THE POLYADIC GENERALIZATION OF THE BOOLEAN AXIOMATIZATION OF FIELDS OF SETS

MIKLÓS FERENCZI

ABSTRACT. A version of the classical representation theorem for Boolean algebras states that the fields of sets form a variety and that a possible axiomatization is the system of Boolean axioms. An important case for fields of sets occurs when the unit V is a subset of an α -power ${}^\alpha U$. Beyond the usual set operations union, intersection, and complement, new operations are needed to describe such a field of sets, e.g., the i th cylindrification C_i , the constant ij th diagonal D_{ij} , the elementary substitution $[i / j]$ and the transposition $[i, j]$ for all $i, j < \alpha$ restricted to the unit V . Here it is proven that such generalized fields of sets being closed under the above operations form a variety; further, a first order finite scheme axiomatization of this variety is presented. In the proof a crucial role is played by the existence of the operator transposition. The foregoing axiomatization is close to that of finitary polyadic equality algebras (or quasi-polyadic equality algebras).

1. INTRODUCTION

A version of the classical representation theorem for Boolean algebras states that the fields of sets form a variety and that a possible axiomatization of it is the system of the Boolean axioms. An important case for the fields of sets (e.g. in measure theory, probability theory, topology, etc.) occurs when the unit V is a subset of an α -power ${}^\alpha U$ for some given set U and ordinal α , i.e., V consists of a certain set of α -sequences. Beyond the usual set operations union, intersection, and complement, new operations are needed to describe such a field of sets. Such operations are e.g. the i th cylindrification C_i , the constant ij th diagonal D_{ij} , the elementary substitution $[i / j]$ and the elementary transposition $[i, j]$ for every $i, j < \alpha$, restricted to the unit V . Let us consider the extended field of sets which is closed under these operations. As it will be shown the class of these fields coincides with the class $\mathbf{Gwp}_\alpha$ of polyadic-like set algebras. Then the following natural problems arise: Do these fields form a variety and what, if anything, is known about its axiomatization?

Here we give positive answers to these questions and also present a possible axiomatization. It turns out that the axiomatization of these fields of sets is that of the abstract class *finitary polyadic equality algebras* (or that of the equivalent class:

Received by the editors September 4, 2009 and, in revised form, February 23, 2010.

2010 *Mathematics Subject Classification.* Primary 03G15, 03G05, 03C05; Secondary 03G25, 03C48, 03C95.

Key words and phrases. Representation, Stone theorem, polyadic algebras, fields of sets, algebraic logic, cylindric algebras.

©2011 American Mathematical Society
 Reverts to public domain 28 years from publication

quasi-polyadic equality algebras; see [15]), except for the axiom of commutativity of the single substitutions and cylindrifications. Instead of this axiom the commutativity of single substitutions has to be postulated. The abstract class obtained in this way is called the *transposition algebras* and denoted by $\mathbf{TA}_\alpha$. Our main theorem states: $\mathcal{A} \in \mathbf{TA}_\alpha$ if and only if $\mathcal{A} \in \mathbf{IsGwp}_\alpha$, $\alpha \geq 4$.

A preliminary part of our theorem is Resek and Thompson's representation theorem (see [11], 3.2.88 or [2]). This theorem answers the above questions with the modification that the operation $[i, j]$ is missing. By that theorem, the axiomatization of this class of fields of sets is that of the abstract class of *cylindric algebras* satisfying the so-called merry-go-round properties and a given weakening (axiom (C4)* of the axiom of commutativity of cylindrifications (class $\mathbf{NA}_\alpha^+$). Ferenczi proved in [9] and [10] that (C4)* is equivalent to the commutativity of single substitutions; further, a possible meaning of the merry-go-round properties is a restricted existence of the abstract transposition operator p_{ij} . Considering the Resek–Thompson theorem and our main theorem together we can establish that in the representability of a cylindric-type algebra by relativized set algebra a crucial role is played by the implicit or explicit existence of the transposition operator.

When Resek and Thompson's theorem was published it was only a conjecture that there was a polyadic version of the theorem as well, but that conjecture had not yet been investigated in detail. The polyadic version is not an immediate consequence of the cylindric one. Finitary polyadic equality algebras (or quasi-polyadic equality algebras) are not as close to cylindric algebras as it was earlier believed. With cylindric algebras cannot always be associated a finitary polyadic equality algebra. For example, the substitution operator s_τ can be introduced in every finitary polyadic equality algebra for finite τ , but not in every cylindric algebra. To be a cylindric reduct of some finitary polyadic equality algebra is quite a strong assumption for a cylindric algebra. Thus the representation class $\mathbf{Gwp}_\alpha$ for $\mathbf{TA}_\alpha$ is simpler than the cylindric representation class $\mathbf{D}_\alpha$ for $\mathbf{NA}_\alpha^+$.

The main problem, i.e., the axiomatization of the extended fields of sets, can also be approached from the *representation theory of universal algebra*. Infinite dimensional polyadic algebras (without equality) are representable (see [6]), but this class is a bit far from classical logic. As is known, *neither* cylindric algebras *nor* finitary polyadic equality algebras (quasi-polyadic equality algebras) can, in general, be representable in the usual universal algebraic sense. That is, they cannot be represented as subdirect products of the corresponding set algebras or, equivalently, as fields of sets in $\mathbf{Gws}_\alpha$ or in $\mathbf{Gwsp}_\alpha$ (see [11] or [14]). The results in the present paper seem to have significance because the representation does work if the representation classes are allowed to be the above fields of sets, i.e., so-called *relativized set algebras*.

In the proof of the Main Theorem (Theorem 3.1) we follow the step-by-step method (see [12] or [2]) applied, for example, in Andrka and Thompson's proof for the Resek–Thompson theorem (Trans. Amer. Math. Soc., vol. 309, no. 2, here [2]). For the sake of convenience of the reader in the Appendix we cite the parts of the Andrka–Thompson proof that we need. Thus in the presented proof below the common parts are not repeated, only the differences are emphasized. In our proof we rely on some lemmas whose proofs are rather elementary. In Theorem 3.12 a variant of a consequence of Theorem 3.1 is formulated and an independent proof for it is given. The reader can find a detailed survey on the topic in [4].

2. CONCEPTS

First we list some classes of *extended fields of sets* and then some classes of *abstract algebras*. Finally, we cite some representation theorems as connections between the concrete and abstract algebras.

The term *cylindric-like algebra* means that the type of algebra is that of cylindric algebras (see Definition 2.11). Similarly, the term *polyadic-like algebra* means that the type of algebra is that of transposition algebras (or that of finitary polyadic equality algebras; see Definition 2.13). It often occurs in the literature that the term cylindric-like algebra or polyadic-like algebra is shortened, and the term *cylindric algebra* or *polyadic algebra* is merely used.

We list some *cylindric extensions* of fields of sets (while later some polyadic extensions are listed). The first one is the class $\mathbf{Crs}_\alpha$, where the unit V of the field is a subset of ${}^\alpha U$ for some set U and the Boolean part is closed under the relativized cylindrifications C_i^V and the relativized diagonals D_{ij}^V for every $i, j < \alpha$. More exactly:

Definition 2.1 (Class $\mathbf{Crs}_\alpha$). An algebra $\mathcal{A}$ is a *cylindric relativized set algebra of dimension α* with unit V if it is of the form

$$\langle A, \cup, \cap, \sim_V, \emptyset, V, C_i^V, D_{ij}^V \rangle_{i,j < \alpha},$$

where V is a set of α -termed sequences (i.e. $V \subseteq {}^\alpha U$) such that A is a non-empty set of subsets of V , closed under the Boolean operations $\cup, \cap, \sim_V$ and under the cylindrifications

$$C_i^V X = \{y \in V : y_u^i \in X \text{ for some } u\},$$

where $i < \alpha$, $X \in A$, and A contain the elements $\emptyset, V$ and the diagonals

$$D_{ij}^V = \{y \in V : y_i = y_j\}$$

(see [11], 3.1).

The meaning of the notation y_u^i is $(y_u^i)_j = y_j$ if $j \neq i$ and $(y_u^i)_j = u$ if $j = i$.

Some *concepts and notation* in $\mathbf{Crs}_\alpha$:

The relativized *substitution operator* ${}^V S_j^i$ is defined as

$$(2.1) \quad {}^V S_j^i X = C_i^V (D_{ij}^V \cap X) \quad (X \in A)$$

(similar to the definition in cylindric algebras; see (2.3)). We often omit the superscript V from ${}^V S_j^i$ and write S_j^i .

The transformation τ defined on α is called *finite* if $\tau i = i$ except for finitely many $i < \alpha$. Sometimes a finite transformation τ is considered to be defined only on a given finite subset of α .

If $y \in V$ and τ is any finite transformation on α , then $S_\tau y$ is defined as $y \circ \tau$ and is often denoted by τy , for short. If $X \in A$, then $S_\tau X$ is defined as $\{y \circ \tau : y \in X\}$ and is often denoted by τX , for short.

Important special instances of the finite τ are the transformations $[i / j]$, called *elementary substitution*, and $[i, j]$, called *transposition*.

Applying the notation introduced, if $y \in V$, then $[i / j] y$ denotes the sequence $y_{y_j}^i$. If $X \in A$, then $[i / j] X$ is defined as $\{[i / j] y : y \in X\}$.

${}^V S_{[i, j]} y$ or $[i, j] y$ is the sequence for which the i th and j th members in y are interchanged; all the other members are the same. If $X \in A$, ${}^V S_{[i, j]} X$ is the set $\{y \circ [i, j] : y \in X\}$; it is often denoted by $[i, j] X$, for short.

In the following subclass of $\mathbf{Crs}_\alpha$, ${}^V S_j^i$ preserves the unit V :

Definition 2.2 (Class $\mathbf{D}_\alpha$). It is the subclass of $\mathbf{Crs}_\alpha$ for which

$$(2.2) \quad {}^V S_j^i V = V$$

for every $i, j < \alpha$, where V is the unit of the algebra (see [2]).

Remark 2.3. Notice that $[i / j]$ may be considered as a defined operation in $\mathbf{Crs}_\alpha$ because $[i / j]X = {}^V S_j^i X$ if $X \in A$. The critical part of (2.2) is the inclusion ${}^V S_j^i V \supseteq V$, of course.

Given a set U and a mapping $p \in {}^\alpha U$, then the set

$${}^\alpha U^{(p)} = \{x \in {}^\alpha U : x \text{ and } p \text{ are different only in finitely many places}\}$$

is called the *weak space* determined by p and U .

Definition 2.4 (Class $\mathbf{Gw}_\alpha$). A set algebra in $\mathbf{Crs}_\alpha$ is called a *generalized weak cylindric relativized set algebra* if there are sets U_k , $k \in K$, and mappings $p_k \in {}^\alpha U_k$ such that $V = \bigcup_{k \in K} {}^\alpha U_k^{(p_k)}$, where V is the unit.

Remark 2.5. The decomposition of V as the foregoing union in the definition is not unique. The disjointness of the members of the decompositions of V is *not required* here. If it is required we get the definition of the well-known class, the class $\mathbf{Gws}_\alpha$ of the *generalized weak cylindric set algebras* (as is known, cylindric algebras are representable in the classical sense if they are isomorphic to some algebra in $\mathbf{Gws}_\alpha$; see [11], II., 3.1.107).

Then we introduce some *polyadic extensions* of the fields of sets:

If an algebra in $\mathbf{Crs}_\alpha$ is extended by the substitution operator ${}^V S_j^i$ defined in (2.1) and by the operator transposition ${}^V S_{[i, j]}$, then we get the polyadic version of $\mathbf{Crs}_\alpha$; more exactly:

Definition 2.6 (Class $\mathbf{Prs}_\alpha$). An algebra

$$\mathcal{B} = \langle A, \cup, \cap, \sim_V, \emptyset, V, C_i^V, {}^V S_j^i, {}^V S_{[i, j]}, D_{ij}^V \rangle_{i, j < \alpha}$$

is an α -dimensional *polyadic relativized set algebra* if for the cylindric reduct, $\text{Rd}_{ca} \mathcal{B} \in \mathbf{Crs}_\alpha$; further, A is closed under the transposition ${}^V S_{[i, j]}$.

Remark 2.7. An obvious consequence of the definition is that $[i, j]V = V$. This latter relation means that $y \in V$ if and only if $y \circ [i, j] \in V$.

With the classes of cylindric set algebras $\mathbf{D}_\alpha$ and $\mathbf{Gw}_\alpha$ we can associate the classes of polyadic set algebras $\mathbf{Dp}_\alpha$ and $\mathbf{Gwp}_\alpha$, respectively (the former are the cylindric reducts of $\mathbf{Dp}_\alpha$ and $\mathbf{Gwp}_\alpha$, respectively):

Definition 2.8 (Class $\mathbf{Dp}_\alpha$). It is the subclass of $\mathbf{Prs}_\alpha$ for which ${}^V S_j^i V = V$ for every $i, j < \alpha$, where V is the unit of the algebra.

Definition 2.9 (Class $\mathbf{Gwp}_\alpha$). A set algebra in $\mathbf{Prs}_\alpha$ is called a *generalized weak polyadic relativized set algebra* if there are sets U_k , $k \in K$, and mappings $p_k \in {}^\alpha U_k$ such that $V = \bigcup_{k \in K} {}^\alpha U_k^{(p_k)}$, where V is the unit.

A known characterization of the class $\mathbf{Gwp}_\alpha$ is: If V is the unit of an $\mathcal{A} \in \mathbf{Prs}_\alpha$, then $\mathcal{A} \in \mathbf{Gwp}_\alpha$ if and only if $y \in V$ implies $y \circ \tau \in V$ for every *finite*

transformation τ (see [13], 5.35). Using this property we shall prove that $\mathbf{Gwp}_\alpha = \mathbf{Dp}_\alpha$ (Lemma 3.10).

The following is a survey of the classes of set algebras listed above:

<i>cylindric-like set algebras</i>	<i>polyadic-like set algebras</i>	<i>about the unit V</i>
$\mathbf{Crs}_\alpha$	$\mathbf{Prs}_\alpha$	$V \subseteq {}^\alpha U$
$\mathbf{D}_\alpha$	$\mathbf{Dp}_\alpha$	${}^V S_j^i V = V$
$\mathbf{Gw}_\alpha$	$\mathbf{Gwp}_\alpha$	$V = \bigcup_{k \in K} {}^\alpha U_k^{(p_k)}$

Among these classes, in the sequel, explicitly only the classes $\mathbf{D}_\alpha$ and $\mathbf{Gwp}_\alpha$ ($\mathbf{Dp}_\alpha$) will be used.

Remark 2.10. We note that lower cases are used for denoting classes of set algebras (except for the first character), while upper cases are used for denoting classes of abstract algebras (i.e. axiomatically defined algebras). Further, upper cases are used for operations defined on set algebras, while lower cases are used for operations defined on abstract algebras.

Now we list some classes of *abstract algebras*. First we define *cylindric-like* abstract algebras (while later polyadic-like abstract algebras are listed):

Definition 2.11 (Class $\mathbf{CA}_\alpha$). A Boolean algebra $\langle A, +, \cdot, -, 0, 1 \rangle$ enriched with a set of additional unary operations c_i ($i < \alpha$) and constants d_{ij} ($i, j < \alpha$), where α is a fixed ordinal, is said to be a *cylindric algebra $\mathcal{A}$ of dimension α* ($\mathcal{A} \in \mathbf{CA}_\alpha$) if it satisfies the following axioms:

- (C1) $c_i 0 = 0$,
 - (C2) $x \leq c_i x$,
 - (C3) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$,
 - (C4) $c_i c_j x = c_j c_i x$,
 - (C5) $d_{ii} = 1$,
 - (C6) $c_j(d_{ji} \cdot d_{jk}) = d_{ik}$, $j \notin \{i, k\}$,
 - (C7) $d_{ij} \cdot c_i(d_{ij} \cdot x) = d_{ij} \cdot x$, $i \neq j$
- (see [11], 1.1.1).

In cylindric algebras, as it is known, the *substitution operator* s_m^j is a unary operation introduced as follows:

$$(2.3) \quad s_m^j x = c_j(d_{jm} \cdot x) \quad \text{if } j \neq m \quad \text{and} \quad s_m^j x = x \quad \text{if } j = m.$$

The following properties are called *merry-go-round axioms* (or *MGR properties*; see [11], 3.2.88):

$$(2.4) \quad s_i^k s_j^i s_k^j c_k x = s_j^k s_i^j s_k^i c_k x,$$

$$(2.5) \quad s_i^k s_j^i s_n^j s_k^n c_k x = s_j^k s_n^j s_i^n s_k^i c_k x$$

for distinct i, j, k and n , $\alpha \geq 4$.

The cylindric axiom (C4) proved to be a quite strong property. In the following axiomatizations, instead of (C4), a weaker property is assumed. Further, the MGR axioms will be postulated. In [10] it is proven that the MGR axioms mean the existence of an operator “weak transposition” (or “partial transposition”). So

by Resek–Thompson theorem (see (2.9)), the existence of such an operator yields representability by relativized set algebras.

Definition 2.12 (Class $\mathbf{F}_\alpha$). If the cylindric axiom (C4) is replaced by the commutativity of the single substitutions, i.e., by the property

$$(2.6) \quad s_j^i s_m^k x = s_m^k s_j^i x$$

if $i, j \notin \{k, m\}$, and further, if the MGR properties are postulated as axioms, then the axiomatization of the class $\mathbf{F}_\alpha$ is obtained (see [9]).

A version of the axiomatization of the class $\mathbf{F}_\alpha$ is that of the class $\mathbf{NA}_\alpha^+$. Here the axiom

$$(C4)^*: c_i c_j x \geq c_j c_i x \cdot d_{jk} \text{ if } k \notin \{i, j\}$$

is postulated instead of (C4) (see [2]) in addition to the MGR axioms.

The inclusion $\mathbf{D}_\alpha \subset \mathbf{F}_\alpha \cup \mathbf{NA}_\alpha^+$ is true for these algebras.

Then we define some *polyadic-like* abstract algebras.

We assume the knowledge of the concepts of polyadic equality and quasi-polyadic equality algebras (see [11], p. 266).

The so-called *finitary polyadic equality algebras*, i.e., the class $\mathbf{FPEA}_\alpha$, is term-definitionally equivalent to the quasi-polyadic equality algebras (see [15], Theorem 1). The axiomatization of $\mathbf{FPEA}_\alpha$ and that of the class $\mathbf{TA}_\alpha$ to be introduced are different in only one axiom of $\mathbf{FPEA}_\alpha$, the axiom

$$(F5) \quad s_j^i c_k x = c_k s_j^i x \text{ if } k \notin \{i, j\}.$$

Definition 2.13 (Class $\mathbf{TA}_\alpha$). A *transposition algebra of dimension α* ($\alpha \geq 4$) is an algebra

$$\mathcal{A} = \langle A, +, \cdot, -, 0, 1, c_i, s_j^i, p_{ij}, d_{ij} \rangle_{i,j < \alpha},$$

where c_i, s_j^i, p_{ij} are unary operations, d_{ij} are constants, and the axioms (F0)–(F9) below are valid for every $i, j, k < \alpha$:

- (F0) $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra, $s_i^i = p_{ii} = d_{ii} = \text{Id} \upharpoonright A$ and $p_{ij} = p_{ji}$,
- (F1) $x \leq c_i x$,
- (F2) $c_i(x + y) = c_i x + c_i y$,
- (F3) $s_j^i c_i x = c_i x$,
- (F4) $c_i s_j^i x = s_j^i x$, $i \neq j$,
- (F5)* $s_j^i s_m^k x = s_m^k s_j^i x$ if $i, j \notin \{k, m\}$,
- (F6) s_j^i and p_{ij} are Boolean endomorphisms (i.e. $s_j^i(-x) = -s_j^i x$, etc.),
- (F7) $p_{ij} p_{ij} x = x$,
- (F8) $p_{ij} p_{ik} x = p_{jk} p_{ij} x$, i, j, k are distinct,
- (F9) $p_{ij} s_j^i x = s_i^j x$,
- (F10) $s_j^i d_{ij} = 1$,
- (F11) $x \cdot d_{ij} \leq s_j^i x$.

(F5)* is obviously a weakening of (F5). The operators c_i, s_j^i and p_{ij} ($i, j < \alpha$) are the (abstract) cylindrification, substitution and transposition operators, respectively.

The inclusion $\mathbf{Gwp}_\alpha \subset \mathbf{TA}_\alpha$ is valid (see Lemma 3.2).

We consider as known the concept of the *substitution operator* s_τ defined for any finite transformation τ on α . s_τ is an undefined (basic) concept in quasi-polyadic equality algebras. But it is easy to see that s_τ can be introduced uniquely in $\mathbf{FPEA}_\alpha$ or in $\mathbf{TA}_\alpha$, too. The existence of such an s_τ follows from the proof of Theorem 1 (ii) in [15], namely, it is easy to check that the proof works by supposing (F5)* instead of (F5) (e.g. the inequality $s_j^k p_{ij} x \leq s_j^k p_{ij} s_k^i x$ in (16) on p. 553 follows easily from the $\mathbf{TA}_\alpha$ axioms).

Throughout this paper we assume that the polyadic-like algebras occurring here are equipped with the operator s_τ , where τ is finite. Further, s_τ is assumed to have the following properties for arbitrary finite transformations τ and λ and ordinals $i, j < \alpha$ (by [15], p. 547):

$$(2.7) \quad \begin{aligned} s_{\tau \circ \lambda} &= s_\tau \circ s_\lambda, \\ p_{ij} &= s_{[i, j]}, \\ s_j^i &= s_{[i / j]}, \\ s_\tau d_{ij} &= d_{\tau i \tau j}, \\ c_i s_\tau &\leq s_\tau c_{\tau^{-1} i}, \text{ here } \tau \text{ is finite permutation.} \end{aligned}$$

The following is a survey of the classes of abstract algebras listed above:

<i>cylindric-like abstract algebras</i>	<i>property commutativity</i>
$\mathbf{CA}_\alpha$	$c_i c_j = c_j c_i$, axiom (C4)
$\mathbf{F}_\alpha$	$s_j^i s_m^k x = s_m^k s_j^i x$, axiom $^-(\text{C4})$
$\mathbf{NA}_\alpha^+$	$c_i c_j x \geq c_j c_i x \cdot d_{jk}$, axiom (C4)*

In $\mathbf{F}_\alpha$ and $\mathbf{NA}_\alpha^+$ the MGR properties are assumed, while in $\mathbf{CA}_\alpha$ there is no MGR.

<i>polyadic-like abstract algebras</i>	<i>property commutativity</i>
$\mathbf{FPEA}_\alpha$	$s_j^i c_k x = c_k s_j^i x$, axiom (F5)
$\mathbf{TA}_\alpha$	$s_j^i s_m^k x = s_m^k s_j^i x$, axiom (F5)*

We note that in the sequel only the classes $\mathbf{F}_\alpha$ ($\mathbf{NA}_\alpha^+$) and $\mathbf{TA}_\alpha$ are used from the tables above.

Completing this section we remind the reader of a representation theorem, the Resek–Thompson–Ferenczi theorem:

$$(2.8) \quad \mathcal{A} \in \mathbf{F}_\alpha \text{ if and only if } \mathcal{A} \in \mathbf{IsD}_\alpha,$$

where $\alpha \geq 4$ (see [9], Corollary 3.2 or [8], Theorem 2.6).

So the class of fields of sets containing the diagonals, closed under cylindrifications (hence closed under single substitutions, too) and satisfying (2.2), i.e., the class $\mathbf{D}_\alpha$, forms a variety and its axioms are the $\mathbf{F}_\alpha$ axioms.

This representation theorem is a version of the Resek–Thompson theorem which states that

$$(2.9) \quad \mathcal{A} \in \mathbf{NA}_\alpha^+ \text{ if and only if } \mathcal{A} \in \mathbf{IsD}_\alpha$$

if $\alpha \geq 4$ (see [2], Theorem 1).

Our Main Theorem (Theorem 3.1) is closely connected to the Resek–Thompson–Ferenczi theorem and includes the classes $\mathbf{TA}_\alpha$ and $\mathbf{Gwp}_\alpha$. The following table includes the four important classes of algebras connected with this theorem:

	<i>cylindric-like algebras</i>	<i>polyadic-like algebras</i>
<i>abstract algebras</i>	$\mathbf{F}_\alpha$	$\mathbf{TA}_\alpha$
<i>set algebras</i>	$\mathbf{D}_\alpha$	$\mathbf{Gwp}_\alpha$

As regards the representation theory of cylindric-like or polyadic-like algebras by relativized set algebras, see e.g., the references [2], [3], [4], [8], [9], [10], [13]. The references [1], [5], [6], [7], [12], [14], [15], [16] are related indirectly to the topic or are related to the applications.

3. RESULTS

Theorem 3.1 (Main Theorem). $\mathcal{A} \in \mathbf{TA}_\alpha$ if and only if $\mathcal{A} \in \mathbf{IsGwp}_\alpha$, where $\alpha \geq 4$.

Notice that $\mathbf{Gwp}_\alpha$ is a canonical variety (see [13], 2.69). The theorem is valid for finite α 's, while, in general, the classical representation theorems are not. The following lemma states the easy part of the theorem:

Lemma 3.2. *If $\mathcal{A} \in \mathbf{Gwp}_\alpha$, then $\mathcal{A} \in \mathbf{TA}_\alpha$, where $\alpha \geq 4$.*

Proof. We assume that $\mathcal{A} \in \mathbf{Gwp}_\alpha$, and we have to check the axioms (F1)–(F11). As examples we check the axioms (F4), (F9) and (F10):

Axiom (F4): $c_i s_j^i x = s_j^i x$, $i \neq j$.

$z \in C_i S_j^i X \Leftrightarrow z_u^i \in S_j^i X$ for some $u \Leftrightarrow z_{z_j}^i \in X$.

$z \in S_j^i X \Leftrightarrow z_{z_j}^i \in X$.

Axiom (F9): $p_{ij} s_j^i x = s_i^j x$.

$z \in [i, j] S_j^i X \Leftrightarrow z \circ [i, j] \in S_j^i X \Leftrightarrow z_{z_i}^j \in X$.

$z \in S_i^j X \Leftrightarrow z_{z_i}^j \in X$.

Axiom (F10): $s_j^i d_{ij} = 1$.

We show that $z \in V$ implies $z \in S_j^i D_{ij}$. Namely, if $z \in V$, then $z_{z_j}^i \in V$ by the definition of a $\mathbf{Gwp}$ unit V . But this implies that $z_{z_j}^i \in D_{ij}$, i.e., $z \in S_j^i D_{ij}$. $\square$

In proving the other part of our Main Theorem we apply the so-called step-by-step method (or iteration method; see [12]) to construct the suitable representation. The published proof of the Resek–Thompson theorem is due to Andrka and Thompson (see [2]). We are going to follow Andrka and Thompson's proof, modifying it in accordance with the polyadic type of the algebras, and we assure some additional requirements. We are going to emphasize the differences and the common features of the two proofs. In order to make the checking of the proof easier the critical parts of Andrka and Thompson's proof are quoted in the Appendix.

First, let us consider the *framework* of Andrka and Thompson's proof (see Part 1 in the Appendix). *On the modification of that framework:*

The only necessary change is that a property (vi) is needed which states the preservation of the operation p_{ij} . By (2.7) p_{ij} may be considered as $s_{[i, j]}$. We are going to use $s_{[i, j]}$ rather than p_{ij} . So we have to prove:

(vi) $\text{rep}(s_{[i, j]}a) = [i, j] \text{rep}(a)$.

We are going to prove the following more general property:

$$(3.1) \quad (\text{vi})' \text{ rep}(s_\sigma a) = S_\sigma \text{ rep}(a),$$

where σ is an arbitrary finite permutation on α .

We note that the original representation is *complete* (see (A.1)); this will be transmitted to our construction, too.

The next part of the original proof is the *definition of the 0th step*, i.e., the definition of the function rep_0 (see Part 2 in the Appendix).

We have to essentially change the definition of rep_0 to handle property (vi)'.

First, as a preparation, we introduce two equivalence relations:

1. Let a be an arbitrary *fixed* atom. The definition of the relation $\equiv_a$ ($\equiv$, for short) on α is

$$(3.2) \quad i \equiv j \text{ if and only if } s_{[i, j]}a = a.$$

$\equiv$ is an equivalence relation. For example, if $i \equiv j$ and $j \equiv k$, then $i \equiv k$, because $s_{[i, j]}a = a$ and $s_{[j, k]}a = a$ imply $s_{[i, k]}a = a$. Namely, by (2.7), $[i, k] = [i, j] \circ [j, k] \circ [i, j]$ implies that $s_{[i, k]}a = (s_{[i, j]} \circ s_{[j, k]} \circ s_{[i, j]})a$.

Notice that

$$(3.3) \quad (i, j) \in \text{Ker}(a) \text{ implies that } i \equiv j.$$

Namely, $a \leq d_{ij}$ implies that $a = s_{[i, j]}a$. (C7) is equivalent to

$$(C7)^* : d_{ij} \cdot c_i(d_{ij} \cdot x) = d_{ij} \cdot x.$$

If $x = a$, then $a \leq d_{ij}$ implies that $d_{ij} \cdot c_i a = a$. Applying $s_{[i, j]}$ to this equality we get that $s_{[i, j]}(d_{ij} \cdot c_i a) = s_{[i, j]}a$, i.e., $d_{ji} \cdot s_i^j(c_i a) = s_{[i, j]}a$.

Replacing $c_i a$ for x in $(C7)^*$ and changing i and j , we get that $d_{ji} \cdot c_j(d_{ij} \cdot c_i a) = d_{ji} \cdot c_i a$, i.e., $d_{ji} \cdot s_i^j c_i a = d_{ji} \cdot c_i a$. Comparing this equality with $d_{ij} \cdot c_i a = a$ and with $d_{ji} \cdot s_i^j(c_i a) = s_{[i, j]}a$, we get that $a = s_{[i, j]}a$.

2. Let us consider the following equivalence relation $\sim$ on $\text{At}\mathcal{A}$:

$$(3.4) \quad a \sim b \text{ if and only if } b = s_\tau a \text{ for some finite permutation } \tau,$$

$a, b \in \text{At}\mathcal{A}$.

In fact, the relation $\sim$ is an *equivalence relation*: it is reflexive because $a = s_I a$. It is symmetrical because $b = s_\tau a$ implies $s_{\tau^{-1}} b = a$. It is transitive because $b = s_\tau a$ and $c = s_\sigma b$ imply that $c = s_\sigma(s_\tau a) = s_{\sigma \circ \tau} a$, where $\sigma \circ \tau$ is also a finite permutation.

Let us choose and fix representative points for the equivalence classes concerning $\sim$ and let Rp denote this fixed set of representative points.

We define the function rep_0 :

Definition 3.3. If $c \in Rp$, then let

$$(3.5) \quad \text{rep}_0(c) = \{S_\tau f_c : s_\tau c = c\},$$

where f_c is the sequence defined in the original proof and τ is a finite permutation on α . If $b = s_\sigma c$, then let

$$(3.6) \quad \text{rep}_0(b) = S_\sigma \text{rep}_0(c).$$

Lemma 3.4. *The above definition is unique.*

Proof. We have to prove that if

$$(3.7) \quad s_\tau c = s_\sigma c$$

for some $c \in Rp$ and finite permutations τ and σ , then

$$(3.8) \quad \text{rep}_0(s_\tau c) = \text{rep}_0(s_\sigma c).$$

(3.7) is equivalent to $c = (s_{\tau^{-1}} \circ s_\sigma)c = s_{\tau^{-1} \circ \sigma}c$, and so to

$$(3.9) \quad c = s_\beta c,$$

where $\beta = \tau^{-1} \circ \sigma$. Similarly, using (3.6), (3.8) is equivalent to

$$(3.10) \quad \text{rep}_0(c) = S_\beta \text{rep}_0(c).$$

By (3.5), (3.10) is equivalent to $\{S_{\tau_1} f_c : s_{\tau_1} c = c\} = S_\beta \{S_{\tau_2} f_c : s_{\tau_2} c = c\}$.

But $S_\beta \{S_{\tau_2} f_c : s_{\tau_2} c = c\} = \{(S_\beta S_{\tau_2}) f_c : s_{\tau_2} c = c\}$, so we have to prove that

$$(3.11) \quad \{S_{\tau_1} f_c : s_{\tau_1} c = c\} = \{(S_\beta S_{\tau_2}) f_c : s_{\tau_2} c = c\}.$$

We show that the left-hand side of (3.11) is a subset of the right-hand side, and conversely. Assume that $S_\tau f_c \in \{S_{\tau_1} f_c : s_{\tau_1} c = c\}$ for some fixed $\tau_1 = \tau$. Then let us choose τ_2 on the right-hand side as $\beta^{-1} \circ \tau$. We have to prove that $s_{\beta^{-1} \circ \tau} c = c$. But $s_{\beta^{-1} \circ \tau} c = (s_{\beta^{-1}} s_\tau) c = s_{\beta^{-1}}(s_\tau c)$; $s_\tau c = c$ by condition and $s_{\beta^{-1}} c = c$ by (3.9), so, really $s_{\beta^{-1} \circ \tau} c = c$. The proof of the converse inclusion in (3.11) is completely similar. $\square$

Lemma 3.5. $\text{rep}_0(s_\sigma a) = S_\sigma \text{rep}_0(a)$, where σ is an arbitrary finite permutation on α and a is an arbitrary atom, i.e., the property (vi)' in (3.1) is satisfied.

Proof. We have to prove that (3.6) is true for arbitrary atoms b and a with $b = s_\sigma a$, not only for representative points c ; i.e. we have to prove that

$$(3.12) \quad \text{rep}_0(b) = S_\sigma \text{rep}_0(a).$$

Namely, if the representative point representing a is c and $a = s_\tau c$ for τ , then $\text{rep}_0(b) = \text{rep}_0(s_\sigma a) = \text{rep}_0(s_\sigma s_\tau c) = \text{rep}_0(s_{\sigma \circ \tau} c)$. But $\text{rep}_0(s_{\sigma \circ \tau} c) = S_{\sigma \circ \tau} \text{rep}_0(c)$ by (3.6) and $S_{\sigma \circ \tau} \text{rep}_0(c) = (S_\sigma S_\tau) \text{rep}_0(c) = S_\sigma \text{rep}_0(s_\tau c) = S_\sigma \text{rep}_0(a)$ by (3.6), so, really, $\text{rep}_0(b) = S_\sigma \text{rep}_0(a)$, and we are ready. $\square$

Similar to the original proof, we show that rep_0 satisfies conditions (i), (ii) and (iv) in (A.2). The proof requires a bit more complex consideration than the original proof.

Lemma 3.6. $\text{rep}_0(a) \cap \text{rep}_0(b) = \emptyset$ if $a \neq b$, $a, b \in \text{At } \mathcal{A}$, i.e., property (i) in (A.2) is true.

Proof. If $a \approx b$ and $a = s_\sigma c$, $b = s_\lambda d$ for some $c, d \in Rp$ and finite permutations σ and λ , then condition b) $\text{Rg}(f_c) \cap \text{Rg}(f_d) = \emptyset$ in (A.3) and (3.5) imply that $\text{rep}(a) \cap \text{rep}(b) = \emptyset$.

Assume that $a \sim b$ and $a = s_\sigma c$, $b = s_\eta c$ for some $c \in Rp$ and finite permutations σ and η , $a \neq b$, i.e., $s_\sigma c \neq s_\eta c$. We have to prove that $\text{rep}_0(s_\sigma c) \cap \text{rep}_0(s_\eta c) = \emptyset$.

Indirectly, assume that $\text{rep}_0(s_\sigma c) \cap \text{rep}_0(s_\eta c) \neq \emptyset$. We show that $a = b$, i.e., $s_\sigma c = s_\eta c$, and this contradicts the condition $a \neq b$.

Taking into consideration (3.6) we get that

$$S_\sigma \{S_{\tau_1} f_c : s_{\tau_1} c = c\} \cap S_\eta \{S_{\tau_2} f_c : s_{\tau_2} c = c\} \neq \emptyset,$$

i.e., $S_\sigma S_{\tau_1} f_c = S_\eta S_{\tau_2} f_c$ for some finite permutations τ_1 and τ_2 . This latter equality is equivalent to $f_c = S_{\tau_1^{-1}} S_{\sigma^{-1}} S_\eta S_{\tau_2} f_c$, i.e., to $f_c = S_{\tau_1^{-1} \circ \sigma^{-1} \circ \eta \circ \tau_2} f_c$. Let γ denote the permutation $\tau_1^{-1} \circ \sigma^{-1} \circ \eta \circ \tau_2$; then $f_c = S_\gamma f_c$.

Using the fact that $s_{\tau_1}c = c$ and $s_{\tau_2}c = c$, $a = b$ (i.e. $s_{\sigma}c = s_{\eta}c$) is equivalent to $s_{\sigma}s_{\tau_1}c = s_{\eta}s_{\tau_2}c$. Similar to the equivalences above, this latter equality is equivalent to $c = s_{\tau_1^{-1} \circ \sigma^{-1} \circ \eta \circ \tau_2}c$, i.e., to $c = s_{\gamma}c$.

Generally, we prove that

$$(3.13) \quad f_c = S_{\beta}f_c \text{ implies } c = s_{\beta}c,$$

where β is an arbitrary finite permutation on α .

We remind the reader that every finite permutation β can be formulated as a composition of finitely many cyclic permutations. Further, a cyclic permutation δ of length n can be formulated as a composition $[\delta^{n-1}i, \delta^n i] \circ \dots \circ [\delta i, \delta^2 i] \circ [i, \delta i]$ of transpositions, where $\delta^n i = i$. This obviously implies that every finite permutation β can be formulated as finitely many compositions of transpositions of the form $[j, \beta j]$. To prove (3.13) let us decompose β in this form:

$$(3.14) \quad \beta = [j_m, \beta j_m] \circ \dots \circ [j_2, \beta j_2] \circ [j_1, \beta j_1].$$

$f_c = S_{\beta}f_c$ implies that $(f_c)_j = (S_{\beta}f_c)_j$ for every $j < \alpha$. $(S_{\beta}f_c)_j = (f_c)_{\beta^{-1}j}$, therefore $(f_c)_j = (f_c)_{\beta^{-1}j}$ for every $j < \alpha$. Here β is an arbitrary finite permutation, so we can write that $(f_c)_{\beta i} = (f_c)_i$ for arbitrary $j < \alpha$ and permutation β . This latter is equivalent to $(f_c)_{\beta j} = (f_c)_j$. This means that $(j, \beta j) \in \ker(f)$. The property a) in (A.3), i.e., $\text{Ker}(c) = \ker(f_c)$, implies that $(j, \beta j) \in \text{Ker}(c)$. (3.3) implies that $j \equiv \beta j$, i.e.,

$$(3.15) \quad s_{[j, \beta j]}c = c$$

for every $j < \alpha$.

Applying (3.14) we get that

$$\begin{aligned} s_{\beta}c &= s_{[j_m, \beta j_m]} \circ \dots \circ s_{[j_2, \beta j_2]} \circ s_{[j_1, \beta j_1]}c \\ &= s_{[j_m, \beta j_m]} \dots s_{[j_2, \beta j_2]} s_{[j_1, \beta j_1]}c. \end{aligned}$$

Using (3.15) step by step we get that

$$(3.16) \quad s_{\beta}c = c$$

and (3.13) is proven.

Applying (3.13) to the transformation γ we get a contradiction, and we are ready. $\square$

Lemma 3.7. $\text{rep}_0(a) \subseteq D_{ij}$ if $a \leq d_{ij}$ and $\text{rep}_0(a) \cap D_{ij} = \emptyset$ if $a \cdot d_{ij} = 0$ for every $i, j < \alpha$, i.e., property (ii) in (A.2) is true.

Proof. First we prove that $\text{rep}_0(a) \subseteq D_{ij}$ if $a \leq d_{ij}$.

Assume that $a = s_{\sigma}c$ for some $c \in Rp$. Then the condition $a \leq d_{ij}$ is of the form $s_{\sigma}c \leq d_{ij}$. By (3.6) we have to prove that

$$(3.17) \quad S_{\sigma}\{S_{\tau}f_c : s_{\tau}c = c\} \subseteq D_{ij},$$

i.e.,

$$(3.18) \quad (S_{\sigma}S_{\tau}f_c)_i = (S_{\sigma}S_{\tau}f_c)_j.$$

Let us consider the following equivalences:

$$\begin{aligned} (S_{\sigma}S_{\tau}f_c)_i &= (S_{\sigma}S_{\tau}f_c)_j \Leftrightarrow (S_{\sigma \circ \tau}f_c)_i \\ &= (S_{\sigma \circ \tau}f_c)_j \Leftrightarrow (f_c)_{(\sigma \circ \tau)^{-1}i} = (f_c)_{(\sigma \circ \tau)^{-1}j} \Leftrightarrow (\lambda^{-1}i, \lambda^{-1}j) \in \ker(f_c) = \text{Ker}(c) \end{aligned}$$

by (A.3), where λ denotes the permutation $\sigma \circ \tau$.

$(\lambda^{-1}i, \lambda^{-1}j) \in \text{Ker}(c)$ means that $c \leq d_{\lambda^{-1}i \lambda^{-1}j}$. Applying s_λ to this inequality we get that $s_\lambda c \leq d_{ij}$, so (3.18) is equivalent to

$$(3.19) \quad s_\lambda c \leq d_{ij}.$$

But $s_\lambda c = s_{\sigma \circ \tau} c = s_\sigma s_\tau c = s_\sigma c = a$ using the fact that $s_\tau c = c$ in (3.17) and $a = s_\sigma c$. Therefore the condition $s_\sigma c \leq d_{ij}$ is equivalent to $s_\lambda c \leq d_{ij}$.

Applying the equivalences above we get that $a = s_\sigma c \leq d_{ij}$ really implies (3.17).

The other case: we have to prove that $\text{rep}_0(a) \cap D_{ij} = \emptyset$ if $a \cdot d_{ij} = 0$. Indirectly, assume that $\text{rep}_0(a) \cap D_{ij} \neq \emptyset$ for some $i, j < \alpha$. Similar to the first case, (3.18) is true for some $i, j < \alpha$. $(S_\sigma S_\tau f_c)_i = (S_\sigma S_\tau f_c)_j$ for *some* finite permutation τ and $i, j < \alpha$. By the argument above, $a \leq d_{ij}$ follows, and this contradicts the condition $a \cdot d_{ij} = 0$. $\square$

Lemma 3.8. $\text{rep}_0(a) \cap C_i \text{rep}_0(b) = \emptyset$ if $a \cdot c_i^A b = 0$, i.e., property (iv) in (A.2) is true.

Proof. If $a \approx b$, then (3.6) and (A.3) b) imply that $\text{rep}_0(a) \cap C_i \text{rep}_0(b) = \emptyset$.

Assume that $a \sim b$ ($a \neq b$) and the representative point is c , so $a = s_\sigma c$ and $b = s_\eta c$ for some permutations σ and η .

Indirectly, assume that

$$(3.20) \quad \text{rep}_0(a) \cap C_i \text{rep}_0(b) \neq \emptyset.$$

This means that there exists a $g \in \text{rep}_0(b)$ for which $g_u^i \in \text{rep}_0(a)$ for some u . By (3.6), this means that $g = S_\eta S_{\tau_2} f_c$ for some τ_2 and $g_u^i = S_\sigma S_{\tau_1} f_c$ for some τ_1 , that is, $(S_\eta S_{\tau_2} f_c)_u^i = S_\sigma S_{\tau_1} f_c$. Therefore

$$(3.21) \quad f_c = S_{\tau_1^{-1}} S_{\sigma^{-1}} (S_\eta S_{\tau_2} f_c)_u^i = (S_{\tau_1^{-1}} S_{\sigma^{-1}} S_\eta S_{\tau_2} f_c)_u^{\delta i},$$

where $\delta = \tau_1^{-1} \circ \sigma^{-1}$.

Let us denote the permutation $\tau_1^{-1} \circ \sigma^{-1} \circ \eta \circ \tau_2$ by β . Then $f_c = (S_\beta f_c)_u^{\delta i}$ implies that $(f_c)_j = (S_\beta f_c)_j$ for every $j \neq \delta i$, i.e.,

$$(3.22) \quad (f_c)_j = (f_c)_{\beta^{-1}j} \text{ if } j \neq \delta i.$$

Let us consider the decomposition of β^{-1} being analogous with (3.14): $\beta^{-1} = [k_m, \beta^{-1}k_m] \circ \dots \circ [k_2, \beta^{-1}k_2] \circ [k_1, \beta^{-1}k_1]$, where $k_m = \delta i$ can be assumed without loss of generality. Similar to (3.16) we get that

$$(3.23) \quad s_{\beta^{-1}} c = s_{[\delta i, \beta^{-1}(\delta i)]} c.$$

By definition of β , $\beta^{-1} = \tau_2^{-1} \circ \eta^{-1} \circ \sigma \circ \tau_1$, therefore $s_{\beta^{-1}} c = s_{\tau_2^{-1} \circ \eta^{-1}} (s_{\sigma \circ \tau_1} c) = s_{\tau_2^{-1} \circ \eta^{-1}} a$ because $s_{\tau_1} c = c$ and $s_\sigma = a$.

(3.23) implies that $s_{\tau_2^{-1} \circ \eta^{-1}} a = s_{[\delta i, \beta^{-1}(\delta i)]} c$. Applying $s_{\eta \circ \tau_2}$ to this equality we get that

$$(3.24) \quad a = s_{\eta \circ \tau_2} s_{[\delta i, \beta^{-1}(\delta i)]} c = s_{[(\eta \circ \tau_2 \circ \delta) i, (\eta \circ \tau_2 \circ \beta^{-1} \circ \delta) i]} s_{\eta \circ \tau_2} c.$$

The second equality follows from

$$(\eta \circ \tau_2) \circ [\delta, \beta^{-1} \circ \delta] = [\eta \circ \tau_2 \circ \delta, \eta \circ \tau_2 \circ \beta^{-1} \circ \delta] \circ (\eta \circ \tau_2)$$

on α and from the property $s_{\tau \circ \lambda} = s_\tau \circ s_\lambda$ in (2.7), applying it to both sides.

The transformation $\eta \circ \tau_2 \circ \beta^{-1} \circ \delta$ is the identity, namely $\eta \circ \tau_2 \circ \beta^{-1} \circ \delta = \eta \circ \tau_2 \circ \tau_2^{-1} \circ \eta^{-1} \circ \sigma \circ \tau_1 \circ \tau_1^{-1} \circ \sigma^{-1} = I$. So (3.24), $c = s_{\tau_2}$ and $b = s_{\eta}c$ imply that

$$(3.25) \quad a = s_{[m, i]}b,$$

where m denotes $(\eta \circ \tau_2 \circ \delta)i$.

(3.12) and (3.25) imply that $\text{rep}_0(a) = [m, i] \text{rep}_0(b)$.

Notice that $a = s_{[m, i]}b$, $a \neq b$, and Lemma 3.6 imply that

$$(3.26) \quad [m, i] \text{rep}_0(b) \cap \text{rep}_0(b) = \emptyset.$$

Further, the (indirect) condition in (3.20) is of the form

$$[m, i] \text{rep}_0(b) \cap C_i \text{rep}_0(b) \neq \emptyset.$$

Then on the one hand, there exists a $g \in \text{rep}_0(b)$ such that $g \circ [m, i] \in C_i \text{rep}_0(b)$, i.e., $(g \circ [m, i])_w^i \in \text{rep}_0(b)$, for some w . Let h denote the sequence $(g \circ [m, i])_w^i$.

On the other hand, both g and h are elements of $\text{rep}(b)$, so (3.5) implies that both of them are finite permutations of the representative sequence f_c . Therefore they are finite permutations of each other, too. For example, let $h = S_{\tau}g$ be for some finite permutation τ . Hence if $\tau k = i$ for some $k < \alpha$, then $g_k = w$. For the sake of simplicity let us consider here the finite permutation τ to be defined on some finite subset of α .

Let us denote g_i and g_m by u and v , so $g_i = u$, $g_m = v$ and $g_k = w$. Then $h = (g \circ [m, i])_w^i$, $h_i = w$, $h_m = u$ and

$$(3.27) \quad g_j = h_j$$

for every $j \notin \{i, m\}$. We state that the expected finite permutation τ between the sequences g and h , having the above properties, cannot exist.

We will now discuss the problem. First notice that $u \neq v$. Namely $u = v$ implies $g = g \circ [m, i]$, so $g \circ [m, i] \in \text{rep}_0(b)$, and this contradicts (3.26). For a similar reason, $u \neq w$.

First, let us consider the case $v \neq w$. We show that this case is impossible. Assume that $\tau m = t$ for some $t < \alpha$. Then $h_t = v$, $v \notin \{u, w\}$, imply that $t \notin \{i, m\}$, and $h_t = v$ and $g_t = h_t$ imply that $g_t = v$ by (3.27). Assume that $\tau t = p$ for some $p < \alpha$. $u \neq v$ and the τ is finite permutation, therefore $p \notin \{i, m, t\}$. Similar to the previous step we get that $g_p = h_p = v$. τ is a *finite* permutation, so there are only finitely many n_j and q_j for which $\tau n_j = q_j$ and $g_{n_j} = h_{q_j} = g_{q_j} = v$. Let q_n be the last q_j with this property in this sequence. τ is a permutation, so $\tau q_n = i$ or $\tau q_n = m$. Therefore $g_{q_n} = v$ implies that $h_i = v$ or $h_m = v$, which contradicts the conditions $h_i = w$, $h_m = u$ and $v \notin \{u, w\}$.

Then assume that $v = w$. In this case $h \circ [m, i] = h$, i.e., $g \circ [m, i] = g$. This contradicts (3.25).

We get a contradiction, so the original proposition is true. $\square$

Notice that in the 0th step, similar to the original proof, the condition $a \cdot c_i^A b = 0$ is not used.

In the original construction, induction and the step-by-step technique are applied. Notice that the foregoing representation class is elementary; therefore we would not lose any generality if the algebra is assumed to be countable. But this does not imply an essential simplification of the proof; we consider the original proof as the standard one.

As regards the $(n + 1)$ th step of the proof, i.e., the definition of the function rep_{n+1} , let us consider Andreka and Thompson's proof again (see Part 3 in the Appendix). *The modified construction is:*

In order to assure the validity of property (vi)' in (3.1), we modify the original construction. Here we consider equivalence classes of triples instead of single triples. From the point of view of the original proof this means that we classify the single triples according to an equivalence relation to be introduced.

The original construction uses an *arbitrary fixed* free transfinite enumeration of the (a, b, i) triples, where $a, b \in \text{At}\mathcal{A}$, $i < \alpha$. In contrast with this we are going to assume certain restrictions for this enumeration, the triples will be classified in a sense. The function rep_n will be defined in accordance with this classification. Beyond this little change we *do not change* the original procedure, so the original proof works. We shall prove that property (vi)' in (3.1) is *preserved* in every step.

Let us consider the following relation $\approx$ on R :

$$(3.28) \quad (a_1, b_1, i_1) \approx (a_2, b_2, i_2) \text{ if and only if } a_2 = s_\sigma a_1, b_2 = s_\sigma b_1, i_2 = \sigma i_1$$

for some permutation σ . $\approx$ is obviously an equivalence relation. Let us fix representative points in the equivalence classes and denote R' the class of the representative points.

We note that the relation $\approx$ preserves the inequalities $a \leq c_i b$ and $b \leq d_{ij}$ in the following sense: if $(a_1, b_1, i_1) \approx (a_2, b_2, i_2)$, then

$$(3.29) \quad a_1 \leq c_{i_1} b_1 \text{ if and only if } a_2 \leq c_{i_2} b_2$$

and

$$(3.30) \quad b \leq d_{ij} \text{ if and only if } s_\sigma b \leq d_{\sigma i \sigma j}.$$

Namely, if $a_1 \leq c_{i_1} b_1$, i.e., $s_{\sigma^{-1}} s_\sigma a_1 \leq c_{i_1} s_{\sigma^{-1}} s_\sigma b_1$, then by the last property in (2.7) $c_{i_1} s_{\sigma^{-1}} s_\sigma b_1 \leq s_{\sigma^{-1}} c_{\sigma i_1} s_\sigma b_1$, therefore, applying s_σ , $s_\sigma a_1 \leq c_{\sigma i_1} s_\sigma b_1$ so $a_2 \leq c_{i_2} b_2$. This argument is symmetrical. The second property is trivial.

Now we are ready to define a special enumeration of R . If $p \in R'$, then let R_p be the members of the $\approx$ -equivalence class with representative point p . So R obviously equals the union of the sets R_p ($p \in R'$).

Let us fix an ordering $\leq^*$ of R' and fix the following lexicographic extension of $\leq^*$ to R :

$$(3.31) \quad \text{if } q \in R_p, \text{ then let } p \leq^* q, \\ \text{if } p_1 \leq^* p_2 \text{ } (p_1, p_2 \in R') \text{ and } p_1 \approx \gamma, p_2 \approx \lambda (\gamma, \lambda \in R), \text{ then let } \gamma \leq^* \lambda.$$

Let ρ be an ordinal and let $r : \rho \rightarrow R$ be an enumeration of R such that r preserves the lexicographic ordering $\leq^*$ and for all $n \in \rho$ and $(a, b, i) \in R$ there is an $m \in \rho$, $m > n$, such that $r(m) = (a, b, i)$. Such ρ and r clearly exist.

Now the *definition of the function* rep_{n+1} is:

Assume that n is a successor ordinal. We are going to define rep_{n+1} for this case. Let the case of the limit ordinal and the general definition of the function rep be the same as the original one in (A.7) and (A.8).

For the *representative point* $p = (a, b, i)$ ($p \in R'$) let the definition of rep_{n+1} be the same as the original one, and so be the same as the one included in (A.4), (A.5) or (A.6), depending on the cases discussed there.

Then we extend the definition of rep_{n+1} for the members of the equivalence class including the respective representative points depending on the cases included in

the original definition. The motivation of these definitions is that $\approx$ preserves the respective inequalities (see (3.29) and (3.30)).

If $a \not\leq c_i b$ let

$$(3.32) \quad \text{rep}_{n+1} = \text{rep}_n$$

for all the members of the equivalence class containing (a, b, i) .

If $a \leq c_i b$, we define the function rep_{n+1} simultaneously for all the triples (a_1, b_1, i_1) for which $(a_1, b_1, i_1) \approx (a, b, i)$.

Assume that $a_1 = s_\tau a$, $b_1 = s_\tau b$, $i_1 = \tau i$ for some permutation τ .

If $b \leq d_{ij}$ for some $j < \alpha$, $j \neq i$, then let

$$(3.33) \quad \text{rep}_{n+1}(s_\tau b) = \text{rep}_n(s_\tau b) \cup \{g(\tau i / g_{\tau j}) : g \in \text{rep}_n(s_\tau a)\}.$$

If $b \not\leq d_{ij}$ for all $j < \alpha$, $j \neq i$, then let

$$(3.34) \quad \text{rep}_{n+1}(s_\tau b) = \text{rep}_n(s_\tau b) \cup \{g(\tau i / u_h) : g \in \text{rep}_n(s_\tau a)\},$$

where u_h is the constant in (A.6) and h denotes the sequence $S_{\tau^{-1}}g$.

That is, $\text{rep}_{n+1}(s_\tau b) = S_\tau \text{rep}_{n+1}(b)$ in (3.33) and (3.34) by definition. In both cases infinitely many steps of the original proof can be reduced into one scheme. The definition of rep_{n+1} for the representative triple (a, b, i) assures the desired properties of rep_{n+1} for the members of the equivalence class determined by (a, b, i) . For example, such a property is the one denoted by (iv)' in the original proof.

The definition of the function rep let be the original (A.8).

We check property (vi)' in (3.1) for the function rep :

Lemma 3.9. $\text{rep}(s_\sigma e) = S_\sigma \text{rep}(e)$ for every $e \in \text{At}\mathcal{A}$, where S_σ is the substitution on the unit V and σ is a permutation on α ; i.e. the property (vi)' in (3.1) is true for the function rep .

Proof. We prove that if $\text{rep}_n(s_\sigma e) = S_\sigma \text{rep}_n(e)$ for every $e \in \text{At}\mathcal{A}$, then

$$(3.35) \quad \text{rep}_{n+1}(s_\sigma e) = S_\sigma \text{rep}_{n+1}(e)$$

for every $e \in \text{At}\mathcal{A}$ and successor ordinal n .

If this implication is proven, then by the definition in (A.7) and the induction condition, (3.35) is true for every ordinal n . From this and from (3.6) we get that $\text{rep}(s_\sigma e) = S_\sigma \text{rep}(e)$ for every $e \in \text{At}\mathcal{A}$, i.e., the proposition of the lemma is true.

In proving (3.35), we shall use the definition of rep_{n+1} . Let us consider the representative point (a, b, i) for the equivalence relation $\approx$ and consider an arbitrary point $(s_\tau a, s_\tau b, \tau i)$ being $\approx$ -equivalent to (a, b, i) . Let us consider the cases listed in the definition of rep_{n+1} :

Case 1. If $a \not\leq c_i b$ (i.e. $s_\tau a \not\leq c_{\tau i} s_\tau b$), then by the definition in (3.32), $\text{rep}_{n+1} = \text{rep}_n$ for all the members of the class containing (a, b, i) . Therefore the property (3.35) is transmitted from n to $n+1$.

Case 2. $a \leq c_i b$ (i.e. $s_\tau a \leq c_{\tau i} s_\tau b$) and $b \leq d_{ij}$ for some j and for every $i \neq j$. We have to prove (3.35) for $e = s_\tau b$, i.e., that

$$(3.36) \quad \text{rep}_{n+1}(s_\sigma(s_\tau b)) = S_\sigma \text{rep}_{n+1}(s_\tau b)$$

for any permutation σ .

Let us consider the left-hand side of (3.36):

$$\text{rep}_{n+1}(s_\sigma(s_\tau b)) = \text{rep}_{n+1}(s_{\sigma \circ \tau} b) = \text{rep}_n(s_\alpha b) \cup \{g(\alpha i / g_{\alpha j}) : g \in \text{rep}_n(s_\alpha a)\}$$

by (3.33), where $\alpha = \sigma \circ \tau$. Here $\text{rep}_n(s_\alpha b) = \text{rep}_n(s_\sigma(s_\tau b)) = S_\sigma \text{rep}_n(s_\tau b)$ by induction.

For the right-hand side of (3.36):

$$\begin{aligned} S_\sigma \text{rep}_{n+1}(s_\tau b) &= S_\sigma(\text{rep}_n(s_\tau b) \cup \{g(\tau i / g_{\tau j}) : g \in \text{rep}_n(s_\tau a)\}) \\ &= S_\sigma \text{rep}_n(s_\tau b) \cup S_\sigma \{g(\tau i / g_{\tau j}) : g \in \text{rep}_n(s_\tau a)\}. \end{aligned}$$

Comparing the above reformulations of the left- and right-hand sides it is sufficient to prove that

$$(3.37) \quad \{g(\alpha i / g_{\alpha j}) : g \in \text{rep}_n(s_\alpha a)\} = S_\sigma \{g(\tau i / g_{\tau j}) : g \in \text{rep}_n(s_\tau a)\}.$$

To prove (3.37), first let us consider the left-hand side. We show that for any finite permutation β the following is true:

$$(3.38) \quad \{g(\beta i / g_{\beta j}) : g \in \text{rep}_n(s_\beta a)\} = S_\beta \{f(i / f_j) : f \in \text{rep}_n(a)\}.$$

But

$$S_\beta \{f(i / f_j) : f \in \text{rep}_n(a)\} = \{(S_\beta f)(\beta i / f_j) : f \in \text{rep}_n(a)\}.$$

Denoting $S_\beta f$ by g we get that $f = S_{\beta^{-1}}g$. Further,

$$(3.39) \quad (S_\beta f)(\beta i / f_j) = g(\beta i / g_{\beta j}).$$

So

$$(3.40) \quad S_\beta f(i / f_j) = g(\beta i / g_{\beta j}).$$

Considering (3.38), if $f = S_{\beta^{-1}}g$, then $f \in \text{rep}_n(a)$ is equivalent to

$$g \in S_\beta \text{rep}_n(a) = \text{rep}_n(s_\beta a).$$

So (3.38) is true.

Now let us consider (3.37). On the one hand,

$$\begin{aligned} \{g(\alpha i / g_{\alpha j}) : g \in \text{rep}_n(s_\alpha a)\} \\ = S_\alpha \{f(i / f_j) : f \in \text{rep}_n(a)\} = S_\sigma S_\tau \{f(i / f_j) : f \in \text{rep}_n(a)\}, \end{aligned}$$

applying (3.38) for $\beta = \alpha$. On the other hand,

$$S_\sigma \{g(\tau i / g_{\tau j}) : g \in \text{rep}_n(s_\tau a)\} = S_\sigma(S_\tau(\{f(i / f_j) : f \in \text{rep}_n(a)\})),$$

applying (3.38) for $\beta = \tau$. Therefore (3.37), so (3.36) is proven.

Case 3.

$a \leq c_i b$ (i.e. $s_\tau a \leq c_{\tau i} s_\tau b$) and $b \not\leq d_{ij}$ (i.e. $s_\tau b \not\leq d_{\tau i} s_\tau j$) for all $j < \alpha$, $j \neq i$.

Similar to the above arguments, considering the definition in (3.34) instead of (3.33), we have to prove the following equality rather than (3.37):

$$(3.41) \quad \{g(\alpha i / u_{h_1}) : g \in \text{rep}_n(s_\alpha a)\} = S_\sigma \{g(\tau i / u_{h_2}) : g \in \text{rep}_n(s_\tau a)\},$$

where $\alpha = \tau \circ \tau_1$, $h_1 = S_{\alpha^{-1}}g$ and $h_2 = S_{\tau^{-1}}g$.

We can prove the following equality (which is analogous to (3.38)) for an arbitrary finite permutation β :

$$(3.42) \quad \{g(\beta i / u_h) : g \in \text{rep}_n(s_\beta a)\} = S_\beta \{f(i / u_f) : f \in \text{rep}_n(a)\},$$

where u_h is the constant in (A.6) and h denotes $S_{\beta^{-1}}g$.

Namely, let us apply the same argument as in the proof of (3.38), but in (3.40) let us use $S_\beta f(i / u_f) = g(\beta i / u_h)$ instead of $S_\beta f(i / f_j) = g(\beta i / g_{\beta j})$, where $h = S_{\beta^{-1}}g$.

The proof of (3.41):

$$\begin{aligned} \{g(\alpha i / u_{h_1}) : g \in \text{rep}_n(s_\alpha a)\} &= S_\alpha \{f(i / u_f) : f \in \text{rep}_n(a)\} \\ &= S_\sigma S_\tau \{f(i / u_f) : f \in \text{rep}_n(a)\}, \end{aligned}$$

applying (3.42) for β and $\alpha = \sigma \circ \tau$. Further,

$$S_\sigma \{g(\tau i / u_{h_2}) : g \in \text{rep}_n(s_\tau a)\} = S_\sigma(S_\tau(\{f(i / u_f) : f \in \text{rep}_n(a)\})),$$

applying (3.42) for $\beta = \tau$. Therefore (3.41), so (3.36) is proven. $\square$

Lemma 3.10. $\mathbf{Dp}_\alpha = \mathbf{Gwp}_\alpha$.

Proof. We use the notation introduced in Section 2. $\mathbf{Gwp}_\alpha \subseteq \mathbf{Dp}_\alpha$ is trivial. To prove the converse inclusion we use the characterization of $\mathbf{Gwp}_\alpha$: $y \in V$ implies $y \circ \tau \in V$ for every finite transformation τ . But τ can be composed in terms of finitely many elementary transformation substitutions $[i / j]$ and transposition $[i, j]$. It is sufficient to prove that V is closed under these transformations. But V is closed under $[i / j]$ because V is a $\mathbf{Dp}_\alpha$ unit; further, V is a $\mathbf{Prs}_\alpha$ unit and therefore it is closed under $[i, j]$, too. $\square$

Then the proof of Theorem 3.1 is:

Proof. In [9] it is proven that $(F5)^*$ and $(C4)^*$ are equivalent under the other $\mathbf{F}_\alpha$ axioms. Andrka and Thompson proved that there is an isomorphism, denoted by rep' , between the algebra $\text{Rd}_{ca}\mathcal{A}$ and some algebra $\mathcal{B}' \in \mathbf{D}_\alpha$. We proved in Lemma 3.9 that this isomorphism preserves the operations s_σ for any finite permutations σ on α . Therefore $\mathcal{B}'$ can be considered as an algebra $\mathcal{B}$ in $\mathbf{Dp}_\alpha$. Lemma 3.10 implies that $\mathcal{B} \in \mathbf{Gwp}_\alpha$. So rep' is an isomorphism between $\mathcal{A}$ and a $\mathcal{B} \in \mathbf{Gwp}_\alpha$. $\square$

Theorem 3.1 and the definition of $\mathbf{FPEA}_\alpha$ imply that $\mathcal{A} \in \mathbf{FPEA}_\alpha$ if and only if $\mathcal{A} \in \text{Is}(\mathbf{Gwp}_\alpha \cap \text{Mod}(F5))$, $\alpha \geq 4$.

Recall that algebras in $\mathbf{FPEA}_\alpha$ are definitionally equivalent to quasi-polyadic equality algebras (see [15]).

$\mathbf{Gw}_\alpha$ is the class $\{\text{Rd}_{ca}\mathcal{B} : \mathcal{B} \in \mathbf{Gwp}_\alpha\}$, by definition, where $\text{Rd}_{ca}\mathcal{A}$ denotes the reduct of $\mathcal{A}$ of cylindric type (see [11], p. 226). The following claim obviously follows from Theorem 3.1:

Corollary. *If $\mathcal{A} \in \mathbf{TA}_\alpha$, then $\text{Rd}_{ca}\mathcal{A} \in \text{Is } \mathbf{Gw}_\alpha$, $\alpha \geq 4$.*

To formulate the following theorem we need a definition:

Definition 3.11 (Class $\mathbf{G}_\alpha$). A set algebra $\mathcal{A}$ in $\mathbf{Cr}_\alpha$ is in $\mathbf{G}_\alpha$ if there are sets U_k , $k \in K$, such that $V = \bigcup_{k \in K} {}^\alpha U_k$, where V is the unit (see [3]).

This class is closely related to the important cylindric algebra class, *generalized cylindric set algebras* $\mathbf{Gs}_\alpha$, where the sets U_k 's are required to be disjoint.

The following theorem is a consequence of the Corollary, making use of the equality $\mathbf{Gw}_\alpha = \mathbf{G}_\alpha$ if α is finite. But below we give a proof which is *independent* of the proof of Theorem 3.1.

Theorem 3.12. *If $\mathcal{A} \in \mathbf{TA}_n$, then $\text{Rd}_{ca}\mathcal{A} \in \text{Is } \mathbf{G}_n$, where $n \geq 4$ and n is finite.*

Proof. We use the main result of [3], i.e., Theorem 3.1 (i) there:

If n is finite and an algebra $\mathcal{A} \in \text{Is}\mathbf{D}_n$ satisfies the following axiom denoted by Ax

$$(3.43) \quad x \leq c_i c_j \left(s_j^i c_j x \cdot s_i^j c_i x \cdot \prod_{k < n, k \neq i, j} s_i^k s_j^i s_k^j c_k x \right),$$

then $\mathcal{A} \in \text{Is}\mathbf{G}_n$.

If $\mathcal{A} \in \mathbf{TA}_n$, then $\text{Rd}_{ca}\mathcal{A} \in \text{Is}\mathbf{D}_n$ by the Resek–Thompson theorem. By the above theorem it is sufficient to show that $\text{Rd}_{ca}\mathcal{A}$ satisfies the axiom Ax . $p_{ij}y \leq c_i c_j y$ because $p_{ij}y \leq p_{ij} c_j y \stackrel{(F3)}{=} p_{ij} s_i^j c_j y \stackrel{(F9)}{=} s_j^i c_j y \leq c_i c_j y$, so it is sufficient to check (3.43) by replacing $c_i c_j$ by p_{ij} . By axiom (F6), p_{ij} is an endomorphism, so

$$\begin{aligned} & p_{ij} \left(s_j^i c_j x \cdot s_i^j c_i x \cdot \prod_{k < n, k \neq i, j} s_i^k s_j^i s_k^j c_k x \right) \\ &= p_{ij} s_j^i c_j x \cdot p_{ij} s_i^j c_i x \cdot p_{ij} \left(\prod_{k < n, k \neq i, j} s_i^k s_j^i s_k^j c_k x \right). \end{aligned}$$

To prove (3.43) it is sufficient to check that x is less than or equal to all the factors of the latter product.

$x \leq p_{ij} s_j^i c_j x$, namely $p_{ij} s_j^i c_j x = s_i^j c_j x = c_j x$ by axiom (F9). Similarly, $x \leq p_{ij} s_i^j c_i x$. $x \leq p_{ij} s_i^k s_j^i s_k^j c_k x$, namely

$$\begin{aligned} & p_{ij} c_k (d_{ki} \cdot s_j^i s_k^j c_k x) \stackrel{[15], \text{Lemma 1.6}}{=} c_k (p_{ij} (d_{ki} \cdot s_j^i s_k^j c_k x)) \\ &= c_k (p_{ij} d_{ki} \cdot p_{ij} s_j^i s_k^j c_k x) = c_k (d_{kj} \cdot p_{ij} s_j^i s_k^j c_k x) \\ &= s_j^k p_{ij} s_j^i s_k^j c_k x \stackrel{[15], (F9)}{=} s_j^k s_i^j s_k^j c_k x = s_j^k s_k^j c_k x \stackrel{[11], \text{Thm. 1.5.10}}{=} s_j^k c_k x = c_k x. \end{aligned}$$

□

APPENDIX

This Appendix includes citations from Andreka and Thompson’s paper ([2]), from the proof of the Resek–Thompson theorem:

Part 1. Citation concerning the framework of the proof.

From now on we assume that $\mathcal{A}$ is atomic, satisfying the axioms.

Let $\text{At}\mathcal{A}$ denote the set of all atoms of $\mathcal{A}$. We want to “build” an isomorphism $\text{rep}: \mathcal{A} \rightarrow \mathcal{B}$, for some $\mathcal{B} \in \mathbf{Crs}_\alpha$, for which the equality below holds:

$$(A.1) \quad \text{rep}(x) = \bigcup \{ \text{rep}(a) : a \in \text{At}\mathcal{A}, a \leq x \} \text{ for every } x \in \mathcal{A}.$$

Let V be a set of α -sequences, and for every $X \subseteq V$ and $i, j < \alpha$ let $C_i X \stackrel{d}{=} \{ f \in V : (\exists u) f(i / u) \in X \}$, $D_{ij} \stackrel{d}{=} \{ f \in V : f_i = f_j \}$. Assume that $\text{rep} : \mathcal{A} \rightarrow \{ X : X \subseteq V \}$ is a function for which (A.1) holds. Then it is easy to check that rep is an isomorphism onto a $\mathcal{B} \in \mathbf{Crs}_\alpha$ with $1^\mathcal{B} \subseteq V$ if and only if conditions (i)–(v)

below hold for every $a, b \in \text{At}\mathcal{A}$ and $i, j < \alpha$:

(A.2)

- (i) $\text{rep}(a) \cap \text{rep}(b) = 0$ if $a \neq b$,
- (ii) $\text{rep}(a) \subseteq D_{ij}$ if $a \leq d_{ij}^{\mathcal{A}}$ and $\text{rep}(a) \cap D_{ij} = \emptyset$ if $a \cdot d_{ij}^{\mathcal{A}} = 0$,
- (iii) $\text{rep}(a) \subseteq C_i \text{rep}(b)$ if $a \leq c_i^{\mathcal{A}} b$,
- (iv) $\text{rep}(a) \cap C_i \text{rep}(b) = \emptyset$ if $a \cdot c_i^{\mathcal{A}} b = 0$,
- (v) $\text{rep}(a) \neq \emptyset$.

We shall construct (a set V of α -sequences and) a function rep with the above properties, step by step.

Part 2. Citation concerning the 0th step.

For every α -sequence f let $\ker(f) \stackrel{d}{=} \{(i, j) \in {}^2\alpha : f_i = f_j\}$.

For every $a \in \text{At}\mathcal{A}$ let $\text{Ker}(a) \stackrel{d}{=} \{(i, j) \in {}^2\alpha : a \leq d_{ij}^{\mathcal{A}}\}$.

Then $\text{Ker}(a)$ is an equivalence relation on α by our axioms (C5)–(C7). For every $a \in \text{At}\mathcal{A}$ let f_a be an α -sequence such that for every $a, b \in \text{At}\mathcal{A}$ we have

$$(A.3) \quad \text{a) } \ker(f_a) = \text{Ker}(a), \quad \text{b) } \text{Rg}(f_a) \cap \text{Rg}(f_b) = \emptyset \text{ if } a \neq b.$$

Such a system $\{f_a : a \in \text{At}\mathcal{A}\}$ of α -sequences does exist. Define

$$\text{rep}_0(a) \stackrel{d}{=} \{f_a\}, \text{ for every } a \in \text{At}\mathcal{A}.$$

Then the function rep_0 satisfies conditions (i), (ii), (iv) and (v), but it does not satisfy condition (iii). Below, we shall make condition (iii) become true step by step, and later we shall check that conditions (i), (ii), (iv), (v) remain true in each step.

Part 3. Citation concerning the $(n+1)$ th step, i.e., about the definition of the function rep_{n+1} .

Let $R = \text{At}\mathcal{A} \times \text{At}\mathcal{A} \times \alpha$, ρ be an ordinal and let $r : \rho \rightarrow R$ be an enumeration of R such that for all $n \in \rho$ and $(a, b, i) \in R$ there is $m \in \rho$, $m > n$ such that $r(m) = (a, b, i)$. Such ρ and r clearly exist.

Assume that $n \in \rho$ and $\text{rep}_n : \text{At}\mathcal{A} \rightarrow \{X : X \subseteq V'\}$ is already defined, where V' is a set of α -sequences. We define $\text{rep}_{n+1} : \text{At}\mathcal{A} \rightarrow \{X : X \subseteq V''\}$, where V'' is a set of α -sequences. Let $r(n) = (a, b, i)$. If $a \not\leq c_i b$, then

$$(A.4) \quad \text{rep}_{n+1} \stackrel{d}{=} \text{rep}_n.$$

Assume $a \leq c_i b$. Then $\text{rep}_{n+1}(e) \stackrel{d}{=} \text{rep}_n(e)$ for all $e \in \text{At}\mathcal{A}$, $e \neq b$.

Further,

Case 1. $b \leq d_{ij}$ for some $j < \alpha$, $j \neq i$. Then

$$(A.5) \quad \text{rep}_{n+1}(b) = \text{rep}_n(b) \cup \{f \mid (i / f_j) : f \in \text{rep}_n(a)\}.$$

Case 2. $b \not\leq d_{ij}$ for all $j < \alpha$, $j \neq i$. For every $f \in \text{rep}_n(a)$ let u_f be such that

- (i) $u_f \notin \bigcup \{\text{Rg}(h) : h \in \bigcup \{\text{rep}_n(e) : e \in \text{At}\mathcal{A}\}\}$,
- (ii) $u_f \neq u_h$ if $f \neq h$, $f, h \in \text{rep}_n(a)$.

Now

$$(A.6) \quad \text{rep}_{n+1}(b) = \text{rep}_n(b) \cup \{f \mid (i / u_f) : f \in \text{rep}_n(a)\}.$$

Let $n \in \rho$ be a limit ordinal and assume that rep_m is defined for all $m < n$. Then

$$(A.7) \quad \text{rep}_n(e) \stackrel{d}{=} \bigcup \{\text{rep}_m(e) : m < n\}$$

for all $e \in \text{At}\mathcal{A}$.

By this, $\langle \text{rep}_n : n \in \rho \rangle$ is defined. Now we define

$$(A.8) \quad \text{rep}(a) \stackrel{d}{=} \bigcup \{\text{rep}_n(a) : n \in \rho\}$$

for all $a \in \text{At}\mathcal{A}$. Let

$$(A.9) \quad V \stackrel{d}{=} \bigcup \{\text{rep}(a) : a \in \text{At}\mathcal{A}\}.$$

REFERENCES

1. H. Andrka, S. D. Comer, J. X. Madarsz, I. Nmeti and T. Sayed Ahmed, *Epimorphisms in cylindric algebras and definability in finite variable logic*, Algebra Universalis, **61** (3–4) (2009), 261–282. MR2565854 (2011a:03070)
2. H. Andrka and R. J. Thompson, *A Stone type representation theorem for algebras of relations of higher rank*, Transaction of Amer. Math. Soc., **309** (2) (1988), 671–682. MR961607 (90d:03135)
3. H. Andrka, *A finite axiomatization of locally square cylindric-relativized set algebras*, Studia Sci. Math. Hung., **38** (1–4) (2001), 1–11. MR1877766 (2002m:03097)
4. H. Andrka, M. Ferenczi and I. Nmeti, *Cylindric-like Algebras and Algebraic Logic*, Bolyai Society Mathematical Studies, Springer, to appear.
5. S. D. Comer, *Galois theory of cylindric algebras and its applications*, Transaction of Amer. Math. Soc., **286** (1984), 771–785. MR760986 (85m:03044)
6. A. Daigneault and J. D. Monk, *Representation theory for polyadic algebras*, Fund. Math., **52** (1963), 151–176. MR0151373 (27:1358)
7. M. Ferenczi, *On the representability of neatly embeddable CA’s by cylindric relativised algebras*, Algebra Universalis, **63** (4) (2010), 331–350. MR2734301
8. M. Ferenczi and G. Sgi, *On some developments in the representation theory of cylindric-like algebras*, Algebra Universalis, **55** (2–3) (2006), 345–353. MR2280236 (2007j:03095)
9. M. Ferenczi, *On cylindric algebras satisfying merry-go-round properties*, Logic Journal of IGPL, **15** (2) (2007), 183–197. MR2333809 (2008i:03073)
10. M. Ferenczi, *Partial transposition implies representability in cylindric algebras*, Mathematical Logic Quarterly, **57** (1) (2011), 87–94.
11. L. Henkin, J. D. Monk and A. Tarski, *Cylindric Algebras I-II*, North Holland, 1985. MR0781929 (86m:03095a); MR0781930 (86m:03095b)
12. R. Hirsch and I. Hodkinson, *Step by step building representations in algebraic logic*, J. Symbolic Logic, **62** (1) (1997), 225–279. MR1450522 (98g:03145)
13. R. Hirsch and I. Hodkinson, *Relation Algebras by Games*, North Holland, 2002. MR1935083 (2003m:03001)
14. J. D. Monk, *Nonfinitizability of classes of representable cylindric algebras*, J. Symbolic Logic, **34** (1969), 331–343. MR0256861 (41:1517)
15. I. Sain and R. J. Thompson, *Strictly finite schema axiomatization of quasi-polyadic algebras*, in Algebraic Logic, Coll. Math. Soc. J. Bolyai, 1988, pp. 539–571. MR1153440 (93a:03072)
16. Tarek Sayed Ahmed, *On complete representations of reducts of polyadic algebras*, Studia Logica, **89** (3) (2008), 325–332. MR2438547 (2009g:03103)

DEPARTMENT OF ALGEBRA, BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, H–1521 BUDAPEST, HUNGARY

E-mail address: ferenczi@math.bme.hu