INVERSE SPECTRAL PROBLEM WITH PARTIAL INFORMATION GIVEN ON THE POTENTIAL AND NORMING CONSTANTS

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ABSTRACT. The inverse spectral problem for a Sturm-Liouville equation in Liouville form with separated self-adjoint boundary conditions on the unit interval $[0, 1]$ is considered. Some uniqueness results are obtained which imply that the potential q can be completely determined even if only partial information is given on q together with partial information on the spectral data, consisting of either one full spectrum and a subset of norming constants or a subset of pairs of eigenvalues and the corresponding norming constants. Moreover, the problem of missing eigenvalues and norming constants is also investigated in the situation where the potential q is C^{2k-1} and the boundary conditions at the endpoints 0 and 1 are fixed.

1. INTRODUCTION

We are concerned with an inverse problem for the Sturm-Liouville equation. Consider the operator $L := L(q, h_0, h_1)$ defined by

$$(1.1) \quad Lu = -u'' + qu$$

on $[0, 1]$ with the boundary conditions

$$(1.2) \quad u'(0) - h_0 u(0) = 0,$$

$$(1.3) \quad u'(1) - h_1 u(1) = 0.$$

Here the potential $q \in L^1[0, 1]$ is real-valued and $h_0, h_1 \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. Note that, in an obvious notation, $h_0 = \infty$ singles out the Dirichlet boundary condition $u(0) = 0$.

It is well known [36] that the operator L subject to the boundary conditions (1.2) and (1.3) is self-adjoint and bounded below in the space $L^2[0, 1]$ and that it has a discrete spectrum consisting of simple real eigenvalues. We use $\sigma(L)$ to denote this spectrum; thus we can write

$$\sigma(L) = \{\lambda_j : j \in \mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}\}.$$

For problems (1.1)-(1.3) there are three main sets of spectral data which may be given to determine a unique triple (q, h_0, h_1) , and each of which includes the spectrum $\sigma(L)$.

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If q is symmetric about $x = 1/2$ and $h_0 = -h_1$, then $\sigma(L)$ alone is sufficient to recover q uniquely (see [17, 19]). However, in general, an additional infinite sequence of spectral data must be given for the unique determination of (q, h_0, h_1) . Three different possibilities for these data include:

- (i) a second set of eigenvalues $\sigma(\tilde{L})$, where (1.1) and (1.2) remain the same but where in (1.3) h_1 is replaced by some $\tilde{h}_1 \neq h_1$;
- (ii) the set of norming constants $\alpha_n = \|v_n\|^2/|v_n(1)|^2$, $n \in \mathbb{N}_0$, when $h_1 \in \mathbb{R}$ or $\alpha_n = \|v_n\|^2/|v'_n(1)|^2$, $n \in \mathbb{N}_0$, when $h_1 = \infty$, where $v_n = v(x, \lambda_n)$ is the eigenfunction of problems (1.1)-(1.3) corresponding to the eigenvalue λ_n and $\|v_n\|$ is the L^2 -norm of v_n ;
- (iii) the set of ratios $\kappa_n = v_n(0)/v_n(1)$ when $h_1 \in \mathbb{R}$ or $\kappa_n = v'_n(0)/v'_n(1)$ when $h_1 = \infty$.

Note that in some literature the ratios κ_n may be written as the form of $\ln |v_n(0)/v_n(1)|$ if $h_1 \in \mathbb{R}$ or $\ln |v'_n(0)/v'_n(1)|$ if $h_1 = \infty$, which are called the *terminal velocities* or *norming constants*; see [6, 7, 8, 21, 22, 32]. Indeed, based on the following relation between κ_n and α_n ,

$$(1.4) \quad \kappa_n \alpha_n = \omega'(\lambda_n),$$

where $\omega(\cdot)$ is the characteristic function of L defined by (3.6) below (see [12], p. 18; [26]), the pair of sequences $\Gamma_1 := \{\lambda_n, \alpha_n; n \in \mathbb{N}_0\}$ is equivalent to the pair of sequences $\Gamma_2 := \{\lambda_n, \kappa_n; n \in \mathbb{N}_0\}$. It is also known [26] that knowing the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}_0}$ and the norming constants $\{\alpha_n\}_{n \in \mathbb{N}_0}$ is equivalent to knowing a singular measure defined by the spectral function for problems (1.1)-(1.3). Moreover, the norming constants $\{\alpha_n\}_{n \in \mathbb{N}_0}$ can be constructed from the two sequences of eigenvalues, $\Gamma_3 := \{\lambda_n, \tilde{\lambda}_n\}_{n \in \mathbb{N}_0}$.

The uniqueness problem of determining the potential q in terms of one of the above-mentioned three sets of spectral data, Γ_j ($j = 1, 2, 3$), was first investigated by Borg [4, 5], Levitan [25, 26], and Marchenko [29, 30]. Further investigation has been carried out by Hald [17, 18], Hochstadt [19], Trubowitz and his co-authors in a series of papers [8, 21, 22, 32], and others. A comprehensive review for the inverse problem in these cases is presented by McLaughlin [31].

The present paper will mainly investigate the uniqueness problem of the determination of the potential q under the circumstances where only partial information of q , of the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}_0}$, and of the norming constants $\{\kappa_n\}_{n \in \mathbb{N}_0}$ is available.

In 1978, Hochstadt and Lieberman [20] proved a remarkable result which says that mixed given data (e.g., given one spectrum and q known on $[0, 1/2]$) can determine q uniquely on $[0, 1]$. More precisely, if $q_1 = q_2$ on $[0, 1/2]$ and if $\sigma(L_1(q_1; h_0, h_1)) = \sigma(L_2(q_2; h_0, h_1))$, then $q_1 = q_2$ on $[0, 1]$. Later, refinements by Hald [17] and Suzuki [34] of Hochstadt and Lieberman's theorem showed that the boundary conditions at $x = 1$ for L_1 and L_2 need not be assumed a priori to be the same and that if q is continuous, then one only needs $\lambda_{1,n} = \lambda_{2,m(n)}$ for all values of n but one, that is, one eigenvalue can be missing. However, this is no longer true if the boundary condition at $x = 1$ is different and q is discontinuous for all values of n but one (see [16]). The same boundary condition for L_1 and L_2 at $x = 0$, however, is crucial for Hochstadt and Lieberman's theorem to hold (see [33]).

Several important generalizations of Hochstadt and Lieberman's theorem are given by Gesztesy and Simon [16] who consider the case where the $L^1[0, 1]$ potential q is known on a larger interval $[0, a]$ with $a \in [1/2, 1)$ and assume that the set of

common eigenvalues of L_1 and L_2 is sufficiently large. Another result in [16] is to assume that the potential q belongs to C^{2k} for some $k \in \mathbf{N}_0$ near $1/2$ so that C^{2k} -smoothness can replace the knowledge of some $(k+1)$ eigenvalues, that is, $(k+1)$ eigenvalues can be missing. In [10], del Rio, Gesztesy and Simon further study the case where a can be any number in the interval $(0, 1)$. In other words, they consider the problem of uniquely determining the potential q if q is only known on part of the interval $(0, 1)$ (which may be a subinterval of $(0, \frac{1}{2})$). In such a case, since less information is given on q , to uniquely determine the potential q on the entire interval $[0, 1]$, one naturally needs more information on the spectra $\sigma(L(q, h_0, h_1))$ for several different values of h_0 . For instance, q can be completely determined if either of the following two conditions is satisfied [10]:

- (a) given the spectrum $\sigma(L(q, h_0, h_1))$, half of the spectrum $\sigma(L(q, h'_0, h_1))$ for $h_0 \neq h'_0$, and q on $[0, 1/4]$,
- (b) given two-thirds of the spectra (q, h_0, h_1) for three different values of h_0 .

These results have been generalized and improved in a variety of ways; see [2, 3, 23, 24].

The purpose of this paper is twofold. First we provide some new results, analogous to the theorems of Gesztesy and Simon [15], [16], on the unique determination problem of the potential q and h_1 for the Sturm-Liouville problem (problems (1.1)-(1.3)) with only partial information available on q and partial spectral data consisting of one full spectrum and a subsequence of norming constants, or a subset of the pairs of eigenvalues and the corresponding norming constants. Furthermore, if the potential q is C^{2k-1} -smooth near the endpoint $x = 0$, then the potential q and h_1 can be determined uniquely from the values of $q^{(j)}(0)$, for $j = 0, 1, \dots, 2k-1$, one full spectrum and the norming constants except for $(k+1)$, that is, $(k+1)$ norming constants can be missing. In particular, if $q \in L^1[0, 1]$, then one norming constant can be missing.

The above results enlighten us to realize that, for the question of uniqueness for the Sturm-Liouville problem, norming constants play an equal role as eigenvalues. In other words, the number of norming constants but one is, in a sense, equivalent to the number of eigenvalues, which has a connection with the interesting open question raised by Gesztesy and Simon [16, p. 2775]: Can one replace information on the missing eigenvalue by knowledge of the boundary condition h_1 ? (This question was recently solved in the affirmative [35] in the case of $q \in L^1[0, 1]$.)

It is therefore the second purpose of this paper to show that if the boundary conditions at the endpoints $x = 0$ and 1 are fixed and if q is C^{2k-1} near $x = 0$, then the values of q on the entire interval $[0, 1]$ can be uniquely determined by the values of $q^{(j)}(0)$ for $j = 0, 1, \dots, 2k-1$, a set of eigenvalues except for one, and a set of the norming constants except for $(k+1)$, that is, one eigenvalue and $(k+1)$ norming constants can be missing. In particular, in the case of $q \in L^1[0, 1]$, the boundary condition h_1 can replace the knowledge of one eigenvalue. This shows that the answer to the above-mentioned open question of Gesztesy and Simon is affirmative in more general spectral data.

The main results presented in this paper are based on the uniqueness theorem of the Weyl m -function developed by Marchenko [29, 30]. This approach has been employed skillfully by del Rio, Gesztesy and Simon in a series of papers [10, 13, 14, 15, 16] to deal with inverse problems. The key technique relies on

the asymptotic expansion of an m -function. However, since the asymptotic expansion of an m -function is independent of boundary conditions, the technique of the Weyl m -functions fails to treat the case where boundary conditions are fixed. In Section 5 we shall provide an alternative technique which is similar to a method of Hochstadt [19] and Hald [18].

The paper is organized as follows. In the next section, we recall a uniqueness theorem due to Marchenko [29]. In Section 3, we discuss the case where one full spectrum and a subset of norming constants are known. In Section 4, we treat the case where a subset of pairs of the eigenvalues and of the norming constants are given. In Section 5, we deal with the case where the boundary conditions at the endpoints are the same. Finally, Section 6 is devoted to the case of the Dirichlet boundary condition.

We adopt the notation:

- $\mathbb{R}$: the set of real numbers;
- $\mathbb{N}$: the set of positive integers;
- $\mathbb{N}_0$: $\{0\} \cup \mathbb{N}$, i.e., the set of nonnegative integers;
- $\mathbb{C}$: the complex plane;
- $\mathbb{C}_+$: the open complex upper half-plane.

2. PRELIMINARIES

Throughout this paper, by the statement “ q on $[0, a]$, eigenvalues λ_n , and norming constants κ_n determine uniquely q and h_1 ” we mean that there are no two distinct potentials q_1 and q_2 on $[0, 1]$ with the two properties: (i) $q_1 = q_2$ a.e. on $[0, a]$, and (ii) λ_n and κ_n are common eigenvalues and norming constants for q_1 and q_2 . Unless explicitly stated otherwise, h_0 will be fixed and all potentials q , q_1 , and q_2 will be real-valued and in $L^1[0, 1]$ for the rest of this paper.

Moreover, to avoid too many case distinctions in the proofs below, we shall assume $h_0, h_1 \in \mathbb{R}$ in (1.1)-(1.3) throughout the main body of this paper. In particular, for $h_0, h_1 \in \mathbb{R}$ we index the eigenvalues λ_n of the operator L by $n \in \mathbb{N}_0$. The case of Dirichlet boundary conditions, where $h_0 = \infty$ and/or $h_1 = \infty$, will be dealt with in Section 6.

For a real-valued potential $q \in L^1[0, 1]$, consider the initial-value problem

$$(2.1) \quad -u'' + qu = zu$$

on $[0, 1]$ with initial conditions

$$(2.2) \quad u_-(0) = 1, \quad u'_-(0) = h_0,$$

$$(2.3) \quad u_+(1) = 1, \quad u'_+(1) = h_1.$$

Let $u_- := u_-(x, z)$ and $u_+ := u_+(x, z)$ be the solutions of problems (2.1)-(2.2) and problems (2.1) and (2.3), respectively. If $z = \lambda_j \in \sigma(L)$, where the operator L is defined by problems (1.1)-(1.3), then both $u_-(x, \lambda_j) =: u_{-,j}$ and $u_+(x, \lambda_j) =: u_{+,j}$ are eigenfunctions, corresponding to the eigenvalue λ_j , of the operator L , and there holds the relation

$$(2.4) \quad u_{+,j} = \kappa_j u_{-,j},$$

where $\kappa_j = u_{+,j}(0) = u_{-,j}^{-1}(1)$ is the norming constant (or ratio) corresponding to the eigenvalue λ_j ; hence $\kappa_j \neq 0, \infty$.

We next formulate the main uniqueness theorem in the literature, due to Marchenko [29]. For the solution $u_+(x, z)$ of (2.1), the Weyl m_+ -function is defined by

$$(2.5) \quad m_+(a, z) = \frac{u'_+(a, z)}{u_+(a, z)}$$

for any $a \in [0, 1]$. Similarly, the Weyl m_- -function is defined by

$$(2.6) \quad m_-(b, z) = -\frac{u'_-(b, z)}{u_-(b, z)}$$

for any $b \in (0, 1]$. The differing signs in (2.5) and (2.6) are picked so that both m_+ and m_- are Herglotz functions, that is, $m_{\pm} : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ are analytic.

Marchenko's [29] fundamental uniqueness theorem of inverse spectral theory then reads as follows:

Theorem 2.1. $m_+(a, z)$ (resp., $m_-(b, z)$) uniquely determines h_1 (resp., h_0) as well as q (a.e.) on $[a, 1]$ (resp., $[0, b]$).

Consider a problem with boundary condition (1.3) at $x = 1$. We need to know the high-energy asymptotic behavior of the m_+ -function with $x \in [0, 1]$. It is known [1, 11] that, under the general hypothesis $q \in L^1[0, 1]$,

$$(2.7) \quad m_+(x, z) = i\sqrt{z} + o(1), \quad \frac{1}{m_+(x, z)} = \frac{-i}{\sqrt{z}} + o\left(\frac{1}{z}\right)$$

uniformly in $x \in [0, 1 - \delta]$ for $\delta > 0$ as $|z| \rightarrow \infty$ in any sector $\varepsilon < \text{Arg}(z) < \pi - \varepsilon$ for $\varepsilon > 0$, where $\sqrt{z}$ is the square root branch with $\text{Im}(\sqrt{z}) \geq 0$. It is also known [9] that if q is C^n near $x_0 \in [0, 1]$, for some $n \in \mathbb{N}_0$, then $m_+(x_0, z)$ and $m_+(x_0, z)^{-1}$ have asymptotic expansions of the form

$$(2.8) \quad m_+(x_0, z) = i\sqrt{z} + \sum_{l=1}^{n+1} c_l(x_0) \frac{1}{z^{(l+1)/2}} + o\left(\frac{1}{z^{(n+1)/2}}\right),$$

$$(2.9) \quad \frac{1}{m_+(x_0, z)} = \frac{-i}{\sqrt{z}} + \sum_{l=1}^{n+1} d_l(x_0) \frac{1}{z^{(l+2)/2}} + o\left(\frac{1}{z^{(n+3)/2}}\right)$$

as $|z| \rightarrow \infty$ in any sector $\varepsilon < \text{Arg}(z) < \pi - \varepsilon$ for $\varepsilon > 0$. Here $c_l(x_0)$ and $d_l(x_0)$ are the universal functions of $q(x_0), q'(x_0), \dots, q^{(l-2)}(x_0)$ and can be computed recursively as follows. Consider the Riccati-type equations for $m_+(x, z)$ and $m_+(x, z)^{-1}$,

$$(2.10) \quad m'_+(x, z) = -m_+(x, z)^2 q(x) - z,$$

$$(2.11) \quad (m_+(x, z)^{-1})' = -m_+(x, z)^{-2} (q(x) - z) + 1.$$

Inserting the asymptotic expansions (2.8) and (2.9) into (2.10) and (2.11) yields the recursive relations

(2.12)

$$\begin{aligned} c_0(x) &= 1, \quad c_1(x) = 0, \quad c_2(x) = -\frac{1}{2}q(x), \\ c_j(x) &= \frac{i}{2}c'_{j-1}(x) - \frac{1}{2}\sum_{l=1}^{j-1} c_l(x)c_{j-l}(x), \quad j \geq 3, \end{aligned}$$

(2.13)

$$\begin{aligned} d_0(x) &= 1, \quad d_1(x) = 0, \quad d_2(x) = -\frac{1}{2}q(x), \\ d_j(x) &= \frac{i}{2}d'_{j-1}(x) - \frac{1}{2}q(x)\sum_{l=1}^{j-1} d_l(x)d_{j-l-2}(x) - \frac{1}{2}\sum_{l=1}^{j-1} d_l(x)d_{j-l}(x), \quad j \geq 3. \end{aligned}$$

Finally, we introduce the Green's formula (see [36]) associating with (2.1) for a real-valued $q \in L^1[0, 1]$. Let

$$D_{\max} := \{y \in L^2[0, 1] : y, y' \in AC[0, 1], -y'' + qy \in L^2[0, 1]\},$$

where $AC[0, 1]$ is the space of absolutely continuous functions on $[0, 1]$. Then the Green's formula is stated as follows:

$$(2.14) \quad \int_0^1 (yL(z) - zL(y))dt = [y, z](1) - [y, z](0),$$

where $y, z \in D_{\max}$ and $[y, z](t) := y(t)z'(t) - y'(t)z(t)$ for $t \in [0, 1]$. It is obvious that if both u_1 and u_2 are solutions of (2.1), then $[u_1, u_2](x) (= [u_1, u_2](0))$ is a constant for all $x \in [0, 1]$. Here, we mention the precise definition of the Sturm-Liouville operator L in $L^2[0, 1] : Ly = -y'' + qy$, where

$$(2.15) \quad y \in D(L) := \{y \in D_{\max} : y \text{ satisfies the boundary conditions (1.2)-(1.3)}\}.$$

3. THE CASE OF ONE FULL SPECTRUM

In this section, we treat the case where the potential q and the coefficient h_1 can be uniquely determined if we are given a set of eigenvalues λ_j and a subset of the corresponding norming constants κ_j . Our results show that if the potential q satisfies a local smoothness condition at $x = 0$, then we require less knowledge on the norming constants κ_j . We shall first formulate some asymptotic expansions of solutions of (2.1), which will be used later on to establish our principal results of the section.

Let q be given on $[0, a]$ with some $a \in [0, 1]$. Let q_1 and q_2 be two candidates for q extended to all of $[0, 1]$. Let $u_{1,+}(x, z)$ and $u_{2,+}(x, z)$ be solutions of (2.1) corresponding to q_1 and q_2 , respectively, where $u_{j,+}(x, z)$ satisfies the initial conditions

$$(3.1) \quad u_{j,+}(1, z) = 1, \quad u'_{j,+}(1, z) = h_j, \quad j = 1, 2,$$

with $h_1, h_2 \in \mathbb{R}$. It is well known [32, p. 10; [12], [28]] that for each $x \in [0, 1]$, $u_{j,+}(x, z)$ and $u'_{j,+}(x, z)$ are entire functions of z and satisfy the following asymptotic expansions:

$$(3.2) \quad u_{j,+}(x, z) = \cos(\sqrt{z}(1-x)) + O\left(e^{\operatorname{Im}(\sqrt{z})(1-x)}\sqrt{z}\right),$$

$$(3.3) \quad u'_{j,+}(x, z) = \sqrt{z} \sin(\sqrt{z}(1-x)) + O\left(e^{\operatorname{Im}(\sqrt{z})(1-x)}\right)$$

as $|z| \rightarrow \infty$ for all $x \in [0, 1]$. The zeros of $u_{j,+}(x, \cdot)$ and of $u'_{j,+}(x, \cdot)$ are all real for any $x \in [0, 1]$, and they are uniformly bounded below. In particular, we infer that if $z = iy$ with $y \in \mathbb{R}$, then (3.2) and (3.3) imply

$$(3.4) \quad |u_{j,+}(a, iy)| = \frac{1}{2} e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}} (1 + o(1)),$$

$$(3.5) \quad |u'_{j,+}(a, iy)| = \frac{1}{2} |y|^{1/2} e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}} (1 + o(1))$$

as $y \rightarrow \infty$. For $j = 1, 2$, let

$$(3.6) \quad \omega_j(z) = u'_{j,+}(0, z) - h_0 u_{j,+}(0, z),$$

which is called the *characteristic* function of the operator L_j . Then $\sigma(L_j) = \{\lambda_{j,n}\}_{n=0}^\infty$ are precisely the zeros of $\omega_j(z)$. By (3.4) and (3.5) we also infer that

$$(3.7) \quad |\omega_j(iy)| = \frac{1}{2} |y|^{1/2} e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}} (1 + o(1)) \quad \text{as } y \rightarrow \infty$$

and, for sufficiently large n ,

$$(3.8) \quad \inf_{\theta \in [0, 2\pi]} |\omega_j((\pi(n + \frac{1}{2}))^2 e^{i\theta})| \geq \pi n + O(1).$$

Since the zeros of $u_{j,+}(a, \cdot)$ and of $u'_{j,+}(a, \cdot)$ are all real and uniformly bounded below, by adding (if necessary) a sufficiently large constant to q_1 and q_2 , we may assume that all zeros of $u_{j,+}(a, \cdot)$, $u'_{j,+}(a, \cdot)$ and $\omega_j(\cdot)$ are in $[1, \infty)$. In this case, all these six functions are of m -type (see [16, p. 2781]). Therefore, $u_{j,+}(a, \cdot)$, $u'_{j,+}(a, \cdot)$ and $\omega_j(\cdot)$ are bounded by $C_1 \exp(C_2 |z|^{1/2})$ for some constant C_1 , $C_2 > 0$ and are of the form (see [16])

$$(3.9) \quad c \prod_{n=0}^{\infty} \left(1 - \frac{z}{x_n}\right)$$

for suitable $\{x_n\}_{n=0}^\infty \subset [1, \infty)$.

For any $\alpha = \{\alpha_j\}_{j \in \mathbb{N}_0} \subset \mathbb{C}$, set

$$(3.10) \quad n_\alpha(t) = \#\{j \in \mathbb{N}_0 : |\alpha_j| \leq t\} \quad \text{for each } t \geq 0.$$

Here is one of our main results of this section.

Theorem 3.1. *Let $\Lambda_0 = \{j_1, j_2, \dots, j_{k+1}\} \subset \mathbb{N}_0$. Suppose that $q_1, q_2 \in C^{2k-1}[0, \varepsilon]$ for some $k \in \mathbb{N}_0$ and for some $\varepsilon > 0$. Assume that*

$$(3.11) \quad q_1^{(j)}(0) = q_2^{(j)}(0), \quad j = 0, 1, \dots, 2k-1,$$

$$(3.12) \quad \sigma(L_1) = \sigma(L_2) \quad \text{and} \quad \kappa_{1,j} = \kappa_{2,j}, \quad j \in \mathbb{N}_0 \setminus \Lambda_0.$$

Then $h_1 = h_2$ and $q_1 = q_2$ a.e. on $[0, 1]$.

Remark 3.2. Under the assumption that $\sigma(L_1) = \sigma(L_2)$, it is known [12, p. 8] that $\omega_1(z) = \omega_2(z)$; see also (3.13) below. By the fact that $\kappa_{l,j}\alpha_{l,j} = \omega'_l(\lambda_j)$ for $l = 1, 2$ and $j \in \mathbb{N}_0$, we easily find that the result of Theorem 3.1 remains valid if the condition $\kappa_{1,j} = \kappa_{2,j}$ is replaced with the condition $\alpha_{1,j} = \alpha_{2,j}$, for all $j \in \mathbb{N}_0 \setminus \Lambda_0$.

Comparing the result of Theorem 3.1 with that of [16, Theorem 1.2], one may see that the $(k+1)$ missed eigenvalues (as $q \in C^{2k}[0, \varepsilon)$) replaced by the $(k+1)$ missed norming constants (as $q \in C^{2k-1}[0, \varepsilon)$) can also uniquely determine q and h_1 . In particular, if $q \in L^1[0, 1]$, by Theorem 3.1, we have

Corollary 3.3. *Let $q \in L^1[0, 1]$. Then one full spectral data $\{\lambda_j, \kappa_j\}_{j \in \mathbb{N}_0}$ except for one norming constant determines the potential q and h_1 uniquely.*

Proof of Theorem 3.1. Let $\sigma(L_1) = \sigma(L_2) =: \{\lambda_j\}_{j \in \mathbb{N}_0}$ be the set of all the common eigenvalues of L_1 and L_2 . Then $\{\lambda_j\}_{j \in \mathbb{N}_0}$ are precisely the zeros of $\omega_l(z)$ for $l = 1, 2$, and by [12, p. 8] we have

$$(3.13) \quad \omega_1(z) = (\lambda_0 - z) \prod_{j=1}^{\infty} \frac{\lambda_j - z}{j^2} = \omega_2(z).$$

Let $v_{2,+}(x, z)$ be another solution of the equation $-u'' + q_2u = zu$ subject to the initial condition

$$(3.14) \quad v_{2,+}(1, z) = 1, \quad v'_{2,+}(1, z) = h$$

with $h \in \mathbb{R}$ and $h \neq h_1, h_2$. It is easy to verify that $v_{2,+}$ satisfies the boundary condition at $x = 1$,

$$v'_{2,+}(1, z) - hv_{2,+}(1, z) = 0.$$

Let

$$(3.15) \quad \tilde{\omega}(z) := v'_{2,+}(0, z) - h_0v_{2,+}(0, z).$$

Consider the operator $L(q_2, h_0, h)$ in $L^2[0, 1]$ with boundary conditions (1.2) and (1.3), where h_1 is replaced by h . Then $L(q_2, h_0, h)$ is self-adjoint and its spectrum consists of eigenvalues, denoted by $\{\mu_j\}_{j \in \mathbb{N}_0}$, which are real, simple and bounded from below. It is also known [36, Ch. 4] that the spectrum of $L(q_2, h_0, h)$ and $L(q_2, h_0, h_2)$ are interlaced and disjoint, since $h \neq h_2$.

Now we consider the function $F(z)$ defined by

$$(3.16) \quad F(z) = (u_{1,+}(0, z) - u_{2,+}(0, z))\tilde{\omega}(z).$$

It follows from (3.15) and (3.14) that $F(\mu_j) = 0$ for all $j \in \mathbb{N}_0$. Furthermore, it is known (see (2.4)) that $u_{l,+}(0, \lambda_j) = \kappa_{l,j}$ for $l = 1, 2$ and thus, $F(\lambda_j) = 0$ if $\lambda_j \in \mathbb{N}_0 \setminus \Lambda_0$. Noting that $\omega_1(z) = \omega_2(z)$ and by (3.15) and (2.5), we get

$$(3.17) \quad \begin{aligned} F(z) &= [(u_{1,+}(0, z)(v'_{2,+}(0, z) - h_0v_{2,+}(0, z)) - v_{2,+}(0, z)\omega_1(z))] \\ &\quad - [u_{2,+}(0, z)(v'_{2,+}(0, z) - h_0v_{2,+}(0, z)) - v_{2,+}(0, z)\omega_2(z)] \\ &= [u_{1,+}, v_{2,+}](0, z) - [u_{2,+}, v_{2,+}](0, z) \end{aligned}$$

$$(3.18) \quad \begin{aligned} &= u'_{1,+}(0, z)v'_{2,+}(0, z) (m_{+,v_{2,+}}(0, z)^{-1} - m_{+,u_{1,+}}(0, z)^{-1}) \\ &\quad - u'_{2,+}(0, z)v'_{2,+}(0, z) (m_{+,v_{2,+}}(0, z)^{-1} - m_{+,u_{2,+}}(0, z)^{-1}), \end{aligned}$$

where $m_{+,v_{2,+}}(0, z) = v'_{2,+}(0, z)/v_{2,+}(0, z)$ is the Weyl m -function associated with the solution $v_{2,+}$ (see (2.5)). Define

$$(3.19) \quad H(z) = \frac{F(z)}{G(z)} \prod_{l=1}^{k+1} (z - \lambda_{j_l}),$$

where $j_l \in \Lambda_0$ and $G(z) = \omega_2(z)\tilde{\omega}(z)$. By the above discussions, the cross ratio $F(z) \prod_{l=1}^{k+1} (z - \lambda_{j_l})$ vanishes at each point where G vanishes. Also, G necessarily has simple zeros, since $L(q_2, h_0, h_2)$ and $L(q_2, h_0, h)$ have simple spectrum, respectively, and their spectrum are interlaced and disjoint. Thus, H is an entire function. In addition, from (3.8) and the fact that the functions $u_{j,+}(0, z)$ and $v_{2,+}(0, z)$ both are of m -type, we conclude that $H(z)$ satisfies

$$|H(z)| \leq C_1 e^{C_2 |z|^{1/2}}.$$

As a matter of fact, it follows from (3.8) that the above inequality holds whenever $|z| = (\pi(n+1/2))^2$ for n sufficiently large; it then extends to all z by the maximum modulus principle. Furthermore, since $q_1^{(j)}(0) = q_2^{(j)}(0)$ for $j = 0, 1, \dots, 2k-1$, by (2.9) we infer that

$$(3.20) \quad \begin{aligned} |m_{+,v_{2,+}}(0, iy)^{-1} - m_{+,u_{1,+}}(0, iy)^{-1}| &= o(|y|^{-(k+1)}), \\ |m_{+,v_{2,+}}(0, iy)^{-1} - m_{+,u_{2,+}}(0, iy)^{-1}| &= o(|y|^{-(k+1)}). \end{aligned}$$

This together with (3.5), (3.18) and (3.7) shows that

$$(3.21) \quad \begin{aligned} |H(iy)| &\leq \left| \frac{(u'_{1,+}v'_{2,+})(0, iy) (m_{+,v_{2,+}}(0, iy)^{-1} - m_{+,u_{1,+}}(0, iy)^{-1})}{G(iy)} \prod_{l=1}^{k+1} (iy - \lambda_{j_l}) \right| \\ &\quad + \left| \frac{(u'_{2,+}v'_{2,+})(0, iy) (m_{+,v_{2,+}}(0, iy)^{-1} - m_{+,u_{2,+}}(0, iy)^{-1})}{G(iy)} \prod_{l=1}^{k+1} (iy - \lambda_{j_l}) \right| \\ &= \frac{|y| e^{\operatorname{Im}(\sqrt{i})2|y|^{1/2}} (1 + o(1)) o(|y|^{-(k+1)})}{|y| e^{\operatorname{Im}(\sqrt{i})2|y|^{1/2}} (1 + o(1))} O(|y|^{(k+1)}) \\ &= o(1) \quad (y \text{ (real)} \rightarrow \infty). \end{aligned}$$

It turns out that $|H(iy)| \rightarrow 0$ as y (real) $\rightarrow \infty$. By [16, Proposition B.6] and (3.21), we obtain $H \equiv 0$. We can multiply H by $(\omega\tilde{\omega})(z)/\prod_{l=1}^{k+1} (z - \lambda_{j_l})$, which has isolated zeros and poles, to conclude that $u_{1,+}(0, z) = u_{2,+}(0, z)$ for all $z \in \mathbb{C}$. This together with (3.13) and (3.6) yields $u'_{1,+}(0, z) = u'_{2,+}(0, z)$, and therefore $m_{+,u_{2,+}}(z) = m_{+,u_{1,+}}(z)$. By Theorem 2.1, $q_1 = q_2$ a.e. on $[0, 1]$, as well as $h_1 = h_2$. The proof is therefore complete. $\square$

The following theorem treats the case where partial information is given on the set of the norming constants when q is known a priori on $[0, a]$ with $a \in (0, 1/2)$.

Theorem 3.4. *Let $a \in (0, 1/2)$ and $q_1, q_2 \in C^n(a - \varepsilon, a + \varepsilon)$ for some $n \in \mathbb{N}_0$ and for some $\varepsilon > 0$. Consider a subset $S \subseteq \sigma(L_2)$ such that $\kappa_{1,j} = \kappa_{2,j}$ for $\lambda_j \in S$. Assume that $q_1 = q_2$ a.e. on $[0, a]$, $\sigma(L_1) = \sigma(L_2)$ and*

$$(3.22) \quad n_S(t) \geq (1 - 2a)n_{\sigma(L_2)}(t) + a - \frac{n+3}{2}$$

for all sufficiently large $t \in \mathbb{R}$. Then $h_1 = h_2$ and $q_1 = q_2$ a.e. on $[0, 1]$.

Remark 3.5. If both q_1 and q_2 are assumed to be in $L^1[0, 1]$, then, instead of condition (3.22), we only need the following condition:

$$(3.23) \quad n_S(t) \geq (1 - 2a)n_{\sigma(L_1)}(t) + a - 1.$$

Remark 3.6. As a typical example, knowing one full spectrum and a half set of the norming constants and also knowing q on $[0, \frac{1}{4}]$ uniquely determines q on all of $[0, 1]$ and h_1 . To the best of our knowledge, Theorem 3.4 solves a new type of inverse spectral problem involving fractions of the set of norming constants as in (3.22), parallel to Gesztesy and Simon's result [16, Theorem 1.3] on inverse spectral problems involving fractions of the set of eigenvalues.

Proof of Theorem 3.4. Consider the function F defined by

$$(3.24) \quad F(z) = (u_{1,+}(0, z) - u_{2,+}(0, z))\tilde{\omega}(z),$$

where $\tilde{\omega}(z)$ is defined by (3.15). Then by the hypothesis that $\sigma(L_1) = \sigma(L_2) =: \{\lambda_j\}_{j \in \mathbb{N}_0}$ and the argument of the proof of Theorem 3.1, we get

$$F(\lambda_j) = 0 \text{ if } \lambda_j \in S, \quad F(\mu_j) = 0 \text{ if } \mu_j \in \sigma(L(q_2, h_0, h)).$$

Let $a \in [0, 1/2)$. Since $q_1 = q_2$ on $[0, a]$ we infer by the Green's formula (2.14) that

$$\begin{aligned} [u_{2,+}, v_{2,+}](0, z) &= [u_{2,+}, v_{2,+}](a, z), \\ [u_{1,+}, v_{2,+}](0, z) &= [u_{1,+}, v_{2,+}](a, z) - \int_0^a (q_1 - q_2)(t)(u_{1,+}v_{2,+})(t, z)dt \\ &= [u_{1,+}, v_{2,+}](a, z). \end{aligned}$$

This, combined with (3.17) and (2.5), shows that

$$\begin{aligned} F(z) &= [u_{1,+}, v_{2,+}](a, z) - [u_{2,+}, v_{2,+}](a, z) \\ &= u'_{1,+}(a, z)v'_{2,+}(a, z)(m_{+,v_{2,+}}(a, z)^{-1} - m_{+,u_{1,+}}(a, z)^{-1}) \\ (3.25) \quad &\quad - u'_{2,+}(a, z)v'_{2,+}(a, z)(m_{+,v_{2,+}}(a, z)^{-1} - m_{+,u_{2,+}}(a, z)^{-1}). \end{aligned}$$

Define

$$(3.26) \quad G_S(z) = \prod_{\lambda_j \in S} \left(1 - \frac{z}{\lambda_j}\right), \quad G_{\sigma(L_2)}(z) = \prod_{\lambda_j \in \sigma(L_2)} \left(1 - \frac{z}{\lambda_j}\right),$$

where $S \subset \sigma(L_2)$. By the hypothesis (3.22) on S we have, by integration by parts,

$$\begin{aligned} \ln |G_S(iy)| &= \sum_{\lambda_j \in S} \frac{1}{2} \ln \left(1 + \frac{y^2}{\lambda_j^2}\right) \\ &= \frac{1}{2} \int_0^\infty \ln \left(1 + \frac{y^2}{t^2}\right) dn_S(t) \\ &= \int_0^\infty \frac{y^2}{t^3 + ty^2} n_S(t) dt \quad (\text{as } n_S(0) = 0) \\ (3.27) \quad &= \int_1^\infty \frac{y^2}{t^3 + ty^2} n_S(t) dt \quad (n_S(t) = 0 \text{ if } t \in [0, 1)). \end{aligned}$$

Furthermore, by hypothesis (3.22) of the theorem, there are constants $t_0 \geq 1$ and $C \geq 0$ such that

$$n_S(t) \geq \begin{cases} (1 - 2a)n_{\sigma(L_2)}(t) + a - (n + 3)/2, & \text{if } t > t_0, \\ (1 - 2a)n_{\sigma(L_2)}(t) - C, & \text{if } t \leq t_0. \end{cases}$$

Hence, by (3.27) and (3.22), and noting the relation

$$\frac{y^2}{t^3 + ty^2} = -\frac{d}{dt} \left(\frac{1}{2} \ln(1 + \frac{y^2}{t^2}) \right),$$

we deduce that

$$\begin{aligned} \ln |G_S(iy)| &= \int_1^{t_0} \frac{y^2}{t^3 + ty^2} n_S(t) dt + \int_{t_0}^{\infty} \frac{y^2}{t^3 + ty^2} n_S(t) dt \\ &\geq (1 - 2a) \int_1^{\infty} \frac{y^2}{t^3 + ty^2} n_{\sigma(L_2)}(t) dt \\ &\quad + (a - \frac{n+3}{2}) \int_1^{\infty} \frac{y^2}{t^3 + ty^2} dt + C_0 \\ (3.28) \quad &= (1 - 2a) \ln |G_{\sigma(L_2)}(iy)| + \frac{1}{2} (a - (n+3)/2) \ln(1 + y^2) + C_0, \end{aligned}$$

where $C_0 = |a - (n+3)/2| + C \ln(t_0)$.

Because $\sigma(L_2)$ is the full set of the eigenvalues of the self-adjoint operator L_2 on $[0, 1]$, we know that asymptotically

$$|G_{\sigma(L_2)}(iy)| = \frac{1}{2} |y|^{1/2} e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}} (1 + o(1)) \quad \text{as } y \text{ (real)} \rightarrow \infty.$$

It thus turns out from (3.28) that there exists a positive constant C_1 such that

$$(3.29) \quad |G_S(iy)| \geq C_1 |y|^{-(n+2)/2} e^{\operatorname{Im}(\sqrt{i})(1-2a)|y|^{1/2}}$$

for $|y|$ sufficiently large. Let

$$(3.30) \quad H(z) = \frac{F(z)}{G(z)},$$

where $G(z) = G_S(z) \tilde{\omega}(z)$. Recall that

$$|u'_{j,+}(a, iy)| = \frac{1}{2} e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}} (1 + o(1)), \quad j = 1, 2.$$

Note that $v'_{2,+}(a, iy)$ has the same asymptotics as $u'_{j,+}(a, iy)$. Thus, by (3.25), (3.20) and (3.29), we have

$$\begin{aligned} |H(iy)| &\leq \left| \frac{u'_{1,+}(a, iy) v'_{2,+}(a, iy) (m_{+,v_{2,+}}(a, iy)^{-1} - m_{+,u_{1,+}}(a, iy)^{-1})}{G(z)} \right| \\ &\quad + \left| \frac{u'_{2,+}(a, iy) v'_{2,+}(a, iy) (m_{+,v_{2,+}}(a, iy)^{-1} - m_{+,u_{2,+}}(a, iy)^{-1})}{G(z)} \right| \\ &\leq \frac{|y| e^{\operatorname{Im}(\sqrt{i})2(1-a)|y|^{1/2}} (1 + o(1)) o(|y|^{-(n+3)/2})}{|y|^{1/2} e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}} (1 + o(1)) |y|^{-(n+2)/2} e^{\operatorname{Im}(\sqrt{i})(1-2a)|y|^{1/2}}} \\ (3.31) \quad &= o(1). \end{aligned}$$

This yields $H(z) = 0$ and therefore $F(z) = 0$ for all $z \in \mathbb{C}$ by the argument of the proof of Theorem 3.1. This together with (3.24) and (3.13) yields $m_{+,u_{2,+}}(z) = m_{+,u_{1,+}}(z)$. By Theorem 2.1, $q_1 = q_2$ a.e. on $[0, 1]$ and $h_1 = h_2$. The proof is complete. $\square$

Remark 3.7. By checking the proofs of Theorems 3.1 and 3.4, we easily see that the results of Theorems 3.1 and 3.4 are independent of the boundary condition h at $x = 1$ of the solution $v_{2,+}(x, z)$, even if the boundary condition is the Dirichlet boundary condition (i.e., $h = \infty$).

4. PAIRS OF EIGENVALUES AND NORMING CONSTANTS

In this section, we study the case where we can use a subset $\{\lambda_j, \kappa_j : j \in \Lambda \subset \mathbb{N}_0\}$ of the (full) set $\{\lambda_j, \kappa_j : j \in \mathbb{N}_0\}$ of eigenvalues and the corresponding norming constants to uniquely determine q and h_1 . The method used here is similar to the method used in the last section.

Theorem 4.1. *Let $a \in [0, 1]$ and $q_1, q_2 \in L^1[0, 1]$. Consider a subset $S \subseteq \sigma(L_1) \cap \sigma(L_2)$ such that $\kappa_{1,j} = \kappa_{2,j}$ for $\lambda_j \in S$. Assume $q_1 = q_2$ a.e. on $[0, a]$ and*

$$(4.1) \quad n_S(t) \geq (1-a)n_{\sigma(L_2)}(t) - \frac{1}{2}(1-a)$$

for all sufficiently large $t \in \mathbb{R}$. Then $h_1 = h_2$ and $q_1 = q_2$ a.e. on $[0, 1]$.

Remark 4.2. Let $a = 1/2$ and let $K \geq 0$ be any fixed integer. A consequence of Theorem 4.1 is the following. To uniquely determine q on $[0, 1]$ and h_1 , we need half data from the set $\{\lambda_j, \kappa_j : 0 \leq j \leq K\}$ (the other half can be missing) plus either $\{\lambda_{2j}, \kappa_{2j} : 2j \geq K\}$ or $\{\lambda_{2j-1}, \kappa_{2j-1} : 2j-1 \geq K\}$. In particular, we have the following result.

Corollary 4.3. *Let $a = 1/2$. Then we need half of the (full) set $\{\lambda_j, \kappa_j : j \in \mathbb{N}_0\}$ of eigenvalues and the corresponding norming constants to uniquely determine q on $[0, 1]$ and h_1 . That is, the even (resp., odd) spectral data $\{\lambda_{2j}, \kappa_{2j}\}_{j \geq 0}$ (resp., $\{\lambda_{2j+1}, \kappa_{2j+1}\}_{j \geq 0}$) and q on $[0, 1/2]$ can uniquely determine h_1 and q on $[0, 1]$.*

Proof of Theorem 4.1. Let us consider the function

$$(4.2) \quad F_v(z) := (h_1 - h_2)v_2(1, z) - \int_0^1 (q_1 - q_2)(x)(u_{1,+}v_2)(x, z)dx,$$

where $v_2(x, z)$ is a solution of the equation

$$(4.3) \quad -u'' + q_2u = zu \quad \text{on } [0, 1].$$

Then by the Green's formula (2.14) and the initial conditions (3.1) of $u_{1,+}$ and $u_{2,+}$ we have

$$\begin{aligned} F_v(z) &= (h_1 - h_2)v_2(1, z) - \int_0^1 [(u_{1,+}' + zu_{1,+})v_2 - (v_2'' + zv_2)u_{1,+}](x, z)dx \\ &= (h_1 - h_2)v_2(1, z) + \int_0^1 \frac{d}{dx}([u_{1,+}, v_2](x, z))dx \\ &= (u_{1,+}' - u_{2,+}')(1, z)v_2(1, z) + [u_{1,+}, v_2](1, z) - [u_{1,+}, v_2](0, z) \\ &= [u_{2,+}, v_2](1, z) - [u_{1,+}, v_2](0, z) \\ &= [u_{1,+} - u_{2,+}, v_2](0, z) \\ &= \begin{vmatrix} u_{1,+}(0, z) - u_{2,+}(0, z) & v_2(0, z) \\ \omega_1(z) - \omega_2(z) & v_2'(0, z) - h_0v_2(0, z) \end{vmatrix}. \end{aligned}$$

Note that $u_{l,+}(0, \lambda_j) = \kappa_{l,j}$ for $l = 1, 2$. This shows that if $\lambda_j \in S$, then $\omega_1(\lambda_j) = \omega_2(\lambda_j) (= 0)$ and $u_{1,+}(0, \lambda_j) = u_{2,+}(0, \lambda_j)$, and therefore $F_v(\lambda_j) = 0$. Furthermore, since $q_1 = q_2$ on $[0, a]$, it follows from (4.2) and the Green's formula that

$$\begin{aligned} F_v(z) &= (u'_{1,+} - u'_{2,+})(1, z)v_2(1, z) + \int_a^1 \frac{d}{dx}([u_{1,+}, v_2](x, z))dx \\ &= [u_{2,+}, v_2](1, z) - [u_{1,+}, v_2](a, z) \\ (4.4) \quad &= [u_{1,+} - u_{2,+}, v_2](a, z). \end{aligned}$$

Let $v_D(x, z) := v_2$ be the solution of (4.3) with the initial conditions $v(a, z) = 0$ and $v'(a, z) = 1$. Then

$$(4.5) \quad F_{v_D}(z) = (u_{1,+} - u_{2,+})(a, z),$$

and by (3.2) we have

$$(4.6) \quad |(u_{1,+} - u_{2,+})(a, iy)| = O(|y|^{-1/2} e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}}).$$

Define

$$(4.7) \quad G_S(z) = \prod_{\lambda_j \in S} \left(1 - \frac{z}{\lambda_j}\right), \quad H_D(z) = \frac{F_{v_D}(z)}{G_S(z)},$$

where $S \subset \sigma(L_2)$. By the hypothesis (4.1) on S and the arguments of the proof of Theorem 3.4 (see (3.27) and (3.28)), we have

$$(4.8) \quad |G_S(iy)| \geq C_0 e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}},$$

and, in addition, from (4.6),

$$(4.9) \quad |H_D(iy)| \leq \frac{O(|y|^{-1/2} e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}})}{e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}}} = O(|y|^{-1/2}).$$

This yields $H_D(z) = 0$; hence $F_{v_D}(z) = (u_{1,+} - u_{2,+})(a, z) = 0$ for all $z \in \mathbb{C}$.

Let $v_N(x, z) := v_2$ be the solution of (4.3) with the initial conditions $v(a, z) = 1$ and $v'(a, z) = 0$. Then from (4.4) we have

$$(4.10) \quad F_{v_N}(z) = (u'_{1,+} - u'_{2,+})(a, z).$$

Define

$$(4.11) \quad H_N(z) = \frac{F_{v_N}(z)}{G_S(z)}.$$

By the hypothesis (4.1) on S and the fact that $u_{1,+}(a, z) = u_{2,+}(a, z)$, we have from (2.7) and (3.4) that

$$\begin{aligned} |H(iy)| &= \frac{|m_{+,1}(a, iy) - m_{+,2}(a, iy)|}{G_S(z)} |u_{1,+}(a, iy)| \\ &\leq \frac{o(1) e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}} (1 + o(1))}{e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}}} \\ (4.12) \quad &= o(1). \end{aligned}$$

This gives us that $H_N(z) = 0$ and $F_{v_N}(z) = (u'_{1,+} - u'_{2,+})(a, z) = 0$ for all $z \in \mathbb{C}$. Thus, we conclude that $m_{+,1}(a, z) = m_{+,2}(a, z)$. By Theorem 2.1, we have $h_1 = h_2$ and $q_1 = q_2$ a.e. on $[a, 1]$ and (hence on $[0, 1]$). The proof is thus complete. $\square$

5. THE CASE WHERE $h_1 = h_2$

In this section, we attack the case where $h_1 = h_2$. In this case we show that it is possible to miss from the spectral data not only one eigenvalue but also some norming constants when q is smooth near $x = 0$.

The technique which we use in this section is based on the method discussed by Hochstadt and Hald; see [19, Theorem 1] and [18, Theorem 1.1].

For simplicity, we shall denote by $u_{l,j}$ ($l = 1, 2$) the eigenfunctions of the Sturm-Liouville problems consisting of the equations

$$u'' = (q_l - \lambda_{l,j})u$$

with the boundary conditions (1.2) and (1.3), and the initial conditions $u(0) = 1$ and $u'(0) = h_0$. In terms of this notation, the norming constants $\kappa_{l,j} = u_{l,j}^{-1}(1)$.

Our main results of this section are the following two theorems.

Theorem 5.1. *Let $\Lambda_0 = \{j_1, j_2, \dots, j_{k+1}\} \subset \mathbb{N}_0$ and $\sigma(L_l) = \{\lambda_{l,j}\}_{j \in \mathbb{N}_0}$ for $l = 1, 2$. Suppose that $q_1, q_2 \in C^{2k-1}[0, \varepsilon]$ for some $k \in \mathbb{N}$ and for some $\varepsilon > 0$. Assume that $h_1 = h_2$,*

$$(5.1) \quad \begin{aligned} & q_1^{(j)}(0) = q_2^{(j)}(0), \quad j = 0, 1, \dots, 2k-1, \\ & \lambda_{1,j} = \lambda_{2,j}, \quad j \in \mathbb{N}_0 \setminus \{m\} \quad \text{and} \quad \kappa_{1,j} = \kappa_{2,j}, \quad j \in \mathbb{N}_0 \setminus \Lambda_0 \end{aligned}$$

with some $m \in \Lambda_0$. Then $q_1 = q_2$ a.e. on $[0, 1]$.

Theorem 5.2. *Let $q_1, q_2 \in L^1[0, 1]$. Assume that $h_1 = h_2$ and*

$$(5.2) \quad \lambda_{1,j} = \lambda_{2,j} \quad \text{and} \quad \kappa_{1,j} = \kappa_{2,j}, \quad j \in \mathbb{N}.$$

Then $q_1 = q_2$ a.e. on $[0, 1]$.

Remark 5.3. Comparing the results of Theorems 5.1 and 5.2 with those of [16, Theorem 1.3] and Theorem 3.1 of this present paper, we observe that one can replace information on one eigenvalue by knowledge of the boundary condition h_1 . This shows that the answer to the open problem of Gesztesy and Simon [16] is affirmative under the circumstance of general spectral data.

Before proving Theorems 5.1 and 5.2, we mention a lemma which will be needed subsequently. Its proof is similar to the proofs of [19, Theorem 1] and [18, Theorem 1.1] and is therefore omitted.

Lemma 5.4. *Let $h_1 = h_2$ and $\Lambda_0 = \{j_1, j_2, \dots, j_{k+1}\} \subset \mathbb{N}_0$. Define the function $\tilde{y}_j$ by*

$$(5.3) \quad \tilde{y}_j = \frac{v_{+,j} - \kappa_{1,j}v_{-,j}}{\omega'_1(\lambda_j)},$$

where $v_{\pm,j}$, for each j , is the solution of (2.1) with $q = q_2$ and $z = \lambda_{1,j}$, and with the initial conditions (2.2)-(2.3). If $\lambda_{1,j} = \lambda_{2,j}$ and $\kappa_{1,j} = \kappa_{2,j}$ for $j \in \mathbb{N}_0 \setminus \Lambda_0$, then

$$(5.4) \quad 0 = \sum_{j \in \Lambda_0} \tilde{y}_j(0),$$

$$(5.5) \quad q_1 - q_2 = 2 \sum_{j \in \Lambda_0} (\tilde{y}_j u_{1,j})' \quad \text{a.e.}$$

Note that if $\lambda_{1,j} = \lambda_{2,j}$ for some $j \in \Lambda_0$, then $v_{+,j} = \kappa_{2,j}v_{-,j}$ and $\kappa_{2,j}$ are the eigenfunction and, respectively, the norming constant of the operator L_2 corresponding to the eigenvalue $\lambda_{2,j}$.

Proof of Theorem 5.1. We first prove that $\lambda_{1,m} = \lambda_{2,m}$. We have from (5.1) that $\lambda_{1,j} = \lambda_{2,j}$ for all $j \in \mathbb{N}_0 \setminus \{m\}$. Note that if $j \neq m$, then $v_{-,j}$ and $v_{+,j}$ are the eigenfunctions, corresponding to the eigenvalue $\lambda_{2,j} (= \lambda_{1,j})$, of the operator L_2 . In this case, by (5.3) and (2.2)-(2.3), we find $v_{+,j} = \kappa_{2,j}v_{-,j}$ with $\kappa_{2,j} = v_{+,j}(0)$ and

$$(5.6) \quad \tilde{y}_j = \frac{v_{+,j} - \kappa_{1,j}v_{-,j}}{\omega'_1(\lambda_j)} = \begin{cases} \frac{v_{+,j} - \kappa_{1,j}v_{-,j}}{\omega'_1(\lambda_j)} & \text{if } j = m, \\ \frac{\kappa_{2,j} - \kappa_{1,j}}{\omega'_1(\lambda_j)} u_{2,j} & \text{if } j \neq m. \end{cases}$$

Since $q_1(0) = q_2(0)$ and both $u_{1,j}$ and $u_{2,j} (= v_{-,j})$ satisfy the same initial condition at $x = 0$, by (5.5) and (5.6) we have

$$(5.7) \quad \begin{aligned} 0 &= \left(\frac{v_{+,m} - \kappa_{1,m}v_{-,m}}{\omega'_1(\lambda_m)} u_{1,m} \right)'(0) + \sum_{j \in \Lambda_0 \setminus \{m\}} \frac{\kappa_{2,j} - \kappa_{1,j}}{\omega'_1(\lambda_j)} (u_{1,j} u_{2,j})'(0) \\ &= \frac{(v'_{+,m} u_{1,m} + v_{+,m} u'_{1,m})(0) - \kappa_{1,m}(v'_{-,m} u_{1,m} + v_{-,m} u'_{1,m})(0)}{\omega'_1(\lambda_m)} \\ &\quad + \sum_{j \in \Lambda_0 \setminus \{m\}} (u'_{1,j} u_{2,j} + u_{1,j} u'_{2,j})(0) \frac{\kappa_{2,j} - \kappa_{1,j}}{\omega'_1(\lambda_j)} \\ &= \frac{v'_{+,m}(0) + h_0 v_{+,m}(0) - 2\kappa_{1,m} h_0}{\omega'_1(\lambda_m)} + 2h_0 \sum_{j \in \Lambda_0 \setminus \{m\}} \frac{\kappa_{2,j} - \kappa_{1,j}}{\omega'_1(\lambda_j)}. \end{aligned}$$

By (5.4) and (5.6) we have

$$(5.8) \quad 0 = \sum_{j \in \Lambda_0} \tilde{y}_j(0) = \frac{v_{+,m}(0) - \kappa_m}{\omega'_1(\lambda_m)} + \sum_{j \in \Lambda_0 \setminus \{m\}} \frac{\kappa_{2,j} - \kappa_{1,j}}{\omega'_1(\lambda_j)}.$$

Substituting (5.8) into (5.7), we get

$$\frac{v'_{+,m}(0) - h_0 v_{+,m}(0)}{\omega'_1(\lambda_m)} = 0.$$

This shows that $\lambda_{1,m}$ is an eigenvalue of the operator L_2 . If $\lambda_{1,m} \neq \lambda_{2,m}$, then we may assume without loss of generality that $\lambda_{1,m} < \lambda_{2,m}$. Thus $\lambda_{1,m} < \lambda_{2,m} < \lambda_{1,m+1} = \lambda_{2,m+1}$ and

$$\lambda_{2,0} < \lambda_{2,1} < \cdots < \lambda_{2,m} < \lambda_{1,m} < \lambda_{2,m+1} \cdots$$

But this contradicts the asymptotic result (see [12, p. 5]) for the eigenvalues of the operator L_2 . Hence $\lambda_{1,m} = \lambda_{2,m}$, and we get $\lambda_{1,j} = \lambda_{2,j}$ for all $j \in \mathbb{N}_0$.

Based on the above result (that is, $\lambda_{1,j} = \lambda_{2,j}$ for all $j \in \mathbb{N}_0$), we next prove that $\kappa_{1,j} = \kappa_{2,j}$ for all $j \in \Lambda_0$. Write $\lambda_{1,j} (= \lambda_{2,j})$ as λ_j , and without loss of generality we assume that $\Lambda_0 = \{0, 1, \dots, k\}$. Set $A_j := (\kappa_{1,j} - \kappa_{2,j})/\omega'(\lambda_j)$. It follows from (5.6) that $\tilde{y}_j = A_j u_{2,j}$. Hence (5.5) can be rewritten as

$$(5.9) \quad (q_1 - q_2)(x) = 2 \sum_{j=0}^k A_j (u_{1,j} u_{2,j})'(x) \quad \text{for } x \in [0, 1].$$

It is easy to observe that $q_1 - q_2 \in C^{2k}[0, \varepsilon]$, since $u_{i,j}$ are the solutions of (2.1) for $q = q_i$, $i = 1, 2$. We next show that

$$(5.10) \quad \sum_{j=0}^k \lambda_j^l A_j = 0, \quad \text{for all } l = 0, 1, \dots, k.$$

We use induction to prove this fact. It is easy to check that if $l = 0, 1$, then (5.10) follows from (5.4) and the following computation:

$$\begin{aligned}
 0 &= 1/2(q'_1(0) - q'_2(0)) \\
 &= \sum_{j=0}^k A_j(u_{1,j}u_{2,j})''(0) \\
 &= \sum_{j=0}^k A_j[(u''_{1,j}u_{2,j} + u_{1,j}u''_{2,j}) + 2u'_{1,j}u'_{2,j}](0) \\
 &= \sum_{j=0}^k A_j[(q_1(0) + q_2(0) - 2\lambda_j)(u_{1,j}u_{2,j})(0) + 2(u'_{1,j}u'_{2,j})(0)] \\
 &= (q_1(0) + q_2(0) + 2h_0^2) \sum_{j=0}^k A_j - 2 \sum_{j=0}^k \lambda_j A_j.
 \end{aligned}$$

Now suppose that (5.10) holds for $l = 0, 1, \dots, t$ ($t < k$). We shall verify that (5.10) also holds for $l = t+1$. Note that $q_1, q_2 \in C^{2k-1}[0, \varepsilon]$ for some $\varepsilon > 0$ and $(q_1 - q_2)^{(j)}(0) = 0$ for $j = 0, 1, \dots, 2k-1$. Differentiating the identity (5.5) $(2t+1)$ times with respect to x and using (2.1) and its coefficients, we find that, by letting $x = 0$,

$$\begin{aligned}
 0 &= 1/2(q_1 - q_2)^{(2t+1)}(0) \\
 &= \sum_{j=0}^k A_j(u_{1,j}u_{2,j})^{(2t+2)}(0) \\
 &= \sum_{j=0}^k A_j[(u''_{1,j}u_{2,j} + u_{1,j}u''_{2,j}) + 2u'_{1,j}u'_{2,j}]^{(2t)}(0) \\
 &= -2 \sum_{j=0}^k \lambda_j A_j(u_{1,j}u_{2,j})^{(2t)}(0) + \sum_{j=0}^k A_j[(q_1 + q_2)(u_{1,j}u_{2,j})]^{(2t)}(0) \\
 (5.11) \quad &+ \sum_{j=0}^k 2A_j(u'_{1,j}u'_{2,j})^{(2t)}(0).
 \end{aligned}$$

Next, we calculate $u_{1,j}^{(l)}(0)$ and $u_{2,j}^{(l)}(0)$ for $l = 0, 1, 2, \dots, 2t$. Note that

$$\begin{aligned}
 u_{1,j}(0) &= 1, \quad u'_{1,j}(0) = h_0, \quad u''_{1,j}(0) = q_1(0) - \lambda_j, \\
 u'''_{1,j}(0) &= (q'_1 u_{1,j})(0) + ((q_1 - \lambda_j)u'_{1,j})(0) \\
 &= q'_1(0) + q_1(0)h_0 - h_0\lambda_j, \\
 u_{1,j}^{(4)}(0) &= (q''_1 u_{1,j})(0) + 2(q'_1 u'_{1,j})(0) + ((q_1 - \lambda_j)^2 u_{1,j})(0) \\
 &= (q''_1 + q_1^2 + 2h_0 q'_1)(0) - 2q_1(0)\lambda_j + \lambda_j^2, \\
 u_{1,j}^{(5)}(0) &= [(q''_1 + q_1^2 - 2\lambda_j q_1)u_{1,j}]'(0) + 2(q'_1 u'_{1,j})'(0) + \lambda_j^2 u'_{1,j}(0) \\
 &= (q'''_1 + 3q''_1 + 3q_1 q'_1 + (1 + h_0)q_1^2)(0) \\
 &\quad - 2(2q'_1 - 2h_0 q_1)(0)\lambda_j + h_0 \lambda_j^2.
 \end{aligned}$$

Proceeding by induction, we obtain that $u_{1,j}^{(l)}(0)$ and $u_{2,j}^{(l)}(0)$ for $l = 2, 3, \dots, 2t$ can be expressed as the following forms, for $l = 2s - 1$ and $2s$, respectively:

$$(5.12) \quad \begin{aligned} u_{1,j}^{(2s-1)}(0) &= (-1)^{s-1} h_0 \lambda_j^{s-1} + a_{s-2} \lambda_j^{s-2} + a_{s-3} \lambda_j^{s-3} + \dots + a_0, \\ u_{1,j}^{(2s)}(0) &= (-1)^s \lambda_j^s + b_{s-1} \lambda_j^{s-1} + b_{s-2} \lambda_j^{s-2} + \dots + b_0, \end{aligned}$$

where all the coefficients a_l and b_l of λ_j^l depend only on $h_0, q_1(0), q_1'(0), \dots, q_1^{(2t-2)}(0)$. We observe that $u_{1,j}^{(2s)}(0)$ is a polynomial of λ_j with degree s and that $u_{1,j}^{(2s-1)}(0)$ is also a polynomial of λ_j with degree at most $s - 1$. By (5.12), condition (5.1) and the fact that $\sigma(L_1) = \sigma(L_2)$, we obtain that $u_{1,j}^{(l)}(0) = u_{2,j}^{(l)}(0)$. Thus, using Leibnitz's formula, we have

$$(5.13) \quad \begin{aligned} (u_{1,j} u_{2,j})^{(2t)}(0) &= \sum_{n=0}^{2t} C_{2t}^n \left(u_{1,j}^{(n)} u_{2,j}^{(2t-n)} \right) (0) \\ &= \sum_{n=0}^t C_{2t}^{2n} \left(u_{1,j}^{(2n)} u_{2,j}^{(2t-2n)} \right) (0) \\ &\quad + \sum_{n=1}^t C_{2t}^{2n-1} \left(u_{1,j}^{(2n-1)} u_{2,j}^{(2t-2n+1)} \right) (0) \\ &= (-1)^t \left(\sum_{n=0}^t C_{2t}^{2n} \right) \lambda_j^t + (-1)^{t-1} h_0^2 \left(\sum_{n=1}^t C_{2t}^{2n-1} \right) \lambda_j^{t-1} \\ &\quad + c_{t-2} \lambda_j^{t-2} + c_{t-3} \lambda_j^{t-3} + \dots + c_0 \end{aligned}$$

and

$$(5.14) \quad \begin{aligned} (u'_{1,j} u'_{2,j})^{(2t)}(0) &= \sum_{n=0}^{2t} C_{2t}^n \left(u_{1,j}^{(n+1)} u_{2,j}^{(2t-n+1)} \right) (0) \\ &= (-1)^t h_0^2 \left(\sum_{n=0}^t C_{2t}^{2n} \right) \lambda_j^t + (-1)^t h_0 \left(\sum_{n=1}^t C_{2t}^{2n-1} \right) \lambda_j^t \\ &\quad + c'_{t-1} \lambda_j^{t-1} + c'_{t-2} \lambda_j^{t-2} + \dots + c'_0, \end{aligned}$$

where $C_{2t}^n = 2t(2t-1) \dots (2t-n+1)/n!$ and all the coefficients c_n and c'_n depend only on $h_0, q_1(0), q_1'(0), \dots, q_1^{(2t-2)}(0)$.

Similarly, using Leibnitz's formula again we easily obtain that

$$(5.15) \quad [(q_1 + q_2)(u_{1,j} u_{2,j})]^{(2t)}(0) = 2 \sum_{n=0}^{2t} C_{2t}^n q_1^{(n)}(0) (u_{1,j} u_{2,j})^{(2t-n)}(0)$$

is a polynomial of λ_j with degree at most t for which the coefficients are only associated with h_0 and $q_1^{(n)}(0)$ ($= q_2^{(n)}(0)$) for $0 \leq n \leq 2t$.

By the relations (5.13)-(5.15), the identity (5.11) can be rewritten as the form

$$(5.16) \quad \begin{aligned} 0 &= \sum_{j=0}^k A_j (u_{1,j} u_{2,j})^{(2t+2)}(0) \\ &= \sum_{j=0}^k 2(-1)^{t+1} \left(\sum_{n=0}^t C_{2t}^{2n} \right) \lambda_j^{t+1} A_j + \sum_{j=0}^k c_{t,j} \lambda_j^t A_j + \dots + \sum_{j=0}^k c_{0,j} A_j, \end{aligned}$$

where the coefficients $c_{i,j}$ for $i = 0, \dots, t$ and $j = 0, \dots, k$ are the linear combinations of the coefficients of polynomials (5.13)-(5.15). It turns out from the presupposition that (5.10) holds for $l = t + 1$ and, hence, for $l = 0, \dots, k$.

By the fact that $\lambda_0 < \lambda_1 < \dots < \lambda_k$ and the fact that the Vandermonde determinant associating with $\{\lambda_j\}_{j=0}^k$ does not vanish, we conclude that $A_j = 0$ from (5.10) and therefore $\kappa_{1,j} = \kappa_{2,j}$ for all $j \in \Lambda_0$. By (5.9) we further conclude that $q_1 = q_2$ a.e. on $[0, 1]$. The proof is therefore complete. $\square$

Next, we prove Theorem 5.2.

Proof of Theorem 5.2. All we need to prove is that $\lambda_{1,0} = \lambda_{2,0}$ and $\kappa_{1,0} = \kappa_{2,0}$. Suppose on the contrary that $\lambda_{1,0} \neq \lambda_{2,0}$; then $\tilde{y}_0 \not\equiv 0$ on $[0, 1]$ (see (5.3) for the definition of $\tilde{y}_0$). By the asymptotic expansions of $\lambda_{i,n}$,

$$\lambda_{i,n} = (n\pi)^2 + 2(h_i - h_0) + \int_0^1 q_i(t)dt + o(1), \quad i = 1, 2,$$

and the assumptions that $h_1 = h_2$ and $\lambda_{1,n} = \lambda_{2,n}$ for all $n \in \mathbb{N}$, we immediately get

$$\int_0^1 (q_1 - q_2)(t)dt = 0.$$

Moreover, we have, by (5.4), $\tilde{y}_0(0) = 0$ and, by (5.5),

$$0 = \int_0^1 (q_1 - q_2)(t)dt = \tilde{y}_0(1)u_{-,0}(1) = \tilde{y}_0(1).$$

This shows that $\lambda_{1,0}$ is an eigenvalue of (2.1) with $q = q_2$ and the Dirichlet boundary conditions, i.e., $u(0) = 0 = u(1)$. By [36, Chapter 4] we have $\lambda_{1,0} > \lambda_{2,0}$. If we interchange the positions of q_1 and q_2 in Lemma 5.4, then, by repeating the argument above, we get $\lambda_{1,0} < \lambda_{2,0}$. This contradicts the assumption that $\tilde{y}_0 \not\equiv 0$. Consequently, we must have $\lambda_{1,0} = \lambda_{2,0}$ which together with (5.4) implies that $\kappa_{1,0} = \kappa_{2,0}$. Moreover, by (5.5), we have $q_1 = q_2$ on $[0, 1]$. $\square$

6. DIRICHLET BOUNDARY CONDITION

In this section, we shall provide some details in the remaining cases, for which the operators L in $L^2[0, 1]$ involve the Dirichlet boundary conditions at $x = 0$ and/or $x = 1$. More precisely, we shall consider some analogs to Theorem 3.4 and Theorem 4.1 in the Dirichlet context. We need to distinguish three cases:

Case I. L has a Dirichlet boundary condition at $x = 0$ and $x = 1$, that is,

$$(6.1) \quad u(0) = 0 = u(1).$$

Case II. L has a non-Dirichlet boundary condition at $x = 0$ and a Dirichlet boundary condition at $x = 1$, that is,

$$(6.2) \quad u'(0) - h_0 u(0) = 0 = u(1), \quad h_0 \in \mathbb{R}.$$

Case III. L has a Dirichlet boundary condition at $x = 0$ and a non-Dirichlet boundary condition at $x = 1$, that is,

$$(6.3) \quad u(0) = 0 = u'(1) - h_1 u(1), \quad h_0 \in \mathbb{R}.$$

We index the corresponding eigenvalues λ_j of the operator L by

$$(6.4) \quad \{\lambda_j\}_{j=1}^\infty \quad \text{in Case I} \quad \text{and} \quad \{\lambda_j\}_{j=0}^\infty \quad \text{in Cases II and III.}$$

Let $u_{1,+}(x, z)$ and $u_{2,+}(x, z)$ be the solutions of (2.1) corresponding to q_1 and q_2 , respectively, where $u_{j,+}(x, z)$ satisfies the boundary conditions at $x = 1$ and normalization, for $j = 1, 2$,

$$(6.5) \quad u_{j,+}(1, z) = 0, \quad u'_{j,+}(1, z) = 1 \quad \text{in Cases I and II,}$$

$$(6.6) \quad u_{j,+}(1, z) = 1, \quad u'_{j,+}(1, z) = h_1 \quad \text{in Case III.}$$

Note that the asymptotic expansions of $u_{j,+}$ and $u'_{j,+}$ are given in (3.2) and (3.3) in Case III, and in Cases I and II we have

$$(6.7) \quad u_{j,+}(x, z) = -\frac{\sin(\sqrt{z}(1-x))}{\sqrt{z}} + O\left(\frac{e^{\operatorname{Im}(\sqrt{z})(1-x)}}{z}\right),$$

$$(6.8) \quad u'_{j,+}(x, z) = \cos(\sqrt{z}(1-x)) + O\left(\frac{e^{\operatorname{Im}(\sqrt{z})(1-x)}}{\sqrt{z}}\right)$$

as $|z| \rightarrow \infty$ for all $x \in [0, 1]$. When $a \in [0, 1)$ and if $z = iy$ with $y \in \mathbb{R}$, then (6.7) and (6.8) imply

$$(6.9) \quad |u_{j,+}(a, iy)| = \frac{1}{2}|y|^{-1/2}e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}}(1 + o(1)),$$

$$(6.10) \quad |u'_{j,+}(a, iy)| = \frac{1}{2}e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}}(1 + o(1))$$

as $y \rightarrow \infty$. For $j = 1, 2$, let

$$(6.11) \quad \omega_j(z) = u_{j,+}(0, z) \quad \text{in Cases I and III,}$$

$$(6.12) \quad \omega_j(z) = u'_{j,+}(0, z) - h_0 u_{j,+}(0, z) \quad \text{in Case II.}$$

Then $\sigma(L_j) = \{\lambda_{j,n}\}_{n=1}^\infty$ in Case I, and $\sigma(L_j) = \{\lambda_{j,n}\}_{n=0}^\infty$ in Cases II and III are precisely the set of zeros of the characteristic functions $\omega_j(z)$. By (6.9) and (6.10) we also infer that, as y (real) $\rightarrow \infty$,

$$(6.13) \quad |\omega_j(iy)| = \frac{1}{2}|y|^{-1/2}e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}}(1 + o(1)) \quad \text{in Case I,}$$

$$(6.14) \quad |\omega_j(iy)| = \frac{1}{2}e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}}(1 + o(1)) \quad \text{in Cases II and III}$$

and for sufficiently large n ,

$$(6.15) \quad \inf_{\theta \in [0, 2\pi]} |\omega_j(((\pi(n + \frac{1}{2}))^2 e^{i\theta}))| \geq \frac{1}{\pi n} + O\left(\frac{1}{n^2}\right) \quad \text{in Case I,}$$

$$(6.16) \quad \inf_{\theta \in [0, 2\pi]} |\omega_j(((\pi(n + 1))^2 e^{i\theta}))| \geq 1 + O\left(\frac{1}{n}\right) \quad \text{in Cases II and III.}$$

It is easy to see that the norming constants $\kappa_{j,n}$ for $j = 1, 2$ can be expressed as

$$(6.17) \quad \kappa_{j,n} = u'_{j,+}(0, \lambda_{j,n}) \quad \text{in Cases I and III,}$$

$$(6.18) \quad \kappa_{j,n} = u_{j,+}(0, \lambda_{j,n}) \quad \text{in Case II,}$$

where $n \in \mathbb{N}$ in Case I and $n \in \mathbb{N}_0$ in Cases II and III.

Based upon the above preliminaries, we give an analog of Theorem 3.3 in all of the Cases I, II and III.

Theorem 6.1. *Let the operators L_1 and L_2 be with one of the boundary conditions (6.1), (6.2), or (6.3). Let $a \in [0, 1/2)$ and $q_1, q_2 \in C^n(a - \varepsilon, a + \varepsilon)$ for some $n \in \mathbb{N}_0$*

and for some $\varepsilon > 0$. Consider a subset $S \subseteq \sigma(L_2)$ such that $\kappa_{1,j} = \kappa_{2,j}$ for all $\lambda_j \in S$. Assume that $q_1 = q_2$ a.e. on $[0, a]$, $\sigma(L_1) = \sigma(L_2)$ and

$$(6.19) \quad n_S(t) \geq (1 - 2a)n_{\sigma(L_2)}(t) - a - \frac{n+1}{2} \quad \text{in Case I,}$$

$$(6.20) \quad n_S(t) \geq (1 - 2a)n_{\sigma(L_2)}(t) - \frac{n+3}{2} \quad \text{in Case II,}$$

$$(6.21) \quad n_S(t) \geq (1 - 2a)n_{\sigma(L_2)}(t) - \frac{n+1}{2} \quad \text{in Case III,}$$

for all sufficiently large $t \in \mathbb{R}$. Then $h_1 = h_2$ (i.e., $h_1 = \infty$ in Cases I and II and $h_1 \in \mathbb{R}$ in Case III) and $q_1 = q_2$ a.e. on $[0, 1]$.

Proof. Consider the function defined by

$$(6.22) \quad F(z) := (u_{1,+}(0, z) - u_{2,+}(0, z))\tilde{\omega}(z),$$

where $\tilde{\omega}(z) = v'_{2,+}(0, z) - h_0 v_{2,+}(0, z)$ and where $v_{2,+}(x, z)$ is the solution of (2.1) for q_2 with the initial condition (6.6) in Cases I and II or the initial condition (6.5) in Case III. Note that in Cases I and III (i.e., $h_0 = \infty$), $\tilde{\omega}(z) = v_{2,+}(0, z)$. Thus, by (3.25) and Green's formula (2.14), we get

$$(6.23) \quad \begin{aligned} F(z) &= [u_{1,+}, v_{2,+}](a, z) - [u_{2,+}, v_{2,+}](a, z) \\ &= u_{1,+}(a, z)v_{2,+}(a, z) (m_{+,v_{2,+}}(a, z) - m_{+,u_{1,+}}(a, z)) - C_0, \end{aligned}$$

where $C_0 = [u_{2,+}, v_{2,+}](1, z) = (u_{2,+}v'_{2,+})(1, z) - (u'_{2,+}v_{2,+})(1, z)$ is independent of z by the initial conditions. Without loss of generality we may assume that the initial condition for $u_{1,+}$ is the Dirichlet condition in Cases I and II and do not assume a priori that $u_{2,+}$ has the Dirichlet condition at $x = 1$. Note that, in all the cases, one of the initial conditions at $x = 1$ for $u_{1,+}$ and $v_{2,+}$ is the Dirichlet condition and the other one is a non-Dirichlet condition. Then by (2.8), (3.4) and (6.9) we infer

$$(6.24) \quad \begin{aligned} |F(iy)| &\leq |y|^{-1/2} e^{\operatorname{Im}(\sqrt{i})2(1-a)|y|^{1/2}} (1 + o(1)) o(|y|^{-(n+1)/2}) + |C_0| \\ &\leq |y|^{-(n+2)/2} e^{\operatorname{Im}(\sqrt{i})2(1-a)|y|^{1/2}} o(1). \end{aligned}$$

Following the argument employed in the proof of Theorem 3.4, we again introduce the functions

$$(6.25) \quad \left. \begin{aligned} G_S(z) &= \prod_{\lambda_j \in S} \left(1 - \frac{z}{\lambda_j}\right) & S \subseteq \sigma(L_1), \\ G_{\sigma(L_1)}(z) &= \prod_{j=1}^{\infty} \left(1 - \frac{z}{\lambda_j}\right) & \text{in Case I,} \\ G_{\sigma(L_1)}(z) &= \prod_{j=0}^{\infty} \left(1 - \frac{z}{\lambda_j}\right) & \text{in Cases II and III.} \end{aligned} \right\}$$

Then from (6.19)-(6.21) and the proof of Theorem 3.4 we have

$$(6.26) \quad \left. \begin{aligned} |G_S(iy)| &\geq C_0 |y|^{-(n+2)/2} e^{\operatorname{Im}(\sqrt{i})(1-2a)|y|^{1/2}} & \text{in Case I,} \\ |G_S(iy)| &\geq C_0 |y|^{-(n+3)/2} e^{\operatorname{Im}(\sqrt{i})(1-2a)|y|^{1/2}} & \text{in Case II,} \\ |G_S(iy)| &\geq C_0 |y|^{-(n+1)/2} e^{\operatorname{Im}(\sqrt{i})(1-2a)|y|^{1/2}} & \text{in Case III.} \end{aligned} \right\}$$

By the definition of $\tilde{\omega}(\cdot)$ and the boundary conditions of the solution v_2 , we deduce that

$$(6.27) \quad \left. \begin{aligned} |\tilde{\omega}(iy)| &= \frac{1}{2} e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}} (1 + o(1)) && \text{in Case I,} \\ |\tilde{\omega}(iy)| &= \frac{1}{2} |y|^{1/2} e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}} (1 + o(1)) && \text{in Case II,} \\ |\tilde{\omega}(iy)| &= \frac{1}{2} |y|^{-1/2} e^{\operatorname{Im}(\sqrt{i})|y|^{1/2}} (1 + o(1)) && \text{in Case III.} \end{aligned} \right\}$$

Let $G(z) = G_S(z)\tilde{\omega}(z)$. By (6.12)-(6.14), we find that

$$(6.28) \quad |G(iy)| \geq C_0 |y|^{-(n+2)/2} e^{\operatorname{Im}(\sqrt{i})2(1-a)|y|^{1/2}}.$$

Now using (6.24) and (6.28), one can complete the proof of Theorem 6.1 by repeating the proof of Theorem 3.4 in Section 3. $\square$

Remark 6.2. If q is assumed to be $C^{2k-1}[0, \varepsilon)$, then $q^{(j)}(0), j = 0, 1, \dots, 2k-1$, together with the knowledge of $h_0 \in \mathbb{R}$, and all the eigenvalues of L and all the corresponding norming constants except for k in Cases I and III, and for $(k+1)$ in Case II uniquely determine h_1 and q on all of $[0, 1]$. In particular, if $q \in L^1[0, 1]$, then one norming constant can be missing in Case II.

Finally, we give an analog of Theorem 4.1 in the Dirichlet context.

Theorem 6.3. *Let the operators L_1 and L_2 be with one of the boundary conditions (6.1), (6.2), or (6.3). Let $a \in [0, 1)$. Consider a subset $S \subseteq \sigma(L_1) \cap \sigma(L_2)$ such that $\kappa_{1,j} = \kappa_{2,j}$ for all $\lambda_j \in S$. Assume that $q_1 = q_2$ a.e. on $[0, a]$ and*

$$(6.29) \quad n_S(t) \geq (1-a)n_{\sigma(L_2)}(t) - \frac{a+1}{2} \quad \text{in Case I,}$$

$$(6.30) \quad n_S(t) \geq (1-2a)n_{\sigma(L_2)}(t) - \frac{1}{2} \quad \text{in Case II.}$$

$$(6.31) \quad n_S(t) \geq (1-2a)n_{\sigma(L_2)}(t) \quad \text{in Case III}$$

for all sufficiently large t . Then $h_1 = h_2$ (i.e., $h_1 = \infty$ in Cases I and II and $h_1 \in \mathbb{R}$ in Case III) and $q_1 = q_2$ a.e. on $[0, 1]$.

Proof. Let us consider the following function:

$$(6.32) \quad F_v(z) := (h_1 - h_2)v_2(1, z) + \int_0^1 (q_1 - q_2)(x)(u_{1,+}v_2)(x, z)dx.$$

Let $v_D := v_2(x, z)$ (resp., $v_N := v_2(x, z)$) be the solution of (2.1) for q_2 satisfying the initial condition (6.5) (resp., (6.6) with $h_1 = 0$) at point $x = a$, respectively. By virtue of the proof of Theorem 4.1, we have

$$(6.33) \quad F_{v_D}(z) = (u_{1,+} - u_{2,+})(a, z), \quad F_{v_N}(z) = (u'_{1,+} - u'_{2,+})(a, z).$$

Since h_0 is fixed and there is an infinite sequence of common eigenvalues of the operators L_1 and L_2 (see (6.29)-(6.31)), and by the asymptotic expansions of $\{\lambda_{j,n}\}$ for $j = 1, 2$,

$$(6.34) \quad \sqrt{\lambda_{j,n}} = n\pi + O\left(\frac{1}{n}\right) \quad \text{in Case I,}$$

$$(6.35) \quad \sqrt{\lambda_{j,n}} = \left(n + \frac{1}{2}\right)\pi + O\left(\frac{1}{n}\right) \quad \text{in Cases II and III}$$

as $n \rightarrow \infty$, we infer that the initial conditions for $u_{1,+}$ and $u_{2,+}$ at $x = 1$ are of the same type, that is, both initial conditions are the Dirichlet condition in Cases I and II and in the non-Dirichlet condition in Case III. This ensures that $h_1 = h_2 = \infty$ in Cases I and II and $h_1, h_2 \in \mathbb{R}$ in Case III. Thus, by (6.7) and (3.2) we have

$$(6.36) \quad \left. \begin{aligned} |F_{v_D}(iy)| &= O(|y|^{-1} e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}}) && \text{in Cases I and II,} \\ |F_{v_D}(iy)| &= O(|y|^{-1/2} e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}}) && \text{in Case III.} \end{aligned} \right\}$$

Following the argument employed in the proof of Theorem 4.1, we again introduce the function

$$(6.37) \quad G_S(z) = \prod_{\lambda_j \in S} \left(1 - \frac{z}{\lambda_j} \right), \quad S \subseteq \sigma(L_2).$$

By the hypotheses (6.29)-(6.31) on S we have

$$(6.38) \quad \left. \begin{aligned} |G_S(iy)| &\geq C_0 |y|^{-1/2} e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}} && \text{in Cases I and II,} \\ |G_S(iy)| &\geq C_0 e^{\operatorname{Im}(\sqrt{i})(1-a)|y|^{1/2}} && \text{in Case III.} \end{aligned} \right\}$$

This yields that $F_{v_D}(z) = (u_{1,+} - u_{2,+})(a, z) = 0$ for all $z \in \mathbb{C}$. In this case,

$$F_{v_N}(z) = (m_{+,1}(a, z) - m_{+,2}(a, z))u_{1,+}(a, z).$$

Using (2.7) and (6.9) we can complete the proof of Theorem 6.3 by following the same method as that of Theorem 4.1 in Section 4. $\square$

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