GENERATORS FOR THE EUCLIDEAN PICARD MODULAR GROUPS

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ABSTRACT. The goal of this article is to show that five explicitly given transformations, a rotation, two screw Heisenberg rotations, a vertical translation and an involution generate the Euclidean Picard modular groups with coefficient in the Euclidean ring of integers of a quadratic imaginary number field. We also obtain a presentation of the isotropy subgroup fixing infinity by analysis of the combinatorics of the fundamental domain in the Heisenberg group.

1. INTRODUCTION

Let $\mathcal{K} = \mathbb{Q}(\sqrt{-d})$ be a quadratic imaginary number field. Let $\mathcal{O}_d$ be the ring of algebraic integers of $\mathcal{K}$. The Bianchi groups $PSL_2(\mathcal{O}_d)$ are the simplest arithmetically defined discrete groups. In number theory they have been used to study the zeta-functions of binary Hermitian forms over the rings $\mathcal{O}_d$. As isometry groups acting on the upper half-space, they are of interest in the theory of Kleinian groups and the related theory of hyperbolic orbifolds. Bianchi groups can be considered as the natural algebraic generalization of the classical modular group $PSL_2(\mathbb{Z})$. A good general reference for the Bianchi groups and their relation to the modular group is [7]. As a natural generalization of the Bianchi groups, the subgroups of $PU(2, 1)$ with coefficients in $\mathcal{O}_d$ are called *Picard modular groups*, denoted by $PU(2, 1; \mathcal{O}_d)$. These groups have attracted a great deal of attention both for their intrinsic interest as discrete groups (see Holzapfel's book [14]) and also for their applications in complex hyperbolic geometry (as holomorphic automorphism subgroups).

A general method to determine finite presentations for each of the Bianchi group $PSL_2(\mathcal{O}_d)$ was developed by Swan [20] based on geometrical work of Bianchi, while a separate purely algebraic method was given by Cohn [1]. In general, fundamental domains for Lie groups were studied by [13], but complex hyperbolic space is a particularly challenging case since there exists no totally geodesic hypersurface. So far very few examples of complex hyperbolic lattices have been constructed explicitly. Due to Mostow's famous paper [15], other explicit constructions of fundamental domains for lattices in $PU(2, 1)$ were obtained; see, for example, the work of Deraux, Falbel and Paupert [3], Parker [16] and his survey paper [17]. In particular, the group $PU(2, 1; \mathcal{O}_3) = PU(2, 1; \mathbb{Z}[\omega])$, where ω is a cube root of unity was studied by Falbel and Parker in [8] and its sister was treated recently in [22]. Analogously a fundamental domain for the Gauss-Picard group $PU(2, 1; \mathbb{Z}[i])$ was described in

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[9], [10], [11] and analysis of the combinatorics of the fundamental domain gives rise to a presentation of the group in [9].

In this paper we give a description of generators for certain Picard modular groups $PU(2, 1; \mathcal{O}_d)$ where the ring $\mathcal{O}_d$ is Euclidean except for $d = 1, 3$ (these two exceptional cases have been studied in many aspects). Among the quadratic imaginary number rings $\mathcal{O}_d$ only $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_7, \mathcal{O}_{11}$ have a Euclidean algorithm; see [19], although there is a larger finite collection of $\mathcal{O}_d$'s ($d = 1, 2, 3, 7, 11, 19, 43, 67, 163$; see [21]) which have class number one. For these values of d the orbifold $\mathbf{H}_{\mathbb{C}}^2/PU(2, 1; \mathcal{O}_d)$ has only one cusp.

The method of this article, also used in [8], [9], [22], is to start with finding suitable generators of the stabiliser of infinity of $PU(2, 1; \mathcal{O}_d)$ and then construct a fundamental domain for the stabiliser acting on the boundary of complex hyperbolic space $\partial\mathbf{H}_{\mathbb{C}}^2$. The generators of the groups in [8], [9], [22] are easily obtained since the fundamental domain constructed lies completely inside the boundary of the largest isometric sphere centred at origin. The real difficulty for the Picard groups studied here is to determine more isometric spheres such that the intersection of these isometric spheres and the fundamental domain for the stabiliser we construct has only one cusp. Again, this reflects the underlying number theory; $\mathcal{O}_1$ and $\mathcal{O}_3$ have non-trivial units while the other three do not. A simple algorithm to decompose any transformation in the Picard group $PU(2, 1; \mathcal{O}_1)$ as a product of the generators was given in [5]. It would be interesting to extend their method to other Picard modular groups. However, it would also be important to find the generators in a geometric way. This will provide more important information for future research on the construction of an explicit fundamental domain for each of the Picard modular groups.

2. COMPLEX HYPERBOLIC SPACE

2.1. The Siegel domain. In this section we give the necessary background material on complex hyperbolic space. For more details on this material we refer the reader to [12].

Let $\mathbb{C}^{2,1}$ denote the complex vector space equipped with the Hermitian form defined by

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1,$$

where $\mathbf{z}$ and $\mathbf{w}$ are the column vectors $[z_1, z_2, z_3]^t$ and $[w_1, w_2, w_3]^t$, respectively. The projective model of complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^2$ is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$; namely, those points $\mathbf{z}$ satisfying $\langle \mathbf{z}, \mathbf{z} \rangle < 0$.

We mainly take the *Siegel domain* $\mathfrak{S}$ as an upper half-space model for the complex hyperbolic space that is given by

$$\mathfrak{S} = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_1 + |z_2|^2 < 0\}.$$

The boundary of the *Siegel domain* $\mathfrak{S}$ is identified with the one-point compactification of the Heisenberg group. The Heisenberg group $\mathfrak{H}$ is $\mathbb{C} \times \mathbb{R}$ with the group law

$$(\zeta_1, t_1) \diamond (\zeta_2, t_2) = (\zeta_1 + \zeta_2, t_1 + t_2 + 2\Im m(\zeta_1 \bar{\zeta}_2)).$$

The Cygan metric on $\mathfrak{H}$ is given by

$$\rho_0((\zeta_1, t_1), (\zeta_2, t_2)) = ||\zeta_1 - \zeta_2|^2 - it_1 + it_2 - 2i\Im m(\zeta_1 \bar{\zeta}_2)|,$$

which is $|(\zeta_1, t_1)^{-1} \diamond (\zeta_2, t_2)|$ in terms of the operation of Heisenberg group.

We can extend the Cygan metric to an incomplete metric on $\tilde{\mathfrak{S}} - \{\infty\}$ as follows:

$$\tilde{\rho}_0 = ||\zeta_1 - \zeta_2|^2 + |u_1 - u_2| - it_1 + it_2 - 2i\Im(\zeta_1 \bar{\zeta}_2)|.$$

The Siegel domain $\mathfrak{S}$ is parametrised in horospherical coordinates by

$$(2.1) \quad (\zeta, t, u) \longrightarrow \begin{bmatrix} (-|\zeta|^2 - u + it)/2 \\ \zeta \\ 1 \end{bmatrix}$$

with the point at infinity being

$$q_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $\mathfrak{S} = \mathfrak{R} \times \mathbb{R}_+$ and $\partial\mathfrak{S} = (\mathfrak{R} \times \{0\}) \cup \{q_\infty\}$.

2.2. Complex hyperbolic isometries. Let $U(2, 1)$ be the group of matrices that are unitary with respect to the form $\langle \cdot, \cdot \rangle$. The group of holomorphic isometries of complex hyperbolic space is the projective unitary group $PU(2, 1) = U(2, 1)/U(1)$, with a natural identification $U(1) = \{e^{i\theta}I, \theta \in [0, 2\pi)\}$. We now describe the action of the stabiliser of q_∞ on the Heisenberg group.

The Heisenberg group acts on itself by *Heisenberg translations*. For $(\tau, v) \in \mathfrak{R}$, this is

$$T_{(\tau, v)} : (\zeta, t) \mapsto (\zeta + \tau, t + v + 2\Im(\tau \bar{\zeta})) = (\tau, v) \diamond (\zeta, t).$$

Heisenberg translation by $(0, v)$ for any $v \in \mathbb{R}$ is called *vertical translation* by v .

The unitary group $U(1)$ acts on the Heisenberg group by *Heisenberg rotations*. For $e^{i\theta} \in U(1)$, the rotation fixing $q_0 = (0, 0, 0)$ is given by

$$R_\theta : (\zeta, t) \mapsto (e^{i\theta}\zeta, t).$$

All other Heisenberg rotations may be obtained from these by conjugating by a Heisenberg translation.

For $\lambda \in \mathbb{R}_+$, *Heisenberg dilation* by λ fixing q_∞ and $q_0 = (0, 0, 0) \in \partial\mathbf{H}_{\mathbb{C}}^2$ is given by

$$D_\lambda : (\zeta, t) \mapsto (\lambda\zeta, \lambda^2 t).$$

All other Heisenberg dilations fixing q_∞ may be obtained by conjugating by a Heisenberg translation.

The stabiliser of q_∞ in $PU(2, 1)$ is generated by all Heisenberg translations, rotations and dilations. However, only Heisenberg translations and rotations are isometric with respect to various natural metrics on $\mathfrak{R}$. For this reason the group generated by all Heisenberg translations and rotations, which is the semidirect product $U(1) \ltimes \mathfrak{R}$, is called the *Heisenberg isometry group* $Isom(\mathfrak{R})$. The non-trivial central elements of the Heisenberg isometry group are precisely the vertical translations. In particular, each element of $Isom(\mathfrak{R})$ preserves every horosphere.

There is a canonical projection from $\mathfrak{R}$ to $\mathbb{C}$ called *vertical projection* and denoted by Π , given by $\Pi : (\zeta, t) \mapsto \zeta$. Using the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{R} \xrightarrow{\Pi} \mathbb{C} \longrightarrow 0,$$

we obtain the exact sequence (see Scott [18] page 467)

$$(2.2) \quad 0 \longrightarrow \mathbb{R} \longrightarrow Isom(\mathfrak{R}) \xrightarrow{\Pi_*} Isom(\mathbb{C}) \longrightarrow 1.$$

Here $Isom(\mathbb{C})$ is the group of orientation preserving Euclidean isometries of $\mathbb{C}$.

Observe that the elements in $Isom(\mathbb{C})$ can be represented by matrices in $GL(2, \mathbb{C})$ of the form

$$\begin{bmatrix} e^{i\theta} & \zeta_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta \\ 1 \end{bmatrix} = \begin{bmatrix} e^{i\theta}\zeta + \zeta_0 \\ 1 \end{bmatrix}.$$

Therefore, the map Π_* can be given by

$$(2.3) \quad \Pi_* : \begin{bmatrix} 1 & -\bar{\zeta}_0 e^{i\theta} & (-|\zeta_0|^2 + it_0)/2 \\ 0 & e^{i\theta} & \zeta_0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} e^{i\theta} & \zeta_0 \\ 0 & 1 \end{bmatrix}.$$

It is clear that

$$Ker(\Pi_*) = \left\{ \begin{bmatrix} 1 & 0 & it_0/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : t_0 \in \mathbb{R} \right\}$$

is the group of vertical translations fixing q_∞ .

2.3. Isometric spheres. Given an element $G \in PU(2, 1)$ with satisfying $G(q_\infty) \neq q_\infty$, we define the isometric sphere of G to be the hypersurface

$$\{\mathbf{z} \in \mathbf{H}_{\mathbb{C}}^2 : |\langle \mathbf{z}, q_\infty \rangle| = |\langle \mathbf{z}, G^{-1}(q_\infty) \rangle|\}.$$

For example, the isometric sphere of

$$I_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is

$$(2.4) \quad \mathcal{B}_0 = \{(\zeta, t, u) \in \mathfrak{S} : ||\zeta|^2 + u + it| = 2\}$$

in horospherical coordinates or

$$(2.5) \quad \mathcal{B}_0 = \{[z_1, z_2, z_3] \in \mathbf{H}_{\mathbb{C}}^2 : |z_1| = |z_3|\}$$

in homogeneous coordinates.

All other isometric spheres are images of $\mathcal{B}_0$ by Heisenberg dilations, rotations and translations. Thus the isometric sphere with radius r and centre $(\zeta_0, t_0, 0)$ is given by

$$\{(\zeta, t, u) : ||\zeta - \zeta_0|^2 + u + it - it_0 + 2i\Im m(\zeta \bar{\zeta}_0)| = r^2\}.$$

If G has the matrix form

$$(2.6) \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix},$$

then $G(q_\infty) \neq q_\infty$ if and only if $g \neq 0$. The isometric sphere of G has radius $r = \sqrt{2/|g|}$ and centre $G^{-1}(q_\infty)$, which in horospherical coordinates is

$$(\zeta_0, t_0, 0) = (\bar{h}/\bar{g}, 2\Im m(\bar{j}/\bar{g}), 0).$$

Isometric spheres are examples of bisectors. The intersection of a bisector with the absolute is a smooth hypersurface in $\partial \mathbf{H}_{\mathbb{C}}^2$ diffeomorphic to a sphere. Such hypersurfaces are called *spinal spheres* in [12]. Mostow [15] described the structure of bisectors in terms of a foliation by complex lines orthogonal to a geodesic, called spine. Bisectors enjoy another decomposition into totally real, totally geodesic 2-submanifolds containing the spine; see [12].

3. ON THE STRUCTURE OF THE STABILISER

In this section we will obtain the generators and a presentation of the stabiliser of Picard modular groups by analysis of the fundamental domain in the Heisenberg group.

Let $\mathcal{O}_d$ be the ring of integers in the quadratic imaginary number field $\mathbb{Q}(i\sqrt{d})$, where d is a positive square-free integer. If $d \equiv 1, 2 \pmod{4}$, then $\mathcal{O}_d = \mathbb{Z}[i\sqrt{d}]$ and if $d \equiv 3 \pmod{4}$, then $\mathcal{O}_d = \mathbb{Z}[\omega_d]$, where $\omega_d = (1 + i\sqrt{d})/2$. The group $\Gamma_d = PU(2, 1; \mathcal{O}_d)$ is called *Euclidean Picard modular group* if the ring $\mathcal{O}_d$ is Euclidean; namely, only the rings $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_7, \mathcal{O}_{11}$. Further, relative to the amalgamation property, these five groups can be subclassified into three groupings: $\{\Gamma_1\}$, $\{\Gamma_3\}$, $\{\Gamma_2, \Gamma_7, \Gamma_{11}\}$. Since the two classes $\{\Gamma_1\}, \{\Gamma_3\}$ (cf. [8], [9]) have been studied in detail, we mainly describe the remaining class $\{\Gamma_2, \Gamma_7, \Gamma_{11}\}$.

3.1. The stabiliser of q_∞ . First we want to analyse $(\Gamma_d)_\infty$ with $d = 2, 7, 11$, the stabiliser of q_∞ . Every element of $(\Gamma_d)_\infty$ is upper triangular and its diagonal entries are units in $\mathcal{O}_d$. Recall that the units of $\mathcal{O}_1$ are $\pm 1, \pm i$, they are $\pm 1, \pm \omega, \pm \omega^2$ for $\mathcal{O}_3$ and they are ± 1 for others. Therefore $(\Gamma_d)_\infty$ contains no dilations and so it is a subgroup of $Isom(\mathfrak{R})$ and fits into the exact sequence as

$$0 \longrightarrow \mathbb{R} \cap (\Gamma_d)_\infty \longrightarrow (\Gamma_d)_\infty \xrightarrow{\Pi_*} \Pi_*((\Gamma_d)_\infty) \longrightarrow 1.$$

We can write the isometry group of the integer lattice as

$$Isom(\mathcal{O}_d) = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix} : \alpha, \beta \in \mathcal{O}_d, \alpha \text{ is a unit} \right\}.$$

We now find the image and kernel in this exact sequence.

Proposition 3.1. *The stabiliser $(\Gamma_d)_\infty$ of q_∞ in Γ_d satisfies*

$$0 \longrightarrow 2\sqrt{d}\mathbb{Z} \longrightarrow (\Gamma_d)_\infty \xrightarrow{\Pi_*} \Delta \longrightarrow 1,$$

where $\Delta \subset Isom(\mathcal{O}_d)$ is of index 2 if $d \equiv 2 \pmod{4}$ and $\Delta = Isom(\mathcal{O}_d)$ if $d \equiv 3 \pmod{4}$.

Proof. Although we only consider the cases $d = 2, 7, 11$, the ring $\mathcal{O}_2$ represents those for the values of d with $d \equiv 2 \pmod{4}$ and the rings $\mathcal{O}_7, \mathcal{O}_{11}$ represent those of the values $d \equiv 3 \pmod{4}$. The remaining case is the same as $\mathcal{O}_1$, which has been done in [9]. Observe that $Isom(\mathcal{O}_d)$ is generated by the subgroup of translations

$$\left\{ \hat{T}_\beta = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} : \beta \in \mathcal{O}_d \right\}$$

and the finite subgroup of order two

$$\left\{ \hat{R}_\alpha = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} : \alpha \in \mathcal{O}_d, \alpha \text{ is a unit} \right\}.$$

Then, to understand $\Delta \subset Isom(\mathcal{O}_d)$, it suffices to determine which translations can be lifted. We divide into two cases to complete the proof.

(i) **The case $\mathcal{O}_d$ with $d \equiv 2 \pmod{4}$.** Suppose that $\beta \in \mathcal{O}_d = \mathbb{Z}[i\sqrt{d}]$ and consider the translation $\hat{T}_\beta$ by β in $\mathbb{Z}[i\sqrt{d}]$ given above. The preimage of $\hat{T}_\beta$ under Π_* has the form

$$T_{\beta,t} = \begin{bmatrix} 1 & -\bar{\beta} & \frac{-|\beta|^2 + it}{2} \\ 0 & 1 & \bar{\beta} \\ 0 & 0 & 1 \end{bmatrix}.$$

This map is in $PU(2, 1; \mathbb{Z}[i\sqrt{d}])$ if and only if $|\beta|^2$ is an even integer and $t \in 2\sqrt{d}\mathbb{Z}$. Writing $\beta = m + i\sqrt{d}n$ for $m, n \in \mathbb{Z}$, we can obtain $m \equiv 0 \pmod{2}$ from the conditions $|\beta|^2 = m^2 + dn^2 \in 2\mathbb{Z}$ and $d \equiv 2 \pmod{4}$. Therefore, we conclude that $\Delta \subset Isom(\mathbb{Z}[i\sqrt{d}])$ is of index 2. Also, the kernel of Π_* is generated by

$$\begin{bmatrix} 1 & 0 & i\sqrt{d} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is a vertical translation of $(0, 2\sqrt{d})$.

(ii) **The case $\mathcal{O}_d$ with $d \equiv 3 \pmod{4}$.** Suppose that $\beta = m + n\frac{1+i\sqrt{d}}{2} \in \mathcal{O}_d$ with $m, n \in \mathbb{Z}$ for $d \equiv 3 \pmod{4}$. By the same argument of (i), it only suffices to determine m, n such that $|\beta|^2$ is an integer. For $d \equiv 3 \pmod{4}$, it is easy to show that $|\beta|^2 = m^2 + mn + n^2(d+1)/4 \in \mathbb{Z}$ for any $m, n \in \mathbb{Z}$, which implies that $\Delta = Isom(\mathcal{O}_d)$. Obviously, the kernel of Π_* is generated by a vertical translation of $(0, 2\sqrt{d})$. $\square$

3.2. Fundamental domain for the stabiliser. As the first step towards the construction of a fundamental domain for the action of $(\Gamma_d)_\infty$ on $\mathfrak{H}$ for $d = 2, 7, 11$, we shall find the suitable generators of $Isom(\mathcal{O}_d)$ to construct a fundamental domain in $\mathbb{C}$. This was already done by Feustel and Hozapfel in [4] and [6], but we state it again for the convenience of the reader.

In the proof of Proposition 3.1 we saw that $\Delta = \Pi_*((\Gamma_2)_\infty)$ is a subgroup of index 2 in $Isom(\mathcal{O}_2)$ consisting of elements of $GL(2, \mathcal{O}_2)$ of the form

$$\left\{ \begin{bmatrix} (-1)^j & m + i\sqrt{2}n \\ 0 & 1 \end{bmatrix} : j = 0, 1, m, n \in \mathbb{Z}, m \equiv 0 \pmod{2} \right\}.$$

A fundamental domain for this group is the triangle in $\mathbb{C}$ with vertices at $-1 + \sqrt{2}i/2$ and $1 \pm \sqrt{2}i/2$; see (a) in Figure 3.1. Side paring maps are given by

$$r_1^{(2)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, r_2^{(2)} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, r_3^{(2)} = \begin{bmatrix} -1 & \sqrt{2}i \\ 0 & 1 \end{bmatrix}.$$

The first of these is a rotation of order 2 fixing origin, the second is a rotation of order 2 fixing 1 and the third is a rotation of order 2 fixing $\sqrt{2}i/2$. Indeed, every element of $\Delta = GL(2, \mathcal{O}_2)$ is generated by $r_1^{(2)}, r_2^{(2)}, r_3^{(2)}$ as follows:

$$\begin{aligned} \begin{bmatrix} (-1)^j & 2m + \sqrt{2}ni \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^m \begin{bmatrix} 1 & \sqrt{2}i \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^j \\ &= \left(r_2^{(2)} r_1^{(2)} \right)^m \left(r_3^{(2)} r_1^{(2)} \right)^n \left(r_1^{(2)} \right)^j. \end{aligned}$$

As the same argument, a fundamental domain for $Isom(\mathcal{O}_d)$ with $d = 7$ or 11 is the triangle in $\mathbb{C}$ with vertices at $(-1 + i\sqrt{d})/4$, $(1 - i\sqrt{d})/4$ and $(3 + i\sqrt{d})/4$; see (b) in Figure 3.1. Side paring maps are given by

$$r_1^{(d)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, r_2^{(d)} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, r_3^{(d)} = \begin{bmatrix} -1 & (1 + i\sqrt{d})/2 \\ 0 & 1 \end{bmatrix}.$$

All of these maps are rotations by π fixing $0, 1/2$ and $(1 + i\sqrt{d})/4$, respectively.

In order to produce a fundamental domain for $(\Gamma_d)_\infty$ we look at all the preimages of the triangle (that is a fundamental domain of $\Pi_*((\Gamma_d)_\infty)$) under vertical projection Π and we intersect this with a fundamental domain for $\ker(\Pi_*)$. The

FIGURE 3.1. (a) Fundamental domain for a subgroup Δ of $Isom(\mathcal{O}_2)$ with index 2. (b) Fundamental domain for $Isom(\mathcal{O}_d)$ with $d = 7, 11$. This is also valid for all the values of d with $d \equiv 3 \pmod{4}$.

inverse of image of the triangle under Π is an infinite prism. The kernel of Π_* is the infinite cyclic group generated by T ; the vertical translation by $(0, 2\sqrt{d})$. We give the generators and geometric presentations for the isotropy subgroups $(\Gamma_d)_\infty$ by analysis of the combinatorics of the fundamental domain in the Heisenberg group and compare on presentations with those given by Dekimpe [2].

Proposition 3.2. $(\Gamma_2)_\infty$ is generated by

$$R_1^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2^{(2)} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_3^{(2)} = \begin{bmatrix} 1 & -i\sqrt{2} & -1 \\ 0 & -1 & i\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{(2)} = \begin{bmatrix} 1 & 0 & i\sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A presentation is given by

$$(\Gamma_2)_\infty = \langle R_j^{(2)}, T^{(2)} | R_j^{(2)2} = [T^{(2)}, R_j^{(2)}] = (T^{(2)2} R_1^{(2)} R_3^{(2)} R_2^{(2)})^2 = Id \rangle.$$

Proof. Those matrices are constructed by lifting generators of the subgroup $\Delta \subset Isom(\mathcal{O}_2)$ with index 2 and also $T^{(2)}$ is a generator of the kernel of the map Π_* . A fundamental domain can be constructed with side pairings as in Figure 3.2, where the vertices of the prism are $v_3^+ = (-1 + \sqrt{2}i/2, \sqrt{2})$, $v_2^+ = (1 + \sqrt{2}i/2, \sqrt{2})$, $v_1^+ = (1 - \sqrt{2}i/2, \sqrt{2})$ for the upper cap of the prism and $v_3^- = (-1 + \sqrt{2}i/2, -\sqrt{2})$, $v_2^- = (1 + \sqrt{2}i/2, -\sqrt{2})$, $v_1^- = (1 - \sqrt{2}i/2, -\sqrt{2})$ for the base. In particular, the points $v_4^\pm, v_5^\pm, v_6^\pm$ are the middle points of the edges $(v_1^\pm, v_2^\pm)$, $(v_2^\pm, v_3^\pm)$ and $(v_3^\pm, v_1^\pm)$, respectively.

FIGURE 3.2. A fundamental domain Σ_2 for $(\Gamma_2)_\infty$ in the Heisenberg group: the map $R_1^{(2)}$ rotates through π about $\zeta = 0$, the map $R_2^{(2)}$ is a Heisenberg rotation through π about $\zeta = 1$ and the map $R_3^{(2)}$ is a Heisenberg rotation through π about $\zeta = \sqrt{2}i/2$.

The actions of side-pairing maps on $\mathfrak{R}$ are given by

$$\begin{aligned} R_1^{(2)}(\zeta, t) &= (-\zeta, t), \\ R_2^{(2)}(\zeta, t) &= (-\zeta + 2, t + 4\Im m\zeta), \\ R_3^{(2)}(\zeta, t) &= (-\zeta + i\sqrt{2}, t - 2\sqrt{2}\Re e\zeta), \\ T^{(2)}(\zeta, t) &= (\zeta, t + 2\sqrt{2}). \end{aligned}$$

We describe the side pairing in terms of the action on the vertices:

$$\begin{aligned} R_1^{(2)} &: (v_6^+, v_1^+, v_1^-, v_6^-) \longrightarrow (v_6^+, v_3^+, v_3^-, v_6^-), \\ R_2^{(2)} &: (v_1^+, v_4^+, v_4^-, v_1^-) \longrightarrow (v_2^-, v_4^+, v_4^-, v_2^+), \\ T^{(2)}R_2^{(2)} &: (v_1^+, v_1^-, v_4^-) \longrightarrow (v_2^+, v_2^-, v_4^+), \\ R_3^{(2)} &: (v_2^+, v_5^+, v_5^-, v_2^-) \longrightarrow (v_3^-, v_5^+, v_5^-, v_3^+), \\ T^{(2)}R_3^{(2)} &: (v_2^+, v_2^-, v_5^-) \longrightarrow (v_3^+, v_3^-, v_5^+), \\ T^{(2)} &: (v_1^-, v_4^-, v_2^-, v_5^-, v_3^-, v_6^-) \longrightarrow (v_1^+, v_4^+, v_2^+, v_5^+, v_3^+, v_6^+). \end{aligned}$$

The presentation can be obtained following from the edge cycles of the fundamental domain.

Writing $\alpha = R_1^{(2)}$, $a = R_1^{(2)}R_2^{(2)}$, $b = R_1^{(2)}R_3^{(2)}$ and $c = T^{(2)}$, it is easy to see that this presentation is equivalent to the presentation of the almost-Bieberbach group of type $Q = p2$ given on page 160 of [2] with $k_1 = 4$ and $k_2 = k_3 = k_4 = 0$. $\square$

FIGURE 3.3. A fundamental domain Σ_7 for $(\Gamma_7)_\infty$ in the Heisenberg group: the map $R_1^{(7)}$ rotates through π about $\zeta = 0$, the action of parabolic $R_2^{(7)}$ is a Heisenberg rotation through π about $\zeta = 1/2$ followed by an upward vertical translation by $\sqrt{7}$ and the map $R_3^{(7)}$ is a Heisenberg rotation through π about $\zeta = (1+i\sqrt{7})/4$.

Proposition 3.3. $(\Gamma_7)_\infty$ is generated by

$$R_1^{(7)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2^{(7)} = \begin{bmatrix} 1 & 1 & -\overline{\omega}_7 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_3^{(7)} = \begin{bmatrix} 1 & \overline{\omega}_7 & -1 \\ 0 & -1 & \omega_7 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{(7)} = \begin{bmatrix} 1 & 0 & i\sqrt{7} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A presentation is given by

$$\begin{aligned} (\Gamma_7)_\infty &= \langle R_j^{(7)}, T^{(7)} | R_1^{(7)2} = R_3^{(7)2} = [T^{(7)}, R_1^{(7)}] = [T^{(7)}, R_3^{(7)}] \\ &= T^{(7)} R_2^{(7)-2} = (R_1^{(7)} R_3^{(7)} R_2^{(7)})^2 = Id \rangle. \end{aligned}$$

Proof. Those matrices are constructed by lifting generators of $Isom(\mathcal{O}_7)$ and also $T^{(7)}$ is a generator of the kernel of the map Π_* . A fundamental domain can be constructed with side pairings as in Figure 3.3, where the vertices of the prism are $v_1^+ = ((1 - i\sqrt{7})/4, \sqrt{7})$, $v_2^+ = ((3 + i\sqrt{7})/4, \sqrt{7})$, $v_4^+ = ((-1 + i\sqrt{7})/4, \sqrt{7})$ for the upper cap of the prism and $v_1^- = ((1 - i\sqrt{7})/4, -\sqrt{7})$, $v_2^- = ((3 + i\sqrt{7})/4, -\sqrt{7})$, $v_4^- = ((-1 + i\sqrt{7})/4, -\sqrt{7})$ for the base. The points $v_3^\pm$ and $v_5^\pm$ are the middle points of the edges $(v_2^\pm, v_4^\pm)$ and $(v_4^\pm, v_1^\pm)$. In particular, we introduce three more

points: $w_1^+ = ((1 - i\sqrt{7})/4, \sqrt{7}/2)$, $w_2^- = ((3 + i\sqrt{7})/4, -\sqrt{7}/2)$ and $w_3^+ = ((-1 + i\sqrt{7})/4, \sqrt{7}/2)$. The actions of side-pairing maps on $\mathfrak{R}$ are given by

$$\begin{aligned} R_1^{(7)}(\zeta, t) &= (-\zeta, t), \\ R_2^{(7)}(\zeta, t) &= (-\zeta + 1, t + 2\Im m\zeta + \sqrt{7}), \\ R_3^{(7)}(\zeta, t) &= (-\zeta + \omega_7, t + 2\Im m(\bar{\omega}_7\zeta)), \\ T^{(7)}(\zeta, t) &= (\zeta, t + 2\sqrt{7}). \end{aligned}$$

We describe the side pairing in terms of the action on the vertices:

$$\begin{aligned} R_1^{(7)} &: (v_5^+, v_1^+, v_1^-, v_5^-) \longrightarrow (v_5^+, v_4^+, v_4^-, v_5^-), \\ R_2^{(7)} &: (v_1^-, v_2^-, w_1^-, w_1^+) \longrightarrow (w_1^-, w_1^+, v_1^+, v_2^+), \\ R_3^{(7)} &: (v_2^+, w_1^-, v_3^-, v_3^+) \longrightarrow (w_2^+, v_4^-, v_3^-, v_3^+), \\ T^{(7)} R_3^{(7)} &: (w_1^-, v_2^-, v_3^-) \longrightarrow (v_4^+, w_2^+, v_3^+), \\ T^{(7)} &: (v_1^-, v_2^-, v_3^-, v_4^-, v_5^-) \longrightarrow (v_1^+, v_2^+, v_3^+, v_4^+, v_5^+). \end{aligned}$$

The presentation can be obtained following from the edge cycles of the fundamental domain.

Writing $\alpha = R_1^{(7)}$, $a = R_1^{(7)} R_2^{(7)}$, $b = R_1^{(7)} R_3^{(7)}$ and $c = T^{(7)}$, it is easy to see that this presentation is equivalent to the presentation of the almost-Bieberbach group of type $Q = p2$ given on page 160 of [2] with $k_1 = k_2 = 1$ and $k_3 = k_4 = 0$. $\square$

Proposition 3.4. $(\Gamma_{11})_\infty$ is generated by

$$\begin{aligned} R_1^{(11)} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2^{(11)} = \begin{bmatrix} 1 & 1 & -\bar{\omega}_{11} \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \\ R_3^{(11)} &= \begin{bmatrix} 1 & \bar{\omega}_{11} & -1 - \bar{\omega}_{11} \\ 0 & -1 & \omega_{11} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{(11)} = \begin{bmatrix} 1 & 0 & i\sqrt{11} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

A presentation is given by

$$\begin{aligned} (\Gamma_{11})_\infty &= \langle R_j^{(11)}, T^{(11)} | R_1^{(11)2} = [T^{(11)}, R_1^{(11)}] = T^{(11)} R_2^{(11)-2} \\ &= T^{(11)} R_3^{(11)-2} = T^{(11)} (R_1^{(11)} R_3^{(11)} R_2^{(11)})^{-2} = Id \rangle. \end{aligned}$$

Proof. Those matrices are constructed by lifting generators of $Isom(\mathcal{O}_{11})$ and also $T^{(11)}$ is a generator of the kernel of the map Π_* . A fundamental domain can be constructed with side pairings in Figure 3.4, where the vertices of the prism are $v_1^+ = ((1 - i\sqrt{11})/4, \sqrt{11})$, $v_2^+ = ((3 + i\sqrt{11})/4, 3\sqrt{11}/2)$, $v_3^+ = ((-1 + i\sqrt{11})/4, 2\sqrt{11})$ for the upper cap of the prism and $v_1^- = ((1 - i\sqrt{11})/4, -\sqrt{11})$, $v_2^- = ((3 + i\sqrt{11})/4, -\sqrt{11}/2)$, $v_3^- = ((-1 + i\sqrt{11})/4, 0)$ for the base. The points $v_0^\pm$ are the middle points of the edges $(v_1^\pm, v_3^\pm)$. In particular, we introduce three more points: $w_1 = ((1 - i\sqrt{11})/4, 0)$, $w_2 = ((3 + i\sqrt{11})/4, \sqrt{11}/2)$ and $w_3 = ((-1 + i\sqrt{11})/4, \sqrt{11})$.

FIGURE 3.4. A fundamental domain Σ_{11} for $(\Gamma_{11})_\infty$ in the Heisenberg group: the map $R_1^{(11)}$ rotates through π about $\zeta = 0$, the action of parabolic $R_2^{(11)}$ is a screw Heisenberg rotation through π about $\zeta = 1/2$ followed by an upward vertical translation by $\sqrt{11}$ and the map $R_3^{(11)}$ is a screw Heisenberg rotation through π about $\zeta = (1 + i\sqrt{11})/4$ followed by an upward vertical translation by $\sqrt{11}$.

The actions of side-pairing maps on $\mathfrak{R}$ are given by

$$\begin{aligned} R_1^{(11)}(\zeta, t) &= (-\zeta, t), \\ R_2^{(11)}(\zeta, t) &= (-\zeta + 1, t + 2\Im m\zeta + \sqrt{11}), \\ R_3^{(11)}(\zeta, t) &= (-\zeta + \omega_{11}, t + 2\Im m(\bar{\omega}_{11}\zeta) + \sqrt{11}), \\ T^{(11)}(\zeta, t) &= (\zeta, t + 2\sqrt{11}). \end{aligned}$$

We describe the side pairing in terms of the action on the vertices:

$$\begin{aligned} R_1^{(11)} &: (v_0^+, v_1^+, w_1, v_0^-) \longrightarrow (v_0^+, w_3, v_3^-, v_0^-), \\ T^{(11)}R_1^{(11)} &: (w_1, v_1^-, v_0^-) \longrightarrow (v_3^+, w_3, v_0^+), \\ R_2^{(11)} &: (v_1^+, w_1, v_1^-, v_2^-) \longrightarrow (v_2^+, w_2, v_2^-, v_1^+), \\ R_3^{(11)} &: (v_2^+, w_2, v_2^-, v_3^-) \longrightarrow (v_3^+, w_3, v_3^-, v_2^+), \\ T^{(11)} &: (v_0^-, v_1^-, v_2^-, v_3^-) \longrightarrow (v_0^+, v_1^+, v_2^+, v_3^+). \end{aligned}$$

The presentation can be obtained following from the edge cycles of the fundamental domain.

Writing $\alpha = R_1^{(11)}$, $a = R_1^{(11)}R_2^{(11)}$, $b = R_1^{(11)}R_3^{(11)}$ and $c = T^{(11)}$, it is easy to see that this presentation is equivalent to the presentation of the almost-Bieberbach group of type $Q = p2$ given on page 160 of [2] with $k_1 = k_4 = 0$ and $k_2 = k_3 = 0$. $\square$

4. STATEMENT OF THE RESULTS

In this section, we introduce the method used in [8], [9] and [22] to determine the generators of the Euclidean Picard modular groups and then state our results.

Recall the map

$$I_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

defined in Section 2.3. We consider the isometric sphere $\mathcal{B}_0$ of I_0 given by (2.4), which is a Cygan sphere centred at $o = (0, 0, 0)$ with radius $\sqrt{2}$. Observe that I_0 maps $\mathcal{B}_0$ to itself and swaps the inside and the outside of $\mathcal{B}_0$. Given an element of Γ_d of the form (2.6), we know that the radius of its isometric sphere is $\sqrt{2}/|g|$. For each case $\mathcal{O}_d$, the radius of isometric sphere is not greater than $\sqrt{2}$ since the absolute of g is not smaller than 1 for $g \in \mathcal{O}_d$. We show that the largest isometric spheres are all images of $\mathcal{B}_0$ under the elements in $(\Gamma_d)_\infty$.

Proposition 4.1. *An isometric sphere has the largest radius if and only if it is the image of $\mathcal{B}_0$ under an element in $(\Gamma_d)_\infty$.*

Proof. Obviously, the image of $\mathcal{B}_0$ under an element in $(\Gamma_d)_\infty$ has the same radius $\sqrt{2}$. Conversely, given an element G of the form (2.6) such that $G(q_\infty) \neq q_\infty$, the isometric sphere of G has the largest radius only if $g = 1$. So the centre of the isometric sphere of G in horospherical coordinates is $G^{-1}(\infty) = (\bar{h}, 2\Im m \bar{j}, 0)$. Since $\bar{h}$ and $2\Im m \bar{j} \in \mathcal{O}_d$, we can take a Heisenberg translation $T \in (\Gamma_d)_\infty$ mapping the origin to $(\bar{h}, 2\Im m \bar{j})$. Writing $T' = GTI_0$, we know that T' fixes ∞ . We conclude explicitly that the isometric sphere of G is

$$\{\mathbf{z} \in \mathbf{H}_{\mathbb{C}}^2 : |\langle \mathbf{z}, q_\infty \rangle| = |\langle \mathbf{z}, G^{-1}(q_\infty) \rangle| = |\langle \mathbf{z}, TI_0(q_\infty) \rangle|\},$$

which is the image of $\mathcal{B}_0$ under T . □

Our method is based on the special feature that the orbifold $\mathbf{H}_{\mathbb{C}}^2/\Gamma_d$ has only one cusp for $d = 2, 7, 11$. For these types of orbifolds, one would like to construct a fundamental domain using the Ford domain (that is, the intersection of the exteriors of isometric spheres of all elements not fixing infinity); namely, the intersection of the Ford domain and a fundamental domain for the stabiliser of infinity. The Ford domain is canonical, but we can choose a fundamental domain for the stabiliser freely. In the previous section, we found suitable generators of the stabiliser and constructed a fundamental domain in the Heisenberg group. We will show that adjoining I_0 to $(\Gamma_d)_\infty$ gives the Euclidean Picard modular groups Γ_d . The basic idea of the proof can be described easily. Analogously to Theorem 3.5 of [8] we shall prove that $\langle R_1^{(d)}, R_2^{(d)}, R_3^{(d)}, T^{(d)}, I_0 \rangle$ has only one cusp. The fact that $PU(2, 1; \mathcal{O}_d)$ has the same cusp and the same stabiliser of infinity as the group generated by $R_1^{(d)}, R_2^{(d)}, R_3^{(d)}, T^{(d)}, I_0$ shows they are the same. The key step is to find a union of isometric spheres such that a fundamental domain for $\langle R_1^{(d)}, R_2^{(d)}, R_3^{(d)}, T^{(d)}, I_0 \rangle$ is contained in the intersection of their exteriors and a fundamental domain for the stabiliser. This implies that the group $\langle R_1^{(d)}, R_2^{(d)}, R_3^{(d)}, T^{(d)}, I_0 \rangle$ has only one cusp. In other words, we want to show that each of the prisms we constructed above lies

inside the union of the interiors of these spinal spheres in the Heisenberg group. The problem of determining the isometric spheres will be discussed in the next section.

A simple lemma will be used in the proof of our theorems many times and we state it as follows.

Lemma 4.2 (cf. [9]). *All Cygan balls are affinely convex.*

Our aim is to prove the following results. We summarise them as three theorems.

Theorem 4.3. *Let $\mathcal{K} = \mathbb{Q}(\sqrt{-2})$ and let $\mathcal{O}_2 = \mathbb{Z}[i\sqrt{2}]$. Then the group $PU(2, 1, \mathcal{O}_2)$ is generated by the elements*

$$I_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, R_1^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_2^{(2)} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_3^{(2)} = \begin{bmatrix} 1 & -i\sqrt{2} & -1 \\ 0 & -1 & i\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{(2)} = \begin{bmatrix} 1 & 0 & i\sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem 4.4. *Let $\mathcal{K} = \mathbb{Q}(\sqrt{-7})$ and let $\mathcal{O}_7 = \mathbb{Z}[\omega_7]$, where $\omega_7 = \frac{1}{2}(1 + i\sqrt{7})$, be the ring of integers of $\mathcal{K}$. Then the group $PU(2, 1, \mathcal{O}_7)$ is generated by the elements*

$$I_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, R_1^{(7)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_2^{(7)} = \begin{bmatrix} 1 & 1 & -\bar{\omega}_7 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_3^{(7)} = \begin{bmatrix} 1 & \bar{\omega}_7 & -1 \\ 0 & -1 & \omega_7 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{(7)} = \begin{bmatrix} 1 & 0 & i\sqrt{7} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem 4.5. *Let $\mathcal{K} = \mathbb{Q}(\sqrt{-11})$ and let $\mathcal{O}_{11} = \mathbb{Z}[\omega_{11}]$, where $\omega_{11} = \frac{1}{2}(1 + i\sqrt{11})$, be the ring of integers of $\mathcal{K}$. Then the group $PU(2, 1, \mathcal{O}_{11})$ is generated by the elements*

$$I_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, R_1^{(11)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_2^{(11)} = \begin{bmatrix} 1 & 1 & -\bar{\omega}_{11} \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_3^{(11)} = \begin{bmatrix} 1 & \bar{\omega}_{11} & -1 - \bar{\omega}_{11} \\ 0 & -1 & \omega_{11} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{(11)} = \begin{bmatrix} 1 & 0 & i\sqrt{11} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark. For other values of d such that $\mathcal{O}_d$ has class number one, namely $d = 19, 43, 67, 163$, we can construct the same type of fundamental domain for $(\Gamma_d)_\infty$ in the Heisenberg group as $(\Gamma_{11})_\infty$. All generators as the above types lie in $PU(2, 1; \mathcal{O}_d)$, but we don't know whether adjoining the element I_0 to $(\Gamma_d)_\infty$ generates the full group $PU(2, 1; \mathcal{O}_d)$. Furthermore, the method of [5] could not be extended to non-Euclidean Picard modular groups.

5. DETERMINATION OF THE ISOMETRIC SPHERES

Recall that the Cygan sphere $\mathcal{B}_0$ is the isometric sphere of I_0 . The boundary of $\mathcal{B}_0$ is a spinal sphere denoted by $\mathcal{S}_0$ in the Heisenberg group, which is defined by

$$(5.1) \quad \mathcal{S}_0 = \{(\zeta, t) : |\zeta|^2 + it = 2\}.$$

Indeed, we only need to consider the boundaries of isometric spheres in the Heisenberg group because two isometric spheres have a non-empty interior intersection if and only if their boundaries have a non-empty interior intersection.

5.1. The case $\mathcal{O}_2$. In the cases of $PU(2, 1; \mathcal{O}_1)$ and $PU(2, 1; \mathcal{O}_3)$, all the vertices of the fundamental domain for the stabiliser of q_∞ acting on $\partial\mathbf{H}_{\mathbb{C}}^2$ lie inside $\mathcal{S}_0$. For the group $PU(2, 1; \mathcal{O}_2)$, it is not hard to show that six vertices of the prism Σ_2 lie outside of $\mathcal{S}_0$. Therefore, we need to find more isometric spheres whose boundaries together with $\mathcal{S}_0$ contain the prism Σ_2 .

We consider the map

$$I_0 R_2^{(2)} I_0 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ -2 & -2 & 1 \end{bmatrix},$$

whose isometric sphere which we denote by $\mathcal{B}_1$ is a Cygan sphere centred at the point $(1, 0, 0)$ (in horospherical coordinates) with radius 1. The boundary of $\mathcal{B}_1$ is given by

$$(5.2) \quad \mathcal{S}_1 = \{(\zeta, t) : |\zeta - 1|^2 + it + 2i\Im m\zeta = 1\}.$$

Minimising the number of spinal spheres by the symmetry of $R_1^{(2)}$, it suffices to consider $\mathcal{S}_0$ and several images of $\mathcal{S}_1$ under some suitable elements in $(\Gamma_2)_\infty$. In Heisenberg coordinates these are given by

$$\begin{aligned} T^{(2)}(\mathcal{S}_1) &= \{(\zeta, t) : |\zeta - 1|^2 + it - 2i\sqrt{2} + 2i\Im m\zeta = 1\}, \\ T^{(2)^{-1}}(\mathcal{S}_1) &= \{(\zeta, t) : |\zeta - 1|^2 + it + 2i\sqrt{2} + 2i\Im m\zeta = 1\}, \\ R_1^{(2)}(\mathcal{S}_1) &= \{(\zeta, t) : |\zeta + 1|^2 + it - 2i\Im m\zeta = 1\}, \\ T^{(2)^{-1}}R_1^{(2)}(\mathcal{S}_1) &= \{(\zeta, t) : |\zeta + 1|^2 + it + 2i\sqrt{2} - 2i\Im m\zeta = 1\}. \end{aligned}$$

We claim that the prism Σ_2 lies inside the union of $\mathcal{S}_0$ and these images of $\mathcal{S}_1$; see Figure 5.1 for viewing these spinal spheres.

Proposition 5.1. *The prism Σ_2 is contained in the union of the interiors of the spinal spheres $\mathcal{S}_0$, $\mathcal{S}_1$, $T^{(2)}(\mathcal{S}_1)$, $T^{(2)^{-1}}(\mathcal{S}_1)$, $R_1^{(2)}(\mathcal{S}_1)$ and $T^{(2)^{-1}}R_1^{(2)}(\mathcal{S}_1)$.*

Proof. It suffices to show that there exist three points $(v_1^+)^{(j)}$ ($j = 1, 2, 3$) on the edges (v_1^+, v_1^-) , (v_1^+, v_2^+) and (v_1^+, v_3^+) which lie in the intersection of the interiors of $\mathcal{S}_0$ and $\mathcal{S}_1$ such that the tetrahedron $\mathbb{T}(v_1^+)$ with vertices v_1^+ , $(v_1^+)^{(1)}$, $(v_1^+)^{(2)}$, $(v_1^+)^{(3)}$ lies inside $\mathcal{S}_1$. By the same argument, we can also obtain five other tetrahedra $\mathbb{T}(v_2^+)$, $\mathbb{T}(v_3^+)$, $\mathbb{T}(v_1^-)$, $\mathbb{T}(v_2^-)$, $\mathbb{T}(v_3^-)$ with apex v_2^+ , v_3^+ , v_1^- , v_2^- , v_3^- , respectively, such that $\mathbb{T}(v_2^+) \in \text{Int}(T^{(2)}(\mathcal{S}_1))$, $\mathbb{T}(v_3^+) \in \text{Int}(R_1^{(2)}(\mathcal{S}_1))$, $\mathbb{T}(v_1^-) \in \text{Int}(T^{(2)^{-1}}(\mathcal{S}_1))$, $\mathbb{T}(v_2^-) \in \text{Int}(\mathcal{S}_1)$ and $\mathbb{T}(v_3^-) \in \text{Int}(T^{(2)^{-1}}R_1^{(2)}(\mathcal{S}_1))$. Moreover, the core part obtained by cutting off six of the tetrahedra from the prism lies inside $\mathcal{S}_0$.

FIGURE 5.1. (a) The shading view of neighboring spinal spheres containing the fundamental domain for $(\Gamma_2)_\infty$. (b) Another view for these spinal spheres.

We shall prove the existence of the tetrahedron $\mathbb{T}(v_1^+)$ and the others follow similarly. The edge joining v_1^+ and v_1^- is contained in the complex line $\zeta = 1 - \sqrt{2}i/2$ which is given by points with Heisenberg coordinates

$$\zeta = 1 - \sqrt{2}i/2, \quad -\sqrt{2} \leq t \leq \sqrt{2}.$$

The edge joining v_1^+ and v_2^+ is given by points with Heisenberg coordinates

$$\Re \zeta = 1, \quad -\sqrt{2}/2 \leq \Im m \zeta \leq \sqrt{2}/2, \quad t = \sqrt{2}.$$

The edge joining v_1^+ and v_3^+ is given by points with Heisenberg coordinates

$$\Re \zeta = -\sqrt{2}\Im m \zeta, \quad t = \sqrt{2}.$$

From equations (5.1) and (5.2), the points on the edge (v_1^+, v_1^-) lie in the intersection of the interiors of $\mathcal{S}_0$ and $\mathcal{S}_1$ if and only if

$$(5.3) \quad |3/2 + it| < 2 \quad \text{and} \quad |1/2 - (t - \sqrt{2})i| < 1.$$

By easy calculations, the inequalities (5.3) are equivalent to

$$\sqrt{2} - \sqrt{3}/2 < t < \sqrt{7}/2.$$

Using the same argument as above, we obtain that the points on the edge (v_1^+, v_2^+) lie in the intersection of the interiors of $\mathcal{S}_0$ and $\mathcal{S}_1$ if and only if $\Re \zeta = 1$ and $-\sqrt{\sqrt{2}-1} < \Im m \zeta < \delta_1$, where $\delta_1 \approx -0.208$ is the largest real root of the equation $x^4 + 4x^2 + 4\sqrt{2}x + 1 = 0$. The points on the edge (v_1^+, v_3^+) lie in the intersection of the interiors of $\mathcal{S}_0$ and $\mathcal{S}_1$ if and only if $\Re \zeta = -\sqrt{2}\Im m \zeta$ and $-2^{1/4}/\sqrt{3} < \Im m \zeta < \delta_2$, where $\delta_2 \approx -0.264$ is the largest real root of the equation $9x^4 + 12\sqrt{2}x^3 + 18x^2 + 8\sqrt{2}x + 2 = 0$.

In term of these, we choose three points as $(v_1^+)^{(1)} = (1 - \sqrt{2}i/2, 1)$ on the edge (v_1^+, v_1^-) , $(v_1^+)^{(2)} = (1 - i/2, \sqrt{2})$ on the edge (v_1^+, v_2^+) and $(v_1^+)^{(3)} = (\sqrt{2}/2 - i/2, \sqrt{2})$ on the edge (v_1^+, v_3^+) , which are inside the intersection of the interiors of $\mathcal{S}_0$ and

$\mathcal{S}_1$. Since the vertex v_1^+ lies inside $\mathcal{S}_1$, the tetrahedron $\mathbb{T}(v_1^+)$ with the vertices v_1^+ , $(v_1^+)^{(1)}$, $(v_1^+)^{(2)}$, $(v_1^+)^{(3)}$ lies inside $\mathcal{S}_1$ by Lemma 4.2. $\square$

5.2. The case $\mathcal{O}_7$. In this case, the distance between the top and base of the fundamental domain for the stabiliser $(\Gamma_7)_\infty$ is greater than the diameter of $\mathcal{S}_0$, which implies that the prism Σ_7 cannot be contained inside $\mathcal{S}_0$ completely. Due to the increasing length of Heisenberg translations, only the images of $\mathcal{S}_0$ under the elements in $(\Gamma_7)_\infty$ could not cover the entire prism. We show that there are also isometric spheres with Cygan radius smaller than $\sqrt{2}$ whose centres are near to the origin.

Therefore, we consider the map

$$Q = I_0 R_2^{(7)} I_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \bar{\omega}_7 & 1 & 1 \end{bmatrix}.$$

Consider the isometric spheres of Q and Q^{-1} , which we denote by $\mathcal{B}_2$ and $\mathcal{B}_3$, respectively. The centre of $\mathcal{B}_2$ is $Q^{-1}(\infty)$, which is the point with horospherical coordinates $(1/4 + i\sqrt{7}/4, \sqrt{7}/2, 0)$ and the centre of $\mathcal{B}_3$, is $Q(\infty)$ which has horospherical coordinates $(1/4 - i\sqrt{7}/4, \sqrt{7}/2, 0)$. Both these isometric spheres have Cygan radius $\sqrt{2/|\omega_7|} = 2^{1/4}$. The boundaries of these isometric spheres $\mathcal{B}_2$ and $\mathcal{B}_3$ are in Heisenberg coordinates given by

$$(5.4) \quad \mathcal{S}_2 = \left\{ (\zeta, t) : \left| |\zeta - \omega_7/2|^2 + it + i\sqrt{7}/2 + i\Im(\bar{\omega}_7\zeta) \right| = \sqrt{2} \right\},$$

$$(5.5) \quad \mathcal{S}_3 = \left\{ (\zeta, t) : \left| |\zeta - \bar{\omega}_7/2|^2 + it - i\sqrt{7}/2 + i\Im(\omega_7\zeta) \right| = \sqrt{2} \right\}.$$

In order to cover the prism Σ_7 by the spinal spheres, we use the symmetry property of $R_1^{(7)}$. It suffices to consider $\mathcal{S}_0, \mathcal{S}_2$ and the images of $\mathcal{S}_0$ and $\mathcal{S}_3$ under suitable elements in $(\Gamma_7)_\infty$. These spinal spheres are points with Heisenberg coordinates given by

$$\begin{aligned} R_2^{(7)}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta - 1|^2 + it - i\sqrt{7} + 2i\Im m\zeta \right| = 4 \right\}, \\ R_2^{(7)^{-1}}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta - 1|^2 + it + i\sqrt{7} + 2i\Im m\zeta \right| = 4 \right\}, \\ R_2^{(7)^{-1}}(\mathcal{S}_3) &= \left\{ (\zeta, t) : \left| |\zeta - (1 + \omega_7)/2|^2 + it + i\sqrt{7} + i\Im((1 + \bar{\omega}_7)\zeta) \right| = \sqrt{2} \right\}, \\ R_3^{(7)} R_2^{(7)}(\mathcal{S}_3) &= \left\{ (\zeta, t) : \left| |\zeta + \bar{\omega}_7/2|^2 + it - i\sqrt{7}/2 - i\Im(\omega_7\zeta) \right| = \sqrt{2} \right\}, \\ R_1^{(7)} R_3^{(7)} R_2^{(7)}(\mathcal{S}_3) &= \left\{ (\zeta, t) : \left| |\zeta - \bar{\omega}_7/2|^2 + it - i\sqrt{7}/2 + i\Im(\omega_7\zeta) \right| = \sqrt{2} \right\}. \end{aligned}$$

We claim that the prism Σ_7 lies inside the union of $\mathcal{S}_0, \mathcal{S}_2$ and the images $R_2^{(7)}(\mathcal{S}_0), R_2^{(7)^{-1}}(\mathcal{S}_0), R_2^{(7)^{-1}}(\mathcal{S}_3), R_3^{(7)} R_2^{(7)}(\mathcal{S}_3), R_1^{(7)} R_3^{(7)} R_2^{(7)}(\mathcal{S}_3)$; see Figure 5.2 for viewing these spinal spheres.

Proposition 5.2. *The prism Σ_7 is contained in the union of the interiors of the spinal spheres $\mathcal{S}_0, \mathcal{S}_2, R_2^{(7)}(\mathcal{S}_0), R_2^{(7)^{-1}}(\mathcal{S}_0), R_2^{(7)^{-1}}(\mathcal{S}_3), R_3^{(7)} R_2^{(7)}(\mathcal{S}_3)$ and $R_1^{(7)} R_3^{(7)} R_2^{(7)}(\mathcal{S}_3)$.*

FIGURE 5.2. (a) The shading view of neighboring spinal spheres containing the fundamental domain for $(\Gamma_7)_\infty$. (b) Another view for these spinal spheres.

Proof. It suffices to show that the prism Σ_7 can be decomposed into several pieces as polyhedra such that each polyhedron lies inside a spinal sphere which is described in the proposition and the common face of two adjacent polyhedra lie in the intersection of the interior of two spinal spheres which contain these two polyhedra.

We need to add sixteen points on the faces of the prism Σ_7 in order to decompose the prism into seven polyhedra, in Heisenberg coordinates. These are given by

$$\begin{array}{ll}
 p_1 = (1/4 - i\sqrt{7}/4, 3/2), & p_2 = (0.11 - i11\sqrt{7}/100, 1.44 + \sqrt{7}/50), \\
 p_3 = (1/2, 8/5), & p_4 = (-1/10 + i\sqrt{7}/10, \sqrt{7}), \\
 p_5 = (-1/10 + i\sqrt{7}/10, \sqrt{7}/2), & p_6 = (3/4 + i\sqrt{7}/4, 1.7), \\
 p_7 = (-1/4 + i\sqrt{7}/4, 1), & p_8 = (1/4 - i\sqrt{7}/4, -1), \\
 p_9 = (1/60 - i\sqrt{7}/60, -2\sqrt{7}/3), & p_{10} = (-1/20 + i\sqrt{7}/20, -\sqrt{7}), \\
 p_{11} = (3/5 + i\sqrt{7}/10, -2\sqrt{7}/3), & p_{12} = (7/10 + i\sqrt{7}/5, -\sqrt{7}), \\
 p_{13} = (3/4 + i\sqrt{7}/4, -2\sqrt{7}/3), & p_{14} = (5/12 + i\sqrt{7}/4, -2\sqrt{7}/3), \\
 p_{15} = (1/4 + i\sqrt{7}/4, -\sqrt{7}), & p_{16} = (-1/4 + i\sqrt{7}/4, -1).
 \end{array}$$

We describe these polyhedra as follows:

- (i) The tetrahedron $\mathbb{T}$ with the vertex v_1^+ , p_1 , p_2 , p_3 ;
- (ii) The hexahedron $\mathbb{H}_1$ with the vertex v_1^+ , v_2^+ , p_2 , p_3 , p_4 , p_5 , p_6 ;
- (iii) The pentahedron $\mathbb{P}_1$ with the vertex v_2^+ , p_4 , p_5 , p_6 , v_4^+ , p_7 ;
- (iv) The pentahedron $\mathbb{P}_2$ with the vertex v_1^- , p_8 , p_9 , p_{10} , p_{11} , p_{12} ;
- (v) The hexahedron $\mathbb{H}_2$ with the vertex p_9 , p_{10} , p_{11} , p_{12} , p_{13} , v_2^- , p_{14} , p_{15} ;
- (vi) The pentahedron $\mathbb{P}_3$ with the vertex p_9 , p_{10} , p_{14} , p_{15} , p_{16} , v_4^+ ;
- (vii) The octahedron $\mathbb{O}$ with the vertex p_1 , p_2 , p_3 , p_5 , p_6 , p_7 , p_8 , p_9 , p_{11} , p_{13} , p_{14} , p_{16} .

FIGURE 5.3. A view of the decomposition for the prism Σ_7 as several polyhedra.

Note that the face (p_1, p_2, p_3) of $\mathbb{T}$ and the face (p_2, p_3, p_5, p_6) of $\mathbb{H}_1$ are on the face (p_1, p_5, p_6) of $\mathbb{O}$; the common face (v_2^+, p_4, p_5, p_6) of $\mathbb{H}_1$ and $\mathbb{P}_1$ is a vertical plane; the face $(p_9, p_{11}, p_{13}, p_{14})$ of $\mathbb{H}_2$ is parallel to the base of the prism. Furthermore, the faces $(p_9, p_{10}, p_{11}, p_{12})$ and $(p_9, p_{10}, p_{14}, p_{15})$ are the trapeziums since the edge (p_9, p_{11}) is parallel to (p_{10}, p_{12}) and the edge (p_9, p_{14}) is parallel to (p_{10}, p_{15}) .

By examining the location of the points and applying Lemma 4.2, we conclude that the tetrahedron $\mathbb{T}$ is inside the spinal sphere $R_1^{(7)} R_3^{(7)} R_2^{(7)}(\mathcal{S}_3)$; the hexahedron $\mathbb{H}_1$ is contained inside the spinal sphere $R_2^{(7)}(\mathcal{S}_0)$; the pentahedron $\mathbb{P}_1$ is inside $R_3^{(7)} R_2^{(7)}(\mathcal{S}_3)$; the pentahedron $\mathbb{P}_2$ is contained inside $R_2^{(7)-1}(\mathcal{S}_0)$; the hexahedron $\mathbb{H}_2$ is contained inside $R_2^{(7)-1}(\mathcal{S}_3)$; the pentahedron $\mathbb{P}_3$ is inside $\mathcal{S}_2$; the remaining octahedron $\mathbb{O}$ is inside $\mathcal{S}_0$; see Figure 5.3 for viewing the polyhedral decomposition of the prism. $\square$

5.3. The case $\mathcal{O}_{11}$. In this case, we again know that the fundamental domain for the stabiliser $(\Gamma_{11})_\infty$ cannot be inside $\mathcal{S}_0$ completely. The radius of spinal spheres other than the largest are so small that these spinal spheres do not contribute much to covering the prism Σ_{11} . Due to the different shape of the prism Σ_{11} with the case $\mathcal{O}_7$, we only need to consider the largest spinal spheres which are the images of $\mathcal{S}_0$ under the elements of $(\Gamma_{11})_\infty$. In order to determine a union of the spinal spheres which covers the prism Σ_{11} , we minimise their number by the symmetry of $R_1^{(11)}$. It suffices to consider $\mathcal{S}_0$ and the images of $\mathcal{S}_0$ under suitable elements in $(\Gamma_{11})_\infty$. In Heisenberg coordinates they are given by

$$\begin{aligned} T^{(11)}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta|^2 + it - 2i\sqrt{11} + 2i\Im m\zeta \right| = 4 \right\}, \\ R_2^{(11)}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta - 1|^2 + it - i\sqrt{11} + 2i\Im m\zeta \right| = 4 \right\}, \end{aligned}$$

FIGURE 5.4. (a) The shading view of neighboring spinal spheres containing the fundamental domain for $(\Gamma_{11})_\infty$. (b) Another view for these spinal spheres.

$$\begin{aligned}
 R_1^{(11)} R_2^{(11)}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta + 1|^2 + it - i\sqrt{11} - 2i\Im m \zeta \right| = 4 \right\}, \\
 R_2^{(11)-1}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta - 1|^2 + it + i\sqrt{11} + 2i\Im m \zeta \right| = 4 \right\}, \\
 R_3^{(11)}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta - \omega_{11}|^2 + it - i\sqrt{11} - 2i\Im m(\bar{\omega}_{11}\zeta) \right| = 4 \right\}, \\
 R_3^{(11)-1}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta - \omega_{11}|^2 + it + i\sqrt{11} - 2i\Im m(\bar{\omega}_{11}\zeta) \right| = 4 \right\}, \\
 R_3^{(11)} R_2^{(11)}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta + \bar{\omega}_{11}|^2 + it - i\sqrt{11} - 2i\Im m(\omega_{11}\zeta) \right| = 4 \right\}, \\
 R_1^{(11)} R_3^{(11)} R_2^{(11)}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta - \bar{\omega}_{11}|^2 + it - i\sqrt{11} + 2i\Im m(\omega_{11}\zeta) \right| = 4 \right\}, \\
 R_1^{(11)} R_3^{(11)-1} R_2^{(11)}(\mathcal{S}_0) &= \left\{ (\zeta, t) : \left| |\zeta - \bar{\omega}_{11}|^2 + it + i\sqrt{11} + 2i\Im m(\omega_{11}\zeta) \right| = 4 \right\}.
 \end{aligned}$$

Definition 5.3. Let X be a closed polygonal chain (not necessarily in a plane), then a topological disc defined by the cone over X with apex v is called a **cone-polygon**, denoted by $\mathbb{D}_v(X)$.

Note that a polygon in the traditional sense can be interpreted as a cone-polygon. In that case, the boundary of cone-polygon and the apex lie in the same plane and moreover the apex is in the interior of the boundary. We claim that the prism Σ_{11} lies inside the union of $\mathcal{S}_0$ and its images as above; see Figure 5.4 for viewing these spinal spheres.

Proposition 5.4. *The prism Σ_{11} is contained in the union of the interiors of the spinal spheres $\mathcal{S}_0$, $T^{(11)}(\mathcal{S}_0)$, $R_2^{(11)}(\mathcal{S}_0)$, $R_2^{(11)-1}(\mathcal{S}_0)$, $R_3^{(11)}(\mathcal{S}_0)$, $R_3^{(11)-1}(\mathcal{S}_0)$, $R_3^{(11)} R_2^{(11)}(\mathcal{S}_0)$, $R_1^{(11)} R_2^{(11)}(\mathcal{S}_0)$, $R_1^{(11)} R_3^{(11)-1} R_2^{(11)}(\mathcal{S}_0)$ and $R_1^{(11)} R_3^{(11)} R_2^{(11)}(\mathcal{S}_0)$*

Proof. Using the same argument as in Proposition 5.2, we want to decompose the prism Σ_{11} into several polyhedral cells. The difference is the complicated intersection of the spinal spheres, which makes it difficult to decompose into several

polyhedral cells, each of which is contained in one spinal sphere. Observe that a union of interiors of several spinal spheres is a star-convex set if they have a non-empty interior intersection. We shall show that the collection of these spinal spheres can be separated into several parts such that each part contains certain polyhedral cell. All of these polyhedral cells are defined by the cone-polygon as its boundary.

We first define a tetrahedron $\mathbb{T}$ with vertices v_1^-, q_1, q_2, q_3 , where

$$\begin{aligned} q_1 &= (1/4 - i\sqrt{11}/4, -2\sqrt{11}/3), \\ q_2 &= (3/20 - 3i\sqrt{11}/20, -4\sqrt{11}/5), \\ q_3 &= (7/20 - 3i\sqrt{11}/20, -9\sqrt{11}/10). \end{aligned}$$

Observe that the points q_1, q_2, q_3 lie on the edges (v_1^+, v_1^-) , (v_1^-, v_3^-) and (v_1^-, v_2^-) , respectively. By Lemma 4.2 and simple calculations we show that the tetrahedron $\mathbb{T}$ is contained inside $R_1^{(11)} R_3^{(11)-1} R_2^{(11)}(\mathcal{S}_0)$.

Next, we define a hexahedron $\mathbb{H}_1$ with vertices $q_1, q_2, q_3, q_4, q_5, q_6, q_7, v_0^+$ and another hexahedron $\mathbb{H}_2$ with vertices $v_2^-, q_5, q_6, q_7, q_8, q_9$, where

$$\begin{aligned} q_4 &= (1/4 - i\sqrt{11}/4, -1/2), & q_5 &= (0.42 + 0.26i, -0.71\sqrt{11} + 0.39), \\ q_6 &= (0.6 + i\sqrt{11}/10, -0.65\sqrt{11}), & q_7 &= (0.58 + 2i\sqrt{11}/25, -1.92), \\ q_8 &= (3/4 + i\sqrt{11}/4, 0), & q_9 &= (0.55 + i\sqrt{11}/4, -2\sqrt{11}/5). \end{aligned}$$

Observe that the points q_4, q_6, q_8, q_9 lie on the edges (v_1^+, v_1^-) , (v_1^-, v_2^-) , (v_2^+, v_2^-) and (v_2^-, v_3^-) , respectively. The point q_5 lies on the interior of the base of the prism Σ_{11} and the point q_7 lies on the interior of the face $(v_1^+, v_1^-, v_2^-, v_2^+)$. Then we know that the hexahedron $\mathbb{H}_1$ has the faces (q_1, q_2, q_3) , (q_4, q_5, v_0^-) , (q_4, q_5, q_7) , (q_5, q_6, q_7) , (q_1, q_2, v_0^+, q_4) and $(q_1, q_3, q_6, q_7, q_4)$ and the hexahedron $\mathbb{H}_2$ has the faces (q_5, q_6, q_7) , (q_5, q_7, q_8) , (v_2^-, q_8, q_9) , (q_5, q_8, q_9) , (q_6, q_7, q_8, v_2^-) , (q_5, q_6, v_2^-, q_9) . By examining the location of these points and Lemma 4.1, we conclude that the hexahedron $\mathbb{H}_1$ is contained inside $R_2^{(11)-1}(\mathcal{S}_0)$ and the hexahedron $\mathbb{H}_2$ lies inside $R_3^{(11)-1}(\mathcal{S}_0)$.

We focus on describing other polyhedral cells in the decomposition of the prism Σ_{11} . Let $\mathcal{U}_1$ denote the union of $R_2^{(11)}(\mathcal{S}_0)$, $R_1^{(11)} R_2^{(11)}(\mathcal{S}_0)$ and $R_1^{(11)} R_3^{(11)} R_2^{(11)}(\mathcal{S}_0)$. We verify that $q_{10} = (0.2 - 0.4i, 2.4)$ is in the intersection of the interiors of these three spinal spheres, which implies that $\mathcal{U}_1$ is a star-convex set about q_{11} . Analogously, we know $\mathcal{U}_2$, denoted by the union of $T^{(11)}(\mathcal{S}_0)$, $R_3^{(11)}(\mathcal{S}_0)$, $R_1^{(11)} R_2^{(11)}(\mathcal{S}_0)$ and $R_3^{(11)} R_2^{(11)}(\mathcal{S}_0)$, is a star-convex set about $q_{11} = (0.18 + 0.72i, 4.8)$. This can be verified by examining the location of q_{12} which is in the intersection of the interiors of these four spinal spheres. We need to add the following points on the faces of the prism Σ_{11} , each of which is in the intersection of the interiors of at least two spinal spheres:

$$\begin{aligned} q_{12} &= (1/4 - i\sqrt{11}/4, \sqrt{11}/2), & q_{13} &= (0.21 - 0.21i\sqrt{11}, \sqrt{11}/2), \\ q_{14} &= (0, \sqrt{11}/2), & q_{15} &= (-0.21 + 0.21i\sqrt{11}, \sqrt{11}/2), \\ q_{16} &= (i\sqrt{11}/4, 1), & q_{17} &= (3/4 + i\sqrt{11}/4, 1), \\ q_{18} &= (0.42 - 2i\sqrt{11}/25, 1.95), & q_{19} &= (3/4 + i\sqrt{11}/4, \sqrt{11}), \\ q_{20} &= (0.6 + i\sqrt{11}/10, 27\sqrt{11}/20), & q_{21} &= (0.42 + 0.26i, 1.29\sqrt{11} + 0.39), \\ q_{22} &= (-1.4 + 1.4i\sqrt{11}, 4\sqrt{11}/5), & q_{23} &= (-1/4 + i\sqrt{11}/4, \sqrt{11}/2). \end{aligned}$$

Observe that the points q_{12}, q_{20}, q_{23} lie on the edges (v_1^+, v_1^-) , (v_1^+, v_2^+) and (v_3^+, v_3^-) , respectively, and the points q_{17}, q_{19} lie on the edge (v_2^+, v_2^-) . Moreover,

the points $q_{13}, q_{14}, q_{15}, q_{22}$ lie on the interior of the face $(v_1^+, v_1^-, v_3^-, v_3^+)$, the point q_{16} lies on the interior of the face $(v_2^+, v_2^-, v_3^-, v_3^+)$, the point q_{18} lies on the interior of the face $(v_1^+, v_1^-, v_2^-, v_2^+)$ and the point q_{21} lies on the interior of the top (v_1^+, v_2^+, v_3^+) . We need to add other three points in the interior of the prism Σ_{11} which are used to define the cone-polygon,

$$\begin{aligned} q_{24} &= (-0.16 + 0.74i, 1.4), \\ q_{25} &= (0.328 - 0.28i, 1.99), \\ q_{26} &= (0.325 + 0.29i, 4.652). \end{aligned}$$

We verify the location of all these points as follows:

- The point q_{12} is in the intersection of the interiors of $\mathcal{S}_0$ and $R_1^{(11)}R_3^{(11)}R_2^{(11)}(\mathcal{S}_0)$;
- The point q_{13} is in the intersection of the interiors of $\mathcal{S}_0$, $R_1^{(11)}R_3^{(11)}R_2^{(11)}(\mathcal{S}_0)$ and $R_1^{(11)}R_2^{(11)}(\mathcal{S}_0)$;
- The point q_{14} is in the intersection of the interiors of $\mathcal{S}_0$, $R_2^{(11)}(\mathcal{S}_0)$ and $R_1^{(11)}R_2^{(11)}(\mathcal{S}_0)$;
- The point q_{15} is in the intersection of the interiors of $\mathcal{S}_0$, $R_2^{(11)}(\mathcal{S}_0)$, $R_3^{(11)}(\mathcal{S}_0)$ and $R_3^{(11)}R_2^{(11)}(\mathcal{S}_0)$;
- The points q_{16}, q_{19}, q_{20} are in the intersection of the interiors of $R_2^{(11)}(\mathcal{S}_0)$ and $R_3^{(11)}(\mathcal{S}_0)$;
- The point q_{17} is in the intersection of the interiors of $\mathcal{S}_0$ and $R_2^{(11)}(\mathcal{S}_0)$;
- The point q_{18} is in the intersection of the interiors of $\mathcal{S}_0$, $R_1^{(11)}R_3^{(11)}R_2^{(11)}(\mathcal{S}_0)$ and $R_2^{(11)}(\mathcal{S}_0)$;
- The point q_{21} is in the intersection of the interiors of $T^{(11)}(\mathcal{S}_0)$, $R_2^{(11)}(\mathcal{S}_0)$ and $R_3^{(11)}(\mathcal{S}_0)$;
- The point q_{22} is in the intersection of the interiors of $R_1^{(11)}R_2^{(11)}(\mathcal{S}_0)$, $R_2^{(11)}(\mathcal{S}_0)$ and $R_3^{(11)}(\mathcal{S}_0)$;
- The point v_0^+ is in the intersection of the interiors of $R_1^{(11)}R_2^{(11)}(\mathcal{S}_0)$ and $T^{(11)}(\mathcal{S}_0)$, $R_2^{(11)}(\mathcal{S}_0)$;
- The point q_{23} is in the intersection of the interiors of $\mathcal{S}_0$, $R_3^{(11)}(\mathcal{S}_0)$ and $R_3^{(11)}R_2^{(11)}(\mathcal{S}_0)$;
- The point q_{24} is in the intersection of the interiors of $\mathcal{S}_0$, $R_2^{(11)}(\mathcal{S}_0)$, $R_3^{(11)}(\mathcal{S}_0)$ and $R_3^{(11)}R_2^{(11)}(\mathcal{S}_0)$;
- The point q_{25} is in the intersection of the interiors of $\mathcal{S}_0$, $R_2^{(11)}(\mathcal{S}_0)$, $R_1^{(11)}R_2^{(11)}(\mathcal{S}_0)$ and $R_1^{(11)}R_3^{(11)}R_2^{(11)}(\mathcal{S}_0)$;
- The point q_{26} is in the intersection of the interiors of $T^{(11)}(\mathcal{S}_0)$, $R_1^{(11)}R_2^{(11)}(\mathcal{S}_0)$, $R_2^{(11)}(\mathcal{S}_0)$ and $R_3^{(11)}(\mathcal{S}_0)$.

In terms of these, we denote by X_1 a closed polygonal chain joining in order with the points $p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_{17}, p_{18}$ and denote by X_2 a closed polygonal chain joining in order with the points $p_{16}, p_{19}, p_{20}, p_{21}, v_0^+, p_{22}, p_{24}$. So then we can define two cone-polygons $\mathbb{D}_{q_{25}}(X_1)$ and $\mathbb{D}_{q_{26}}(X_2)$. By examining the locations of these points, we show that $\mathbb{D}_{q_{25}}(X_1)$ is in the intersection of the interiors of $\mathcal{S}_0$, $\mathcal{U}_1$ and $\mathbb{D}_{q_{26}}(X_2)$ is in the intersection of the interiors of $R_2^{(11)}(\mathcal{S}_0)$ and $T^{(11)}(\mathcal{S}_0)$, $R_1^{(11)}R_2^{(11)}(\mathcal{S}_0)$, $R_3^{(11)}(\mathcal{S}_0)$; namely, the intersection of the interiors of $\mathcal{U}_1$ and $\mathcal{U}_2$. The remaining faces which are contained inside $\mathcal{S}_0, \mathcal{U}_1$ or $\mathcal{U}_2$ can be easily verified.

Finally, we define three polyhedral cells as follows:

(i) The polyhedral cell $\mathbf{P}_1$ is defined by its boundary as the faces $\mathbb{D}_{q_{25}}(X_1)$, $\mathbb{D}_{q_{26}}(X_2)$, $(v_1^+, q_{20}, q_{21}, v_0^+)$, $(v_1^+, q_{12}, q_{18}, q_{17}, q_{19}, q_{20})$ and $(v_1^+, q_{12}, q_{13}, q_{14}, q_{15}, q_{22}, v_0^+)$;

(ii) The polyhedral cell $\mathbf{P}_2$ is defined by its boundary as the faces $\mathbb{D}_{q_{26}}(X_2)$, (q_{23}, q_{16}, q_{24}) , $(v_1^+, q_{20}, q_{21}, v_0^+)$, (q_{15}, q_{23}, q_{24}) , $(v_1^+, q_{12}, q_{18}, q_{17}, q_{19}, q_{20})$ and $(v_1^+, q_{12}, q_{13}, q_{14}, q_{15}, q_{22}, v_0^+)$;

(iii) The polyhedral cell $\mathbf{P}_3$ is defined by the faces $\mathbb{D}_{q_{25}}(X_1)$, (v_3^-, v_0^-, q_5, q_9) , (q_4, q_5, q_7) , (q_5, q_7, q_8) , (q_4, q_5, v_0^-) , $(q_8, q_9, v_3^-, q_{23}, q_{16}, q_{17})$, (q_{15}, q_{23}, q_{24}) , $(q_4, q_7, q_8, q_{17}, q_{18}, q_{12})$, (q_{23}, q_{16}, q_{24}) and $(q_{12}, q_{13}, q_{14}, q_{15}, q_{23}, v_3^-, v_0^-, q_4)$ as its boundary.

By Lemma 4.2 and the properties of star-convex of $\mathcal{U}_1$ and $\mathcal{U}_2$, we conclude that the polyhedral cell $\mathbf{P}_1$ contained inside $\mathcal{U}_1$, the polyhedral cell $\mathbf{P}_2$ contained inside $\mathcal{U}_2$, and the polyhedral cell $\mathbf{P}_3$ is contained inside $\mathcal{S}_0$. This completes the proof. $\square$

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