ON MULTIPLICATIVE CONDITIONALLY FREE CONVOLUTION

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ABSTRACT. Using the combinatorics of non-crossing partitions, we construct a conditionally free analogue of Voiculescu's S -transform. The result is applied to the analytical description of conditionally free multiplicative convolution and the characterization of infinite divisibility.

1. INTRODUCTION

This paper presents some results in conditionally free (or, shorter, following [8], c -free) probability. The concept of c -freeness was developed in the early 1990s (see [6], [7], [8]) in connection to freeness with amalgamation. Throughout the paper we will use the definition of c -freeness from [7] and [8]. Namely, if $\mathfrak{A}$ is an algebra and $\varphi, \psi : \mathfrak{A} \rightarrow \mathbb{C}$ are two normalized linear functionals, then the family $\{\mathfrak{A}_j\}_{j \in J}$ of subalgebras of $\mathfrak{A}$ is said to be c -free if

- (i) $\psi(a_1 \cdots a_n) = 0$,
- (ii) $\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$

for all $a_j \in \mathfrak{A}_{\varepsilon(j)}$ with $\varepsilon(j) \in J$ such that $\varepsilon(k) \neq \varepsilon(k+1)$, $1 \leq k \leq n-1$, and $\psi(a_j) = 0$, $1 \leq j \leq n$.

Moreover, as shown in [8], Theorem 2.2, if $\mathfrak{A}_1, \mathfrak{A}_2$ are two unital $*$ -algebras, each $\mathfrak{A}_k$ endowed with two states (normalized, positive, linear functionals) ψ_k, φ_k ($k = 1, 2$), then there exist two unique states ψ, φ on $\mathfrak{A}_1 * \mathfrak{A}_2$ (free product of $*$ -algebras with amalgamation over the unit) with respect to which $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are c -free and $\psi|_{\mathfrak{A}_k} = \psi_k$, $\varphi|_{\mathfrak{A}_k} = \varphi_k$.

Two important tools in free probability theory are the R - and S -transforms. Those are power series with the property that if X and Y are free random variables, then $R_{X+Y} = R_X + R_Y$ and $S_{XY} = S_X S_Y$. While a c -free version of the R -transform is constructed in [8] and used in several papers (such as [24], [17], [14]) for the study of c -free additive convolution of measures or of c -freeness with amalgamation, the literature lacked a similar development for the multiplicative case.

The present paper shows the construction of a suitable c -free version of the S -transform (in fact, as in [9], a more natural choice is its multiplicative inverse, called

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the T -transform) and demonstrates some of its applications in limit theorems and characterization of infinite divisibility.

The material is structured in seven sections. The second section contains the notation and preliminary results used throughout the paper, but mostly in the third section. The third section presents the construction of the cT -transform and proves its multiplicative property. The argument is based on enumerative combinatorial techniques, in the spirit of [16]; the lack of a suitable Fock space model makes difficult an analytical procedure, as in [12]. The fourth section defines the multiplicative c -free convolution of two measures on the unit circle and presents its connections to free and Boolean multiplicative convolutions. The fifth and the sixth sections present applications of the multiplicative property of the cT -transform for the study of limit distributions, respectively for the characterization of infinite divisibility. These sections use more analytical techniques, analogous to [3], [4] and [5]. The last section (the appendix) shows an explicit combinatorial formula for computing the coefficients of the T - and cT -transforms. The computations, generalizing a result from [9], demonstrate the use of non-crossing linked partitions in free probability.

2. PRELIMINARIES

2.1. The R -, T - and cR -transforms. Let $\mathfrak{A}$ be a complex unital algebra endowed with two linear functionals, ψ and φ . For $a_1, \dots, a_n \in \mathfrak{A}$, the free cumulant $R^n(a_1, \dots, a_n)$, respectively the c -free cumulant ${}^cR^n(a_1, \dots, a_n)$, is defined by the recurrences:

$$\begin{aligned} \psi(a_1 \cdots a_n) &= \sum_{l=1}^n \sum R^l(a_{i(1)}, \dots, a_{i(l)}) \left[\prod_{k=1}^{l-1} \psi(a_{i(k)+1} \cdots a_{i(k+1)-1}) \right] \psi(a_{i(l)+1} \cdots a_n), \\ \varphi(a_1 \cdots a_n) &= \sum_{l=1}^n \sum {}^cR^l(a_{i(1)}, \dots, a_{i(l)}) \left[\prod_{k=1}^{l-1} \psi(a_{i(k)+1} \cdots a_{i(k+1)-1}) \right] \varphi(a_{i(l)+1} \cdots a_n), \end{aligned}$$

where in both lines the second summation is done over all $i(0), \dots, i(l)$ such that $1 = i(0) < i(1) < \dots < i(l) \leq n$.

For $X \in \mathfrak{A}$, we will write R_X^n and ${}^cR_X^n$ for $R^n(X, \dots, X)$, respectively ${}^cR^n(X, \dots, X)$. The R -transform and the cR -transform of X are, respectively, the formal power series

$$R_X(z) = \sum_{n=1}^{\infty} R_X^n z^n, \quad {}^cR_X(z) = \sum_{n=1}^{\infty} {}^cR_X^n z^n.$$

Let $m_X(z)$, respectively $M_X(z)$, be the moment generating series of X with respect to ψ and φ , i.e.,

$$m_X(z) = \sum_{n=1}^{\infty} \psi(X^n) z^n, \quad M_X(z) = \sum_{n=1}^n \varphi(X^n) z^n.$$

As shown in [16] and [8], the definitions of R^n and ${}^cR^n$ give

$$\begin{aligned} (1) \quad & R(z[1 + m_X(z)]) = m_X(z), \\ (2) \quad & {}^cR(z[1 + m_X(z)])(1 + M_X(z)) = M_X(z)(1 + m_X(z)). \end{aligned}$$

Two elements, X and Y , from $\mathfrak{A}$ are said to be conditionally free (c-free) in $(\mathfrak{A}, \varphi, \psi)$ if the subalgebras generated by them in $\mathfrak{A}$ are conditionally free, as defined in the introduction. The key properties of the R - and cR -transforms are summarized in the following result:

Lemma 2.1. *Let X, Y be c-free in $(\mathfrak{A}, \varphi, \psi)$.*

(i) *Let $a_k \in \{X, Y\}$, $1 \leq k \leq n$. If there exist $k \neq l$ such that $a_k \neq a_l$, then $R^n(a_1, \dots, a_n) = 0$ and ${}^cR^n(a_1, \dots, a_n) = 0$.*

(ii) *Note that $\mathfrak{A}^\circ = \mathfrak{A} \setminus \text{Ker}(\psi)$. For $Z \in \mathfrak{A}^\circ$, we define T_Z as the formal power series given by*

$$T_Z(z) = \left(\frac{1}{z} R_Z(z) \right) \circ \left(R_Z^{(-1)}(z) \right),$$

where $F^{(-1)}(z)$ is the substitutional inverse of the formal series $F(z)$. With the above notation, if $X, Y \in \mathfrak{A}^\circ$, then

$$T_{XY}(z) = T_X(z)T_Y(z).$$

The proofs for the first part of (i) can be found, for example, in [12], Theorem 2.2, and [16], Theorem 11.16. The second part of (i) is proved in [8], Theorem 3.1.

For (ii), in [12], Corollary 2.4. or Theorem 3.5, and [16], Corollary 18.17, the property is proved for $S(z) = [T(z)]^{-1}$ (inverse with respect to multiplication). As in [9], for the purpose of this paper it will be simpler to consider $T(z) = [S(z)]^{-1}$.

2.2. Boolean independence, the η - and B -transforms. For $\mathfrak{A}$ a unital algebra and $\varphi : \mathfrak{A} \rightarrow \mathbb{C}$ a normalized functional, two subalgebras $\mathfrak{A}_1, \mathfrak{A}_2$ are said to be *Boolean independent* if

$$\varphi(x_1 y_1 x_2 y_2 \cdots) = \varphi(x_1) \varphi(y_1) \varphi(x_2) \varphi(y_2) \cdots$$

for all $x_k \in \mathfrak{A}_1$ and $y_k \in \mathfrak{A}_2$. We will say that $X, Y \in \mathfrak{A}$ are Boolean independent if the non-unital algebras generated in $\mathfrak{A}$ by X , respectively Y , are Boolean independent.

If there exists an algebra $\mathfrak{A}_0$ such that $\mathfrak{A} = \mathbb{C} \oplus \mathfrak{A}_0$ (direct sum of vector spaces), and $\psi = Id_{\mathbb{C}} \oplus 0_{\mathfrak{A}_0}$, then conditional freeness with respect to (φ, ψ) is equivalent to Boolean independence with respect to φ .

For the results in Sections 4-6, we will need the definitions and results below (see [18] or [11] for proofs).

Definition 2.2. Let $X \in \mathfrak{A}$ and $M_X(z)$ be the moment generating series of X with respect to φ , that is, $M_X(z) = \sum_{n=1}^{\infty} \varphi(X^n) z^n$. The η -transforms, respectively B -transforms, of X are the formal power series given by the relations $\eta_X(z) = \frac{M_X(z)}{1+M_X(z)}$, respectively $B_X(z) = \frac{1}{z} \eta_X(z)$.

Proposition 2.3. *If X and Y are two Boolean independent elements of $\mathfrak{A}$, then:*

- (i) $\eta_{X+Y}(z) = \eta_X(z) + \eta_Y(z)$ (see [21]);
- (ii) $B_{X+Y}(z) = B_X(z) + B_Y(z)$ (see [21]) and $B_{(1+X)(1+Y)}(z) = B_{(1+X)}(z) \cdot B_{(1+Y)}(z)$ (see [11], [18]).

2.3. Non-crossing partitions. By a partition on the ordered set $\{v_1, v_2, \dots, v_n\}$ we will understand a collection of mutually disjoint subsets $\gamma = (B_1, \dots, B_q)$, called *blocks*, whose union is the entire set $\{v_1, \dots, v_n\}$. A *crossing* is a sequence $v_i < v_j < v_k < v_l$ with the property that there exist two different blocks B_r and B_s such that $v_i, v_k \in B_r$ and $v_j, v_l \in B_s$. A partition that has no crossings will be called non-crossing. The set of all non-crossing partitions on $\{1, 2, \dots, n\}$ will be denoted by $NC(n)$. To simplify the notation, we will also write $\gamma \in NC(n)$ for γ a non-crossing partition of $\{v_1, \dots, v_n\}$ identifying v_i with i .

For $\gamma \in NC(n)$, a block $B = (i_1, \dots, i_k)$ of γ will be called *interior* if there exists another block $D \in \gamma$ and $i, j \in D$ such that $i < i_1, i_2, \dots, i_k < j$. A block will be called *exterior* if it is not interior. The set of all interior, respectively exterior, blocks of γ will be denoted by $Int(\gamma)$, respectively $Ext(\gamma)$.

In this framework, the recurrences defining R^n , respectively ${}^cR^n$, can be written as

$$\begin{aligned}\psi(a_1 \cdots a_n) &= \sum_{\gamma \in NC(n)} \prod_{\substack{B \in \gamma \\ B = (i_1, \dots, i_k)}} R^k(a_{i_1}, \dots, a_{i_k}), \\ \varphi(a_1 \cdots a_n) &= \sum_{\gamma \in NC(n)} \prod_{\substack{D \in Ext(\gamma) \\ D = (i_1, \dots, i_k)}} {}^cR^k(a_{i_1}, \dots, a_{i_k}) \prod_{\substack{B \in Int(\gamma) \\ B = (j_1, \dots, j_s)}} R^s(a_{j_1}, \dots, a_{j_s}).\end{aligned}$$

$NC(n)$ has a lattice structure with respect to the reversed refinement order, with the biggest, respectively smallest, element $\mathbb{1}_n = (1, 2, \dots, n)$, respectively $0_n = (1), \dots, (n)$. For $\pi, \sigma \in NC(n)$ we will denote by $\pi \vee \sigma$ their join (smallest common upper bound).

The Kreweras complement $Kr(\pi)$ of $\pi \in NC(n)$ is defined as follows. Consider the symbols $\bar{1}, \dots, \bar{n}$ such that $1 < \bar{1} < 2 < \dots < n < \bar{n}$. Then $Kr(\pi)$ is the largest element of $NC(\bar{1}, \dots, \bar{n}) \cong NC(n)$ such that

$$\pi \cup Kr(\pi) \in NC(1, \bar{1}, \dots, n, \bar{n}).$$

The following notation will also be used:

$$\begin{aligned}NC_1(n) &= \{\sigma : \sigma \in NC(n), \sigma \text{ has only one exterior block}\}; \\ NC_2(n) &= \{\sigma : \sigma \in NC(n), \sigma \text{ has only two exterior blocks}\}; \\ NC_S(n) &= \{\sigma : \sigma \in NC(2n), \text{the elements from the same block of } \sigma \text{ have the same parity}\}.\end{aligned}$$

For $\sigma \in NC_S(2n)$, denote σ_+ , respectively σ_- , the restriction of σ to the even, respectively odd, numbers from $\{1, 2, \dots, 2n\}$. Define

$$NC_0(n) = \{\sigma : \sigma \in NC_S(n), \sigma_+ = Kr(\sigma_-)\}.$$

Also, we will need to consider the mappings

$$NC(n) \times NC(m) \ni (\pi, \sigma) \mapsto \pi \oplus \sigma \in NC(m+n),$$

the juxtaposition of partitions, and

$$NC(n) \ni \sigma \mapsto \hat{\sigma} \in NC(2n)$$

constructed by doubling the elements; i.e., if $(i_1, i_2, \dots, i_s)$ is a block of σ , then $(2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \dots, 2i_s - 1, 2i_s)$ is a block of $\hat{\sigma}$.

3. THE cT -TRANSFORM

Let $\mathfrak{A}, \varphi, \psi$ be as above and $X \in \mathfrak{A} \setminus \text{Ker}(\psi)$. Denote by $m_X(z)$, respectively $M_X(z)$, the moment generating power series of X with respect to ψ and φ , as in the formulas (1), (2).

In this section we will construct a formal power series ${}^cT_X(z)$ such that:

- (i) for $\varphi = \psi$ one has that $T_X(z) = {}^cT_X(z)$ (see Lemma 2.1 for the definition of $T_X(z)$);
- (ii) $M_X(z)$ can be obtained from $T_X(z)$ and ${}^cT_X(z)$ via substitutional composition, substitutional inverse and algebraic operations;
- (iii) if X and Y are c-free elements from $\mathfrak{A}$, then

$${}^cT_{XY}(z) = {}^cT_X(z){}^cT_Y(z).$$

The construction of ${}^cT_X(z)$ is presented as a natural consequence of the combinatorial properties of the free and c-free cumulants R^n and ${}^cR^n$.

For $a_1, \dots, a_n \in \mathfrak{A}$ we need to consider the multilinear maps

$$\begin{aligned} \kappa_\pi[a_1, \dots, a_n] &= \prod_{\substack{B \in \pi \\ B = (i(1), \dots, i(p))}} R^p(a_{i(1)} \cdots a_{i(p)}), \\ \mathcal{K}_\pi[a_1, \dots, a_n] &= \left[\prod_{\substack{D \in \text{Ext}(\pi) \\ D = (i(1), \dots, i(p))}} {}^cR^p(a_{i(1)} \cdots a_{i(p)}) \right] \left[\prod_{\substack{B \in \text{Int}(\pi) \\ B = (i(1), \dots, i(q))}} R^q(a_{i(1)} \cdots a_{i(q)}) \right]. \end{aligned}$$

First, let us note the following result on free and c-free cumulants with constants among entries:

Lemma 3.1. *For all $a_1, \dots, a_n \in \mathfrak{A}$, one has that*

$$\begin{aligned} R^{n+1}(a_1, \dots, a_j, 1, a_{j+1}, \dots, a_n) &= 0, \\ {}^cR^{n+1}(a_1, \dots, a_j, 1, a_{j+1}, \dots, a_n) &= 0. \end{aligned}$$

Proof. The first part is proved in [16] (Proposition 11.15, p. 182). We will prove the second part by induction. First, since $\varphi(a) = \varphi(a1)$, the definition of ${}^cR^n$ gives

$${}^cR^2(a, 1) + {}^cR^1(a)1 = {}^cR^1(a);$$

therefore ${}^cR^2(a, 1) = 0$.

In general, $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_j 1 a_{j+1} \cdots a_n)$. But

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \mathcal{K}_\pi[a_1, \dots, a_n]$$

while

$$\begin{aligned} \varphi(a_1 \cdots a_j 1 a_{j+1} \cdots a_n) &= \sum_{\pi \in NC(n+1)} \mathcal{K}_\pi[a_1, \dots, a_j, 1, a_{j+1}, \dots, a_n] \\ &= \sum_{\substack{\pi \in NC(n+1) \\ (j) \in \pi}} \mathcal{K}_\pi[a_1, \dots, a_j, 1, a_{j+1}, \dots, a_n] + {}^cR^{n+1}(a_1, \dots, 1, \dots, a_n) \\ &\quad + \sum_{\substack{\pi \in NC(n+1) \\ (j) \notin \pi \\ \pi \neq 1_{n+1}}} \mathcal{K}_\pi[a_1, \dots, a_j, 1, a_{j+1}, \dots, a_n]. \end{aligned}$$

The last term cancels from the induction hypothesis. Consider the bijection $\pi \mapsto \pi'$ between $\{\pi : \pi \in NC(n+1), (j) \in \pi\}$ and $NC(n)$, where π' is obtained by erasing the block (j) from π . It has the property that

$$\mathcal{K}_\pi[a_1, \dots, a_j, 1, a_{j+1}, \dots, a_n] = \mathcal{K}_{\pi'}[a_1, \dots, a_n].$$

Therefore

$$\varphi(a_1 \cdots a_j 1 a_{j+1} \cdots a_n) = \varphi(a_1 \cdots a_n) + {}^c R^{n+1}(a_1, \dots, a_j, 1, a_{j+1}, \dots, a_n),$$

hence the conclusion. $\square$

Next we will focus on free and c-free cumulants with products as entries. The next lemma is a version of the Leonov/Shiryaev formula from [19] and [15], Proposition 3.3.

Lemma 3.2. *Let $X, Y \in \mathfrak{A}$, c-free. For all $\pi \in NC(n)$ one has that:*

$$(a) \quad \kappa_\pi[XY, \dots, XY] = \sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \vee \widehat{0}_n = \widehat{\pi}}} \kappa_\sigma[X, Y, \dots, X, Y].$$

Particularly,

$$R_{XY}^n = \sum_{\sigma \in NC_0(2n)} \kappa_\sigma[X, Y, \dots, X, Y].$$

$$(b) \quad \mathcal{K}_\pi[XY, \dots, XY] = \sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \vee \widehat{0}_n = \widehat{\pi}}} \mathcal{K}_\sigma[X, Y, \dots, X, Y].$$

Particularly,

$${}^c R_{XY}^n = \sum_{\sigma \in NC_0(2n)} \mathcal{K}_\sigma[X, Y, \dots, X, Y].$$

Proof. Lemma 3.2(a) is shown in [16]. More precisely, from [16], Theorem 11.12, it follows that

$$\kappa_\pi[XY, \dots, XY] = \sum_{\substack{\sigma \in NC(2n) \\ \sigma \vee \widehat{0}_n = \widehat{\pi}}} \kappa_\sigma[X, Y, \dots, X, Y]$$

and since mixed cumulants vanish (see Lemma 2.1), only partitions from $NC_S(n)$ contribute to the summation. The second part is just a reformulation of Theorem 14.4 from [16] with the notation from Section 2.3.

The proof of (b) will be done by induction on n . For $n = 1$, the statement is trivial:

$${}^c R^1[XY] = \varphi(XY) = \varphi(X \cdot Y) = {}^c R^2(X, Y) + {}^c R^1(X) {}^c R^1(Y) = {}^c R^1(X) {}^c R^1(Y),$$

since the mixed c-free cumulants vanish (see Lemma 2.1).

For the induction step, we distinguish two cases:

Case 1: $\pi \neq \mathbb{1}_n$.

If π has more than one exterior block, then $\pi = \pi_1 \oplus \pi_2$ for some $m < n$, $\pi_1 \in NC(m)$, $\pi_2 \in NC(n-m)$. One has that

$$\mathcal{K}_\pi[XY, \dots, XY] = \mathcal{K}_{\pi_1}[XY, \dots, XY] \mathcal{K}_{\pi_2}[XY, \dots, XY],$$

and the result follows from the induction hypothesis.

If π has exactly one exterior block, then

$$\mathcal{K}_\pi[XY, \dots, XY] = {}^c R^p[XY, \dots, XY] \mathcal{K}_{\bar{\pi}}[XY, \dots, XY],$$

where p is the length of the exterior block of π , and $\bar{\pi}$ is the non-crossing partition obtained by erasing the exterior block of π . The result now follows from part (a) and the induction hypothesis.

Case 2: $\pi = \mathbb{1}_n$.

We need to show that

$$(3) \quad {}^cR_{XY}^n = \sum_{\sigma \in NC_0(2n)} \mathcal{K}_\sigma[X, Y, \dots, X, Y].$$

Before proceeding with the proof, let us take a better look at the right-hand side of (3). Since any $\sigma \in NC_0(2n)$ has exactly 2 exterior blocks, one containing 1 and one containing $2n$, each $\mathcal{K}_\sigma[X, Y, \dots, X, Y]$ will have exactly two factors of the type ${}^cR^m$, namely ${}^cR_X^p$ and ${}^cR_Y^q$, where p, q are, respectively, the lengths of the first and second exterior blocks of σ . Also, $\sigma \in NC_0(2n)$ implies that all other factors of $\mathcal{K}_\sigma[X, Y, \dots, X, Y]$ are of the form R_X^s or R_Y^t .

One has that

$$\begin{aligned} \varphi((XY)^n) &= \varphi(XY \cdot XY \cdots XY) \\ &= \sum_{\pi \in NC(n)} \mathcal{K}_\pi[XY, \dots, XY] \\ &= {}^cR_{XY}^n + \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbb{1}_n}} \mathcal{K}_\pi[XY, \dots, XY]. \end{aligned}$$

On the other hand,

$$(4) \quad \varphi((XY)^n) = \varphi(X \cdot Y \cdots X \cdot Y) = \sum_{\sigma \in NC(2n)} \mathcal{K}_\sigma[X, Y, \dots, X, Y].$$

Since all the mixed cumulants vanish, (4) becomes:

$$\begin{aligned} \varphi((XY)^n) &= \sum_{\sigma \in NC_S(2n)} \mathcal{K}_\sigma[X, Y, \dots, X, Y] \\ &= \sum_{\sigma \in NC_0(2n)} \mathcal{K}_\sigma[X, Y, \dots, X, Y] + \sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \notin NC_0(2n)}} \mathcal{K}_\sigma[X, Y, \dots, X, Y]. \end{aligned}$$

But

$NC_S(2n) = \bigcup_{\pi \in NC(n)} \{\sigma : \sigma \in NC_S(2n), \sigma \vee \widehat{0_n} = \widehat{\pi}\}$. Also, for $\sigma \in NC_S(2n)$, one has that $\sigma \in NC_0(2n)$ if and only if $\sigma \vee \widehat{0_n} = \mathbb{1}_{2n}$. Therefore:

$$NC_S(2n) \setminus NC_0(2n) = \bigcup_{\substack{\pi \in NC(n) \\ \pi \neq \mathbb{1}_n}} \{\sigma : \sigma \in NC_S(2n), \sigma \vee \widehat{0_n} = \widehat{\pi}\}.$$

It follows that

$$\begin{aligned} \sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \notin NC_0(2n)}} \mathcal{K}_\sigma[X, Y, \dots, X, Y] &= \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbb{1}_n}} \sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \vee \widehat{0_n} = \widehat{\pi}}} \mathcal{K}_\sigma[X, Y, \dots, X, Y] \\ &= \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbb{1}_n}} \mathcal{K}_\pi[XY, \dots, XY], \end{aligned}$$

so the proof is now complete. $\square$

For stating the next result we first need a brief review of the operation $\boxtimes$ (boxed convolution), as defined in [16].

Consider the formal power series $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$. For $\pi \in NC(n)$, we define

$$Cf_{\pi}(f) = \prod_{B \in \pi} \alpha_{|B|}.$$

If $g(z) = \sum_{n=1}^{\infty} \beta_n z^n$ is another formal power series, we define their boxed convolution

$$f(z) \boxtimes g(z) = (f \boxtimes g)(z) = \sum_{n=1}^{\infty} \gamma_n z^n$$

by

$$\gamma_n = \sum_{\pi \in NC(n)} Cf_{\pi}(f) \cdot Cf_{Kr(\pi)}(g).$$

We will need the following two results proved in [16]:

Lemma 3.3 (see [16], Proposition 18.21). *Suppose $\alpha_1 \neq 0$. Then*

$$f^{(-1)} \circ (f \boxtimes g) = \frac{1}{\alpha_1} \left(f \overset{\vee}{\boxtimes} g \right),$$

where

$$\left(f \overset{\vee}{\boxtimes} g \right)(z) = \sum_{n=1}^{\infty} \lambda_n z^n$$

for

$$\lambda_n = \sum_{\substack{\pi \in NC(n) \\ (1) \in \pi}} Cf_{\pi}(f) \cdot Cf_{Kr(\pi)}(g).$$

Lemma 3.4 (see [16], Proposition 17.2). *If X and Y are free, then*

$$R_{XY}(z) = R_X(z) \boxtimes R_Y(z).$$

Now we can proceed with the main theorems of this section. To ease the notation, say that $R_X(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ and $R_Y(z) = \sum_{n=1}^{\infty} \beta_n z^n$.

Theorem 3.5. *If X, Y are c -free, then*

$$\frac{1}{z} {}^c R_{XY}(z) = \left[\left(\frac{1}{z} {}^c R_X \right) \circ \left(\frac{1}{\alpha_1} R_X \overset{\vee}{\boxtimes} R_Y \right) \right] \left[\left(\frac{1}{z} {}^c R_Y \right) \circ \left(\frac{1}{\beta_1} R_Y \overset{\vee}{\boxtimes} R_X \right) \right].$$

Proof. As shown in Lemma 3.2,

$$(5) \quad {}^c R_{XY}^n = \sum_{\sigma \in NC_0(2n)} \mathcal{K}_{\sigma}[X, Y, \dots, X, Y].$$

Now fix $\sigma \in NC_0(2n)$.

First note that, with the notation from Section 2.3, any block of σ is also a block of either σ_- or $\sigma_+ = Kr(\sigma_-)$. Hence

$$\begin{aligned} \kappa_\sigma[X, Y, \dots, X, Y] &= \prod_{B \in \sigma_-} R_X^{|B|} \prod_{B \in \sigma_+} R_Y^{|B|} \\ (6) \qquad \qquad \qquad &= Cf_{\sigma_-}(R_X) \cdot Cf_{Kr(\sigma_-)}(R_Y). \end{aligned}$$

Second, note that any $\sigma \in NC_0(2n)$ has exactly 2 exterior blocks. Indeed, the block $B_1 = (b_1, \dots, b_p)$ with $b_1 = 1$ is exterior, since no other blocks contain elements less than 1. Also, the block $B_2 = (d_1, d_2, \dots, d_q)$ with $d_1 = b_p + 1$ must be exterior, since σ is non-crossing. If $d_q + 1 \leq 2n$, the block containing it is also exterior, but since, with the notation from Section 2.3, we have that $\sigma_+ = Kr(\sigma_-)$, it follows that b_p and $d_q + 1$ must be in the same block, which contradicts the choice of B_1 . Hence $d_q = 2n$ and the only two exterior blocks of σ are B_1 and B_2 .

Now let $1 = b_1 < b_2 < \dots < b_p$ and $b_p + 1 = d_1 < d_2 < \dots < d_q = 2n$ be the exterior blocks of σ . Denote by π_k the restriction of σ to $(b_k + 1, b_k + 2, \dots, b_{k+1} - 1)$ and by ω_l the restriction of σ to $(d_l + 1, d_l + 2, \dots, d_{l+1} - 1)$. From the definition of $\mathcal{K}_\sigma$, it follows that

$$\begin{aligned} \mathcal{K}_\sigma[X, Y, \dots, X, Y] &= {}^cR_X^p \cdot \left(\prod_{k=1}^{p-1} \kappa_{\pi_k}[Y, X, \dots, X, Y] \right) \\ &\quad \cdot {}^cR_Y^q \cdot \left(\prod_{l=1}^{q-1} \kappa_{\omega_l}[X, Y, \dots, X, Y, X] \right). \end{aligned}$$

Since $\sigma \in NC_0(2n)$, one has that $(1) \oplus \pi_k \in NC_0(b_{k+1} - b_k + 1)$ and $(1) \oplus \omega_l \in NC_0(d_{l+1} - d_l + 1)$. Therefore, utilizing (6) and the definition of the functions κ_π , we obtain:

$$\begin{aligned} \kappa_{\pi_k}[Y, X, \dots, Y] &= \frac{1}{\alpha_1} \kappa_{(1)}[X] \kappa_{\pi_k}[Y, X, \dots, Y] \\ &= \frac{1}{\alpha_1} \kappa_{(1) \oplus \pi_k}[X, Y, \dots, X, Y] \\ &= \frac{1}{\alpha_1} Cf_{((1) \oplus \pi_k)_-}(R_X) \cdot Cf_{Kr(((1) \oplus \pi_k)_-)}(R_Y). \end{aligned}$$

Similarly,

$$\kappa_{\omega_l}[X, Y, X, \dots, X] = \frac{1}{\beta_1} Cf_{((1) \oplus \omega_l)_-}(R_Y) \cdot Cf_{Kr(((1) \oplus \omega_l)_-)}(R_X),$$

hence (5) implies the conclusion. $\square$

Theorem 3.6. *If $X, Y \in \mathfrak{A}^\circ$ are c-free, then*

$$\left[\left(\frac{1}{z} {}^cR_{XY} \right) \circ \left(R_{XY}^{(-1)}(z) \right) \right] = \left[\left(\frac{1}{z} {}^cR_X \right) \circ \left(R_X^{(-1)}(z) \right) \right] \cdot \left[\left(\frac{1}{z} {}^cR_Y \right) \circ \left(R_Y^{(-1)}(z) \right) \right].$$

Proof. From Theorem 3.5, one has:

$$\frac{1}{z} {}^cR_{XY}(z) = \left[\left(\frac{1}{z} {}^cR_X \right) \circ \left(\frac{1}{\alpha_1} R_X \boxtimes R_Y \right) \right] \left[\left(\frac{1}{z} {}^cR_Y \right) \circ \left(\frac{1}{\beta_1} R_Y \boxtimes R_X \right) \right].$$

Using Lemma 3.3, the above equality becomes

$$\begin{aligned} \frac{1}{z} {}^c R_{XY}(z) &= \left[\left(\frac{1}{z} {}^c R_X \right) \circ \left(R_X^{(-1)} \circ (R_X \boxtimes R_Y) \right) \right] \\ &\quad \cdot \left[\left(\frac{1}{z} {}^c R_Y \right) \circ \left(R_Y^{(-1)} \circ (R_Y \boxtimes R_X) \right) \right] \\ &= \left[\left(\left(\frac{1}{z} {}^c R_X \right) \circ R_X^{(-1)} \right) \circ (R_X \boxtimes R_Y) \right] \\ &\quad \cdot \left[\left(\left(\frac{1}{z} {}^c R_Y \right) \circ R_Y^{(-1)} \right) \circ (R_Y \boxtimes R_X) \right]. \end{aligned}$$

But Lemma 3.4 gives $R_{XY} = R_X \boxtimes R_Y$, so

$$\frac{1}{z} {}^c R_{XY}(z) = \left[\left(\left(\frac{1}{z} {}^c R_X \right) \circ R_X^{(-1)} \right) \circ (R_{XY}) \right] \left[\left(\left(\frac{1}{z} {}^c R_Y \right) \circ R_Y^{(-1)} \right) \circ (R_{XY}) \right].$$

Finally, composing at the right to $R_{XY}^{(-1)}$ one gets the conclusion. $\square$

Corollary 3.7. *The power series ${}^c T_x(z)$ defined by*

$${}^c T_X(z) = \left(\frac{1}{z} {}^c R_X(z) \right) \circ \left(R^{(-1)}(z) \right)$$

satisfies the properties (i)-(iii) described in the beginning of the section.

Proof. (iii) is proved in Theorem 3.6. (i) is an immediate consequence of comparing the definitions of T_X (as in Lemma 2.1) and of ${}^c T_X$. Finally, (ii) is implied by (1), (2) and the definition of ${}^c T_X$. $\square$

4. MULTIPLICATIVE CONDITIONALLY FREE CONVOLUTION OF MEASURES

Let $\mathcal{M}_{\mathbb{T}}$ be the family of all Borel probability measures supported on the unit circle $\mathbb{T}$ and let $\mathcal{M}_{\mathbb{T}}^{\times}$ be the set of all measures $\nu \in \mathcal{M}_{\mathbb{T}}$ such that $\int_{\mathbb{T}} \zeta d\nu(\zeta) \neq 0$. An element μ from $\mathcal{M}_{\mathbb{T}}$ can be identified with a normalized positive linear functional on the $*$ -algebra $\mathbb{C}\langle X, X^* \rangle$ with $XX^* = 1$ via

$$\mu(X^n) = \int_{\mathbb{T}} \zeta^n d\mu(t) \quad (n \geq 0).$$

Given (μ_1, ν_1) and (μ_2, ν_2) two pairs of measures from $\mathcal{M}_{\mathbb{T}}$, we define their conditionally free multiplicative convolution, $(\mu_1, \nu_1) \boxtimes (\mu_2, \nu_2)$, as the pair of measures (μ, ν) constructed as follows. As above, we identify μ_j, ν_j with functionals on $\mathfrak{A}_j = \mathbb{C}\langle X_j, X_j^* \rangle$ ($j = 1, 2$) and, as stated in the introduction, there are two unique positive, normalized, linear functionals μ, ν on

$$\mathfrak{A} = \mathbb{C}\langle X_1, X_1^*, X_2, X_2^* \rangle = \mathfrak{A}_1 * \mathfrak{A}_2$$

such that $\mu|_{\mathfrak{A}_k} = \mu_k$, $\nu|_{\mathfrak{A}_k} = \nu_k$ and with respect to which $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are c-free. Denote $X = X_1 X_2 \in \mathfrak{A}$. Since $X_1 X_1^* = X_2 X_2^* = 1$, it follows that $XX^* = 1$, so the restrictions of μ, ν to $\mathbb{C}\langle X, X^* \rangle$ are given by elements of $\mathcal{M}_{\mathbb{T}}$.

The next two sections utilize the construction from Section 3 to address some analytical properties of the operation $\boxtimes$. More precisely, Section 5 states some limit theorems and Section 6 describes $\boxtimes$ -infinite divisibility.

First we need some notation. For $\mu \in \mathcal{M}_{\mathbb{T}}$ we denote

$$m_{\mu}(z) = \int_{\mathbb{T}} \frac{z\zeta}{1-z\zeta} d\mu(\zeta),$$

the moment generating function of μ , analytic in the unit disk $\mathbb{D}$.

Now consider $\mu \in \mathcal{M}_{\mathbb{T}}$ and $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$. To the pair (μ, ν) we associate the functions $R_{\nu}(z)$, ${}^cR_{(\mu, \nu)}(z)$, $T_{\nu}(z)$, ${}^cT_{(\mu, \nu)}(z)$, analytic in some neighborhood of zero, and given by the following relations:

$$\begin{aligned} R_{\nu}(z[1+m_{\nu}(z)]) &= m_{\nu}(z), \\ {}^cR_{(\mu, \nu)}(z[1+m_{\nu}(z)] \cdot (1+m_{\mu}(z))) &= m_{\mu}(z) \cdot (1+m_{\nu}(z)), \\ T_{\nu}(z) &= \left(\frac{1}{z} R_{\nu}(z) \right) \circ \left(R_{\nu}^{(-1)}(z) \right), \\ {}^cT_{(\mu, \nu)}(z) &= \left(\frac{1}{z} {}^cR_{(\mu, \nu)}(z) \right) \circ \left(R_{\nu}^{(-1)}(z) \right). \end{aligned}$$

The following is an immediate consequence of the properties of the cT -transform discussed in Section 3 (see Corollary 3.7):

Remark 4.1. If $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}^{\times}$, their multiplicative conditionally free convolution, $(\mu_1, \nu_1) \boxtimes (\mu_2, \nu_2)$, is the unique pair $(\mu, \nu) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}^{\times}$ such that $T_{\nu}(z) = T_{\nu_1}(z) \cdot T_{\nu_2}(z)$ and ${}^cT_{(\mu, \nu)}(z) = {}^cT_{(\mu_1, \nu_1)}(z) \cdot {}^cT_{(\mu_2, \nu_2)}(z)$.

It will be convenient to introduce a variation of the function ${}^cT_{(\mu, \nu)}$ as follows. Namely, for the measures $\mu \in \mathcal{M}_{\mathbb{T}}$ and $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$, we consider the analytic function

$$(7) \quad \Sigma_{(\mu, \nu)}(z) = {}^cT_{(\mu, \nu)}\left(\frac{z}{1-z}\right)$$

in a neighborhood of zero.

Similarly to Definition 2.2, we denote

$$\eta_{\mu}(z) = \frac{m_{\mu}(z)}{1+m_{\mu}(z)}$$

and

$$B_{\mu}(z) = \frac{m_{\mu}(z)}{z(1+m_{\mu}(z))}.$$

Lemma 4.2. For $\eta_{\nu}(z)$ and $B_{\mu}(z)$, defined above, one has

$$\Sigma_{(\mu, \nu)} = \left[\frac{1}{z} \eta_{\mu}(z) \right] \circ \left[\eta_{\nu}^{(-1)}(z) \right] = B_{\mu} \left(\eta_{\nu}^{(-1)}(z) \right).$$

Proof. The relation $R_{\nu}(z[1+m_{\nu}(z)]) = m_{\nu}(z)$ (i.e., the definition of R_{ν}) implies that

$$R_{\nu}^{(-1)}(z) = (1+z)m_{\nu}^{(-1)}(z),$$

hence

$$(8) \quad R_{\nu}^{(-1)}(m_{\nu}(z)) = z(1+m_{\nu}(z)).$$

Let us now consider the definition of ${}^cR_{(\mu, \nu)}$,

$${}^cR_{(\mu, \nu)}(z[1+m_{\nu}(z)] \cdot (1+m_{\mu}(z))) = m_{\mu}(z) \cdot (1+m_{\nu}(z)),$$

which implies that

$$\frac{{}^cR_{(\mu,\nu)}(z(1+m_\nu(z)))}{z(1+m_\nu(z))} = \frac{m_\mu(z)}{z(1+m_\mu(z))}.$$

Taking (8) into consideration it follows that

$$\frac{{}^cR_{(\mu,\nu)}\left(R_\nu^{(-1)}(m_\nu(z))\right)}{R_\nu^{(-1)}(m_\nu(z))} = B_\mu(z),$$

that is,

$${}^cT_{(\mu,\nu)}(m_\nu(z)) = B_\mu(z),$$

and composing at the right with $m_\nu^{(-1)}$ we get the conclusion. $\square$

Note that

$$\Sigma_{(\mu,\nu)}(0) = B_\mu(0) = \int_{\mathbb{T}} \zeta d\mu(\zeta)$$

and the function $\Sigma_{(\delta_\lambda,\nu)}(z)$ is the constant function λ .

As observed in [11], the function B_μ maps $\mathbb{D}$ into $\overline{\mathbb{D}}$, and, conversely, any analytic function $B : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ is of the form B_μ for a unique probability measure μ on $\mathbb{T}$. As a consequence of this observation, the function $\Sigma_{(\mu,\nu)}$ is uniformly bounded by 1. Moreover, if $\nu \in \mathcal{M}_{\mathbb{T}}^\times$ and Σ is an analytic function defined on the set $\eta_\nu(\mathbb{D})$ such that the function Σ is uniformly bounded by 1, then the function $\Sigma \circ \eta_\nu$ is an analytic function taking values in $\overline{\mathbb{D}}$, and hence there exists a unique probability measure μ on $\mathbb{T}$ such that the function $\Sigma(\eta_\nu u(z)) = B_\mu(z)$ in $\mathbb{D}$. Thus we see that the function Σ is of the form $\Sigma_{(\mu,\nu)}$ near the origin.

Before starting Section 5, we will finally mention the following construction from [11]. If μ, ν are two probability measures on $\mathbb{T}$, their *Boolean convolution* $\mu \boxtimes \nu$ is the unique measure on $\mathbb{T}$ given by

$$B_{\mu \boxtimes \nu}(z) = B_\mu(z)B_\nu(z).$$

5. LIMIT THEOREMS

Let $\{k_n\}_{n=1}^\infty$ be a sequence of positive integers. Consider two *infinitesimal* triangular arrays $\{\mu_{nk} : n \in \mathbb{N}, 1 \leq k \leq k_n\}$ and $\{\nu_{nk} : n \in \mathbb{N}, 1 \leq k \leq k_n\}$ in $\mathcal{M}_{\mathbb{T}}^\times$. Here the infinitesimality means that

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} \mu_{nk}(\{\zeta \in \mathbb{T} : |\zeta - 1| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. We say that a sequence $\{\mu_n\}_{n=1}^\infty \subset \mathcal{M}_{\mathbb{T}}$ converges *weakly* to a measure $\mu \in \mathcal{M}_{\mathbb{T}}$ if

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} f(\zeta) d\mu_n(\zeta) = \int_{\mathbb{T}} f(\zeta) d\mu(\zeta)$$

for every continuous function f on $\mathbb{T}$. The weak convergence of a sequence of pairs $\{(\mu_n, \nu_n)\}_{n=1}^\infty$ simply means the componentwise weak convergence. Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\lambda'_n\}_{n=1}^\infty$ be two sequences in $\mathbb{T}$. The aim of the current section is to study the limiting behavior of the sequence of pairs $\{(\mu_n, \nu_n)\}_{n=1}^\infty$, where

$$(\mu_n, \nu_n) = (\delta_{\lambda_n}, \delta_{\lambda'_n}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n})$$

for δ_λ the Dirac point mass supported at $\lambda \in \mathbb{T}$.

We would like to mention at this point that the asymptotic behavior of the Boolean convolution $\boxtimes$ and that of the free convolution $\boxplus$ has been studied thoroughly in [23], where the necessary and sufficient conditions for the weak convergence were found. These conditions show that the sequence $\delta_{\lambda_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$ converges weakly if and only if the sequence $\delta_{\lambda_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ does, provided that the array $\{\mu_{nk}\}_{n,k}$ is infinitesimal. Moreover, the limit laws are proved to be infinitely divisible (see [23]), and the Boolean and free limits are related in a quite explicit manner. In the sequel, we will prove the analogous results for c-free and Boolean convolutions.

For our purposes, we also mention the characterization of infinite divisibility relative to Boolean convolution $\boxtimes$ and that relative to free convolution $\boxplus$. A measure $\nu \in \mathcal{M}_{\mathbb{T}}$ is $\boxtimes$ -infinitely divisible if, for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}_{\mathbb{T}}$ such that

$$\nu = \underbrace{\nu_n \boxtimes \nu_n \boxtimes \cdots \boxtimes \nu_n}_{n \text{ times}}.$$

The notion of $\boxplus$ -infinite divisibility for a measure is defined analogously.

As shown in [11], a measure $\nu \in \mathcal{M}_{\mathbb{T}}$ is $\boxtimes$ -infinitely divisible if and only if either ν is the Haar measure on $\mathbb{T}$, or the function B_ν can be expressed as

$$B_\nu(z) = \gamma \exp \left(- \int_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z} d\sigma(\zeta) \right), \quad z \in \mathbb{D},$$

where $\gamma \in \mathbb{T}$ and σ is a finite positive Borel measure on $\mathbb{T}$. In other words, ν is $\boxtimes$ -infinitely divisible if and only if either the function $B_\nu(z) = 0$ for all z in $\mathbb{D}$, or $0 \notin B_\nu(\mathbb{D})$. The notation $\nu_{\boxtimes}^{\gamma, \sigma}$ will denote the $\boxtimes$ -infinitely divisible measure determined by γ and σ via the above formula.

Analogously, it was proved in [4] that a measure $\nu \in \mathcal{M}_{\mathbb{T}}^\times$ is $\boxplus$ -infinitely divisible if and only if the function $\eta_\nu^{(-1)}$ can be written as

$$\eta_\nu^{(-1)}(z) = z \cdot \gamma \exp \left(\int_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z} d\sigma(\zeta) \right), \quad z \in \mathbb{D},$$

for some $\gamma \in \mathbb{T}$ and a finite positive Borel measure σ on $\mathbb{T}$. The $\boxplus$ -infinitely divisible measure ν described above will be denoted by $\nu_{\boxplus}^{\gamma, \sigma}$. The Haar measure m is the only $\boxplus$ -infinitely divisible measure on $\mathbb{T}$ with zero first moment.

Let us proceed to the proof of the limit theorems for c-free convolution. We first show that the weak convergence of probability measures can be translated into convergence properties of the corresponding functions B and Σ .

Proposition 5.1. *Let $\{\mu_n\}_{n=1}^\infty \subset \mathcal{M}_{\mathbb{T}}$ and $\{\nu_n\}_{n=1}^\infty \subset \mathcal{M}_{\mathbb{T}}^\times$ be two sequences of probability measures, and let μ be a probability measure supported on $\mathbb{T}$.*

- (i) *The sequence μ_n converges weakly to μ if and only if the functions B_{μ_n} converge uniformly on compact subsets of $\mathbb{D}$ to the function B_μ .*
- (ii) *Suppose that $\{\nu_n\}_{n=1}^\infty$ converges weakly to a measure $\nu \in \mathcal{M}_{\mathbb{T}}^\times$. Then the sequence μ_n converges weakly to μ if and only if there exists a neighborhood of zero $\mathcal{D} \subset \mathbb{D}$ such that all functions $\Sigma_{(\mu_n, \nu_n)}$ and $\Sigma_{(\mu, \nu)}$ are defined in $\mathcal{D}$, and the functions $\Sigma_{(\mu_n, \nu_n)}$ converge uniformly on $\mathcal{D}$ to the function $\Sigma_{(\mu, \nu)}$.*

Proof. We first prove (i). Note the following identity:

$$(9) \quad \frac{1 + zB_\mu(z)}{1 - zB_\mu(z)} = \int_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z} d\mu(\zeta), \quad z \in \mathbb{D},$$

which says that the Poisson integral of the measure $d\mu(\bar{\zeta})$ is determined by the function B_μ . Suppose that the B_{μ_n} converge uniformly on the compact subsets of $\mathbb{D}$ to the function B_μ . Then (9) shows that the weak limit μ of $\{\mu_n\}_{n=1}^\infty$ is uniquely determined and hence the full sequence μ_n must converge to μ . Conversely, let us suppose that the sequence μ_n converges weakly to μ . Since $\{B_{\mu_n}\}_{n=1}^\infty$ is a normal family of analytic functions, it has a subsequence that converges uniformly on the compact subsets of $\mathbb{D}$ to an analytic function B . By (9) again, we have $B = B_\mu$ on $\mathbb{D}$. Hence the whole sequence B_{μ_n} must converge to B_μ .

Let us now prove (ii). Assume that $\{\mu_n\}_{n=1}^\infty$ converges weakly to μ . Then the hypothesis on weak convergence of $\{\nu_n\}_{n=1}^\infty$ implies that there exists a neighborhood of zero $\mathcal{D} \subset \mathbb{D}$ such that the functions $\eta_{\nu_n}^{(-1)}$ and $\eta_\nu^{(-1)}$ are defined in $\mathcal{D}$ (hence so are the functions $\Sigma_{(\mu_n, \nu_n)}$ and $\Sigma_{(\mu, \nu)}$), and the sequence $\eta_{\nu_n}^{(-1)}$ converges uniformly on $\mathcal{D}$ to $\eta_\nu^{(-1)}$. It follows that there exist $N \in \mathbb{N}$ and a disk $\mathcal{D}' \subset \mathbb{D}$ such that $\eta_{\nu_n}^{(-1)}(\mathcal{D}) \subset \mathcal{D}'$ and $\eta_{\nu_n}^{(-1)}(\mathcal{D}) \subset \mathcal{D}'$ for every $n > N$. Also, the Cauchy estimate implies that there exists $K = K(\mathcal{D}') > 0$ so that the derivatives $|B'_{\mu_n}(z)| \leq K$ for every $n \in \mathbb{N}$ and $z \in \mathcal{D}'$. Therefore, we have

$$\begin{aligned} |\Sigma_{(\mu_n, \nu_n)}(z) - \Sigma_{(\mu, \nu)}(z)| &\leq \left| B_{\mu_n}(\eta_{\nu_n}^{(-1)}(z)) - B_{\mu_n}(\eta_\nu^{(-1)}(z)) \right| \\ &\quad + \left| B_{\mu_n}(\eta_\nu^{(-1)}(z)) - B_\mu(\eta_\nu^{(-1)}(z)) \right| \\ &\leq K \left| \eta_{\nu_n}^{(-1)}(z) - \eta_\nu^{(-1)}(z) \right| \\ &\quad + \left| B_{\mu_n}(\eta_\nu^{(-1)}(z)) - B_\mu(\eta_\nu^{(-1)}(z)) \right| \end{aligned}$$

for every $n > N$ and $z \in \mathcal{D}$. Then (i) implies that the functions $\Sigma_{(\mu_n, \nu_n)}$ converge uniformly on $\mathcal{D}$ to the function $\Sigma_{(\mu, \nu)}$.

Conversely, suppose that the functions $\Sigma_{(\mu_n, \nu_n)}$ converge uniformly on $\mathcal{D}$ to the function $\Sigma_{(\mu, \nu)}$. Observe that $B_{\mu_n}(z) = \Sigma_{(\mu_n, \nu_n)}(\eta_{\nu_n}(z))$ for all $z \in \mathbb{D}$. A similar argument as in the previous paragraph shows that there exists a positive constant K' such that the estimate

$$|B_{\mu_n}(z) - B_\mu(z)| \leq K' |\eta_{\nu_n}(z) - \eta_\nu(z)| + |\Sigma_{(\mu_n, \nu_n)}(\eta_\nu(z)) - \Sigma_{(\mu, \nu)}(\eta_\nu(z))|$$

holds in a neighborhood of zero for sufficiently large n . Therefore, the weak convergence of $\{\nu_n\}_{n=1}^\infty$ implies that the sequence $B_{\mu_n}(z)$ converges uniformly on a neighborhood of zero to the function $B_\mu(z)$. Moreover, this convergence is actually uniform on any compact subset of $\mathbb{D}$ by an easy application of Montel's theorem. We therefore conclude, from (i), that the sequence μ_n converges weakly to μ . $\square$

For an infinitesimal array $\{\mu_{nk}\}_{n,k}$, we define the complex numbers $b_{nk} \in \mathbb{T}$ by

$$(10) \quad b_{nk} = \exp \left(i \int_{|\arg \zeta| < 1} \arg \zeta \, d\mu_{nk}(\zeta) \right),$$

where $\arg \zeta$ denotes the principal value of the argument of ζ , and the probability measures μ_{nk}° by

$$d\mu_{nk}^\circ(\zeta) = d\mu_{nk}(b_{nk}\zeta).$$

The array $\{\mu_{nk}^\circ\}_{n,k}$ is infinitesimal and $\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} |\arg b_{nk}| = 0$. We also associate to each measure μ_{nk}° an analytic function

$$h_{nk}(z) = -i \int_{\mathbb{T}} \Im \zeta d\mu_{nk}^\circ(\zeta) + \int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z} (1 - \Re \zeta) d\mu_{nk}^\circ(\zeta), \quad z \in \mathbb{D},$$

and observe that $\Re h_{nk}(z) > 0$ for all $z \in \mathbb{D}$ unless the measure $\mu_{nk}^\circ = \delta_1$.

Proposition 5.2. *Suppose that $\mathcal{D} \subset \mathbb{D}$ is a disk centered at zero with radius less than $1/4$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$ be an infinitesimal array in $\mathcal{M}_{\mathbb{T}}^\times$. Then we have:*

- (1) $1 - B_{\mu_{nk}^\circ}(z) = h_{nk}(\overline{b_{nk}z}) (1 + o(1))$ uniformly in k and $z \in \mathcal{D}$ as $n \rightarrow \infty$.
- (2) There exists a constant $L = L(\mathcal{D}) > 0$ such that for every n and k we have

$$|h_{nk}(w) - h_{nk}(z)| \leq L |h_{nk}(z)| |w - z|, \quad z, w \in \mathcal{D}.$$

- (3) The sequence of functions $\{\exp(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} - \sum_{k=1}^{k_n} h_{nk}(z))\}_{n=1}^\infty$ converges uniformly on compact subsets of $\mathbb{D}$ if and only if the sequence of functions $\{\lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z)\}_{n=1}^\infty$ does. Moreover, the two sequences have the same limit function.

Proof. (1) and (3) are proved in [23]. To prove (2), let us consider the analytic function

$$h_\mu(z) = \int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z} (1 - \Re \zeta) d\mu(\zeta), \quad z \in \mathbb{D},$$

for a measure $\mu \in \mathcal{M}_{\mathbb{T}}$. For $z, w \in \mathcal{D}$, we have

$$\begin{aligned} (11) \quad |h_\mu(w) - h_\mu(z)| &\leq |w - z| \int_{\mathbb{T}} \frac{2}{(1-|z|)(1-|w|)} (1 - \Re \zeta) d\mu(\zeta) \\ &\leq 4 |w - z| \int_{\mathbb{T}} (1 - \Re \zeta) d\mu(\zeta). \end{aligned}$$

In addition, using Harnack's inequality, it is easy to see that there exists $M = M(\mathcal{D}) > 0$ such that

$$\Re \left[\frac{1+\zeta z}{1-\zeta z} \right] \geq M, \quad z \in \mathcal{D}, \zeta \in \mathbb{T}.$$

Therefore, we deduce that

$$(12) \quad M \int_{\mathbb{T}} (1 - \Re \zeta) d\mu(\zeta) \leq \int_{\mathbb{T}} \Re \left[\frac{1+\zeta z}{1-\zeta z} \right] (1 - \Re \zeta) d\mu(\zeta) = \Re h_\mu(z) \leq |h_\mu(z)|$$

for every $z \in \mathcal{D}$. Finally, (2) follows from (11) and (12). $\square$

Lemma 5.3. *For sufficiently large n , there exists a disk $\mathcal{D} \subset \mathbb{D}$ centered at zero such that*

$$1 - \overline{b_{nk}} \Sigma_{(\mu_{nk}, \nu_{nk})}(z) = h_{nk}(z)(1 + u_{nk}(z)), \quad z \in \mathcal{D}, 1 \leq k \leq k_n,$$

where the limit

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} |u_{nk}(z)| = 0$$

holds uniformly in $\mathcal{D}$.

Proof. Introduce measures

$$d\nu_{nk}^\circ(\zeta) = d\nu_{nk}(b_{nk}\zeta),$$

where the complex numbers b_{nk} are defined as in (10). It was shown in [2] that the limits $\lim_{n \rightarrow \infty} \eta_{\nu_{nk}}(z) = z$ and $\lim_{n \rightarrow \infty} \eta_{\nu_{nk}^\circ}(z) = z$ hold uniformly in k and on compact subsets of $\mathbb{D}$. In particular, it follows that, as n tends to infinity, there exists a disk $\mathcal{D} \subset \mathbb{D}$ centered at zero such that the functions $\Sigma_{(\mu_{nk}, \nu_{nk})}$ and $\Sigma_{(\mu_{nk}^\circ, \nu_{nk}^\circ)}$ are both defined in $\mathcal{D}$ and $\eta_{\nu_{nk}}^{(-1)}(z) = z(1 + o(1))$ uniformly in k and $z \in \mathcal{D}$.

Then the desired result follows from Proposition 5.2(1), 5.2(2), and the following observation:

$$\Sigma_{(\mu_{nk}, \nu_{nk})}(z) = b_{nk} \Sigma_{(\mu_{nk}^\circ, \nu_{nk}^\circ)}(z) = b_{nk} B_{\mu_{nk}^\circ} \left(\eta_{\nu_{nk}^\circ}^{(-1)}(z) \right) = b_{nk} B_{\mu_{nk}^\circ} \left(b_{nk} \eta_{\nu_{nk}}^{(-1)}(z) \right).$$

□

As shown in [5], for every neighborhood of zero $\mathcal{D} \subset \mathbb{D}$ there exists $M = M(\mathcal{D}) > 0$ such that

$$|\Im h_{nk}(z)| \leq M \Re h_{nk}(z), \quad z \in \mathcal{D}, 1 \leq k \leq k_n,$$

for sufficiently large n .

Suppose that $\mathcal{D} \subset \mathbb{D}$ is a neighborhood of zero. The infinitesimality of the arrays $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ implies that the functions $\Sigma_{(\mu_{nk}, \nu_{nk})}(z)$ converge uniformly in k and $z \in \mathcal{D}$ to 1 as $n \rightarrow \infty$. It follows that the principal branch of $\log \Sigma_{(\mu_{nk}, \nu_{nk})}(z)$ is defined in $\mathcal{D}$ for large n . Moreover, we have

$$\log \overline{b_{nk}} \Sigma_{(\mu_{nk}, \nu_{nk})}(z) = [\overline{b_{nk}} \Sigma_{(\mu_{nk}, \nu_{nk})}(z) - 1] (1 + o(1))$$

uniformly in k and $z \in \mathcal{D}$ when n is sufficiently large.

To prove Theorem 5.5, we will again need an auxiliary result from [23]:

Lemma 5.4. *Consider a sequence $\{r_n\}_{n=1}^\infty \subset \mathbb{R}$ and two triangular arrays $\{z_{nk}\}_{n,k}$ and $\{w_{nk}\}_{n,k}$ of complex numbers. Suppose that*

- (1) $\Re w_{nk} \leq 0$ and $\Re z_{nk} \leq 0$ for all n and k ;
- (2) $z_{nk} = w_{nk}(1 + \varepsilon_{nk})$, where $\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} |\varepsilon_{nk}| = 0$;
- (3) there exists a constant $M > 0$ such that $|\Im w_{nk}| \leq M |\Re w_{nk}|$ for all n, k .

Then the sequence $\{\exp(ir_n + \sum_{k=1}^{k_n} z_{nk})\}_{n=1}^\infty$ converges if and only if the sequence $\{\exp(ir_n + \sum_{k=1}^{k_n} w_{nk})\}_{n=1}^\infty$ does. Moreover, the two sequences have the same limit.

Theorem 5.5. *Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\lambda'_n\}_{n=1}^\infty$ be two sequences in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}_\mathbb{T}^\times$. Assume that $\mathcal{D} \subset \mathbb{D}$ is a neighborhood of zero. Then the sequence of functions $\{\lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk}, \nu_{nk})}(z)\}_{n=1}^\infty$ converges uniformly on $\mathcal{D}$ if and only if the sequence of functions $\{\lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z)\}_{n=1}^\infty$ does. Moreover, the two sequences have the same limit function.*

Proof. For sufficiently large n and $z \in \mathcal{D}$, let us write

$$\lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk}, \nu_{nk})}(z) = \exp \left(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} + \sum_{k=1}^{k_n} \log \overline{b_{nk}} \Sigma_{(\mu_{nk}, \nu_{nk})}(z) \right).$$

Applying Lemmas 5.3 and 5.4 to the arrays $\{\log \overline{b_{nk}} \Sigma_{(\mu_{nk}, \nu_{nk})}(z)\}_{n,k}$ and $\{-h_{nk}(z)\}_{n,k}$, we conclude that the sequences $\{\exp(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} - \sum_{k=1}^{k_n} h_{nk}(z))\}_{n=1}^\infty$ and $\{\lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk}, \nu_{nk})}(z)\}_{n=1}^\infty$ have the same asymptotic behavior as $n \rightarrow \infty$. The desired result follows from the fact that the above sequences are normal families of analytic functions, and from Proposition 5.2(3). $\square$

Now fix $\gamma, \gamma' \in \mathbb{T}$ and finite positive Borel measures σ, σ' on $\mathbb{T}$. Recall that $\nu_{\boxtimes}^{\gamma, \sigma}$ (resp., $\nu_{\boxtimes}^{\gamma', \sigma'}$) is the $\boxtimes$ (resp., $\boxtimes$)-infinitely divisible measure we have seen earlier.

Theorem 5.6. *Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\lambda'_n\}_{n=1}^\infty$ be two sequences in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}_{\mathbb{T}}^\times$. Define*

$$(\mu_n, \nu_n) = (\delta_{\lambda_n}, \delta_{\lambda'_n}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n})$$

and $\rho_n = \delta_{\lambda_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$, for every $n \in \mathbb{N}$. Suppose that $\{\nu_n\}_{n=1}^\infty$ converges weakly to $\nu_{\boxtimes}^{\gamma', \sigma'}$. Then the following statements are equivalent:

- (1) The sequence μ_n converges weakly to a measure $\mu \in \mathcal{M}_{\mathbb{T}}^\times$.
- (2) The sequence ρ_n converges weakly to $\nu_{\boxtimes}^{\gamma, \sigma}$.
- (3) The sequence of measures

$$d\sigma_n(\zeta) = \sum_{k=1}^{k_n} (1 - \Re \zeta) d\mu_{nk}^\circ(\zeta)$$

converges weakly on $\mathbb{T}$ to the measure σ , and the sequence of complex numbers

$$\gamma_n = \exp \left(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} + i \sum_{k=1}^{k_n} \int_{\mathbb{T}} \Im \zeta d\mu_{nk}^\circ(\zeta) \right)$$

converges to γ as $n \rightarrow \infty$.

Moreover, if (1)-(3) are satisfied, then we have

$$\Sigma_{(\mu, \nu_{\boxtimes}^{\gamma', \sigma'})}(z) = B_{\nu_{\boxtimes}^{\gamma, \sigma}}(z)$$

in a neighborhood of zero.

Proof. The equivalence between (2) and (3) was proved in [23]. We will show the equivalence of (1) and (2). First note that we have

$$\Sigma_{(\mu_n, \nu_n)}(z) = \lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk}, \nu_{nk})}(z)$$

in a neighborhood of zero $\mathcal{D} \subset \mathbb{D}$ and

$$B_{\rho_n}(z) = \lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z), \quad z \in \mathbb{D}.$$

Suppose that (1) holds. By Proposition 5.1, we have

$$\lim_{n \rightarrow \infty} \Sigma_{(\mu_n, \nu_n)}(z) = \Sigma_{(\mu, \nu_{\boxtimes}^{\gamma', \sigma'})}(z)$$

uniformly on $\mathcal{D}$. Then Theorem 5.5 implies that

$$\lim_{n \rightarrow \infty} B_{\rho_n}(z) = \Sigma_{(\mu, \nu_{\boxtimes}^{\gamma', \sigma'})}(z)$$

uniformly on $\mathcal{D}$. Note that the function $\eta_{\nu_{\gamma', \sigma'}}^{(-1)}(z)$ can be analytically continued to the whole disk $\mathbb{D}$. It follows that there exists a unique measure $\nu \in \mathcal{M}_{\mathbb{T}}$ such that

$$B_{\nu}(z) = \Sigma_{(\mu, \nu_{\gamma', \sigma'})}(z), \quad z \in \mathcal{D}.$$

Note that

$$\int_{\mathbb{T}} \zeta d\nu(\zeta) = B_{\nu}(0) = \Sigma_{(\mu, \nu_{\gamma', \sigma'})}(0) = \int_{\mathbb{T}} \zeta d\mu(\zeta) \neq 0.$$

We conclude that $\{\rho_n\}_{n=1}^{\infty}$ converges weakly to the measure $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$ by Proposition 5.1, and hence the limit law ν is $\boxtimes$ -infinitely divisible as we have mentioned earlier. Consequently, the measure ν is of the form $\nu_{\gamma, \sigma}^{\gamma, \sigma}$ for some $\gamma \in \mathbb{T}$ and a finite Borel measure σ on $\mathbb{T}$. Hence (2) holds.

Now assume (2). Then we have

$$\lim_{n \rightarrow \infty} B_{\rho_n}(z) = B_{\nu_{\gamma, \sigma}}(z)$$

uniformly on compact subsets of $\mathbb{D}$. Theorem 5.5 shows that

$$\lim_{n \rightarrow \infty} \Sigma_{(\mu_n, \nu_n)}(z) = B_{\nu_{\gamma, \sigma}}(z)$$

uniformly in a neighborhood of zero. Observe now there exists a measure $\mu \in \mathcal{M}_{\mathbb{T}}^{\times}$ such that

$$B_{\mu}(z) = B_{\nu_{\gamma, \sigma}}(\eta_{\nu_{\gamma', \sigma'}}(z)), \quad z \in \mathbb{D}.$$

Therefore the function $B_{\nu_{\gamma, \sigma}}(z)$ has the form $\Sigma_{(\mu, \nu_{\gamma', \sigma'})}(z)$ in a neighborhood of zero, and (1) follows from Proposition 5.1. $\square$

Next, we address the case that the measures μ_n converge to m , for m the Haar measure on $\mathbb{T}$.

Theorem 5.7. *Let $\{\lambda_n\}_{n=1}^{\infty}, \{\lambda'_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}, \{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}_{\mathbb{T}}^{\times}$. As in the statement of Theorem 5.6, define*

$$(\mu_n, \nu_n) = (\delta_{\lambda_n}, \delta_{\lambda'_n}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n})$$

and $\rho_n = \delta_{\lambda_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$, for every $n \in \mathbb{N}$, and suppose that $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to ν . Then the following assertions are equivalent:

- (1) *The sequence μ_n converges weakly to m .*
- (2) *The sequence ρ_n converges weakly to m .*

Proof. Assume (1) holds. Observe that $\Sigma_{(\mu_n, \nu_n)}(0) = B_{\mu_n}(0) = \int_{\mathbb{T}} \zeta d\mu_n(\zeta)$ and

$$\Sigma_{(\mu_n, \nu_n)}(0) = \lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk}, \nu_{nk})}(0) = \lambda_n \prod_{k=1}^{k_n} \int_{\mathbb{T}} \zeta d\mu_{nk}(\zeta), \quad n \in \mathbb{N}.$$

Since the sequence μ_n converges weakly to m , the above product converges to zero as n tends to infinity. As shown in [23], Theorem 4.3, this condition implies that (2) holds.

Conversely, assume that (2) holds. Then the limit $\lim_{n \rightarrow \infty} B_{\rho_n}(z) = 0$ holds uniformly on the compact subsets of $\mathbb{D}$, and therefore Theorem 5.5 shows that

$$\lim_{n \rightarrow \infty} \Sigma_{(\mu_n, \nu_n)}(z) = 0$$

uniformly in a neighborhood of zero $\mathcal{D} \subset \mathbb{D}$. First note that the measure ν must be $\boxtimes$ -infinitely divisible (see [2]). If the measure ν has non-zero first moment, then we conclude, by Proposition 5.1, that (1) holds since the zero function is of the form $\Sigma_{(m,\nu)}$. If the first moment of the measure ν is zero, then $\nu = m$ and the sequence $\{\eta_{\nu_n}(z)\}_{n=1}^{\infty}$ converges uniformly on $\mathcal{D}$ to zero. In particular, the set $\eta_{\nu_n}(\mathcal{D})$ is contained in $\mathcal{D}$ when n is sufficiently large. We conclude in this case that the functions $B_{\mu_n}(z) = \Sigma_{(\mu_n, \nu_n)}(\eta_{\nu_n}(z))$ converge uniformly on $\mathcal{D}$ to zero as well, and therefore the sequence μ_n converges weakly to m by Proposition 5.1. $\square$

Remark 5.8. We remark here for further use that the proof of Theorem 5.7 gives a convenient criterion of determining whether a limit law is the Haar measure m or not. Namely, if the free convolutions ν_n converge weakly to ν , and the c -free convolutions μ_n converge to a measure μ with $\int_{\mathbb{T}} \zeta d\mu(\zeta) = 0$, then the measure μ must be the Haar measure m .

6. INFINITE DIVISIBILITY

Definition 6.1. A pair $(\mu, \nu) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}$ is said to be $\boxtimes$ -infinitely divisible if for any $n \in \mathbb{N}$ there exists $(\mu_n, \nu_n) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}$ such that

$$(\mu, \nu) = \underbrace{(\mu_n, \nu_n) \boxtimes \cdots \boxtimes (\mu_n, \nu_n)}_{n \text{ times}}.$$

In this section, we give a complete characterization of $\boxtimes$ -infinite divisibility for a pair $(\mu, \nu) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}$. We begin with the case when μ has zero first moment. We will see that these are the pairs (m, ν) , where m is the Haar measure on $\mathbb{T}$ and the measure ν is $\boxtimes$ -infinitely divisible.

Proposition 6.2. Suppose that the pair (μ, ν) is $\boxtimes$ -infinitely divisible and that $\int_{\mathbb{T}} \zeta d\mu(\zeta) = 0$. Then $\mu = m$.

Proof. First we assume that $\int_{\mathbb{T}} \zeta d\nu(\zeta) \neq 0$. In this case, for every $n \in \mathbb{N}$, there exist measures $\mu_n \in \mathcal{M}_{\mathbb{T}}$ and $\nu_n \in \mathcal{M}_{\mathbb{T}}^{\times}$ such that

$$(\mu, \nu) = \underbrace{(\mu_n, \nu_n) \boxtimes \cdots \boxtimes (\mu_n, \nu_n)}_{n \text{ times}}.$$

Thus we have that $\Sigma_{(\mu, \nu)} = (\Sigma_{(\mu_n, \nu_n)})^n$. Or, equivalently, we have that

$$B_{\mu}(z) = \left(B_{\mu_n} \left(\eta_{\nu_n}^{(-1)}(\eta_{\nu}(z)) \right) \right)^n, \quad z \in \mathbb{D}.$$

In [1], Proposition 3.3, it is shown that the function $\eta_{\nu_n}^{(-1)}(\eta_{\nu}(z))$ extends analytically to the entire disk $\mathbb{D}$. Therefore the function B_{μ} is the n -th power of an analytic function in $\mathbb{D}$ for any $n \in \mathbb{N}$. This happens if and only if either B_{μ} is identically zero in $\mathbb{D}$ or $0 \notin B_{\mu}(\mathbb{D})$. Hence the measure μ is the $\boxtimes$ -infinitely divisible measure with first moment zero, that is, $\mu = m$.

Now suppose $\int_{\mathbb{T}} \zeta d\nu(\zeta) = 0$. Then we have that

$$(\mu, \nu) = (\mu, m) = (\mu_1, m) \boxtimes (\mu_1, m)$$

for some measure $\mu_1 \in \mathcal{M}_{\mathbb{T}}$. In this case we can view the pair (μ, m) as the distribution of XY in a non-commutative C^* -probability space $(\mathcal{A}, \varphi, \psi)$, where X

and Y are Boolean independent random variables with the common distribution (μ_1, m) . Then, for any $n \in \mathbb{N}$, the n -th moment of the measure μ is given by

$$\varphi(\underbrace{(XY) \cdots (XY)}_{n \text{ times}}) = \varphi(X)^n \varphi(Y)^n = \varphi(XY)^n = \left(\int_{\mathbb{T}} \zeta d\mu(\zeta) \right)^n = 0.$$

Therefore the measure μ is the Haar measure m . $\square$

Next we focus on the $\boxtimes$ -infinitely divisible pair (μ, m) such that $\mu \in \mathcal{M}_{\mathbb{T}}^{\times}$.

Proposition 6.3. *The above measure μ is either a point mass δ_{λ} for some $\lambda \in \mathbb{T}$ or the harmonic measure for the unit disc relative to some point $\alpha \in \mathbb{D} \setminus \{0\}$, i.e., $d\mu = P_{\alpha} dm$, where P_{α} is the Poisson kernel at α .*

Proof. By virtue of (9), it suffices to show that the function B_{μ} is a constant function. Set $c = \int_{\mathbb{T}} \zeta d\mu(\zeta)$. Then the same argument as in the proof of Proposition 6.2 shows that the n -th moment of the measure μ is c^n . We conclude that the function

$$m_{\mu}(z) = cz + c^2 z^2 + \cdots = \frac{cz}{1 - cz}$$

in a neighborhood of zero. This fact implies that $B_{\mu}(z) = c$ for all $z \in \mathbb{D}$ as desired. $\square$

It is easy to see that if the measure ν is $\boxtimes$ -infinitely divisible, then a pair of measures of the form (m, ν) , (δ_{λ}, m) or $(P_{\alpha} dm, m)$ is also $\boxtimes$ -infinitely divisible.

Finally, we characterize the $\boxtimes$ -infinite divisibility in the class $\mathcal{M}_{\mathbb{T}}^{\times} \times \mathcal{M}_{\mathbb{T}}^{\times}$. A family of pairs $\{(\mu_t, \nu_t)\}_{t \geq 0}$ of probability measures on $\mathbb{T}$ is a *weakly continuous semigroup* relative to the convolution $\boxtimes$ if $(\mu_t, \nu_t) \boxtimes (\mu_s, \nu_s) = (\mu_{t+s}, \nu_{t+s})$ for $t, s \geq 0$, and the maps $t \mapsto \mu_t$ and $t \mapsto \nu_t$ are continuous.

Theorem 6.4. *Given a measure $\mu \in \mathcal{M}_{\mathbb{T}}^{\times}$ and a $\boxtimes$ -infinitely divisible measure $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$, the following statements are equivalent:*

- (1) *The pair (μ, ν) is $\boxtimes$ -infinitely divisible.*
- (2) *There exists a complex number $\gamma \in \mathbb{T}$ and a finite positive Borel measure σ on $\mathbb{T}$ such that*

$$\Sigma_{(\mu, \nu)}(z) = \gamma \exp \left(- \int_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z} d\sigma(\zeta) \right)$$

in a neighborhood of zero.

- (3) *There exists a weakly continuous semigroup $\{(\mu_t, \nu_t)\}_{t \geq 0}$ relative to $\boxtimes$ such that $(\mu_1, \nu_1) = (\mu, \nu)$ and $(\mu_0, \nu_0) = (\delta_1, \delta_1)$.*

Moreover, if statements (1) to (3) are all satisfied, then the limit

$$\gamma = \lim_{t \rightarrow 0^+} \exp \left(\frac{1}{t} \int_{\mathbb{T}} i \Im \zeta d\mu_t(\zeta) \right)$$

exists and the measure σ is the weak limit of measures

$$\frac{1}{t} (1 - \Re \zeta) d\mu_t(\zeta)$$

as $t \rightarrow 0^+$.

Proof. We first prove that (1) implies (2). Assume that (1) holds. For every $n \in \mathbb{N}$, we have

$$(\mu, \nu) = \underbrace{(\mu_n, \nu_n) \boxtimes \cdots \boxtimes (\mu_n, \nu_n)}_{n \text{ times}},$$

where $\mu_n \in \mathcal{M}_{\mathbb{T}}$ and $\nu_n \in \mathcal{M}_{\mathbb{T}}^{\times}$. The $\boxtimes$ -infinite divisibility of the measure ν implies that there exists an analytic function $u(z)$ in $\mathbb{D}$ such that $\Re u(z) \geq 0$ and

$$\eta_{\nu}^{(-1)}(z) = z \exp(u(z))$$

for all $z \in \mathbb{D}$ (see [4]). It follows that $\eta_{\nu_n}^{(-1)}(z) = z \exp(u(z)/n)$, and hence we deduce that the measures ν_n converge weakly to δ_1 . On the other hand, the identity $(B_{\mu_n}(z))^n = \Sigma_{(\mu, \nu)}(\eta_{\nu_n}(z))$ and Proposition 5.1 imply that the measures μ_n converge weakly to δ_1 as well. Define two infinitesimal arrays $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ by setting $\mu_{nk} = \mu_n$ and $\nu_{nk} = \nu_n$, where $1 \leq k \leq n$. Then the measures (μ, ν) can be viewed as the weak limit of c-free convolutions $(\mu_{n1}, \mu_{n1}) \boxtimes \cdots \boxtimes (\mu_{nn}, \nu_{nn})$. Hence (2) follows from Theorem 5.6.

Now, assume that (2) holds. It was also proved in [4] that there exists a weakly continuous semigroup $\{\nu_t\}_{t \geq 0}$ relative to $\boxtimes$ so that $\nu_0 = \delta_1$ and $\nu_1 = \nu$. Note that, for every $t \geq 0$, there exists a unique probability measure μ_t on $\mathbb{T}$ such that $B_{\mu_t}(z) = (\Sigma_{(\mu, \nu)}(\eta_{\nu_t}(z)))^t$ for all $z \in \mathbb{D}$, where $\mu_0 = \delta_1$. Then it is easy to see that the convolution semigroup $\{(\mu_t, \nu_t)\}_{t \geq 0}$ has the desired properties.

The implication from (3) to (1) is obvious. To conclude, we only need to show the assertions about the measure σ and the number γ . Assume that the pair (μ, ν) is $\boxtimes$ -infinitely divisible, and let $\{(\mu_t, \nu_t)\}_{t \geq 0}$ be the corresponding convolution semigroup as in (3). Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\lim_{n \rightarrow \infty} t_n = 0$. Let $k_n = [1/t_n]$ for every $n \in \mathbb{N}$, where $[x]$ denotes the largest integer that is no greater than the real number x . Observe that

$$1 - t_n < t_n k_n \leq 1, \quad n \in \mathbb{N}.$$

Hence we have $\lim_{n \rightarrow \infty} t_n k_n = 1$, and further the properties of the semigroup $\{(\mu_t, \nu_t)\}_{t \geq 0}$ show that the c-free convolutions

$$\underbrace{(\mu_{t_n}, \nu_{t_n}) \boxtimes (\mu_{t_n}, \nu_{t_n}) \boxtimes \cdots \boxtimes (\mu_{t_n}, \nu_{t_n})}_{k_n \text{ times}} = (\mu_{t_n k_n}, \nu_{t_n k_n})$$

converge weakly to $(\mu_1, \nu_1) = (\mu, \nu)$ as $n \rightarrow \infty$. Then Theorem 5.6 implies that the measures

$$\frac{1}{t_n} (1 - \Re \zeta) d\mu_{t_n}^{\circ}(\zeta) = \frac{1}{t_n k_n} k_n (1 - \Re \zeta) d\mu_{t_n}^{\circ}(\zeta)$$

converge weakly to the measure σ and

$$\gamma = \lim_{n \rightarrow \infty} \exp \left(\frac{1}{t_n} \int_{\mathbb{T}} i \Im \zeta d\mu_{t_n}^{\circ}(\zeta) \right),$$

where the centered measures $d\mu_{t_n}^{\circ}(\zeta) = d\mu_{t_n}(b_n \zeta)$ and the numbers b_n are defined as in (10). The desired result follows immediately from the fact that $\lim_{n \rightarrow \infty} b_n = 1$ and that the topology on the set $\mathcal{M}_{\mathbb{T}}$ determined by the weak convergence of measures is actually metrizable. $\square$

We conclude this section by showing a c-free analogue of Hinčin's classical theorem on the infinite divisibility of real random variables [13].

Corollary 6.5. *Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\lambda'_n\}_{n=1}^\infty$ be two sequences in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}_\mathbb{T}^\times$. Suppose that the sequence*

$$\{(\delta_{\lambda_n}, \delta_{\lambda'_n}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n})\}_n$$

converges weakly to (μ, ν) . Then the pair (μ, ν) is $\boxtimes$ -infinitely divisible.

Proof. First note that the measure ν is $\boxtimes$ -infinitely divisible as we have seen earlier. The case $\mu, \nu \in \mathcal{M}_\mathbb{T}^\times$ is an application of Theorems 5.6 and 6.4. Remark 5.8 shows that $\mu = m$ when $\int_\mathbb{T} \zeta d\mu(\zeta) = 0$. Only the case $\mu \in \mathcal{M}_\mathbb{T}^\times$ and $\nu = m$ requires a proof. In this case we set

$$(\mu_n, \nu_n) = (\delta_{\lambda_n}, \delta_{\lambda'_n}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n}).$$

Observe that $B_{\mu_n}(z) = \Sigma_{(\mu_n, \nu_n)}(\eta_{\nu_n}(z))$ and that the family $\{\Sigma_{(\mu_n, \nu_n)}(z)\}_{n=1}^\infty$ is uniformly Lipschitz in a neighborhood of zero $\mathcal{D}$ with a Lipschitz constant $K > 0$. We have

$$\begin{aligned} \left| B_{\mu_n}(z) - \int_\mathbb{T} \zeta d\mu(\zeta) \right| &\leq |B_{\mu_n}(z) - B_{\mu_n}(0)| + |B_{\mu_n}(0) - B_\mu(0)| \\ &\leq K |\eta_{\nu_n}(z)| + |B_{\mu_n}(0) - B_\mu(0)|. \end{aligned}$$

It follows that the functions B_{μ_n} converge uniformly on compact subsets of $\mathbb{D}$ to the constant function $\int_\mathbb{T} \zeta d\mu(\zeta)$, and hence the function B_μ is the same constant function. The measure μ in this case is either a point mass concentrated at a point on $\mathbb{T}$ or the harmonic measure for $\mathbb{D}$ relative to a point in $\mathbb{D} \setminus \{0\}$. Therefore (μ, m) is $\boxtimes$ -infinitely divisible. $\square$

7. APPENDIX: NON-CROSSING LINKED PARTITIONS AND A FORMULA FOR THE COEFFICIENTS OF THE T - AND cT -TRANSFORMS

7.1. Non-crossing linked partitions. The notion of non-crossing linked partitions, that we will largely use in this section, is a generalization of non-crossing partitions. It was first discussed in [9], in connection with the T -transform.

By a non-crossing linked partition γ of the ordered set $\{1, 2, \dots, n\}$ we will understand a collection $B_1, \dots, B_k$ of subsets of $\{1, 2, \dots, n\}$, called blocks, with the following properties:

- (1) $\bigcup_{l=1}^k B_l = \{1, \dots, n\}$;
- (2) $B_1, \dots, B_k$ are non-crossing, in the sense that there are no two blocks B_l, B_s and $i < k < p < q$ such that $i, p \in B_l$ and $k, q \in B_s$;
- (3) for any $1 \leq l, s \leq k$, the intersection $B_l \cap B_s$ is either void or contains only one element. If $\{j\} = B_l \cap B_s$, then $|B_s|, |B_l| \geq 2$ and j is the minimal element of only one of the blocks B_l and B_s .

An element will be said to be *singly*, respectively *doubly*, covered by γ if it is contained in exactly one, respectively exactly two, blocks of γ . The set of all non-crossing linked partitions on $\{1, \dots, n\}$ will be denoted by $NCL(n)$. If $\gamma \in NCL(n)$ and $B = i_1 < i_2 < \cdots < i_p$ is a block of γ , the element i_1 will be denoted $\min(B)$. The block B will be called *exterior* if there is no other block D of γ such that D contains $\min(B)$ or contains two elements l, s with $l < i_1 < i_p < s$. The set of all exterior blocks of γ will be denoted by $\text{ext}(\gamma)$; the set of all blocks of γ which are not exterior will be denoted by $\text{int}(\gamma)$. We will use the notation $NCL_1(n)$ for the set of all elements in $NCL(n)$ with only one exterior block.

Example. Below is represented graphically the non-crossing linked partition $\pi = (1, 4, 6, 9), (2, 3), (4, 5), (6, 7, 8), (10, 11), (11, 12) \in NCL(12)$. Its exterior blocks (drawn with thick lines) are $(1, 4, 6, 9)$ and $(10, 11)$.

Note that $NC(n) \subset NCL(n)$ and, for a partition in $NC(n)$, hence also in $NCL(n)$, blocks that are exterior, respectively interior, in $NC(n)$ are also in $NCL(n)$.

If $\pi \in NCL(p)$ and $\sigma \in NCL(s)$, then $\pi \oplus \sigma$ is the partition from $NCL(p + s)$ obtained by the juxtaposition of π and σ .

Finally, if $\gamma \in NCL(n)$ and A is an (ordered) subset of $\{1, 2, \dots, n\}$ with m elements, then by $\gamma|_A$ we will understand the partition in $NCL(m)$ obtained by intersecting the blocks of γ with A , then shifting the elements of the ordered set A into $\{1, 2, \dots, m\}$.

7.2. A formula for the coefficients of the T - and cT -transforms.

Let $X \in \mathfrak{A}^\circ$ and $m(z) = \sum_{n=1}^{\infty} m_n z^n$, $M(z) = \sum_{n=1}^{\infty} M_n z^n$ be the moment generating series of X with respect to ψ , respectively φ , while $T(z) = \sum_{n=0}^{\infty} t_n z^n$ and ${}^cT(z) = \sum_{n=0}^{\infty} {}^c t_n z^n$ are the T -, respectively the cT -transform of X .

Lemma 7.1. *With the above notation, one has that:*

- (i) $[T \circ (m(z))](1 + m(z)) = \frac{m(z)}{z}$.
- (ii) $[{}^cT \circ (m(z))](1 + M(z)) = \frac{\tilde{M}(z)}{z}$.

Proof. (i): By definition, $T(z) = \left[\frac{1}{z} R(z) \right] \circ [R^{(-1)}(z)]$.

Since $m(z) = R(z[1 + m(z)])$, composing at the left with $R^{(-1)}$ and at the right with $m^{(-1)}$, we get

$$R^{(-1)}(z) = m^{(-1)}(z)(1 + z);$$

therefore

$$T(z) = \left[\frac{1}{z} R(z) \right] \circ [(1 + z)m^{(-1)}(z)].$$

It then suffices to show that

$$\left[\frac{1}{z} R(z) \right] \circ [(1 + z)m^{(-1)}(z)] \circ m(z) = \frac{m(z)}{z[1 + m(z)]}.$$

But, from (1),

$$\frac{m(z)}{z[1 + m(z)]} = \left[\frac{1}{z} R(z) \right] \circ (z[1 + m(z)])$$

and, since,

$$[(1 + z)m^{(-1)}(z)] \circ m(z) = z[1 + m(z)]$$

we have q.e.d.

(ii): Analogously, it suffices to show that

$$\left[\frac{1}{z} {}^c R(z)\right] \circ \left[(1+z)m^{(-1)}(z)\right] \circ m(z) = \frac{M(z)}{z[1+M(z)]}.$$

(2) implies

$$\frac{M(z)}{z[1+M(z)]} = \left(\frac{1}{z} {}^c R(z)\right) \circ (z[1+m(z)])$$

and, since,

$$\left[(1+z)m^{(-1)}(z)\right] \circ m(z) = z[1+m(z)]$$

we again have q.e.d.. $\square$

Consequence 7.2. Let us denote, to ease the writing, $m_0 = M_0 = 1$. Then:

$$\begin{aligned} \text{(i)} \quad m_n &= t_0 m_{n-1} + \sum_{k=1}^{n-1} \sum_{\substack{p_1+\dots+p_k \leq n-1 \\ p_1, \dots, p_k \geq 1}} (t_k m_{p_1} \cdots m_{p_k}) m_{n-1-(p_1+\dots+p_k)}. \\ \text{(ii)} \quad M_n &= {}^c t_0 M_{n-1} + \sum_{k=1}^{n-1} \sum_{\substack{p_1+\dots+p_k \leq n-1 \\ p_1, \dots, p_k \geq 1}} ({}^c t_k m_{p_1} \cdots m_{p_k}) M_{n-1-(p_1+\dots+p_k)}. \end{aligned}$$

Theorem 7.3. With the above notation and with the convention that the product over the void set is 1, one has that:

$$\begin{aligned} \text{(i)} \quad m_n &= \sum_{\gamma \in NCL(n)} t_0^{n-|\gamma|} \cdot \prod_{B \in \gamma} t_{|B|-1}. \\ \text{(ii)} \quad M_n &= \sum_{\gamma \in NCL(n)} t_0^{n-|\gamma|} \cdot \prod_{B \in \text{ext}(\gamma)} {}^c t_{|B|-1} \cdot \prod_{B \in \text{int}(\gamma)} t_{|B|-1}. \end{aligned}$$

Proof. Part (i) of the result is also shown in [9]. The proof presented below is shorter and employs a different approach.

We will use induction on n . For $n = 1$, the result is just $m_1 = t_0$ and $M_1 = {}^c t_0$.

For $\gamma \in NCL(n)$, let us denote

$$\begin{aligned} \mathcal{E}(\gamma) &= t_0^{n-|\gamma|} \prod_{B \in \gamma} t_{|B|-1}, \\ {}^c \mathcal{E}(\gamma) &= t_0^{n-|\gamma|} \prod_{B \in \text{ext}(\gamma)} {}^c t_{|B|-1} \prod_{B \in \text{int}(\gamma)} t_{|B|-1}. \end{aligned}$$

Note that if $\gamma = \gamma_1 \oplus \gamma_2$, then $\mathcal{E}(\gamma) = \mathcal{E}(\gamma_1)\mathcal{E}(\gamma_2)$ and ${}^c \mathcal{E}(\gamma) = {}^c \mathcal{E}(\gamma_1){}^c \mathcal{E}(\gamma_2)$.

Fix $\gamma \in NCL(n)$ and let $F = 1 < i_2 < \dots < i_k$ be the first block of γ . Particularly, $F \in \text{ext}(\gamma)$. Let $\overline{F}$ be the smallest subset of $\{1, 2, \dots, n\}$ with the following properties:

- (i) $F \subset \overline{F}$.
- (ii) If $j \in \overline{F}$, then $\{1, 2, \dots, j\} \subset \overline{F}$.
- (iii) If $j \in F$ and $j \in D \in \gamma$, then $D \in \overline{F}$.

Since $\gamma = \gamma_{|\overline{F}|} \oplus \gamma_{\{1, \dots, n\} \setminus \overline{F}}$, it follows that

$$NCL(n) = \bigsqcup_{k=1}^n NCL_1(k) \oplus NCL(n-k),$$

hence using the convention that a sum over a void set is 1

$$\sum_{\gamma \in NCL(n)} \mathcal{E}(\gamma) = \sum_{k=1}^n \sum_{\sigma \in NCL_1(k)} \mathcal{E}(\sigma) \sum_{\pi \in NCL(n-k)} \mathcal{E}(\pi).$$

Applying the induction hypothesis, the above equality becomes

$$(13) \quad \sum_{\gamma \in NCL(n)} \mathcal{E}(\gamma) = \sum_{k=1}^n \sum_{\sigma \in NCL_1(k)} \mathcal{E}(\sigma) m_{n-k}.$$

Now suppose $\sigma \in NCL_1(p)$ and $F = 1 < i_2 < \dots < i_k$ is the exterior block of σ . For $2 \leq l \leq k-1$ define

$$\sigma(l) = \begin{cases} \sigma|_{\{i_l, i_l+1, \dots, i_{l+1}-1\}} & \text{if } i_l \text{ is singly covered,} \\ \sigma|_{\{i_l, i_l+1, \dots, i_{l+1}-1\}} \setminus (i_l) & \text{if } i_l \text{ is doubly covered;} \end{cases}$$

also let

$$\sigma'(k) = \begin{cases} \sigma|_{\{i_k, i_k+1, \dots, p\}} & \text{if } i_k \text{ is singly covered,} \\ \sigma|_{\{i_k, i_k+1, \dots, p\}} \setminus (i_k) & \text{if } i_k \text{ is doubly covered} \end{cases}$$

and

$$\sigma(k) = \sigma'(k) \oplus \sigma|_{\{2, 3, \dots, i_2-1\}}.$$

Example: The partition $\sigma = (1, 4, 5, 9), (2, 3), (5, 6, 7), (8), (9, 10)$ from $NCL_1(10)$ is represented graphically below:

and we have that

$$\begin{aligned} \sigma(2) &= (1), \\ \sigma(3) &= (1, 2, 3)(4), \\ \sigma(4) &= (1, 2)(3, 4). \end{aligned}$$

Note that, since $\sigma \in NCL_1(p)$, we have that $\sigma'(k) \in NCL(p-k+1)$. Furthermore, the knowledge of $\sigma(2), \dots, \sigma(k)$ determines σ uniquely and

$${}^c\mathcal{E}(\sigma) = {}^c t_{k-1} \prod_{l=2}^k \mathcal{E}(\sigma(l)).$$

Combining the above result with (13) and the induction hypothesis, we get

$$\begin{aligned}
 \sum_{\gamma \in NCL(n)} {}^c\mathcal{E}(\gamma) &= \sum_{k=1}^n M_{n-k} \sum_{\sigma \in NCL_1(k)} {}^c\mathcal{E}(\sigma) \\
 &= M_{n-1} \cdot {}^c t_0 + \sum_{k=1}^{n-1} M_{n-1-k} \sum_{\substack{p_1+\dots+p_k \leq n-1 \\ p_1, \dots, p_k \geq 1}} \sum_{\substack{\gamma_l \in NCL(p_l) \\ 1 \leq l \leq k}} {}^c t_k \prod_{l=1}^k \mathcal{E}(\gamma_k) \\
 &= M_{n-1} \cdot {}^c t_0 + \sum_{k=1}^{n-1} M_{n-1-k} \sum_{\substack{p_1+\dots+p_k \leq n-1 \\ p_1, \dots, p_k \geq 1}} {}^c t_k \prod_{l=1}^k \left(\sum_{\gamma \in NCL(p_k)} \mathcal{E}(\gamma) \right) \\
 &= M_{n-1} \cdot {}^c t_0 + \sum_{k=1}^{n-1} M_{n-1-k} \sum_{\substack{p_1+\dots+p_k \leq n-1 \\ p_1, \dots, p_k \geq 1}} {}^c t_k m_{p_1} \cdots m_{p_k} \\
 &= M_n.
 \end{aligned}$$

The inductive step for m_n is analogous (for $\varphi = \psi$, the sequences $\{m_n\}_n$ and $\{M_n\}_n$ coincide). $\square$

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REFERENCES

- [1] S. Belinschi and H. Bercovici. *Partially defined semigroups relative to multiplicative free convolution*, Int. Math. Res. Not. 2005, no. 2, 65–101. MR2128863 (2006f:46061)
- [2] S. Belinschi and H. Bercovici. *Hinčin's theorem for multiplicative free convolution*, Canadian Math. Bulletin, 2008, Vol. 51, no. 1, 26–31. MR2384735 (2008m:46130)
- [3] H. Bercovici and V. Pata. *Stable laws and domains of attraction in free probability theory*, Ann. of Math. (2), 1449 (1999), 1023–1060. MR1709310 (2000i:46061)
- [4] H. Bercovici and D. Voiculescu. *Lévy-Hinčin type theorems for multiplicative and additive free convolution*, Pacific Journal of Mathematics, (1992), Vol. 153, No. 2, 217–248. MR1151559 (93k:46052)
- [5] H. Bercovici and J.C. Wang. *Limit theorems for free multiplicative convolutions*, Trans. Amer. Math. Soc. 360 (2008), no. 11, 6089–6102. MR2425704
- [6] F. Boca. *Free products of completely positive maps and spectral sets*, J. Funct. Anal. 97 (1991), no. 2, 251–263. MR1111181 (92f:46064)
- [7] M. Bożejko and R. Speicher. *ψ -independent and symmetrized white noises*, Quantum Probability and Related Topics, (L. Accardi, ed.), World Scientific, Singapore, **VI** (1991), 219–236. MR1149828
- [8] M. Bożejko, M. Leinert and R. Speicher. *Convolution and Limit Theorems for Conditionally free Random Variables*, Pac. J. Math. 175 (1996), 357–388. MR1432836 (98j:46069)
- [9] K. Dykema. *Multilinear function series and transforms in free probability theory*, Adv. Math. 208 (2007), no. 1, 351–407. MR2304321 (2008k:46193)
- [10] K. Dykema. *On the S -transform over a Banach algebra*, J. Func. Anal. 231 (2006), no. 1, 90–110. MR2190164 (2006k:46106)
- [11] U. Franz. *Boolean convolution of probability measures on the unit circle*, arXiv:math/0403243.
- [12] U. Haagerup. *On Voiculescu's R - and S -transforms for Free non-commuting Random Variables*, Fields Institute Communications, vol. 12(1997), 127–148. MR1426838 (98c:46137)
- [13] A. Hinčin. *Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze*, Mat. Sb. 2 (1937), 79–119.

- [14] A.D. Krystek. *Infinite divisibility for conditionally free convolution*, *Infin. Dim. Anal. Quantum Prob. Rel. Top.* 10 (2007), no. 4, 499–522. MR2376439 (2009d:46118)
- [15] F. Lehner. *Cumulants in noncommutative probability theory. I. Noncommutative exchangeability systems*, *Math. Z.* 248 (2004), no. 1, 67–100. MR2092722 (2005h:46091)
- [16] A. Nica and R. Speicher. *Lectures on the Combinatorics of the Free Probability*, *London Mathematical Society Lecture Note Series* 335, Cambridge University Press, 2006. MR2266879 (2008k:46198)
- [17] M. Popa. *A combinatorial approach to monotonic independence over a C^* -algebra*, *Pacific J. Math.* 237 (2008), no. 2, 299–325. MR2421124 (2009k:46119)
- [18] M. Popa. *A new proof for the multiplicative property of the Boolean cumulants with applications to the operator-valued case*, *Colloq. Math.*, vol. 117 (2009), no. 1, 81–93. MR2539549 (2010k:46067)
- [19] T. Speed. *Cumulants and partition lattices*, *Austral. J. Stat.* 25 (1983) 378–388. MR725217 (85f:62020)
- [20] R. Speicher. *Combinatorial Theory of the Free Product with Amalgamation and Operator-Valued Free Probability Theory*, *Mem. AMS*, Vol. 132, No 627 (1998). MR1407898 (98i:46071)
- [21] R. Speicher and R. Waroudi. *Boolean convolution*, *Fields Institute Communications*, Vol. 12 (D. Voiculescu, ed.), AMS, 1997, pp. 267–279. MR1426845 (98b:46084)
- [22] D.V. Voiculescu, K. Dykema, and A. Nica. *Free random variables*, CRM Monograph Series, 1. AMS, Providence, RI, 1992. MR1217253 (94c:46133)
- [23] J.C. Wang. *Limit laws for Boolean convolutions*, *Pacific J. Math.* 237 (2008), no. 2, 349–371. MR2421126 (2009h:46128)
- [24] J.C. Wang. *Limit theorems for additive c-free convolution*, (2008), preprint.

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