NONCOMMUTATIVE SEMIALGEBRAIC SETS AND ASSOCIATED LIFTING PROBLEMS

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ABSTRACT. We solve a class of lifting problems involving approximate polynomial relations (soft polynomial relations). Various associated C^* -algebras are therefore projective. The technical lemma we need is a new manifestation of Akemann and Pedersen's discovery of the norm adjusting power of quasi-central approximate units.

A projective C^* -algebra is the analog of an absolute retract. Thus we can say that various noncommutative semialgebraic sets turn out to be absolute retracts. In particular we show that a noncommutative absolute retract results from the intersection of the approximate locus of a noncommutative homogeneous polynomial with the noncommutative unit ball. By unit ball we are referring to the C^* -algebra of the universal row contraction. We show that various alternative noncommutative unit balls are also projective.

Sufficiently many C^* -algebras are now known to be projective so that we are able to show that the cone over any separable C^* -algebra is the inductive limit of C^* -algebras that are projective.

1. INTRODUCTION

Lifting problems for relations in C^* -algebras have tended to have ad hoc solutions. Olsen and Pedersen prove in [23] that a nilpotent always has a nilpotent lift, specifically that given x in a C^* -algebra quotient A/I with

$$x^n = 0$$

there is always X in A with $\pi(X) = x$ and $X^n = 0$. Their proof is rather different from the techniques Akemann and Pedersen used in [2] to show that for x in A/I with

$$\|x^n\| \leq \epsilon \quad (\epsilon > 0)$$

there is always a lift X with $\|X^n\| \leq \epsilon$. Different still are the techniques used in [22] to show that the relations describing $\mathbf{CM}_n = C_0((0, 1], \mathbf{M}_n)$ are liftable: given

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$x_1, \dots, x_n$ in A/I satisfying all the relations

$$\begin{aligned}\|x_j\| &\leq 1 \quad (\forall j), \\ x_j^* x_k &= 0 \quad (j \neq k), \\ x_j^* x_j &= x_k^* x_k \quad (\forall j, \forall k), \\ x_1 x_1^* &= x_1^* x_1,\end{aligned}$$

there are lifts $X_1, \dots, X_n$ in A that also satisfy these relations. More recently, M -ideals showed up in [24] to settle the lifting problem for the relations

$$\begin{aligned}\|x\| &\leq 1, \\ x^n &= 0.\end{aligned}$$

The lifting results above (most of them, anyway) show various C^* -algebras are projective. Projectivity was introduced by Effros and Kaminker in [10]. A C^* -algebra P is *projective* if the map

$$\rho \circ - : \text{hom}(P, B) \rightarrow \text{hom}(P, C)$$

is onto whenever $\rho : B \rightarrow C$ is onto.

Projectivity was shown by Blackadar in [4] to be the noncommutative analog of a space being an absolute retract (AR). The analog of absolute neighborhood retract (ANR) is semiprojectivity, which we will not discuss in detail in this paper except in Section 6.

Systematic investigations of projectivity exist, but only in the case of at most one-dimensional spectrum. There was a study of $C_0(X)$ for X^+ a tree in [17]. In [7], Chigogidze and Dranishnikov solved the general question for $C_0(X)$ being projective. The answer is that $C_0(X)$ is projective if and only if X^+ is a dendrite. The finite mapping telescopes associated to inclusions of finite-dimensional C^* -algebras were shown to be projective in [22]. In the later terminology of [13], this says we have projectivity for a large class of one-dimensional noncommutative CW complexes.

“NC” will stand for “noncommutative.” Thus noncommutative CW complex becomes NCCW.

In the commutative case, very sweeping statements can be made about what spaces are AR or ANR. For example, every compact semialgebraic set in finite-dimensional Euclidean space is an absolute neighborhood retract; see [25, p. 79] and [16] for precise results and definitions. A subset of Euclidean space is said to be *semialgebraic* if it is the union of solution sets of polynomial equations and polynomial inequalities. As we are interested in closed and connected sets, it will suffice to have in mind sets of the form

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid p_j(x_1, \dots, x_n) \leq \epsilon_j \text{ for } j = 1 \dots J\},$$

where the p_j are polynomials.

This general result about semialgebraic sets being ANR cannot translate directly to C^* -algebras. We know that for the unit disk $\mathbb{D}$, the C^* -algebra

$$C_0(\mathbb{D} \setminus \{0\}) \cong C^* \langle x \mid x^* x = x x^*, \|x\| \leq 1 \rangle$$

fails to be projective. Normals do not generally lift to normals. Some normals fail to have partial lifts and are bounded away from other normals that have partial lifts. To get technical, $C_0(\mathbb{D} \setminus \{0\})$ is not even weakly semiprojective ([12]).

TABLE 1. Some known projective C^* -algebras / liftable relations

Gen- erators	Individual restrictions	Other Relations	Name or Comment	Credit
x	$\ x\ \leq 1$ $x^n = 0$			Shulman [24] (Olsen, Pedersen, Loring for $\ x\ < 1$.)
x	$\ x\ \leq 1$ $\ x^n\ \leq C$			Akemann, Pedersen [1]
x, y	$\ x\ \leq 1$ $\ y\ \leq 1$	$x^*x = y^*y$ $x^*y = y^*x = 0$ $x^2 = y^2 = 0$	$C_0((0, 1], \mathbf{M}_3)$	Loring, Pedersen [22]
x, y	$-1 \leq x \leq 1$ $-1 \leq y \leq 1$	$xy = 0$	$C_0((0, 1], \mathbb{C}^4)$	Loring [17]
$x_1, \dots, x_n$	$\ x_j\ \leq 1$	$x_i^*x_i = x_j^*x_j \quad (\forall i, \forall j)$ $x_i^*x_j = 0 \quad (\text{if } i \neq j)$		Loring, Pedersen [22, Example 3.11]
h, k, x		$hk = 0$ $0 \leq \begin{bmatrix} 1-h & x^* \\ x & k \end{bmatrix} \leq 1$	Useful in the $q\mathbb{C}$ picture of K -theory.	Loring [19]
x, y, z, w	$0 \leq x \leq 1$ $0 \leq y \leq 1$ $0 \leq z \leq 1$ $0 \leq w \leq 1$	$xy = 0$ $zw = 0$ $(1-x)z(1-x) = 0$ $(1-x)w(1-x) = 0$	$C_0(X)$ where X^+ is a tree with four edges.	Loring [17]
$x_1, \dots, x_r$		$\left\ \sum_{k=1}^r x_k x_k^* \right\ \leq 1$		Folklore, functional calculus

There is a way to avoid the difficulty posed by this nonliftable example other than keeping to small dimension. We will *avoid exact relations*.

An important instance of Theorem 3.2 is the fact that for any *positive* ϵ , an element x in a C^* -algebra quotient A/I with

$$\begin{aligned} \|x\| &\leq 1, \\ \|x^*x - xx^*\| &\leq \epsilon \end{aligned}$$

has a lift to X , so $\pi(X) = x$, with $\|X\| \leq 1$ and $\|X^*X - XX^*\| \leq \epsilon$. Put another way, we show

$$A_\epsilon = C^* \langle x \mid \|x^*x - xx^*\| \leq \epsilon, \|x\| \leq 1 \rangle$$

is projective for all positive ϵ . Since

$$C_0(\mathbb{D} \setminus \{0\}) \cong \varinjlim A_{\frac{1}{k}},$$

we have shown $C_0(\mathbb{D} \setminus \{0\})$ has a shape system (cf. [4, 6, 10]) that is trivial in the sense that all the C^* -algebras in the system are projective. It was previously unknown whether $C_0(\mathbb{D} \setminus \{0\})$ could be written as an inductive limit of semiprojective C^* -algebras.

TABLE 2. Some new projective C^* -algebras/liftable relations

Gen- erators	Individual restrictions	Other Relations	Name / Remark	Reference
h, k	$-1 \leq h \leq 1$ $-1 \leq k \leq 1$	$\ [h, k]\ \leq C$	Soft Square	Theorem 3.1
h, k	$-1 \leq h \leq 1$ $-1 \leq k \leq 1$	$\ [h, k]\ \leq C$ $\ h + ik\ \leq 1$	Soft Disk I	Theorem 3.1
x	$\ x\ \leq 1$	$\ [x^*, x]\ \leq 2C$	Soft Disk I a second presentation	Theorem 3.1
h, k	$-1 \leq h \leq 1$ $-1 \leq k \leq 1$	$\ [h, k]\ \leq C$ $\ h^2 + k^2\ \leq 1$	Soft Disk II	Theorem 3.1
a, k	$\ a\ \leq 1$ $0 \leq k \leq 1$	$\ [a, k]\ \leq C$	Might be useful inves- tigating commuta- tors and square roots	Theorem 3.1
h, k, x		$\ hk\ \leq C$ $0 \leq \begin{bmatrix} 1-h & x^* \\ x & k \end{bmatrix} \leq 1$		Theorem 3.1
x, y, z, w	$0 \leq x \leq 1$ $0 \leq y \leq 1$ $0 \leq z \leq 1$ $0 \leq w \leq 1$	$\ xy\ \leq C$ $\ zw\ \leq C$ $\ (1-x)z(1-x)\ \leq C$ $\ (1-x)w(1-x)\ \leq C$		Theorem 4.2
$x_1, \dots, x_r$		$\left\ \sum_{k=1}^r (x_k x_k^*)^{\frac{q}{2}} \right\ \leq 1$	$1 < q < \infty$	Theorem 5.3

An important special case that we study is the approximate zero locus of a homogeneous NC $*$ -polynomial intersected with the NC unit ball. The homogeneity is imposed to give contractability, and so gives us an expectation of finding not only semiprojectivity, but projectivity. By approximate zero locus we mean the universal C^* -algebra

$$A_\epsilon = C^* \left\langle x_1, \dots, x_r \left| \|p(x_1, \dots, x_r)\| \leq \epsilon, \left\| \sum_{j=1}^r x_j x_j^* \right\| \leq 1 \right. \right\rangle,$$

and the “row contraction” condition $\sum x_j x_j^* \leq 1$ (cf. [3, 9]) is implementing the intersection with the unit ball. A special case of Theorem 3.2 states that A_ϵ is projective for all $\epsilon > 0$.

In many cases the relations we can handle have as their universal C^* -algebra something that is very unfamiliar. In these cases, it is perhaps best to see our results as lifting results for the relations rather than projectivity results for the C^* -algebras.

Table 1 lists some known liftable relations. It is somewhat representative. There are closure results, such as when A is projective $\mathbf{M}_n(A)$ is also projective, which leads to many more liftable sets of relations, so no table can be complete.

Table 2 lists some of the new examples. It is not certain these are new projective C^* -algebras, as projective C^* -algebras are contractible and so invariants such as K -theory are of no avail. We can say with some certainty the relations were not known to lift.

A useful result in topological shape theory is that every compact metric space is the projective limit of ANRs [6, IX.1.4]. Blackadar writes in [4, 4.4]:

It is not clear that every C^* -algebra has a strong shape system [is an inductive limit of semiprojective C^* -algebras].

We show in Section 7 that every cone over a separable C^* -algebra is the inductive limit of projective C^* -algebras.

2. QUASI-CENTRAL APPROXIMATE UNITS FIX NORMS

Our key tool for lifting is Theorem 2.3. It was extracted from the difficult terrain that is page 127 of Akemann and Pedersen's paper [2].

Approximate units are assumed to satisfy $0 \leq u_\lambda \leq 1$. If I is an ideal in A , we let $\pi : A \rightarrow A/I$ denote the quotient map.

Lemma 2.1. *Suppose $I \triangleleft A$. For any approximate unit u_λ of I , any h in A_+ , and any real $0 \leq \delta \leq 1$,*

$$\limsup_{\lambda} \left\| (1 - u_\lambda)^{\frac{1}{2}} h (1 - u_\lambda)^{\frac{1}{2}} + (1 - \delta) u_\lambda^{\frac{1}{2}} h u_\lambda^{\frac{1}{2}} \right\| \leq \max(\|\pi(h)\|, (1 - \delta) \|h\|).$$

Proof. We can lift $\pi(h)$ to k with $0 \leq k \leq \|\pi(h)\|$. Setting $x = h - k$, we have x in I and

$$\begin{aligned} \limsup_{\lambda} \left\| (1 - u_\lambda)^{\frac{1}{2}} h (1 - u_\lambda)^{\frac{1}{2}} + (1 - \delta) u_\lambda^{\frac{1}{2}} h u_\lambda^{\frac{1}{2}} \right\| \\ = \limsup_{\lambda} \left\| (1 - u_\lambda)^{\frac{1}{2}} k (1 - u_\lambda)^{\frac{1}{2}} + (1 - \delta) u_\lambda^{\frac{1}{2}} h u_\lambda^{\frac{1}{2}} \right\|. \end{aligned}$$

Now we use the order structure in A and find

$$\begin{aligned} (1 - u_\lambda)^{\frac{1}{2}} k (1 - u_\lambda)^{\frac{1}{2}} + (1 - \delta) u_\lambda^{\frac{1}{2}} h u_\lambda^{\frac{1}{2}} &\leq \|k\| (1 - u_\lambda) + (1 - \delta) \|h\| u_\lambda \\ &\leq \max(\|\pi(h)\|, (1 - \delta) \|h\|). \end{aligned}$$

□

Lemma 2.2. *Suppose $I \triangleleft A$. For any approximate unit u_λ of I that is quasi-central for A , any a in A , and any real $0 \leq \delta \leq 1$,*

$$\limsup_{\lambda} \left\| a (1 - \delta u_\lambda)^{\frac{1}{2}} \right\| \leq \max(\|\pi(a)\|, (1 - \delta)^{\frac{1}{2}} \|a\|).$$

Proof. Using the quasi-central property and Lemma 2.1,

$$\begin{aligned}
\limsup_{\lambda} \left\| a(1 - \delta u_{\lambda})^{\frac{1}{2}} \right\|^2 &= \limsup_{\lambda} \|a^* a(1 - \delta u_{\lambda})\| \\
&= \limsup_{\lambda} \|a^* a(1 - u_{\lambda}) + (1 - \delta)a^* a u_{\lambda}\| \\
&= \limsup_{\lambda} \left\| (1 - u_{\lambda})^{\frac{1}{2}} a^* a(1 - u_{\lambda})^{\frac{1}{2}} + (1 - \delta)u_{\lambda}^{\frac{1}{2}} a^* a u_{\lambda}^{\frac{1}{2}} \right\| \\
&\leq \max(\|\pi(a^* a)\|, (1 - \delta)\|a^* a\|) \\
&= (\max(\|\pi(a)\|, (1 - \delta)\|a\|))^2.
\end{aligned}$$

□

Theorem 2.3. *Suppose $I \triangleleft A$, that u_{λ} is a approximate unit u_{λ} for I that is quasi-central for A , and a is in A . If f is a continuous function on $[0, 1]$ so that*

$$1 = f(0) \geq f(t) \geq f(1) \geq 0,$$

then

$$\limsup_{\lambda} \|af(u_{\lambda})\| \leq \max(\|\pi(a)\|, f(1)\|a\|).$$

Proof. Let $\delta = 1 - f(1)^2$ and

$$g(t) = \delta^{-1} (1 - f(t)^2).$$

This function is continuous and

$$0 = g(0) \leq g(t) \leq g(1) = 1,$$

so $g(u_{\lambda})$ is also a quasi-central approximate unit. By Lemma 2.2,

$$\begin{aligned}
\limsup_{\lambda} \|af(u_{\lambda})\| &= \limsup_{\lambda} \left\| a(1 - \delta g(u_{\lambda}))^{\frac{1}{2}} \right\| \\
&\leq \max(\|\pi(a)\|, (1 - \delta)^{\frac{1}{2}}\|a\|) \\
&= \max(\|\pi(a)\|, f(1)\|a\|).
\end{aligned}$$

□

3. LIFTING SOFTENED HOMOGENEOUS RELATIONS

We will consider $*$ -polynomials in infinitely many variables that are homogeneous in some finite subset of the variables. These we take to be the first r -variables, which we label $x_1, \dots, x_r$, and the remaining variables we label $y_1, y_2, \dots$. We also use the n -tuple notation $\mathbf{x} = (x_1, \dots, x_r)$ and $\mathbf{y} = (y_1, y_2, \dots)$, and with an NC $*$ -polynomial p we use the notation

$$p(\mathbf{x}, \mathbf{y}) = p(x_1, \dots, x_r, y_1, y_2, \dots).$$

For scalar t we use

$$t\mathbf{x} = (tx_1, \dots, tx_r).$$

We will say p is d -homogeneous in the first r variables if

$$p(t\mathbf{x}, \mathbf{y}) = t^d p(\mathbf{x}, \mathbf{y})$$

for all real scalars t . In other words, in each monomial the x_j and x_j^* appear collectively d times. As d is not necessarily the degree of p , we call d the *degree of homogeneity* of p .

We will rather quickly run out of letters if we insist on other symbols when evaluating p on specific elements of a C^* -algebra A . Given $x_1, \dots, x_r$ in A and m in A or $\tilde{A}$ we define

$$m\mathbf{x} = (mx_1, \dots, mx_r).$$

(We use $\tilde{A}$ to denote the unitization of A .) If $\varphi : A \rightarrow B$, then

$$\varphi(\mathbf{x}) = (\varphi(x_1), \dots, \varphi(x_r))$$

and

$$\varphi(\mathbf{y}) = (\varphi(y_1), \varphi(y_2), \dots).$$

If $z_1, \dots, z_r$ are in A , then $\mathbf{z} \leq \mathbf{y}$ shall mean $z_j \leq y_j$ for $j = 1, \dots, r$. If $\mathbf{0} \leq \mathbf{x}$, then $\mathbf{x}^{\frac{1}{2}}$ shall denote $(x_1^{\frac{1}{2}}, \dots, x_r^{\frac{1}{2}})$. For $a_1, \dots, a_n$ and $b_1, \dots, b_n$ in A , we use the notation

$$a \bullet b = a_1 b_1 + \dots + a_n b_n.$$

Recall π is our generic notation for the quotient map $A \rightarrow A/I$.

Theorem 3.1. *Suppose $p_1, \dots, p_J$ are NC $*$ -polynomials in infinitely many variables that are homogeneous in the first r variables, with each degree of homogeneity d_j at least one. Suppose $C_j > 0$ are constants. For every C^* -algebra A and $I \triangleleft A$ an ideal, given $x_1, \dots, x_r$ and $y_1, y_2, \dots$ in A with $\mathbf{0} \leq \mathbf{x}$ and*

$$\|p_j(\pi(\mathbf{x}), \pi(\mathbf{y}))\| \leq C_j,$$

there are $z_1, \dots, z_r$ in A with $\mathbf{0} \leq \mathbf{z} \leq \mathbf{x}$ and $\pi(\mathbf{z}) = \pi(\mathbf{x})$ and

$$(3.1) \quad \|p_j(\mathbf{z}, \mathbf{y})\| \leq C_j.$$

Proof. Our proof is modeled on that of [2, Theorem 3.7].

We start by performing the easier lifting where C_j in (3.1) is replaced by $(1 + \epsilon_1)C_j$. We pick ϵ_1 later, but it will be positive. Since C_j is not allowed to be zero, $(1 + \epsilon_1)C_j$ will be strictly larger than C_j .

Let u_λ be any approximate unit u_λ for I that is quasi-central for A . By quasi-centrality and the homogeneity in $\mathbf{x}$, we have

$$\begin{aligned} \lim_{\lambda} \left\| p_j \left(\mathbf{x}^{\frac{1}{2}} \bullet (1 - u_\lambda) \mathbf{x}^{\frac{1}{2}}, \mathbf{y} \right) \right\| &= \lim_{\lambda} \left\| p_j(\mathbf{x}, \mathbf{y}) (1 - u_\lambda)^{d_j} \right\| \\ &\leq \lim_{\lambda} \|p_j(\mathbf{x}, \mathbf{y}) (1 - u_\lambda)\| \\ &= \|\pi(p_j(\mathbf{x}, \mathbf{y}))\| \\ &= \|p_j(\pi(\mathbf{x}), \pi(\mathbf{y}))\| \\ &\leq C_j. \end{aligned}$$

We define

$$z_k^{(1)} = x_k^{\frac{1}{2}} (1 - u_{\lambda_1}) x_k^{\frac{1}{2}},$$

where λ_1 is large enough to give us

$$\left\| p_j \left(\mathbf{z}^{(1)}, \mathbf{y} \right) \right\| \leq (1 + \epsilon_1) C_j$$

for $j = 1, \dots, J$. Clearly, $\mathbf{0} \leq \mathbf{z}^{(1)} \leq \mathbf{x}$ and $\pi(\mathbf{z}^{(1)}) = \pi(\mathbf{x})$.

We will create ever better lifts by defining

$$z_k^{(2)} = \left(z_k^{(1)} \right)^{\frac{1}{2}} (1 - \delta_2 u_{\lambda_2}) \left(z_k^{(1)} \right)^{\frac{1}{2}},$$

and so forth. For consistency, we let $\delta_1 = 1$ and $\mathbf{z}^{(0)} = \mathbf{x}$. We choose $\delta_1 > \delta_2 > \dots$ all positive with

$$(3.2) \quad \sum_{c=1}^{\infty} \delta_c < \infty.$$

We set ϵ_1 and the rest of a sequence ϵ_c by requiring

$$(1 - \delta_{c+1})(1 + \epsilon_c) = 1.$$

Notice the ϵ_c are positive and decreasing to zero.

Assume we have found $\mathbf{z}^{(1)}$ through $\mathbf{z}^{(c-1)}$ with

$$(3.3) \quad \mathbf{0} \leq \mathbf{z}^{(c-1)} \leq \mathbf{z}^{(c-2)} \leq \dots \leq \mathbf{z}^{(1)} \leq \mathbf{x},$$

$$(3.4) \quad \pi(\mathbf{z}^{(c-1)}) = \pi(\mathbf{z}^{(c-2)}) = \dots = \pi(\mathbf{z}^{(1)}) = \pi(\mathbf{x}),$$

and

$$(3.5) \quad \|p_j(\mathbf{z}^{(w)}, \mathbf{y})\| \leq (1 + \epsilon_w) C_j, \quad (w = 1, \dots, c-1).$$

Moreover, assume the $\mathbf{z}^{(w)}$ have been constructed via the formula

$$(3.6) \quad \mathbf{z}^{(w)} = \left(\mathbf{z}^{(w-1)}\right)^{\frac{1}{2}} (1 - \delta_w u_{\lambda_w}) \left(\mathbf{z}^{(w-1)}\right)^{\frac{1}{2}}, \quad (w = 1, \dots, c-1).$$

Theorem 2.3 tells us

$$\begin{aligned} \limsup_{\lambda} \left\| p_j \left(\left(\mathbf{z}^{(c-1)}\right)^{\frac{1}{2}} \bullet (1 - \delta_c u_{\lambda}) \left(\mathbf{z}^{(c-1)}\right)^{\frac{1}{2}}, \mathbf{y} \right) \right\| \\ &= \limsup_{\lambda} \left\| p_j \left(\mathbf{z}^{(c-1)}, \mathbf{y} \right) (1 - \delta_c u_{\lambda})^{d_j} \right\| \\ &\leq \limsup_{\lambda} \left\| p_j \left(\mathbf{z}^{(c-1)}, \mathbf{y} \right) (1 - \delta_c u_{\lambda}) \right\| \\ &\leq \max \left(\left\| \pi \left(p_j \left(\mathbf{z}^{(c-1)}, \mathbf{y} \right) \right) \right\|, (1 - \delta_c) \left\| p_j \left(\mathbf{z}^{(c-1)}, \mathbf{y} \right) \right\| \right) \\ &= \max \left(\left\| p_j \left(\pi(\mathbf{x}), \pi(\mathbf{y}) \right) \right\|, (1 - \delta_c) \left\| p_j \left(\mathbf{z}^{(c-1)}, \mathbf{y} \right) \right\| \right) \\ &\leq \max(C_j, (1 - \delta_c)((1 + \epsilon_{c-1}) C_j)) \\ &= C_j, \end{aligned}$$

so we may choose λ_c with

$$\left\| p_j \left(\left(\mathbf{z}^{(c-1)}\right)^{\frac{1}{2}} \bullet (1 - \delta_c u_{\lambda_c}) \left(\mathbf{z}^{(c-1)}\right)^{\frac{1}{2}}, \mathbf{y} \right) \right\| \leq (1 + \epsilon_c) C_j.$$

We set

$$\mathbf{z}^{(c)} = \left(\mathbf{z}^{(c-1)}\right)^{\frac{1}{2}} (1 - \delta_c u_{\lambda_c}) \left(\mathbf{z}^{(c-1)}\right)^{\frac{1}{2}},$$

and the construction continues.

We wish to set $z_k = \lim_c z_j^{(c)}$, and we may because

$$\begin{aligned} \|z_k^{(c)} - z_k^{(c-1)}\| &= \left\| \left(\mathbf{z}_k^{(c-1)}\right)^{\frac{1}{2}} (\delta_c u_{\lambda_c}) \left(\mathbf{z}_k^{(c-1)}\right)^{\frac{1}{2}} \right\| \\ &\leq \left\| \mathbf{z}_k^{(c-1)} \right\| \|\delta_c u_{\lambda_c}\| \\ &\leq 2 \|x_k\| \delta_c. \end{aligned}$$

Equations (3.3), (3.4), and (3.5) give us $\mathbf{0} \leq \mathbf{z} \leq \mathbf{x}$ and $\pi(\mathbf{z}) = \mathbf{x}$ and finally the norm conditions

$$\|p_j(\mathbf{z}, \mathbf{y})\| = \lim_c \|p_j(\mathbf{z}^{(c)}, \mathbf{y})\| \leq C_j.$$

□

If we have soft $*$ -polynomial relations involving self-adjoint variables, we can replace each by two positive variables. A variable that is a contraction can be replaced by four positive variables. These replacements will preserve any homogeneity in a subset of the variables. Thus we can have a more flexible version of Theorem 3.1. As stated, Theorem 3.2 is not a corollary as we are very specific in how the lifts are adjusted for the different types of variables.

Theorem 3.2. *Suppose $p_1, \dots, p_J$ are NC $*$ -polynomials in infinitely many variables that are homogeneous in the first r variables, with each degree of homogeneity d_j at least one. Suppose $S = \{1, \dots, r\}$ is partitioned as*

$$S = S_+ \cup S_h \cup S_g,$$

we have positive constants C_j , nonnegative constants D_k , E_k , F_k and G_k , and consider the relations

$$(3.7) \quad 0 \leq x_k \leq D_k \quad (k \in S_+),$$

$$(3.8) \quad -E_k \leq x_k \leq F_k \quad (k \in S_h),$$

$$(3.9) \quad \|x_k\| \leq G_k \quad (k \in S_g),$$

$$(3.10) \quad \|p_j(\mathbf{x}, \mathbf{y})\| \leq C_j.$$

For every C^* -algebra A and $I \triangleleft A$ an ideal, given $x_1, \dots, x_r$ and $y_1, y_2, \dots$ in A so that $(\mathbf{x}, \mathbf{y})$ satisfies (3.7)–(3.9) and $(\pi(\mathbf{x}), \pi(\mathbf{y}))$ satisfies (3.10), there are elements $z_1, \dots, z_r$ in A so that $(\mathbf{z}, \mathbf{y})$ satisfy (3.7)–(3.10) and $\pi(\mathbf{z}) = \pi(\mathbf{x})$. Moreover, it is possible to do so with

$$z_k = (x_k)^{\frac{1}{2}} m^2 (x_k)^{\frac{1}{2}} \quad (k \in S_+),$$

$$z_k = m x_k m \quad (k \in S_h),$$

$$z_k = x_k m^2 \quad (k \in S_g),$$

for some m in $1 + I$ with $0 \leq m \leq 1$.

Proof. Let ϵ_c , δ_c , and u_λ be as before. We modify the construction used for Theorem 3.1 by requiring $m_0 = 1$ and

$$m_c = (1 - \delta_c u_{\lambda_c}) m_{c-1} (1 - \delta_c u_{\lambda_c})$$

and

$$z_k^{(c)} = x_k^{\frac{1}{2}} m_c^2 x_k^{\frac{1}{2}} \quad (k \in S_+),$$

$$z_k^{(c)} = m_c x_k m_c \quad (k \in S_h),$$

$$z_k^{(c)} = x_k m_c^2 \quad (k \in S_g).$$

We want

$$\|p_j(\mathbf{z}^{(c)}, \mathbf{y})\| \leq (1 + \epsilon_c) C_j,$$

given that we have already defined m_{c-1} in $1 + I$ with

$$(3.11) \quad \|p_j(\mathbf{z}^{(c-1)}, \mathbf{y})\| \leq (1 + \epsilon_{c-1}) C_j.$$

To unify the initial step and subsequent steps, we take ϵ_0 large enough to force (3.11) when $c = 1$. We need to find the right λ_c to define $z_j^{(c)} = w_j^{(\lambda_c)}$, where

$$\begin{aligned} w_k^{(\lambda)} &= x_k^{\frac{1}{2}} ((1 - \delta_c u_\lambda) m_{c-1} (1 - \delta_c u_\lambda))^2 x_k^{\frac{1}{2}} \quad (k \in S_+), \\ w_k^{(\lambda)} &= ((1 - \delta_c u_\lambda) m_{c-1} (1 - \delta_c u_\lambda)) x_k ((1 - \delta_c u_\lambda) m_{c-1} (1 - \delta_c u_\lambda)) \quad (k \in S_h), \\ w_k^{(\lambda)} &= x_k ((1 - \delta_c u_\lambda) m_{c-1} (1 - \delta_c u_\lambda))^2 \quad (k \in S_g), \end{aligned}$$

and see

$$\lim_{\lambda} \|w_j^{(\lambda)} - (1 - \delta_c u_\lambda)^4 z_j^{(c-1)}\| = 0.$$

Therefore

$$\begin{aligned} \limsup_{\lambda} \|p_j(\mathbf{w}^{(\lambda)}, \mathbf{y})\| &= \limsup_{\lambda} \|p_j((1 - \delta_c u_\lambda)^4 \mathbf{z}^{(c-1)}, \mathbf{y})\| \\ &= \limsup_{\lambda} \|p_j(\mathbf{z}^{(c-1)}, \mathbf{y}) (1 - \delta_c u_\lambda)^{4d_j}\| \\ &\leq \limsup_{\lambda} \|p_j(\mathbf{z}^{(c-1)}, \mathbf{y}) (1 - \delta_c u_\lambda)\| \\ &\leq \max \left(\|p_j(\pi(\mathbf{x}), \pi(\mathbf{y}))\|, (1 - \delta_c) \|p_j(\mathbf{z}^{(c-1)}, \mathbf{y})\| \right) \\ &\leq C_j, \end{aligned}$$

and it is possible to choose the needed λ_c .

It is clear that m_c stays in $1 + I$, so these are all lifts of the original $\mathbf{x}$. What is left to check is that $m = \lim_c m_c$ exists. Indeed it does, as

$$\begin{aligned} \|m_c - m_{c-1}\| &= \|(1 - \delta_c u_{\lambda_c}) m_{c-1} (1 - \delta_c u_{\lambda_c}) - m_{c-1}\| \\ &\leq \|(\delta_c u_{\lambda_c}) m_{c-1} (1 - \delta_c u_{\lambda_c})\| + \|m_{c-1} (\delta_c u_{\lambda_c})\| \\ &\leq 2 \|\delta_c u_{\lambda_c} m_{c-1}\| \\ &\leq 2\delta_c. \end{aligned}$$

□

From Theorem 3.2 we get a myriad of projective C^* -algebras, simply by adding relations such as $-1 \leq y_j \leq 1$ that are liftable and that impose a norm restriction forcing the universal C^* -algebra to exist. We generally add the relation $\|y_j\| = 0$ to most of the y_j so as to be working with a finitely generated projective C^* -algebra.

4. SOFT VERSIONS OF KNOWN PROJECTIVES

Consider $C_0(X)$ where X^+ is a (finite) tree. The presentation in [17] for the projective C^* -algebra $C_0(X)$ was based on a partial order $\preceq$ on $\{1, \dots, s\}$. This was not a general partial order, it had to be the partial order on the nonroot vertices determined by paths away from the root. Let us call such a relation a tree order.

The presentation associated to the tree order $\preceq$ had generators $h_1, \dots, h_s$ and relations

$$\begin{aligned} 0 \leq h_j \leq 1 \quad (j = 1, \dots, s), \\ h_i h_j = h_j \quad (\text{if } i \prec j), \\ h_i h_j = 0 \quad (\text{if } i \not\preceq j \text{ and } j \not\preceq i). \end{aligned}$$

The last two lines of relations are not generally homogeneous in any subset of the variables. Taking advantage of the precise way the liftings are modified in our main theorem, we can still lift the soft version of these.

The concrete function in $C_0(X)$ that corresponds to the abstract h_j only varies along one edge, the edge incident to the j th vertex on a path from that vertex to the root. From this point of view, it makes more sense to index the generators by the edges (as in [17]) but in graph theory, tree orders are on the vertices.

We do need to make two changes to the relations. Some are redundant, since for positive elements

$$h_1 h_2 = h_2 \ \& \ h_3 h_1 = 0 \implies h_3 h_2 = 0$$

and

$$h_1 h_2 = h_2 \ \& \ h_3 h_1 = h_1 \implies h_3 h_2 = h_2.$$

We adopt the usual terminology of i being a child of j whenever $j \preceq i$ and

$$j \preceq k \preceq i \implies k = j \text{ or } k = i.$$

The only relations we need are those that ask that the parent act as a unit on the child and that two children of the same parent must be orthogonal. We call two children of the same vertex siblings, of course. Children of children, and so forth, we call descendants.

A second change is that we replace $h_i h_j = h_j$ by

$$(h_i - 1) h_j (h_i - 1) = 0.$$

Theorem 4.1. *Suppose $\preceq$ is a tree order on $\{1, \dots, s\}$. The relations*

$$\begin{aligned} 0 \leq h_j \leq 1 \quad (j = 1, \dots, s), \\ \|(h_i - 1) h_j (h_i - 1)\| \leq \epsilon \quad (\text{if } j \text{ is a child of } i), \\ \|h_i h_j\| \leq \epsilon \quad (\text{if } i \text{ and } j \text{ are siblings}) \end{aligned}$$

are liftable.

The following, slightly stronger result is more easily proven.

Theorem 4.2. *Suppose $\preceq$ is a tree order on $\{1, \dots, s\}$ and $\epsilon > 0$. For every C^* -algebra A and $I \triangleleft A$ an ideal, given $h_1, \dots, h_s$ in A with $\mathbf{0} \leq \mathbf{h} \leq \mathbf{1}$ and*

$$\begin{aligned} \|\pi((h_i - 1) h_j (h_i - 1))\| \leq \epsilon \quad (\text{if } j \text{ is a child of } i), \\ \|\pi(h_i) \pi(h_j)\| \leq \epsilon \quad (\text{if } i \text{ and } j \text{ are siblings}), \end{aligned}$$

there are $k_1, \dots, k_s$ in A with $\mathbf{0} \leq \mathbf{k} \leq \mathbf{h}$ and $\pi(\mathbf{k}) = \pi(\mathbf{h})$ and

$$\begin{aligned} \|(k_i - 1) k_j (k_i - 1)\| \leq \epsilon \quad (\text{if } j \text{ is a child of } i), \\ \|k_i k_j\| \leq \epsilon \quad (\text{if } i \text{ and } j \text{ are siblings}). \end{aligned}$$

Proof. The very trivial base case for our proof by induction is the case of zero generators.

Reindexing, we may assume the minimal elements (i.e., elders or vertices closest to the root) in this partial order are $\{1, \dots, r\}$. If m and n are descendants of different minimal elements i and j , then there are no relations involving both h_m

and h_n . The relations not involving the minimal elements are a disjoint union of relations of the type in the statement of the theorem.

The relations we need that involve the minimal elements are $0 \leq h_j \leq 1$ for $j = 1, \dots, r$ and

$$\begin{aligned} \|(h_i - 1) h_m (h_i - 1)\| &\leq \epsilon \quad (m \text{ is the child of } i), \\ \|k_i k_j\| &\leq \epsilon \quad (1 \leq i < j \leq r). \end{aligned}$$

These are homogeneous in $\{h_1, \dots, h_r\}$. By Theorem 3.1 there are $k_1, \dots, k_r$ in A with $0 \leq k_j \leq h_j$ and $\pi(h_j) = \pi(k_j)$ for $j \leq r$ and

$$\begin{aligned} \|(k_i - 1) h_m (k_i - 1)\| &\leq \epsilon \quad (m \text{ is the child of } i), \\ \|k_i k_j\| &\leq \epsilon \quad (1 \leq i < j \leq r). \end{aligned}$$

The induction hypothesis tells us there are $k_{r+1}, \dots, k_s$ with $0 \leq k_m \leq h_m$ and $\pi(k_m) = \pi(h_m)$ for $m > r$ with all the relations not involving indices $\{1, \dots, r\}$. We might have lost the relations between some h_i and h_m with m a child of i , but we have not, since

$$(h_i - 1) k_m (h_i - 1) \leq (h_i - 1) h_m (h_i - 1).$$

□

A rather different example with a similar proof is a soft version of the projective C^* -algebra

$$C^* \left\langle h, k, x \left| \begin{array}{l} hk = 0 \\ 0 \leq \begin{bmatrix} 1-h & x^* \\ x & k \end{bmatrix} \leq 1 \end{array} \right. \right\rangle$$

considered in [19]. (For a detailed explanation of how the second relation is valid, see [20].)

Theorem 4.3. *For any positive ϵ , the C^* -algebra*

$$C^* \left\langle h, k, x \left| \begin{array}{l} \|hk\| \leq \epsilon \\ 0 \leq \begin{bmatrix} 1-h & x^* \\ x & k \end{bmatrix} \leq 1 \end{array} \right. \right\rangle$$

is projective.

Proof. Suppose h, k and x are in A , which we may assume is unital, such that

$$\|\pi(h) \pi(k)\| \leq \epsilon$$

and

$$0 \leq \begin{bmatrix} 1 - \pi(h) & \pi(x)^* \\ \pi(x) & \pi(k) \end{bmatrix} \leq 1.$$

We know positive contractions lift to positive contractions from $\mathbf{M}_2(A/I)$ to $\mathbf{M}_2(A)$, and so we can find $\hat{h}$, $\hat{k}$, and $\hat{x}$ in A so that $\pi(\hat{h}) = \pi(h)$, $\pi(\hat{k}) = \pi(k)$, $\pi(\hat{x}) = \pi(x)$ and

$$0 \leq \begin{bmatrix} 1 - \hat{h} & \hat{x}^* \\ \hat{x} & \hat{k} \end{bmatrix} \leq 1.$$

The polynomial hk is homogeneous in k so Theorem 3.2 tells us there is $0 \leq m \leq 1$ in $1 + I$ so that $\|\hat{h}m\hat{k}m\| \leq \epsilon$. Let $\bar{h} = \hat{h}$, $\bar{x} = m\hat{x}$ and $\bar{k} = m\hat{k}m$. These are still lifts of h , x and k , and now $\|\bar{h}\bar{k}\| \leq \epsilon$ and

$$\begin{bmatrix} 1 - \bar{h} & \bar{x}^* \\ \bar{h} & \bar{h} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 - \hat{h} & \hat{x}^* \\ \hat{x} & \hat{k} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}$$

implies

$$0 \leq \begin{bmatrix} 1 - \bar{h} & \bar{x}^* \\ \bar{h} & \bar{h} \end{bmatrix} \leq 1.$$

□

5. FATTENED CURVES IN VARIOUS NC UNIT BALLS

Theorem 5.1. *Suppose $p_1, \dots, p_J$ are NC $*$ -polynomials in $x_1, \dots, x_s$. Suppose $1 \leq r \leq s$ and each p_j is homogeneous in $\{x_1, \dots, x_r\}$ with degree of homogeneity $d_j \geq 1$. For $\epsilon > 0$, the C^* -algebra*

$$A_\epsilon = C^* \left\langle x_1, \dots, x_s \left| \begin{array}{l} \|x_k\| \leq 1, k = 1, \dots, s \\ \|p_j(x_1, \dots, x_s)\| \leq \epsilon, j = 1, \dots, J \end{array} \right. \right\rangle$$

is projective.

Proof. This is an immediate consequence of Theorem 3.2. □

For a single NC $*$ -polynomial p , we can think of

$$(5.1) \quad C^* \left\langle x_1, \dots, x_s \left| \begin{array}{l} \|x_k\| \leq 1 \\ \|p(x_1, \dots, x_s)\| \leq \epsilon \end{array} \right. \right\rangle$$

as an approximate zero locus of an NC curve in the NC unit square. Likewise, we can think of

$$(5.2) \quad C^* \left\langle x_1, \dots, x_s \left| \begin{array}{l} \left\| \sum_{k=1}^s x_k^* x_k \right\| \leq 1 \\ \|p(x_1, \dots, x_s)\| \leq \epsilon \end{array} \right. \right\rangle$$

as an approximate zero locus of an NC curve in the NC unit ball. Notice that the “row contraction” condition $\left\| \sum x_k^* x_k \right\| \leq 1$ implies $\|x_k\| \leq 1$, so we can still apply Theorem 5.1. For $\epsilon > 0$, and with p homogeneous in $x_1, \dots, x_s$, we find (5.1) and (5.2) define projective C^* -algebras.

We will see that it is possible to work with other unit balls, not just the ones corresponding to the ℓ^2 - and ℓ^∞ -norms.

Lemma 5.2. *Suppose $0 < \alpha < \infty$ is a scalar. For every $\epsilon > 0$ there is a $\delta > 0$ so that for any two positive contractions in any C^* -algebra,*

$$\|hk - kh\| \leq \delta \implies \|(hkh)^\alpha - k^\alpha h^{2\alpha}\| \leq \epsilon.$$

Proof. This can be rephrased so it becomes a special case of Lemma 10 of [19], but it is easier to revise the proof. We know that for nonnegative scalars, $(xyx)^\alpha = y^\alpha x^{2\alpha}$, so by spectral theory,

$$hk = kh \implies (hkh)^\alpha = k^\alpha h^{2\alpha}.$$

If the lemma is false, there must be some ϵ_0 and h_n and k_n in A_n with $0 \leq h_n \leq 1$ and $0 \leq k_n \leq 1$ and

$$\|h_n k_n - k_n h_n\| \leq \frac{1}{n}$$

and

$$\|(h_n k_n h_n)^\alpha - k_n^\alpha h_n^{2\alpha}\| \geq \epsilon_0.$$

This creates an element in

$$\prod A_n / \bigoplus A_n$$

with $hk = kh$ and $(hkh)^\alpha - k^\alpha h^{2\alpha} \neq 0$, a contradiction. $\square$

Theorem 5.3. *Suppose r is a natural number. For $0 < p < \infty$, define*

$$B_p = C^* \left\langle x_1, \dots, x_r \left\| \left\| \sum_{k=1}^r (x_k x_k^*)^{\frac{p}{2}} \right\| \leq 1 \right\rangle \right.$$

If $0 < p \leq \infty$, then B_p is projective.

Proof. Suppose $I \triangleleft A$ with quasi-central approximate unit u_λ . Suppose x_k are in A with $\pi(x_k)$ in A/I satisfying

$$\left\| \sum_{k=1}^r (\pi(x_k) (\pi(x_k))^*)^{\frac{p}{2}} \right\| \leq 1.$$

Let $z_k^{(0)} = x_k$, and let ϵ_0 be sufficiently large so as to have

$$\left\| \sum_{k=1}^r (z_k^{(0)} (z_k^{(0)})^*)^{\frac{p}{2}} \right\| \leq 1 + \epsilon_0.$$

Choose δ_c a positive sequence decreasing to zero with $\delta_1 = 1$ and

$$\sum_{c=1}^{\infty} \left(1 - (1 - \delta_c)^{\frac{1}{p}}\right) < \infty.$$

Define $\epsilon_c \searrow 0$ for $c \geq 1$ by the formula

$$(1 - \delta_{c+1}) (1 + \epsilon_c) = 1.$$

Assume we have found $\mathbf{z}^{(c-1)}$ with

$$\pi(\mathbf{z}^{(c-1)}) = \pi(\mathbf{x})$$

and

$$\left\| \sum_{k=1}^r (z_k^{(c-1)} (z_k^{(c-1)})^*)^{\frac{p}{2}} \right\| \leq 1 + \epsilon_c.$$

Using Lemma 5.2 and Theorem 2.3 we find

$$\begin{aligned} \limsup_{\lambda} \left\| \sum_{k=1}^r \left((1 - \delta_c u_\lambda)^{\frac{1}{p}} z_k^{(c-1)} (z_k^{(c-1)})^* (1 - \delta_c u_\lambda)^{\frac{1}{p}} \right)^{\frac{p}{2}} \right\| \\ = \limsup_{\lambda} \left\| \sum_{k=1}^r (z_k^{(c-1)} (z_k^{(c-1)})^*)^{\frac{p}{2}} (1 - \delta_c u_\lambda) \right\| \\ \leq 1. \end{aligned}$$

We can choose λ_c and set

$$z_k^{(c)} = (1 - \delta_c u_{\lambda_c})^{\frac{1}{p}} z_k^{(c-1)},$$

where λ_c is large enough to ensure

$$\left\| \sum_{k=1}^r \left(z_k^{(c)} \left(z_k^{(c)} \right)^* \right)^{\frac{p}{2}} \right\| \leq 1 + \epsilon_c.$$

The $z_j^{(c)}$ converge because

$$\begin{aligned} \left\| z_k^{(c)} - z_k^{(c-1)} \right\| &= \left\| (1 - \delta_c u_{\lambda_c})^{\frac{1}{p}} z_k^{(c-1)} - z_k^{(c-1)} \right\| \\ &\leq \left\| (1 - \delta_c u_{\lambda_c})^{\frac{1}{p}} - 1 \right\| \left\| z_k^{(c-1)} \right\| \\ &\leq \left(1 - (1 - \delta_c)^{\frac{1}{p}} \right) \|x_k\|. \end{aligned}$$

□

We still cannot tell if all the B_p are isomorphic. They do interact with homogeneous $*$ -polynomials in about the same fashion as the usual unit ball. The set of NC $*$ -polynomials that we know we can mix with the nonstandard unit ball condition depends on p . We have no idea if this is a limitation of our methods, or a real limitation.

Theorem 5.4. *Suppose $r \leq s$ and $p_1, \dots, p_J$ are NC $*$ -polynomials in $x_1, \dots, x_s$, each homogeneous in $\{x_1, \dots, x_r\}$ with degree of homogeneity d_j at least one. For $C_j > 0$ and $0 < q \leq 2$ the C^* -algebra*

$$A_\epsilon = C^* \left\langle x_1, \dots, x_s \left| \begin{array}{l} \left\| \sum_{k=1}^s (x_k x_k^*)^{\frac{q}{2}} \right\| \leq 1 \\ \|p_j(x_1, \dots, x_s)\| \leq C_j, j = 1, \dots, J \end{array} \right. \right\rangle$$

is projective.

Proof. Suppose we are given $\pi : A \rightarrow A/I$ with $x_1, \dots, x_s$ in A with

$$\left\| \sum_{k=1}^s (\pi(x_k x_k^*))^{\frac{q}{2}} \right\| \leq 1$$

and

$$\|p_j(\pi(x_1), \dots, \pi(x_s))\| \leq C_j.$$

We first apply Theorem 5.3 to find $y_1, \dots, y_s$ in A with $\pi(y_k) = \pi(x_k)$ and

$$\left\| \sum_{k=1}^s (y_k y_k^*)^{\frac{q}{2}} \right\| \leq 1.$$

Theorem 3.2 gives us $z_1, \dots, z_s$ in A with $\pi(z_j) = \pi(x_j)$ and

$$\|p_j(z_1, \dots, z_s)\| \leq C_j,$$

but also with $z_k = y_k m$ for $k \leq r$ and $z_k = y_k$ for $k > r$, where $0 \leq m \leq 1$. Therefore

$$z_k z_k^* = y_k m^2 y_k^* \leq y_k y_k^*$$

for $1 \leq k \leq r$. Since for $q \leq 2$ the function $t^{q/2}$ is operator-monotone, we get

$$\sum_{k=1}^s (z_k z_k^*)^{\frac{q}{2}} \leq \sum_{k=1}^s (y_k y_k^*)^{\frac{q}{2}} \leq 1.$$

□

Theorem 5.5. *Suppose $p_1, \dots, p_J$ are homogeneous, degree- d_j NC $*$ -polynomials in $x_1, \dots, x_r$ with $d_j \geq 1$. For $C_j > 0$ and $2 < q < \infty$, the C^* -algebra*

$$A_\epsilon = C^* \left\langle x_1, \dots, x_r \left| \begin{array}{l} \left\| \sum_{k=1}^r (x_k x_k^*)^{\frac{q}{2}} \right\| \leq 1 \\ \|p_j(x_1, \dots, x_r)\| \leq C_j, j = 1, \dots, J \end{array} \right. \right\rangle$$

is projective.

Proof. Suppose we are given $\pi : A \rightarrow A/I$ with $x_1, \dots, x_r$ in A with

$$\left\| \sum_{k=1}^r (\pi(x_k x_k^*))^{\frac{q}{2}} \right\| \leq 1$$

and

$$\|p_j(\pi(x_1), \dots, \pi(x_r))\| \leq C_j.$$

Choose δ_c and ϵ_c as before, with the δ_c summable. Keeping with our earlier notation, we are going to define $\mathbf{z}^{(c)}$ from $\mathbf{z}^{(c-1)}$ by

$$z_k^{(c)} = (1 - \delta_c u_{\lambda_c}) z_k^{(c-1)}.$$

Lemma 5.2 and Theorem 2.3 give us

$$\begin{aligned} & \limsup_{\lambda} \left\| \sum_{k=1}^r \left((1 - \delta_c u_{\lambda}) z_k^{(c-1)} \left(z_k^{(c-1)} \right)^* (1 - \delta_c u_{\lambda}) \right)^{\frac{q}{2}} \right\| \\ &= \limsup_{\lambda} \left\| \sum_{k=1}^r \left(z_k^{(c-1)} \left(z_k^{(c-1)} \right)^* \right)^{\frac{q}{2}} (1 - \delta_c u_{\lambda})^q \right\| \\ &\leq \limsup_{\lambda} \left\| \sum_{k=1}^r \left(z_k^{(c-1)} \left(z_k^{(c-1)} \right)^* \right)^{\frac{q}{2}} (1 - \delta_c u_{\lambda}) \right\| \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} & \limsup_{\lambda} \left\| p_j \left((1 - \delta_1 u_{\lambda}) \mathbf{z}^{(c-1)} \right) \right\| \\ &= \limsup_{\lambda} \left\| p_j \left(\mathbf{z}^{(c-1)} \right) (1 - \delta_c u_{\lambda})^{d_j} \right\| \\ &\leq \limsup_{\lambda} \left\| p_j \left(\mathbf{z}^{(c-1)} \right) (1 - \delta_c u_{\lambda}) \right\| \\ &= C_j. \end{aligned}$$

The limit of the $z_j^{(c)}$ will exist because

$$\begin{aligned} \left\| z_k^{(c)} - z_k^{(c-1)} \right\| &= \left\| (1 - \delta_c u_{\lambda_c}) z_k^{(c-1)} - z_k^{(c-1)} \right\| \\ &\leq \left\| (1 - \delta_c u_{\lambda_c}) - 1 \right\| \left\| z_k^{(c-1)} \right\| \\ &\leq \delta_c \|x_k\|. \end{aligned}$$

□

6. SOFT CYLINDERS

When we stray from homogeneous relations, we come across K -theoretical obstructions to projectivity. To illustrate what properties can still hold, we offer the example of the “soft cylinder”. The weaker properties are semiprojectivity (as in [4]) and the residually finite dimensional (RFD) property. Projectivity implies semiprojectivity and also RFD ([22, §1]).

For $\epsilon \geq 0$ we define the soft cylinder almost like Exel’s soft torus ([14]):

$$A_\epsilon = C_1^* \langle u, h \mid u^*u = uu^* = 1, -1 \leq h \leq 1, \|uh - hu\| \leq \epsilon \rangle.$$

Notice we retained some homogeneity.

Theorem 6.1. *For positive ϵ , the soft cylinder A_ϵ is semiprojective.*

Proof. Suppose B is a unital C^* -algebra, with ideal $I = \bigcup I_n$ for some increasing sequence of ideals I_n . Suppose we are given u and h in B/I where u is unitary, $-1 \leq h \leq 1$, and

$$\|uh - hu\| \leq \epsilon.$$

For some n it is possible to lift u to v in B/I_n that is a unitary ([4, Prop. 2.21]). Take any lift of h to $-1 \leq k \leq 1$ in B/I_n . Theorem 3.2 tells us there is $\hat{k}$ in A/I with $-1 \leq \hat{k} \leq 1$ and

$$\|v\hat{k} - \hat{k}v\| \leq \epsilon.$$

□

Eilers and Exel ([11]) have shown that the soft torus is RFD. The same can be said, and proven much more easily, for the soft cylinder.

Theorem 6.2. *For positive ϵ , the soft cylinder A_ϵ is RFD.*

Proof. Consider the surjection

$$\rho : C(S^1) *_C C[0, 1] \twoheadrightarrow \widetilde{A}_\epsilon$$

that sends the obvious unitary generator to u and the obvious positive, norm-1 generator to h . By [15, Theorem 3.2] the free product is RFD. Theorem 3.2 tells us that ρ is split. Thus A_ϵ can be embedded in an RFD C^* -algebra and so is itself RFD. □

Our lifting theorems can be used to determine whether many more C^* -algebras are RFD. The study of weak projectivity ([21]) and RFD of the C^* -algebras associated to rather general relations that have some homogeneity might lead to some interesting examples.

7. CONES ARE LIMITS OF PROJECTIVE C^* -ALGEBRAS

We end with a tantalizing result: every cone is the limit of projectives. As a C^* -algebra with a projective cone must be semiprojective ([5, II.8.3.10]) it would seem that we are close to proving that every separable C^* -algebra is a limit of semiprojective C^* -algebras.

We say definitively, projectivity is not “extremely rare”, cf. [18, p. 73].

Lemma 7.1. *Suppose A is the unital C^* -algebra*

$$A = C_1^* \left\langle x_1, x_2, \dots \left| \begin{array}{l} -C_k \leq x_k \leq C_k \quad (\forall k) \\ p_j(\mathbf{x}) = 0 \quad (\forall j) \end{array} \right. \right\rangle,$$

where the $p_1, p_2, \dots$ are NC polynomials in the x_k of degrees D_j with zero constant term. Then the cone $\mathbf{C}A$ has presentation

$$\mathbf{C}A = C^* \left\langle h, x_1, x_2, \dots \left| \begin{array}{l} 0 \leq h \leq 1 \\ hx_k = x_k h \quad (\forall k) \\ -C_k h \leq x_k \leq C_k h \quad (\forall k) \\ q_j(h, \mathbf{x}) = 0 \quad (\forall j) \end{array} \right. \right\rangle,$$

where q_j is the NC polynomial derived from the p_j by padding monomials on the left with various powers of h so that q_j is homogeneous with degree D_j .

Proof. To illustrate the construction of the q_j , if

$$p_1 = x_1 + 3x_1x_2^*x_1,$$

then

$$q_1 = h^2x_1 + 3x_1x_2^*x_1.$$

In general, we can break up p_j into homogeneous summands

$$p_j = \sum_{d=1}^{D_j} p_{j,d}$$

and then describe the q_j as

$$q_j(h, \mathbf{x}) = \sum_{d=1}^{D_j} h^{D_j-d} p_{j,d}(\mathbf{x}).$$

Let the universal unital C^* -algebra for these relations be denoted $\mathcal{U}$. This exists, as the relations satisfy the needed four axioms as in [20]. One of the axioms is that setting all variables to the zero elements in $\{0\}$ leads to a representation of the relations, which is true because we require the constant terms to be zero.

To define a $*$ -homomorphism $\mathcal{U} \rightarrow \mathbf{C}A$, we define in $\mathbf{C}A = C_0((0, 1], A)$ elements $\tilde{x}_k = tx_k$ and $\tilde{h} = t$, shorthand for $\tilde{x}(t) = tx$ and so forth. It is obvious that $0 \leq \tilde{h} \leq 1$ and that $\tilde{h}$ commutes with each $\tilde{x}_k$. Also

$$-C_k \leq x_k \leq C_k \implies -tC_k \leq tx_k \leq tC_k,$$

so $-C_j \tilde{h} \leq \tilde{x}_j \leq C_j \tilde{h}$. The last relation holds as well since

$$\begin{aligned} \left(q_j \left(\tilde{h}, \tilde{\mathbf{x}} \right) \right) (t) &= \sum_{d=1}^{D_j} t^{D_j-d} p_{j,d}(t\mathbf{x}) \\ &= t^{D_j} \sum_{d=1}^{D_j} p_{j,d}(\mathbf{x}) \\ &= t^{D_j} p_j(\mathbf{x}) \\ &= 0. \end{aligned}$$

Next we will show this map is onto. Basic algebra and the usual isomorphism of $\mathbf{CA}$ with $C_0(0, 1] \otimes A$ tells us that functions of the form $t^m w$ generate the cone, where w ranges over words in the x_k . Suppose $w = w_1 w_2 \cdots w_n$. If $m \geq n$, then this is easily in the image, as

$$t^m w = t^{m-n} (tw_1) (tw_2) \cdots (tw_n).$$

If $1 \leq m < n$, then the Stone-Weierstrass theorem tells us we can approximate in $C_0(0, 1]$ the function t^m by a polynomial in $t^n, t^{n+1}, \dots$, and so we can approximate $t^m w$ by a polynomial in $t^n w, t^{n+1} w, \dots$ and the map is indeed onto. We turn to proving it is one-to-one.

Consider an irreducible representation in $\mathbb{B}(\mathbb{H})$ of the relations defining $\mathcal{U}$ by H and $X_1, X_2, \dots$. Since $0 \leq H \leq 1$ and $HX_k = X_k H$, we find that H is central and so $H = \lambda I$ for some scalar λ with $0 \leq \lambda \leq 1$. If $\lambda = 0$, then $H = 0$ and

$$-C_k \lambda \leq X_k \leq C_k \lambda \implies X_k = 0.$$

This is the zero representation, which is the pullback of the zero representation of $\mathbf{CA}$. If λ is positive, then

$$-C_k \lambda \leq X_k \leq C_k \lambda \implies -C_k \leq \lambda^{-1} X_k \leq C_k,$$

and $q_j(H, \mathbf{X}) = 0$ implies

$$\begin{aligned} p_j(\lambda^{-1} \mathbf{X}) &= \sum_{d=1}^{D_j} \lambda^{-d} p_{j,d}(\mathbf{X}) \\ &= \lambda^{-D_j} \sum_{d=1}^{D_j} H^{D_j-d} p_{j,d}(\mathbf{X}) \\ &= \lambda^{-D_j} q_j(H, \mathbf{X}) \\ &= 0. \end{aligned}$$

Thus the $\lambda^{-1} X_k$ form a representation of A on $\mathbb{H}$ and so a representation of $\mathbf{CA}$ via the composition

$$\mathbf{CA} \xrightarrow{\delta_\lambda} A \longrightarrow \mathbb{B}(\mathbb{H}).$$

This sends $\tilde{h}$ to $\lambda I = H$ and $\tilde{x}_k$ to X_k , finishing the proof. $\square$

Theorem 7.2. *If $q_1, q_2, \dots$ are homogeneous NC polynomials, each of degree at least one, in noncommuting variables $h, x_1, x_2, \dots$, then for positive constants $C_1, \dots, C_J$*

and $D_1, \dots, D_K$, the C^* -algebra

$$C^* \left\langle h, x_1, x_2, \dots \left| \begin{array}{l} 0 \leq h \leq 1, \\ -C_k h \leq x_k \leq C_k h, \quad (1 \leq k \leq K) \\ \|q_j(h, x_1, x_2, \dots)\| \leq D_j \quad (1 \leq j \leq J) \end{array} \right. \right\rangle$$

is projective.

Proof. We can find some r so that $x_{r+1}, x_{r+2}, \dots$ are not in any of the polynomials $q_1, \dots, q_J$. If we relabel these $y_1, y_2, \dots$, our lifting problem becomes

$$\begin{aligned} 0 &\leq h \leq 1, \\ -C_k h &\leq x_k \leq C_k h, \\ -C'_k h &\leq y_k \leq C'_k h, \\ \|q_j(h, \mathbf{x})\| &\leq D_j \quad (1 \leq j \leq J), \end{aligned}$$

where now the q_j are homogeneous in $\{h, x_1, \dots, x_r\}$. We are using $\mathbf{x}$ for $(x_1, \dots, x_r)$.

Given h, x_k and y_k in A with

$$\begin{aligned} 0 &\leq \pi(h) \leq 1, \\ -C_k \pi(h) &\leq \pi(x_k) \leq C_k \pi(h), \\ -C'_k \pi(h) &\leq \pi(y_k) \leq C'_k \pi(h), \\ \|q_j(\pi(h), \pi(\mathbf{x}))\| &\leq D_j \quad (1 \leq j \leq J), \end{aligned}$$

we first find a new lift $\hat{h}$ of $\pi(h)$ with

$$0 \leq \hat{h} \leq 1.$$

Using [8, Corollary 2.2] we find $\hat{x}_k$ and $\hat{y}_k$ with

$$\begin{aligned} -C_k \hat{h} &\leq \hat{x}_k \leq C_k \hat{h}, \\ -C'_k \hat{h} &\leq \hat{y}_k \leq C'_k \hat{h} \end{aligned}$$

and $\pi(\hat{x}_k) = \pi(x_k)$ and $\pi(\hat{y}_k) = \pi(y_k)$. By Theorem 3.2 there is an m with $0 \leq m \leq 1$ in $1 + I$ so that

$$\|q_j(m\hat{h}m, m\hat{\mathbf{x}}m)\| \leq D_j.$$

Our desired lifts are $m\hat{h}m$, $m\hat{\mathbf{x}}m$ and $m\hat{\mathbf{y}}m$. □

Lemma 7.3. *Let D be a separable C^* -algebra. Then*

$$D \cong C^* \left\langle x_1, x_2, \dots \left| \begin{array}{l} -C_j \leq x_j \leq C_j \quad (\forall j) \\ p_k(x_1, x_2, \dots) = 0 \quad (\forall k) \end{array} \right. \right\rangle$$

for a countable collection of NC polynomials.

Proof. Example 1.3(b) in [4] tells us that D has a presentation with countably many generators, countably many relations in the form of an NC $*$ -polynomial set to zero and countably many norm conditions. We will modify Blackadar's method a bit.

Let $\mathbb{F} = \mathbb{Q} + i\mathbb{Q}$, which is a countable dense subfield of $\mathbb{C}$. Select a countable dense sequence in D and apply to this sequence all polynomials over $\mathbb{F}$ in countably many variables. This results in a countable, dense $\mathbb{F}$ - $*$ -subalgebra B of D . Enumerate B as $x_1, x_2, \dots$. The algebraic operations for B can be encoded in $*$ -polynomial

relations. For example, if $\alpha x_j = x_k$ for some α in $\mathbb{F}$, then we use the relation $\alpha x_j - x_k = 0$. If $x_j^* = x_k$, then we use the relation $x_j^* - x_k = 0$, and so forth. This means B is the universal $\mathbb{F}$ -*-algebra for generators $x_1, x_2, \dots$ and some countable set of *-polynomial relations $p_j(x_1, x_2, \dots) = 0$. We now add to these relations the C^* -relations $\|x_k\| \leq C_k$ where C_k is the norm of the element x_k in D . Then any function $f : B \rightarrow G$, for G a C^* -algebra, that satisfies these relations is first of all an $\mathbb{F}$ -linear *-algebra homomorphism. It is continuous with respect to the norm on D since $x_j - x_k$ will equal some x_ℓ so we have the relation $\|f(x_\ell)\| \leq \|x_\ell\|$ and so

$$\|f(x_j) - f(x_k)\| = \|f(x_\ell)\| \leq \|x_\ell\| = \|x_j - x_k\|.$$

It therefore extends to a continuous function $\varphi : D \rightarrow G$. This extended function will be linear over $\mathbb{C}$. To verify this, consider $\alpha = \lim \alpha_n$, a limit of scalars from $\mathbb{F}$, and $d = \lim d_j$, a limit of elements in B . Then

$$\varphi(\alpha_n d) = \varphi(\lim_j \alpha_n d_j) = \lim_j f(\alpha_n d_j) = \alpha_n \lim_j f(d_j) = \alpha_n \varphi(d)$$

and

$$\varphi(\alpha d) = \varphi(\lim_n \alpha_n d) = \lim_n \varphi(\alpha_n d) = \lim_n \alpha_n \varphi(d) = \alpha \varphi(d).$$

Finally, continuity implies that φ is a *-homomorphism. It is uniquely determined by f and so D is universal for these relations.

We can eliminate many of the norm conditions. Suppose we keep only the norm restrictions $\|x_k\| \leq C_k$ for those x_k that are self-adjoint. Then the estimate that gave continuity changes a little. Any $x_j - x_k$ will equal some x_ℓ and for some r and s we will have $x_r = \frac{1}{2}x_\ell + \frac{1}{2}x_\ell^*$ and $x_s = \frac{-i}{2}x_\ell - \frac{i}{2}x_\ell^*$. Of course these are the Hermitian and anti-Hermitian parts of x_ℓ , and as they are self-adjoint we have the relations $\|f(x_r)\| \leq \|x_r\|$ and $\|f(x_s)\| \leq \|x_s\|$. Therefore

$$\begin{aligned} \|f(x_j) - f(x_k)\| &= \|f(x_r) + if(x_s)\| \\ &\leq \|f(x_r)\| + \|f(x_s)\| \\ &\leq \|x_r\| + \|x_s\| \\ &\leq 2\|x_\ell\| \\ &= 2\|x_j - x_k\|. \end{aligned}$$

This still gives us continuity and so the rest of the proof goes through.

We can toss the generators that are not self-adjoint if we modify each polynomial by the evaluating x_k at $x_r + ix_s$ whenever x_r and x_s are the real and imaginary parts of x_k . Among the polynomial relations will be $x_j^* - x_j = 0$ for the generators we are keeping. Given this, it is our option to use the relation $\|x_j\| \leq C_j$ or $-C_j \leq x_j \leq C_j$. $\square$

Theorem 7.4. *If A is a separable C^* -algebra, then its cone $\mathbf{CA}$ is isomorphic to the inductive limit of a countable system of projective C^* -algebras with surjective bonding maps.*

Proof. We start with the case where $A = \tilde{D}$ for some separable, possibly unital C^* -algebra.

Lemma 7.3 tells us

$$A \cong C_1^* \left\langle x_1, x_2, \dots \left| \begin{array}{l} -C_k \leq x_k \leq C_k \quad (\forall k) \\ p_j(x_1, x_2, \dots) = 0 \quad (\forall j) \end{array} \right. \right\rangle,$$

and then Lemma 7.1 tells us

$$\mathbf{CA} \cong C^* \left\langle h, x_1, x_2, \dots \left| \begin{array}{l} 0 \leq h \leq 1 \\ hx_k = x_k h \quad (\forall k) \\ -C_k h \leq x_k \leq C_k h \quad (\forall k) \\ q_j(h, x_1, x_2, \dots) = 0 \quad (\forall j) \end{array} \right. \right\rangle,$$

where the q_k are homogeneous. Clearly,

$$\mathbf{CA} \cong \varinjlim P_n,$$

where

$$P_n = C^* \left\langle h, x_1, x_2, \dots \left| \begin{array}{l} 0 \leq h \leq 1 \\ -C_k h \leq x_k \leq C_k h, \quad (\forall k) \\ \|hx_k - x_k h\| \leq \frac{1}{n}, \quad (k = 1, \dots, n) \\ \|q_j(x_1, x_2, \dots)\| \leq \frac{1}{n} \quad (j = 1, \dots, n) \end{array} \right. \right\rangle.$$

Since the commutators are homogeneous NC polynomials, Theorem 7.2 applies and the P_n are projective. We are done for $\mathbf{CA} = \mathbf{C}(\tilde{D})$. What about $\mathbf{CD}$?

We have the exact sequence

$$0 \longrightarrow \mathbf{CD} \longrightarrow \mathbf{CA} \longrightarrow \mathbf{CC} \longrightarrow 0.$$

Of course $\mathbf{CC}$ equals $C_0(0, 1]$ and is projective. Let Q_n be the kernel of the map of P_n onto $C_0(0, 1]$ that sends h to $t \mapsto t$ and x_k to zero. Then we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q_n & \longrightarrow & P_n & \longrightarrow & \mathbf{CC} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & Q_{n+1} & \longrightarrow & P_{n+1} & \longrightarrow & \mathbf{CC} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbf{CD} & \longrightarrow & \mathbf{CA} & \longrightarrow & \mathbf{CC} \longrightarrow 0 \end{array}$$

with the rows exact. Also, $\mathbf{CD}$ is isomorphic to $\varinjlim Q_n$, which we can see as follows.

There is a $*$ -homomorphism

$$\varphi : \varinjlim Q_n \rightarrow \mathbf{CD}$$

induced by the maps $Q_n \rightarrow \mathbf{CD}$. The maps $Q_n \rightarrow P_n$ are inclusions and hence isometries. Theorem 13.1.2.2 in [18] implies that the induced map $\varinjlim Q_n \rightarrow \varinjlim P_n$ is also an isometry. From the commutative diagram

$$\begin{array}{ccc} \varinjlim Q_n & \hookrightarrow & \varinjlim P_n \\ \downarrow \varphi & & \downarrow \cong \\ \mathbf{CD} & \hookrightarrow & \mathbf{CA} \end{array}$$

we conclude φ is injective. As to surjectivity, consider x in $\mathbf{CD}$. This gets sent to zero in $\mathbf{CC}$. Any lift of x to y in P_1 is also sent to zero in $\mathbf{CC}$, and so is Q_n . This shows φ is onto.

By Theorem 5.3 of [22] the Q_n are projective. $\square$

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