EXPONENTIAL SELF-IMPROVEMENT OF GENERALIZED POINCARÉ INEQUALITIES ASSOCIATED WITH APPROXIMATIONS OF THE IDENTITY AND SEMIGROUPS

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ABSTRACT. The purpose of this paper is to present a general method that allows us to study exponential self-improving properties of generalized Poincaré inequalities associated with an approximation of the identity or a semigroup. In particular, we show the connection between our results and the John-Nirenberg theorem for the space BMO associated with approximations of the identity and semigroups.

1. INTRODUCTION

It is well known that the theory of self-improving properties of Poincaré inequalities plays a fundamental role in the study of partial differential equations. The classical Sobolev embedding (see for instance [SC3]) says that $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ for $1 \leq p < n$ and $p^* = np/(n-p)$. That is, every function in $L^p(\mathbb{R}^n)$ with gradient in $L^p(\mathbb{R}^n)$ improves its integrability and it belongs to $L^{p^*}(\mathbb{R}^n)$. This can also be expressed via the following Poincaré-Sobolev inequality:

$$(1.1) \quad \left(\int_Q |f - f_Q|^{p^*} dx \right)^{1/p^*} \lesssim \ell(Q) \left(\int_Q |\nabla f|^p dx \right)^{1/p}.$$

If $p = n$, $W^{1,p}(\mathbb{R}^n) \subset \exp L^{n'}(\mathbb{R}^n)$ with $n' = n/(n-1)$ and the corresponding inequality, called Trudinger's inequality (first derived by Yudovich in [Yud]), is

$$(1.2) \quad \|f - f_Q\|_{\exp L^{n'}, Q} \lesssim \left(\int_Q |\nabla f|^n dx \right)^{1/n}.$$

If $n < p \leq \infty$, we have $W^{1,p}(\mathbb{R}^n) \subset L(1 - n/p)(\mathbb{R}^n)$. Here $L(\alpha)$, $\alpha \geq 0$, is the set of functions such that $\int_Q |f - f_Q| \lesssim |Q|^\alpha$ and we also have the exponential self-improvement:

$$\|f - f_Q\|_{\exp L, Q} \lesssim |Q|^\alpha.$$

The case $\alpha = 0$ corresponds to BMO and the John and Nirenberg inequality; see [JN].

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Saloff-Coste considered further self-improvement properties of Poincaré inequalities in more general spaces such as the Riemannian manifolds; see [BCLS], [SC1], [SC2], [SC3]. This turns out to be useful when one studies large scale behavior of solutions of partial differential equations such as the Laplace and heat equations in these contexts. See for instance [HK1], [HK2], [Jer], [SC2], [SC3], [VSCC].

Another important contribution to this field was made by Franchi, Pérez and Wheeden in [FPW]. These authors showed that this self-improving phenomenon holds for general spaces of homogeneous type (X, d, μ) (see Section 5). Also they considered wider classes of functionals on the right-hand side and studied self-improving properties of the following estimates:

$$(1.3) \quad \oint_B |f - f_B| dx \leq a(B, f),$$

with a being a functional depending on the ball B , and sometimes on the function f . By analogy to the classical $(1, 1)$ -Poincaré inequality, the authors referred to (1.3) as a generalized Poincaré inequality. [FPW] introduced a general method based on the Calderón-Zygmund theory and the good- λ inequalities discovered by Burkholder and Gundy ([BG]) that allows one to establish that under mild geometric conditions on the functional a ($a \in D_r$; see Section 2), inequality (1.3) encodes an intrinsic self-improvement of $L^{p, \infty}$ -type for $p > 1$. Moreover, it turns out that (under certain conditions) the weak-type estimate yields strong-type results. This point of view also provides the L^p -version of the John-Nirenberg inequality for functions in BMO .

Under a stronger condition on the functional a , in [MP], exponential self-improving is obtained and this covers in particular the John-Nirenberg inequality for BMO functions. They also extended some results from [HK2], [Fra], [GN], [Lu], [Bus]. It should be pointed out that the approach used by these authors does not use any differentiability structure.

We would like to point out that in these last approaches it is key to exploit the intimate relation between the maximal operator and the sharp maximal operator. On the other hand, a new sharp maximal operator associated with an approximation of the identity $\{S_t\}_{t>0}$ is introduced in [Ma1],

$$M_S^\# f(x) = \sup_{B \ni x} \oint_B |f - S_{t_B} f| dy,$$

where t_B is a parameter depending on the radius of the ball B . This operator allows one to define the BMO_S space, for which the John-Nirenberg theorem is also valid (see [DY]). In this way, starting with an estimate as (1.3) with $a(B, f) = C$ and where the oscillation $f - f_B$ is replaced by $f - S_{t_B} f$, a self-improving property is obtained. This new way of measuring the oscillation allows one to define new function spaces as the BMO_S of [DY] and the Morrey-Campanato associated with an approximation of the identity of [DDY].

In the Euclidean setting $\mathbb{R}^n$ with the Lebesgue measure dx (sometimes the Lebesgue measure is replaced by a Muckenhoupt weight), [JM] presents a method based on the Whitney covering lemma and the technique of good- λ inequalities which allows one to obtain self-improvement in the scale of Lebesgue spaces of Poincaré-type inequalities with respect to an approximation of the identity or a

semigroup. More precisely, let f satisfy the following generalized Poincaré inequality:

$$(1.4) \quad \int_Q |f - S_{t_Q} f| dx \leq a(Q),$$

where $t_Q = \ell(Q)^m$ and $\{S_t\}_{t>0}$ is an approximation of the identity or a semigroup whose kernel decays fast enough (see Section 2) and the functional a satisfies a certain geometric condition ($a \in D_r$; see Section 2). Then $L^{r,\infty}(Q)$ estimates are obtained for the generalized oscillation $f - S_{t_Q} f$. In particular, some expanded Poincaré estimates that take into account the lack of localization of the approximation of the identity or the semigroup are proposed. As a consequence of this method, global pseudo-Poincaré inequalities and strong-type inequalities are obtained. Let us also mention [BJM], where this theory is extended to the so-called spaces of homogeneous type.

In this paper we continue along the lines in [JM], [BJM] and the study of the exponential self-improving properties of the generalized Poincaré inequalities. That is, starting from (1.4) with a functional a that is quasi-increasing (as in [MP] we write $a \in T_\infty$), we show an exponential-type estimate for the generalized oscillations. Also, the method yields estimates with respect to Muckenhoupt weights. We consider this problem both in the Euclidean setting and in the so-called spaces of homogeneous type. We state here our main result in the Euclidean setting and postpone the space of homogeneous type version until Part 2. The main result goes as follows (see Section 2 for the needed background and the precise definitions):

Theorem 1.1. *Let $\{S_t\}_{t>0}$ be an approximation of the identity or a semigroup and let $a \in T_\infty$. Assume that $f \in \mathcal{M}$ satisfies*

$$(1.5) \quad \int_Q |f - S_{t_Q} f| dx \leq a(Q)$$

for all cubes Q and where $t_Q = \ell(Q)^m$. Then, for every cube Q ,

$$(1.6) \quad \|f - S_{t_Q} f\|_{\exp L, Q} \leq C \sum_{k \geq 0} 2^{nk} g(2^{m(k-6)}) a(2^k Q).$$

Furthermore, for every $w \in A_\infty$ and every cube Q ,

$$(1.7) \quad \|f - S_{t_Q} f\|_{\exp L(w), Q} \leq C \sum_{k \geq 0} 2^{nk} g(c 2^{mk}) a(2^k Q)$$

with $C \geq 1$ and $0 < c < 1$.

Remark 1.2. We would like to call attention to the fact that (1.5) is an unweighted estimate and that from it we obtain a weighted estimate for the oscillation $f - S_{t_Q} f$. Also, we observe that if a is doubling, then the decay of g easily implies that in the right-hand side of both estimates one can write $C a(Q)$.

Compare this with the approach in [MP] where a similar estimate is proved starting from (1.3). Here we use the Whitney covering lemma, in place of the Calderón-Zygmund decomposition. Since we have to take into account that the operators S_t do not localize, this leads to some tail effects that can be found on the right-hand side of (1.6). The exponential decay assumed on the kernels gives some kind of localization, in the sense that the operators S_t are “essentially” local, but still the values of f away from a given cube matter and one needs to control

them. Let us also point out that in our argument we need to be always localized to the given cube Q . Thus, when using the Whitney covering lemma and restricting to the given cube Q , all the involved Whitney cubes should be contained in Q . To do this we perform a Whitney covering lemma with respect to the dyadic grid induced by Q and this is key for our purposes. Notice that in the setting of spaces of homogeneous type, we need a similar approach. Then we use Christ's dyadic sets and a Whitney covering lemma related to this dyadic structure (see Section 8.1).

The paper is organized as follows. In the first part we work in $\mathbb{R}^n$. The preliminaries and main definitions are in Section 2. Applications are in Section 3, and the proof of Theorem 1.1 is in Section 4. In the second part of the paper we extend the previous result to the setting of spaces of homogeneous type (see Theorem 6.1). The difference between both parts is technical: we write in detail the first part, and in the second part we omit many of the already used arguments. Preliminaries, applications and the proof of Theorem 6.1 are given respectively in Sections 5, 7 and 8.

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Part 1. Exponential self-improvement of Poincaré inequalities in $\mathbb{R}^n$

2. PRELIMINARIES

Consider the Euclidean space $\mathbb{R}^n$ with the Lebesgue measure dx (we sometimes replace the Lebesgue measure by the measure associated to a Muckenhoupt weight). We work with the infinity distance, since we manipulate cubes, that is, $|x - y| = |x - y|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|$. Obviously, all the results can be easily adapted to the Euclidean distance. We consider, without loss of generality, cubes of the form $\prod_{i=1}^n [a_i, a_i + \ell(Q))$, with $a_i \in \mathbb{R}$ and where $\ell(Q)$ denotes the side-length of Q .

Given a cube $Q \subset \mathbb{R}^n$ and $\lambda > 0$, we denote by λQ the cube concentric with Q so that $\ell(\lambda Q) = \lambda \ell(Q)$. Also, we write

$$f_Q = \int_Q f(x) dx = \frac{1}{|Q|} \int_Q f(x) dx.$$

We will use several times the following decomposition of $\mathbb{R}^n$ in dyadic annuli: $\mathbb{R}^n = \bigcup_{k \geq 1} C_k(Q)$ with $C_1(Q) = 2Q$ and $C_k(Q) = 2^k Q \setminus 2^{k-1} Q$, $k \geq 2$.

Finally, we make some conventions: $A \lesssim B$ means that the ratio A/B is bounded by a constant that does not depend on the relevant variables in A and B . Throughout this paper, the letter C denotes a constant that is independent of the essential variables, but it may vary from line to line.

2.1. The space $\exp L$. This was introduced by A. Zygmund (and E. C. Titchmarsh) in 1928. See [BS] and the references therein. It appears as a natural limit of L^p when $p \rightarrow \infty$. The space $\exp L(Q)$ is defined as the set of all measurable functions f on Q such that for some $C > 0$,

$$\int_Q \exp(C|f|) dx < \infty.$$

The localized and normalized Luxemburg norm of $\exp L(Q)$ is given by

$$\|f\|_{\exp L, Q} = \inf \left\{ \lambda > 0 : \int_Q \left(\exp \frac{|f|}{\lambda} - 1 \right) dx \leq 1 \right\}.$$

We refer to [BS] and [RR] for more details on $\exp L$ and Orlicz spaces.

2.2. Approximations of the identity and semigroups. We work with families of linear operators $\{S_t\}_{t>0}$ that play the role of generalized approximations of the identity. We assume from now on that these operators commute, that is, $S_t \circ S_s = S_s \circ S_t$ for every $s, t > 0$. Families of operators that form a semigroup (that is, $S_s S_t = S_{s+t}$, for all $s, t > 0$) satisfy this property. We assume that these operators admit the following integral representation:

$$S_t f(x) = \int_{\mathbb{R}^n} s_t(x, y) f(y) dy,$$

where $s_t(x, y)$ is a measurable function such that

$$(2.1) \quad |s_t(x, y)| \leq \frac{1}{t^{n/m}} g\left(\frac{|x - y|^m}{t}\right)$$

for some positive constant m and for some positive, bounded and non-increasing function g . We assume that for all $N \geq 0$ we have

$$\lim_{r \rightarrow \infty} r^N g(r) = 0.$$

We notice that one can assume less decay on g by fixing $N > 0$ in the previous expression. This exponent would be taken large enough so that the estimates obtained below are not trivial (further details are left to the reader).

Let us note that the decay of g yields that the integral representation of S_t makes sense for all functions $f \in L^p(\mathbb{R}^n)$ and that the operators S_t are uniformly bounded on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$. As in [DY], we introduce a wider class of functions for which S_t is well defined. We define $\mathcal{M} = \bigcup_{\beta > 0} \mathcal{M}_\beta$, where $\mathcal{M}_\beta$ is the set of measurable functions f such that

$$\|f\|_{\mathcal{M}_\beta} = \int_{\mathbb{R}^n} |f(x)| (1 + |x|)^{-\beta} dx < \infty.$$

In [DY], the authors establish that $(\mathcal{M}_\beta, \|\cdot\|_{\mathcal{M}_\beta})$ is a Banach space. Moreover, if $f \in \mathcal{M}$, then, for all $t, s > 0$, $S_t f$ and $S_s(S_t f)$ are well defined and finite almost everywhere.

We observe that (2.1) leads to a rescaling between the parameter t and the space variables. Thus, in what follows, given a cube Q , we write $t_Q = \ell(Q)^m$ in such a way that the parameters t and S_t are “adapted” or “scaled” to Q .

We present some examples of approximations of the identity and semigroups (see [JM]). We consider second order elliptic form operators: let A be an elliptic $n \times n$ matrix of real and L^∞ -valued coefficients defined on $\mathbb{R}^n$. Associated with this matrix we define the second order divergence form operator $Lf = -\operatorname{div}(A \nabla f)$, which is understood in the standard weak sense as a maximal-accretive operator on $L^2(\mathbb{R}^n, dx)$. The operator $-L$ generates a semigroup $\{e^{-tL}\}_{t>0}$ whose heat kernel has Gaussian bounds, that is, the above estimates hold with $m = 2$ and $g(t) = C e^{-c t}$. In dimensions $n = 1, 2$ one can also consider complex-valued matrices. The easiest example is given by the identity matrix in which case $L = -\Delta$ and therefore $\{e^{t\Delta}\}_{t>0}$ is the heat semigroup. We can also take the operators $S_t = I - (I - e^{-tL})^m$

for some fixed $m \geq 1$, in which case we lose the semigroup property. However, we still have the commutation rule and the Gaussian decay. Thus we can apply our results to the family $\{S_t\}_{t>0}$. In some applications it is interesting to have m large enough so that one obtains extra decay in the resulting estimates (see [JM] and the references therein).

2.3. Functionals. We consider functionals $a : \mathcal{Q} \times \mathcal{F} \rightarrow [0, +\infty)$, where $\mathcal{Q}$ is the family of cubes in $\mathbb{R}^n$ and $\mathcal{F}$ is a certain family of functions. When the dependence on the functions is not of our interest, we simply write $a(Q)$.

We say that the functional a is quasi-increasing or satisfies the T_∞ condition (we write $a \in T_\infty$) if there exists a finite positive constant C_a such that

$$a(Q_1) \leq C_a a(Q_2)$$

for every $Q_1 \subset Q_2$. The infimum of the constants C_a is denoted by $\|a\|_{T_\infty}$ ($\|a\|_{T_\infty} \geq 1$). When $\|a\|_{T_\infty} = 1$, we say that a is increasing. This condition was introduced in [MP] in order to study exponential self-improving of classical Poincaré-type inequalities.

Observe that T_∞ is stronger than D_r (considered in [FPW] or [JM] to obtain L^r -type self-improving properties). Recall that given a Borel measure μ and $1 \leq r < \infty$, we say that $a \in D_r(\mu)$ if there exists a constant $1 \leq C_a < \infty$ such that for each cube Q and any family $\{Q_i\}_i$ of pairwise disjoint subcubes of Q , the following holds:

$$\sum_i a(Q_i)^r \mu(Q_i) \leq C_a^r a(Q)^r \mu(Q).$$

We say that a is doubling if there exists some finite positive constant C_a such that

$$a(2Q) \leq C_a a(Q)$$

for each cube Q .

2.4. Weights. A weight w is a non-negative locally integrable function. For any measurable set E , we write $w(E) = \int_E w(x) dx$. Also, we set

$$\int_Q f dw = \int_Q f(x) dw(x) = \frac{1}{w(Q)} \int_Q f(x) w(x) dx.$$

We say that w is doubling if for some finite positive constant C_w and any cube Q ,

$$w(2Q) \leq C_w w(Q) < \infty.$$

A well-known collection of doubling weights is given by the Muckenhoupt classes, which are defined as follows. We say that a weight $w \in A_p$, $1 < p < \infty$, if there exists a constant $0 < C < \infty$ such that for every cube Q ,

$$\left(\int_Q w dx \right) \left(\int_Q w^{1-p'} dx \right)^{p-1} \leq C.$$

For $p = 1$, we say that $w \in A_1$ if there exists a constant $0 < C < \infty$ such that for every cube Q ,

$$\int_Q w dx \leq C w(y), \quad \text{for a.e. } y \in Q.$$

We write $A_\infty = \bigcup_{p \geq 1} A_p$. We say that $w \in RH_p$, $1 < p < \infty$, if there exists a constant $0 < C < \infty$ such that for every cube Q ,

$$\left(\int_Q w^p dx \right)^{1/p} \leq C \int_Q w dx.$$

We also consider the weighted exponential space $\exp L(Q, w)$ whose norm is

$$\|f\|_{\exp L(w), Q} = \inf \left\{ \lambda > 0 : \int_Q \left(\exp \frac{|f|}{\lambda} - 1 \right) dw \leq 1 \right\}.$$

3. APPLICATIONS

Example 1 (BMO and Morrey-Campanato spaces). As in [JM], given $\alpha \geq 0$ and $\{S_t\}_{t>0}$ as above (we notice that in [DY] and [DDY] these authors only consider the semigroup case), the space of Morrey-Campanato $L_S(\alpha)$ is defined as follows:

$$L_S(\alpha) = \left\{ f \in \mathcal{M} : \sup_Q \frac{1}{|Q|^\alpha} \int_Q |f - S_{t_Q} f| dx < \infty \right\}.$$

Notice that when $\alpha = 0$ this space coincides with the bounded mean oscillation space BMO_S . The previous definition generalizes the classical spaces $L(\alpha)$ and BMO on which $S_{t_Q} f$ is replaced by f_Q .

Given $f \in L_S(\alpha)$ with $\alpha \geq 0$, for every cube Q we have that

$$\int_Q |f - S_{t_Q} f| dx \leq C |Q|^\alpha.$$

In this way we choose $a(Q) = |Q|^\alpha$ (we intentionally drop the constant as it is harmless); a is clearly increasing and doubling. Then, Theorem 1.1 yields

$$\|f - S_{t_Q} f\|_{\exp L, Q} \lesssim |Q|^\alpha$$

for every Q . Also, we obtain that for every $w \in A_\infty$ and cube Q ,

$$\|f - S_{t_Q} f\|_{\exp L(w), Q} \lesssim |Q|^\alpha.$$

On the other hand, S. Spanne [Spa] introduces new spaces that contain the image of L^p ($p > n/\alpha$) via the fractional integral I_α . These are generalizations of the classical BMO . We extend this definition and replace f_Q by $S_{t_Q} f$. Thus, as in [JM], given a non-decreasing and positive function φ defined in $(0, \infty)$, we define $BMO_{\varphi, S}$ as follows:

$$BMO_{\varphi, S} = \left\{ f \in \mathcal{M} : \sup_Q \frac{1}{\varphi(\ell(Q))} \int_Q |f - S_{t_Q} f| dx < \infty \right\}.$$

Using Theorem 1.1 with $a(Q) = \varphi(\ell(Q))$, which is increasing (since φ is not decreasing), we obtain that for every $f \in BMO_{\varphi, S}$ and for all cubes Q ,

$$\|f - S_{t_Q} f\|_{\exp L, Q} \leq C \sum_{k \geq 0} 2^{nk} g(e 2^{mk}) \varphi(2^k \ell(Q)).$$

In addition, if φ is doubling (that is, $\varphi(2t) \lesssim \varphi(t)$, $t > 0$), then so is a . This and the decay of g give

$$\|f - S_{t_Q} f\|_{\exp L, Q} \lesssim \varphi(\ell(Q)).$$

Notice that we can also get weighted versions for A_∞ weights.

Example 2 (Poincaré-Sobolev inequality). As in [JM] we consider the following expanded Poincaré-type inequality for $f \in \mathcal{M}$:

$$(3.1) \quad \int_Q |f - S_{t_Q} f| dx \leq \sum_{k \geq 0} \alpha(k) \ell(2^k Q) \left(\int_{2^k Q} h^p dx \right)^{1/p}$$

for all cubes Q , where $\{\alpha(k)\}_{k \geq 0}$ is a sequence of non-negative numbers and where h is some non-negative measurable function; typically h depends on f , for instance $h = C |\nabla f|$. Observe that in the classical situation, replacing $S_{t_Q} f$ by f_Q and taking $h = |\nabla f|$ and $\alpha(k) = 0$ for $k \geq 1$, (3.1) is nothing but a classical Poincaré-Sobolev inequality. Let us observe that if h^p is doubling and $\alpha(k)$ decays fast enough, then (3.1) leads us to a reduced Poincaré-type inequality:

$$(3.2) \quad \int_Q |f - S_{t_Q} f| dx \leq \ell(Q) \left(\int_Q h^p dx \right)^{1/p}.$$

However, we believe that expanded Poincaré-type inequalities are more natural in the sense that they take into account the tail effects of the approximation of the identity, in place of looking only at a somehow local term. Moreover, under certain conditions the classical Poincaré-Sobolev inequality implies (3.1) with $h = |\nabla f|$ (see [JM]).

Self-improving properties of these expanded Poincaré inequalities have been studied in [JM]. In the range $1 < p < n$, $L^{q,\infty}$ self-improving estimates are shown for every $1 < q < p^*$. Consequently, Kolmogorov's inequality gives strong-type estimates in the same range. For $q = p^*$ a weak-type estimate in an Orlicz space is obtained. For the range $p \geq n$, it is pointed out that all the D_r conditions are satisfied (since a is quasi-increasing) and therefore one obtains weak and strong-type estimates in the full range $1 < q < \infty$.

Here we consider the case $p \geq n$ and look for exponential estimates. As a consequence of Theorem 1.1, if $f \in \mathcal{M}$ satisfies (3.1), then

$$\|f - S_{t_Q} f\|_{\exp L, Q} \lesssim \sum_{k \geq 0} \hat{\alpha}(k) \ell(2^k Q) \left(\int_{2^k Q} h^p dx \right)^{1/p},$$

with $\hat{\alpha}(k) = \sum_{0 \leq l \leq k} 2^{nl} g(2^{m(l-6)}) \alpha(k-l)$ for all $k \geq 0$. Note that to use Theorem 1.1, it is enough to observe that if $p \geq n$, then

$$a(Q) = \sum_{k \geq 0} \alpha(k) \ell(2^k Q) \left(\int_{2^k Q} h^p dx \right)^{1/p} \in T_\infty.$$

Applying Theorem 1.1, we get the corresponding weighted version for A_∞ weights. Note that for $p = n$, this gives a weaker version of Trudinger's inequality (1.2) with this new way of measuring the oscillation.

4. PROOF OF THEOREM 1.1

4.1. Proof of 1.6. We use ideas of [MP]. We can assume that $\|a\|_{T_\infty} = 1$. Indeed, if we take $\hat{a}(Q) = \sup_{P \subset Q} a(P)$, then $a(Q) \leq \hat{a}(Q) \leq \|a\|_{T_\infty} a(Q)$, $\|\hat{a}\|_{T_\infty} = 1$ and (1.5) holds with a replaced by $\hat{a}$.

We define a new functional $\tilde{a} : \mathcal{Q} \times \mathcal{F} \rightarrow [0, +\infty)$ given by

$$(4.1) \quad \tilde{a}(Q) = \frac{2^{2n}}{g(1)} \sum_{k \geq 0} 2^{nk} g(2^{m(k-6)}) a(2^k Q).$$

Observe that $a \in T_\infty$ implies $\tilde{a} \in T_\infty$ with $\|\tilde{a}\|_{T_\infty} = \|a\|_{T_\infty} = 1$. Fix a cube Q and assume that $\tilde{a}(Q) < \infty$. Otherwise, there is nothing to prove. We define the function

$$(4.2) \quad G(x) = \frac{|f(x) - S_{t_Q} f(x)|}{\tilde{a}(Q)} \chi_{4Q}(x).$$

The following result allows us to show that $G \in L^1(\mathbb{R}^n)$.

Lemma 4.1. *Under the hypotheses of Theorem 1.1, for each cube R and $k \geq 1$, we have*

$$\int_{2^k R} |f - S_{t_R} f| dy \leq \|a\|_{T_\infty} a(2^k R).$$

Applying this lemma and $\tilde{a}(Q) \geq 16^n a(4Q)$, we get

$$\|G\|_{L^1(\mathbb{R}^n)} = \frac{1}{\tilde{a}(Q)} \int_{4Q} |f - S_{t_Q} f| dx \leq \frac{a(4Q)}{\tilde{a}(Q)} |4Q| \leq \frac{1}{4^n} |Q| < \infty.$$

Consider the level set

$$(4.3) \quad \Omega = \{x \in \mathbb{R}^n : MG(x) > e\},$$

where M is the Hardy-Littlewood maximal function. Given that M is of weak type $(1, 1)$ with constant 3^n , we have

$$(4.4) \quad |\Omega| \leq \frac{3^n}{e} \|G\|_{L^1(\mathbb{R}^n)} < e^{-1} |Q| < \infty.$$

Note that Ω is a level set of the lower semicontinuous function MG . Moreover, as we have already seen, $G \in L^1(\mathbb{R}^n)$ and $|\Omega| < \infty$. Thus, $\Omega \subsetneq \mathbb{R}^n$ is a proper open set.

Next we need to use the dyadic grid in $\mathbb{R}^n$ adapted to the cube Q that consists of translations and dilations of the dyadic classical structure (see [JM]). Given $Q = \prod_{i=1}^n [a_i, a_i + \ell(Q))$, the collection of dyadic cubes induced by Q is denoted by $\mathcal{D}_Q$ and its k -th generation by $\mathcal{D}_{Q,k}$. That is,

$$\mathcal{D}_Q = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_{Q,k},$$

where, for each $k \in \mathbb{Z}$, $\mathcal{D}_{Q,k}$ is the set of cubes of the form

$$\prod_{i=1}^n [a_i + 2^k m_i \ell(Q), a_i + 2^k (m_i + 1) \ell(Q))$$

with $m_1, m_2, \dots, m_n \in \mathbb{Z}$. By construction, this dyadic decomposition has the following properties:

- (i) If $R \in \mathcal{D}_{Q,k}$, then $\ell(R) = 2^k \ell(Q)$.
- (ii) If $R_1, R_2 \in \mathcal{D}_Q$ and $R_1 \cap R_2 \neq \emptyset$, then either $R_1 \subset R_2$ or $R_1 \supset R_2$.
- (iii) $\mathbb{R}^n = \bigcup_{R \in \mathcal{D}_{Q,k}} R$, and these cubes are pairwise disjoint.

The proofs of the following results are in [JM]. First we present a version of the Whitney covering lemma adapted to $\mathcal{D}_Q$.

Theorem 4.2 ([JM]). *Given a cube Q in $\mathbb{R}^n$ and Ω a proper open subset of $\mathbb{R}^n$, there is a family of Whitney cubes $\{Q_i\}_i$ such that*

- (a) $\Omega = \bigcup_i Q_i$.
- (b) $\{Q_i\}_i \subset \mathcal{D}_Q$ and these cubes are maximal with respect to the inclusion and therefore pairwise disjoint.
- (c) $\ell(Q_i) < d(Q_i, \Omega^c) \leq 4\ell(Q_i)$ and $10Q_i \cap \Omega^c \neq \emptyset$.

When Ω is a level set of the Hardy-Littlewood maximal function, the following property holds:

Lemma 4.3 ([JM]). *Fix Q , $t > 0$ and $G \in L^1(\mathbb{R}^n)$. If $\Omega = \{x \in \mathbb{R}^n : MG(x) > t\}$ and $\{Q_i\}_i$ is its family of Whitney cubes, then, for every $k \geq 1$, we have*

$$\int_{2^k Q_i} G \, dx \leq 10^n t.$$

With these results, the set Ω can be covered by the family of Whitney cubes $\{Q_i\}_i \subset \mathcal{D}_Q$. Furthermore, as a consequence of (4.4), for every i we have

$$(4.5) \quad \ell(Q_i) < \ell(Q).$$

For each $t > 0$ and each cube R , we consider the set

$$(4.6) \quad E(R, t) = \{x \in R : |f(x) - S_{t_R} f(x)| > t \tilde{a}(R)\}.$$

Take $t > e$. Note that except for a set of null Lebesgue measure, we can write

$$E(Q, t) = E(Q, t) \cap \{x \in Q : MG(x) > e\} = \bigcup_i \{x \in Q_i \cap Q : G(x) > t\}.$$

From now on, we only consider those cubes Q_i such that $Q_i \cap Q \neq \emptyset$. Therefore, since the cubes Q_i are dyadic with respect to Q , (4.5) implies that $Q_i \subset Q$. Thus,

$$(4.7) \quad E(Q, t) = \bigcup_{i: Q_i \subset Q} \{x \in Q_i : |f(x) - S_{t_{Q_i}} f(x)| > t \tilde{a}(Q)\}.$$

Next, we replace $S_{t_Q} f$ by $S_{t_{Q_i}} f$ by using the following auxiliary result, whose proof is in Section 4.2.

Proposition 4.4. *Under the hypotheses of Theorem 1.1, for all $x \in Q_i$, we have*

$$|S_{t_{Q_i}} f(x) - S_{t_Q} f(x)| \leq c_0 \tilde{a}(Q).$$

Hence, if $x \in Q_i$, we get

$$|f(x) - S_{t_Q} f(x)| \leq |f(x) - S_{t_{Q_i}} f(x)| + c_0 \tilde{a}(Q).$$

Fix $t > C_0$ with $C_0 = \max\{e, c_0\}$, where c_0 is the constant that appears in the previous proposition. Using that $\tilde{a}(Q_i) \leq \tilde{a}(Q)$ (since a is increasing and $Q_i \subset Q$), we get

$$(4.8) \quad \begin{aligned} |E(Q, t)| &\leq \sum_{i: Q_i \subset Q} |\{x \in Q_i : |f(x) - S_{t_{Q_i}} f(x)| > (t - C_0) \tilde{a}(Q_i)\}| \\ &= \sum_{i: Q_i \subset Q} |E(Q_i, t - C_0)|. \end{aligned}$$

Let us define the function $\varphi : [0, +\infty) \rightarrow [0, 1]$ given by

$$\varphi(t) = \sup_R \frac{|E(R, t)|}{|R|}.$$

Note that $\varphi(t) \leq 1$ for all $t \geq 0$. Furthermore, when $t > C_0$, (4.8) and (4.4) imply that

$$|E(Q, t)| \leq \varphi(t - C_0) \sum_{i: Q_i \subset Q} |Q_i| \leq \varphi(t - C_0) |\Omega| < e^{-1} \varphi(t - C_0) |Q|.$$

Thus, $\varphi(t) < e^{-1} \varphi(t - C_0)$ for all $t > C_0$. Therefore, iterating this estimate and using that $\varphi(t) \leq 1$ for all $t \geq 0$, we conclude that $\varphi(t) \leq e^{-(t/C_0 - 1)}$ for all $t \geq 0$. In particular, for every $t \geq 0$ we have

$$\frac{|E(Q, t)|}{|Q|} \leq \varphi(t) \leq e^{-(t/C_0 - 1)}.$$

Hence, we pick $A > C_0$ and use the previous estimate to get

$$\int_Q \exp\left(\frac{|f - S_{t_Q} f|}{A \tilde{a}(Q)}\right) dx \leq \int_0^\infty e^t \frac{|E(Q, A t)|}{|Q|} dt \lesssim \int_0^\infty e^{-t(A/C_0 - 1)} dt < \infty.$$

This immediately gives the desired estimate.

4.2. Proofs of the auxiliary results.

Proof of Lemma 4.1. Cover $2^k R$ with a family of pairwise disjoint subcubes $\{R_i\}_{i=1}^{2^{kn}}$ with side-length $\ell(R)$; thus $t_R = t_{R_i}$. As f satisfies (1.5) and a is increasing, we obtain

$$\begin{aligned} \int_{2^k R} |f - S_{t_R} f| dy &= \sum_{i=1}^{2^{kn}} \int_{R_i} |f - S_{t_{R_i}} f| dy \leq \sum_{i=1}^{2^{kn}} a(R_i) |R_i| \leq \|a\|_{T_\infty} a(2^k R) \sum_{i=1}^{2^{kn}} |R_i| \\ &= \|a\|_{T_\infty} a(2^k R) |2^k R|. \end{aligned} \quad \square$$

Proof of Proposition 4.4. The commutation rule implies that for every $x \in Q_i$,

$$|S_{t_{Q_i}} f(x) - S_{t_Q} f(x)| \leq |S_{t_{Q_i}}(f - S_{t_Q} f)(x)| + |S_{t_Q}(f - S_{t_{Q_i}} f)(x)| = I + II,$$

and we study each term in turn.

Pick $k_i \in \mathbb{Z}$ such that

$$(4.9) \quad 2^{k_i} \ell(Q_i) \leq \ell(Q) < 2^{k_i+1} \ell(Q_i).$$

Then

$$(4.10) \quad k_i \geq 1 \quad \text{and} \quad 2^{k_i} Q_i \subset 2Q.$$

The first claim is a consequence of (4.9) together with the fact that $Q_i \in \mathcal{D}_Q$ and $\ell(Q_i) \leq \ell(Q)/2$ (by (4.5)). The second claim follows from (4.9) and $Q_i \subset Q$.

We decompose $\mathbb{R}^n$ as the union of the annuli $\{C_k(Q_i)\}_{k \geq 1}$. If $x \in Q_i$ and $y \in C_k(Q_i)$, then

$$\frac{|x - y|^m}{t_{Q_i}} \geq \lambda_k, \quad \text{where} \quad \lambda_k = \begin{cases} 0, & \text{if } k = 1, \\ 2^{m(k-3)}, & \text{if } k \geq 2. \end{cases}$$

In order to study I , we first observe that, as a consequence of (4.10), if $y \in 2^{k_i} Q_i$, then $|f(y) - S_{t_Q} f(y)| = G(y) \tilde{a}(Q)$ and we obtain

$$(4.11) \quad \begin{aligned} I &= |S_{t_{Q_i}}(f - S_{t_Q} f)(x)| \\ &\leq \frac{1}{|Q_i|} \int_{\mathbb{R}^n} g\left(\frac{|x-y|^m}{t_{Q_i}}\right) |f(y) - S_{t_Q} f(y)| dy \\ &\leq \frac{\tilde{a}(Q)}{|Q_i|} \int_{\mathbb{R}^n} g\left(\frac{|x-y|^m}{t_{Q_i}}\right) G(y) dy + \frac{1}{|Q_i|} \int_{(2^{k_i} Q_i)^c} \cdots dy = I_1 \tilde{a}(Q) + I_2. \end{aligned}$$

Let us estimate I_1 : applying Lemma 4.3,

$$I_1 \leq \frac{1}{|Q_i|} \sum_{k \geq 1} g(\lambda_k) \int_{C_k(Q_i)} G dy \leq \sum_{k \geq 1} 2^{n k} g(\lambda_k) \int_{2^k Q_i} G dy \lesssim 1.$$

For I_2 , we apply Lemma 4.1 and the fact that $C_k(Q_i) \subset 2^k Q_i \subset 2^{k-k_i+1} Q \subset 2^k Q$, for every $k \geq k_i$:

$$\begin{aligned} I_2 &= \frac{1}{|Q_i|} \sum_{k \geq k_i+1} \int_{C_k(Q_i)} g\left(\frac{|x-y|^m}{t_{Q_i}}\right) |f(y) - S_{t_Q} f(y)| dy \\ &\lesssim \sum_{k \geq k_i+1} 2^{n k} g(\lambda_k) \int_{2^{k-k_i+1} Q} |f - S_{t_Q} f| dy \\ &\lesssim \sum_{k \geq k_i+1} 2^{n k} g(\lambda_k) a(2^{k-k_i+1} Q) \\ &\lesssim \sum_{k \geq k_i+1} 2^{n k} g(\lambda_k) a(2^k Q) \lesssim \tilde{a}(Q). \end{aligned}$$

Plugging these into (4.11) we get $I \lesssim \tilde{a}(Q)$.

We are going to show that $II \lesssim \tilde{a}(Q)$. Proceeding as above, by the computations in Lemma 4.1, (4.9) and (4.10), we obtain

$$\begin{aligned} II &\leq \frac{1}{|Q|} \int_{\mathbb{R}^n} g\left(\frac{|x-y|^m}{t_Q}\right) |f(y) - S_{t_{Q_i}} f(y)| dy \\ &\leq \frac{g(0)}{|Q|} \int_{2^{k_i+2} Q_i} |f - S_{t_{Q_i}} f| dy + \frac{1}{|Q|} \sum_{k \geq k_i+3} g\left(\frac{\lambda_k t_{Q_i}}{t_Q}\right) \int_{2^k Q_i} |f - S_{t_{Q_i}} f| dy \\ &\lesssim a(2^{k_i+2} Q_i) + \sum_{k \geq k_i+3} 2^{n(k-k_i)} g(\lambda_k 2^{-m(k_i+1)}) a(2^k Q_i) \\ &\lesssim a(2^4 Q) + \sum_{k \geq k_i+3} 2^{n(k-k_i)} g(\lambda_k 2^{-m(k_i+1)}) a(2^{k-k_i+1} Q) \\ &\lesssim a(2^4 Q) + \sum_{k \geq 4} 2^{n k} g(2^{m(k-5)}) a(2^k Q) \lesssim \tilde{a}(Q). \end{aligned} \quad \square$$

4.3. Proof of (1.7). We follow the arguments of the proof of (1.6) and so we only point out those steps where both proofs differ. First we recall that $w \in A_\infty$ implies that there are $1 \leq s_0 < \infty$ and $1 < s_1 \leq \infty$ such that $w \in A_{s_0} \cap RH_{s_1}$. In particular, for any cube Q and any measurable set $S \subset Q$,

$$(4.12) \quad \frac{1}{C_w} \left(\frac{|S|}{|Q|} \right)^{s_0} \leq \frac{w(S)}{w(Q)} \leq C_w \left(\frac{|S|}{|Q|} \right)^{1/s_1'}.$$

The first inequality is a consequence of $w \in A_{s_0}$, and the second one of $w \in RH_{s_1}$ (see [GR] or [Gra]).

As before, we consider $\tilde{a}$ defined in (4.1). Fix Q such that $\tilde{a}(Q) < \infty$. Set G to be the function given by (4.2) and Ω to be the level set of MG at level $e^{s'_1(1+\log C_w)}$. Applying Theorem 4.2 and Lemma 4.3, we cover Ω with the family of Whitney cubes $\{Q_i\}_i \subset \mathcal{D}_Q$. Take the functional $\varphi : [0, +\infty) \rightarrow [0, 1]$ given by

$$\varphi(t) = \sup_R \frac{w(E(R, t))}{w(R)},$$

where $E(R, t)$ is defined in (4.6). It is clear that $\varphi(t) \leq 1$ for all $t \geq 0$. Let $t > C_0$ with $C_0 = \max\{e^{s'_1(1+\log C_w)}, c_0\}$ and c_0 the constant in Proposition 4.4. Arguing as in (4.8), using (4.12) and $|\Omega| < e^{-s'_1(1+\log C_w)} |Q|$, we obtain

$$\begin{aligned} w(E(Q, t)) &\leq \sum_{i: Q_i \subset Q} w(E(Q_i, t - C_0)) \leq \varphi(t - C_0) \sum_{i: Q_i \subset Q} w(Q_i) \\ &= \varphi(t - C_0) w\left(\bigcup_{i: Q_i \subset Q} Q_i\right) \\ &\leq C_w \varphi(t - C_0) w(Q) \left(\frac{|\bigcup_{i: Q_i \subset Q} Q_i|}{|Q|}\right)^{1/s'_1} \\ &< e^{-1} \varphi(t - C_0) w(Q). \end{aligned}$$

Thus, $\varphi(t) < e^{-1} \varphi(t - C_0)$ for $t > C_0$. Iterating the previous estimate, we obtain $\varphi(t) \leq e^{-(t/C_0 - 1)}$ when $t \geq 0$. As in the proof of Theorem 1.1, the resulting estimate of the functional φ allows us to complete the proof of (1.7).

Part 2. Exponential self-improvement of Poincaré inequalities in spaces of homogeneous type

5. PRELIMINARIES

5.1. Spaces of homogeneous type. For full details and references we refer the reader to [CW] and [Chr].

Let (X, d, μ) be a space of homogeneous type. That is, X is a set equipped with a quasi-metric d and a non-negative Borel measure μ satisfying the doubling condition

$$\mu(B(x, 2r)) \leq c_\mu \mu(B(x, r)) < \infty$$

for some $c_\mu \geq 1$, uniformly for all $x \in X$ and $r > 0$. The doubling property implies that for some $c_\mu, n > 0$ and for all $x, y \in X$, $r > 0$ and $\lambda \geq 1$,

$$(5.1) \quad \mu(B(x, \lambda r)) \leq c_\mu \lambda^n \mu(B(x, r))$$

and also

$$(5.2) \quad \mu(B_2) \leq c_\mu \left(\frac{r(B_2)}{r(B_1)}\right)^n \mu(B_1)$$

for all balls B_1 and B_2 with $B_1 \subsetneq B_2$.

In order to simplify the computations, we assume that X is unbounded and therefore $\mu(X) = \infty$.

Let us recall that d being a quasi-metric on X means that d is a function from $X \times X$ to $[0, +\infty)$ satisfying the same conditions as a metric, except for the triangle inequality that is weakened to

$$d(x, y) \leq D_0 (d(x, z) + d(z, y))$$

for all $x, y, z \in X$ and where $1 \leq D_0 < \infty$ is a constant independent of x, y, z . Unfortunately, when $D_0 > 1$, it does not follow in general that the balls are open sets. However, Macías and Segovia ([MS]) proved that given any quasi-metric d , there exists another quasi-metric d' equivalent to d such that the metric balls defined with respect to d' are open. Thus, without loss of generality, from now on we assume that the metric balls are open sets.

We use the same notation as in the first part of the paper. We replace the Lebesgue measure by the doubling and non-negative Borel measure μ and the infinity distance by the quasi-metric d . Given a ball $B \subset X$ and $\lambda > 0$, we denote by λB the ball concentric with B so that $r(\lambda B) = \lambda r(B)$ and by x_B the center of B and λB .

5.1.1. Approximations of the identity and semigroups. We work with families of linear operators $\{S_t\}_{t>0}$ with the established properties in the first part, with the difference that now (in place of (2.1)) we assume

$$(5.3) \quad |s_t(x, y)| \leq \frac{1}{\mu(B(x, t^{1/m}))} g\left(\frac{d(x, y)^m}{t}\right).$$

As before, for any ball B we write $t_B = r(B)^m$. Next, as in [DY], we define $\mathcal{M} = \bigcup_{x \in X} \bigcup_{\beta > 0} \mathcal{M}_{(x, \beta)}$, where $\mathcal{M}_{(x, \beta)}$ is the set of measurable functions f such that

$$\|f\|_{\mathcal{M}_{(x, \beta)}} = \int_X \frac{|f(y)|}{(1 + d(x, y))^{2n+\beta} \mu(B(x, 1 + d(x, y)))} d\mu(y) < \infty.$$

In [DY], the authors establish that $(\mathcal{M}_{(x, \beta)}, \|\cdot\|_{\mathcal{M}_{(x, \beta)}})$ is a Banach space. Moreover, if $f \in \mathcal{M}$, then, for all $t, s > 0$, $S_t f$ and $S_s(S_t f)$ are well defined and finite almost everywhere.

5.1.2. Functionals. We consider $a : \mathcal{B} \times \mathcal{F} \rightarrow [0, +\infty)$, where $\mathcal{B}$ is the family of balls in X and $\mathcal{F}$ is a certain family of functions. We say that a is quasi-increasing or satisfies the T_∞ condition ($a \in T_\infty$) if $a(B_1) \leq C_a a(B_2)$ when $B_1 \subset B_2$. The infimum of the positive constants C_a is denoted by $\|a\|_{T_\infty}$ ($\|a\|_{T_\infty} \geq 1$). If $\|a\|_{T_\infty} = 1$, a is said to be increasing.

5.1.3. Weights. A weight w is a non-negative μ -locally integrable function. For any measurable set E , we write $w(E) = \int_E w(x) d\mu(x)$. Also, we set

$$\int_Q f dw = \int_Q f(x) dw(x) = \frac{1}{w(Q)} \int_Q f(x) w(x) d\mu(x).$$

The definitions and properties of weights that appear in the first part can be extended to this setting, replacing the Lebesgue measure by μ (see [ST]).

6. MAIN RESULT

As mentioned above, the goal of this part is to obtain the following extension of Theorem 1.1. In what follows, $\sigma > 1$ is a sufficiently large fixed parameter (see Theorem 8.1 below).

Theorem 6.1. *Let $\{S_t\}_{t>0}$ be as above and $a \in T_\infty$. Let $f \in \mathcal{M}$ be such that*

$$(6.1) \quad \int_B |f - S_{t_B} f| d\mu \leq a(B)$$

for all balls B . Then,

$$(6.2) \quad \|f - S_{t_B} f\|_{\exp L, B} \leq C \sum_{k \geq 0} \sigma^{nk} g(c \sigma^{mk}) a(\sigma^k B)$$

holds for all balls B . Furthermore, for all $w \in A_\infty$, we have

$$(6.3) \quad \|f - S_{t_B} f\|_{\exp L(w), B} \leq C \sum_{k \geq 0} \sigma^{nk} g(c \sigma^{mk}) a(\sigma^k B).$$

Notice that $C \geq 1$ and $0 < c < 1$. If a is doubling, then $a(\sigma B) \lesssim a(B)$ for all B , and we can write $a(B)$ on the right-hand side of the previous estimates.

7. APPLICATIONS

In what follows, $\sigma > 1$ is the sufficiently large parameter that appears in Theorem 8.1 below.

Example 3 (BMO and Morrey-Campanato spaces). As in [DY] and [Tan], given $\alpha \geq 0$ and $\{S_t\}_{t>0}$ as above (these authors only consider the semigroup case), the space of Morrey-Campanato $L_S(\alpha)$ is defined as follows:

$$L_S(\alpha) = \left\{ f \in \mathcal{M} : \sup_B \frac{1}{\mu(B)^\alpha} \int_B |f - S_{t_B} f| d\mu < \infty \right\}.$$

When $\alpha = 0$ this space coincides with BMO_S .

Applying Theorem 6.1 with $a(B) = \mu(B)^\alpha$ (which is increasing and doubling), we have that every $f \in L_S(\alpha)$, $\alpha \geq 0$, satisfies

$$\|f - S_{t_B} f\|_{\exp L, B} \lesssim \mu(B)^\alpha$$

for all balls B . Also, a weighted version of this inequality holds.

As in Example 1, one can also obtain in this setting self-improving results for $f \in BMO_{\varphi, S}(\mu)$ (where $BMO_{\varphi, S}(\mu)$ is defined as there, replacing the Lebesgue measure by μ).

Example 4 (Poincaré-Sobolev inequality). To avoid some technicalities, we assume that all annuli are non-empty. Let $p \geq 1$. A space of homogeneous type (X, d, μ) is said to support a generalized L^p -Poincaré inequality if there exists $\mathcal{M}_0 \subset \mathcal{M}$ such that every $f \in \mathcal{M}_0$ satisfies

$$(7.1) \quad \int_B |f - S_{t_B} f| d\mu \leq \sum_{k \geq 0} \alpha(k) r(\sigma^k B) \left(\int_{\sigma^k B} h^p dx \right)^{1/p}$$

for all balls $B \subset X$, where $\{\alpha(k)\}_{k \geq 0}$ is a sequence of non-negative numbers and h is some non-negative measurable function.

Theorem 6.1 yields that if a space of homogeneous type supports a generalized L^p -Poincaré inequality with $p \geq n$ (where n is the constant that appears in (5.1) and (5.2)), then

$$\|f - S_{t_B} f\|_{\exp L, B} \lesssim \sum_{k \geq 0} \hat{\alpha}(k) r(\sigma^k B) \left(\int_{\sigma^k B} h^p d\mu \right)^{1/p}$$

holds for all balls $B \subset X$, where $\hat{\alpha}(k) = \sum_{0 \leq l \leq k} \sigma^{nl} g(\sigma^{m(l-10)}) \alpha(k-l)$, for all $k \geq 0$.

To apply Theorem 6.1, it is enough to show that if $B_1 \subset B_2$, then

$$\frac{\mu(B_2)}{\mu(B_1)} \lesssim \left(\frac{r(B_2)}{r(B_1)} \right)^p$$

as this implies that $a(B) = \sum_{k \geq 0} \alpha(k) r(\sigma^k B) \left(\int_{\sigma^k B} h^p dx \right)^{1/p} \in T_\infty$. As μ is doubling and $p \geq n$, we obtain

$$\frac{\mu(B_2)}{\mu(B_1)} \leq c_\mu \left(\frac{r(B_2)}{r(B_1)} \right)^n \leq c_\mu \left(\frac{r(B_2)}{r(B_1)} \right)^p.$$

These inequalities appear naturally in some applications. For instance, following the notation in [ACDH], let M be a non-compact Riemannian manifold with d its geodesic distance and μ its volume form, which is assumed to be doubling. Let ∇ and Δ be respectively the associated Riemannian gradient and Laplace-Beltrami operator. Assume that the heat kernel $p_t(x, y)$ of the semigroup $e^{-t\Delta}$ has Gaussian upper bounds, that is, (5.3) holds with $g(r) = Ce^{-cr}$ and $m = 2$. Then all our results apply in this setting. We notice that without assuming any classical Poincaré inequality, one can prove estimates such as (7.1) with $h = |\nabla f|$ or $h = |\sqrt{L}|$. For the first one, we take $S_t = e^{-t\Delta}$ and impose that $\sqrt{t} \nabla e^{-t\Delta}$ satisfies L^p off-diagonal estimates. In that case one shows (7.1) with $h = |\nabla f|$ and $\alpha(k) = e^{-c4^k}$. On the other hand, taking $S_t = I - (I - e^{-t\Delta})^m$ with m large enough one can obtain (7.1) with $h = |\sqrt{L}f|$ and $\alpha(k) = 2^{-k\theta_m}$ with $\theta_m > 0$ (by taking m large, we get more decay on $\alpha(k)$). For more details we refer the reader to [BJM].

8. PROOF OF THEOREM 6.1

We follow the proof of Theorem 1.1. We prove (6.3) as (6.2), using the same ideas.

8.1. Step I: Dyadic case. First, we consider the following dyadic structure:

Theorem 8.1 ([Chr]). *There exist $\sigma > 4D_0^3 > 1$ large enough, positive and finite constants c_1, C_1, C_2 and a countable collection $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$ of open sets Q (we will refer to Q as dyadic cubes and to $\mathcal{D}_k$ as the k -th generation of $\mathcal{D}$, for all $k \in \mathbb{Z}$) with the following properties:*

- (i) $\mathcal{D}_k$ is a countable collection of disjoint sets such that $X = \bigcup_{Q \in \mathcal{D}_k} Q$ μ -a.e.
- (ii) Each $Q \in \mathcal{D}_k$ satisfies $\text{diam}(Q) \leq C_1 \sigma^k$, for all $k \in \mathbb{Z}$.
- (iii) For each $Q \in \mathcal{D}_k$, there exist $x_Q \in Q$ and balls $B_Q = B(x_Q, c_1 \sigma^k)$ and $\hat{B}_Q = B(x_Q, C_1 \sigma^k)$ such that $B_Q \subset Q \subset \hat{B}_Q$, for all $k \in \mathbb{Z}$.
- (iv) If $Q_1 \in \mathcal{D}_{k_1}$ and $Q_2 \in \mathcal{D}_{k_2}$ with $k_1 \leq k_2$, then either $Q_1 \cap Q_2 \neq \emptyset$ or $Q_1 \subseteq Q_2$.

These sets are the analogs of the Euclidean dyadic cubes. It may help to think of Q as being essentially a cube of side-length σ^k and with center x_Q . The previous result (with a different notation) appears in [Chr].

From now on, we fix $\sigma > 4D_0^3$ large enough and take the dyadic structure given by Theorem 8.1. We will use the following decomposition of X in dyadic annuli $X = \bigcup_{k \geq 1} C_k(Q)$ with $C_1(Q) = \sigma \hat{B}_Q$ and $C_k(Q) = \sigma^k \hat{B}_Q \setminus \sigma^{k-1} \hat{B}_Q$, $k \geq 2$ several times. We state the following version of the Whitney covering lemma.

Theorem 8.2. *Given Ω a proper open subset of X , there is a family of Whitney cubes $\{Q_i\}_i$ such that*

- (a) $\Omega = \bigcup_i Q_i$, μ -almost everywhere.
- (b) $\{Q_i\}_i \subset \mathcal{D}$ and these cubes are maximal with respect to the inclusion and pairwise disjoint.
- (c) $0 < (\sigma - 1) r(\hat{B}_{Q_i}) < d(Q_i, \Omega^c) \leq \sigma^2 r(\hat{B}_{Q_i})$ and $(C_1/c_1) \sigma^3 B_{Q_i} \cap \Omega^c \neq \emptyset$.

If Ω is a level set of the maximal Hardy-Littlewood function, the following result holds.

Lemma 8.3. *Given $t > 0$ and $G \in L^1(X)$, if $\Omega = \{x \in X : MG(x) > t\}$ and $\{Q_i\}_i$ is its family of Whitney cubes, then, for every $k \geq 1$, we have*

$$\int_{\sigma^k \hat{B}_{Q_i}} G d\mu \leq c_\mu (C_1/c_1)^n \sigma^{3n} t.$$

After these preliminaries, in this part of the proof, we show that for every $1 \leq \tau < \sigma^m$ and for every $Q \in \mathcal{D}$,

$$\|f - S_{\tau t_{\hat{B}_Q}} f\|_{\exp L, Q} \lesssim \sum_{k \geq 0} \sigma^{nk} g(\sigma^{m(k-8)}) a(\sigma^k \hat{B}_Q).$$

As before, it is enough to assume that $\|a\|_{T_\infty} = 1$. We present an auxiliary result; its proof is postponed until Section 8.3.

Lemma 8.4. *Under the hypotheses of Theorem 6.1, for every $1 \leq \tau < \sigma^m$, $k \geq 0$ and $R \in \mathcal{D}$, we have*

$$\int_{\sigma^k \hat{B}_R} |f - S_{\tau t_{\hat{B}_R}} f| d\mu \leq \|a\|_{T_\infty} c_\mu^2 (C_1/c_1)^n \sigma^{2n} a(\sigma^{2+k} \hat{B}_R).$$

Consider the increasing functional $\tilde{a} : \mathcal{B} \times \mathcal{F} \rightarrow [0, +\infty)$ given by

$$(8.1) \quad \tilde{a}(B) = \sum_{k \geq 0} \sigma^{nk} g(\sigma^{m(k-6)}) a(\sigma^k B).$$

Fix $Q \in \mathcal{D}$ and assume that $\tilde{a}(\sigma^2 \hat{B}_Q) < \infty$. Thus, the following function,

$$(8.2) \quad G(x) = \frac{|f(x) - S_{\tau t_{\hat{B}_Q}} f(x)|}{\tilde{a}(\sigma^2 \hat{B}_Q)} \chi_{\sigma^2 \hat{B}_Q}(x),$$

is well defined. By Lemma 8.4 and by the doubling property of the measure, we have

$$\|G\|_{L^1(X)} \leq \frac{c_\mu^3 (C_1/c_1)^n \sigma^{2n}}{g(1)} \mu(\hat{B}_Q) < \infty.$$

Let $\lambda_0 = \frac{c_M c_\mu^4}{g(1)} \left(\frac{C_1}{c_1}\right)^{2n} \sigma^{4n} > 0$, where c_M is the constant of the weak type (1, 1) of M . We consider the level set

$$(8.3) \quad \Omega = \{x \in X : MG(x) > \lambda_0\}$$

and obtain

$$(8.4) \quad \mu(\Omega) \leq \frac{c_M}{\lambda_0} \|G\|_{L^1(X)} \leq \sigma^{-2n} \mu(Q).$$

By Theorem 8.2, Ω can be covered with the family of Whitney cubes $\{Q_i\}_i$ (here we use that Ω is open and $\Omega \subsetneq X$ as $\mu(X) = \infty$). Furthermore, if $Q_i \cap Q \neq \emptyset$, then, for every i , we have $\mu(Q_i) \leq \sigma^{-2n} \mu(Q)$, $Q_i \subsetneq Q$,

$$(8.5) \quad r(\hat{B}_{Q_i}) \leq \sigma^{-2} r(\hat{B}_Q) \quad \text{and} \quad \sigma^2 \hat{B}_{Q_i} \subset \sigma \hat{B}_Q.$$

The first claim follows as an immediate consequence of (8.4) and $Q_i \subset \Omega$. In particular, it implies that $Q_i \subset Q$ and $r(\hat{B}_{Q_i}) \leq r(\hat{B}_Q)$ (by (iv) in Theorem 8.1, since $Q_i, Q \in \mathcal{D}$ and $Q_i \cap Q \neq \emptyset$). From this, (8.4), (5.1), (5.2) and $\sigma \geq 4D_0^3$, we obtain

$$\begin{aligned} \mu(Q_i) &\leq \frac{c_M c_\mu^3 (C_1/c_1)^n \sigma^{2n}}{\lambda_0 g(1)} \mu(\hat{B}_Q) \leq \frac{c_M c_\mu^4 (C_1/c_1)^n \sigma^{2n}}{\lambda_0 g(1)} \left(\frac{r(\hat{B}_Q)}{r(\hat{B}_{Q_i})}\right)^n \mu(B_{Q_i}) \\ &\leq \frac{c_M c_\mu^4 (C_1/c_1)^{2n} \sigma^{2n}}{\lambda_0 g(1)} \left(\frac{r(\hat{B}_Q)}{r(\hat{B}_{Q_i})}\right)^n \mu(Q_i) \leq \sigma^{-2n} \left(\frac{r(\hat{B}_Q)}{r(\hat{B}_{Q_i})}\right)^n \mu(Q_i), \end{aligned}$$

which gives $r(\hat{B}_{Q_i}) \leq \sigma^{-2} r(\hat{B}_Q)$. The last claim in (8.5) follows from the previous one using that $\sigma \geq 4D_0^3$.

Next, for each $t > 0$ and $R \in \mathcal{D}$, we consider the set

$$(8.6) \quad E(R, t) = \{x \in R : |f(x) - S_{\tau t_{\hat{B}_R}} f(x)| > t \tilde{a}(\sigma^2 \hat{B}_R)\}.$$

Fix $t > \lambda_0$. Note that μ -almost everywhere we have

$$E(Q, t) = E(Q, t) \cap \{x \in Q : MG(x) > \lambda_0\} = \bigcup_i \{x \in Q_i \cap Q : G(x) > t\}.$$

From now on, we only consider those cubes Q_i such that $Q_i \cap Q \neq \emptyset$ and therefore $Q_i \subset Q$. Thus, we write

$$E(Q, t) = \bigcup_{i: Q_i \subset Q} \{x \in Q_i : |f(x) - S_{\tau t_{\hat{B}_Q}} f(x)| > t \tilde{a}(\sigma^2 \hat{B}_Q)\}.$$

Next, in order to replace $S_{\tau t_{\hat{B}_Q}} f$ by $S_{\tau t_{\hat{B}_{Q_i}}} f$, we use the following auxiliary result whose proof is in Section 8.3.

Proposition 8.5. *Under the hypotheses of Theorem 6.1, for all $x \in Q_i$, we have*

$$|S_{\tau t_{\hat{B}_{Q_i}}} f(x) - S_{\tau t_{\hat{B}_Q}} f(x)| \leq c_0 \tilde{a}(\sigma^2 \hat{B}_Q).$$

Fix $t > C_0$ with $C_0 = \max\{\lambda_0, c_0\}$. As $\tilde{a}(\sigma^2 \hat{B}_{Q_i}) \leq \tilde{a}(\sigma^2 \hat{B}_Q)$ (because $\tilde{a}$ is increasing and (8.5) holds), we obtain

$$\begin{aligned} (8.7) \quad \mu(E(Q, t)) &\leq \sum_{i: Q_i \subset Q} \mu(\{x \in Q_i : |f(x) - S_{\tau t_{\hat{B}_{Q_i}}} f(x)| > (t - C_0) \tilde{a}(\sigma^2 \hat{B}_{Q_i})\}) \\ &= \sum_{i: Q_i \subset Q} \mu(E(Q_i, t - C_0)). \end{aligned}$$

As before, we define $\varphi : [0, +\infty) \rightarrow [0, 1]$ given by

$$\varphi(t) = \sup_{R \in \mathcal{D}} \frac{\mu(E(R, t))}{\mu(R)}.$$

Note that $\varphi(t) \leq 1$ for all $t \geq 0$. This, together with (8.7), (8.4) and $\sigma > e$, implies that $\varphi(t) \leq e^{-2n} \varphi(t - C_0)$ for all $t > C_0$. Iterating, we get $\varphi(t) \leq e^{-2n(t/C_0 - 1)}$ for all $t \geq 0$.

To finish, we pick $A > C_0/(2n)$ and using the previous estimate, we conclude that

$$\int_Q \exp\left(\frac{|f - S_{\tau t_{\hat{B}_Q}} f|}{A \tilde{a}(\sigma^2 \hat{B}_Q)}\right) d\mu \leq \int_0^\infty e^t \frac{\mu(E(Q, At))}{\mu(Q)} dt \leq \int_0^\infty e^{t-2n(tA/C_0 - 1)} dt < \infty,$$

which leads to the desired estimate. $\square$

8.2. Step II: General case. Fix a ball B . Let $k_0 \in \mathbb{Z}$ be such that $C_1 \sigma^{k_0} \leq r(B) < C_1 \sigma^{k_0+1}$ and $\mathcal{I} = \{Q \in \mathcal{D}_{k_0} : Q \cap B \neq \emptyset\}$. Following the computations in Lemma 8.6 below, we get $\#\mathcal{I} \lesssim 1$ and μ -almost everywhere we can write

$$B \subset \bigcup_{Q \in \mathcal{I}} Q \subset \bigcup_{Q \in \mathcal{I}} \hat{B}_Q \subset \sigma B.$$

Also, each $Q \in \mathcal{I}$ satisfies $\mu(\hat{B}_Q) \approx \mu(B)$. Note that $\sigma^k \hat{B}_Q \subset \sigma^{k+1} B$ implies that $a(\sigma^k \hat{B}_Q) \lesssim a(\sigma^{k+1} B)$ for all $k \geq 0$. Since $t_B = \tau t_Q$ with $1 \leq \tau < \sigma^m$, from the preceding arguments and the Step I in the proof, we obtain

$$\begin{aligned} \|f - S_{t_B} f\|_{\exp L, B} &\lesssim \sum_{Q \in \mathcal{I}} \|f - S_{t_B} f\|_{\exp L, Q} \\ &\lesssim \sum_{Q \in \mathcal{I}} \sum_{k \geq 0} \sigma^{nk} g(\sigma^{m(k-8)}) a(\sigma^k \hat{B}_Q) \\ &\lesssim \sum_{k \geq 0} \sigma^{nk} g(\sigma^{m(k-9)}) a(\sigma^k B). \end{aligned} \quad \square$$

8.3. Proofs of the auxiliary results.

Proof of Theorem 8.2. We put $\Omega = \bigcup_{k \in \mathbb{Z}} \Omega_k$ with

$$\Omega_k = \{x \in \Omega : C_1 D_0 \sigma^k < d(x, \Omega^c) \leq C_1 D_0 \sigma^{k+1}\}.$$

Let us consider the family $\mathcal{F}$ given by

$$\mathcal{F} = \bigcup_{k \in \mathbb{Z}} \{R \in \mathcal{D}_k : R \cap \Omega_{k+1} \neq \emptyset\}.$$

We see that the cubes of $\mathcal{F}$ have the desired properties. First, let us check that they satisfy (c). If $R \in \mathcal{F}$, then, for some $k \in \mathbb{Z}$, $R \in \mathcal{D}_k$ and $R \cap \Omega_{k+1} \neq \emptyset$. Thus, there is some $x \in R \cap \Omega_{k+1}$ and

$$C_1 D_0 \sigma^{k+1} < d(x, \Omega^c) \leq D_0 (d(R, \Omega^c) + \text{diam}(R)) \leq D_0 (d(R, \Omega^c) + C_1 \sigma^k),$$

which implies that $d(R, \Omega^c) > C_1 \sigma^k (\sigma - 1)$. Besides,

$$d(R, \Omega^c) \leq d(x, \Omega^c) \leq C_1 D_0 \sigma^{k+1}.$$

Consequently,

$$(8.8) \quad 0 < (\sigma - 1) r(\hat{B}_R) < d(R, \Omega^c) \leq \sigma^2 r(\hat{B}_R).$$

Next, we see that $(C_1/c_1) \sigma^3 B_R \cap \Omega^c \neq \emptyset$. The fundamental property of the infimum implies that there is $y_0 \in \Omega^c$ such that

$$d(y_0, R) < d(R, \Omega^c) + r(\hat{B}_R).$$

Then, we use the previous inequality and (8.8) to obtain

$$\begin{aligned} d(y_0, x_R) &\leq D_0 (d(y_0, R) + r(\hat{B}_R)) < D_0 (d(R, \Omega^c) + 2 r(\hat{B}_R)) \\ &\leq D_0 (\sigma^2 + 2) r(\hat{B}_R) \leq (C_1/c_1) \sigma^3 r(B_R), \end{aligned}$$

which gives $y_0 \in (C_1/c_1) \sigma^3 B_R \cap \Omega^c$.

We now show that $\Omega = \bigcup_{R \in \mathcal{F}} R$ μ -almost everywhere. Note that (c) implies that the distance of each cube $R \in \mathcal{F}$ to Ω^c is strictly positive; therefore, $\bigcup_{R \in \mathcal{F}} R \subset \Omega$. The other inclusion is a consequence of $\Omega = \bigcup_k \Omega_k$ together with the fact that each generation of dyadic cubes provides a μ -almost everywhere partition of the space.

The problem is that the collection $\mathcal{F}$ consists of cubes that are not necessarily pairwise disjoint. In order to solve it, we select the maximal cubes of $\mathcal{F}$, eliminating those that are contained elsewhere in the collection. That is, for each cube $R \in \mathcal{F}$ select the unique maximal cube $R^{\max} \in \mathcal{F}$ which contains R . Such a cube exists since otherwise there exists an infinite increasing sequence of cubes $\{R_k\}_{k \geq 1} \subset \mathcal{F}$ containing R . As these cubes are in $\mathcal{F}$, they satisfy (c). In particular, for every $k \geq 1$ we have

$$r(\hat{B}_{R_k}) < \frac{1}{\sigma - 1} d(R_k, \Omega^c) \leq \frac{1}{\sigma - 1} d(R, \Omega^c) \leq \frac{\sigma^2}{\sigma - 1} r(\hat{B}_R)$$

and this leads to a contradiction. Therefore, such maximal cubes do exist and $\{Q_i\}_i$ is precisely the family of all the maximal cubes. Note that by maximality these cubes are pairwise disjoint. $\square$

Proof of Lemma 8.3. Recall that Ω is open and $\Omega \subsetneq X$ as $\mu(X) = \infty$. So Ω is a proper open set of X . Therefore, applying Theorem 8.2, Ω can be covered with the family of Whitney cubes $\{Q_i\}_i$ and there exists $z \in (C_1/c_1) \sigma^3 \hat{B}_{Q_i} \cap \Omega^c$. Then, if $k \geq 1$ is such that $\sigma^k \leq (C_1/c_1) \sigma^3$, using that μ is a doubling measure, we write

$$\int_{\sigma^k \hat{B}_{Q_i}} G d\mu \leq c_\mu (C_1/c_1)^n \sigma^{(3-k)n} MG(z) \leq c_\mu (C_1/c_1)^n \sigma^{2n} t.$$

Otherwise, when $\sigma^k > (C_1/c_1) \sigma^3$ we have

$$\int_{\sigma^k \hat{B}_{Q_i}} G d\mu \leq MG(z) \leq t. \quad \square$$

Lemma 8.6. *Let $R \in \mathcal{D}_{k_0}$ for some $k_0 \in \mathbb{Z}$, and set $\mathcal{J}_k = \{Q \in \mathcal{D}_{k_0} : Q \cap \sigma^k \hat{B}_R \neq \emptyset\}$ with $k \geq 0$. Then*

$$(8.9) \quad \sigma^k \hat{B}_R \subset \bigcup_{Q \in \mathcal{J}_k} Q \subset \bigcup_{Q \in \mathcal{J}_k} \hat{B}_Q \subset \sigma^{k+1} \hat{B}_R, \quad \mu\text{-a.e.},$$

and

$$(8.10) \quad \#\mathcal{J}_k \leq c_\mu \sigma^{(k+2)n} (C_1/c_1)^n.$$

Proof. Note that (8.9) follows easily from Theorem 8.1. It is trivial to see that for every $Q \in \mathcal{J}_k$ we have $\sigma^{k+1} \hat{B}_R \subset \sigma^{k+2} \hat{B}_Q$. Then, all these give

$$\begin{aligned} \mu(\sigma^{k+1} \hat{B}_R) \# \mathcal{J}_k &\leq \sum_{Q \in \mathcal{J}_k} \mu(\sigma^{k+2} \hat{B}_Q) \leq c_\mu \sigma^{(k+2)n} (C_1/c_1)^n \sum_{Q \in \mathcal{J}_k} \mu(Q) \\ &\leq c_\mu \sigma^{(k+2)n} (C_1/c_1)^n \mu\left(\bigcup_{Q \in \mathcal{J}_k} Q\right) \leq c_\mu \sigma^{(k+2)n} (C_1/c_1)^n \mu(\sigma^{k+1} \hat{B}_R), \end{aligned}$$

and this readily implies (8.10). $\square$

Proof of Lemma 8.4. This follows as a consequence of Lemma 8.6. Fix $R \in \mathcal{D}_{k_0}$ for some $k_0 \in \mathbb{Z}$, $k \geq 0$ and $1 \leq \tau < \sigma^m$. Then, by (8.9) and the fact that a is increasing, we obtain

$$\begin{aligned} \int_{\sigma^k \hat{B}_R} |f - S_{\tau t_{\hat{B}_R}} f| d\mu &\leq \sum_{Q \in \mathcal{J}_k} \int_{\tau^{1/m} \hat{B}_Q} |f - S_{t_{\tau^{1/m} \hat{B}_Q}} f| d\mu \\ &\leq \sum_{Q \in \mathcal{J}_k} a(\tau^{1/m} \hat{B}_Q) \mu(\tau^{1/m} \hat{B}_Q) \\ &\leq \|a\|_{T_\infty} c_\mu (C_1/c_1)^n \sigma^n a(\sigma^{2+k} \hat{B}_R) \sum_{Q \in \mathcal{J}_k} \mu(Q) \\ &\leq \|a\|_{T_\infty} c_\mu^2 (C_1/c_1)^n \sigma^{2n} a(\sigma^{2+k} \hat{B}_R) \mu(\sigma^k \hat{B}_R). \quad \square \end{aligned}$$

Proof of Proposition 8.5. We follow the proof in the Euclidean setting, so we skip many details. The commutation rule implies that for every $x \in Q_i$,

$$\begin{aligned} |S_{\tau t_{\hat{B}_{Q_i}}} f(x) - S_{\tau t_{\hat{B}_Q}} f(x)| &\leq |S_{\tau t_{\hat{B}_{Q_i}}} (f - S_{\tau t_{\hat{B}_Q}} f)(x)| + |S_{\tau t_{\hat{B}_Q}} (f - S_{\tau t_{\hat{B}_{Q_i}}} f)(x)| \\ &= I + II. \end{aligned}$$

We study each term in turn. First we pick $k_i \in \mathbb{Z}$ such that

$$(8.11) \quad \sigma^{k_i} r(\hat{B}_{Q_i}) \leq r(\hat{B}_Q) < \sigma^{k_i+1} r(\hat{B}_{Q_i}).$$

Then, by (8.5) one can see that

$$(8.12) \quad k_i \geq 2 \quad \text{and} \quad \sigma^{k_i} \hat{B}_{Q_i} \subset \sigma^2 \hat{B}_Q.$$

Next, for I , we observe that as a consequence of (8.12), if $y \in \sigma^{k_i} \hat{B}_{Q_i}$, $|f(y) - S_{\tau t_{\hat{B}_Q}} f(y)| = G(y) \tilde{a}(\sigma^2 \hat{B}_Q)$ and so, arguing as in (4.11), we obtain

$$\begin{aligned} (8.13) \quad I &\leq \frac{\tilde{a}(\sigma^2 \hat{B}_Q)}{\mu(B(x, \tau^{1/m} r(\hat{B}_{Q_i})))} \int_X g\left(\frac{d(x, y)}{\tau^{1/m} r(\hat{B}_{Q_i})}\right)^m G(y) d\mu(y) \\ &\quad + \frac{1}{\mu(B(x, \tau^{1/m} r(\hat{B}_{Q_i})))} \int_{(\sigma^{k_i} \hat{B}_{Q_i})^c} g\left(\frac{d(x, y)}{\tau^{1/m} r(\hat{B}_{Q_i})}\right)^m |f(y) - S_{\tau t_{\hat{B}_Q}} f(y)| d\mu(y) \\ &= I_1 \tilde{a}(\sigma^2 \hat{B}_Q) + I_2. \end{aligned}$$

In order to estimate I_1 and I_2 , we decompose X as the union of dyadic annuli $\{C_k(Q_i)\}_{k \geq 1}$. Then, if $x \in Q_i$ and $y \in C_k(Q_i)$, we have

$$\left(\frac{d(x, y)}{\tau^{1/m} r(\hat{B}_{Q_i})}\right)^m \geq \lambda_k, \quad \text{where} \quad \lambda_k = \begin{cases} 0, & \text{if } k = 1, \\ \sigma^{m(k-4)}, & \text{if } k \geq 2. \end{cases}$$

On the other hand, note that $\sigma^k \hat{B}_{Q_i} \subset B(x, \sigma^{k+1} r(\hat{B}_{Q_i}))$ and, by (5.1) and Lemma 8.3, we get

$$I_1 \leq \sum_{k \geq 1} g(\lambda_k) \frac{\mu(\sigma^k \hat{B}_{Q_i})}{\mu(B(x, \tau^{1/m} r(\hat{B}_{Q_i})))} \int_{\sigma^k \hat{B}_{Q_i}} G d\mu \lesssim \lambda_0 \sum_{k \geq 1} g(\lambda_k) \sigma^{nk} \lesssim \lambda_0.$$

We continue with I_2 . Observe that $Q_i \subset Q$, and (8.11) and (8.12) imply that for every $k \geq k_i + 1$,

$$C_k(Q_i) \subset \sigma^k \hat{B}_{Q_i} \subset \sigma^{k-k_i+1} \hat{B}_Q \subset B(x, \sigma^{k+3} r(\hat{B}_{Q_i})).$$

Then, using the previous observations, the facts that μ is doubling and $a \in T_\infty$ and Lemma 8.4, we write

$$\begin{aligned} I_2 &\leq \frac{1}{\mu(B(x, \tau^{1/m} r(\hat{B}_{Q_i})))} \sum_{k \geq k_i+1} g(\lambda_k) \int_{\sigma^{k-k_i+1} \hat{B}_Q} |f - S_{\tau t_{\hat{B}_Q}} f| d\mu \\ &\lesssim \sum_{k \geq k_i+1} g(\lambda_k) \sigma^{nk} a(\sigma^{k-k_i+3} \hat{B}_Q) \lesssim \tilde{a}(\sigma^2 \hat{B}_Q). \end{aligned}$$

Plugging these into (8.13), we get $I \lesssim \tilde{a}(\sigma^2 \hat{B}_Q)$.

Next, we obtain the same estimate for II . Notice that if $k \geq k_i + 1$, by (8.11),

$$\sigma^k \hat{B}_{Q_i} \subset B(x, \sigma^{k-k_i+1} r(\hat{B}_Q)) \quad \text{and} \quad \sigma^{k+2} \hat{B}_{Q_i} \subset \sigma^{k-k_i+3} \hat{B}_Q.$$

We use these inclusions, the fact that μ is doubling, Lemma 8.4, (8.11) and (8.12) and we have

$$\begin{aligned} II &\leq \frac{g(0)}{\mu(B(x, \tau^{1/m} r(\hat{B}_Q)))} \int_{\sigma^{k_i+1} \hat{B}_{Q_i}} |f - S_{\tau t_{\hat{B}_{Q_i}}} f| d\mu \\ &\quad + \frac{1}{\mu(B(x, \tau^{1/m} r(\hat{B}_Q)))} \sum_{k \geq k_i+2} g\left(\lambda_k \frac{t_{\hat{B}_{Q_i}}}{t_{\hat{B}_Q}}\right) \int_{\sigma^k \hat{B}_{Q_i}} |f - S_{\tau t_{\hat{B}_{Q_i}}} f| d\mu \\ &\lesssim a(\sigma^4 \hat{B}_Q) + \sum_{k \geq k_i+2} g(\sigma^{m(k-k_i-5)}) \sigma^{n(k-k_i)} a(\sigma^{k-k_i+3} \hat{B}_Q) \lesssim \tilde{a}(\sigma^2 \hat{B}_Q). \quad \square \end{aligned}$$

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