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ON GENERALIZED WHITEHEAD PRODUCTS

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ABSTRACT. We define a symmetric monodical pairing $G \circ H$ among simply connected co-H spaces G and H with the property that $S(G \circ H)$ is equivalent to the smash product $G \wedge H$ as co-H spaces. We further generalize the Whitehead product map to a map $G \circ H \rightarrow G \vee H$ whose mapping cone is the cartesian product.

Whitehead products have played an important role in unstable homotopy. They were originally introduced [Whi41] as a bilinear pairing of homotopy groups:

$$\pi_m(X) \otimes \pi_n(X) \to \pi_{m+n-1}(X), \quad m, n > 1.$$

This was generalized ([Ark62], [Coh57], [Hil59]) by constructing a map

 $W: S(A \land B) \to SA \lor SB.$

Precomposition with W defines a function on based homotopy classes:

 $[SA, X] \times [SB, X] \to [S(A \land B), X],$

which is bilinear in case A and B are suspensions.

The case where A and B are Moore spaces was central to the work of Cohen, Moore and Neisendorfer ([CMN79]). In [Ani93] and in particular [AG95], this work was generalized. Much of this has since been simplified in [GT10], but further understanding will require a generalization from suspensions to co-H spaces.

The purpose of this work is to carry out and study such a generalization. Let \mathcal{CO} be the category of simply connected co-H spaces and co-H maps. We define a functor

$$\begin{aligned} \mathcal{CO} \times \mathcal{CO} \to \mathcal{CO}, \\ (G, H) \to G \circ H, \end{aligned}$$

and a natural transformation

(1) $W: G \circ H \to G \lor H$

generalizing the Whitehead product map. The existence of $G \circ H$ generalizes a result of Theriault [The03] who showed that the smash product of two simply connected co-associative co-H spaces is the suspension of a co-H space. We do not need the co-H spaces to be co-associative and require only one of them to be simply connected.¹ We call $G \circ H$ the Theriault product of G and H.

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¹In fact we can define $G \circ H$ for any two co-H spaces but require at least one of them to be either simply connected or a suspension in order to obtain the co-H space structure map on $G \circ H$. Recently Grbić, Theriault and Wu have shown that the smash product of any two co-H spaces is the suspension of a co-H space, but their construction cannot satisfy Theorem 1(a) below [GTW].

We summarize our results in the following theorems.

Theorem 1. There is a functor $CO \times CO \rightarrow CO$ given by

$$(G,H) \to G \circ H$$

and equivalences in CO:

- (a) $(SX) \circ H \simeq X \wedge H$,
- (b) $S(G \circ H) \simeq G \wedge H$,
- (c) $(G_1 \vee G_2) \circ H \simeq G_1 \circ H \vee G_2 \circ H$

and homotopy equivalences:

- (d) $G \circ H \simeq H \circ G$,
- (e) $(G \circ H) \circ K \simeq G \circ (H \circ K)$.

Theorem 2. There is a natural transformation

$$W\colon G\circ H\to G\vee H$$

which is the Whitehead product map in case G and H are both suspensions. Furthermore, there is a homotopy equivalence

$$G \times H \simeq G \vee H \cup_{\mathcal{W}} C(G \circ H).$$

The next theorem concerns the inclusion of the fiber in certain standard fibration sequences [Gra71]:

$$\begin{split} G \rtimes \Omega H \xrightarrow{\iota_1} G \vee H \xrightarrow{\pi_2} H, \\ \Omega G \ast \Omega H \xrightarrow{\iota_2} G \vee H \to G \times H. \end{split}$$

Define $\operatorname{ad}^{n}(H)(G)$ inductively by $\operatorname{ad}^{0}(H)(G) = G$ and

$$\mathrm{ad}^{n}(H)(G) = [ad^{n-1}(H)(G)] \circ H$$

and by an iterated Whitehead product

$$ad^n \colon ad^n(H)(G) \to G \lor H$$

as the composition

$$ad^{n}(H)(G) \xrightarrow{\mathrm{W}} ad^{n-1}(H)(G) \vee H \xrightarrow{ad^{n-1} \vee 1} G \vee H \vee H \to G \vee H.$$

Theorem 3. Suppose G and H are simply connected co-H spaces. Then there are homotopy equivalences:

- (a) $G \rtimes \Omega H \simeq \bigvee_{n \ge 0} ad^n(H)(G)$, where ι_1 corresponds to ad^n on the appropriate factor,
- (b) $\Omega G * \Omega H \simeq \bigvee_{\substack{i \ge 0 \\ j \ge 1}} ad^j(H)(ad^i(G)(G)), \text{ where } \iota_2 \text{ corresponds to } ad^j(ad^i) \text{ on }$

to the appropriate factor,

(c) $S\Omega G \simeq \bigvee_{n \ge 0} ad^n(G)(G)$, where the composition $S\Omega G \to S\Omega G \lor S\Omega G \to G \lor G$ corresponds to the appropriate iterated Whitehead product on each factor.

It should be pointed out that equivalence (c) generalizes the result of Theriault [The03, 1.1] where it is shown that a simply connected co-associative co-H space decomposes

$$\Sigma\Omega G\simeq \bigvee_{n\geqslant 1}M_n$$

for some spaces M_n , which are not further decomposed.

Theorem 4. Suppose X is finite dimensional and $f: SX \to G \lor H$. Then f is the sum of the projections onto G and H and a finite sum of iterated Whitehead products.

Throughout this work we will assume that all spaces are of the homotopy type of a CW complex. All homology and cohomology will be with a field of coefficients. We will often show that a map between simply connected CW complexes is a homotopy equivalence by showing that it induces an isomorphism in homology with an arbitrary field of coefficients, without further comment.

Section will be devoted to some general remarks about telescopes, and we will construct the Theriault product in section. Theorem 1 will follow from Propositions 2.3, 2.5 and 2.7. The functor in Theorem 2 is defined after Corollary 3.2, and the equivalence follows from Proposition 3.8. The proof of the first part of Theorem 3 occurs just prior to Corollary 3.5 and the rest occurs following Corollary 3.5. Theorem 4 follows from Corollary 3.7.

Section 1

In this section we will discuss some general properties of telescopes of a self map $e: G \to G$, where G is a co-H space. We do not assume that e is idempotent. We will call e quasi-idempotent if the induced homomorphism in homology satisfies the equation

$$(e_*)^2 = -e_*$$

where u is a unit. We construct two telescopes:

$$T(e): G \xrightarrow{e} G \xrightarrow{e} G \to \dots,$$
$$T(1+e): G \xrightarrow{1+e} G \xrightarrow{1+e} G \to \dots$$

and a map:

$$\Gamma \colon G \to G \lor G \xrightarrow{\Gamma_1 \lor \Gamma_2} T(e) \lor T(1+e).$$

Proposition 1.1. If G is simply connected and e is a quasi-idempotent, Γ is a homotopy equivalence. Furthermore $H_*(T(e)) = im \ e_*$ and $\widetilde{H}_*(T(1+e)) \cong \ker e_*$.

Proof. Suppose $\Gamma_*(\xi) = 0$. Since $(\Gamma_1)_*(\xi) = 0$ $(e_*)^k(\xi) = 0$ for some k, $e_*(\xi) = 0$. Since $(\Gamma_2)_*(\xi) = 0$, $(1+e)_*{}^k(\xi) = 0$. However, $(1+e)_*{}^2 = (1+e)_*$, so $\xi = -e_*(\xi) = 0$. Clearly Γ_* is onto. Moreover, $H_*(T(e)) \cong im \ e_*$ and $\widetilde{H}_*(T(1+e)) \cong im \ (1+e_*) = \ker e_*$.

Corollary 1.2. Suppose G is a simply connected co-H space and $A \subset G$ is a retract of G. Let e be the composition

$$G \xrightarrow{r} A \xrightarrow{i} G.$$

Then $T(e) \simeq A$, $T(1-e) \simeq G/A$, and the identity map of T(e) can be factored:

$$T(e) \xrightarrow{\xi} A \xrightarrow{i} G \xrightarrow{r} A \xrightarrow{\eta} T(e).$$

where ξ and η are inverse homotopy equivalences.

Proof. The telescope T(e) and A are both simply connected, and there are maps $T(e) \to A$ and $A \to T(e)$ making A a retract of $T(\xi)$; these maps are homotopy equivalences. By the Van Kampen theorem $G/A \simeq G \cup CA$ is simply connected. Since $1 - e \colon G \to G$ factors through the projection $\pi \colon G \to G/A$, we can factor the identity map up to homotopy,

$$G \to G \lor G \xrightarrow{r \lor \pi} A \lor G/A \to G,$$

and hence $G \simeq A \lor G/A$. The factorization of the identity map of T(e) is obtained by replacing each space by a telescope where the three in the center are constant.

Now consider two maps $f_1, f_2: X \to X$.

Proposition 1.3. $T(f_1f_2) \simeq T(f_2f_1)$.

Proof. We define maps between the telescopes:



The composition is the shift map which is a homotopy equivalence.

Section 2

In this section we will consider a pair of co-H spaces in which at least one is simply connected. Let G and H be two such co-H spaces with their structure determined by maps $\nu_1: G \to S\Omega G$ and $\nu_2: H \to S\Omega H$, each of which is a right inverse to the respective evaluation maps (see [Gan70]), which we label as ϵ_1, ϵ_2 . We define self maps of $S(\Omega G \wedge \Omega H)$ as follows:

 $e_1 \colon S(\Omega G \land \Omega H) \xrightarrow{\epsilon_1 \land 1} G \land \Omega H \xrightarrow{\nu_1 \land 1} S(\Omega G \land \Omega H),$ $e_2 \colon S(\Omega G \land \Omega H) \xrightarrow{1 \land \epsilon_2} \Omega G \land H \xrightarrow{1 \land \nu_2} S(\Omega G \land \Omega H).$

Here we freely move the suspension coordinate to wherever it is needed. Clearly e_1 and e_2 are idempotents but e_1e_2 is not an idempotent; however it is a quasiidempotent. To see this we need to pay careful attention to the order of the suspension coordinates. Let

$$\Gamma \colon S^1 \wedge S^1 \wedge \Omega G \wedge \Omega H \to S^1 \wedge S^1 \wedge \Omega G \wedge \Omega H$$

be the map switching the suspension coordinates. What we will show is that the composition

$$e = (Se_2) \circ T \circ (Se_1)$$

is an idempotent. Since $T \sim (-1)$ and (Se_1) is a suspension,

$$e \sim -(Se_2) \circ (Se_1).$$

Once we see that e is an idempotent it follows that $(Se_1) \circ (Se_2)$ is a quasiidempotent.

Now we factor e and note that it contains the composition

$$S(1 \wedge \epsilon_2) \circ T \circ S(\nu_1 \wedge 1)$$

However, this composition is equal (no homotopies) to the composition

$$SG \land \Omega H \xrightarrow{1 \land \epsilon_2} G \land H \xrightarrow{\nu_1 \land 1} S\Omega G \land H.$$

Consequently, e is equal to the composition

$$\begin{split} S^{1} \wedge S^{1} \wedge \Omega G \Omega H \simeq S \Omega G \wedge S \Omega H \xrightarrow{\epsilon_{1} \wedge \epsilon_{2}} G \wedge H \\ \xrightarrow{\nu_{1} \wedge \nu_{2}} S \Omega G \wedge S \Omega H \simeq S^{1} \wedge S^{1} \wedge \Omega G \wedge \Omega H, \end{split}$$

and this map is clearly an idempotent.

Now assuming that one of G, H is simply connected, it follows that $\Omega G \wedge \Omega H$ is connected, so $S(\Omega G \wedge \Omega H)$ is simply connected. Consequently,

Proposition 2.1. If one of G and H is simply connected, there is a homotopy equivalence,

$$S(\Omega G \wedge \Omega H) \simeq T(e_1 e_2) \vee T(1 + e_1 e_2).$$

Let

$$\theta \colon S(\Omega G \land \Omega H) \to T(e_1 e_2)$$

be the projection and

$$\psi \colon T(e_1 e_2) \to S(\Omega G \land \Omega H)$$

be the unique right inverse to θ which projects trivially onto $T(1 + e_1e_2)$. These maps determine a co-H space structure on $T(e_1e_2)$.

Definition 2.2. $G \circ H = T(e_1e_2)$.

Proposition 2.3. Given co-H maps $f: G \to G'$ and $g: H \to H'$, there is an induced co-H map

$$f \circ g \colon G \circ H \to G' \circ H'$$

making $G \circ H$ a functor of two variables. In addition there are equivalences of co-H spaces:

- (a) $SX \circ H \simeq X \wedge H$, (b) $S(G \circ H) \simeq G \wedge H$, (c) $S^1 \circ H \simeq H$,

and there is a homotopy equivalence $G \circ H \simeq H \circ G$.

Proof. Since f and q are co-H maps, the squares

$$\begin{array}{ccc} G & \stackrel{\nu_1}{\longrightarrow} S\Omega G & H & \stackrel{\nu_2}{\longrightarrow} S\Omega H \\ f & & \\ f & & \\ G' & \stackrel{\nu_1'}{\longrightarrow} S\Omega G' & H' & \stackrel{\nu_2'}{\longrightarrow} S\Omega H' \end{array}$$

commute up to homotopy. It follows that f and g induce maps that commute with e_1 and e_2 and hence with the equivalences of Proposition 2.1, θ and ψ . For part (a), observe that the composition e_2e_1 factors

$$S(\Omega SX \land \Omega H) \xrightarrow{\epsilon_1 \land 1} SX \land \Omega H \xrightarrow{S\iota \land 1} S\Omega SX \land \Omega H \xrightarrow{1 \land \epsilon_2} \Omega SX \land H \xrightarrow{1 \land \nu_2} S(\Omega SX \land \Omega H)$$

where $(1 \wedge \epsilon_2)(\epsilon_1 \wedge 1)$ is a right universe to $(1 \wedge \nu_2)(\iota \wedge 1)$. Thus we can apply Corollary 1.2 to see that $SX \circ H \simeq X \wedge H$ with co-H structure given by the composite $(1 \wedge \nu_2)(\iota \wedge 1)$. This is precisely the co-H structure induced by ν_2 . Part (b) follows since $S(G \circ H)$ is the telescope of \overline{e} with co-H structure given by $\nu_1 \wedge \nu_2$. Part (c) is a special case of part (a): The last statement follows directly from Proposition 1.3.

To complete the proof of Theorem 1 it remains to prove parts (c), (d), and (e) by Proposition 2.3. Part (d) follows directly from Proposition 1.3. For the other two parts, it will be convenient to describe an alternative definition of $G \circ H$. For this we assume that G is a retract of a space SX and H is a retract of SY:

$$G \xrightarrow{\nu_1} SX \xrightarrow{\epsilon_1} G \qquad H \xrightarrow{\nu_2} SY \xrightarrow{\epsilon_2} H.$$

We can then replace the telescope in the definition by the telescope of the composition:

$$T \colon SX \land Y \xrightarrow{1 \land \epsilon_2} X \land H \xrightarrow{1 \land \nu_2} SX \land Y \xrightarrow{\epsilon_1 \land 1} G \land Y \xrightarrow{\nu_1 \land 1} SX \land Y$$

The co-H structures defined by these maps are equivalent to the structures defined by

$$\widetilde{\nu_2} \colon G \xrightarrow{\nu_1} SX \xrightarrow{S\widetilde{\epsilon_1}} S\Omega G, \qquad \widetilde{\nu_1} \colon H \xrightarrow{\nu_2} SY \xrightarrow{S\Omega\widetilde{\epsilon_2}} S\Omega H,$$

and we have a homotopy commutative ladder:

$$\begin{array}{c} SX \wedge Y \xrightarrow{1 \wedge \epsilon_{2}} X \wedge H \xrightarrow{1 \wedge \nu_{2}} SX \wedge Y \xrightarrow{\epsilon_{1} \wedge 1} G \wedge Y \xrightarrow{\nu_{1} \wedge 1} SX \wedge Y \\ & \downarrow S\tilde{\epsilon}_{1} \wedge \tilde{\epsilon}_{2} \qquad \qquad \downarrow \tilde{\epsilon}_{1} \wedge 1 \qquad \qquad \downarrow S\tilde{\epsilon}_{1} \wedge \tilde{\epsilon}_{2} \qquad \qquad \downarrow 1 \wedge \tilde{\epsilon}_{2} \qquad \qquad \downarrow S\tilde{\epsilon}_{1} \wedge \tilde{\epsilon}_{2} \\ S\Omega G \wedge \Omega H \xrightarrow{1 \wedge \epsilon} \Omega G \wedge H \xrightarrow{1 \wedge \tilde{\nu}_{2}} S\Omega G \wedge \Omega H \xrightarrow{\epsilon \wedge 1} G \wedge \Omega H \xrightarrow{\tilde{\nu}_{1} \wedge 1} S\Omega G \wedge \Omega H. \end{array}$$

Hence we have a commutative diagram:

$$\begin{array}{c|c} SX \wedge Y & \longrightarrow T \vee \overline{T} \\ \\ S\widetilde{\epsilon}_{1} \wedge \widetilde{\epsilon}_{2} & & \\ S\Omega G \wedge \Omega H & \longrightarrow G \circ H \vee \operatorname{Tel}(1 + e_{1}e_{2}) \end{array}$$

where \overline{T} is the telescope defined by $1 + (\nu_1 \wedge 1)(\epsilon_1 \wedge 1)(1 \wedge \nu_2)(1 \wedge \epsilon_2)$. The map $\alpha: T \to G \circ H$ is compatible with the homotopy equivalence $ST \simeq G \wedge H \simeq S(G \circ H)$, so is itself a homotopy equivalence. Now choose a right homotopy inverse for the map $SX \wedge Y \to T$ which projects trivially to \overline{T} . Its composite with $S\tilde{\epsilon}_1 \wedge \tilde{\epsilon}_2$ will then project trivially to $T(1 + \epsilon_1 \epsilon_2)$. Hence we have a homotopy commutative diagram:



and consequently the co-H structure on T is compatible under α with the co-H structure on $G \circ H$. We have proven the proposition.

Proposition 2.4. Suppose G is represented as a retract of SX and H as a retract of SY. Then $G \circ H$ is homotopy equivalent to the telescope T of the composition

$$SX \wedge Y \to G \wedge Y \to SX \wedge Y \to X \wedge H \to SX \wedge Y$$

as co-H spaces, where the co-H structure on T is given by the equivalence $SX \wedge Y \simeq T \vee \overline{T}$.

Proposition 2.5. $(G_1 \lor G_2) \circ H \simeq G_1 \circ H \lor G_2 \circ H$ as co-H spaces.

Proof. Write G_i as a retract of SX_i , i = 1, 2. Then $G_1 \lor G_2$ is a retract of $S(X_1 \lor X_2)$. Thus the telescope for $(G_1 \lor G_2) \circ H$ is at each point the wedge of the telescopes for $G_1 \circ H$ and $G_2 \circ H$.

At this point we will apply Proposition 2.4 to prove Theorem 1(e), the associativity formula. We will make repeated use of

Lemma 2.6. There is a homotopy commutative square

$$\begin{array}{c|c} S(G \circ H) & \xrightarrow{S\psi} & S^2X \wedge Y \\ \simeq & & \swarrow & & & \\ G \wedge H & \xrightarrow{\simeq} & S(G \circ H). \end{array}$$

Proof. $G \wedge H$ is a retract of $S^2X \wedge Y$, so we may apply Corollary 1.2.

Proposition 2.7. There is a homotopy equivalence²

 $(G \circ H) \circ K \simeq G \circ (H \circ K).$

Proof. We suppose that G, H, and K are presented by retractions

$$\begin{array}{l} G \xrightarrow{\psi_1} SX \xrightarrow{\theta_1} G, \\ H \xrightarrow{\psi_2} SY \xrightarrow{\theta_2} H, \\ K \xrightarrow{\psi_3} SZ \xrightarrow{\theta_3} K, \end{array}$$

and we then construct retractions for $G \circ H$ and $H \circ K$:

$$\begin{split} G \circ H \xrightarrow{\psi_3} SX \wedge Y \xrightarrow{\theta_3} G \circ H, \\ H \circ K \xrightarrow{\psi_4} SY \wedge Z \xrightarrow{\theta_4} H \circ K. \end{split}$$

Using these we construct retractions for $G \circ (H \circ K)$ and $(G \circ H) \circ K$:

$$(G \circ H) \circ K \xrightarrow{\psi} S(X \wedge Y \wedge Z) \xrightarrow{\theta} (G \circ H) \circ K,$$
$$G \circ (H \circ K) \xrightarrow{\psi'} S(X \wedge Y \wedge Z) \xrightarrow{\theta'} (G \circ (H \circ K)).$$

$$G \circ H \to G \circ S\Omega H \simeq G \wedge \Omega H$$

²Note that we are not asserting a co-H equivalence here and in (c) of Theorem 1. However, this holds if one of the spaces involved is co-associative. If, for example, H is co-associative, the map

is a co-H map by Proposition 2.3. Since this map also has a left homotopy inverse, the co-H structure is determined by that on G.

We will show that $\theta'\psi: (G \circ H) \circ K \to G \circ (H \circ K)$ is a homotopy equivalence. By Lemma 2.6, we have homotopy commutative diagrams:



Suspending and applying Lemma 2.6 again, we obtain a homotopy commutative diagram:



From these diagrams it follows that $S^2(\theta'\psi)$ is a homotopy equivalence and hence $\theta'\psi$ is as well.

Section 3

In this section we generalize the clutching construction [Gra88, Proposition 1] for fibrations over a suspension to fibrations over a co-H space. This allows for the decomposition results in Theorems 2 and 3.

Proposition 3.1. Suppose $F \to E \to G$ is a fibration where G is a co-H space. Then $E/F \simeq G \rtimes F$.

Proof. In the case G = SX, we have by [Gra88, Proposition 1]

$$E \simeq F \cup_{\theta} (CX) \times F.$$

So $E/F \simeq SX \rtimes F$. It is easy to construct a map $G \rtimes F \to E/F$ in general. Consider the sequence of pull backs:



Then we consider the composite

$$E/F \to E'/F \simeq S\Omega G \rtimes F \to G \rtimes F,$$

where the middle equivalence follows since the base is a suspension. Showing that the composite is a homotopy equivalence will take some work.

Since $\nu \epsilon \colon S\Omega G \to S\Omega G$ is an idempotent, we can decompose $S\Omega G$:

$$S\Omega G \simeq G \lor G'.$$

We now observe that we can construct a quasi-fibration model for a fibration over a one point union. Suppose we have such a fibration



with pull backs E_A and E_B and fiber F. Then we can construct

$$E_A \cup_F E_B,$$

the union of E_A and E_B with the subspace F identified. Then

 ϕ is a homotopy equivalence. In our case $S\Omega G \simeq G \lor G'$ and $E_G = E, E_{G'} = G' \times F$, so

$$E' \simeq E \cup_F G' \times F$$

$$\downarrow$$

$$G \lor G'$$

is a quasi-fibering by [DT58, 2.10]. On the other hand $E'/F \simeq S\Omega G \rtimes F \simeq G \rtimes F \lor G' \rtimes F$, while $E \cup_F G' \rtimes F \simeq E/F \lor G' \times F$. Since the map between E' and $E \cup_F G' \rtimes F$ is a map over $G \lor G'$, we see that $E/F \simeq G \times F$.

Corollary 3.2. $S\Omega G \simeq G \rtimes \Omega G$.

Proof. Apply Proposition 3.1 to the path space fibration over G.

Proof of Theorem 2. Construction: We now describe our generalization of the Whitehead product. Suppose G and H are co-H spaces and one of them is simply connected. The Whitehead product

$$W\colon G\circ H\to G\vee H$$

is then defined as the composition

$$G \circ H \xrightarrow{\psi} S\Omega G \wedge \Omega H \simeq \Omega G * \Omega H \xrightarrow{\omega} G \vee H,$$

where ω is the inclusion of the fiber in the fibration sequence

$$\Omega G * \Omega H \xrightarrow{\omega} G \lor H \to G \times H$$

Clearly ψ and ω are natural transformations, so W is as well.

Before we prove the homotopy equivalence in Theorem 2, we need to establish some results in Theorem 3. We begin by constructing maps

$$ad^n \colon ad^n(H)(G) \to G \lor H$$

inductively. For n = 0 this is just the inclusion of G in $G \vee H$. For n > 0 we define ad^n as the composition

$$ad^{n}(H)(G) = \left(ad^{n-1}(H)(G)\right) \circ H$$

$$\rightarrow \left(ad^{n-1}(H)(G)\right) \lor H \rightarrow (G \lor H) \lor H = G \lor H.$$

Next we calculate the effect of ad^n in loop space homology:

$$\Omega(ad^n)_* \colon H_*(\Omega(ad^n(H)(G))) \to H_*(\Omega(G \lor H)).$$

To do this we need some notation. For each co-H space G, write

$$\sigma^{-1} \colon \widetilde{H}_r(G) \to \widetilde{H}_{r-1}(\Omega G)$$

for the composition

$$\widetilde{H}_r(G) \xrightarrow{\nu_+} \widetilde{H}_r(S\Omega G) \simeq \widetilde{H}_{r-1}(\Omega G).$$

Let $\{x_i\}$ be a basis for $\widetilde{H}_*(G)$. Then $H_*(\Omega G)$ is the tensor algebra on the classes $\{\sigma^{-1}(x_i)\}$. Given two classes $x \in \widetilde{H}_r(G), y \in \widehat{H}_s(H)$, we will write

$$x \circ y \in H_{r+s-1}(G \circ H)$$

for the class that corresponds to $x \ {}_{\bigtriangleup} y \in \widetilde{H}_{r+s}(G \land H)$ under the isomorphism

$$\widetilde{H}_{r-1}(G \circ H) \cong \widetilde{H}_r(S(G \circ H)) \simeq \widetilde{H}_r(G \wedge H).$$

Then the classes $\{x_i \circ y_j\}$ form a basis for $\widetilde{H}_*(G \circ H)$, where $\{x_i\}$ and $\{y_i\}$, respectively, are bases for $\widetilde{H}_*(G)$ and $\widetilde{H}_*(H)$.

Proposition 3.3. $(\Omega W)_*(\sigma^{-1}(x \circ y)) = \pm [\sigma^{-1}x, \sigma^{-1}y],$ where

 $(\Omega W)_* \colon H_*(\Omega(G \circ H)) \to H_*(\Omega(G \lor H)).$

Proof. By Lemma 2.6

$$\psi_*(x \circ y) = e \vartriangle \sigma^{-1}(x) \vartriangle \sigma^{-1}(y) \in H_*(S\Omega G \land \Omega H),$$

 \mathbf{SO}

$$(\Omega\psi)_*(\sigma^{-1}(x\circ y)) = \sigma^{-1}(x) \land \sigma^{-1}(y) \in H_*(\Omega G \land \Omega H)$$

regarded as a submodule of $H_*(S\Omega G \wedge \Omega H).$ It now suffices to evaluate the composition

$$\Omega G \land \Omega H \to \Omega(S\Omega G \land \Omega H) \xrightarrow{\Omega \xi} \Omega(\Omega G * \Omega H) \to \Omega(G \lor H),$$

where ξ is the standard homotopy equivalence $SX \wedge Y \simeq X * Y$.

Lemma 3.4. The composition

$$\Omega G \land \Omega H \to \Omega S(\Omega G \land \Omega H) \xrightarrow{\Omega \xi} \Omega(\Omega G \ast \Omega H) \to \Omega(G \lor H)$$

carries $\sigma^{-1}(x) \leq \sigma^{-1}(y) \in H_*(\Omega G \wedge \Omega H)$ to $\pm \left[\sigma^{-1}(x), \sigma^{-1}(y)\right]$.

Proof. We first need to describe the homotopy equivalence

$$SX \wedge Y \xrightarrow{\xi} X * Y$$

Here we write points of the join as tx + (1 - t)y, $0 \le t \le 1$. Therefore, X * Y is the quotient of $X \times I \times Y$ given by the identifications $(x, 0, y) \sim (x', 0, y)$ and $(x, 1, y) \sim (x, 1, y')$. Then ξ is given by the formula

$$\xi(t, x, y) = \begin{cases} (*, 1 - 3t, y), & 0 \le 3t \le 1, \\ (x, 3t - 1, y), & 1 \le 3t \le 2, \\ (x, 3 - 3t, *), & 2 \le 3t \le 3. \end{cases}$$

The map $\Omega X * \Omega Y \xrightarrow{\omega} X \lor Y$ is given by

$$(\omega_1, t, \omega_2) \to \begin{cases} \omega_1(2t), & 0 \le 2t \le 1, \\ \omega_2(2-2t), & 1 \le 2t \le 2. \end{cases}$$

Combining these we get

$$S\Omega X * \Omega Y \xrightarrow{\xi} \Omega X * \Omega Y \xrightarrow{\omega} X \lor Y$$

with a 6-part formula:

$$(t,\omega_1,\omega_2) = \begin{cases} (*,\omega_2(6t)), & 0 \le 6t \le 1, \\ *, & 1 \le 6t \le 2, \\ (\omega_1(6t-2),*), & 2 \le 6t \le 3, \\ (*,\omega_2(4-6t)), & 3 \le 6t \le 4, \\ *, & 4 \le 6t \le 5, \\ (\omega_1(6-6t),*), & 5 \le 6t \le 6, \end{cases}$$

so the adjoint takes the pair (ω_1, ω_2) to the product of loops $\omega_1^{-1} \omega_2^{-1} \omega_1 \omega_2$. The effect of this on a primitive element is the graded commutator.

Now the iterated circle product $ad^{n}(H)(G)$ has homology generated by classes of the form

$$(\cdots ((x \circ y_1) \circ y_2) \cdots \circ y_n),$$

where $x \in \widetilde{H}_*(G)$ and $y_i \in \widehat{H}_*(H)$. By Proposition 3.3,

$$(\Omega W)_* \left(\sigma^{-1} \left(\cdots \left(x \circ y_1 \right) \circ y_2 \right) \cdots \circ y_n \right)$$

is \pm the graded commutator

$$\left[\cdots\left[\left[\sigma^{-1}(x),\sigma^{-1}(y_1)\right],\sigma^{-1}(y_2)\right]\cdots\sigma^{-1}(y_*)\right],$$

where the classes x and y_i are thought of as classes in $\widetilde{H}_*(G \vee H)$.

Proof of Theorem 3(a). Now let $G_* = \widetilde{H}_*(G)$ and $H_* = \widetilde{H}_*(H)$. Let $L(G_* \oplus H_*)$ be the free Lie algebra generated by G_* and H_* , and let $L(H_*)$ be the free Lie algebra generated by H_* . Then Neisendorfer has analyzed the kernel

$$L(G_* \vee H_*) \to L(H_*)$$

([Nei09, 8.7.4]). He has shown that this is the free Lie algebra

$$L\left(\bigoplus_{n\geqslant 0}ad^n(H_*)(G_*)\right).$$

The universal enveloping algebra is thus the tensor algebra generated by the elements $ad^n(H_*)(G_*)$ for $n \ge 0$. Consequently the fiber of the projection $\Omega(G \lor H) \to \Omega H$ is

$$\Omega\left(\bigvee_{n\geqslant 0}ad^n(H)(G)\right).$$

This is homotopy equivalent to $\Omega(G \rtimes \Omega H)$, and the map

$$\bigvee_{n \ge 0} ad^n(H)(G) \to G \lor H$$

which factors through $G \rtimes \Omega H$ establishes the homotopy equivalence in Theorem 3.

Corollary 3.5. (a) If G is simply connected, $S\Omega G \simeq \bigvee_{n \ge 0} ad^n(G)(G)$,

(b) if both G and H are simply connected,

$$\Omega G * \Omega H \simeq \bigvee_{\substack{i \ge 0 \\ j \ge 1}} ad^j(H) \left(ad^i(G)(G) \right).$$

Proof. For (a) apply Corollary 3.2 and Theorem 3(a). For (b), we expand

$$\begin{split} \Omega G * \Omega H &\simeq (S\Omega G) \wedge \Omega H \\ &\simeq \left[\bigvee_{i \geqslant 0} \left(ad^i(G)(G) \right) \right] \wedge \Omega H \\ &\simeq \bigvee_{i \geqslant 0} \left[ad^i(G)(G) \wedge \Omega H \right] \\ &\simeq \bigvee_{\substack{i \geqslant 0 \\ j \geqslant 1}} ad^j(H) \left(ad^i(G)(G) \right) \end{split}$$

using Proposition 2.5 and Theorem 3(a).

Given a Theriault product $P = G_1 \circ \cdots \circ G_s$ with some fixed association, let us write $\ell(P) = s$ for the length of P.

Theorem 3.6. Suppose G and H are both simply connected co-H spaces and $k \ge 1$. Then there is a locally finite collection of iterated Theriault products $\{P_{\alpha}(k)\}$ of length ℓ_{α} and iterated Whitehead product maps

$$\omega_{\alpha}(k) \colon P_{\alpha}(k) \to G \lor H$$

such that

$$\Omega(G \vee H) \simeq \Omega\left(\bigvee_{\ell_{\alpha} > k} P_{\alpha}(k)\right) \times \Omega\left(\prod_{\ell_{\alpha} \leqslant k} P_{\alpha}(k)\right),$$

and the factors of the right-hand side are mapped to the left-hand side by the $\omega_{\alpha}(k)$.

Proof. For k = 1 we use the decomposition

$$\Omega(G \lor H) \simeq \Omega(\Omega G * \Omega H) \times \Omega(G \times H),$$

where $\Omega G * \Omega H$ is a bouquet of iterated Theriault products of length at least 2 by Corollary 3.5(b). Now we proceed by induction on k. Among the finite list of products $P_{\alpha}(k)$ of length k + 1, choose one which we label P. Then

$$\Omega(\bigvee_{\ell_{\alpha}>k} P_{\alpha}) \simeq \Omega(P \lor \bigvee_{\substack{\ell_{\alpha}>k\\P_{\alpha}\neq P}} P_{\alpha})$$
$$\simeq \Omega P \times \Omega(\bigvee_{\substack{\ell_{\alpha}>k\\P_{\alpha}\neq P}} P_{\alpha} \rtimes \Omega P).$$

The second factor has one less product of length k + 1. If we repeat this process once for each $P_{\alpha}(k)$ of length k + 1, we obtain

$$\Omega(\bigvee_{\ell(P_{\alpha})>k} P_{\alpha}) = \Omega P_j \times \dots \times \Omega P_m \times \Omega\Big(\bigvee_{\ell(P_{\alpha})>k+1} P_{\alpha}(k+1)\Big).$$

Now add the $P_1 \cdots P_m$ to the list of P_α with $\ell(P_\alpha) \leq k$ to obtain all $P_\alpha(k+1)$ with length $\leq k+1$.

Corollary 3.7. Suppose X is a finite dimensional co-associative co-H space and $f: X \to G \lor H$, where G and H are simply connected co-H spaces. Then f is a sum of iterated Whitehead products.

Proof. Suppose dim X = k and $f: X \to G \lor H$ is given. Decompose $\Omega(G \lor H)$ as in Theorem 3.6 and note that any product P_{α} of length k is at least k connected. Consequently the restriction of Ωf ,

$$(\Omega X)^{k-1} \xrightarrow{\Omega f} \Omega(G \vee H),$$
 factors through the product $\Omega(\prod_{\ell_{\alpha} \leq k} P_{\alpha}(k))$ and the adjoint,

$$S\left[(\Omega X)^{k-1}\right] \to G \lor H,$$

is a sum of iterated Whitehead products. However, f is the composition of this map with the co-H space structure map

$$X \to S\left[(\Omega X)^{k-1}\right]$$

which is a co-H map, so f is such a sum as well.

Proposition 3.8. If G and H are simply connected, then there is a homotopy equivalence

$$\phi \colon (G \lor H) \cup_W C(G \circ H) \to G \times H.$$

Proof. Since the composition

$$G \circ H \to \Omega G * \Omega H \to G \lor H \to G \times H$$

is null homotopic, there is an extension

$$C = (G \lor H) \cup C(G \circ H) \xrightarrow{\phi} G \times H.$$

The problem is to show that this map is a homotopy equivalence. We begin by observing that we can construct a right inverse ζ to $\Omega \phi$ as the sum of the loops on the inclusions of G and H into C:

$$\Omega G \times \Omega H \xrightarrow{\zeta} \Omega C \xrightarrow{\Omega \phi} \Omega G \times \Omega H,$$

so $(\Omega \phi)_*$: $H_*(\Omega C) \to H_*(\Omega G \times \Omega H)$ is an epimorphism. We will complete the proof by showing that the rank of $H_k(\Omega C)$ is less than or equal to the rank of $H_k(\Omega G \times \Omega H)$. We will need several lemmas.

Lemma 3.9. Write $\Omega G * \Omega H \simeq G \circ H \lor SQ$. Then the restriction

 $SQ \to \Omega G * \Omega H \xrightarrow{\omega} G \lor H \to C$

is null homotopic

Proof. We first look at the homotopy commutative diagram

Applying Theorem 3(a) we see that the composition

$$SQ \to \Omega G * \Omega H \to G \rtimes \Omega H \to G \lor H$$

is a sum of maps γ_i factoring through $ad^i(H)(G) \to G \lor H$ for $i \leq 2$. By induction on i we see that

$$ad^{i}(H)(G) \to G \lor H \to C$$

is null homotopic for $i \ge 1$. This follows since ad^i factors

$$ad^{i}(H)(G) \to ad^{i-1}(H)(G) \lor H \xrightarrow{ad^{i-1} \lor 1} G \lor H \lor H \to G \lor H \to C.$$

It follows from Lemma 3.9 that the mapping cone of ω is homotopy equivalent to $C \vee S^2 Q$. Recall that Ganea proved [Gan70] that given a fibration sequence $F \to E \to B$, one can construct a fibration sequence

$$F * \Omega B \to E \cup CF \xrightarrow{\pi} B$$
,

where π pinches the cone on F to a point. Apply this to the fibration sequence

$$\Omega G * \Omega H \xrightarrow{\omega} G \lor H \to G \times H$$

to obtain

$$\Omega G * \Omega H * \Omega (G * H) \to C \vee S^2 Q \xrightarrow{\pi} G \times H.$$

It is possible that the map $\pi|_{S^2Q}$ is nontrivial. However, $\pi|_{S^2Q}$ is the sum of the projections onto G and H, so it factors through C up to homotopy. Using such a factorization we can construct a homotopy equivalence

$$\Gamma \colon C \lor S^2 Q \to C \lor S^2 Q$$

such that $\pi\Gamma|_{S^2Q}$ is null homotopic. Replacing π with $\pi\Gamma$ does not alter the homotopy type of the fiber of π , so we can form the following diagram of fibrations:



The left-hand verticle fibration has a cross section since π_1 does, hence K is a co-H space and we have a splitting:

$$\Omega(\Omega G * \Omega H * \Omega(G \times H)) \simeq \Omega(S^2 Q \rtimes \Omega C) \times \Omega K.$$

We now study the homology algebra with a field of coefficients. Each space here is the loop space on a co-H space, so all the homology algebras are tensor algebras. It follows that for each $i \ge 0$

$$\operatorname{rank} H_i(\Omega G * \Omega H * \Omega(G \times H)) \geq \operatorname{rank} H_i(S^2 Q \rtimes \Omega C).$$

Suppose now that we have two power series $f(t) = \Sigma a_n t^n$ and $g(t) = \Sigma b_n t^n$ with a_i, b_i nonnegative integers. We will say that $f \ge g$ iff $a_i \ge b_i$ for each *i*. Write $\mathcal{X}(X)$ for the Poincaré series of a space X. In these terms, we have shown that

$$\mathcal{X}(\Omega G * \Omega H * \Omega(G \times H)) \ge \mathcal{X}(S^2 Q \rtimes \Omega C).$$

We now calculate the Poincaré series for each of these spaces. Suppose $\mathcal{X}(G) = 1+gt$ and $\mathcal{X}(H) = 1 + ht$, where g and h are polynomial in t with positive integral coefficient. We then have the following consequences:

$$\begin{split} \mathcal{X}(\Omega G) &= 1 + \frac{g}{1-g}, \\ \mathcal{X}(\Omega H) &= 1 + \frac{h}{1-h}, \\ \mathcal{X}(\Omega G * \Omega H) &= 1 + \frac{ght}{(1-g)(1-h)}, \\ \mathcal{X}(G \circ H) &= 1 + ght, \\ \mathcal{X}(SQ) &= 1 + ght \frac{g+h-gh}{(1-g)(1-h)}, \\ \mathcal{X}(S^2Q \rtimes \Omega C) &= 1 + ght^2 \frac{g+h-gh}{(1-g)(1-h)} \cdot \mathcal{X}(\Omega C) \end{split}$$

On the other hand,

$$\mathcal{X}(\Omega G \times \Omega H) = 1 + \frac{g + h - gh}{(1 - g)(1 - h)}$$

Consequently,

$$\mathcal{X}(\Omega G * \Omega H * (\Omega G \times \Omega H)) = 1 + \frac{ght^2(g+h-gh)}{(1-g)^2(1-h)^2}$$

Observe that if $h(t) = \Sigma c_n t^n \neq 0$ also has nonnegative coefficients, $f(t) \geq g(t)$ iff $h(t)f(t) \geq h(t)g(t)$. It follows that $\mathcal{X}(\Omega C) \leq \frac{1}{(1-g)(1-h)} = \mathcal{X}(\Omega G \times \Omega H)$, so rank $H_i(\Omega C) \leq \operatorname{rank} H_i(\Omega G \times \Omega H)$ for all *i*. Thus $(\Omega \phi)_*$ is an isomorphism, and hence ϕ_* is as well. Since this is true for any field of coefficients, ϕ_* induces isomorphisms in integral homology, and hence ϕ is a homotopy equivalence.

References

- [AG95] D. Anick and B. Gray, Small H spaces related to Moore spaces, Topology 34 (1995), no. 4, 859–881. MR1362790 (97a:55011)
- [Ani93] D. Anick, Differential algebras in topology, Research Notes in Mathematics, vol. 3, A K Peters, Ltd., Wellesley, MA, 1993, xxxvi+274 pp. MR1213682 (94h:55020)

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- [Ark62] M. Arkowicz, The generalized Whitehead product, Pacific J. Math 12 (1962), no. 1, 7–23. MR0155328 (27:5262)
- [CMN79] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, *Torsion in homotopy groups*, Ann. of Math. (2) **109** (1979), no. 1, 121–168. MR519355 (80e:55024)
- [Coh57] D. E. Cohen, Products and carrier theory, Proc. London Math. Soc. 7 (1957), no. 3, 219–248. MR0087941 (19:441a)
- [DT58] A. Dold and R. Thom, Quasifasserungen und unendliche symmetrische Producte, Ann. Math. 67 (1958), 239–281. MR0097062 (20:3542)
- [Gan70] T. Ganea, Cogroups and suspensions, Invent. Math. 9 (1970), 185–197. MR0267582 (42:2484)
- [Gra71] B. Gray, A note on the Hilton-Milnor Theorem, Topology 10 (1971), 199–201. MR0281202 (43:6921)
- [Gra88] _____, On the iterated suspension, Topology **27** (1988), no. 3, 301–310. MR963632 (89h:55016)
- [GT10] B. Gray and S. Theriault, An elementary construction of Anick's fibration, Geometry and Topology 14 (2010), 243–276. MR2578305
- [GTW] J. Grbić, S. Theriault, and J. Wu, Suspension splittings and James-Hopf Invariants for retracts of the loops on co-H spaces (private communication).
- [Hil59] P. J. Hilton, Homotopy theory and duality, mimeographed notes, 1959, Cornell University.
- [Nei09] J. Neisendorfer, Algebraic methods in unstable homotopy theory, New Mathematical Monographs, no. 12, Cambridge University Press, 2009. MR2604913
- [The03] S. D. Theriault, Homotopy decompositions involving the loops on a co-associative co-H space, Canada J. Math. 55 (2003), no. 1, 181–203. MR1952331 (2003k:55006)
- [Whi41] J. H. C. Whitehead, On adding relations to homotopy groups, Annals of Math. 42 (1941), no. 2, 409–428. MR0004123 (2:323c)

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