ON RESTRICTION OF MAXIMAL MULTIPLIERS IN WEIGHTED SETTINGS

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ABSTRACT. We obtain restriction results of K. De Leeuw's type for maximal operators defined through Fourier multipliers of either strong or weak type for weighted L^p spaces with $1 \leq p < \infty$. Applications to the case of Hörmander-Mihlin multipliers, singular integral operators and Bochner-Riesz sums are given.

1. INTRODUCTION

In 1965 K. De Leeuw proved that if $\mathbf{m}$ is a continuous function on $\mathbb{R}$ such that $\mathbf{m}$ is a Fourier multiplier on $L^p(\mathbb{R})$, its restriction to the integers $\mathbf{m}|_{\mathbb{Z}}$ is a Fourier multiplier on $L^p(\mathbb{T})$. Moreover, its norm does not exceed the norm of $\mathbf{m}$ as a multiplier on $L^p(\mathbb{R})$ (see [8, Proposition 3.3] and Jodeit's article [12]).

In 1980 C. Kenig and P. Tomas extended De Leeuw's result to maximal operators associated to a family of multipliers given by the dilations of a given one. More precisely, they proved that if $\mathbf{m}$ is a continuous function and if T_r denotes the multiplier operator associated to $\mathbf{m}_r(\xi) = \mathbf{m}(\xi/r)$, whenever $T^\sharp f(x) = \sup_{r>0} |T_r f(x)|$ is a bounded operator on $L^p(\mathbb{R}^d)$ the same holds for the maximal operator on $L^p(\mathbb{T}^d)$ associated to the multipliers $\mathbf{m}_r|_{\mathbb{Z}}$. Furthermore, its norm does not exceed a constant times the norm of $T^\sharp$. They also obtained similar results for operators of weak type for $p > 1$ (see [13]).

In 2003, E. Berkson and T.A. Gillespie extended De Leeuw's restriction result for multipliers on $L^p(\mathbb{R}, w)$ with w a 1-periodic weight belonging to $A_p(\mathbb{R})$ and $1 < p < \infty$. Such weights are said to be in the class $A_p(\mathbb{T})$. Their result is the following.

Theorem 1.1 ([4, Theorem 1.2]). *Let $1 < p < \infty$ and let $w \in A_p(\mathbb{T})$. If $\mathbf{m}$ is a continuous function on $\mathbb{R}$ such that it is a Fourier multiplier for $L^p(\mathbb{R}, w)$, then $\mathbf{m}|_{\mathbb{Z}}$ is a Fourier multiplier on $L^p(\mathbb{T}, w)$. Moreover, there is a constant $\mathbf{c}_{p,w}$ depending only on p and the A_p -constant of w , such that the norm of $\mathbf{m}|_{\mathbb{Z}}$ as a multiplier on $L^p(\mathbb{T}, w)$ does not exceed $\mathbf{c}_{p,w}$ times the norm of $\mathbf{m}$ as a multiplier on $L^p(\mathbb{R}, w)$.*

This theorem has been recently improved by K. Andersen and P. Mohanty as follows.

Theorem 1.2 ([1, Theorem 1.1]). *Let $1 < p < \infty$ and let $w \in L^1(\mathbb{T}^d)$. If $\mathbf{m}$ is a continuous function on $\mathbb{R}^d$ such that it is a Fourier multiplier on $L^p(\mathbb{R}^d, w)$,*

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then $\mathbf{m}|_{\mathbb{Z}}$ is a Fourier multiplier on $L^p(\mathbb{T}^d, w)$. Moreover, the norm of $\mathbf{m}|_{\mathbb{Z}}$ as a multiplier on $L^p(\mathbb{T}^d, w)$ does not exceed the norm of $\mathbf{m}$ as a multiplier on $L^p(\mathbb{R}^d, w)$.

The purpose of this paper is twofold:

i) To give restriction results from $\mathbb{R}^d$ to $\mathbb{T}^d$ for Fourier multipliers and for associated maximal operators of weak type (and strong type) in any dimension and for $1 \leq p < \infty$. In particular, we shall prove the following.

Theorem 1.3. *Let $1 \leq p < \infty$ and let w be a periodic weight on $\mathbb{R}^d$ satisfying $w \in L^1(\mathbb{T}^d)$. Suppose that $\{\mathbf{m}_j\}_j$ is a family of multipliers that are continuous functions satisfying that the associated maximal operator (see Definition 2.2 below) is bounded from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$ (or to $L^p(\mathbb{R}^d, w)$). Then the maximal operator associated to their restriction to the integers $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j$ (see Definition 3.1 below) is bounded from $L^p(\mathbb{T}^d, w)$ to $L^{p,\infty}(\mathbb{T}^d, w)$ (resp. to $L^p(\mathbb{T}^d, w)$) and its operator norm does not exceed c_p times the norm of the maximal operator associated to $\{\mathbf{m}_j\}_j$, where c_p is a constant that depends only on p .*

ii) K. De Leeuw in [8] and J. Jodeit in [12] also proved some restriction results for strong Fourier multipliers on $L^p(\mathbb{R}^d)$ to a lower dimensional space. In [7] a counterpart for Fourier multipliers on $L^p(\mathbb{R}^d, w)$ with w a suitable weight in $A_p(\mathbb{R}^d)$ was given. Namely, [7, Corollary 4.13] states that if $\mathbf{m}$ is a continuous and bounded function in $\mathbb{R}^d$ that is a Fourier multiplier on $L^p(\mathbb{R}^d, w)$ where $w = u \otimes v$ with $u \in A_p(\mathbb{R}^{d_1})$, $v \in A_p(\mathbb{R}^{d_2})$, then, for any $\xi \in \mathbb{R}^{d_1}$, the function $\mathbf{m}(\xi, \cdot)$ is a Fourier multiplier on $L^p(\mathbb{R}^{d_2}, v)$. In this setting, we shall prove the following.

Theorem 1.4. *Let $d = d_1 + d_2$, $1 \leq p < \infty$, $u \in A_p(\mathbb{R}^{d_1})$, $v \in A_p(\mathbb{R}^{d_2})$ and define $w(x, y) = u(x)v(y)$. Suppose that $\{\mathbf{m}_j\}_j$ is a family of multipliers that are continuous functions satisfying that the associated maximal operator is bounded from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$ (or to $L^p(\mathbb{R}^d, w)$). Then, fixed $\xi \in \mathbb{R}^{d_1}$, the maximal operator associated to the family $\{\mathbf{m}_j(\xi, \cdot)\}_j$ is bounded from $L^p(\mathbb{R}^{d_2}, v)$ to $L^{p,\infty}(\mathbb{R}^{d_2}, v)$ (resp. to $L^p(\mathbb{R}^{d_2}, v)$) and its operator norm does not exceed $c_{p,w}$ times the norm of the maximal operator associated to $\{\mathbf{m}_j\}_j$, where $c_{p,w}$ is a constant that depends only on p , d and the A_p -constant of w .*

We want to emphasize that the techniques developed in this paper are different from those in [1, 4, 7] where duality properties of Lebesgue spaces are strongly used. Our approach allows us to also consider the case of maximal multipliers of weak type $(1, 1)$, and deal with the difficulties derived from the fact that $L^{1,\infty}$ is not a Banach space. The endpoint case $p = 1$ is the weighted analogue of the results in [2, 15].

2. DEFINITIONS AND NOTATION

In this section we present some basic definitions needed for our consideration. Let $0 < p < \infty$ and let $(\mathcal{M}, \mu)$ be a σ -finite measure space. The space $L^{p,\infty}(\mu)$ is defined by the quasinorm $\|f\|_{L^{p,\infty}} = \sup_{t>0} t\mu_f(s)^{1/p}$, where $\mu_f(s) = \mu\{x : |f(x)| > s\}$. It is known (see [10, p. 485]) that, for every $q < p$,

$$(2.1) \quad \|f\|_{L^{p,\infty}(\mu)} \leq \sup \|f\chi_E\|_{L^q(\mu)} \mu(E)^{1/p-1/q} \leq c_{p,q} \|f\|_{L^{p,\infty}(\mu)},$$

where the supremum is taken on the family of sets of finite measure and $c_{p,q}^q = \frac{p}{p-q}$. The finiteness of the middle expression is called Kolmogorov's condition.

If ν is a positive measure absolutely continuous with respect to μ and w denotes the Radon-Nykodym derivative of ν with respect to μ , we shall write $L^p(w)$ for $L^p(\nu)$. If any confusion can arise, we shall write $L^p(\mathcal{M}, \mu)$ and $L^{p,\infty}(\mathcal{M}, \mu)$ to indicate the underlying measure space $\mathcal{M}$.

Let $\mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ denote the class of infinitely differentiable functions with compact support and the Schwartz class of test functions, respectively. As usual, $\mathfrak{B}(X, Y)$ indicates the set of bounded operators on X into Y and $\mathfrak{B}(X) = \mathfrak{B}(X, X)$.

A weight on $\mathbb{R}^d$ is a locally integrable function $w : \mathbb{R}^d \rightarrow [0, \infty)$ such that $0 < w < \infty$ a.e.

Definition 2.1. We say that a weight w belongs to the class $A_p(\mathbb{R}^d)$, and we write $w \in A_p(\mathbb{R}^d)$ if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1/p-1} dx \right)^{p-1} < \infty,$$

for $1 < p < \infty$, and

$$[w]_{A_1} = \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \|w^{-1} \chi_Q\|_\infty < +\infty,$$

where the supremum is taken over the family of cubes Q with sides parallel to the coordinate axis. These quantities will be referred to as the A_p -constant of w .

It is well known that, for $1 \leq p < \infty$ and $w \in A_p(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d, w)$ and $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, w)$. We refer the reader to [10, 11] for other properties and generalities of A_p -weights.

For any function f , we shall denote by $\hat{f}$ ($f^\vee$) the Fourier transform (resp. the inverse Fourier transform) of f , whenever it is well defined.

Definition 2.2. Let $1 \leq p < \infty$. A function $\mathbf{m} \in L^\infty(\mathbb{R}^d)$ is called a weak type multiplier on $L^p(\mathbb{R}^d, w)$ (in symbols, $\mathbf{m} \in M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)$) if the mapping $f \in \mathcal{S}(\mathbb{R}^d) \mapsto (\mathbf{m}\hat{f})^\vee$ can be extended from $\mathcal{S}(\mathbb{R}^d)$ to a continuous linear mapping $S_{\mathbf{m}}$ from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$. In this case we write

$$\|\mathbf{m}\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)} = \|S_{\mathbf{m}}\|_{\mathfrak{B}(L^p(\mathbb{R}^d, w), L^{p,\infty}(\mathbb{R}^d, w))}.$$

If $S_{\mathbf{m}} \in \mathfrak{B}(L^p(\mathbb{R}^d, w))$, we say that $\mathbf{m}$ is a Fourier multiplier on $L^p(\mathbb{R}^d, w)$ and write

$$\|\mathbf{m}\|_{M_{p,w}(\mathbb{R}^d)} = \|S_{\mathbf{m}}\|_{\mathfrak{B}(L^p(\mathbb{R}^d, w))}.$$

If $\{\mathbf{m}_j\}_j$ is a sequence in $M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)$, we denote by $\|\{\mathbf{m}_j\}_j\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)}$ the norm of the operator defined for $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$S_{\{\mathbf{m}_j\}_j}^\# f(x) = \sup_j |S_{\mathbf{m}_j} f(x)|,$$

provided it defines a continuous mapping from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$. If it extends to a bounded mapping on $L^p(\mathbb{R}^d, w)$, we write its norm by $\|\{\mathbf{m}_j\}_j\|_{M_{p,w}(\mathbb{R}^d)}$.

We shall denote by $\mathbb{T}^d$ the topological group $\mathbb{R}^d/\mathbb{Z}^d$, which can be identified with the cube $[0, 1)^d$ or eventually with $[-1/2, 1/2)^d$ in $\mathbb{R}^d$. Functions on $\mathbb{T}^d$ will be identified with functions on $\mathbb{R}^d$ which are 1-periodic in each variable. A function

$f : \mathbb{T}^d \rightarrow \mathbb{C}$ such that for a finitely supported sequence $\{a_k\}_{k \in \mathbb{Z}^d}$ of complex numbers written as

$$f(x) = \sum_{k \in \mathbb{Z}^d} a_k e^{2\pi i k x}$$

is called a trigonometric polynomial, and we write $f \in P(\mathbb{T}^d)$. Let us recall that $P(\mathbb{T}^d)$ is dense in $L^p(\mathbb{T}^d, \mu)$ for any Radon measure μ on $\mathbb{T}^d$.

From now on, we work in the range

$$1 \leq p < \infty,$$

and w is a weight in $\mathbb{R}^d$. Observe that if in addition w is 1-periodic, then $w \in L^1(\mathbb{T}^d)$.

3. RESTRICTION OF FOURIER MULTIPLIERS FROM $\mathbb{R}^d$ TO $\mathbb{T}^d$

Definition 3.1. A function $\mathbf{m} \in \ell^\infty(\mathbb{Z}^d)$ is a weak type multiplier on $L^p(\mathbb{T}^d, w)$ (in symbols, $\mathbf{m} \in M_{p,w}^{(\mathbf{w})}(\mathbb{T}^d)$) if the mapping

$$\sum_{k \in \mathbb{Z}^d} a_k e^{2\pi i k \theta} \in P(\mathbb{T}^d) \longrightarrow \sum_{k \in \mathbb{Z}^d} \mathbf{m}(k) a_k e^{2\pi i k \theta}$$

extends to a continuous operator $T_{\mathbf{m}} \in \mathfrak{B}(L^p(\mathbb{T}^d, w), L^{p,\infty}(\mathbb{T}^d, w))$. In this case,

$$\|\mathbf{m}\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{T}^d)} = \|T_{\mathbf{m}}\|_{\mathfrak{B}(L^p(\mathbb{T}^d, w), L^{p,\infty}(\mathbb{T}^d, w))}.$$

If $T_{\mathbf{m}} \in \mathfrak{B}(L^p(\mathbb{T}^d, w))$, $\mathbf{m}$ is said to be a multiplier on $L^p(\mathbb{T}^d, w)$, we denote it by $\mathbf{m} \in M_{p,w}(\mathbb{T}^d)$ and

$$\|\mathbf{m}\|_{M_{p,w}(\mathbb{T}^d)} = \|T_{\mathbf{m}}\|_{\mathfrak{B}(L^p(\mathbb{T}^d, w))}.$$

If $\{\mathbf{m}_j\}_j$ is a sequence in $M_{p,w}^{(\mathbf{w})}(\mathbb{T}^d)$ we denote by $\|\{\mathbf{m}_j\}_j\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{T}^d)}$ the norm of the operator defined for every $f \in P(\mathbb{T}^d)$ by

$$T_{\{\mathbf{m}_j\}_j}^\# f(x) = \sup_j |T_{\mathbf{m}_j} f(x)|,$$

provided it extends to a continuous mapping from $L^p(\mathbb{T}^d, w)$ to $L^{p,\infty}(\mathbb{T}^d, w)$. We shall write $\|\{\mathbf{m}_j\}_j\|_{M_{p,w}(\mathbb{T}^d)}$ in the case that it extends to a continuous operator on $L^p(\mathbb{T}^d, w)$.

3.1. Restriction results for weak type maximal multipliers.

Theorem 3.2. Let w be 1-periodic and let $\{\mathbf{m}_j\}_j \in M_{p,w}^{(\mathbf{w})}(\mathbb{R}^d)$ satisfying that, for each j , there exists $K_j \in L^1(\mathbb{R}^d)$ with compact support such that $\hat{K}_j(x) = \mathbf{m}_j(x)$ for every $x \in \mathbb{R}^d$. Then $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \in M_{p,w}^{(\mathbf{w})}(\mathbb{T}^d)$ and

$$\left\| \{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \right\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{T}^d)} \leq c_p \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{R}^d)},$$

where c_p depends only on p .

Proof. Let $\mathfrak{N} = \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{R}^d)}$. Since convolution operators commute with translations, it follows that for every $\theta \in [0, 1)^d$ and every $N \in \mathbb{N}$,

$$(3.1) \quad \left\| \sup_{1 \leq j \leq N} |K_j * g| \right\|_{L^{p,\infty}(\mathbb{R}^d, w(\cdot + \theta))} \leq \mathfrak{N} \|g\|_{L^p(\mathbb{R}^d, w(\cdot + \theta))}.$$

Now, given $f(\theta) = \sum_k a_k e^{2\pi i k \theta} \in P(\mathbb{T}^d)$, let us consider

$$\begin{aligned} \tilde{T}_{K_j} f(\theta) &= \int_{\mathbb{R}^d} K_j(x) f(\theta - x) dx = \sum_k a_k \int_{\mathbb{R}^d} K_j(x) e^{2\pi i k(\theta - x)} dx \\ &= \sum_{k \in \mathbb{Z}^d} a_k \mathbf{m}_j(k) e^{2\pi i k \theta}, \end{aligned}$$

that is, $\tilde{T}_{K_j}$ coincides with the multiplier operator $T_{\mathbf{m}_j|_{\mathbb{Z}^d}}$.

Let $Q_r = (-r, r)^d$ with $r > 0$ such that $\text{supp } K_j \subset Q_r$ for $j = 1, \dots, N$. Let $q < p$, and for any measurable $E \subset [0, 1)^d$, let $\tilde{E} = \bigcup_{k \in \mathbb{Z}^d} E + k$ be its periodic extension. Set $E_\theta = \{x \in \mathbb{R}^d : x + \theta \in \tilde{E}\}$ with $\theta \in \mathbb{T}^d$ and $R_x f(\theta) = f(\theta + x)$. Then, by translation invariance, we have that, for every $x \in \mathbb{R}^d$,

$$(3.2) \quad \left\| \sup_{1 \leq j \leq N} |\tilde{T}_{K_j} f| \chi_E \right\|_{L^q(\mathbb{T}^d, w)}^q = \int_{\mathbb{T}^d} \sup_{1 \leq j \leq N} |R_x \tilde{T}_{K_j} f(\theta)|^q w(x + \theta) \chi_{\tilde{E}}(x + \theta) d\theta.$$

Therefore, for every $s > 0$,

$$\begin{aligned} &\left\| \sup_{1 \leq j \leq N} |\tilde{T}_{K_j} f| \chi_E \right\|_{L^q(\mathbb{T}^d, w)}^q \\ &= \frac{1}{(2s)^d} \int_{Q_s} \int_{\mathbb{T}^d} \sup_{1 \leq j \leq N} |R_x T_{K_j} f(\theta)|^q w(x + \theta) \chi_{\tilde{E}}(x + \theta) d\theta dx. \end{aligned}$$

Now, using that $\text{supp } K_j \subset Q_r$ for $j = 1, \dots, N$, one can easily see that, if $x \in Q_s$,

$$R_x \tilde{T}_{K_j} f(\theta) = B_{K_j} (R_{(\cdot)} f(\theta) \chi_{Q_{r+s}})(x),$$

where $B_{K_j}(h)(x) = (K_j * h)(x)$, and hence,

$$\begin{aligned} &\left\| \sup_{1 \leq j \leq N} |\tilde{T}_{K_j} f| \chi_E \right\|_{L^q(\mathbb{T}^d, w)}^q \\ &\leq \frac{1}{(2s)^d} \int_{\mathbb{T}^d} \left\{ \int_{E_\theta \cap Q_s} \sup_{1 \leq j \leq N} |B_{K_j} (R_{(\cdot)} f(\theta) \chi_{Q_{r+s}})(x)|^q w(x + \theta) dx \right\} d\theta. \end{aligned}$$

By (2.1) and (3.1), the term inside curly brackets is bounded by

$$(c_{p,q} \mathfrak{N})^q \left\{ \int_{Q_{r+s}} |R_x f(\theta)|^p w(x + \theta) dx \right\}^{\frac{q}{p}} \left\{ \int_{E_\theta \cap Q_s} w(x + \theta) dx \right\}^{1 - \frac{q}{p}}.$$

Also, using Hölder's inequality, it follows that

$$\begin{aligned} &\left\| \sup_{1 \leq j \leq N} |\tilde{T}_{K_j} f| \chi_E \right\|_{L^q(\mathbb{T}^d, w)}^q \\ &\leq \frac{c_{p,q}^q \mathfrak{N}^q}{(2s)^d} \left\{ \int_{\mathbb{T}^d} \int_{Q_{r+s}} |R_x f(\theta)|^p w(x + \theta) dt d\theta \right\}^{\frac{q}{p}} \left\{ \int_{\mathbb{T}^d} \int_{Q_s \cap E_\theta} w(x + \theta) dx d\theta \right\}^{1 - \frac{q}{p}} \\ &\leq \frac{c_{p,q}^q \mathfrak{N}^q}{(2s)^d} (2(r+s))^{\frac{dq}{p}} (2s)^{d(1 - \frac{q}{p})} w(E)^{1 - \frac{q}{p}} \|f\|_{L^p(\mathbb{T}^d, w)}^q \\ &\leq c_{p,q}^q \mathfrak{N}^q \left(\frac{r+s}{s} \right)^{\frac{dq}{q}} w(E)^{1 - \frac{q}{p}} \|f\|_{L^p(\mathbb{T}^d, w)}^q. \end{aligned}$$

Thus, taking $s \rightarrow +\infty$, and using Kolmogorov's condition (2.1), we obtain that

$$\left\| \sup_{1 \leq j \leq N} \left| \tilde{T}_{K_j} f \right| \right\|_{L^{p,\infty}(\mathbb{T}^d, w)} \leq c_{p,q} \mathfrak{N} \|f\|_{L^p(\mathbb{T}^d, w)}.$$

Now, considering $c_p = \inf_{q < p} c_{p,q}$, the result easily follows by Fatou's Lemma and the density of $P(\mathbb{T}^d)$ in $L^p(\mathbb{T}^d, w)$. $\square$

The next step is to weaken the hypothesis assumed on $\mathbf{m}_j$ in the previous theorem as is done both in [4] and [1]. As usually happens, this is the technical part of the work.

Definition 3.3. A bounded function $\mathbf{m}$ defined in $\mathbb{R}^d$ is *normalized* if for any $x \in \mathbb{R}^d$,

$$\lim_n \widehat{\varphi_n} * \mathbf{m}(x) = \mathbf{m}(x),$$

where $\varphi_n(x) = \varphi(x/n)$, $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\widehat{\varphi} \geq 0$ and $\|\widehat{\varphi}\|_1 = 1$.

It is easy to see that $\lim_n \widehat{\varphi_n} * \mathbf{m}(x) = \mathbf{m}(x)$ for every Lebesgue point x of $\mathbf{m}$. In particular, any continuous and bounded function is normalized.

In order to extend Theorem 3.2 to the class of normalized multipliers, we shall need some previous lemmas. The following one is a direct consequence of the proof of [15, Lemma 2.6] for $G = \mathbb{R}^d$.

Lemma 3.4. Let $J \in \mathbb{N}$ and let $\{\mathbf{m}_j\}_{j=1}^J$ be a family of $L^\infty(\mathbb{R}^d)$ functions. For $f \in \mathcal{S}(\mathbb{R}^d)$, $j = 1, \dots, J$ and $x \in \mathbb{R}^d$, let

$$F_{j,x}(\xi) = S_{\mathbf{m}_j}(e^{-2\pi i \xi \cdot} f)(x), \quad \xi \in \mathbb{R}^d.$$

Let $\mathcal{K}$ be a compact set. Then, for each $k \in \mathbb{N} \setminus \{0\}$, there exists a finite family $\{V_l^k\}_{l=1}^{I_k}$ of pairwise disjoint measurable sets in $\mathbb{R}^d$ such that

- (1) $\mathcal{K} \subset \bigcup_{l=1}^{I_k} V_l^k$,
- (2) if $l = 1, \dots, I_k$ and $\xi, \zeta \in V_l^k$, then

$$|F_{j,x}(\xi) - F_{j,x}(\zeta)| \leq 1/k,$$

uniformly on $j \in \{1, \dots, J\}$ and $x \in \mathbb{R}$.

Another key ingredient is the following version of Marcinkiewicz-Zygmund's inequality, whose proof is analogous to that given in [10, Theorem V.2.9] for $p = q = 1$ for linear operators.

Theorem 3.5. Let $\{T_j\}_j$ be a countable family of linear operators such that

$$\left\| \sup_j |T_j f| \right\|_{L^{1,\infty}(\mathbb{R}^d, w)} \leq \|\{T_j\}_j\| \|f\|_{L^1(\mathbb{R}^d, w)}.$$

Then

$$\left\| \sup_j \left(\sum_l |T_j f_l|^2 \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R}^d, w)} \leq \mathfrak{c}_1 \|\{T_j\}_j\| \left\| \left(\sum_l |f_l|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d, w)},$$

where

$$(3.3) \quad \mathfrak{c}_1 := \inf_{0 < r < 1} \frac{\sqrt{\pi}}{2 \left((1-r) \Gamma \left(1 + \frac{r}{2} \right) \right)^{1/r}}.$$

For $p > 1$, the next lemma is an immediate consequence of Minkowski's inequality, as $L^{p,\infty}$ is normable, but for $p = 1$ the convexity of the space $L^{1,\infty}$ fails. Similar results in the unweighted setting are given by [3, Lemma 2.1] and [4, Theorem 1.2].

Lemma 3.6. *Let $\varphi \in L^1(\mathbb{R}^d)$ and $\{\mathbf{m}_j\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)$. Then $\{\varphi * \mathbf{m}_j\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)$ and*

$$(3.4) \quad \left\| \{\varphi * \mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \leq \mathfrak{c}_p \|\varphi\|_{L^1(\mathbb{R}^d)} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)},$$

where $\mathfrak{c}_p = p'$ if $p > 1$ and $\mathfrak{c}_1$ is the constant given in (3.3).

Proof. We shall only prove the case $p = 1$. Without loss of generality, we can assume that $\{\mathbf{m}_j\}_j$ is a finite family of multipliers of cardinality, say $J \in \mathbb{N}$. For $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int (\varphi * \mathbf{m}_j)(\xi) \widehat{g}(\xi) e^{2\pi i \xi x} d\xi = \int \varphi(y) e^{2\pi i x y} S_{\mathbf{m}_j}(e^{-2\pi i y \cdot} g)(x) dy.$$

Hence,

$$(3.5) \quad |S_{\varphi * \mathbf{m}_j} g(x)| \leq \int |\varphi(y)| |S_{\mathbf{m}_j}(e^{-2\pi i y \cdot} g)(x)| dy,$$

and thus

$$\sup_{1 \leq j \leq J} |S_{\varphi * \mathbf{m}_j} g(x)| \leq \int |\varphi(y)| \sup_{1 \leq j \leq J} |S_{\mathbf{m}_j}(e^{-2\pi i y \cdot} g)(x)| dy.$$

Let us first assume that $\varphi \in L^1(\mathbb{R}^d)$ is supported on a compact set $\mathcal{K}$. For each $k \geq 1$ let $\{V_l^k\}_{l=1}^{I_k}$ be the family of pairwise disjoint sets given by Lemma 3.4, and for each l , select $y_l^k \in V_l^k$. Then, for every $y \in \mathcal{K}$ and any $k \geq 1$, there exists a unique $l \in \{1, \dots, I_k\}$ such that $y \in V_l^k$, and hence

$$\left| S_{\mathbf{m}_j}(e^{-2\pi i y \cdot} g)(x) - S_{\mathbf{m}_j}(e^{-2\pi i y_l^k \cdot} g)(x) \right| \leq \frac{1}{k},$$

uniformly on $j = 1, \dots, J$ and $x \in \mathbb{R}^d$. It follows that for every $x \in \mathbb{R}^d$, any $j \in \{1, \dots, J\}$ and all $y \in \mathcal{K}$,

$$\lim_k \sum_{l=1}^{I_k} S_{\mathbf{m}_j}(e^{-2\pi i y_l^k \cdot} g)(x) \chi_{V_l^k}(y) = S_{\mathbf{m}_j}(e^{-2\pi i y \cdot} g)(x).$$

Then, by Fatou's Lemma on (3.5),

$$\sup_{1 \leq j \leq J} |S_{\varphi * \mathbf{m}_j} g(x)| \leq \liminf_k \sup_{1 \leq j \leq J} \left(\sum_{l=1}^{I_k} |S_{\mathbf{m}_j}(e^{-2\pi i y_l^k \cdot} g)(x)| \lambda_l^k \right),$$

where $\lambda_l^k = \int_{V_l^k} |\varphi(y)| dy$. Observe that the term inside brackets is less than or equal to

$$\|\varphi\|_{L^1(\mathbb{R}^d)}^{1/2} \left(\sum_{l=1}^{I_k} \left| S_{\mathbf{m}_j} \left(\sqrt{\lambda_l^k} e^{-2\pi i y_l^k \cdot} g \right)(x) \right|^2 \right)^{1/2},$$

where we have used that $\sum_{l=1}^{I_k} \lambda_l^k = \int_{\mathfrak{I}_{l=1}^{I_k}} V_l^k |\varphi(y)| \, dy = \|\varphi\|_{L^1(\mathbb{R}^d)}$. Then,

$$\begin{aligned} & \left\| \sup_{1 \leq j \leq J} |S_{\varphi * \mathbf{m}_j} g| \right\|_{L^{1,\infty}(\mathbb{R}^d, w)} \\ & \leq \|\varphi\|_{L^1(\mathbb{R}^d)}^{1/2} \liminf_k \left\| \sup_{1 \leq j \leq J} \left(\sum_{l=1}^{I_k} \left| S_{\mathbf{m}_j} \left(\sqrt{\lambda_l^k} e^{-2\pi i y_l^k \cdot} g \right) (x) \right|^2 \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R}^d, w)}. \end{aligned}$$

Applying Theorem 3.5 with the family of operators $\{S_{\mathbf{m}_j}\}_j$ to the functions $f_l = \sqrt{\lambda_l^k} e^{-2\pi i y_l^k \cdot} g$, we obtain that

$$\begin{aligned} & \left\| \sup_{1 \leq j \leq J} \left(\sum_{l=1}^{I_k} \left| S_{\mathbf{m}_j} \left(\sqrt{\lambda_l^k} e^{-2\pi i y_l^k \cdot} g \right) \right|^2 \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R}^d, w)} \\ & \leq \mathfrak{c}_1 \left\| \{\mathbf{m}_j\}_j \right\|_{M_{1,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \left\| \left(\sum_{l=1}^{I_k} \left| \sqrt{\lambda_l^k} e^{-2\pi i y_l^k \cdot} g \right|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d, w)} \\ & = \mathfrak{c}_1 \left\| \{\mathbf{m}_j\}_j \right\|_{M_{1,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \|\varphi\|_{L^1(\mathbb{R}^d)}^{1/2} \|g\|_{L^1(\mathbb{R}^d, w)}. \end{aligned}$$

Therefore,

$$(3.6) \quad \left\| \sup_{1 \leq j \leq J} |S_{\varphi * \mathbf{m}_j} g| \right\|_{L^{1,\infty}(\mathbb{R}^d, w)} \leq \mathfrak{c}_1 \|\varphi\|_{L^1(\mathbb{R}^d)} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{1,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d, w)}.$$

In the case that φ is not compactly supported, considering $\varphi_n = \varphi \chi_{B(0,n)}$, we can write

$$\sup_{1 \leq j \leq J} |S_{\varphi * \mathbf{m}_j} g(x)| \leq \lim_n \int |\varphi_n(y)| \sup_{1 \leq j \leq J} |S_{\mathbf{m}_j} (e^{-2\pi i y \cdot} g)(x)| \, dy,$$

and using the previous argument we obtain that

$$\left\| \sup_{1 \leq j \leq J} |S_{\varphi * \mathbf{m}_j} g| \right\|_{L^{1,\infty}(\mathbb{R}^d, w)} \leq \mathfrak{c}_1 \liminf_n \|\varphi_n\|_{L^1(\mathbb{R}^d)} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{1,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d, w)},$$

from where it follows that (3.6) holds for any $\varphi \in L^1(\mathbb{R}^d)$. The result now follows by the density of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d, w)$. $\square$

Lemma 3.7. *Let $w \in \mathcal{C}(\mathbb{T}^d)$ such that $\inf_{x \in \mathbb{T}^d} w(x) > 0$. Consider $h \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ satisfying $0 \leq h \leq 1$ and $\int_{\mathbb{R}^d} h = 1$ and define $h_n(x) = n^d h(nx)$. Then,*

- (1) *There exists $n_0 = n_0(w) \in \mathbb{N}$ such that, for any $p \in [1, \infty)$,*

$$\sup_{n \geq n_0} \|\widehat{h_n}\|_{M_{p,w}(\mathbb{R}^d)} \leq 2^{1/p}.$$

- (2) $\sup_n \|\widehat{h_n}\|_{L^\infty(\mathbb{R}^d)} \leq 1$.

- (3) *For every $\xi \in \mathbb{R}^d$, $\lim_n \widehat{h_n}(\xi) = 1$.*

Proof. Since $\|h_n\|_{L^1} = 1$, it follows that $\|\widehat{h_n}\|_\infty \leq 1$. On the other hand, for every $\xi \in \mathbb{R}^d$ and for every $\epsilon > 0$ there exists n_0 such that for all $|x| < \frac{1}{n_0}$, $|1 - e^{2\pi i x \xi}| < \epsilon$.

Hence, for every $n \geq n_0$,

$$\left| 1 - \widehat{h_n}(\xi) \right| \leq \int h_n(x) |1 - e^{2\pi i x \xi}| dx \leq \epsilon.$$

Then, it follows that $\widehat{h_n} \rightarrow 1$ pointwise. It remains to show that $\|\widehat{h_n}\|_{M_{p,w}(\mathbb{R}^d)}$ are uniformly bounded on n .

Observe that $\|f\|_{L^\infty(w)} = \|f\|_{L^\infty}$ and hence, for any $n \geq 1$,

$$\|h_n * f\|_{L^\infty(\mathbb{R}^d, w)} \leq \|f\|_{L^\infty(\mathbb{R}^d, w)}.$$

Let $\delta = \inf_{x \in \mathbb{T}^d} w(x) > 0$. Since $w \in \mathcal{C}(\mathbb{T}^d)$, there exists $n_0 = n_0(\delta)$ such that, for any $n \geq n_0$, for any x and any $y \in \text{supp } h_n$,

$$|w(x) - w(x - y)| \leq \delta,$$

which implies that, for any $x \in \mathbb{T}^d$,

$$h_n * w(x) \leq \delta + w(x) \leq 2w(x).$$

Then, for $n \geq n_0$,

$$\|h_n * f\|_{L^1(\mathbb{R}^d, w)} \leq \int_{\mathbb{R}^d} |f(y)| h_n * w(y) dy \leq 2 \|f\|_{L^1(\mathbb{R}^d, w)}.$$

In other words, we have seen that for $n \geq n_0$ the linear operator defined by $T_n f = h_n * f$ is uniformly bounded on $L^1(\mathbb{R}^d, w)$ and $L^\infty(\mathbb{R}^d, w)$ with norm respectively bounded by 2 and 1. Riesz-Thorin's Theorem implies the result. $\square$

Lemma 3.8. *Let w be 1-periodic. If $g \in \mathcal{C}_c(\mathbb{R}^d)$ is nonnegative, $\int_{\mathbb{R}^d} g = 1$ and $\text{supp } g \subset [-1/2, 1/2]^d$, then $\inf_{x \in \mathbb{R}^d} g * w(x) > 0$.*

Proof. Clearly $g * w \in \mathcal{C}(\mathbb{T}^d)$, and hence there exists $x_0 \in [-1/2, 1/2]^d$ such that $\inf_{x \in \mathbb{R}^d} g * w(x) = \min_{|x|_\infty \leq 1/2} g * w(x) = g * w(x_0)$. Since $g \in \mathcal{C}_c(\mathbb{R}^d)$, there exists a set of positive Lebesgue measure Q where $g(y) > 0$ for all $y \in Q$. Thus if $0 = g * w(x_0) = \int g(y) w(x_0 - y) dy$, then $g(y) w(x_0 - y) = 0$ a.e. $y \in Q$, which implies that $w(z) = 0$ a.e. $z \in x_0 - Q$, but this contradicts the fact that the set $\{x : w(x) = 0\}$ is null. $\square$

Lemma 3.9. *Let T be any bounded operator from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$ that commutes with translations. Then, for any nonnegative function $g \in \mathcal{C}_c(\mathbb{R}^d)$, T is bounded from $L^p(\mathbb{R}^d, g * w)$ to $L^{p,\infty}(\mathbb{R}^d, g * w)$ and*

$$\|T\|_{\mathcal{B}(L^p(\mathbb{R}^d, g * w), L^{p,\infty}(\mathbb{R}^d, g * w))} \leq c_p \|T\|_{\mathcal{B}(L^p(\mathbb{R}^d, w), L^{p,\infty}(\mathbb{R}^d, w))},$$

where $c_p = \inf_{q < p} \left(\frac{p}{p-q} \right)^{\frac{1}{q}}$.

Proof. Let E be any measurable set in $\mathbb{R}^d$ such that $0 < g * w(E) < +\infty$. Then, for any $q < p$,

$$\begin{aligned} \|Tf \chi_E\|_{L^q(\mathbb{R}^d, g * w)}^q &= \int_E |Tf(x)|^q g * w(x) dx = \int g(y) \int_E |Tf(x)|^q w(x - y) dx dy \\ &= \int g(y) \int_{E-y} |Tf_y(x)|^q w(x) dx dy, \end{aligned}$$

with $f_y(z) = f(z + y)$. Thus, by the boundedness hypothesis, Kolmogorov's condition and Hölder's inequality,

$$\begin{aligned} \|Tf\chi_E\|_{L^q(\mathbb{R}^d, g*w)}^q &\leq \|T\|^q \int g(y)w(E-y)^{1-\frac{q}{p}} \left(\int |f_y(x)|^p w(x) dx \right)^{q/p} dy \\ &\leq c_{p,q}^q \|T\|^q (g*w(E))^{1-\frac{q}{p}} \|f\|_{L^p(\mathbb{R}^d, g*w)}^q, \end{aligned}$$

where $c_{p,q}^q = p/(p-q)$. Then, the result follows by Kolmogorov's condition and by taking the infimum for $q < p$. $\square$

Theorem 3.10. *Let w be a 1-periodic weight on $\mathbb{R}^d$. Suppose that $\{\mathbf{m}_j\}_j$ are normalized functions and $\{\mathbf{m}_j\}_j \in M_{p,w}^{(\mathbf{w})}(\mathbb{R}^d)$. Then $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \in M_{p,w}^{(\mathbf{w})}(\mathbb{T}^d)$ and*

$$\left\| \{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \right\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{T}^d)} \leq c_p \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{R}^d)},$$

where c_p depends only on p .

Proof. Let $\{g_l\}_l$ be a family of nonnegative functions in $\mathcal{C}_c^\infty(\mathbb{R}^d)$, supported in $[-1/2, 1/2]^d$ such that it is an approximation of the identity in $L^1(\mathbb{T}^d)$. We can also assume that $\lim_l g_l * w(x) = w(x)$ a.e. $x \in [-1/2, 1/2]^d$.

For a fixed $l \in \mathbb{N}$, by Lemma 3.9,

$$\left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,g_l*w}^{(\mathbf{w})}(\mathbb{R}^d)} \leq c_p \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{R}^d)}.$$

By Lemma 3.8 it follows that for any $h \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $0 \leq h \leq 1$ and that $\int_{\mathbb{R}^d} h = 1$, there exists an n_l such that, for any $n \geq n_l$, the conclusions of Lemma 3.7 hold for the periodic weight $g_l * w$.

Consider, for $j, n \in \mathbb{N}$,

$$\mathbf{m}_{j,n}(\xi) = \widehat{K_{j,n}}(\xi) = (\widehat{\varphi_n} * \mathbf{m}_j)(\xi) \widehat{h_n}(\xi),$$

where φ_n are the functions given by the normalized condition. First observe that $\widehat{K_{j,n}} \in \mathcal{S}(\mathbb{R}^d)$ and hence $K_{j,n} \in \mathcal{S}(\mathbb{R}^d)$. Moreover, since

$$K_{j,n}(x) = (\varphi_n \mathbf{m}_j^\vee)(h_n(x - \cdot)) = \mathbf{m}_j^\vee(\varphi_n(\cdot) h_n(x - \cdot)),$$

and φ_n, h_n are compactly supported, it follows that $K_{j,n} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. On the other hand, since $\mathbf{m}_j$ is normalized and $\widehat{h_n} \rightarrow 1$, it holds that for every $\xi \in \mathbb{R}^d$,

$$\lim_n \widehat{K_{j,n}}(\xi) = \mathbf{m}_j(\xi).$$

Since $\|\widehat{h_n}\|_{L^\infty(\mathbb{R}^d)} \leq 1$ and $\|\widehat{\varphi_n}\|_{L^1(\mathbb{R}^d)} \leq 1$, then $\|\mathbf{m}_{j,n}\|_{L^\infty(\mathbb{R}^d)} \leq \|\mathbf{m}_j\|_{L^\infty(\mathbb{R}^d)}$.

Let us fix $J \in \mathbb{N}$. Since for any $f \in \mathcal{C}_c^\infty(\mathbb{R})$

$$K_{j,n} * f = T_{\widehat{\varphi_n} * \mathbf{m}_j}(h_n * f),$$

it follows that for every $n \geq n_l$,

$$\begin{aligned} \left\| \sup_{1 \leq j \leq J} |K_{j,n} * f| \right\|_{L^{p,\infty}(\mathbb{R}^d, g_l * w)} &\leq \left\| \{\widehat{\varphi_n} * \mathbf{m}_j\}_j \right\|_{M_{p,g_l*w}^{(\mathbf{w})}(\mathbb{R}^d)} \|h_n * f\|_{L^p(\mathbb{R}^d, g_l * w)} \\ &\leq c_p 2^{\frac{1}{p}} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d, g_l * w)}, \end{aligned}$$

where we have used that, by Lemma 3.6,

$$\left\| \{\widehat{\varphi}_n * \mathbf{m}_j\}_j \right\|_{M_{p, g_l * w}^{(\mathbf{w})}(\mathbb{R}^d)} \leq c_p \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p, g_l * w}^{(\mathbf{w})}(\mathbb{R}^d)} \leq c_p^2 \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p, w}^{(\mathbf{w})}(\mathbb{R}^d)}.$$

We can now apply Theorem 3.2 to deduce that for any $n \geq n_l$,

$$\left\| \{\mathbf{m}_{j,n}|_{\mathbb{Z}^d}\}_{j=1}^J \right\|_{M_{p, g_l * w}^{(\mathbf{w})}(\mathbb{T}^d)} \leq 2^{\frac{1}{p}} c_p^2 \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p, w}^{(\mathbf{w})}(\mathbb{R}^d)}.$$

Since, for any $f \in P(\mathbb{T}^d)$,

$$\lim_n S_{\mathbf{m}_{j,n}} f(s) = \lim_n \sum_{k \in \mathbb{Z}^d} \mathbf{m}_{j,n}(k) \widehat{f}(k) e^{2\pi i k s} = \sum_{k \in \mathbb{Z}^d} \mathbf{m}_j(k) \widehat{f}(k) e^{2\pi i k s} = S_{\mathbf{m}_j} f(s),$$

by Fatou's Lemma, the following inequality holds:

$$\begin{aligned} \left\| \sup_{1 \leq j \leq J} |S_{\mathbf{m}_j} f| \right\|_{L^{p, \infty}(\mathbb{T}^d, g_l * w)} &\leq \liminf_n \left\| \sup_{1 \leq j \leq J} |S_{\mathbf{m}_{j,n}} f| \right\|_{L^{p, \infty}(\mathbb{T}^d, g_l * w)} \\ &\leq 2^{\frac{1}{p}} c_p^2 \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p, w}^{(\mathbf{w})}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{T}^d, g_l * w)}, \end{aligned}$$

which implies

$$\left\| \sup_{1 \leq j} |S_{\mathbf{m}_j} f| \right\|_{L^{p, \infty}(\mathbb{T}^d, w)} \leq 2^{\frac{1}{p}} c_p^2 \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p, w}^{(\mathbf{w})}(\mathbb{R}^d)} \liminf_{l \rightarrow \infty} \|f\|_{L^p(\mathbb{T}^d, g_l * w)}.$$

Observe that

$$\|f\|_{L^p(\mathbb{T}^d, g_l * w)} \leq \|f\|_{L^\infty(\mathbb{T}^d)} \|g_l * w - w\|_{L^1(\mathbb{T}^d)} + \|f\|_{L^p(\mathbb{T}^d, w)},$$

and since $\lim_l \|g_l * w - w\|_{L^1(\mathbb{T}^d)} = 0$, it follows that

$$\left\| \sup_{1 \leq j} |S_{\mathbf{m}_j} f| \right\|_{L^{p, \infty}(\mathbb{T}^d, w)} \leq 2^{\frac{1}{p}} c_p^2 \|f\|_{L^p(\mathbb{T}^d, w)}.$$

The result follows by the density of $P(\mathbb{T}^d)$ in $L^p(\mathbb{T}^d, w)$. $\square$

With minor modifications in the proofs, the analogous result for operators of strong type can be proved. In the particular case of a single multiplier, we recover K. Andersen and P. Mohanty's [1, Theorem 1.1].

Theorem 3.11. *Let w be 1-periodic. Suppose that $\{\mathbf{m}_j\}_j \subset M_{p, w}(\mathbb{R}^d)$ and that they are normalized functions. Then $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \subset M_{p, w}(\mathbb{T}^d)$ and*

$$\left\| \{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \right\|_{M_{p, w}(\mathbb{T}^d)} \leq 2^{1/p} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p, w}(\mathbb{R}^d)}.$$

3.2. An improvement for nonperiodic weights. A similar approach to that in the previous section allows us to obtain a more general version of Theorem 3.10 (and also of Theorem 3.11) for a class of nonnecessarily periodic weights which includes those in $A_p(\mathbb{T}^d)$.

Definition 3.12. We say that a weight $v \in W(\mathbb{R}^d)$ if it satisfies the following conditions:

i) For every $x \in \mathbb{R}^d$, $\theta \in [0, 1)^d$,

$$\frac{1}{\zeta} \leq \frac{v(x)}{v(x + \theta)} \leq \zeta.$$

ii)

$$\lim_{s \rightarrow \infty} \frac{v(Q_{r+s})}{v(Q_s)} = 1.$$

Theorem 3.13. *Let u be a periodic weight in $\mathbb{R}^d$, let $v \in W$ and set $w = uv$. Assume that $\{\mathbf{m}_j\}_j \in M_{p,w}^{(\mathbf{w})}(\mathbb{R}^d)$ (respectively $M_{p,w}(\mathbb{R}^d)$) and that they are normalized functions in $\mathbb{R}^d$. Then $\{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \in M_{p,w}^{(\mathbf{w})}(\mathbb{T}^d)$ (respectively $M_{p,w}(\mathbb{T}^d)$) and*

$$\left\| \{\mathbf{m}_j|_{\mathbb{Z}^d}\}_j \right\|_{M_{p,u}^{(\mathbf{w})}(\mathbb{T}^d)} \leq c_{p,w} c_{p,v} \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,u}^{(\mathbf{w})}(\mathbb{R}^d)}$$

(respectively replacing $M_{p,u}^{(\mathbf{w})}$ with $M_{p,u}$ in the previous inequality), where $c_{p,w}$ and $c_{p,v}$ depend only on p .

Proof. We shall prove the weak case. The proof for the strong case is similar and we leave the details to the reader. Assume first that $\{\mathbf{m}_j\}_j$ is a finite sequence. The argument is similar to that for Theorem 3.2, and we shall sketch the major changes to be done in the proof.

Let $\mathfrak{N} = \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathbf{w})}(\mathbb{R}^d)}$. Since $v \in W$,

$$(3.7) \quad \frac{1}{\zeta} w(x + \theta) \leq u(x + \theta) v(x) \leq \zeta w(x + \theta).$$

By (3.2), for every $f \in P(\mathbb{T})$ and every measurable set $E \subset \mathbb{T}$,

$$\begin{aligned} & \left\| \sup_{1 \leq j \leq N} |T_{K_j} f| \chi_E \right\|_{L^q(\mathbb{T}^d, u)}^q \\ &= \frac{1}{v(Q_s)} \int_{Q_s} \int_{\mathbb{T}^d} \sup_{1 \leq j \leq N} |R_x T_{K_j} f(\theta)|^q u(x + \theta) v(x) \chi_{\tilde{E}}(x + \theta) d\theta dx \\ &\leq \frac{\zeta}{v(Q_s)} \int_{\mathbb{T}^d} \left\{ \int_{E_\theta \cap Q_s} \sup_{1 \leq j \leq N} |B_{K_j}(R_{(\cdot)} f(\theta) \chi_{Q_{r+s}})(x)|^q w(x + \theta) dx \right\} d\theta, \end{aligned}$$

where $E_\theta = \{x \in \mathbb{R}^d : x + \theta \in \tilde{E}\}$ and $\tilde{E}$ is the periodic extension of E . By (2.1) and (3.1), the term inside curly brackets is bounded by

$$(c_{p,q} \mathfrak{N})^q \left\{ \int_{Q_{r+s}} |R_x f(\theta)|^p w(x + \theta) dt \right\}^{\frac{q}{p}} \left\{ \int_{E_\theta \cap Q_s} w(x + \theta) dt \right\}^{1 - \frac{q}{p}}.$$

Hence, by Hölder's inequality, it follows that

$$\begin{aligned} & \left\| \sup_{1 \leq j \leq N} |T_{K_j} f| \chi_E \right\|_{L^q(\mathbb{T}^d, u)}^q \\ &\leq \frac{c_{p,q}^q \mathfrak{N}^q \zeta}{v(Q_s)} \left[\int_{\mathbb{T}^d} \int_{Q_{r+s}} |R_x f(\theta)|^p w(x + \theta) dt d\theta \right]^{\frac{q}{p}} \left[\int_{\mathbb{T}^d} \int_{Q_s \cap E_\theta} w(x + \theta) dx d\theta \right]^{1 - \frac{q}{p}}. \end{aligned}$$

By (3.7), the first term is bounded by $[\zeta v(Q_{r+s})]^{\frac{q}{p}} \|f\|_{L^p(\mathbb{T}, u)}^q$, and the second one by $[\zeta v(Q_s) u(E)]^{1 - \frac{q}{p}}$. Hence,

$$u(E)^{\frac{q}{p} - 1} \left\| \sup_{1 \leq j \leq N} |T_{K_j} f| \chi_E \right\|_{L^q(\mathbb{T}^d, u)}^q \leq c_{p,q}^q \mathfrak{N}^q \zeta^2 \left(\frac{v(Q_{r+s})}{v(Q_s)} \right)^{\frac{q}{p}} \|f\|_{L^p(\mathbb{T}, u)}^q.$$

Letting $s \rightarrow \infty$ and using (2.1) we obtain that

$$\left\| \sup_{1 \leq j \leq N} |T_{K_j} f| \right\|_{L^{p,\infty}(\mathbb{T}^d, u)} \leq c_{p,q} \zeta^{\frac{2}{q}} \mathfrak{N} \|f\|_{L^p(\mathbb{T}^d, u)}.$$

Considering $c_{p,v} = \inf_{q < p} \zeta^{2/q} c_{p,q}$, the result easily follows by Fatou's Lemma and the density of $P(\mathbb{T}^d)$ in $L^p(\mathbb{T}^d, u)$. $\square$

4. RESTRICTION OF FOURIER MULTIPLIERS TO LOWER DIMENSION

Restriction of Fourier multipliers of strong type to a lower dimensional space was studied in [7, Corollary 4.13]. Here we shall give a weak counterpart to that result.

We have to mention here that in this section we work with $A_p(\mathbb{R}^d)$ weights mainly because, under this condition, we can prove the analogue to Lemma 3.7 (see Lemma 4.2 below). Other conditions that we can assume in w in order to have an approximation lemma are, for example, that w is uniformly continuous and $\inf_{x \in \mathbb{R}^d} w(x) > 0$. In this case the proof is a simple modification of the proof of Lemma 3.7.

Lemma 4.1. *If $w \in \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^d)$, then, for any $s > 0$, $\lim_{s \rightarrow \infty} \frac{w(Q_{r+s})}{w(Q_s)} = 1$.*

Proof. By the A_∞ -condition [10, Theorem IV.2.9], there exist $\delta, C > 0$ such that

$$0 \leq 1 - \frac{w(Q_s)}{w(Q_{r+s})} = \frac{w(Q_{r+s} \setminus Q_s)}{w(Q_{r+s})} \leq C \left(1 - \frac{s^d}{(r+s)^d} \right)^\delta,$$

from where the result easily follows. $\square$

Lemma 4.2. *If $w \in A_p(\mathbb{R}^d)$, there exists $\{h_n\}_n \subset C_c^\infty(\mathbb{R}^d)$, such that*

- (1) $\mathfrak{s}_{p,w} := \sup_n \|\widehat{h_n}\|_{M_{p,w}(\mathbb{R}^d)} < \infty$,
- (2) $\sup_n \|\widehat{h_n}\|_{L^\infty(\mathbb{R}^d)} \leq 1$,
- (3) *for every $\xi \in \mathbb{R}^d$, $\lim_n \widehat{h_n}(\xi) = 1$.*

Proof. Properties (2) and (3) are proved as in Lemma 3.7. To prove (1), we first observe that clearly

$$\sup_n |h_n * f|(x) \lesssim Mf(x)$$

and hence the case $p > 1$ is trivial.

To prove the case $p = 1$, fix $\beta \in \mathbb{N}$, $\beta > d$. Then,

$$\|h_n * f\|_{L^1(w)} \leq \int |f(y)| \int h_n(x-y) w(x) dx dy.$$

For a fixed $y \in \mathbb{R}^d$ and $n > 0$, the inner integral can be split into

$$\int_{|x-y| < n^{-1}} + \sum_{j \geq 0} \int_{2^j n^{-1} < |x-y| \leq 2^{j+1} n^{-1}} h(n(x-y)) n^d w(x) dx.$$

The first term can be bounded by

$$\|h\|_\infty n^d \int_{|x-y| < n^{-1}} w(x) dx \leq [w]_{A_1} 2^d w(y),$$

as the ball $|x - y| < 1/n$ is included in $y + [-1/n, 1/n]^d$. On the other hand, if $p_{0,\beta}(h) = \sup_{x \in \mathbb{R}^d} |h(x)| |x|^\beta$, each term on the sum can be bounded from above by

$$p_{0,\beta}(h) n^{d-\beta} \int_{2^j < n|x-y| \leq 2^{j+1}} |x-y|^{-\beta} w(x) dx \leq p_{0,\beta}(h) 4^d 2^{j(d-\beta)} [w]_{A_1} w(y).$$

Thus, the sum is bounded from above by $\frac{p_{0,\beta}(h) 4^d}{1-2^{d-\beta}} [w]_{A_1} w(y)$. Hence

$$\|h_n * f\|_{L^1(w)} \leq [w]_{A_1} c_{d,h} \|f\|_{L^1(w)},$$

where $c_{d,h} = 2^d \left(1 + \inf_{\beta > d} p_{0,\beta}(h) \frac{2^d}{1-2^{d-\beta}}\right)$. $\square$

The following result is the weighted version of [6, Lemma 2] and the weak type maximal counterpart of [7, Proposition 4.10].

Proposition 4.3. *Let $w \in A_p(\mathbb{R}^d)$ and let $\{\mathbf{m}_j\}_j \subset M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ that are normalized functions. Then, there exist $\{\mathbf{m}_{j,n}\}_{j,n} \subset L^\infty(\mathbb{R}^d)$ satisfying:*

- (1) *For any j and every $\xi \in \mathbb{R}^d$,*
- (4.1)
$$\mathbf{m}_j(\xi) = \lim_n \mathbf{m}_{j,n}(\xi).$$
- (2) *$K_{j,n} = \mathbf{m}_{j,n}^\vee \in L^1(\mathbb{R}^d)$, and it is compactly supported.*
- (3) $\sup_n \|\mathbf{m}_{j,n}\|_{L^\infty(\mathbb{R}^d)} \leq \|\mathbf{m}_j\|_{L^\infty(\mathbb{R}^d)}.$
- (4) $\sup_n \|\{\mathbf{m}_{j,n}\}_j\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)} \leq \mathfrak{d}_{p,w} \|\{\mathbf{m}_j\}_j\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)},$ *where $\mathfrak{d}_{p,w}$ depends only on p, d and the A_p -constant of w .*

Proof. Let $\{h_n\}$ be the functions given by Lemma 4.2 and $\varphi_n(x)$ as in Definition 3.3. Consider, for $j, n \in \mathbb{N}$,

$$m_{j,n}(\xi) = \widehat{K_{j,n}}(\xi) = (\widehat{\varphi_n} * \mathbf{m}_j)(\xi) \widehat{h_n}(\xi),$$

and proceed as in the proof of Theorem 3.10. $\square$

Theorem 4.4. *Let $d = d_1 + d_2$, $u \in A_p(\mathbb{R}^{d_1})$, $v \in A_p(\mathbb{R}^{d_2})$ and define $w(x, y) = u(x)v(y)$. Suppose that $\{\mathbf{m}_j\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)$ and are normalized functions. Then, for a fixed $\xi \in \mathbb{R}^{d_1}$, $\{\mathbf{m}_j(\xi, \cdot)\}_j \in M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^{d_2})$ and*

$$\sup_{\xi \in \mathbb{R}^{d_1}} \left\| \{\mathbf{m}_j(\xi, \cdot)\}_j \right\|_{M_{p,u}^{(\mathfrak{w})}(\mathbb{R}^{d_2})} \leq \mathfrak{c}_{p,w} \left\| \{\mathbf{m}_j\}_j \right\|_{\mathbf{m} \in M_{p,u}^{(\mathfrak{w})}(\mathbb{R}^d)},$$

where $\mathfrak{c}_{p,w}$ depends only on p, d and the A_p -constant of w .

Proof. Since $u \in A_p(\mathbb{R}^{d_1})$ and $v \in A_p(\mathbb{R}^{d_2})$ we have that $w \in A_p(\mathbb{R}^d)$ and $[w]_{A_p(\mathbb{R}^d)} \leq [v]_{A_p(\mathbb{R}^{d_2})} [u]_{A_p(\mathbb{R}^{d_1})}$. Then, by Proposition 4.3, we can assume that $\{\mathbf{m}_j\}_{j=1}^J$ is a finite family such that $K_j = \mathbf{m}_j^\vee \in L^1$ with compact support.

Let $\mathfrak{N} = \left\| \{\mathbf{m}_j\}_j \right\|_{M_{p,w}^{(\mathfrak{w})}(\mathbb{R}^d)}$. Since translations and convolution commute, it follows that for every $z \in \mathbb{R}^{d_2}$,

$$(4.2) \quad \left\| \sup_{1 \leq j \leq J} |B_{K_j} g| \right\|_{L^{p,\infty}(\mathbb{R}^d, u(\cdot)v(\cdot+z))} \leq \mathfrak{N} \|g\|_{L^p(\mathbb{R}^d, u(\cdot)v(\cdot+z))}.$$

Fix $\xi \in \mathbb{R}^{d_1}$. For any $f \in C_c^\infty(\mathbb{R}^{d_2})$, write

$$(4.3) \quad R_{(x,y)} f(z) = e^{2\pi i x \xi} f(z + y), \quad (x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}.$$

Observe that in this way,

$$\begin{aligned}\tilde{T}_{K_j} f(z) &= \int_{\mathbb{R}^d} K_j(x, y) R_{-(x, y)} f(z) dx dy \\ &= \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} K_j(x, y) e^{-2\pi i \xi x} dx \right) f(z - y) dy \\ &= \int_{\mathbb{R}^{d_2}} \mathbf{m}_j(\xi, \eta) \hat{f}(\eta) e^{-2\pi i z \eta} d\eta.\end{aligned}$$

Fix $q < p$ and fix $E \subset \mathbb{R}^{d_1}$ a set of finite measure. For any $z \in \mathbb{R}^{d_2}$, let $A_z = \{(x, y) \in \mathbb{R}^d : y + z \in E\}$. Let $r > 0$ such that $\text{supp } K_j \subset (-r, r)^d = Q_r$ for $j = 1, \dots, J$.

Let $s > 0$. For any $(x, y) \in Q_s = (-s, s)^d$,

$$\left\| \sup_{1 \leq j \leq J} |\tilde{T}_{K_j} f| \chi_E \right\|_{L^q(\mathbb{R}^{d_2}, w)}^q = \int_{\mathbb{R}^{d_2}} \sup_{1 \leq j \leq J} |R_{(x, y)} T_{K_j} f(z)|^q v(y + z) \chi_E(y + z) dz.$$

If we consider the weight $\omega = u \otimes 1$ on $\mathbb{R}^d$, it follows that

$$\begin{aligned}& \left\| \sup_{1 \leq j \leq J} |\tilde{T}_{K_j} f| \chi_E \right\|_{L^q(\mathbb{T}^d, w)}^q \\ &= \frac{1}{\omega(Q_s)} \int_{Q_s} \int_{\mathbb{R}^{d_2}} \sup_{1 \leq j \leq J} |R_{(x, y)} \tilde{T}_{K_j} f(z)|^q u(x) v(y + z) \chi_E(y + z) dx dy dz \\ &\leq \frac{1}{\omega(Q_s)} \int_{\mathbb{R}^{d_2}} \left\{ \int_{A_z \cap Q_s} \sup_{1 \leq j \leq J} |B_{K_j}(R_{(\cdot)} f(z) \chi_{Q_{r+s}})(x, y)|^q u(x) v(y + z) dx dy \right\} dz.\end{aligned}$$

By Kolmogorov's condition (2.1) and (4.2), the term inside curly brackets is bounded by

$$(c_{p, q} \mathfrak{N})^q \left\{ \int_{Q_{r+s}} |R_{(x, y)} f(z)|^p u(x) v(y + z) dt \right\}^{\frac{q}{p}} \left\{ \int_{A_z \cap Q_s} u(x) v(y + z) dt \right\}^{1 - \frac{q}{p}}.$$

Then, by Hölder's inequality, it follows that

$$\begin{aligned}\left\| \sup_{1 \leq j \leq J} |T_{K_j} f| \chi_E \right\|_{L^q(\mathbb{R}^{d_2}, v)}^q &\leq \frac{c_{p, q}^q \mathfrak{N}^q}{\omega(Q_s)} \left\{ \int_{\mathbb{R}^{d_2}} \int_{Q_s \cap A_z} u(x) v(y + z) dt dz \right\}^{1 - \frac{q}{p}} \\ &\quad \times \left\{ \int_{\mathbb{R}^{d_2}} \int_{Q_{r+s}} |R_{(x, y)} f(z)|^p u(x) v(y + z) dx dy dz \right\}^{\frac{q}{p}} \\ &\leq c_{p, q}^q \mathfrak{N}^q \left(\frac{\omega(Q_{r+s})}{\omega(Q_s)} \right)^{\frac{q}{p}} v(E)^{1 - \frac{q}{p}} \|f\|_{L^p(\mathbb{R}^{d_2}, w)}^q.\end{aligned}$$

Since $u \in A_p(\mathbb{R}^{d_1})$, $\omega \in A_p(\mathbb{R}^d)$. Then by Lemma 4.1 and Kolmogorov's condition (2.1), it follows that

$$\left\| \sup_{1 \leq j \leq J} |T_{K_j} f| \right\|_{L^{p, \infty}(\mathbb{R}^{d_2}, w)} \leq c_{p, q} \mathfrak{N} \|f\|_{L^p(\mathbb{R}^{d_2}, w)}.$$

Finally, considering $c_p = \inf_{q < p} c_{p, q}$, the result easily follows by Fatou's Lemma and the density of $\mathcal{C}_c^\infty(\mathbb{R}^{d_2})$ in $L^p(\mathbb{R}^{d_2}, w)$. $\square$

5. CONSEQUENCES AND APPLICATIONS

5.1. Hörmander-Mihlin type multipliers. The first application involves multipliers satisfying a Hörmander-Mihlin type condition.

Definition 5.1 (see [14]). Let $\mathbf{m} \in L^\infty(\mathbb{R}^d) \cap \mathcal{C}^d(\mathbb{R}^d \setminus \{0\})$, $l \in \mathbb{N}$ and $s \geq 1$. We say $\mathbf{m} \in M(s, l)$ if it satisfies

$$(5.1) \quad c_{\mathbf{m}, s, l} = \sup_{\substack{|\alpha| \leq l \\ \alpha = (\alpha_1, \dots, \alpha_d)}} \sup_{r > 0} \left(r^{s|\alpha| - d} \int_{r < |x| < 2r} \left| \frac{\partial^{|\alpha|} \mathbf{m}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}(x) \right|^s dx \right)^{1/s} < \infty.$$

In 1979, D. Kurtz and R. Wheeden proved the following result.

Theorem 5.2 ([14, Theorem 1]). Let $1 < s \leq 2$, $\frac{d}{s} < l \leq d$ and $\mathbf{m} \in M(s, l)$. If

- (1) $d/l < p < \infty$ and $w \in A_{p, l/d}(\mathbb{R}^d)$ or
- (2) $1 < p < (d/l)'$ and $w^{-1/(p-1)} \in A_{p', l/d}(\mathbb{R}^d)$,

then $\mathbf{m} \in M_{p, w}(\mathbb{R}^d)$. When $l < d$ it can be taken $p = d/l$ in (1) or $p = (d/l)'$ in (2).

Moreover, if $w^{d/l} \in A_1(\mathbb{R}^d)$, then $\mathbf{m} \in M_{1, w}^{(w)}(\mathbb{R}^d)$.

Corollary 5.3. Under the hypothesis of Theorem 5.2 and assuming that $\mathbf{m}$ is a normalized function, the following holds: If

- (1) $d/l < p < \infty$ and $w \in A_{p, l/d}(\mathbb{T}^d)$ or
- (2) $1 < p < (d/l)'$ and $w^{-1/(p-1)} \in A_{p', l/d}(\mathbb{T}^d)$,

then $\mathbf{m}|_{\mathbb{Z}^d} \in M_{p, w}(\mathbb{T}^d)$. When $l < d$ it can be taken $p = d/l$ in (1) or $p = (d/l)'$ in (2).

Moreover, if $w^{d/l} \in A_1(\mathbb{T}^d)$, then $\mathbf{m}|_{\mathbb{Z}^d} \in M_{1, w}^{(w)}(\mathbb{T}^d)$.

Proof. The result follows by applying Theorems 3.10 and 3.11 to $\mathbf{m}$. $\square$

5.2. Singular integral operators. Our second example involves the classical theory of Calderón-Zygmund singular integrals.

Definition 5.4 ([10, Definition II.5.17]). A function $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ is said to be a regular kernel if $\widehat{K} \in L^\infty(\mathbb{R}^d)$ and it satisfies

$$(5.2) \quad |K(x)| \leq C |x|^{-d}, \quad \forall x \in \mathbb{R}^d \setminus \{0\},$$

$$(5.3) \quad |K(x - y) - K(x)| \leq C |y| |x|^{-d-1}, \quad |x| > 2|y|.$$

Corollary 5.5. Let K be a regular kernel and consider for any $0 < r < s < \infty$, $K_{r, s} = K \chi_{r < |x| < s}$ and $\mathbf{m}_{r, s} = \widehat{K_{r, s}}$. If $1 < p < \infty$ and $w \in A_p(\mathbb{T}^d)$ there exists a constant c such that

$$\|\{\mathbf{m}_{r, s}|_{\mathbb{Z}^d}\}_{r < s}\|_{M_{p, w}(\mathbb{T}^d)} \leq c.$$

If $w \in A_1(\mathbb{T}^d)$, then there exists a constant c such that

$$\|\{\mathbf{m}_{r, s}|_{\mathbb{Z}^d}\}_{r < s}\|_{M_{1, w}^{(w)}(\mathbb{T}^d)} \leq c.$$

Proof. It is easy to see that $T_{\{\mathbf{m}_{r, s}|_{\mathbb{Z}^d}\}_{r < s}}^\sharp f(x) = T_{\{\mathbf{m}_{r, s}|_{\mathbb{Z}^d}\}_{r, s \in \mathbb{Q}_+, r < s}}^\sharp f(x)$ for every $f \in P(\mathbb{T}^d)$. Then, the result follows by the known corresponding result for functions in $\mathbb{R}^d$ (see [10, Theorem IV.3.6 and V.4.11]) by applying Theorem 3.2 and its corresponding strong version. $\square$

5.3. Bochner-Riesz partial sums. Our third application involves Bochner-Riesz partial sums. Let us recall that the Bochner-Riesz operators in $\mathbb{R}^d$ are defined as

$$(B_\lambda^r f)^\wedge(\xi) = \mathbf{m}_r(\xi) \widehat{f}(\xi), \quad \text{where} \quad \mathbf{m}_r(x) = \left(1 - \frac{|x|^2}{r^2}\right)_+^\lambda,$$

$t_+ = \max(t, 0)$, and the associated maximal operator is defined by

$$B_\lambda^\# f(x) = \sup_{r>0} |B_\lambda^r f(x)|$$

for $\lambda > 0$. It is known that for $\lambda > \frac{d-1}{2}$, $B_\lambda^\# f$ is pointwise majorized by the Hardy-Littlewood maximal operator; then it inherits its boundedness properties. For the critical index the following is known (S. Shi and Q. Sun [17, Theorem 1] and A. Vargas [16, Theorem 1]).

Theorem 5.6. *Let $\lambda = \frac{d-1}{2}$. If $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d)$, then*

$$\|B_\lambda^\# f\|_{L^p(\mathbb{R}^d, w)} \leq C \|f\|_{L^p(\mathbb{R}^d, w)},$$

and if $w \in A_1(\mathbb{R}^d)$, there is a constant C such that for each $r > 0$,

$$\|B_\lambda^r f\|_{L^{1,\infty}(\mathbb{R}^d, w)} \leq C \|f\|_{L^1(\mathbb{R}^d, w)},$$

where the constants depend only on the A_p -constant of w and the dimension d .

Let us observe that for $\lambda = (d-1)/2$, the kernel of the operator B_λ^r , say K , satisfies the size condition $|K(x)| \lesssim |x|^{-d}$, but it does not satisfy any Hörmander type condition such as (5.3) above. Then we can't apply the result obtained in the previous example.

In the periodic case, for $r > 0$, the Bochner-Riesz partial sum of order $\lambda > 0$ is defined for every $f \in P(\mathbb{T}^d)$ by

$$S_\lambda^r f(\theta) = \sum_{|n| \leq r} \left(1 - \frac{|n|^2}{r^2}\right)_+^\lambda \widehat{f}(n) e^{2\pi i n \theta},$$

and we denote by $S_\lambda^\#$ the associated maximal operator. Observe that since the function $(1 - |x|^2)_+^\lambda$ is continuous, for every $f \in P(\mathbb{T}^d)$,

$$S_\lambda^\# f(x) = \sup_{r \in \mathbb{Q}_+} |S_\lambda^r f(x)|.$$

Then, as a consequence of our results, the following counterpart to Theorem 5.6 is obtained.

Corollary 5.7. *Let $\lambda \geq \frac{d-1}{2}$. If $1 < p < \infty$ and $w \in A_p(\mathbb{T}^d)$, then there exists $C > 0$ such that*

$$\|S_\lambda^\# f\|_{L^p(\mathbb{T}^d, w)} \leq C \|f\|_{L^p(\mathbb{T}^d, w)}.$$

If $w \in A_1(\mathbb{T}^d)$ and $\lambda = \frac{d-1}{2}$ there exists $C > 0$ such that for any $r > 0$,

$$\|S_\lambda^r f\|_{L^{1,\infty}(\mathbb{T}^d, w)} \leq C \|f\|_{L^1(\mathbb{T}^d, w)},$$

and if $\lambda > \frac{d-1}{2}$, there exists $C > 0$ such that

$$\|S_\lambda^\# f\|_{L^{1,\infty}(\mathbb{T}^d, w)} \leq C \|f\|_{L^1(\mathbb{T}^d, w)}.$$

By standard arguments the theorem implies:

Corollary 5.8. *Let $\lambda \geq \frac{d-1}{2}$, $1 \leq p < \infty$ and $w \in A_p(\mathbb{T}^d)$. For any $f \in L^p(\mathbb{T}^d, w)$,*

$$\lim_{r \rightarrow 0^+} S_\lambda^r f = f,$$

where the convergence is considered in measure for $p = 1$ and $\lambda = \frac{d-1}{2}$ and pointwise almost everywhere in the other cases.

For λ below the critical index the study of the boundedness properties of the Bochner-Riesz operators constitutes an active area of research (see [9] for instance and the references therein). In this setting, the following is a direct consequence of [9, Theorem 5.1] (taking $u_0 = 1$ with the notation therein).

Theorem 5.9. *Let $0 < \lambda < (d-1)/2$. If $w(x) = v(x)^{2\lambda/d-1}$ with $v \in A_2(\mathbb{R}^d)$, then, for any $r > 0$, B_λ^r is bounded in $L^2(\mathbb{R}^d, w)$ uniformly on r .*

Theorem 3.11 leads to obtain the following periodic counterpart result.

Corollary 5.10. *Let $0 < \lambda < (d-1)/2$. If $w(x) = v(x)^{2\lambda/d-1}$ with $v \in A_2(\mathbb{T}^d)$, then, for any $r > 0$, S_λ^r is bounded in $L^2(\mathbb{T}^d, w)$ uniformly on r .*

5.4. Extension of multipliers from $L^p(\mathbb{T})$ to $L^p(\mathbb{R}, w)$. In this section we are going to show how Theorem 3.13 allows us to see the strong ties between $M_p(\mathbb{T})$ and a subspace of $M_{p,w}(\mathbb{R})$ for a subclass of weights in $A_p(\mathbb{R})$ (see Corollary 5.12 below).

Following M. Jodeit's ideas in [12], E. Berkson, M. Paluszyński and G. Weiss in [5] gave a way to extend multipliers from $L^p(\mathbb{T})$ to $L^p(\mathbb{R}, w)$ with $w \in A_p(\mathbb{R})$ satisfying that there exists a constant $\rho \geq 1$ such that for each $k \in \mathbb{Z}$

$$(5.4) \quad \rho^{-1}w(k) \leq w(x) \leq \rho w(k), \quad \text{for all } x \in [k, k+1).$$

These weights are said to be in W_p .

In this framework, E. Berkson, M. Paluszyński and G. Weiss proved the following result.

Theorem 5.11 ([5, Theorem 4.21]). *Let $1 < p < \infty$, $w \in W_p$, $\Psi \in M_{p,w}(\mathbb{R})$ and the support of Ψ is contained in $[-1/2, 1/2]$. Then, if $\{\phi_n\}_n \in M_p(\mathbb{T})$, we have that*

$$\mathcal{W}_{\phi, \Psi}(t) = \sum_{m \in \mathbb{Z}} \phi(m) \Psi(t - m) \in M_{p,w}(\mathbb{R})$$

and

$$\|\{\mathcal{W}_{\phi_n, \Psi}\}_n\|_{M_{p,w}(\mathbb{R})} \leq K_{p,w} \|\Psi\|_{M_{p,w}(\mathbb{R})} \|\{\phi_n\}_n\|_{M_p(\mathbb{T})}.$$

Since $\mathcal{W}_{\phi_n, \Psi}|_{\mathbb{Z}} = \Psi(0)\phi_n|_{\mathbb{Z}}$, a direct consequence of Theorem 3.13 with $u = 1$ and $v = w$ is that the converse of Theorem 5.11 also holds:

Corollary 5.12. *Let $\{\phi_n\}_n \subset \ell^\infty(\mathbb{Z})$. Then, under the hypothesis of Theorem 5.11, we have that if $\Psi(0) \neq 0$,*

$$\|\{\mathcal{W}_{\phi_n, \Psi}\}_n\|_{M_{p,w}(\mathbb{R})} < +\infty \quad \text{if and only if} \quad \|\{\phi_n\}_n\|_{M_p(\mathbb{T})} < +\infty.$$

Moreover,

$$(5.5) \quad \frac{C_{p,w} \|\{\phi_n\}_n\|_{M_p(\mathbb{T})}}{|\Psi(0)|} \leq \|\{\mathcal{W}_{\phi_n, \Psi}\}_n\|_{M_{p,w}(\mathbb{R})} \leq K_{p,w} \|\Psi\|_{M_{p,w}(\mathbb{R})} \|\{\phi_n\}_n\|_{M_p(\mathbb{T})}.$$

Observation 5.13. In the particular case of a single multiplier, inequality (5.5) yields that, for any $w \in W_p$, the map $\phi \mapsto \mathcal{W}_{\phi, \Psi}$ induces an isomorphism between $M_p(\mathbb{T})$ and a subspace of $M_{p,w}(\mathbb{R})$. This result is a one dimensional weighted generalization of the unweighted result in [12, p. 225] for Ψ the characteristic function of the interval $[-1/2, 1/2)$.

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