SIGN CHANGES OF THE ERROR TERM IN WEYL’S LAW FOR HEISENBERG MANIFOLDS

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ABSTRACT. Let $R(T)$ be the error term in Weyl’s law for the $(2l + 1)$ -dimensional Heisenberg manifold $(H_l/\Gamma, g_l)$. In this paper, several results on the sign changes and odd moments of $R(t)$ are proved. In particular, it is proved that for some sufficiently large constant c , $R(t)$ changes sign in the interval $[T, T + c\sqrt{T}]$ for all large T . Moreover, for a small constant c_1 there exist infinitely many subintervals in $[T, 2T]$ of length $c_1\sqrt{T}\log^{-5}T$ such that $\pm R(t) > c_1t^{l-1/4}$ holds on each of these subintervals.

1. INTRODUCTION

Let (M, g) be a closed n -dimensional Riemannian manifold with metric g and Laplace-Beltrami operator Δ . Let $N(t)$ denote its spectral counting function, which is defined as the number of the eigenvalues of Δ not exceeding t . Hörmander [13] proved that Weyl’s law

$$(1.1) \quad N(t) = \frac{\text{vol}(B_n)\text{vol}(M)}{(2\pi)^n}t^{n/2} + O(t^{(n-1)/2})$$

holds, where $\text{vol}(B_n)$ is the volume of the n -dimensional unit ball.

Let

$$R(t) = N(t) - \frac{\text{vol}(B_n)\text{vol}(M)}{(2\pi)^n}t^{n/2}.$$

Hörmander’s estimate (1.1) in general is sharp, as the well-known example of the sphere S^n with its canonical metric shows [13]. However, it is a very difficult problem to determine the optimal bound of $R(t)$ in any given manifold, which depends on the properties of the associated geodesic flow. Many improvements have been obtained for certain types of manifolds; see [1, 2, 3, 4, 7, 10, 14, 17, 20, 22, 25, 29, 30, 31].

1.1. Weyl’s law for $\mathbb{T}^2$: The Gauss circle problem. The simplest compact manifold with integrable geodesic flow is the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The exponential functions $e(mx + ny)$, $m, n \in \mathbb{Z}$ form a basis of eigenfunctions of the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$, which acts on functions on $\mathbb{T}^2$. The corresponding eigenvalues are $4\pi^2(m^2 + n^2)$, $m, n \in \mathbb{Z}$. The spectral counting function

$$N_I(t) = \{\lambda_j \in \text{Spec}(\Delta) : \lambda_j \leq t\}$$

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is equal to the number of lattice points of $\mathbb{Z}^2$ inside a circle of radius $\sqrt{t}/2\pi$. The well-known Gauss circle problem is the study of the properties of the error term of the function $N_I(t)$.

In this case, formula (1.1) becomes

$$(1.2) \quad N_I(t) = \frac{t}{4\pi} + O(t^{1/2}),$$

which is the classical result of Gauss. Let $R_I(t)$ denote the error term in (1.2). Many authors improved the upper bound estimate of $R_I(t)$. The latest result is due to Huxley [14], which reads

$$(1.3) \quad R_I(t) \ll t^{131/416} \log^{26947/8320} t.$$

Hardy [11] conjectured that

$$(1.4) \quad R_I(t) \ll t^{1/4+\varepsilon}.$$

Cramér [5] proved that

$$\lim_{T \rightarrow \infty} T^{-3/2} \int_1^T |R_I(t)|^2 dt = C, \quad C = \frac{1}{6\pi^3} \sum_{n=1}^{\infty} \frac{r^2(n)}{n^{3/2}},$$

which is consistent with Hardy's conjecture. Here $r(n)$ denotes the number of ways in which n can be written as a sum of two squares.

Ivić [15] first used the large value technique to study the higher power moments of $R_I(t)$. He proved that the estimate

$$(1.5) \quad \int_1^T |R_I(t)|^A dt \ll T^{1+A/4+\varepsilon}$$

holds for each fixed $0 \leq A \leq 35/4$. The value of A for which (1.5) holds is closely related to the upper bound of $R_I(t)$. If we insert the estimate (1.3) into Ivić's machinery, we get that (1.5) holds for $0 \leq A \leq 262/27$.

The first author [28] studied the third and the fourth moments of $R_I(t)$. He proved the following two asymptotic formulas:

$$(1.6) \quad \int_1^T R_I^3(t) dt = c_3 T^{7/4} + O(T^{7/4-1/14+\varepsilon}),$$

$$(1.7) \quad \int_1^T R_I^4(t) dt = c_4 T^2 + O(T^{2-1/23+\varepsilon}),$$

where c_3 and c_4 are explicit constants.

In [30], the second author proved by a unified method that the asymptotic formula

$$(1.8) \quad \int_1^T R_I^k(t) dt = c_k T^{1+k/4} + O(T^{1+k/4-\delta_k+\varepsilon})$$

holds for $3 \leq k \leq 9$, where c_k and $\delta_k > 0$ are explicit constants.

1.2. Weyl's law for $(2l+1)$ -dimensional Heisenberg manifold. Let $l \geq 1$ be a fixed integer and $(H_l/\Gamma, g)$ be a $(2l+1)$ -dimensional Heisenberg manifold with a metric g . When $l = 1$, Petridis and Toth [25] proved that $R(t) = O(t^{5/6} \log t)$ for a special metric. Later in [4] this bound was improved to $O(t^{119/146+\varepsilon})$ for all left-invariant Heisenberg metrics. For $l > 1$ Khosravi and Petridis [20] proved that $R(t) = O(t^{l-7/41})$ holds for rational Heisenberg manifolds. In both [4] and [20] they

first established a ψ -expression of $R(t)$ and then used the van der Corput method of exponential sums. Substituting Huxley's result of [14] into the arguments of [4] and [20], we can get that the estimate

$$(1.9) \quad R(t) = O(t^{l-77/416}(\log t)^{26947/8320})$$

holds for all rational $(2l+1)$ -dimensional Heisenberg manifolds.

It was conjectured that for rational Heisenberg manifolds, the pointwise estimate

$$(1.10) \quad R(t) \ll t^{l-1/4+\varepsilon}$$

holds, which was proposed in Petridis and Toth [25] for the case $l = 1$ and in Khosravi and Petridis [20] for the case $l > 1$. As an evidence of this conjecture, Petridis and Toth proved the following L^2 result:

$$\int_{I^3} \left| N(t; \vec{u}) - \frac{1}{6\pi^2} \text{vol}(M(\vec{u})) t^{3/2} \right|^2 d\vec{u} \leq C_\delta t^{3/2+\delta}$$

for the 3-dimensional Heisenberg manifold H_1 , where $N(t; \vec{u})$ is the counting function for H_1 with the metric

$$g(\vec{u}) = \begin{pmatrix} u_1^{-1} & 0 & 0 \\ 0 & u_2^{-1} & 0 \\ 0 & 0 & u_3^{-1} \end{pmatrix}$$

for any $\vec{u} = (u_1, u_2, u_3) \in I^3$, and $I = [1 - \varepsilon, 1 + \varepsilon]$. They also proved

$$\frac{1}{T} \int_T^{2T} \left| N(t) - \frac{1}{6\pi^2} \text{vol}(M) t^{3/2} \right| dt \gg T^{3/4}.$$

Now let $M = (H_l/\Gamma, g_l)$ be a $(2l+1)$ -dimensional Heisenberg manifold with the metric

$$g_l := \begin{pmatrix} I_{2l \times 2l} & 0 \\ 0 & 2\pi \end{pmatrix},$$

where $I_{2l \times 2l}$ is the identity matrix.

Khosravi and Toth [21] proved that

$$(1.11) \quad \int_1^T R(t)^2 dt = C_{2,l} T^{2l+1/2} + O(T^{2l+1/4+\varepsilon}),$$

where $C_{2,l}$ is an explicit constant.

Khosravi [19] proved that the asymptotic formula

$$(1.12) \quad \int_1^T R^3(t) dt = C_{3,l} T^{3l+1/4} + O(T^{3l+3/14+\varepsilon})$$

is true for some explicit constant $C_{3,l}$.

In [32] the second author proved that the asymptotic formula

$$(1.13) \quad \int_1^T R^k(t) dt = C_{k,l} T^{k(l-1/4)+1} + O(T^{k(l-1/4)+1-\eta_k+\varepsilon})$$

holds true for any $3 \leq k \leq 9$, where $C_{k,l}$ and $\eta_k > 0$ are explicit constants.

Recently, Nowak [23, 24] proved that the estimate

$$\limsup_{t \rightarrow \infty} \frac{R(t)}{t^{l-1/4} \omega_l(t)} > 0$$

holds with

$$\omega_l(t) = \begin{cases} (\log t)^{1/4}, & \text{if } l \text{ is even,} \\ (\log_2 t \log_3 t)^{1/4}, & \text{if } l \text{ is odd,} \end{cases}$$

where $\log_r t = \log \log_{r-1} t$, $\log_1 t = \log t$.

Notation. For a real number t , let $[t]$ denote the integer part of t , $\{t\} = t - [t]$, $\|t\| = \min(\{t\}, 1 - \{t\})$, $e(t) = e^{2\pi i t}$. ε always denotes a sufficiently small positive constant. $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ denote the set of real numbers, the set of integers, and the set of positive integers, respectively. $d(n)$ denotes the Dirichlet divisor function. Throughout this paper, $\mathcal{L}$ always denotes $\log T$.

2. SIGN CHANGES OF $R(t)$

From now on, we always suppose that $R(t)$ denotes the error term in Weyl's law for the $(2l + 1)$ -dimensional Heisenberg manifold $(H_l/\Gamma, g_l)$.

In [12], Heath-Brown and the first author studied the sign changes of the error term $R_I(t)$. They proved that for a suitable constant $C > 0$, $R_I(t)$ changes sign on the interval $[T, T + C\sqrt{T}]$ for every sufficiently large T . Here the length $\sqrt{T}$ is almost best possible since they proved that in the interval $[T, 2T]$ there are many subintervals of length $\gg \sqrt{T} \log^{-5} T$ such that $R_I(t)$ does not change sign in any of these subintervals.

In this paper we shall show that similar results hold for $R(t)$. More precisely, we have the following theorems.

Theorem 1. *Let $c_1 > 0$ be a sufficiently small constant and $c_2 > 0$ be a sufficiently large constant. For any real-valued function $g(t)$ satisfying $|g(t)| \leq c_1 t^{l-1/4}$, the function $R(t) + g(t)$ changes sign at least once in the interval $[T, T + c_2\sqrt{T}]$ for every sufficiently large T . In particular, there exist $t_1, t_2 \in [T, T + c_2\sqrt{T}]$ such that $R(t_1) \geq c_1 t_1^{l-1/4}$ and $R(t_2) \leq -c_1 t_2^{l-1/4}$.*

Theorem 2. *There exist three positive absolute constants c_3, c_4, c_5 such that, for any large parameter T , there are at least $c_3\sqrt{T} \log^5 T$ disjoint subintervals of length $c_4\sqrt{T} \log^{-5} T$ in $[T, 2T]$ such that $\pm R(t) > c_5 t^{l-1/4}$ whenever t lies in any of these subintervals. We also have the estimate*

$$\text{meas}\{t \in [T, 2T] : \pm R(t) > c_5 t^{l-1/4}\} \gg T.$$

Remark 1. Our proof of Theorem 2 is a variant of the proof of Theorem 2 in Section 3 of [12]. However, our approach can prove that $R(t)$ (respectively $-R(t)$) has large values on long intervals of length $\gg \sqrt{T} \log^{-5} T$.

As an application of Theorem 2, we study the Ω -result of the error term in the asymptotic formula (1.13) for odd k . For any integer $k \geq 2$, define

$$\mathcal{F}_{k,l}(T) := \int_1^T R^k(t) dt - C_{k,l} T^{k(l-1/4)+1}.$$

We then have the following

Theorem 3. *The estimate*

$$\mathcal{F}_{k,l}(T) = \Omega(T^{k(l-1/4)+1/2} \log^{-5} T)$$

holds for any fixed odd integer $k \geq 3$.

Remark 2. The results of [32] show that (1.13) should be true for any integer $k \geq 3$. However, up to the present we can only prove it for $3 \leq k \leq 9$. Theorem 3 provides an Ω -result for any odd $k \geq 3$.

The corresponding result on $R_I(t)$ proved in [12] can be improved slightly via the same approach. We state it as the following theorem.

Theorem 4. *There exist three positive absolute constants c_6, c_7, c_8 such that, for any large parameter T , there are at least $c_6\sqrt{T}\log^3 T$ disjoint subintervals of length $c_7\sqrt{T}\log^{-3} T$ in $[T, 2T]$ such that $\pm R_I(t) > c_8 t^{1/4}$ whenever t lies in any of these subintervals. We also have the estimate*

$$\text{meas}\{t \in [T, 2T] : \pm R_I(t) > c_8 t^{1/4}\} \gg T.$$

Remark 3. By Theorem 4, the argument of Theorem 3 proves that the formula

$$\int_1^T R_I^k(t) dt = c_k T^{1+k/4} + \Omega(T^{(k+2)/4} \log^{-3} T)$$

holds for any odd integer $k \geq 3$.

For the error term $\Delta(x)$ in the divisor problem, the asymptotic formula (see [28] and [30])

$$(2.1) \quad \int_1^T \Delta^k(x) dx = C_k T^{k/4+1} + O(T^{k/4+1-\eta_k})$$

holds for any integer $3 \leq k \leq 9$, where C_k and η_k are explicit constants. In [16], Ivić and the second author proved the estimate

$$\int_1^T \Delta^k(x) dx - C_k T^{k/4+1} = \Omega(G^{k+1}(T) \log^{-1} T)$$

for any $k \geq 2$, where

$$G(x) = (x \log x)^{1/4} (\log \log x)^{\frac{3}{4}(2^{4/3}-1)} (\log \log \log x)^{-5/8}$$

is the Ω -estimate of $\Delta(x)$ proved by Soundararajan [26]. In view of the work in [12], the proof of Theorem 3 implies that, for any odd integer $k \geq 3$, the estimate $\Omega(G^{k+1}(T) \log^{-1} T)$ can be substantially improved to $\Omega(T^{(k+2)/4} \log^{-5} T)$. A similar result also holds for $E(t)$, the error term in the mean square of the Riemann zeta-function $\zeta(s)$ over the critical line.

3. BACKGROUND OF HEISENBERG MANIFOLDS AND THE ANALOGUE VORONOI FORMULA FOR $R(2\pi x)$

In this section, we first review some background of Heisenberg manifolds. The reader can refer to [6], [9], [27] for more details.

3.1. Heisenberg manifolds. Suppose $x \in \mathbb{R}^l$ is a row vector and $y \in \mathbb{R}^l$ is a column vector. Define

$$\gamma(x, y, t) = \begin{pmatrix} 1 & x & t \\ 0 & I_l & y \\ 0 & 0 & 1 \end{pmatrix}, \quad X(x, y, t) = \begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

The $(2l+1)$ -dimensional Heisenberg group H_l is defined by

$$H_l = \{\gamma(x, y, t) : x, y \in \mathbb{R}^l, t \in \mathbb{R}\},$$

and its Lie algebra is

$$\mathfrak{H}_l = \{X(x, y, t) : x, y \in \mathbb{R}^l, t \in \mathbb{R}\}.$$

We say Γ is a uniform discrete subgroup of H_l if H_l/Γ is compact. A $(2l+1)$ -dimensional Heisenberg manifold is a pair $(H_l/\Gamma, g)$ for which Γ is a uniform discrete subgroup of H_l and g is a left H_l -invariant metric.

For every l -tuple $r = (r_1, r_2, \dots, r_l) \in \mathbb{N}^l$ such that $r_j | r_{j+1}$ ($j = 1, 2, \dots, l-1$), let $r\mathbb{Z}^l$ denote the l -tuple $x = (x_1, x_2, \dots, x_l)$ with $x_j \in r_j\mathbb{Z}$. Define

$$\Gamma_r = \{\gamma(x, y, t) : x \in r\mathbb{Z}^l, y \in r\mathbb{Z}^l, t \in \mathbb{Z}\}.$$

It is clear that Γ_r is a uniform discrete subgroup of H_l . According to Theorem 2.4 of [9], the subgroup Γ_r classifies all the uniform discrete subgroups of H_l up to automorphisms. Thus (see [9], Corollary 2.5) given any Riemannian Heisenberg manifold $M = (H_l/\Gamma, g)$, there exists a unique l -tuple r as before and a left-invariant metric $\tilde{g}$ on H_l such that M is isometric to $(H_l/\Gamma, \tilde{g})$. So (see [9], 2.6(b)) we can replace the metric g by ϕ^*g , where ϕ is an inner automorphism such that the direct sum split of the Lie algebra $\mathfrak{H}_l = \mathbb{R}^{2l} \oplus \mathfrak{Z}$ is orthogonal. Here $\mathfrak{Z}$ is the center of the Lie algebra and

$$\mathbb{R}^{2l} = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y \in \mathbb{R}^l \right\}.$$

With respect to this orthogonal split of H_l the metric g has the form

$$\begin{pmatrix} h & 0 \\ 0 & g_{2l+1} \end{pmatrix},$$

where h is a positive-definite $2l \times 2l$ matrix and $g_{2l+1} > 0$ is a real number.

The volume of the Heisenberg manifold is given by

$$\text{vol}(H_l/\Gamma, g) = |\Gamma_r| \sqrt{\det(g)}$$

with $|\Gamma_r| = r_1 r_2 \cdots r_l$ for $r = (r_1, r_2, \dots, r_l)$.

3.2. The spectrum of Heisenberg manifolds. Let Σ be the spectrum of the Laplacian on $M = (H_l/\Gamma, g_l)$, where the eigenvalues are counted with multiplicities. According to [9] (p. 258), Σ can be divided into two parts, Σ_1 and Σ_2 , where Σ_1 is the spectrum of $2l$ -dimensional torus and Σ_2 contains all eigenvalues of the form

$$2\pi m^2 + 2\pi m(2n_1 + \cdots + 2n_l + l), m \in \mathbb{N}, n_j \in \mathbb{N} \cup \{0\},$$

each eigenvalue counted with the multiplicity $2m^l$.

3.3. The Voronoi-type formula for $R(2\pi x)$. In [32] the second author proved an analogue Voronoi formula for $R^*(x) := R(2\pi x)$. Suppose $T \geq 10$ is a large parameter, $\mathcal{L} = \log T$. Suppose $T \leq x \leq 2T$, $H \geq T$, and $J = [(\mathcal{L} - \log \mathcal{L})/2 \log 2]$. Then we have

$$(3.1) \quad R^*(x) = \frac{2^{2-l} x^{l-1/4}}{(l-1)! \pi} \sum_{1 \leq n \leq H^2(2^{2J+1}+1/2)} \frac{\tau_l(n; H, T)}{n^{3/4}} \cos\left(2\pi\sqrt{xn} - \frac{\pi}{4}\right) \\ + O(T^{l-1/2} G(x) + T^{l-1/2} \mathcal{L}^2),$$

where

$$(3.2) \quad \tau_l(n; H, T) : = \sum_{\substack{n=h(2r-h), 1 \leq h \leq H \\ h \leq r \leq h(2^{2J+1}+1/2)}} \frac{e(lh/2)h^{1/2}}{(2r-h)^{1/2}} \left(1 - \frac{h}{2r-h}\right)^{l-1},$$

$$(3.3) \quad G(x) = \sum_{m \leq \sqrt{2T}} \min \left(1, \frac{1}{H \left\| \frac{x}{2m} - \frac{m}{2} + \frac{l}{2} \right\|} \right).$$

We note that if $n \leq T\mathcal{L}^{-1}$, then

$$(3.4) \quad \tau_l(n; H, T) = \tau_l(n) := \sum_{\substack{n=h(2r-h) \\ h \leq r}} \frac{e(lh/2)h^{1/2}}{(2r-h)^{1/2}} \left(1 - \frac{h}{2r-h}\right)^{l-1}.$$

Remark 3.1. There is an error in the definition of $\tau_l(n)$ in [32], where the important condition $h \leq r$ was omitted.

Remark 3.2. The term $T^{l-1/2}\mathcal{L}^2$ in (3.1) reads as $T^{l-1/2}\mathcal{L}^3$ in Proposition 6.1 of [32]. However, by a little more analysis in Section 6.2 of [32], we see that $T^{l-1/2}\mathcal{L}^3$ can be replaced by $T^{l-1/2}\mathcal{L}^2$.

4. PROOF OF THEOREM 1

In this section we prove Theorem 1. We follow the approach of [12].

Let n_0 denote the smallest integer n such that $\tau_l(n) \neq 0$. From the definition of $\tau_l(n)$ it is easy to see that $n_0 = 1$ if $l = 1$ and $n_0 = 3$ if $l > 1$, and indeed

$$\tau_l(n_0) = \begin{cases} -1, & \text{if } l = 1, \\ e(l/2)3^{1/2-l}2^{l-1}, & \text{if } l > 1. \end{cases}$$

Suppose $|g(t)| \leq c_1 t^{l-1/4}$. Let

$$(4.1) \quad R^{**}(t) = t^{-(2l-1/2)}(R(2\pi t^2) + g(2\pi t^2)), \quad t \geq 1,$$

and define

$$(4.2) \quad K_\zeta(u) := (1 - |u|)(1 + \zeta \sin 2\pi\alpha\sqrt{n_0}u), \quad u \geq 1,$$

where $\zeta = 1$ or -1 and $\alpha > 2$ is a large constant.

It is easy to see that Theorem 1 follows from Lemma 4.1 below.

Lemma 4.1. *Suppose $T \geq 10$ is a large parameter. Then for each $\sqrt{T} \leq t \leq \sqrt{2T}$, we have*

$$(4.3) \quad \int_{-1}^1 R^{**}(t + \alpha u) K_\zeta(u) du = -\frac{\zeta 2^{1-l} \tau_l(n_0)}{(l-1)! \pi n_0^{3/4}} \sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}) \\ + O(t^{-(2l-1/2)} \sup_{|u| \leq 1} |g(2\pi(t + \alpha u)^2)| + t^{-1/2} \log^2 t).$$

Proof. From (3.1) and the definition of n_0 we have

$$(4.4) \quad t^{-(2l-1/2)} R^*(t^2) = \frac{2^{2-l}}{(l-1)!\pi} \sum_{n_0 \leq n \leq H^2(2^{2J+1}+1/2)} \frac{\tau_l(n; H, T)}{n^{3/4}} \\ \times \cos\left(2\pi t\sqrt{n} - \frac{\pi}{4}\right) + O(t^{-1/2}G_1(t) + t^{-1/2}\log^2 t),$$

$$(4.5) \quad G_1(t) = \sum_{m \leq \sqrt{2T}} \min\left(1, \frac{1}{H\|\frac{t^2}{2m} - \frac{m}{2} + \frac{l}{2}\|}\right).$$

We first estimate the integral $\int_{-1}^1 G_1(t + \alpha u) du$. It is well known that

$$(4.6) \quad \min(1, \frac{1}{H\|r\|}) = \sum_{h=-\infty}^{\infty} a(h)e(hr)$$

with

$$a(0) \ll \frac{\log H}{H}, \quad a(h) \ll \min\left(\frac{\log H}{H}, \frac{H}{h^2}\right), \quad h \neq 0.$$

Thus we have

$$(4.7) \quad \int_{-1}^1 G_1(t + \alpha u) du = \sum_{h=-\infty}^{\infty} a(h) \sum_{m \leq \sqrt{2T}} e\left(\frac{ht^2}{2m} - \frac{hm}{2} + \frac{hl}{2}\right) \\ \times \int_{-1}^1 e\left(\frac{2ht\alpha u + h\alpha^2 u^2}{2m}\right) du \\ \ll \sqrt{T}|a(0)| + \sum_{h=1}^{\infty} |a(h)| \sum_{m \leq \sqrt{2T}} \frac{m}{ht\alpha} \\ \ll \sqrt{T}H^{-1}\log^2 H,$$

where the first derivative test was used.

Let

$$J_{\zeta}(\alpha, t, n) := \int_{-1}^1 \cos(2\pi(t + \alpha u)\sqrt{n} - \frac{\pi}{4}) K_{\zeta}(u) du.$$

Then we have

$$(4.8) \quad J_{\zeta}(\alpha, t, n) = J_1 - J_2 + J_3 - J_4,$$

where

$$J_1 = \cos(2\pi t\sqrt{n} - \frac{\pi}{4}) \int_{-1}^1 (1 - |u|) \cos(2\pi\alpha u\sqrt{n}) du, \\ J_2 = \sin(2\pi t\sqrt{n} - \frac{\pi}{4}) \int_{-1}^1 (1 - |u|) \sin(2\pi\alpha u\sqrt{n}) du, \\ J_3 = \zeta \cos(2\pi t\sqrt{n} - \frac{\pi}{4}) \int_{-1}^1 (1 - |u|) \cos(2\pi\alpha u\sqrt{n}) \sin(2\pi\alpha\sqrt{n_0}u) du, \\ J_4 = \zeta \sin(2\pi t\sqrt{n} - \frac{\pi}{4}) \int_{-1}^1 (1 - |u|) \sin(2\pi\alpha u\sqrt{n}) \sin(2\pi\alpha\sqrt{n_0}u) du.$$

It is easy to see that $J_2 = J_3 = 0$. By the first derivative test we get that

$$(4.9) \quad J_1 \ll \alpha^{-1}n^{-1/2}.$$

For J_4 we have

$$J_4 = \frac{\zeta}{2} \sin(2\pi t\sqrt{n} - \frac{\pi}{4}) \int_{-1}^1 (1 - |u|) \\ \times (\cos(2\pi\alpha(\sqrt{n} - \sqrt{n_0})u) - \cos(2\pi\alpha(\sqrt{n} + \sqrt{n_0})u)) du.$$

So by the first derivative test again we get

$$J_4 = \begin{cases} -\frac{\zeta}{2} \sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}), & \text{if } n = n_0, \\ \ll \alpha^{-1}n^{-1/2}, & \text{if } n \neq n_0, \end{cases}$$

which combining (4.8) and (4.9) gives

$$(4.10) \quad J_\zeta(\alpha, t, n) = \begin{cases} -\frac{\zeta}{2} \sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}), & \text{if } n = n_0, \\ \ll \alpha^{-1}n^{-1/2}, & \text{if } n \neq n_0. \end{cases}$$

From (4.4), (4.5), (4.7) and (4.10) we get (taking $H = T^2$)

$$(4.11) \quad \begin{aligned} & \int_{-1}^1 R^{**}(t + \alpha u) K_\zeta(u) du \\ &= \frac{2^{2-l}}{(l-1)! \pi} \sum_{n_0 \leq n \leq H^2(2^{2J+1}+1/2)} \frac{\tau_l(n; H, T)}{n^{3/4}} J_\zeta(\alpha, t, n) \\ & \quad + O(t^{-(2l-1/2)} \sup_{|u| \leq 1} |g(2\pi(t + \alpha u)^2)| + T^{1/2} H^{-1} \mathcal{L}^2 + t^{-1/2} \log^2 t) \\ &= -\frac{\zeta 2^{1-l} \tau_l(n_0)}{(l-1)! \pi n_0^{3/4}} \sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}) + \sum_{n_0+1 \leq n \leq H^2(2^{2J+1}+1/2)} \frac{|\tau_l(n)|}{\alpha n^{5/4}} \\ & \quad + O(t^{-(2l-1/2)} \sup_{|u| \leq 1} |g(2\pi(t + \alpha u)^2)| + t^{-1/2} \log^2 t) \\ &= -\frac{\zeta 2^{1-l} \tau_l(n_0)}{(l-1)! \pi n_0^{3/4}} \sin(2\pi\sqrt{n_0}t - \frac{\pi}{4}) + O(\alpha^{-1}) \\ & \quad + O(t^{-(2l-1/2)} \sup_{|u| \leq 1} |g(2\pi(t + \alpha u)^2)| + t^{-1/2} \log^2 t). \end{aligned} \quad \square$$

5. THE MEAN VALUE OF $R(t)$ IN SHORT INTERVALS

Suppose $T \geq 10$ is a large parameter, $1 \leq h \leq \frac{1}{2}\sqrt{T}$. In this section we shall estimate the integral

$$I(T, h) = \int_1^T (R(x+h) - R(x))^2 dx,$$

which would play an important role in the proof of Theorem 2. This type of integral was studied for the error term in the mean square of $\zeta(1/2 + it)$ by Good [8] and for the error term in the Dirichlet divisor problem by Jutila [18]. Our approach is based on Jutila [18], but with some modifications.

Without loss of generality, we shall estimate the integral

$$(5.1) \quad I^*(T, h) = \int_1^T (R^*(x+h) - R^*(x))^2 dx,$$

where $R^*(x)$ was defined in (3.1). We shall prove the following

Lemma 5.1. *The estimate*

$$(5.2) \quad I^*(T, h) \ll T^{2l} h \log^3 \frac{\sqrt{T}}{h} + T^{2l} \mathcal{L}^4$$

holds uniformly for $1 \leq h \leq \frac{1}{2}\sqrt{T}$.

Remark. Lemma 5.1 is also true for $I(T, h)$.

Proof. Write

$$(5.3) \quad I^*(T, h) = \int_1 + \int_2,$$

where

$$\begin{aligned} \int_1 &:= \int_1^{100 \max(h^2, T^{2/3})} (R^*(x+h) - R^*(x))^2 dx, \\ \int_2 &:= \int_{100 \max(h^2, T^{2/3})}^T (R^*(x+h) - R^*(x))^2 dx. \end{aligned}$$

From (1.11) we have

$$(5.4) \quad \int_1 \ll h^{2(2l+1/2)} + T^{\frac{2}{3}(2l+1/2)} \ll T^{2l} h.$$

In order to bound $\int_2$, we first estimate the integral

$$J(U, h) = \int_U^{2U} (R^*(x+h) - R^*(x))^2 dx, \quad 100 \max(h^2, T^{2/3}) \leq U \leq T.$$

In (3.1) we use U in place of T and then take $H = U^{100}$, $J = [(\log U - \log \log U)/2 \log 2]$. Let $z := \min(\varepsilon U h^{-1}, U \log^{-1} U)$. Define

$$\begin{aligned} R_1(x) &:= \frac{2^{2-l} x^{l-1/4}}{(l-1)! \pi} \sum_{1 \leq n \leq z} \frac{\tau_l(n)}{n^{3/4}} \cos\left(2\pi \sqrt{nx} - \frac{\pi}{4}\right) \\ R_2(x) &:= \frac{2^{2-l} x^{l-1/4}}{(l-1)! \pi} \sum_{z \leq n \leq H^{2(2J+1+1/2)}} \frac{\tau_l(n; H, T)}{n^{3/4}} \cos\left(2\pi \sqrt{nx} - \frac{\pi}{4}\right). \end{aligned}$$

Then we have

$$(5.5) \quad R^*(x) = R_1(x) + R_2(x) + O(U^{l-1/2} G_2(x) + U^{l-1/2} \log^2 U),$$

where

$$G_2(x) := \sum_{m \leq \sqrt{2U}} \min\left(1, \frac{1}{H \left\| \frac{x}{2m} - \frac{m}{2} + \frac{l}{2} \right\|}\right).$$

From (6.30) of [32] we have

$$(5.6) \quad \int_U^{2U} |R_2(x)|^2 dx \ll U^{2l+1/2} z^{-1/2} \log^3 z.$$

Lemma 6.1 of [32] implies that (trivially $G_2(x) \ll \sqrt{U}$)

$$(5.7) \quad \int_U^{2U} |U^{l-1/2} G_2(x)|^2 dx \ll U^{2l-1/2} \int_U^{2U} G_2(x) dx \ll U^{2l-99} \log H.$$

Let

$$M(x) = R_2(x) + O(U^{l-1/2}G_2(x) + U^{l-1/2}\log^2 U).$$

Then (5.6) and (5.7) implies

$$(5.8) \quad \int_U^{2U} |M(x)|^2 dx \ll U^{2l+1/2} z^{-1/2} \log^3 z + U^{2l} \log^4 U \\ \ll h^{1/2} U^{2l} \log^3 z + U^{2l} \log^4 U.$$

Now we estimate $\int_U^{2U} (R_1(x+h) - R_1(x))^2 dx$. From the definition of $R_1(x)$ we have

$$(5.9) \quad R_1(x+h) - R_1(x) = F_1(x) + F_2(x),$$

where

$$F_1(x) = \frac{2^{2-l}}{(l-1)!\pi} \left((x+h)^{l-1/4} - x^{l-1/4} \right) \sum_{1 \leq n \leq z} \frac{\tau_l(n)}{n^{3/4}} \\ \times \cos \left(2\pi \sqrt{n(x+h)} - \frac{\pi}{4} \right) \ll h x^{-1} |R_1(x+h)|, \\ F_2(x) = \frac{2^{2-l}}{(l-1)!\pi} x^{l-1/4} \sum_{1 \leq n \leq z} \frac{\tau_l(n)}{n^{3/4}} \\ \times \left(\cos \left(2\pi \sqrt{n(x+h)} - \frac{\pi}{4} \right) - \cos \left(2\pi \sqrt{nx} - \frac{\pi}{4} \right) \right).$$

For the mean square of $F_1(x)$ we have

$$(5.10) \quad \int_U^{2U} |F_1(x)|^2 dx \ll h^2 U^{-2} U^{2l+1/2} \ll h U^{2l}.$$

We write

$$(5.11) \quad F_2^2(x) = F_{21}(x) + F_{22}(x),$$

where

$$F_{21}(x) = \frac{2^{4-2l}}{(l-1)!^2 \pi^2} x^{2l-1/2} \sum_{1 \leq n \leq z} \frac{\tau_l^2(n)}{n^{3/2}} \\ \times \left(\cos \left(2\pi \sqrt{n(x+h)} - \frac{\pi}{4} \right) - \cos \left(2\pi \sqrt{nx} - \frac{\pi}{4} \right) \right)^2, \\ F_{22}(x) = \frac{2^{4-2l}}{(l-1)!^2 \pi^2} x^{2l-1/2} \sum_{1 \leq m \neq n \leq z} \frac{\tau_l(m) \tau_l(n)}{(mn)^{3/4}} \\ \times \left(\cos \left(2\pi \sqrt{m(x+h)} - \frac{\pi}{4} \right) - \cos \left(2\pi \sqrt{mx} - \frac{\pi}{4} \right) \right) \\ \times \left(\cos \left(2\pi \sqrt{n(x+h)} - \frac{\pi}{4} \right) - \cos \left(2\pi \sqrt{nx} - \frac{\pi}{4} \right) \right).$$

By writing

$$\cos \left(2\pi \sqrt{n(x+h)} - \frac{\pi}{4} \right) - \cos \left(2\pi \sqrt{nx} - \frac{\pi}{4} \right) \\ = \sum_{j=0}^1 (-1)^{j+1} \cos \left(2\pi \sqrt{n(x+jh)} - \frac{\pi}{4} \right)$$

we get

$$F_{22}(x) = \frac{2^{4-2l}x^{2l-1/2}}{(l-1)!^2\pi^2} \sum_{j_1=0}^1 \sum_{j_2=0}^1 (-1)^{j_1+j_2} \sum_{1 \leq m \neq n \leq z} \frac{\tau_l(m)\tau_l(n)}{(mn)^{3/4}} \\ \times \cos\left(2\pi\sqrt{m(x+j_1h)} - \frac{\pi}{4}\right) \times \cos\left(2\pi\sqrt{n(x+j_2h)} - \frac{\pi}{4}\right).$$

By the elementary formula

$$\cos a \cos b = \frac{\cos(a-b) + \cos(a+b)}{2}$$

we have

$$(5.12) \quad F_{22}(x) = F_{221}(x) + F_{222}(x),$$

where

$$F_{221}(x) = \frac{2^{3-2l}x^{2l-1/2}}{(l-1)!^2\pi^2} \sum_{j_1=0}^1 \sum_{j_2=0}^1 (-1)^{j_1+j_2} \sum_{1 \leq m \neq n \leq z} \frac{\tau_l(m)\tau_l(n)}{(mn)^{3/4}} \\ \times \cos\left(2\pi\sqrt{m(x+j_1h)} - 2\pi\sqrt{n(x+j_2h)}\right), \\ F_{222}(x) = \frac{2^{3-2l}x^{2l-1/2}}{(l-1)!^2\pi^2} \sum_{j_1=0}^1 \sum_{j_2=0}^1 (-1)^{j_1+j_2} \sum_{1 \leq m \neq n \leq z} \frac{\tau_l(m)\tau_l(n)}{(mn)^{3/4}} \\ \times \sin\left(2\pi\sqrt{m(x+j_1h)} + 2\pi\sqrt{n(x+j_2h)}\right).$$

Let

$$g_{\pm}(x) = 2\pi\sqrt{m(x+j_1h)} \pm 2\pi\sqrt{n(x+j_2h)}.$$

By the power series expansion

$$(5.13) \quad (1+t)^{1/2} = 1 + \sum_{v=1}^{\infty} d_v t^v \quad (|t| \leq 1/2)$$

we get that

$$g_{\pm}(x) = 2\pi\sqrt{x}(\sqrt{m} \pm \sqrt{n}) + 2\pi \sum_{v=1}^{\infty} \frac{d_v h^v}{x^{v-1/2}} (\sqrt{m}j_1^v \pm \sqrt{n}j_2^v),$$

which implies

$$|g'_{\pm}(x)| \gg x^{-1/2} |\sqrt{m} \pm \sqrt{n}| \quad (m \neq n)$$

by noting that $m, n \leq \varepsilon U h^{-1}$. By the first derivative test we have

$$(5.14) \quad \int_U^{2U} F_{221}(x) dx \ll U^{2l} \sum_{1 \leq m \neq n \leq z} \frac{|\tau_l(m)\tau_l(n)|}{(mn)^{3/4} |\sqrt{m} - \sqrt{n}|} \\ \ll U^{2l} \sum_{1 \leq m \neq n \leq z} \frac{d(m)d(n)}{(mn)^{3/4} |\sqrt{m} - \sqrt{n}|} \\ \ll U^{2l} \log^4 z,$$

where in the last step we have used the well-known estimate

$$\sum_{1 \leq m \neq n \leq y} \frac{d(m)d(n)}{(mn)^{3/4} |\sqrt{m} - \sqrt{n}|} \ll \log^4 y, \quad y \geq 10.$$

We also have

$$\begin{aligned}
 (5.15) \quad \int_U^{2U} F_{222}(x) dx &\ll U^{2l} \sum_{1 \leq m \neq n \leq z} \frac{|\tau_l(m)\tau_l(n)|}{(mn)^{3/4}|\sqrt{m} + \sqrt{n}|} \\
 &\ll U^{2l} \sum_{1 \leq m < n \leq z} \frac{d(m)d(n)}{m^{3/4}n^{5/4}} \\
 &\ll U^{2l} \log^3 z,
 \end{aligned}$$

by the well-known estimate $\sum_{n \leq y} d(n) \ll y \log y$.

From (5.12), (5.14) and (5.15) we have

$$(5.16) \quad \int_U^{2U} F_{22}(x) dx \ll U^{2l} \log^4 z.$$

By using the formulas

$$\cos u - \cos v = -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$

and

$$\sin^2 u = (1 - \cos 2u)/2$$

we have

$$\begin{aligned}
 (5.17) \quad &\int_U^{2U} F_{21}(x) dx \\
 &= \frac{2^{6-2l}}{(l-1)!^2 \pi^2} \sum_{1 \leq n \leq z} \frac{\tau_l^2(n)}{n^{3/2}} \int_U^{2U} x^{2l-\frac{1}{2}} \sin^2 \left(\pi \sqrt{n(x+h)} + \pi \sqrt{nx} - \frac{\pi}{4} \right) \\
 &\quad \times \sin^2 \left(\pi \sqrt{n(x+h)} - \pi \sqrt{nx} \right) dx = S_1 - S_2,
 \end{aligned}$$

for instance, where

$$\begin{aligned}
 S_1 &= \frac{2^{5-2l}}{(l-1)!^2 \pi^2} \sum_{1 \leq n \leq z} \frac{\tau_l^2(n)}{n^{3/2}} \int_U^{2U} x^{2l-\frac{1}{2}} \sin^2 \left(\pi \sqrt{n(x+h)} - \pi \sqrt{nx} \right) dx, \\
 S_2 &= \frac{2^{5-2l}}{(l-1)!^2 \pi^2} \sum_{1 \leq n \leq z} \frac{\tau_l^2(n)}{n^{3/2}} \\
 &\quad \times \int_U^{2U} x^{2l-\frac{1}{2}} \sin \left(2\pi \sqrt{n(x+h)} + 2\pi \sqrt{nx} \right) \sin^2 \left(\pi \sqrt{n(x+h)} - \pi \sqrt{nx} \right) dx.
 \end{aligned}$$

For each $n \leq z$, let $L_n(t) = \int_U^t x^{2l-1/2} \sin \left(2\pi \sqrt{n(x+h)} + 2\pi \sqrt{nx} \right) dx$. By the first derivative test

$$(5.18) \quad L_n(t) \ll U^{2l} n^{-1/2}, U \leq t \leq 2U.$$

So by partial summation

$$\begin{aligned}
 & \int_U^{2U} x^{2l-\frac{1}{2}} \sin \left(2\pi \sqrt{n(x+h)} + 2\pi \sqrt{nx} \right) \sin^2 \left(\pi \sqrt{n(x+h)} - \pi \sqrt{nx} \right) dx \\
 &= \int_U^{2U} \sin^2 \left(\pi \sqrt{n(x+h)} - \pi \sqrt{nx} \right) dL_n(x) \\
 &= L_n(2U) \sin^2 \left(\pi \sqrt{n(2U+h)} - \pi \sqrt{2nU} \right) \\
 &\quad - \int_U^{2U} L_n(x) \sin \left(\pi \sqrt{n(x+h)} - \pi \sqrt{nx} \right) \cos \left(\pi \sqrt{n(x+h)} - \pi \sqrt{nx} \right) \\
 &\quad \times \left(\frac{\pi \sqrt{n}}{\sqrt{x+h}} - \frac{\pi \sqrt{n}}{\sqrt{x}} \right) dx \\
 &\ll U^{2l} n^{-1/2} + hU^{2l-1/2}.
 \end{aligned}$$

Thus we get

$$(5.19) \quad S_2 \ll \sum_{1 \leq n \leq z} \frac{\tau_l^2(n)}{n^{3/2}} (U^{2l} n^{-1/2} + hU^{2l-1/2}) \ll U^{2l} + hU^{2l-1/2} \ll U^{2l}.$$

By (5.13) we have

$$\pi \sqrt{n(x+h)} - \pi \sqrt{nx} = \frac{\pi h \sqrt{n}}{2\sqrt{x}} + O\left(\frac{h^2 \sqrt{n}}{x^{3/2}}\right),$$

which implies that

$$\sin^2 \left(\pi \sqrt{n(x+h)} - \pi \sqrt{nx} \right) = \sin^2 \frac{\pi h \sqrt{n}}{2\sqrt{x}} + O\left(\frac{h^2 \sqrt{n}}{x^{3/2}}\right).$$

Thus

$$\begin{aligned}
 (5.20) \quad S_1 &= \frac{2^{5-2l}}{(l-1)!^2 \pi^2} \sum_{1 \leq n \leq z} \frac{\tau_l^2(n)}{n^{3/2}} \int_U^{2U} x^{2l-\frac{1}{2}} \sin^2 \frac{\pi h \sqrt{n}}{2\sqrt{x}} dx \\
 &\quad + O\left(\sum_{1 \leq n \leq z} \frac{|\tau_l^2(n)|}{n^{3/2}} \int_U^{2U} x^{2l-\frac{1}{2}} \frac{h^2 \sqrt{n}}{x^{3/2}} dx \right) \\
 &= \frac{2^{5-2l}}{(l-1)!^2 \pi^2} \sum_{1 \leq n \leq z} \frac{\tau_l^2(n)}{n^{3/2}} \int_U^{2U} x^{2l-\frac{1}{2}} \sin^2 \frac{\pi h \sqrt{n}}{2\sqrt{x}} dx \\
 &\quad + O\left(h^2 U^{2l-1} \sum_{1 \leq n \leq z} \frac{|\tau_l^2(n)|}{n} \right).
 \end{aligned}$$

Since $|\tau_l(n)| \leq d(n)$, we have the estimate

$$(5.21) \quad \sum_{n \leq y} \tau_l^2(n) \ll \sum_{n \leq y} d^2(n) \ll y \log^3 y \quad (y \geq 2),$$

which immediately implies that

$$(5.22) \quad h^2 U^{2l-1} \sum_{1 \leq n \leq z} \frac{\tau_l^2(n)}{n} \ll h^2 U^{2l-1} \log^4 z \ll hU^{2l-1/2} \log^4 U.$$

From (5.21) we can get

$$\begin{aligned}
 (5.23) \quad & \sum_{1 \leq n \leq z} \frac{\tau_l^2(n)}{n^{3/2}} \int_U x^{2l-\frac{1}{2}} \sin^2 \frac{\pi h \sqrt{n}}{2\sqrt{x}} dx \\
 & \ll \sum_{1 \leq n \leq z} \frac{d^2(n)}{n^{3/2}} \int_U x^{2l-\frac{1}{2}} \min\left(1, \frac{h^2 n}{x}\right) dx \\
 & \ll h^2 U^{2l-1/2} \sum_{1 \leq n \leq U/h^2} \frac{d^2(n)}{n^{1/2}} + U^{2l+1/2} \sum_{U/h^2 < n \leq z} \frac{d^2(n)}{n^{3/2}} \\
 & \ll h U^{2l} \log^3 \frac{U}{h^2} \ll h U^{2l} \log^3 \frac{\sqrt{U}}{h}.
 \end{aligned}$$

Combining (5.20), (5.22) and (5.23) we get

$$(5.24) \quad S_1 \ll h U^{2l} \log^3 \frac{\sqrt{U}}{h},$$

which together with (5.17) gives

$$(5.25) \quad \int_U^{2U} F_{21}(x) dx \ll h U^{2l} \log^3 \frac{\sqrt{U}}{h}.$$

From (5.9)-(5.11), (5.16) and (5.25) we get

$$(5.26) \quad \int_U^{2U} (R_1(x+h) - R_1(x))^2 dx \ll h U^{2l} \log^3 \frac{\sqrt{U}}{h} + U^{2l} \log^4 z.$$

Now combining (5.5), (5.8) and (5.26) we get

$$J(U, h) \ll h U^{2l} \log^3 \frac{\sqrt{U}}{h} + U^{2l} \log^4 U,$$

which immediately implies that

$$(5.27) \quad \int_2 \ll h T^{2l} \log^3 \frac{\sqrt{T}}{h} + T^{2l} \log^4 T$$

via a splitting argument. Finally Lemma 5.1 follows from (5.3), (5.4) and (5.27). $\square$

6. PROOF OF THEOREM 2

In this section we shall prove Theorem 2. Our approach is a variant of the proof of Theorem 2 of [12].

Define

$$R_+(t) = \begin{cases} R(t), & \text{if } R(t) > 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$R_-(t) = |R(t)| - R_+(t).$$

We first prove the following two lemmas.

Lemma 6.1. *The estimate*

$$(6.1) \quad \int_T^{2T} R_{\pm}^2(t) dt \gg T^{2l+1/2}$$

holds.

Proof. From (1.11) and (1.13) with $k = 4$, we get by Hölder's inequality that

$$\begin{aligned} T^{2l+1/2} &\ll \int_T^{2T} R^2(t) dt \leq \left(\int_T^{2T} |R(t)| dt \right)^{2/3} \left(\int_T^{2T} R^4(t) dt \right)^{1/3} \\ &\leq \left(\int_T^{2T} |R(t)| dt \right)^{2/3} T^{4l/3}. \end{aligned}$$

Thus

$$(6.2) \quad \int_T^{2T} |R(t)| dt \gg T^{l+3/4}.$$

From (3.1) and Lemma 6.1 of [32], it is easy to verify that

$$\int_T^{2T} R(t) dt \ll T^{l+1/2} \mathcal{L}^2,$$

which implies

$$(6.3) \quad \int_T^{2T} R_{\pm}(t) dt \gg T^{l+3/4}$$

in view of (6.2). By (6.3) and Cauchy-Schwarz's inequality, it follows that

$$\begin{aligned} T^{l+3/4} &\ll \left(\int_T^{2T} dt \right)^{1/2} \left(\int_T^{2T} R_{\pm}^2(t) dt \right)^{1/2} \\ &\ll T^{1/2} \left(\int_T^{2T} R_{\pm}^2(t) dt \right)^{1/2}, \end{aligned}$$

which immediately implies Lemma 6.1. □

Lemma 6.2. *Suppose $2 \leq H \leq \sqrt{T}$. Then*

$$\int_T^{2T} \max_{h \leq H} (R_{\pm}(t+h) - R_{\pm}(t))^2 dt \ll HT^{2l} \log^5 T.$$

Proof. It is easy to verify that

$$|R_{\pm}(t+h) - R_{\pm}(t)| \leq |R(t+h) - R(t)|,$$

so it is sufficient to prove the estimate

$$(6.4) \quad I = \int_T^{2T} \max_{h \leq H} (R(t+h) - R(t))^2 dt \ll HT^{2l} \log^5 T.$$

Write

$$H = 2^{\lambda} b$$

such that $\lambda \in \mathbb{N}$ and $1 \leq b < 2$. Similar to the argument of the proof of Lemma 2 of [12], we can deduce by using Lemma 5.1 that

$$\begin{aligned}
I &\ll \lambda \sum_{\mu \leq \lambda} \sum_{0 \leq \nu < 2^\mu} \int_{T+\nu 2^{\lambda-\mu} b}^{2T+\nu 2^{\lambda-\mu} b} |R(t+2^{\lambda-\mu} b) - R(t)|^2 dt + T^{2l} \log^2 T \\
&\ll \lambda \sum_{\mu \leq \lambda} \sum_{0 \leq \nu < 2^\mu} (T^{2l} 2^{\lambda-\mu} b \log^3 T + T^{2l} \log^4 T) \\
&\ll \lambda \sum_{\mu \leq \lambda} (T^{2l} 2^\lambda b \log^3 T + 2^\mu T^{2l} \log^4 T) \\
&\ll \lambda^2 H T^{2l} \log^3 T + \lambda H T^{2l} \log^4 T \\
&\ll H T^{2l} \log^5 T,
\end{aligned}$$

namely (6.4) holds. $\square$

Now we finish the proof of Theorem 2. For any function $P(t)$ and $Q(t)$, if

$$\omega(t) = P^2(t) - 4 \max_{h \leq H} (P(t+h) - P(t))^2 - Q^2(t) > 0,$$

then

$$|P(t)| \geq 2 \max_{h \leq H} |P(t+h) - P(t)|$$

and

$$|P(t)| \geq |Q(t)|.$$

The first inequality implies, for any $0 \leq h \leq H$,

$$P(t) - \frac{1}{2}|P(t)| \leq P(t+h) \leq P(t) + \frac{1}{2}|P(t)|,$$

and hence $P(t+h)$ has the same sign as $P(t)$. Moreover, by the second inequality above we get

$$|P(t+h)| \geq \frac{1}{2}|P(t)| \geq \frac{1}{2}|Q(t)|.$$

Now take $P(t) = R_\pm(t)$ and $Q(t) = \delta t^{l-1/4}$ for a sufficiently small $\delta > 0$. By Lemma 6.1 and Lemma 6.2 we get

$$\begin{aligned}
(6.5) \quad \int_T^{2T} \omega(t) dt &\gg T^{2l+1/2} - O(HT^{2l} \log^5 T) - O(\delta^2 T^{2l+1/2}) \\
&\gg T^{2l+1/2}
\end{aligned}$$

by taking $H = \delta \sqrt{T} \log^{-5} T$. Let $\mathcal{S} = \{t \in [T, 2T] : \omega(t) > 0\}$. By (6.5), the Cauchy-Schwarz inequality and (1.13) with $k = 4$ we get

$$\begin{aligned}
T^{2l+1/2} &\ll \int_T^{2T} \omega(t) dt \leq \int_{\mathcal{S}} \omega(t) dt \leq \int_{\mathcal{S}} R_\pm^2(t) dt \\
&\leq |\mathcal{S}|^{1/2} \left(\int_T^{2T} R^4(t) dt \right)^{1/2} \ll |\mathcal{S}|^{1/2} T^{2l}.
\end{aligned}$$

Thus we get

$$|\mathcal{S}| \gg T.$$

This completes the proof of Theorem 2.

Remark for Theorem 4. The proof of Theorem 4 is the same except that we use $\log^3 T$ instead of $\log^5 T$. Here $\log^3 T$ appears since for $R_I(t)$ we can prove that the estimate

$$(6.6) \quad \int_1^T (R_I(t+h) - R_I(t))^2 dt \ll hT \log \frac{\sqrt{T}}{h} + T\mathcal{L} \log \mathcal{L}$$

holds for $1 \leq h \leq \sqrt{T}/2$, which implies that the $\log^5 T$ in Lemma 6.2 can be replaced by $\log^3 T$ if we have $R_I(t)$ in place of $R(t)$.

7. PROOF OF THEOREM 3

In this section we prove Theorem 3. Suppose $k \geq 3$ is a fixed odd integer and T is a large parameter. Define

$$\delta = \begin{cases} -1, & \text{if } C_{k,l} \geq 0, \\ 1, & \text{if } C_{k,l} < 0, \end{cases}$$

where $C_{k,l}$ is defined in the formula (1.13).

By Theorem 2, let $t \in [T, 2T]$ such that

$$\delta R(u) > c_5 t^{l-1/4}, u \in [t, t+H],$$

with $H = c_4 \sqrt{T} \log^{-5} T$. Then

$$\begin{aligned} c_5^k H t^{k(l-1/4)} &< \int_t^{t+H} \delta^k R^k(u) du = \delta^k \int_t^{t+H} R^k(u) du \\ &= C_{k,l} \delta^k \left((t+H)^{k(l-1/4)+1} - t^{k(l-1/4)+1} \right) \\ &\quad + \delta^k (\mathcal{F}_{k,l}(t+H) - \mathcal{F}_{k,l}(t)) \\ &= C_{k,l} \delta^k (k(l-1/4) + 1) H t^{k(l-1/4)} + O(H^2 t^{k(l-1/4)-1}) \\ &\quad + \delta^k (\mathcal{F}_{k,l}(t+H) - \mathcal{F}_{k,l}(t)). \end{aligned}$$

Hence we get

$$(7.1) \quad \delta^k (\mathcal{F}_{k,l}(t) - \mathcal{F}_{k,l}(t+H)) < B_{k,l} H t^{k(l-1/4)} (1 + O(HT^{-1})),$$

where

$$(7.2) \quad B_{k,l} = C_{k,l} \delta^k (k(l-1/4) + 1) - c_5^k \leq -c_5^k < 0.$$

From (7.1) and (7.2) we have

$$|\mathcal{F}_{k,l}(t) - \mathcal{F}_{k,l}(t+H)| \gg H t^{k(l-1/4)},$$

and Theorem 3 hence follows.

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