 5.12.

For the proof of *Theorem* 2.1, the main difficulty comes from the critical nonlinearity and the uniformness w.r.t. $\varepsilon \in [0, 1]$.

Part II: $g(x) \in H^{-1}$.

To prove some asymptotic regularity for this case is more than a dilemma. When $g(x) \in H^{-1}$, we know that in general the solution of the elliptic equation $-\Delta u + f(u) = g(x) \in H^{-1}$ with $u|_{\partial\Omega} = 0$ only belongs to $H_0^1(\Omega)$ when $f(\cdot)$ satisfies (2.1)-(2.2). So, in this case, we cannot expect any higher regularity of the attractor (if it exists) than $H_0^1(\Omega) \times H_0^1(\Omega)$, and indeed we will get a result different from *Theorem* 2.1.

In this part, inspired by more recent results in [11, 12, 30], we show that if we shift the solution $(u(t), u_t(t))$ of (E_ε) by a proper (fixed) point $(\phi(x), 0)$, then $(u(t), u_t(t)) - (\phi(x), 0)$ will be bounded in some regular space for t sufficiently large.

For this, besides (2.1)-(2.2), we need to assume additionally that $f(\cdot) \in \mathcal{C}^2$ and satisfies

$$|f''(s)| \leq \begin{cases} C(1 + |s|^{\frac{N+2}{N-2} - 2}) & N = 3, 4, 5, \\ C & N \geq 6, \end{cases} \quad \text{for all } s \in \mathbb{R}, \tag{2.3}$$

and

$$f'(s) \geq -l \quad \text{for all } s \in \mathbb{R}. \tag{2.4}$$

At the same time, from the estimates and calculations given in §§3 and 4, we observe that we only need to estimate for some fixed $\varepsilon_0 \in (0, 1]$, and for the limit case ($\varepsilon = 0$) we refer the reader to [30]. So, without loss of generality, we fix in this part $\varepsilon \equiv 1$, and take the following notation:

Assumption II. *$\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $g(x) \in H^{-1}$ and f satisfies* (2.1)-(2.2) *with $f(0) = 0$ and* (2.3)-(2.4)*; take $\varepsilon \equiv 1$, and denote $S(t) = S_{\varepsilon=1}(t)$ for simplicity.*

The main result of this part is the following theorem.

Theorem 2.2. *Under Assumption II, for each $0 \leq \alpha < \min\{1, \frac{N}{2} - 1\}$, there exist a subset $\mathcal{B}_\alpha$, a positive constant μ and a monotone increasing function $Q_\alpha(\cdot) : [0, \infty) \to [0, \infty)$ such that, for any bounded set $B \subset \mathcal{H}$,*

$$\mathrm{dist}_{\mathcal{H}}(S(t)B, \mathcal{B}_\alpha) \leq Q_\alpha(\|B\|_{\mathcal{H}}) e^{-\mu t} \quad \textit{for all } t \geq 0,$$

where $\mathcal{B}_\alpha$ and $Q_\alpha(\cdot)$ may depend on α, but μ is independent of α, and where $\mathcal{B}_\alpha$ satisfies

$$\mathcal{B}_\alpha = \{z \in \mathcal{H} : \ \|z - (\phi(x), 0)\|_{\mathcal{H}^\alpha} \leq \Lambda_\alpha < \infty\}$$

for some positive constant Λ_α. $\phi(x)$ is the unique solution of the following elliptic equation:

$$\begin{cases} -\Delta\phi + f(\phi) + \eta_0 \phi = g(x), & \textit{in } \Omega, \\ \phi|_{\partial\Omega} = 0, \end{cases} \tag{2.5}$$

where the constant $\eta_0 > 0$ is large enough (will be given precisely in (6.1)-(6.2)*).*

As an immediate result of *Theorem* 2.2, we know that $\{S(t)\}_{t\geq 0}$ is asymptotically smooth (see [18]) and then has a global attractor $\mathcal{A}'$ in $\mathcal{H}$. Moreover, $\mathcal{A}'$ has the decomposition $\mathcal{A}' = (\phi(x), 0) + \mathcal{A}''$ with $\mathcal{A}''$ bounded in $H^{1+\alpha}(\Omega) \times H^{1+\alpha}(\Omega)$ for any $\alpha \in [0, \min\{1, \frac{N}{2} - 1\})$. Furthermore, we can show that if the initial data belongs to $(\phi(x), 0) + \mathcal{H}^\alpha$, then the corresponding solution $(u(t), u_t(t))$ will also lie in $(\phi(x), 0) + \mathcal{H}^\alpha$ for all $t \geq 0$ and $\|\big(u(t) - \phi(x), u_t(t)\big)\|_{\mathcal{H}^\alpha}$ uniformly (w.r.t. time t and initial data) bounded; see *Lemma* 6.6.

Remark 2.3. Comparing with *Assumption I*, *Assumption II* relaxes the regularity of the forcing term $g(x)$ to H^{-1} (which is the weakest forcing term if we work in the weak energy phase space $\mathcal{H}_\varepsilon^0$), but with the price that we require two additional technical assumptions (2.3)-(2.4). Especially, (2.4) is a restriction and stronger than (2.2) to some extent (although it is reasonable for the critical polynomial nonlinearity case). How to remove such technical assumptions would be interesting. At the same time, we also remark that assumptions (2.1)-(2.2) are sufficient for the existence of a compact global attractor in $\mathcal{H}$ for the case $g(x) \in H^{-1}$ (e.g., see [25, 29, 36]).

Hereafter, we will also use the following notation (see, e.g., [13]): denote by $\mathfrak{J}$ the space of continuous increasing functions $J : \mathbb{R}^+ \to \mathbb{R}^+$, and by $\mathfrak{D}$ the space of continuous decreasing functions $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\beta(\infty) < 1$. Moreover, C, C_i are the generic constants, and $Q(\cdot), Q_i(\cdot) \in \mathfrak{J}$ are generic functions, which are all independent of ε; otherwise we will point out clearly. We also denote $\|A\|_X = \sup_{x\in X} \|x\|_X$ for any $A \subset (X, \|\cdot\|_X)$.

3. Preliminaries

In this section, we recall some results used in the main part of the paper.

The first result comes from [13], which will be used to prove the asymptotic regularity for the case $g \in L^2(\Omega)$.

Lemma 3.1 ([13]). *Let X, V be two Banach spaces and $\{T(t)\}_{t\geq 0}$ be a C^0-semigroup on X with a bounded absorbing set $\bar{B} \subset X$. For every $x \in \bar{B}$, assume that there exist two solution operators $V_x(t)$ on X and $U_x(t)$ on V satisfying the following properties:*

i) *For any two vectors $y \in X$ and $z \in V$ satisfying $y + z = x$,*
$$T(t)x = V_x(t)y + U_x(t)z \quad \text{for any } t \geq 0.$$
ii) *There exists $\alpha \in \mathfrak{D}$ such that*
$$\sup_{x\in\bar{B}} \|V_x(t)y\|_X \leq \alpha(t)\|y\|_X, \quad \forall\, y \in \bar{B}.$$
iii) *There are $\beta \in \mathfrak{D}$ and $J \in \mathfrak{J}$ such that*
$$\sup_{x\in\bar{B}} \|U_x(t)z\|_V \leq \beta(t)\|z\|_V + J(t), \quad \forall\, z \in V.$$

Then, there exist positive constants ρ, K, ω such that
$$\mathrm{dist}_X(T(t)\bar{B}, B_V(\rho)) \leq Ke^{-\omega t} \quad \text{for all } t \geq 0,$$
where $B_V(\rho) = \{z \in V : \ \|z\|_V \leq \rho\}$.

Next we recall a criterion for the upper semicontinuity of attractors.

Lemma 3.2 ([18, 26]). *Let $\{T_\lambda(t)\}_{t\geq 0}$ $(\lambda \in \Lambda)$ be a family of semigroups defined on the Banach space X, and for each $\lambda \in \Lambda$, let $\{T_\lambda(t)\}_{t\geq 0}$ have a global attractor $\mathcal{A}_\lambda$. Assume further that λ_0 is a nonisolated point of Λ and that there exist $s > 0$, $t_0 > 0$ and a compact set $K \subset X$ such that*
$$\bigcup_{\lambda\in\mathcal{N}_\Lambda(\lambda_0,s)} \mathcal{A}_\lambda \subset K, \quad \text{and}$$
$$\text{if } \lambda_n \to \lambda_0 \text{ and } x_n \to x_0 \ (x_n \in \mathcal{A}_{\lambda_n} \text{ as } n \neq 0), \text{ then } T_{\lambda_n}(t_0)x_n \to T_{\lambda_0}(t_0)x_0. \tag{3.1}$$

Then the global attractors $\mathcal{A}_\lambda$ are upper semicontinuous on Λ at $\lambda=\lambda_0$; that is,

$$\lim_{\Lambda\ni\lambda\to\lambda_0} \operatorname{dist}_X(\mathcal{A}_\lambda,\, \mathcal{A}_{\lambda_0}) = 0.$$

We also recall a Gronwall-type inequality; for the proof, please see [17].

Lemma 3.3. *Let $\Lambda:\mathbb{R}^+\to\mathbb{R}^+$ be an absolutely continuous function satisfying*

$$\frac{d}{dt}\Lambda(t)+2\eta\Lambda(t)\le h(t)\Lambda(t)+k,$$

where $\eta>0$, $k\ge 0$ and $\int_s^t h(\tau)d\tau<\eta(t-s)+m$ for all $t\ge s\ge 0$ and some $m\ge 0$. Then,

$$\Lambda(t)\le \Lambda(0)e^m e^{-\eta t}+\frac{ke^m}{\eta},\quad \forall\ t\ge 0.$$

4. Uniformly decaying estimates in $\mathcal{H}$

In this section, **we always assume that only** (2.1)-(2.2) **hold, and** $g(x)$ **only belongs to** H^{-1} (so all results obtained in this section certainly hold for the case $g(x)\in L^2(\Omega)$).

The main purpose of this section is to deduce some dissipative estimates about the semigroups $\{S_\varepsilon(t)\}_{t\ge0}$ ($\varepsilon\in[0,1]$) in $\mathcal{H}$.

The existence of a bounded absorbing set for $\{S_0(t)\}_{t\ge0}$ in $H_0^1(\Omega)\times L^2(\Omega)$ was established in many references under the assumptions (2.1)-(2.2); e.g., see [2, 5, 23]. Recently, Pata and Zelik [25] showed further that indeed there is a bounded absorbing set for $\{S_0(t)\}_{t\ge0}$ in $\mathcal{H}$, and the authors in [6, 33] obtained the existence of a bounded absorbing set for each fixed $\varepsilon\in(0,1]$. Here, using the method in [23, 25] for a strongly damped wave equation, we will show that the radius of the absorbing set of $\{S_\varepsilon(t)\}_{t\ge0}$ in $\mathcal{H}$ can be chosen to be independent of $\varepsilon\in[0,1]$.

Lemma 4.1. *There exists a positive constant M, which depends only on $|\Omega|$, $\|g\|_{H^{-1}}$ and the coefficients of (2.1)-(2.2), satisfying that for any $\varepsilon\in[0,1]$ and any bounded (in $\mathcal{H}_\varepsilon^0$) subset $B\subset\mathcal{H}_\varepsilon^0$, there is a $t_B=t(\|B\|_{\mathcal{H}_\varepsilon^0})>0$ (which depends only on the bound of $\|B\|_{\mathcal{H}_\varepsilon^0}$) such that*

$$\|S_\varepsilon(t)z\|_{\mathcal{H}}^2\le M\quad \textit{for all } t\ge t_B \textit{ and all } z\in B,$$

where both t_B and M are independent of $\varepsilon\in[0,1]$.

Proof. Throughout the proof, the generic constants C, C_j ($j=1,2,\cdots$) are independent of ε, and Π_i denotes the projector from $X_1\times X_2$ to X_i, $i=1,2$.

For clarity, we separate the proof into three claims.

Claim 1. There exists an M_1 (independent of B and ε) such that, $\forall\ \varepsilon\in[0,1]$,

$$\|S_\varepsilon(t)B\|_{\mathcal{H}_\varepsilon^0}^2=\|S_\varepsilon(t)B\|_{H_0^1(\Omega)\times L^2(\Omega)}^2+\varepsilon\|\Pi_2 S_\varepsilon(t)B\|_{H_0^1(\Omega)}^2\le M_1\quad \text{as } t\ge T_{1B}, \tag{4.1}$$

where $T_{1B}=T_1(\|B\|_{\mathcal{H}_\varepsilon^0})$ depends on $\|B\|_{\mathcal{H}_\varepsilon^0}$ but not on ε.

Multiplying (E_ε) by $u_t+\theta u$ (here and after, note that the multiplication holds in a Faedo-Galerkin scheme; however, due to the global well-posed result given in [6], the estimates hold in the limit) with $\theta\ll 1$, which will be determined later, we obtain that

$$\frac{d}{dt}E_{1u}(t)+2G_{1u}(t)\le 2\|\nabla u_t\|\|g\|_{H^{-1}}+2\theta\|\nabla u\|\|g\|_{H^{-1}}, \tag{4.2}$$

where

$$E_{1u}(t)=\|u_t(t)+\theta u(t)\|^2+\varepsilon\|\nabla(u_t(t)+\theta u(t))\|^2+(1+\theta-\varepsilon\theta^2)\|\nabla u(t)\|^2 \\ -\theta^2\|u(t)\|^2+2\int_\Omega F(u(t))dx,$$

$$G_{1u}(t)=(1-\varepsilon\theta)\|\nabla u_t(t)\|^2-\theta\|u_t(t)\|^2+\theta\|\nabla u(t)\|^2+\theta\int_\Omega f(u(t))u(t)dx$$

and

$$F(u)=\int_0^{u(x,t)}f(s)ds.$$

Then, from assumptions (2.1)-(2.2) and using Poincaré's inequality, we have

$$E_{1u}(t)\le C_1\big(\|u_t\|^2+\varepsilon\|\nabla u_t\|^2\big)+C_2\big(1+\|\nabla u\|^{\frac{2N}{N-2}}\big) \tag{4.3}$$

and

$$F(s)\ge -\frac{\lambda}{2}s^2-c_1 \quad \text{for all } s\in\mathbb{R} \text{ with some } \lambda\in(0,\lambda_1). \tag{4.4}$$

Noting that $\|x+y\|^2\ge\frac{1}{2}\|x\|^2-\|y\|^2$ holds for any $x,y\in L^2(\Omega)$, we have

$$E_{1u}(t)\ge\frac{1}{2}(\|u_t\|^2+\varepsilon\|\nabla u_t\|^2)+(1+\theta-\theta^2-\varepsilon\theta^2-\frac{2\theta^2+\lambda}{\lambda_1})\|\nabla u\|^2-2c_1|\Omega|, \tag{4.5}$$

where the positive constant c_1 depends only on $f(\cdot)$ (from (4.4)).

For $G_{1u}(t)$, we have

$$G_{1u}(t)\ge(1-\theta-\frac{\theta}{\lambda_1})\|\nabla u_t\|^2+\theta(1-\frac{\lambda}{\lambda_1})\|\nabla u\|^2-\theta c_2|\Omega|, \tag{4.6}$$

where $\lambda\in(0,\lambda_1)$ comes from (2.2) and the constant c_2 depends only on $f(\cdot)$.

At the same time, by the Cauchy-Schwarz inequality, we have

$$2\|\nabla u_t\|\|g\|_{H^{-1}}+2\theta\|\nabla u\|\|g\|_{H^{-1}}\le(1-\theta-\frac{\theta}{\lambda_1})\|\nabla u_t\|^2+\theta(1-\frac{\lambda}{\lambda_1})\|\nabla u\|^2+C_\theta\|g\|^2_{H^{-1}}. \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.2), we obtain

$$\frac{d}{dt}E_{1u}(t)+(1-\theta-\frac{\theta}{\lambda_1})\|\nabla u_t\|^2+\theta(1-\frac{\lambda}{\lambda_1})\|\nabla u\|^2\le C_\theta\|g\|^2_{H^{-1}}+2\theta c_2|\Omega|. \tag{4.8}$$

Hence, we first take θ small enough such that

$$1+\theta-2\theta^2-\frac{2\theta^2+\lambda}{\lambda_1}>0 \text{ and } 1-\theta-\frac{\theta}{\lambda_1}>0, \tag{4.9}$$

and then applying the Gronwall-type inequality, [23, Lemma 1], to (4.8), and combining with (4.3) and (4.5), we have

$$E_{1u}(t)\le\sup_{\varepsilon\in[0,1]}\sup_{z\in\mathcal{H}^0_\varepsilon}\Big\{E_\varepsilon(z):\delta\|z\|^2_{\mathcal{H}^0_\varepsilon}\le 2C_\theta\|g\|^2_{H^{-1}}+4c_2|\Omega|\Big\} \text{ for all } (u_0,v_0)\in B \tag{4.10}$$

provided that

$$t \geq T_{1B} = \frac{2c_1|\Omega| + C_1\|B\|_{\mathcal{H}_\varepsilon^0} + C_2(1+\|B\|_{\mathcal{H}_\varepsilon^0}^{\frac{2N}{N-2}})}{C_\theta\|g\|_{H^{-1}}^2 + 2c_2|\Omega|},$$

where $\delta = \min\{(1-\theta-\frac{\theta}{\lambda_1}), \theta(1-\frac{\lambda}{\lambda_1})\}$ and (from (4.3))

$$E_\varepsilon(z) := C_1\|z\|_{\mathcal{H}_\varepsilon^0}^2 + C_2(1+\|z\|_{\mathcal{H}_\varepsilon^0}^2)^{\frac{N}{N-2}}$$

for any $z \in \mathcal{H}_\varepsilon^0$.

Then, noticing (4.5) and (4.9), *Claim* 1 follows from (4.10) immediately.

Claim 2. There exists an M_2 (independent of B and ε) such that

$$\forall\ \varepsilon \in [0,1],\ \int_{T_{1B}}^{\infty} \|\Pi_2 S_\varepsilon(s)B\|_{H_0^1(\Omega)}^2 ds \leq M_2, \tag{4.11}$$

where T_{1B} is given in *Claim* 1.

Multiplying (E_ε) by u_t, we have

$$\frac{1}{2}\frac{d}{dt}\big(\|u_t\|^2+\|\nabla u\|^2+\varepsilon\|\nabla u_t\|^2+2\int_\Omega F(u)dx-2\langle g(x),u\rangle_{H_0^1(\Omega),H^{-1}}\big)+\|\nabla u_t\|^2 \leq 0. \tag{4.12}$$

Then, for any $t \geq T_{1B}$, integrating (4.12) over $[T_{1B}, t]$ and using *Claim* 1, we have

$$\int_{T_{1B}}^{t} \|\Pi_2 S_\varepsilon(s)B\|_{H_0^1(\Omega)}^2 ds \leq 2M_1 + 2\|g\|_{H^{-1}}^2 + 2C'(|\Omega| + M_1^{\frac{N}{N-2}}),$$

where the constant C' depends only on the constant C in (2.1).

Claim 3. There exists an M_3 (independent of B and ε) such that

$$\forall\ \varepsilon \in [0,1],\ \|\Pi_2 S_\varepsilon(t)B\|_{H_0^1(\Omega)}^2 \leq M_3 \quad \text{as } t \geq T_{1B}+1. \tag{4.13}$$

Similar to the proof of [25, Lemma 3.5], multiplying (E_ε) by u_{tt}, we have

$$\frac{d}{dt}E_{2u}(t) + 2\|u_{tt}\|^2 + 2\varepsilon\|\nabla u_{tt}\|^2 = 2\|\nabla u_t\|^2 + 2\langle f'(u)u_t, u_t\rangle, \tag{4.14}$$

where $\langle\cdot,\cdot\rangle$ is the L^2-inner product and

$$E_{2u}(t) = \|\nabla u_t(t)\|^2 + 2\langle\nabla u, \nabla u_t\rangle + 2\langle f(u), u_t\rangle - 2\langle g(x), u_t\rangle_{H_0^1(\Omega),H^{-1}}.$$

Then, as $t \geq T_{1B}$, using *Claim* 1 and (2.1), we have

$$\frac{d}{dt}E_{2u}(t) + 2\|u_{tt}\|^2 + 2\varepsilon\|\nabla u_{tt}\|^2 \leq C(1+M_1^{\frac{N}{N-2}})\|\nabla u_t\|^2 \tag{4.15}$$

and

$$\frac{1}{2}\|\nabla u_t\|^2 - C_1(1+M_1^{\frac{N}{N-2}}+\|g\|_{H^{-1}}^2) \leq E_{2u}(t) \leq 2\|\nabla u_t\|^2 + C_2(1+M_1^{\frac{N}{N-2}}+\|g\|_{H^{-1}}^2). \tag{4.16}$$

On the other hand, from *Claim* 2 we know that for each $(u_0, v_0) \in B$, there is a time $t_0 \in [T_{1B}, T_{1B}+1]$ such that

$$\|\Pi_2 S_\varepsilon(t_0)(u_0,v_0)\|_{H_0^1(\Omega)}^2 \leq M_2, \tag{4.17}$$

where t_0 depends on (u_0, v_0).

Therefore, as $t \geq T_{1B}+1$, for each $(u_0, v_0) \in B$, integrating (4.15) over $[t_0, t]$ and applying (4.16)-(4.17), we obtain that

$$\frac{1}{2}\|\nabla u_t(t)\|^2 + 2\int_{T_{1B}+1}^{t} (\|u_{tt}(s)\|^2 + \varepsilon\|\nabla u_{tt}(s)\|^2)ds$$
$$\leq 2M_2 + (C_1 + C_2)(1 + M_1^{\frac{N}{N-2}} + \|g\|^2_{H^{-1}}) + C(1 + M_1^{\frac{N}{N-2}})M_2;$$

this show that *Claim* 3 holds.

Now, we can complete our proof by taking

$$M = M_1 + M_3 \quad \text{and } t_B = T_{1B} + 1.$$

$\square$

Remark 4.2. Observe that from (4.1), (4.8), (4.3) and (4.5) we can also deduce that, for any $\varepsilon \in [0, 1]$ and any $B \subset \mathcal{H}^0_\varepsilon$,

$$\|S_\varepsilon(t)B\|^2_{\mathcal{H}^0_\varepsilon} \leq Q(\|B\|_{\mathcal{H}^0_\varepsilon}) \quad \text{for all } t \geq 0, \tag{4.18}$$

where $Q(\cdot) \in \mathfrak{J}$ is independent of B and ε.

Moreover, if B is bounded in $\mathcal{H}$, then we can obtain

$$\forall\ \varepsilon \in [0, 1], t \geq 0,\ \|S_\varepsilon(t)B\|^2_{\mathcal{H}} \leq C_{\|B\|_{\mathcal{H}}} \tag{4.19}$$

for some constant $C_{\|B\|_{\mathcal{H}}}$ which depends only on $\|B\|_{\mathcal{H}}$. Indeed, from the fact that there is a constant c_1 such that $c_1\|\cdot\|_{\mathcal{H}} \geq \|\cdot\|_{\mathcal{H}^0_\varepsilon}$ for any $\varepsilon \in [0, 1]$, (4.19) can be obtained just by repeating the proof of *Lemma* 4.1 and taking $t_0 = 0$ in (4.17) since B is bounded in $\mathcal{H}$.

On the other hand, from the proof of *Claim* 3 above, we can get further estimates about u_{tt}:

$$\forall\ \varepsilon \in [0, 1],\ \int_{T_{1B}+1}^{\infty} (\|u_{tt}(s)\|^2 + \varepsilon\|\nabla u_{tt}(s)\|^2)ds \leq M_3 \quad \text{for all } (u_0, v_0) \in B. \tag{4.20}$$

Then, similar to [25, Lemma 3.6], we indeed can deduce the following estimates:

Lemma 4.3. *There exists an M_4 such that for any $\varepsilon \in [0, 1]$ and any bounded (in $\mathcal{H}^0_\varepsilon$) subset $B \subset \mathcal{H}^0_\varepsilon$,*

$$\|u_{tt}(t)\|^2 + \varepsilon\|\nabla u_{tt}(t)\|^2 + \int_{T_{1B}+2}^{t} \|\nabla u_{tt}(s)\|^2 ds \leq M_4 \quad \textit{for all } t \geq T_{1B} + 2,$$

where $(u(t), u_t(t)) = S_\varepsilon(t)(u_0, v_0)$ $((u_0, v_0) \in B)$, T_{1B} is the time given in Claim 1, and M_4 is independent of B and ε.

For later applications, we present some Hölder continuity of $\{S_\varepsilon(t)\}_{t\geq 0}$ in $\mathcal{H}^0_\varepsilon$, which has been obtained in [6] for each fixed $\varepsilon \in (0, 1]$ and [23] for $\varepsilon = 0$.

Lemma 4.4. *For any $\varepsilon \in [0, 1]$ and any bounded (in $\mathcal{H}^0_\varepsilon$) subset $B \subset \mathcal{H}^0_\varepsilon$, there is a constant $C_{\|B\|_{\mathcal{H}^0_\varepsilon}}$ which depends only on $\|B\|_{\mathcal{H}^0_\varepsilon}$ such that*

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{\mathcal{H}^0_\varepsilon} \leq e^{C_{\|B\|_{\mathcal{H}^0_\varepsilon}} t}\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon},\ \forall\ t \geq 0,\ z_i \in B \tag{4.21}$$

and

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{\mathcal{H}} \leq e^{C_{\|B\|_{\mathcal{H}^0_\varepsilon}} t}\|z_1 - z_2\|^{\frac{1}{2}}_{\mathcal{H}^0_\varepsilon},\ \forall\ t \geq T_{1B} + 2,\ z_i \in B. \tag{4.22}$$

Proof. Let $(u^i(t), u^i_t(t))$ be the solution of (E_ε) corresponding to the initial data $z_i \in B$. Then the difference $\bar{u} = u^1 - u^2$ satisfies

$$\bar{u}_{tt} - \Delta\bar{u}_t - \Delta\bar{u} - \varepsilon\Delta\bar{u}_{tt} + f(u^1) - f(u^2) = 0 \tag{4.23}$$

with initial data $(\bar{u}(0), \bar{u}_t(0)) = z_1 - z_2$.

Then, as that in [23], we can obtain (4.21) through multiplying (4.23) by $\bar{u}_t$ (where we need to use (4.18)).

For (4.22), when $t \geq T_{1B} + 2$, we have

$$\|\nabla\bar{u}_t(t)\|^2 \leq \|\bar{u}_{tt}\|\|\bar{u}_t\| + \|\nabla\bar{u}\|\|\nabla\bar{u}_t\| + \varepsilon\|\nabla\bar{u}_{tt}\|\|\nabla\bar{u}_t\| + C_M\|\nabla\bar{u}\|\|\nabla\bar{u}_t\|,$$

and then, combining with *Lemma* 4.3 and (4.21), we have

$$\begin{aligned}\frac{1}{4}\|\nabla\bar{u}_t(t)\|^2 &\leq \sqrt{M_4}(\|\bar{u}_t\| + \sqrt{\varepsilon}\|\nabla\bar{u}_t\|) + C_M\|\nabla\bar{u}\|^2\\ &\leq \sqrt{M_4}e^{C_{\|B\|_{\mathcal{H}^0_\varepsilon}}t}\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon} + C_M e^{2C_{\|B\|_{\mathcal{H}^0_\varepsilon}}t}\|z_1 - z_2\|^2_{\mathcal{H}^0_\varepsilon}\\ &\leq C_{M,M_4}e^{C_M t}\|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon} \text{ as } t \geq T_{1B} + 2,\end{aligned}$$

which, combining with (4.21) again, implies (4.22) immediately. □

Hereafter, we denote the uniformly (w.r.t. $\varepsilon \in [0,1]$) bounded absorbing set obtained in *Lemma* 4.1 as B_0, i.e.,

$$B_0 = \{z \in \mathcal{H} : \ \|z\|^2_{\mathcal{H}} \leq M\}, \tag{4.24}$$

and denote by Λ_0 the time such that *Lemma* 4.1 and *Lemma* 4.3 hold for B_0; i.e.,

$$\|S_\varepsilon(t)B_0\|^2_{\mathcal{H}} + \|u_{tt}(t)\|^2 + \varepsilon\|\nabla u_{tt}(t)\|^2 + \int_{T_{1B}+2}^t \|\nabla u_{tt}(s)\|^2 ds \leq \bar{M} = M + M_4 \tag{4.25}$$

holds for any $\varepsilon \in [0,1]$ and all $t \geq \Lambda_0$. Moreover, similar to *Remark* 4.2, noting now that B_0 is bounded in $\mathcal{H}$, we have

$$\forall\ \varepsilon \in [0,1],\ \|S_\varepsilon(t)B_0\|^2_{\mathcal{H}} \leq C_M \quad \text{for all } t \geq 0. \tag{4.26}$$

5. Part I: $g(x) \in L^2(\Omega)$

Throughout this section, we always (only) assume that Assumption I holds.

5.1. **Decomposition of the equation.** For the nonlinear function f satisfying (2.1)-(2.2), from [1] (see also [13, 23, 28, 35] for our situation) we know that f allows the following decomposition $f = f_0 + f_1$, where $f_0, f_1 \in \mathcal{C}^1(\mathbb{R})$ and satisfy

$$|f_0(u)| \leq C|u|^{\frac{N+2}{N-2}} \quad \text{for all } u \in \mathbb{R}, \tag{5.1}$$

$$f_0(u)u \geq 0 \quad \text{for all } u \in \mathbb{R}, \tag{5.2}$$

$$|f_1(u)| \leq C(1 + |u|) \quad \text{for all } u \in \mathbb{R}, \tag{5.3}$$

$$\liminf_{|u|\to\infty} \frac{f_1(u)}{u} > -\lambda_1. \tag{5.4}$$

For example, from (2.2) we know that there are $s_1 \geq 0$ and $\lambda < \lambda_1$ such that

$$f(u)u \geq -\lambda u^2 \quad \text{as } |u| \geq s_1,$$

and from (2.1) we know that there is a constant $C_1 > 0$ such that

$$|f(u)| \leq C_1(1 + |u|^{\frac{N+2}{N-2}}) \leq 2C_1|u|^{\frac{N+2}{N-2}} \quad \text{as } |u| \geq 1.$$

Take the cutoff function $\varphi(\cdot):[0,\infty)\to[0,1]$ as follows:

$$\varphi(s)=\begin{cases}1 & \text{as } s\geq s_1+2;\\ 0 & \text{as } s\leq s_1+1.\end{cases}$$

Then, we can take $f_0(s)=\varphi(|s|)\big(f(s)+\lambda s\big)$ and $f_1(s)=f(s)-f_0(s)$, $\forall\ s\in\mathbb{R}$.

We will follow the idea (method) in [23, 28, 35, 37] to deduce the asymptotic regularity. Decomposing the solution $S_\varepsilon(t)(u_0,v_0)=(u(t),u_t(t))$ into the sum

$$S_\varepsilon(t)(u_0,v_0)=D_\varepsilon(t)(u_0,v_0)+K_\varepsilon(t)(u_0,v_0) \tag{5.5}$$

for any $t\geq 0$ and any $(u_0,v_0)\in\mathcal{H}$, where $D_\varepsilon(t)(u_0,\,v_0)=(v(t),v_t(t))$ and $K_\varepsilon(t)(u_0,\,v_0)=(w(t),w_t(t))$ are the solution of the following equations:

$$\begin{cases}v_{tt}-\Delta v_t-\Delta v-\varepsilon\Delta v_{tt}+f_0(v)=0 \text{ in } \Omega\times[0,\infty),\\ (v(0),\,v_t(0))=(u_0,\,v_0),\ v|_{\partial\Omega}=0,\end{cases} \tag{5.6}$$

and

$$\begin{cases}w_{tt}-\Delta w_t-\Delta w-\varepsilon\Delta w_{tt}+f(u)-f_0(v)=g(x) \text{ in } \Omega\times[0,\infty),\\ (w(0),\,w_t(0))=(0,\,0),\ w|_{\partial\Omega}=0.\end{cases} \tag{5.7}$$

Applying the general results in [6], we know that both (5.6) and (5.7) are global well-posed in $\mathcal{H}$, and $\{D_\varepsilon(t)\}_{t\geq 0}$ also forms a semigroup.

Moreover, as in §3, we can deduce a similar estimate for $\{D_\varepsilon(t)\}_{t\geq 0}$ in $\mathcal{H}$, and so $\{K_\varepsilon(t)\}_{t\geq 0}$ (from (5.5)): There exist constants C_M and Λ_1 such that for any $\varepsilon\in[0,1]$ and any $(u_0,v_0)\in B_0$,

$$\|D_\varepsilon(t)B_0\|_{\mathcal{H}}^2+\|v_{tt}(t)\|^2+\varepsilon\|\nabla v_{tt}(t)\|^2+\int_{T_{1B}+2}^t\|\nabla v_{tt}(s)\|^2ds\leq\bar{M} \text{ as } t\geq\Lambda_1 \tag{5.8}$$

and

$$\forall\ \varepsilon\in[0,1],\ \|D_\varepsilon(t)B_0\|_{\mathcal{H}}^2+\|K_\varepsilon(t)B_0\|_{\mathcal{H}}^2\leq C_M \quad \text{for all } t\geq 0. \tag{5.9}$$

5.2. **The first a priori estimate.** We begin with the decay estimate for the solution of (5.6).

Lemma 5.1. *There exist a constant $k>0$ and $Q(\cdot)\in\mathfrak{J}$ such that*

$$\|D_\varepsilon(t)B_0\|_{\mathcal{H}}^2\leq Q(\|B_0\|_{\mathcal{H}})e^{-kt} \quad \textit{for all } t\geq 0 \textit{ and any } \varepsilon\in[0,1],$$

where both k and $Q(\cdot)$ are independent of $\varepsilon\in[0,1]$.

Proof. Multiplying (5.6) by $v_t+\theta v$, we have

$$\frac{d}{dt}E_{3v}(t)+2\big(\theta\|\nabla v\|^2+\|\nabla v_t\|^2+\theta\int_\Omega vf_0(v)dx\big)=2\langle v_t+\theta v,\theta v_t-\varepsilon\theta\Delta v_t\rangle, \tag{5.10}$$

where $E_{3v}(t)=\|v_t+\theta v\|^2+(1+\theta)\|\nabla v\|^2+\varepsilon\|\nabla(v_t+\theta v)\|^2+2\int_\Omega F_0(v)dx$ and $F_0(v)=\int_0^v f_0(s)ds$.

Then, using (5.2), we have

$$\begin{aligned}\frac{1}{2}\|v_t\|^2&+(1+\theta-\theta^2-\frac{\theta^2}{\lambda_1})\|\nabla v\|^2+\varepsilon\|\nabla v_t\|^2\\ &\leq E_{3v}(t)\leq 2\|v_t\|^2+2\varepsilon\|\nabla v_t\|^2+C_{\theta,\lambda_1}\|\nabla v\|^2+2\int_\Omega F_0(v)dx\end{aligned} \tag{5.11}$$

and
(5.12)
$$|2\langle v_t+\theta v, \theta v_t-\varepsilon\theta\Delta v_t\rangle| \leq 2\theta\big(\|v_t\|^2+\|\nabla v_t\|^2+\theta\|v\|\|v_t\|+\theta\|\nabla v_t\|\|\nabla v_t\|\big).$$

Note that, from (5.1) and (5.9), we have

$$0 \leq \int_\Omega F_0(v)dx \leq C(\|v\|^2+\|v\|^{\frac{2N}{N-2}}) \leq C_M\|\nabla v\|^2. \tag{5.13}$$

Hence, by taking θ small enough, we can deduce from (5.10) that

$$\frac{d}{dt}E_{3v}(t)+C_{M,\lambda_1,\theta}E_{3v}(t) \leq 0,$$

where the constant $C_{M,\lambda_1,\theta}$ depends on M, λ_1 and θ, but not on ε, which, combining with (5.11) and (5.13), implies that

$$\begin{aligned}\frac{1}{2}\|v_t(t)\|^2&+(1+\theta-\theta^2-\frac{\theta^2}{\lambda_1})\|\nabla v(t)\|^2+\varepsilon\|\nabla v_t(t)\|^2\\ &\leq e^{-C_{M,\lambda_1,\theta}t}\big(\frac{4}{\lambda_1}\|\nabla v_t(0)\|^2+(1+\theta+2\theta^2+2C_M)\|\nabla v(0)\|^2\big).\end{aligned} \tag{5.14}$$

Now, to complete our proof, we multiply (5.6) by v_t and obtain

$$\begin{aligned}\|\nabla v_t\|^2 &\leq -\langle v_{tt}, v_t\rangle+\langle\Delta v, v_t\rangle+\varepsilon\langle\Delta v_{tt}, v_t\rangle-\langle f_0(v), v_t\rangle\\ &\leq \|v_{tt}\|\|v_t\|+\|\nabla v\|\|\nabla v_t\|+\varepsilon\|\nabla v_{tt}\|\|\nabla v_t\|+C\|v\|\|v_t\|+C\|\nabla v\|^{\frac{N+2}{N-2}}\|\nabla v_t\|,\end{aligned}$$

which, combining with (5.8)-(5.9), implies that

$$\|\nabla v_t\|^2 \leq 2\sqrt{\bar{M}}(\|v_t\|+\sqrt{\varepsilon}\|\nabla v_t\|)+C_{\lambda_1,M}\|\nabla v\|^2 \quad \text{as } t \geq \Lambda_1.$$

Therefore, combining with the estimates (5.14), we can finally deduce that

$$\|\nabla v_t\|^2 \leq C_{\bar{M},M,\lambda_1,\theta}e^{-\frac{C_{M,\lambda_1,\theta}}{2}t}+C_{\lambda_1,M}\|\nabla v\|^2 \text{ as } t \geq \Lambda_1,$$

which, combining with (5.14) again for the estimate of $\|\nabla v(t)\|^2$ and using *Lemma* 5.2 below with (5.9), allows us to complete our proof by taking $k=\frac{C_{M,\lambda_1,\theta}}{2}$ and some increasing function $Q(\cdot)$. □

Lemma 5.2. *Let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on the Banach space X, satisfying*

$$\|S(t)B\|_X \leq Q_1(\|B\|_X)e^{-\mu t} \text{ as } t \geq t_0, \text{ and } \|\{S(t)B:\ t \geq 0\}\|_X \leq Q_2(\|B\|_X).$$

Then

$$\|S(t)B\|_X \leq Q_3(\|B\|_X, t_0)e^{-\mu t} \quad \text{for all } t \geq 0.$$

Its proof is obvious and we omit it here.

The next estimate is about the solution of (5.7):

Lemma 5.3. *There exist $k_1 > 0$ and $Q(\cdot) \in \mathfrak{J}$ such that for any $t \geq 0$ and any $\varepsilon \in [0,1]$,*

$$\|K_\varepsilon(t)B_0\|^2_{\mathcal{H}^\sigma_\varepsilon} \leq Q(\|B_0\|_{\mathcal{H}})e^{k_1 t},$$

where both k_1 and $Q(\cdot)$ are independent of $\varepsilon \in [0,1]$, and $\sigma=\frac{1}{2}\min\{1, \frac{N}{2}-1\}$.

Proof. Multiplying (5.7) by $A^\sigma w_t(t)$ (recall that $A=-\Delta$), then the proof is the same as that in [28, Lemma 4.2]. □

Based on *Lemma* 5.1 and *Lemma* 5.3, following the idea in Zelik [37], we can now decompose $u(t)$ as follows (the proof is completely similar to that in [28, 35, 37] since the estimates in *Lemmas* 5.1 and 5.3 hold uniformly w.r.t. $\varepsilon \in [0,1]$):

Lemma 5.4. *Let $(u(t), u_t(t))$ be the solution of (E_ε) corresponding to the initial data $(u_0, v_0) \in B_0$. Then, for any $\eta > 0$, we can decompose $(u(t), u_t(t)) = S_\varepsilon(t)(u_0, v_0)$ as*

$$u(t) = v_1(t) + w_1(t), \quad \text{for all } t \geq 0,$$

where $v_1(t)$ and $w_1(t)$ satisfy the following estimates:

$$\int_s^t \|\nabla v_1(\tau)\|^2 d\tau \leq \eta(t-s) + C_\eta \text{ for all } t \geq s \geq 0,$$

and

$$\|A^{\frac{1+\sigma}{2}} w_1(t)\|^2 \leq K_\eta \text{ for all } t \geq 0$$

with the constants C_η and K_η depending on η, $\|B_0\|_{\mathcal{H}}$ and $\|g\|$, but both are independent of $\varepsilon \in [0,1]$.

5.3. **The second a priori estimate.** The main purpose of this subsection is to deduce some uniformly asymptotic (w.r.t. $\varepsilon \in [0,1]$ and time t) a priori estimates about the solution of (E_ε).

Lemma 5.5. *There exist positive constants $\bar{\nu}, \bar{R} > 0$ and $Q_1(\cdot) \in \mathfrak{J}$ such that for each $\varepsilon \in [0,1]$, there is a subset $\bar{B}_\varepsilon \subset \mathcal{H}^1_\varepsilon$ satisfying*

$$\|\bar{B}_\varepsilon\|^2_{\mathcal{H}^1_\varepsilon} = \sup_{(u,v) \in \bar{B}_\varepsilon} \{\|\Delta u\|^2 + \|\nabla v\|^2 + \varepsilon \|\Delta v\|^2\} \leq \bar{R} \tag{5.15}$$

and the exponential attraction

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t) B_0,\, \bar{B}_\varepsilon) \leq Q_1(\|B_0\|_{\mathcal{H}}) e^{-\bar{\nu} t} \quad \text{for all } t \geq 0. \tag{5.16}$$

Here, all $\bar{\nu}, \bar{R}$ and $Q_1(\cdot)$ are independent of $\varepsilon \in [0,1]$, and $\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(\cdot, \cdot)$ denotes the Hausdorff semidistance with respect to the $\mathcal{H}^0_\varepsilon$-norm.

This lemma shows some asymptotic regularity of $\{S_\varepsilon(t)\}_{t \geq 0}$ for each fixed $\varepsilon \in [0,1]$. Combining with the attraction transitivity lemma established in [15, Theorem 5.1], there are at least two ways to prove this lemma: one is as that in [28, 35] to apply the idea introduced in Zelik [37]; another one is the method introduced recently in Conti and Pata [13]. Here we will use the method in [13].

Proof of Lemma 5.5. It is convenient to separate our proof into three steps. We emphasize especially that all the generic constants in the proof are independent of ε.

Step 1. We first claim that (recall $\sigma = \frac{1}{2}\min\{1, \frac{N}{2} - 1\}$)

$\exists\ \nu_\sigma, R_\sigma > 0$ *and $Q_\sigma(\cdot) \in \mathfrak{J}$ such that for each $\varepsilon \in [0,1]$, there is a subset $\bar{B}_{\sigma,\varepsilon} \subset \mathcal{H}^\sigma_\varepsilon$ satisfying*

$$\|\bar{B}_{\sigma,\varepsilon}\|^2_{\mathcal{H}^\sigma_\varepsilon} = \sup_{(u,v) \in \bar{B}_{\sigma,\varepsilon}} \{\|A^{\frac{1+\sigma}{2}} u\|^2 + \|A^{\frac{\sigma}{2}} v\|^2 + \varepsilon \|A^{\frac{1+\sigma}{2}} v\|^2\} \leq R_\sigma$$

and the exponential attraction

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t) B_0,\, \bar{B}_{\sigma,\varepsilon}) \leq Q_\sigma(\|B_0\|_{\mathcal{H}}) e^{-\nu_\sigma t} \quad \text{for all } t \geq 0.$$

We will apply *Lemma* 3.1 with $X = \mathcal{H}^0_\varepsilon$ and $V = \mathcal{H}^\sigma_\varepsilon$ (note that $B_0 \subset \mathcal{H}^0_\varepsilon$ for any $\varepsilon \in [0,1]$).

Based on *Lemmas* 5.1 and 5.3, the proof of the above claim is completely similar to that in [13] for a strongly damped wave equation. From (5.1) we can write

$$f_0(s) = s\varphi(s) \quad \text{with } |\varphi(s)| \le C|s|^{\frac{4}{N-2}}. \tag{5.17}$$

For any $x \in B_0$ and $y \in \mathcal{H}_\varepsilon^0, z \in \mathcal{H}_\varepsilon^\sigma$ satisfying $x = y + z$, we decompose the solution of (E_ε) as $S_\varepsilon(t)x = V_x^\varepsilon(t)y + U_x^\varepsilon(t)z$, where

$$V_x^\varepsilon(t)y = (\bar{v}(t), \bar{v}_t(t)) \quad \text{and} \quad U_x^\varepsilon(t)z = (\bar{w}(t), \bar{w}_t(t)),$$

which uniquely solve the following equations, respectively:

$$\begin{cases} \bar{v}_{tt} - \Delta\bar{v}_t - \Delta\bar{v} - \varepsilon\Delta\bar{v}_{tt} = h_1, \\ \bar{v}|_{\partial\Omega} = 0, \ (\bar{v}(0), \bar{v}_t(0)) = y, \end{cases} \tag{5.18}$$

and

$$\begin{cases} \bar{w}_{tt} - \Delta\bar{w}_t - \Delta\bar{w} - \varepsilon\Delta\bar{w}_{tt} = h_2, \\ \bar{w}|_{\partial\Omega} = 0, \ (\bar{w}(0), \bar{w}_t(0)) = z, \end{cases} \tag{5.19}$$

with

$$h_1 = -\bar{v}\varphi(v) \quad \text{and } h_2 = g(x) - f(u) + \bar{v}\varphi(v), \tag{5.20}$$

and $v(t)$ is the solution of (5.6) corresponding to the initial data x.

From (5.1), (5.3), (5.17) and *Lemmas* 5.1, 5.3, we can directly calculate that

$$\|h_1\|_{L^{\frac{2N}{N+2}}} \le C\|\nabla\bar{v}\|\|\nabla v\|^{\frac{4}{N-2}} \le C_M e^{-k't}\|\nabla\bar{v}\| \tag{5.21}$$

and similarly

$$\|h_2\|_{L^{\frac{2N}{N+2(1-\sigma)}}} \le C_M e^{-k't}\|A^{\frac{1+\sigma}{2}}\bar{w}\| + C_M e^{k_1' t}, \tag{5.22}$$

where we only have used the embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, $D(A^{\frac{1+\sigma}{2}}) \hookrightarrow L^{\frac{2N}{N-2(1+\sigma)}}(\Omega)$ and $D(A^{\frac{1-\sigma}{2}}) \hookrightarrow L^{\frac{2N}{N-2(1-\sigma)}}(\Omega)$ (which are independent of ε).

Hence, multiplying (5.18) and (5.19) respectively by $\bar{v}_t + \theta\bar{v}$ and $A^\sigma(\bar{w}_t + \theta\bar{w})$, through some similar calculations as that in the proof of *Lemma* 4.1 (see also the proof of [13, Theorem 4.3]), we can verify that all the conditions of *Lemma* 3.1 are satisfied for the case $X = \mathcal{H}_\varepsilon^0$, $V = \mathcal{H}_\varepsilon^\sigma$ and $T(t) = S_\varepsilon(t)$. Moreover, since there is a $c_1 > 0$ (independent of ε) such that $c_1\|B_0\|_{\mathcal{H}} \ge \|B_0\|_{\mathcal{H}_\varepsilon^0}$ for any $\varepsilon \in [0,1]$ and the constants in our estimates are all independent of ε, consequently, ν_σ, R_σ and $Q_\sigma(\cdot)$ are all independent of $\varepsilon \in [0,1]$, we can then deduce our claim.

Step 2. We claim that

there exists a positive constant $\bar{R}_\sigma$ which depends only on R_σ such that

$$\forall\ \varepsilon \in [0,1], \ \|S_\varepsilon(t)\bar{B}_{\sigma,\varepsilon}\|^2_{\mathcal{H}_\varepsilon^\sigma} \le \bar{R}_\sigma \quad \textit{for all } t \ge 0.$$

This claim can be proved completely similar to that in [28, Lemma 4.5] via multiplying (E_ε) by $A^\sigma(u_t + \theta u)$, and applying *Lemma* 5.4 to overcome the critical nonlinearity.

Step 3. Based on *Steps 1 and 2*, applying the attraction transitivity lemma given in [15, Theorem 5.1] and noticing the Hölder continuity *Lemma* 4.4, we can prove our lemma by performing a bootstrap argument, whose proof is now simple since *Step 1* makes the nonlinear term become subcritical to some extent (e.g., see [29] for some similar calculations). □

5.4. **Proof of Theorem 2.1.** *Lemma* 5.5 has shown some asymptotic regularity; however, the radius of $\|\bar{B}_\varepsilon\|_{\mathcal{H}^1}$ depends on ε and the distance only under the $\mathcal{H}^0_\varepsilon$-norm.

To prove *Theorem* 2.1, we first give two lemmas as preliminary.

Lemma 5.6. *There exists $R_1 > 0$ such that for any bounded (in $\mathcal{H}^1_\varepsilon$) subset $B \subset \mathcal{H}^1_\varepsilon$, there is a $T_1 = T_1(\|B\|_{\mathcal{H}^1_\varepsilon})$ such that*

$$\forall\ \varepsilon \in [0,1],\ \|S_\varepsilon(t)B\|^2_{\mathcal{H}^1_\varepsilon} \le R_1 \quad \textit{for all } t \ge T_1.$$

Proof. Multiplying (E_ε) by $-\Delta(u_t + \theta u)$ and taking θ small enough, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\|\nabla(u_t+\theta u)\|^2 + (1+\theta)\|\Delta u\|^2 + \varepsilon\|\Delta(u_t+\theta u)\|^2\Big) + C_\theta(\|\Delta u_t\|^2 + \|\Delta u\|^2)\\ &\le C_\theta\|g\|^2 + C\int_\Omega (1+|u|^{\frac{4}{N-2}})|\nabla u|(|\nabla u_t| + |\nabla u|).\end{aligned} \tag{5.23}$$

Then, as that in [29, Lemma 5.5], applying *Lemma* 5.4 we can deal with the nonlinear term and finally complete the proof as an application of the Gronwall inequality. $\square$

Lemma 5.7. *There exists $R_2 > 0$ such that for any bounded (in $\mathcal{H}^1_\varepsilon$) subset $B \subset \mathcal{H}^1_\varepsilon$, there is a $T_2 = T_2(\|B\|_{\mathcal{H}^1_\varepsilon})$ such that*

$$\forall\ \varepsilon \in [0,1],\ \|S_\varepsilon(t)B\|^2_{\mathcal{H}^1} \le R_2 \quad \textit{for all } t \ge T_2.$$

Proof. From *Lemma* 5.6 above, we only need to estimate that the bound of $\|\Delta u_t\|^2$ is independent of $\varepsilon \in [0,1]$.

Multiplying (E_ε) by $-\Delta u_{tt}$, we have

$$\frac{1}{2}\frac{d}{dt}E_{4u}(t) + \|\nabla u_{tt}\|^2 + \varepsilon\|\Delta u_{tt}\|^2 - \|\Delta u_t\|^2 = -\langle f'(u)u_t, \Delta u_t\rangle, \tag{5.24}$$

where $E_{4u}(t) = \|\Delta u_t\|^2 + 2\langle \Delta u, \Delta u_t\rangle + 2\langle \Delta u_t, g\rangle - 2\langle f(u), \Delta u_t\rangle$.

Note that, as $t \ge T_1(\|B\|_{\mathcal{H}^1_\varepsilon})$ (given in *Lemma* 5.6), we have

$$\begin{aligned}|\langle f'(u)u_t, \Delta u_t\rangle| &\le C\int_\Omega (1+|u|^{\frac{4}{N-2}})|u_t||\Delta u_t|dx\\ &\le C(\|u_t\| + \|\nabla u\|\|\Delta u_t\|)\|\Delta u_t\| \le C_{\lambda_1,R_1}\|\Delta u_t\|^2\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{4}\|\Delta u_t(t)\|^2 - 4(\|\Delta u\|^2 + \|g\|^2 + \|f(u)\|^2) \le E_{4u}(t)\\ &\qquad\le 4\|\Delta u_t(t)\|^2 + \|\Delta u\|^2 + \|g\|^2 + \|f(u)\|^2.\end{aligned}$$

At the same time, applying *Lemma* 5.6 again and integrating (5.23) on $[t, t+1]$ yield

$$\int_t^{t+1} \|\Delta u_t(s)\|^2 ds \le C_{R_1} \quad \text{for all } t \ge T_1(\|B\|_{\mathcal{H}^1_\varepsilon}).$$

Hence, we can complete our proof by applying the uniform Gronwall lemma to (5.24). $\square$

Now, we are ready to prove *Theorem* 2.1.

Proof of Theorem 2.1. Set

$$\mathbb{B} = \{z \in \mathcal{H}^1 : \ \|z\|^2_{\mathcal{H}^1} \le R_2\}, \tag{5.25}$$

where the constant R_2 comes from *Lemma* 5.7 above.

From *Lemmas* 5.7 and 5.5, we know that there is a t_0 such that $S_\varepsilon(t)\bar{B}_\varepsilon \subset \mathbb{B}$ (recall that $\bar{B}_\varepsilon$ is given in (5.15)) for all $t \ge t_0$ and any $\varepsilon \in [0,1]$.

On the other hand, note that

$$c_1\|\cdot\|_{\mathcal{H}^0_\varepsilon} \le \|\cdot\|_{\mathcal{H}} \le c_2\|\cdot\|_{\mathcal{H}^1_\varepsilon} \quad \text{for all } \varepsilon \in [0,1], \tag{5.26}$$

where $c_1, c_2 > 0$ are independent of ε. Then, from *Lemma* 4.4, there exists t_1 which depends only on $\|B_0\|_{\mathcal{H}}$ and $\|\bar{B}_\varepsilon\|_{\mathcal{H}^1_\varepsilon}$ (so only on $M, \bar{R}$) such that

$$\forall\ \varepsilon \in [0,1],\ \|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{\mathcal{H}} \le e^{C_{\bar{R}}t}\|z_1 - z_2\|^{\frac{1}{2}}_{\mathcal{H}^0_\varepsilon},\ \forall\ t \ge t_1,\ z_1 \in B_0,\ z_2 \in \bar{B}_\varepsilon \tag{5.27}$$

and

$$\forall\ \varepsilon \in [0,1],\ S_\varepsilon(t)B_0 \subset B_0 \quad \text{for all } t \ge t_1.$$

Therefore, from *Lemma* 5.5, we have

$$\begin{aligned} \mathrm{dist}_{\mathcal{H}}(S_\varepsilon(t+t_0+t_1)B_0, \mathbb{B}) &\le \ \mathrm{dist}_{\mathcal{H}}(S_\varepsilon(t+t_0+t_1)B_0,\, S_\varepsilon(t_0+t_1)\bar{B}_\varepsilon) \\ &\le C_{M,\bar{R},t_0+t_1}\ \mathrm{dist}^{\frac{1}{2}}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)B_0,\, \bar{B}_\varepsilon) \text{ (by (5.27))} \\ &\le C_{M,\bar{R},t_0+t_1}\sqrt{Q_1(\|B_0\|_{\mathcal{H}})}e^{-\frac{\bar{\nu}}{2}t}, \quad \forall\ t \ge 0. \end{aligned}$$

Hence, noting that t_0, t_1 and $\bar{R}$ are all fixed, we can complete the proof by taking $\nu = \frac{\bar{\nu}}{2}$ and applying *Lemma* 5.2. ☐

5.5. **Applications of** *Theorem* **2.1.** As the application of *Theorem* 2.1, in this subsection, we consider the existence of finite dimensional exponential attractors and the upper semicontinuity of global attractors.

5.5.1. *A priori estimates.* For the subset $\mathbb{B}$ defined in (5.25), from *Lemma* 4.1 and *Lemma* 4.3 we know that there is a $t_{\mathbb{B}}$ such that

$$\forall\ \varepsilon \in [0,1],\ \|\nabla u_t(t)\|^2 + \|u_{tt}\|^2 + \varepsilon\|\nabla u_{tt}(t)\|^2 \le M + M_4 \text{ for all } t \ge t_{\mathbb{B}}, (u_0, v_0) \in \mathbb{B}, \tag{5.28}$$

where $(u(t), u_t(t)) = S_\varepsilon(t)(u_0, v_0)$.

Now, for each $\varepsilon \in [0,1]$, define $\hat{B}_\varepsilon$ as follows:

$$\hat{B}_\varepsilon = \bigcup_{t \ge t_{\mathbb{B}} + T_2} S_\varepsilon(t)\mathbb{B}, \tag{5.29}$$

where T_2 is the time given in *Lemma* 5.7 corresponding to $\mathbb{B}$. Then, for each $\varepsilon \in [0,1]$ we have that $\hat{B}_\varepsilon$ is positive invariant under $S_\varepsilon(t)$ (i.e., $S_\varepsilon(t)\hat{B}_\varepsilon = \hat{B}_\varepsilon$, $\forall\, t \ge 0$) and (from *Lemma* 5.7)

$$\forall\ \varepsilon \in [0,1],\ \|\hat{B}_\varepsilon\|^2_{\mathcal{H}^1} \le R_2. \tag{5.30}$$

Moreover, we have the following results:

Lemma 5.8. *There exists a $\mathcal{T} > 0$ such that for every $\varepsilon \in [0,1]$, the semigroup $S_\varepsilon(t)$ satisfies the following properties: $S_\varepsilon(\mathcal{T})$ admits a decomposition of the form*

$$S_\varepsilon(\mathcal{T}) = L_\varepsilon + N_\varepsilon, \quad L_\varepsilon :\ \hat{B}_\varepsilon \to \mathcal{H}^0_\varepsilon, \quad N_\varepsilon :\ \hat{B}_\varepsilon \to \mathcal{H}^\gamma_\varepsilon,$$

where L_ε and N_ε satisfy the estimates

$$\|L_\varepsilon(z_1) - L_\varepsilon(z_2)\|_{\mathcal{H}_\varepsilon^0} \le \frac{1}{4}\|z_1 - z_2\|_{\mathcal{H}_\varepsilon^0}, \quad \forall\, z_1, z_2 \in \hat{B}_\varepsilon \tag{5.31}$$

and

$$\|N_\varepsilon(z_1) - N_\varepsilon(z_2)\|_{\mathcal{H}_\varepsilon^\gamma} \le C_{R_2,\mathcal{T}}\|z_1 - z_2\|_{\mathcal{H}_\varepsilon^0}, \quad \forall\, z_1, z_2 \in \hat{B}_\varepsilon, \tag{5.32}$$

with the constant $C_{R_2,\mathcal{T}}$ which is independent of ε and

$$\gamma = \begin{cases} 1, & N = 3,4,5,6, \\ \frac{4}{N-2}, & N > 6. \end{cases} \tag{5.33}$$

Proof. For any two initial data $z_i \in \hat{B}_\varepsilon$ with solution $S_\varepsilon(t)z_i = (u^i(t), u_t^i(t))$ ($i = 1,2$), we decompose the difference $S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2$ as follows:

$$S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2 = L_\varepsilon(t)(z_1 - z_2) + N_\varepsilon(t)(z_1 - z_2),$$

where $L_\varepsilon(t)(z_1 - z_2) = (\tilde{v}(t), \tilde{v}_t(t))$ solves

$$\begin{cases} \tilde{v}_{tt} - \Delta\tilde{v}_t - \Delta\tilde{v} - \varepsilon\Delta\tilde{v}_{tt} = 0, \\ (\tilde{v}(0),\, \tilde{v}_t(0)) = z_1 - z_2,\ \tilde{v}|_{\partial\Omega} = 0, \end{cases} \tag{5.34}$$

and $N_\varepsilon(t)(z_1 - z_2) = (\tilde{w}(t), \tilde{w}_t(t))$ solves

$$\begin{cases} \tilde{w}_{tt} - \Delta\tilde{w}_t - \Delta\tilde{w} - \varepsilon\Delta\tilde{w}_{tt} + f(u^1) - f(u^2) = 0, \\ (\tilde{w}(0),\, \tilde{w}_t(0)) = (0,0),\ \tilde{w}|_{\partial\Omega} = 0. \end{cases} \tag{5.35}$$

In the following, for clarity, we decompose the remainder proof into two steps.

Step 1. Similar to the proof of *Lemma* 4.4, for (5.34) we can deduce that

$$\|L_\varepsilon(t)z_1 - L_\varepsilon(t)z_2\|^2_{\mathcal{H}_\varepsilon^0} = \|(\tilde{v}(t), \tilde{v}_t(t))\|^2_{\mathcal{H}_\varepsilon^0} \le Q(\|\hat{B}_\varepsilon\|_{\mathcal{H}})\|z_1 - z_2\|^2_{\mathcal{H}_\varepsilon^0} e^{-\mu_1 t},$$

where the constant μ_1 only depends on the first eigenvalue λ_1. Hence, by taking $T' > 0$ large enough, we have

$$\|L_\varepsilon(t+T')z_1 - L_\varepsilon(t+T')z_2\|_{\mathcal{H}_\varepsilon^0} \le \frac{1}{4}\|z_1 - z_2\|_{\mathcal{H}_\varepsilon^0} \text{ for all } t \ge 0. \tag{5.36}$$

Step 2. For $\tilde{w}(t)$, multiplying (5.35) by $A^\gamma\tilde{w}_t(t)$ (where γ is given in (5.33)) we obtain that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|A^{\frac{\gamma}{2}}\tilde{w}_t\|^2 + \|A^{\frac{1+\gamma}{2}}\tilde{w}\|^2 + \varepsilon\|A^{\frac{1+\gamma}{2}}\tilde{w}_t\|^2\big) + \|A^{\frac{1+\gamma}{2}}\tilde{w}_t\|^2 \\ &\quad + \langle f(u^1) - f(u^2), A^\gamma\tilde{w}_t\rangle = 0. \end{aligned}$$

Case 1. $N = 3,4$. Then using the embedding $D(A) \hookrightarrow L^p(\Omega)$ for any $p \ge 1$, we have

$$\begin{aligned} |\langle f(u^1) - f(u^2), A\tilde{w}_t\rangle| &\le C(1 + \|u^1\|_{H^2}^{\frac{4}{N-2}} + \|u^2\|_{H^2}^{\frac{4}{N-2}})\|\nabla(u^1 - u^2)\|\|A\tilde{w}_t\| \\ &\le C_{R_2}\|\nabla(u^1 - u^2)\|\|A\tilde{w}_t\| \\ &\le C_{R_2}e^{C_{R_2}t}\|z_1 - z_2\|_{\mathcal{H}_\varepsilon^0}\|A\tilde{w}_t\| \\ &\le C_{R_2,t}\|z_1 - z_2\|^2_{\mathcal{H}_\varepsilon^0} + \frac{1}{2}\|A\tilde{w}_t\|^2, \end{aligned}$$

where we have used (5.30) and (4.21).

Case 2. $N = 5, 6$. Since $\frac{4N}{N-2} \le \frac{2N}{N-4}$ and embedding $D(A) \hookrightarrow L^{\frac{2N}{N-4}}(\Omega)$, we also have

$$\begin{aligned}|\langle f(u^1) - f(u^2), A\tilde{w}_t\rangle| &\le C(1 + \|u^1\|_{H^2}^{\frac{4}{N-2}} + \|u^2\|_{H^2}^{\frac{4}{N-2}})\|\nabla(u^1 - u^2)\|\|A\tilde{w}_t\| \\ &\le C_{R_2,t}\|z_1 - z_2\|^2_{\mathcal{H}^0_\varepsilon} + \frac{1}{2}\|A\tilde{w}_t\|^2.\end{aligned}$$

Case 3. $N > 6$. Noting that $1 = \frac{N-2}{2N} + \frac{N-2(1-\gamma)}{2N} + \frac{4-2\gamma}{2N}$ and $\frac{4}{N-2}\cdot\frac{2N}{4-2\gamma} = \frac{2N}{N-4}$, we have

$$\begin{aligned}|\langle f(u^1) - f(u^2), A^\gamma\tilde{w}_t\rangle| &\le C(1 + \|u^1\|_{H^2}^{\frac{4}{N-2}} + \|u^2\|_{H^2}^{\frac{4}{N-2}})\|\nabla(u^1 - u^2)\|\|A^{\frac{1+\gamma}{2}}\tilde{w}_t\| \\ &\le C_{R_2,t}\|z_1 - z_2\|^2_{\mathcal{H}^0_\varepsilon} + \frac{1}{2}\|A^{\frac{1+\gamma}{2}}\tilde{w}_t\|^2.\end{aligned}$$

Therefore, for any $N \ge 3$, we have

$$\begin{aligned}&\frac{d}{dt}\big(\|A^{\frac{\gamma}{2}}\tilde{w}_t\|^2 + \|A^{\frac{1+\gamma}{2}}\tilde{w}\|^2 + \varepsilon\|A^{\frac{1+\gamma}{2}}\tilde{w}_t\|^2\big) + \|A^{\frac{1+\gamma}{2}}\tilde{w}_t\|^2 \\ &\quad\le 2C_{R_2,t}\|z_1 - z_2\|^2_{\mathcal{H}^0_\varepsilon} \quad \text{for all } t \ge 0,\end{aligned}$$

which, noting that $(\tilde{w}(0),\, \tilde{w}_t(0)) = (0,0)$, implies that

$$\begin{aligned}\|\tilde{w}(t),\, \tilde{w}_t(t)\|^2_{\mathcal{H}^\gamma_\varepsilon} &= \|A^{\frac{\gamma}{2}}\tilde{w}_t(t)\|^2 + \|A^{\frac{1+\gamma}{2}}\tilde{w}(t)\|^2 + \varepsilon\|A^{\frac{1+\gamma}{2}}\tilde{w}_t(t)\|^2 \\ &\le C_{R_2,t}\|z_1 - z_2\|^2_{\mathcal{H}^0_\varepsilon} \quad \text{for all } t \ge 0.\end{aligned} \tag{5.37}$$

Hence, taking

$$\mathcal{T} = T' \quad \text{and } L_\varepsilon = L_\varepsilon(\mathcal{T}), \quad N_\varepsilon = N_\varepsilon(\mathcal{T}),$$

then, from (5.36) and (5.37), we can see that $\mathcal{T}$, L_ε and N_ε satisfy (5.31)-(5.32), respectively. □

Lemma 5.9. *For an arbitrary fixed time $T > 0$ and any $\varepsilon \in [0,1]$, the semigroup $S_\varepsilon(t)$ is Lipschitz continuous on $[0,T] \times \hat{B}_\varepsilon$ in the following sense: there exists a positive constant $\bar{C}_{T,R_2}$ such that for any $z_i \in \hat{B}_\varepsilon$, $t_i \in [0,T], i = 1,2,$*

$$\|S_\varepsilon(t_1)z_1 - S_\varepsilon(t_2)z_2\|_{\mathcal{H}^0_\varepsilon} \le \bar{C}_{T,R_2}\big(|t_1 - t_2| + \|z_1 - z_2\|_{\mathcal{H}^0_\varepsilon}\big). \tag{5.38}$$

Proof. Obviously, we have

$$\|S_\varepsilon(t_1)z_1 - S_\varepsilon(t_2)z_2\|_{\mathcal{H}^0_\varepsilon} \le \|S_\varepsilon(t_1)z_1 - S_\varepsilon(t_2)z_1\|_{\mathcal{H}^0_\varepsilon} + \|S_\varepsilon(t_2)z_1 - S_\varepsilon(t_2)z_2\|_{\mathcal{H}^0_\varepsilon}.$$

Note that

$$\|S_\varepsilon(t_1)z_1 - S_\varepsilon(t_2)z_1\|_{\mathcal{H}^0_\varepsilon} = \left\|\int_{t_1}^{t_2}\frac{d}{dt}S_\varepsilon(t)z_1 ds\right\|_{\mathcal{H}^0_\varepsilon} \le \left|\int_{t_1}^{t_2}\|\frac{d}{dt}S_\varepsilon(t)z_1\|_{\mathcal{H}^0_\varepsilon} ds\right|.$$

Then from (5.28) and (5.29) we can deduce

$$\|S_\varepsilon(t_1)z_1 - S_\varepsilon(t_2)z_1\|_{\mathcal{H}^0_\varepsilon} \le \sqrt{M + M_4}|t_1 - t_2|,$$

which, combining with (4.21), implies (5.38) immediately. □

5.5.2. *Exponential attractors.* Based on the preliminary lemmas given in §5.5.1, we are now ready to prove the following result about the existence of exponential attractors.

Lemma 5.10. *Under Assumption I, for every $\varepsilon \in [0,1]$, there exists a compact subset $\mathcal{E}_\varepsilon \subset \mathcal{H}^1$, uniformly bounded in $\mathcal{H}^1$, which satisfies the following conditions:*

(i) *$\mathcal{E}_\varepsilon$ is semi-invariant with respect to the semigroup $\{S_\varepsilon(t)\}_{t\geq 0}$, that is,*

$$S_\varepsilon(t)\mathcal{E}_\varepsilon \subset \mathcal{E}_\varepsilon \quad \textit{for all } t \geq 0;$$

(ii) *the fractal dimension of $\mathcal{E}_\varepsilon$ is finite, that is,*

$$\dim_F(\mathcal{E}_\varepsilon, \mathcal{H}) \leq \Lambda_\varepsilon < \infty, \quad \forall\, \varepsilon \in [0,1];$$

(iii) *for each $\varepsilon \in [0,1]$, $\mathcal{E}_\varepsilon$ enjoys a uniform exponential attraction property of the form: for any bounded (in $\mathcal{H}$) subset $B \subset \mathcal{H}$,*

$$\mathrm{dist}_{\mathcal{H}}(S_\varepsilon(t)B,\, \mathcal{E}_\varepsilon) \leq Q_\varepsilon(\|B\|_{\mathcal{H}})e^{-\nu' t}, \quad \forall\, t \geq 0.$$

Here, Λ_ε and $Q_\varepsilon(\cdot)$ may depend on ε, but ν' is independent of ε.

Proof. For each $\varepsilon \in [0,1]$, we know that $\hat{B}_\varepsilon$ is invariant and compact in $\mathcal{H}^0_\varepsilon$. Hence, applying the abstract results established in [9, 14, 22] (or see [23, Lemma 9]), from *Lemmas* 5.8 and 5.9 we can first construct an exponential attractor on $\hat{B}_\varepsilon$ with respect to the $\mathcal{H}^0_\varepsilon$-norm. Then, we can complete the proof, by using the attraction transitivity lemma given in [15, Theorem 5.1] from *Lemma* 5.5 and the Hölder continuity (4.22). □

Remark 5.11. Indeed, as shown in [9, *Proposition* 2.7, *Corollary* 2.8] the upper bounds of the fractal dimension of $\mathcal{E}_\varepsilon$ can be specified explicitly only by $N^{\mathcal{H}^0_\varepsilon}_{\frac{1}{5C_{R_2,\mathcal{T}}}}\big(B^{\mathcal{H}^\gamma_\varepsilon}(0,1)\big)$ and the constant $C_{R_2,\mathcal{T}}$ given in (5.32). Here $N^V_r(C)$ denotes the smallest number of r-balls in V needed to cover C, $B^{\mathcal{H}^\gamma_\varepsilon}(0,1)$ is the unit ball in $\mathcal{H}^\gamma_\varepsilon$ and γ is given in (5.33).

5.5.3. *Upper semicontinuity of global attractors.* Since $\mathcal{A}_\varepsilon \subset \mathcal{E}_\varepsilon$, (ii) of *Lemma* 5.10 implies that the fractal dimension of the global attractor $\mathcal{A}_\varepsilon$ is finite too. Moreover, we have the following upper semicontinuity result of $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$:

Lemma 5.12. *Under Assumption I, the global attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ are upper semicontinuous at $\varepsilon = 0$:*

$$\mathrm{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon,\, \mathcal{A}_0) \to 0 \quad \textit{as } \varepsilon \to 0^+.$$

Since the global attractor $\mathcal{A}_\varepsilon$ is strictly invariant, i.e., $S_\varepsilon(t) = \mathcal{A}_\varepsilon$ for all $t \geq 0$, it is obvious to see that

$$\bigcup_{\varepsilon\in[0,1]} \mathcal{A}_\varepsilon \subset \mathbb{B} \text{ and compact in } \mathcal{H}. \tag{5.39}$$

Therefore, to apply *Lemma* 3.2, we can take $K = cl_{\mathcal{H}^1}(\mathbb{B})$ and we only need to verify condition (3.1).

Let $\varepsilon \in (0,1]$ and $(\hat{u}(t), \hat{u}_t(t)) = S_\varepsilon(t)z_\varepsilon$ with $z_\varepsilon \in \mathcal{A}_\varepsilon$; also let $(\hat{v}(t), \hat{v}_t(t)) = S_0(t)z_0$ with $z_0 \in \mathbb{B}$. Denote $\hat{w}(t) = \hat{u}(t) - \hat{v}(t)$. Then $\hat{w}$ solves the following equation:

$$\begin{cases} \hat{w}_{tt} - \Delta\hat{w} - \Delta\hat{w}_t + f(\hat{u}) - f(\hat{v}) = \varepsilon\Delta\hat{u}_{tt}, \\ (\hat{w}(x,0),\, \hat{w}_t(x,0)) = z_\varepsilon - z_0,\ \hat{w}|_{\partial\Omega} = 0. \end{cases} \tag{5.40}$$

Multiplying (5.40) by $\hat{w}_t$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\big(\|\hat{w}_t\|^2+\|\nabla\hat{w}\|^2\big)+\|\nabla\hat{w}_t\|^2+\langle f(\hat{u})-f(\hat{v}),\hat{w}_t\rangle=-\varepsilon\int_\Omega\nabla\hat{u}_{tt}\cdot\nabla\hat{w}_t.$$

Therefore,

$$\frac{d}{dt}\big(\|\hat{w}_t\|^2+\|\nabla\hat{w}\|^2\big)+\|\nabla\hat{w}_t\|^2\le C_{\|\mathbb{B}\|_{\mathcal{H}}}\|\nabla\hat{w}\|^2+\varepsilon^2\|\nabla\hat{u}_{tt}\|^2. \tag{5.41}$$

Since $z_\varepsilon\in\mathcal{A}_\varepsilon$, from (5.28) we have

$$\varepsilon\|\nabla\hat{u}_{tt}(t)\|^2\le M+M_4\quad\text{for all } t\ge 0.$$

Hence, integrating (5.41) over $[0,t]$, we have

$$\|\hat{w}_t(t)\|^2+\|\nabla\hat{w}(t)\|^2\le C_{\|\mathbb{B}\|_{\mathcal{H}},t}\big(\|z_\varepsilon-z_0\|^2_{H_0^1(\Omega)\times L^2(\Omega)}+\varepsilon\big)\quad\text{for all } t\ge 0. \tag{5.42}$$

Moreover, we also have

$$\|\nabla\hat{w}_t(t)\|^2\le\|\hat{w}_{tt}\|\|\hat{w}_t\|+\|\nabla\hat{w}\|\|\nabla\hat{w}_t\|+\varepsilon\|\nabla\hat{u}_{tt}\|\|\nabla\hat{w}_t\|+C_{\|\mathbb{B}\|_{\mathcal{H}}}\|\nabla\hat{w}\|\|\nabla\hat{w}_t\|.$$

Then, from *Lemma* 4.3 and using (5.42), we know that there is a $t_1=t_1(\|\mathbb{B}\|_{\mathcal{H}})$ (which is independent of ε) such that

$$\|\nabla\hat{w}_t(t_1+1)\|^2\le C_{\|\mathbb{B}\|_{\mathcal{H}},t_1}\big(\|z_\varepsilon-z_0\|_{H_0^1(\Omega)\times L^2(\Omega)}+\varepsilon\big),$$

which, combining with (5.42) again, implies

$$\text{if } \varepsilon_n\to 0^+ \text{ and } \mathcal{A}_{\varepsilon_n}\ni z_n\to z_0, \text{ then } S_{\varepsilon_n}(t_1+1)z_n\to S_0(t_1+1)z_0. \tag{5.43}$$

Proof of Lemma 5.12. From (5.39) and (5.43), the proof is a direct application of *Lemma* 3.2. ☐

6. Part II: $g(x)\in H^{-1}$

Throughout this section, we always assume that Assumption II holds.

We first recall a simple result (its proof can be found in [30]) about an elliptic equation:

Lemma 6.1. *Let $f(\cdot)$ satisfy* (2.1) *and let* (2.4)*, and let u_θ be the solution of the following elliptic equation:*

$$\begin{cases}-\Delta u+f(u)+\theta u=g(x)\in H^{-1} & \text{in } \Omega,\\ u|_{\partial\Omega}=0,\end{cases}$$

where $\theta>l$. Then

$$\|\nabla u_\theta\|\to 0\quad\text{as } \theta\to\infty.$$

Then, as in [30], combining with *Lemma* 6.1, we can take η_0 (in (2.5)) large enough such that (recall that $\phi(x)$ is the unique solution of (2.5))

$$\frac{1}{2}\|\nabla\varphi\|^2+2\langle h(\varphi+\phi)-h(\phi),v\rangle-\langle h'(\phi)\varphi,\varphi\rangle\ge 0\quad\text{for any } \varphi\in H_0^1(\Omega), \tag{6.1}$$

and define

$$h(s)=f(s)+\eta_0 s\quad\text{for all } s\in\mathbb{R}. \tag{6.2}$$

6.1. **Decomposition of the equation.** We first decompose the solution $S(t)(u_0, v_0) = (u(t), u_t(t))$ into the sum

$$S(t)\xi_u(0) = K(t)\xi_u(0) + D(t)\xi_u(0),$$

where $K(t)\xi_u(0) = (w(t), w_t(t))$ and $D(t)\xi_u(0) = (z(t), z_t(t))$ solve the following equations, respectively:

$$\begin{cases} w_{tt} - \Delta w_t - \Delta w - \Delta w_{tt} + f(u) - f(z) = \eta_0 z & \text{in } \Omega \times \mathbb{R}^+, \\ w|_{\partial\Omega} = 0, \quad (w(x,0),\, w_t(x,0)) = (0,0), \end{cases} \tag{6.3}$$

and

$$\begin{cases} z_{tt} - \Delta z_t - \Delta z - \Delta z_{tt} + h(z) = g(x) & \text{in } \Omega \times \mathbb{R}^+, \\ z|_{\partial\Omega} = 0, \quad (z(x,0),\, z_t(x,0)) = \xi_u(0). \end{cases} \tag{6.4}$$

Then, we further decompose the solution $z(x,t)$ of (6.4) as $z(x,t) = v(x,t) + \phi(x)$, where $\phi(x)$ is the unique solution of (2.5) and $v(x,t)$ solves the following equation:

$$\begin{cases} v_{tt} - \Delta v_t - \Delta v - \Delta v_{tt} + h(z) - h(\phi) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ v|_{\partial\Omega} = 0, \quad (v(x,0),\, v_t(x,0)) = \xi_u(0) - (\phi(x), 0). \end{cases} \tag{6.5}$$

6.2. **A priori estimates.** At first, for the solution of (6.4), from *Remark* 4.2, we have the following estimate:

Lemma 6.2. *There exists $Q_3(\cdot) \in \mathfrak{J}$ such that for any bounded set $B \subset \mathcal{H}$, the following estimate holds: for any $t \geq 0$,*

$$\|\nabla z(t)\|^2 + \int_0^t \|\nabla z_t(s)\|^2 ds \leq Q_3(\|B\|_{\mathcal{H}} + \|g\|_{H^{-1}}), \quad \forall\ (z(x,0),\, z_t(x,0)) \in B.$$

Second, for the solution of (6.5) we have the following results:

Lemma 6.3. *There exist a positive constant k_1 and $Q_4(\cdot) \in \mathfrak{J}$ such that for any bounded set $B \subset \mathcal{H}$, the following estimate holds:*

$$\|(v(x,t),\, v_t(x,t))\|_{\mathcal{H}} \leq Q_4(\|B\|_{\mathcal{H}}) e^{-k_1 t}, \quad \forall\ t \geq 0,\ (v(x,0),\, v_t(x,0)) \in B.$$

Consequently, for the solution of (6.4) *the following estimate holds:*

$$\|(z(x,t),\, z_t(x,t)) - (\phi(x), 0)\|_{\mathcal{H}} \leq Q_4(\|B\|_{\mathcal{H}}) e^{-k_1 t}, \quad \forall\ t \geq 0,\ \xi_0 \in B.$$

Proof. Multiplying (6.5) by $v_t + \epsilon v$, we have

$$\frac{d}{dt} E_{5v}(t) + \epsilon E_{5v}(t) + G_{5v}(t) + \frac{\epsilon}{2}\|\nabla v(t)\|^2 = 2\langle \big(h'(z) - h'(\phi)\big) z_t,\, v\rangle, \tag{6.6}$$

where

$$\begin{aligned} E_{5v}(t) = \|v_t(t)\|^2 &+ (1+\epsilon)\|\nabla v(t)\|^2 + \|\nabla v_t(t)\|^2 + 2\epsilon\langle v_t(t), v(t)\rangle \\ &+ 2\epsilon\langle \nabla v_t(t), \nabla v(t)\rangle + 2\langle h(z) - h(\phi), v(t)\rangle - \langle h'(\phi) v, v\rangle \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} G_{5v}(t) = 2\|\nabla v_t(t)\|^2 &+ \frac{\epsilon}{2}\|\nabla v(t)\|^2 - 3\epsilon\|v_t\|^2 - 3\epsilon\|\nabla v_t\|^2 - 2\epsilon^2\langle v_t, v\rangle \\ &- 2\epsilon^2\langle \nabla v_t, \nabla v\rangle - \epsilon^2\|\nabla v\|^2 + \epsilon\langle h'(\phi) v,\, v\rangle. \end{aligned} \tag{6.8}$$

Noticing (6.1), by further taking ϵ small enough, we have

$$E_{5v}(t) \geq \frac{1}{4}\|(v(t), v_t(t))\|^2_{\mathcal{H}} \quad \text{for all } t \geq 0 \tag{6.9}$$

and

$$G_{5v}(t) \geq 0 \quad \text{for all } t \geq 0. \tag{6.10}$$

Moreover, from (2.3) and using the Hölder inequality, we have

$$\begin{aligned}
2\langle\big(h'(z) - h'(\phi)\big)z_t,\, v\rangle &= 2\langle h''\big(rz + (1-r)\phi\big)z_t,\, v^2\rangle \\
&\leq \begin{cases} C\int_\Omega \big(1 + |z|^{\frac{6-N}{N-2}} + |\phi|^{\frac{6-N}{N-2}}\big)|z_t||v|^2 dx & N = 3,4,5, \\ C\int_\Omega |z_t||v|^2 dx & N \geq 6 \end{cases} \\
&\leq c_3\|\nabla z_t\|\|\nabla v\|^2 \leq \frac{\epsilon}{2}\|\nabla v\|^2 + \frac{c_3}{\epsilon}\|\nabla z_t\|^2 E_{5v}(t),
\end{aligned}$$

where $r \in (0,1)$ and the constant c_3 depends only on $\|B\|_{\mathcal{H}} + \|\nabla\phi\|$.

Therefore, from *Lemma* 6.2 and noticing $E_{5v}(0) \leq Q(\|B\|_{\mathcal{H}} + \|\nabla\phi\|)$, applying *Lemma* 3.3 we can complete our proof immediately. □

For the solution of (6.3), the same as *Lemma* 5.3, for each $\alpha \in [0, \min\{1, \frac{N}{2}-1\})$, by multiplying (6.3) by $A^\alpha w_t$, we have

Lemma 6.4. *For any $\alpha \in [0, \min\{1, \frac{N}{2} - 1\})$, there exist $k_\alpha > 0$ and $Q_\alpha(\cdot) \in \mathfrak{J}$ such that for any $t \geq 0$,*

$$\|K(t)\xi_u(0)\|^2_{\mathcal{H}^\alpha} \leq Q_\alpha(\|\xi_u(0)\|_{\mathcal{H}})e^{k_\alpha t},$$

where k_α also depends on $\|\xi_u(0)\|_{\mathcal{H}}$.

Now, similar to *Lemma* 5.4, based on *Lemma* 6.2 and *Lemma* 6.3 we can decompose $u(t)$ as follows:

Lemma 6.5. *Let $(u(t), u_t(t))$ be the solution of (E_1) corresponding to the initial data $\xi_u(0) = (u_0, v_0)$. Then, for any $\eta > 0$, we can decompose $u(t)$ as*

$$u(t) = v_1(t) + w_1(t), \quad \text{for all } t \geq 0,$$

where $v_1(t)$ and $w_1(t)$ satisfy the following estimates:

$$\int_s^t \|\nabla v_1(\tau)\|^2 d\tau \leq \eta(t-s) + C_\eta \text{ for all } t \geq s \geq 0 \tag{6.11}$$

and

$$\|A^{\frac{1+\alpha}{2}} w_1(t)\|^2 \leq K_\eta \text{ for all } t \geq 0, \tag{6.12}$$

with the constants C_η and K_η depending on η, $\|\xi_u(0)\|_{\mathcal{H}}$ and $\|g\|_{H^{-1}}$.

Moreover, $v_1(\cdot,t)$ and $w_1(\cdot,t)$ satisfy the following estimates respectively:

$$\|\nabla v_1(t)\| \leq Q_5(\|B\|_{\mathcal{H}}) \quad \text{for all } t \geq 0 \tag{6.13}$$

and

$$\|\nabla w_1(t)\| = \|\nabla(u(t) - v_1(t))\| \leq Q_6(\|B\|_{\mathcal{H}}) \quad \text{for all } t \geq 0. \tag{6.14}$$

Proof. The proof is the same as that in [30, Lemma 4.5]. □

6.3. **Proof of Theorem 2.2.** We will follow the idea from [37], and the details similar to [28].

Proof of Theorem 2.2. We decompose our proof into two steps for clarity.

Step 1. We first claim that

For each $\alpha \in [0, \min\{1, \frac{N}{2}-1\})$, there exists a constant $J_{B,\alpha}$, which depends only on the $\mathcal{H}$-bounds of $B(\subset \mathcal{H})$ and α, such that

$$\|K(t)\xi_u(0)\|^2_{\mathcal{H}^\alpha} = \|(w(t), w_t(t))\|^2_{\mathcal{H}^\alpha} \le J_{B,\alpha} \quad \text{for all } t \ge 0 \text{ and } \xi_u(0) \in B. \tag{6.15}$$

Multiplying (6.3) by $A^\alpha(w_t(t) + \epsilon w(t))$, we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&\big(\|A^{\frac{\alpha}{2}}(w_t+\epsilon w)\|^2 + \mu\|A^{\frac{1+\alpha}{2}}(w_t+\epsilon w)\|^2\big) - \langle \epsilon w_t, A^\alpha(w_t+\epsilon w)\rangle \\ &\quad - \langle \epsilon w_t, A^{1+\alpha}(w_t+\epsilon w)\rangle - \langle \Delta w_t, A^\alpha(w_t+\epsilon w)\rangle - \langle \Delta w, A^\alpha(w_t+\epsilon w)\rangle \\ &= -\langle f(u) - f(z), A^\alpha(w_t+\epsilon w)\rangle + \langle \eta_0 z, A^\alpha(w_t+\epsilon w)\rangle, \end{aligned}$$

where ϵ (>0) is small enough and will be determined later.

Then, as in [28, *Lemma* 4.4], we can obtain the claim above by applying *Lemma* 6.5 to overcome the difficulty from the critical nonlinearity.

Step 2. Applying *Lemma* 6.3 and *Step 1* to B_0 (recall $B_0 \subset \mathcal{H}$ is the bounded absorbing set given in §3), also using the attraction transitivity lemma devised in [15], we can finish our proof by setting: for each $\alpha \in [0, \min\{1, \frac{N}{2}-1\})$,

$$\mathcal{B}_\alpha = \{z \in \mathcal{H} : \|z - (\phi(x), 0)\|^2_{\mathcal{H}^\alpha} \le J_{B_0,\alpha}\},$$

where $J_{B_0,\alpha}$ is the constant given in (6.15) corresponding to B_0. □

In the following we state a decomposition result about $u(t)$, which can be used to construct a finite dimensional exponential attractor (e.g., see [14, 23, 30]); its proof is the same as that in [30, Lemma 4.9].

Lemma 6.6. *Under the assumption of Theorem* 2.1*, for any bounded (in $\mathcal{H}^\alpha$, $\alpha \in [0, \min\{\frac{N}{2}-1\})$) subset $B_1 \subset \mathcal{H}^\alpha$, if the initial data $\xi_u(0) \in \phi(x) + B_1$, then the solution $u(t)$ of (E_1) also satisfies a similar estimate; more precisely, we have*

$$\|S(t)\xi_u(0) - (\phi(x), 0)\|^2_{\mathcal{H}^\alpha} = \|(u(t), u_t(t)) - (\phi(x), 0)\|^2_{\mathcal{H}^\alpha} \le K_{B_1}$$

for any $t \ge 0$ and any $\xi_u(0) \in \phi(x) + B_1$; where the constant K_{B_1} depends only on α and the $\mathcal{H}^\alpha$-bound of B_1.

Remark 6.7. Based on *Theorem* 2.2 and *Lemma* 6.6, we can construct a finite dimensional exponential attractor $\mathcal{E}$ for $\{S(t)\}_{t\ge 0}$ under *Assumption* II. Moreover, we can decompose $\mathcal{E}$ as $\mathcal{E} = (\phi(x), 0) + \mathcal{E}'$, where $\mathcal{E}'$ is bounded in $\mathcal{H}^\alpha$ for any $\alpha \in [0, \min\{1, \frac{N}{2}-1\})$.

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