THE STRUCTURE AND DEFINABILITY IN THE LATTICE OF EQUATIONAL THEORIES OF STRONGLY PERMUTATIVE SEMIGROUPS

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ABSTRACT. In this paper, we study the structure and the first-order definability in the lattice $\mathcal{L}(SP)$ of equational theories of strongly permutative semigroups, that is, semigroups satisfying a permutation identity

$$x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$$

with $\sigma(1) > 1$ and $\sigma(n) < n$. We show that each equational theory of such semigroups is described by five objects: an order filter, an equivalence relation, and three integers. We fully describe the lattice $\mathcal{L}(SP)$; inclusion, operations $\vee$ and $\wedge$, and covering relation. Using this description, we prove, in particular, that each individual theory of strongly permutative semigroups is definable, up to duality.

1. INTRODUCTION

In [22, 16], A. Tarski and R. McKenzie raised problems of first-order definability in the lattices of equational theories. In a series of papers [8, 9, 10, 11], J. Ježek has solved the most general problems in this area considering the case of lattices of equational theories of a given type. In [12] Ježek and McKenzie have studied the first-order definability in the lattice of equational theories of semigroups. They have proved that in this lattice many interesting sets of theories are definable (e.g. the finitely axiomatizable theories, the locally finite theories, and the theories of finite semigroups). Moreover, they have proved that each finitely axiomatizable locally finite theory is individually definable up to duality (i.e., up to inverting the order of occurrence of letters in defining equations). They conjecture that the assumption on local finiteness in the last quoted result may be omitted and that, in consequence, the lattice in question has no nontrivial automorphisms, but these problems still remain open.

One of the reasons that the problems above seem very hard is that we have very little detailed knowledge about the structure of the lattice of equational theories of semigroups. In contrast, we have very good knowledge about the structure of the lattice $L(Com)$ of equational theories of commutative semigroups ([13] and [7, 4]). Using this, in [14], A. Kisielewicz has proved that many sets and individual theories in $L(Com)$ are definable. However, he also discovered that there exist nontrivial automorphisms of $L(Com)$.

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In [6], we have completed the study of the first-order definability in the lattice $L(Com)$. We have obtained complete descriptions of the set of definable theories and of the group of automorphisms of $L(Com)$, which has appeared to form an uncountable Boolean group.

It seems natural to now consider the automorphism groups of intermediate classes of equational theories of semigroups to find some “minimal examples” of lattices with small automorphism groups.

A *permutative* semigroup is one satisfying a *permutation identity* of the form

$$(1) \quad x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$$

for some nontrivial permutation σ . Perkins [18] proved that such a semigroup necessarily satisfies an equation of the form

$$x_1 \cdots x_k y z x_{k+1} \cdots x_n = x_1 \cdots x_k z y x_{k+1} \cdots x_n,$$

and therefore he used the name *almost commutative* for such semigroups.

The set of equational theories of permutative semigroups forms a lattice that contains the lattice of equational theories of commutative semigroups. An extension of Kisielewicz’s [13] description to the lattice of all equational theories of permutative semigroups does not seem easy. The main obstacle is that, as it was proved in [18], there are equational theories of permutative semigroups that fail to be finitely based. In contrast, it is well known (see [18, 20], for instance) that every theory containing an equation (1) for some permutation σ with $\sigma(1) > 1$ and $\sigma(n) < n$ is finitely based. Such theories (and corresponding semigroups) will be called *strongly permutative*. It is known (see e.g. [20]) that for every strongly permutative theory T there exists $m > 1$ such that T contains all permutation equations (1) for all $n \geq m$.

If we wish to study first-order definability of intermediate lattices of equational theories of semigroups, the case of all strongly permutative semigroups may be considered as an appropriate candidate for the beginning. Every strongly permutative variety is finitely based, and therefore the set of all strongly permutative varieties is countable.

For $k \geq 0$ and $n > 1$, let $B_{k,n}$ denote the set that consists of $n!$ identities of the following form:

$$t_1 \cdots t_k x_1 x_2 \cdots x_n t_{k+1} \cdots t_{2k} = t_1 \cdots t_k x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} t_{k+1} \cdots t_{2k},$$

where σ is a permutation on the set $\{1, \dots, n\}$. If $k \geq 0$, then an equational theory T is called *k-permutative* if for some $n \geq 2$, $T = B_{k,n}$ ([21]). By the result of Perkins [18], each permutative theory is *k-permutative* for some $k \geq 0$. For each n , the set of all *k-permutative* theories with $k \leq n$ forms a sublattice of the lattice P of all permutative semigroup theories, which we denote by $P(n)$. It is easy to see that for each $n \geq 0$, we have $P(n) \subset P(n+1)$. It is also easy to see that the term “strongly permutative variety” is equivalent to the term “0-permutative variety”. Our result can be considered as the first step in proving the following conjecture, which we believe may be true.

Conjecture. *For each $n \geq 0$ and $k \leq n$, every finitely based k -permutative theory is individually first-order definable, up to duality, in the lattice $P(n)$.*

The first part of this paper is aimed to prepare a general machinery to work with strongly permutative theories, which we believe gives a direct insight and may

be useful to formulate and prove further results in the area. This generalizes, in fact, the tools worked out in [13] and results in [4, 7] for commutative semigroups. It seems that it may be the largest part of the lattice of equational theories of all semigroups admitting such a nice description.

The second part, starting from section 7, is devoted to proving the result that every equational theory of strongly permutative semigroups is individually first-order definable, up to duality (Theorem 10.9).

2. PRELIMINARIES

It was J. Almeida [1] who discovered that behind the lattice of varieties of commutative semigroups a combinatorial structure of better-quasi-orderings is hidden. This also can be extended into the strongly permutative case. Once it is not necessary to prove the results in the second part of the paper, it may be useful for attacking more difficult problems in the area.

2.1. Better-quasi-orderings. Here we recall a definition of a better-quasi-ordering (well-quasi-ordering) and some results from [6, 17].

Let ω be the set of nonnegative integers with the usual order. For $s, t \subseteq \omega$ we write $s \leq t$ ($s < t$) if s is a (proper) initial segment of t .

Define $s \triangleleft t$ to hold if there are $n > 0$ and $i_0 < \dots < i_n \in \omega$ such that for some $m < n$, $s = \{i_0, \dots, i_m\}$ and $t = \{i_1, \dots, i_n\}$.

For an infinite set $X \subseteq \omega$, a set B of finite subsets of X will be called a *barrier* on X if the following conditions hold:

- (i) $\emptyset \notin B$,
- (ii) for every infinite $Y \subseteq X$ there is an $s \in B$ such that $s < Y$,
- (iii) if $s, t \in B$ and $s \neq t$, then $s \not\leq t$.

Let $(Q, \leq)$ be a quasi-ordering. For a barrier B , a function $f : B \rightarrow (Q, \leq)$ is *good* if there are $s, t \in B$ such that $s \triangleleft t$ and $f(s) \leq f(t)$, and f is *bad* otherwise. Then, $(Q, \leq)$ is a *better-quasi-ordering* (bqo) if for every barrier B and every $f : B \rightarrow (Q, \leq)$, f is good. The quasi-ordering $(Q, \leq)$ is a *well-quasi-ordering* (wqo) if for every barrier B consisting of one-element sets and every $f : B \rightarrow (Q, \leq)$, f is good.

It is an immediate consequence of definition that if $(Q, \leq)$ is bqo, then $(Q, \leq)$ is wqo. There is also another characterization of well-quasi-orderings. A quasi-ordering $(Q, \leq)$ is a well-quasi-ordering if and only if every order filter is finitely generated.

For the quasi-orderings $(Q_i, \leq_i)$, $i \in \{1, \dots, n\}$, we define $\bigcup (Q_i, \leq_i)$ to be a quasi-ordering $(\bigcup Q_i, \leq)$, where $p \leq q$ if and only if there is i such that $p, q \in Q_i$, and $p \leq_i q$.

Proposition 2.1 ([17]). *If $(Q_i, \leq_i)$, $i \in \{1, \dots, n\}$, are bqo, then $\bigcup (Q_i, \leq_i)$ is bqo.*

Below, we write three easy facts which can be found in [6].

Fact 2.2. Let $(Q, \leq)$ be bqo and $Q_1 \subseteq Q$. Then $(Q_1, \leq)$ is bqo.

Fact 2.3. Let $(Q, \leq_1), (Q, \leq_2)$ be quasi-orderings such that for $q, w \in Q$, if $q \leq_1 w$, then $q \leq_2 w$. Then, if $(Q, \leq_1)$ is bqo, then $(Q, \leq_2)$ is bqo.

Fact 2.4. For quasi-orderings $(Q_1, \leq_1), (Q_2, \leq_2)$ let $f : Q_1 \rightarrow Q_2$ be an onto mapping such that for $q, w \in Q_1$, if $q \leq_1 w$, then $f(q) \leq_2 f(w)$. Then, if $(Q_1, \leq_1)$ is bqo, then $(Q_2, \leq_2)$ is bqo.

Moreover, the following fact can be easily verified.

Fact 2.5. For quasi-orderings $(Q_1, \leq_1), (Q_2, \leq_2)$ let $f : Q_1 \rightarrow Q_2$ be a mapping such that for $q, w \in Q_1$, we have $q \leq_1 w$ if and only if $f(q) \leq_2 f(w)$. Then, if $(Q_2, \leq_2)$ is bqo, then $(Q_1, \leq_1)$ is bqo.

2.2. The lattices $\mathcal{L}(Com)$. Now we recall the terminology and facts concerning the description of the lattice of equational theories of commutative semigroups $\mathcal{L}(Com)$ based on [13], taking into account some improvements from [4, 7, 5]. We recall only those facts that we apply in this paper. For more details the reader is referred to [13, 7, 4, 5] and to the introductions in [14, 6].

Generally, an equational theory of commutative semigroups is determined by four parameters J, m, r, π , where m, r are integers, J is an order filter in a certain ordered set of finite sequences of integers, and π is a special additional set of equations of finite character (called the remainder).

Let Γ be the set of all finite sequences $(a_1, \dots, a_n)$ of nonnegative integers such that at least one $a_i \neq 0$, and let Γ_+ consist of those sequences where $a_i > 0$ for all i . For a sequence $\mathbf{a} = (a_1, \dots, a_n) \in \Gamma$ and a permutation of the indices $p \in S_n$, we put $p(\mathbf{a}) = (a_{p(1)}, \dots, a_{p(n)})$; if $p \in S_m$ with $n \neq m$, then we put $p(\mathbf{a}) = \mathbf{a}$. Further, for every $\gamma \geq 0$ we denote $h_\gamma(\mathbf{a}) = (a_1, \dots, a_n, \gamma)$, and for $1 \leq i, j \leq n$, $i \neq j$, we denote $g_{ij}(\mathbf{a}) = (a_1, \dots, a_i + a_j, \dots, a_n)$, where the latter is the sequence obtained from $\mathbf{a}$ by replacing a_i by $a_i + a_j$ and deleting a_j ; otherwise, we denote $g_{ij}(\mathbf{a}) = \mathbf{a}$. By PGH we denote the smallest class of unary operations on Γ closed under composition and containing all the operations h_γ, g_{ij} and permutations p .

For sequences $\mathbf{a}, \mathbf{b} \in \Gamma$, we define $\mathbf{a} \leq_K \mathbf{b}$ if and only if there exists an operation $f \in PGH$ such that $f(\mathbf{a}) = \mathbf{b}$. In [5], the following is proved.

Proposition 2.6 ([5]). *The quasi-ordering $(\Gamma, \leq_K)$ is bqo.*

For other results in this direction see [1] also.

Proposition 2.6 shows, in particular, that every (order) filter J is finitely generated. The filter generated by the sequences $\mathbf{a}_1, \dots, \mathbf{a}_t$ is denoted $[\mathbf{a}_1, \dots, \mathbf{a}_t]$.

Now, a commutative semigroup equation of the form

$$x_1^{a_1} \cdots x_n^{a_n} = x_1^{b_1} \cdots x_n^{b_n}$$

will be identified with the pair $(\mathbf{a}, \mathbf{b})$ of sequences $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, where $a_i + b_i > 0$, and is denoted by $(\mathbf{a}, \mathbf{b})$ for short. Such an equation is nontrivial if $a_i \neq b_i$ for some i . It is clear that for $f \in PGH$, the equation $(f(\mathbf{a}), f(\mathbf{b}))$ is a consequence of $(\mathbf{a}, \mathbf{b})$. An equational theory of a commutative semigroup (a theory) is a set of equations closed onto natural operations: multiplication, substitution, permutation of variables and transitivity of equality.

For a nonempty filter J in Γ , and integers $m \geq 0, r > 0$, we define $E(J, m, r)$ to be the theory given by the set of those equations $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n)$ are such sequences in J that satisfy the following conditions:

- (J1) for every i , if $a_i \neq b_i$, then $a_i, b_i \geq m$,
- (J2) for every i , $a_i \equiv b_i \pmod{r}$.

Moreover, we consider the following set of conditions for equations with $a_i, b_i > 0$, which belong to an equivalence relation π on the set $\Gamma_+ \setminus J$:

- ($\pi 1$) the conditions (J1), (J2) hold,
- ($\pi 2$) for every permutation $f \in S_n$, $(f(\mathbf{a}), f(\mathbf{b})) \in \pi$,

- ($\pi 3$) for every i such that $a_i \neq b_i$, and $\gamma = \min(a_i, b_i)$, both $h_\gamma(\mathbf{a}), h_\gamma(\mathbf{b}) \in J$,
- ($\pi 4$) if f is one of the operations g_{ij} or h_1 , then either $(f(\mathbf{a}), f(\mathbf{b})) \in \pi$ or both $f(\mathbf{a}), f(\mathbf{b}) \in J$.

Now let $m \geq 0$, $r > 0$ be integers, J a nonempty filter contained in $[(m)]$ and π an equivalence relation on the set $\Gamma_+ \setminus J$. If every block of π contains sequences of one fixed length (regular equations) and if for all pairs $(\mathbf{a}, \mathbf{b}) \in \pi$ the conditions ($\pi 1$) – ($\pi 4$) are satisfied, then π is called a *remainder of the type* (J, m, r) . For such a remainder, which may be viewed as a set of equations, we define $E(J, m, r, \pi)$ to be the set of equations $E(J, m, r) \cup \pi$. Then:

Theorem 2.7 ([13]). *Every set of equation $E(J, m, r, \pi)$ defined above is a theory, and every equational theory of commutative semigroups, except the theory consisting of trivial equations, is of this form.*

Theorem 2.8 ([13]). *If $E_1 = E(J_1, m_1, r_1, \pi_1)$ and $E_2 = E(J_2, m_2, r_2, \pi_2)$ belong to $\mathcal{L}(\text{Com})$, then the following hold:*

- (i) $E_1 \supseteq E_2$ if and only if $J_1 \supseteq J_2$, $m_1 \leq m_2$, r_1 divides r_2 , and $\pi_1 \cup J_1^2 \supseteq \pi_2$.
- (ii) $E_1 \vee E_2 = E(J, \min(m_1, m_2), \gcd(r_1, r_2), \pi)$, where $J \supseteq J_1 \cup J_2$, and π is a minimal equivalence relation on the set $\Gamma_+ \setminus J$ that contains both the relations π_1 and π_2 restricted to the set $\Gamma_+ \setminus J$.
- (iii) $E_1 \wedge E_2 = E(J_1 \cap J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2), \pi)$, where π is an equivalence relation on the set $\Gamma_+ \setminus (J_1 \cap J_2)$ contained in $(\pi_1 \cap \pi_2) \cup (\pi_1 \cap J_2^2) \cup (\pi_2 \cap J_1^2)$.

The following remarks will be used throughout this paper.

Remark 2.9 ([13]). A theory $E(J, m, r, \pi)$ consists of those equations $(\mathbf{a}, \mathbf{b})$ for which either $\mathbf{a}, \mathbf{b} \in J$ and satisfy the conditions (J1) and (J2) or $\mathbf{a}, \mathbf{b} \notin J$ and $(\mathbf{a}, \mathbf{b}) \in \pi$. Moreover:

- (i) if $\mathbf{a} \leq_K \mathbf{b}$, then $\mathbf{a}, \mathbf{b} \in J$,
- (ii) if $(\mathbf{a}, \mathbf{b})$ is an irregular equation, then $\mathbf{a}, \mathbf{b} \in J$,
- (iii) if $(\mathbf{a}, \mathbf{b}) \in \pi$, then $(\mathbf{a}, \mathbf{b})$ is a regular equation and the sequences $\mathbf{a}, \mathbf{b}$ are incomparable or equivalent in the better-quasi-ordering $(\Gamma_+, \leq_K)$.

Remark 2.10 ([4, Theorem 2.5]). For every remainder π there is a finite set $\bar{\pi}$ of the blocks of π such that every block $C \in \pi$ is of the form $f(B)$ for some $B \in \bar{\pi}$ and $f \in PH$.

3. QUASI-ORDERINGS $\leq_n$

Let $\mathcal{W} = F(X)$ be the free semigroup over countable alphabet $X = \{x_1, x_2, \dots\}$ for $i \in \{1, \dots, t\}$. The elements of X are called *letters* or *variables*. The elements of $\mathcal{W}$ are called *words* and will be written in the form

$$v = y_1 y_2 \dots y_t,$$

where $y_i \in X$ for all i . The *support* of v , written $\text{supp}(v)$, is the set of variables occurring in v . The length of v is t , and we write $|v| = t$.

Moreover, $|v|_x$ is the number of occurrences of the letter x in v . The notation $v = v(x_1, \dots, x_k)$ means that $\text{supp}(v) \subseteq \{x_1, \dots, x_k\}$. We also write x^t to denote the word $xx \dots x$ of length t .

We introduce some operations on $\mathcal{W}$. By

$$(2) \quad r_y(v) = vy \quad \text{and} \quad l_y(v) = yv$$

we denote adjoining an additional variable y to the word v , on the right and on the left, respectively. Usually we will assume that y does not occur in v . For identifying variables we introduce

$$(3) \quad g_{x_i x_j}(v) = v(x_1, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_k) \text{ if } 1 \leq i < j \leq k.$$

We put $g_{x_j x_i}(v) = g_{x_i x_j}(v)$ for $j < i$. For substitution we write

$$(4) \quad t_{x_i y}(v) = \begin{cases} v(x_1, \dots, x_{i-1}, x_i y, x_{i+1}, \dots, x_k), & \text{if } 1 \leq i \leq k, \\ v & \text{otherwise.} \end{cases}$$

Finally, for a permutation $p \in S_l$, we define

$$(5) \quad p(v) = \begin{cases} v(x_{p(1)}, \dots, x_{p(l)}), & \text{if } l \geq k, \\ v & \text{otherwise.} \end{cases}$$

We define P to be the set of all operations of the form (5). By $\mathcal{O}$, we denote the set of all operations on $\mathcal{W}$ generated by operations (2) – (5). We define a binary relation $\leq_n$ on the set $\mathcal{W}$. Let $v, w \in \mathcal{W}$. Assume that $\text{supp}(v) \cup \text{supp}(w) \subseteq \{x_1, \dots, x_k\}$ for $k > 0$. We write $w \leq_n v$ if either there is an operation $f \in \mathcal{O}$ such that $f(v) = w$ or for the sequences $(|v|_{x_1}, \dots, |v|_{x_k})$, $(|w|_{x_1}, \dots, |w|_{x_k})$ we have $\sum_{i=1}^k |w|_{x_i} \geq n$ and $(|v|_{x_1}, \dots, |v|_{x_k}) \leq_K (|w|_{x_1}, \dots, |w|_{x_k})$. The latter is equivalent with the fact that there is an operation $f \in \mathcal{O}$ such that $(|f(v)|_{x_1}, \dots, |f(v)|_{x_k}) \equiv_K (|w|_{x_1}, \dots, |w|_{x_k})$.

Lemma 3.1. *Let $v, w \in \mathcal{W}$ be such that $\text{supp}(v) \cup \text{supp}(w) \subseteq \{x_1, \dots, x_k\}$, and there is an operation $f \in \mathcal{O}$ such that $f(v) = w$. Then the following inequality holds: $(|v|_{x_1}, \dots, |v|_{x_k}) \leq_K (|w|_{x_1}, \dots, |w|_{x_k})$.*

Proof. For the operations $r_y, l_y, g_{x_i x_j}, t_{x_i y}, p$, we have an induced action of those operations on the set Γ . In these actions, the operations are equal to $h_1, h_1, g_{ij}, h_{|v|_{x_i}}$, and p , respectively. Now, the statement is obvious. $\square$

It is an immediate consequence of the definition and Lemma 3.1 that the relation $\leq_n$ is transitive. Hence, $(\mathcal{W}, \leq_n)$ is a quasi-ordering. We will write $v <_n w$ if $v \leq_n w$ and $w \not\leq_n v$. Moreover, we write $v \equiv_n w$ if $v \leq_n w$ and $w \leq_n v$.

Lemma 3.2. *Let $v, w \in \mathcal{W}$.*

- (i) *If $v \leq_n w$, then $|v| \leq |w|$.*
- (ii) *If $v <_n w$, and $\text{supp}(v) = \text{supp}(w)$, then $|v| < |w|$.*
- (iii) *$v \equiv_n w$ if and only if there is $k \in \mathbb{N}_+$ and $p \in S_k$ such that $p(v) = w$. In particular, $|v| = |w|$.*

Proof. All of (i), (ii) and (iii) are true when $|w| \geq n$ (see [4], for instance). Thus, we assume that $f(v) = w$ for some $f \in \mathcal{O}$. For (i), it is enough to observe that every operation (2) – (5) does not decrease the length of the words. For (ii), observe that the operations that do not increase the length of the words are $g_{x_i x_j}$ and permutations. It is clear that when we apply a permutation to a word, we obtain an equivalent word. Applying an operation $g_{x_i x_j}$, we decrease the cardinality of the support. Hence, $f(v) = w$, where $f \in \mathcal{O}$ is a composition of operations of the form (2) – (5) such that at least one of the operations of the form (2) or (4) is used. It implies that $|v| < |w|$. By (i), if $v \equiv_n w$, then $|v| = |w|$. Let $f_1, f_2 \in \mathcal{O}$ be such that $f_1(v) = w$ and $f_2(w) = v$. Since the operations (2) and (4) increase the length of the words, f_1 and f_2 are the compositions of the operations $g_{x_i x_j}$ and permutations. Since the operations $g_{x_i x_j}$ decrease the cardinality of supports, f_1

must be a composition of permutations. It is clear that this composition is equal to p , for some $p \in P$. This proves (iii). $\square$

Let Γ_n be the subset of Γ consisting of the sequences with the sum of elements greater than or equal to n . By Proposition 2.6 and Fact 2.2, $(\Gamma_n, \leq_K)$ is bqo. Let $\mathcal{W}_n$ be the subset of $\mathcal{W}$ consisting of the words of the length at least n . There is a natural mapping $\phi : (\mathcal{W}_n, \leq_n) \rightarrow (\Gamma_n, \leq_K)$, $\phi(w) = (|w|_{x_1}, \dots, |w|_{x_t})$, where t is a minimal number such that $\text{supp}(w) \subseteq \{x_1, \dots, x_t\}$. Obviously, for $w, v \in \mathcal{W}$, we have $w \leq_n v$ if and only if $\phi(w) \leq_K \phi(v)$. Hence, by Fact 2.5, $(\mathcal{W}_n, \leq_n)$ is bqo. The set of the words that belong to $\mathcal{W}$ and have length smaller than n , with the quasi-ordering $\leq_n$, consists of finitely many equivalence blocks at the relation $\equiv_n$. Each equivalence block is trivially bqo. From the facts above and by Proposition 2.1 and Fact 2.3, we have

Theorem 3.3. *The quasi-ordering $(\mathcal{W}, \leq_n)$ is bqo. Consequently, every order filter in $(\mathcal{W}, \leq_n)$ is finitely generated.*

4. SUBLATTICE $\mathcal{L}_S(SP)$

By an equational theory of semigroup T , we mean the set of equations of the form

$$(6) \quad v = w,$$

$v, w \in W$, closed under natural operations: substitutions, identifying variables, multiplication by variables and transitivity of equality (i.e., preserved by all the operations from $\mathcal{O}$ and by transitivity of equality). If there is n such that, for every permutation $\sigma \in S_n$, an equation

$$(7) \quad x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

belongs to T , then T is a theory of strongly permutative semigroups. We recall that this is equivalent to the fact that T contains at least one equation of the form (7), for some n , such that $\sigma(1) > 1$ and $\sigma(n) < n$.

The set of all equational theories of strongly permutative semigroups forms a lattice $\mathcal{L}(SP)$. The least element in $\mathcal{L}(SP)$ is the theory $\mathcal{I}$ consisting of the trivial equations. The greatest theory is the theory generated by the equation $x = y$ (the set of all equations).

Let e be an equation of the form $v = w$. Alternatively, we denote this equation as (v, w) and write

$$(8) \quad e = (v, w).$$

We define $m(e)$ to be the least of the integers $|v|_x, |w|_x$, where x runs over the set of those $x \in \text{supp}(v) \cup \text{supp}(w)$ for which $|v|_x \neq |w|_x$. If this set is empty, we put $m(e) = \infty$. For a theory T , by $m(T)$ we denote the least of the integers $m(e)$, with $e \in T$. Similarly, by $r(e)$ we denote the least of the integers $||v|_x - |w|_x|$, where x runs over the same set as above. Again, if this set is empty, then we denote $r(e) = \infty$. For a theory T , by $r(T)$ we denote the least of the integers $r(e)$, with $e \in T$.

Now, let $n = n(T)$ be the last number such that for all $\sigma \in S_n$, the equation of the form (7) belongs to a theory $T \in \mathcal{L}(SP)$. Then, by T_n , we denote the union of the set of all equations of the form (6) such that $|v| \geq n$ and $|w| \geq n$ with the set of all the equations that belong to the theory $\mathcal{I}$. It is easy to see that the set T_n is a

theory that belongs to $\mathcal{L}(SP)$. Moreover, it is obvious that the theory T_n belongs to $\mathcal{L}(Com)$. Hence, there are parameters: an order filter $J_n \subseteq \Gamma$, $m \in \mathbb{N}_+$, $r \in \mathbb{N}$, and an equivalence relation π_n on the set $\Gamma_+ \setminus J_n$ such that $T_n = E(J_n, m, r, \pi_n)$. The set J_n we may identify with the order filter of those words $v(x_1, \dots, x_k) \in \mathcal{W}$ such that $(|v|_{x_1}, \dots, |v|_{x_k}) \in J_n$. It may happen that $T_n = \mathcal{I}$. In this situation, we have $J_n = \emptyset$ and $\pi = \mathcal{I}$. Then we will write $m = \infty$ and $r = \infty$.

Lemma 4.1. *Let $T \in \mathcal{L}(SP)$ and $T_n = E(J_n, m, r, \pi_n)$. Then we have the following:*

- (i) $m(T) = m$,
- (ii) $r(T) = r$.

Moreover, r divides $r(e)$ for every equation $e \in T$.

Proof. We have to show that for the equation e of the form (6), if $e \in T$, then $m(e) \geq m$ and r divides $r(e)$. Let $y \notin \text{supp}(v) \cup \text{supp}(w)$, and consider the equation e_n of the form $v_n = w_n$, where $v_n = v y^n$ and $w_n = w y^n$. Obviously, the equation e_n belongs to T_n . Hence, $m(e_n) \geq m$ and r divides $r(e)$. Moreover, $m(e_n) = r(e)$ and $r(e_n) = r(e)$. This proves the lemma. $\square$

We define $J(T)$ to be the set consisting of the words v such that an equation $v = w$ belongs to T for some $w \in J_n$.

Lemma 4.2. *The set $J(T)$ is an order filter on the bqo $(\mathcal{W}, \leq_n)$.*

Proof. Let $v \in J$ and $u \geq_n v$. Then there is an operation $f \in \mathcal{O}$ such that either $f(v) = u$ or $(|f(v)|_{x_1}, \dots, |f(v)|_{x_k}) \equiv_K (|u|_{x_1}, \dots, |u|_{x_k})$ and $|u| \geq n$. In the latter case the equation $f(v) = u$ belongs to T_n . Take a word $w \in J_n$ such that an equation $v = w$ belongs to T . Obviously, equation $f(v) = f(w)$ belongs to T . Moreover, J_n is an order filter. Hence, $f(w) \in J_n$. Consequently, $u \in J$ and J is an order filter. $\square$

Let $v \in J$ and $m(v) = \max\{|v|_x : x \in \text{supp}(v)\}$. We define $m(J)$ to be $\min\{m(v) : v \in J\}$.

Lemma 4.3. *Let $T \in \mathcal{A}$. Then $m(J(T)) \geq m(T)$.*

Proof. Let $v \in J(T)$ be a word such that $m(v) = m(J(T))$. If $v \in J_{n(T)}(T)$, then $m(v) \geq m(T)$, by assumption on $J_{n(T)}(T)$. If $v \notin J_{n(T)}(T)$, then there is $w \in J_{n(T)}(T)$ such that the equation $v = w$ belongs to T . Moreover, there is a variable $x \in \text{supp}(v) \cup \text{supp}(w)$ such that $|v|_x \neq |w|_x$. Hence, $|v|_x \geq m(T)$. Obviously, $|v|_x \leq m(v)$. This implies the statement. $\square$

Obviously, if $v \in J(T)$ and an equation $w = v$ belongs to T , then $w \in J(T)$. By Theorem 3.3, we know that $J(T)$ is generated by a finite set of words.

For the equation e of the form (6), we consider two conditions:

- (J1) $m(e) \geq m$,
- (J2) $r(e)$ is divisible by r .

It is easy to observe that the conditions (J1) and (J2) generalize the corresponding conditions from subsection 2.2. In the sequel, every reference to (J1) and (J2) concerns conditions formulated above.

Lemma 4.4. *Let $v, w \in J$. Then an equation e of the form $v = w$ belongs to T if and only if e satisfies the conditions (J1) and (J2) with $m = m(T)$ and $r = r(T)$.*

Proof. The “only if” part is an immediate consequence of Lemma 4.1. For the “if” part, we assume that e satisfies the conditions (J1) and (J2). By assumption, there are words $v_n, w_n \in J_n$ such that equations $v = v_n$ and $w = w_n$ belong to T . Let $x \in \text{supp}(v_n) \cup \text{supp}(w_n)$ be such that $|v_n|_x \neq |w_n|_x$. Then, either $|v_n|_x \neq |v|_x$ or $|v|_x \neq |w|_x$, or else $|w|_x \neq |w_n|_x$. In all these cases $\min\{|v|_x, |w|_x, |v_n|_x, |w_n|_x\} \geq m$, and obviously, r divides the numbers $|v_n|_x - |v|_x$, $|v|_x - |w|_x$, and $|w|_x - |w_n|_x$. Hence, r divides $|v_n|_x - |w_n|_x = |v_n|_x - |v|_x + |v|_x - |w|_x + |w|_x - |w_n|_x$. This implies that the equation $v_n = w_n$ satisfies the conditions (J1) and (J2). By Remark 2.9, the equation $v_n = w_n$ belongs to T . Consequently, the equation e belongs to T . $\square$

If a word $x_1 \cdots x_k$ belongs to the filter J , then by Lemma 4.2, for every $\sigma \in S_k$, $x_{\sigma(1)} \cdots x_{\sigma_k}$ belongs to J , too. Moreover, it is easy to see that, for every $\sigma \in S_k$, the equation $x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)}$ belongs to T . In addition, by Lemma 4.4, we have the following.

Theorem 4.5. *For every $T \in \mathcal{L}(SP)$, the word $x_1 \cdots x_{n(T)-1}$ does not belong to $J(T)$.*

For the parameters n, J, m, r , where $m \in \mathbb{N}$, $n, r \in \mathbb{N}_+$ and J is an order filter on the bqo $(W, \leq_n)$ such that $m(J) \geq m$ and no word $x_1 \cdots x_{n-1}$ belongs to J , we define an equivalence relation S on the set $\mathcal{L}(SP)$ as follows. A theory T belongs to the class $S(n, J, m, r)$ if $n(T) = n$, $J(T) = J$, $m(T) = m$, $r(T) = r$.

By $T(n, J, m, r)$ we denote the least element of the class $S(n, J, m, r)$.

Theorem 4.6. *Let $n > 0$, $m \geq 0$, $r > 0$ and J be an order filter on bqo $(W, \leq_n)$ such that $m(J) \geq m$, and the word $x_1 \cdots x_{n-1}$ does not belong to J . Then theory $T(n, J, m, r)$ consists of those equations e of the form $v = w$ for which either v and w belong to J , $m(e) \geq m$ and r divides $r(e)$ or $v = y_1 \cdots y_t$ and $w = y_{\sigma(1)} \cdots y_{\sigma(t)}$, with $t \geq n$, and $\sigma \in S_t$.*

Note that $y_1, y_2, \dots, y_n$ above need not be different.

Proof. Let $T^*(n, J, m, r)$ denote a set consisting of those equations e of the form $v = w$ for which either v and w belong to J , $m(e) \geq m$, and r divides $r(e)$ or $v = y_1 \cdots y_t$, and $w = y_{\sigma(1)} \cdots y_{\sigma(t)}$, where $t \geq n$, $\sigma \in S_t$. It is easy to see that $T^*(n, J, m, r)$ is a theory. We show that $T^*(n, J, m, r) = T(n, J, m, r)$. Obviously, $T^*(n, J, m, r) \in S(n, J, m, r)$. We have to show that $T^*(n, J, m, r)$ is the least element in this class. Since $n(T^*(n, J, m, r)) = n$, all the equations $y_1 \cdots y_t = y_{\sigma(1)} \cdots y_{\sigma(t)}$, where $t \geq n$, $\sigma \in S_t$ belong to $T^*(n, J, m, r)$. By Lemma 4.4, all equations e of the form $v = w$ such that $v, w \in J$, $m(e) \geq m$ and r divides $r(e)$ belong to $T^*(n, J, m, r)$. Hence $T^*(n, J, m, r)$ is the least element in $S(n, J, m, r)$ and $T^*(n, J, m, r) = T(n, J, m, r)$. $\square$

For a subset M of bqo $(W, \leq_n)$, we define $Cl_n(M) = M \cup \{y_{\sigma(1)} \cdots y_{\sigma(k)}; y_1 \cdots y_k \in M, k \geq n, \sigma \in S_k\}$. By $\mathcal{L}_S(SP)$ we denote the set of the theories of the form $T(n, J, m, r)$.

Corollary 4.7. *Let $T_1 = T(n_1, J_1, m_1, r_1)$ and $T_2 = T(n_2, J_2, m_2, r_2)$. Then the following hold:*

- (i) $T_1 \supseteq T_2$ if and only if $n_1 \leq n_2$, $J_1 \supseteq J_2$, $m_1 \leq m_2$, and r_1 divides r_2 ,
- (ii) $T_1 \vee T_2 = T(n, J, m, r)$, where $n = \min(n_1, n_2)$, $J = Cl_n(J_1 \cup J_2)$, $m = \min(m_1, m_2)$, and $r = \gcd(r_1, r_2)$,

- (iii) *the theory $T(\max(n_1, n_2), J_1 \cap J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2))$ is the largest theory in a set $\mathcal{L}_S(SP)$ that is contained in the set $T_1 \cap T_2$.*

Proof. The statement (i) is an immediate consequence of Theorem 4.6. Using (i), we have directly that (iii) holds and that $T(n, J, m, r)$ contains the set $T_1 \cup T_2$. Assume that an equation $v = w$ belongs to $T(n, J, m, r)$. Then, either $v = y_1 \cdots y_k$ and $w = y_{\sigma(1)} \cdots y_{\sigma(k)}$ for some $k \geq n$, and $\sigma \in S_k$ or $v, w \in J$, $m(v = w) \geq m$, and r divides $r(v = w)$.

In the first case the equation $v = w$ belongs to $T_1 \cup T_2$. In the latter, we may assume that J_1 is not empty. Let $v = y_1 \cdots y_k$, $w = z_1 \cdots z_l$. If J_2 is not empty, then there are the permutations $\sigma_1 \in S_k, \sigma_2 \in S_l$ (if $k \leq n$, then σ_1 is trivial, and if $l \leq n$, then σ_2 is trivial), and the word $u \in J_1 \cap J_2$ such that the equations $u = y_{\sigma_1(1)} \cdots y_{\sigma_1(k)}$ and $u = z_{\sigma_2(1)} \cdots z_{\sigma_2(l)}$ belong to $T_1 \cap T_2$. Hence, the equation $v = w$ is a consequence of the set $T_1 \cap T_2$. If J_2 is empty, then the argument is similar. The only difference is that the word u belongs to J_1 . This implies that every equation which belongs to $T(n, J, m, r)$ is a consequence of the set of equations $T_1 \cup T_2$. This proves (ii). $\square$

An immediate consequence of Corollary 4.7 is the following.

Theorem 4.8. *The set $\mathcal{L}_S(SP)$ forms a lattice with operations $\vee$ and $\wedge$. For $T_1 = T(n_1, J_1, m_1, r_1)$, $T_2 = T(n_2, J_2, m_2, r_2)$, and $n = \min(n_1, n_2)$, we have the following:*

- (i) $T_1 \vee T_2 = T(n, Cl_n(J_1 \cup J_2), \min(m_1, m_2), \text{gcd}(r_1, r_2))$,
- (ii) $T_1 \wedge T_2 = T(\max(n_1, n_2), J_1 \cap J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2))$.

Moreover, $\mathcal{L}_S(SP)$ is $\vee$ -subsemilattice of $\mathcal{L}(SP)$.

Since sometimes it may happen that $T_1 \wedge T_2 \notin \mathcal{L}_S(SP)$, the lattice $\mathcal{L}_S(SP)$ is not a sublattice of $\mathcal{L}(SP)$.

We are going to prove that the sets $S(n, J, m, r)$ are intervals in $\mathcal{L}(SP)$. We need the following lemma.

Lemma 4.9. *Let $T_1 \in S(n_1, J_1, m_1, r_1)$ and $T_2 \in S(n_2, J_2, m_2, r_2)$. Then the following hold:*

- (i) $T_1 \wedge T_2 \in S(\max(n_1, n_2), J_1 \cap J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2))$,
- (ii) $T_1 \vee T_2 \in S(n, J, \min(m_1, m_2), \text{gcd}(r_1, r_2))$, for some $n \leq \min(n_1, n_2)$ and $J \supseteq J_1 \cup J_2$.

It may happen that $J \neq J_1 \cup J_2$ (see [13] when $T_1, T_2 \in \mathcal{L}(Com)$). Similarly, there are examples that $n \neq \min(n_1, n_2)$. Consequently, $\equiv_S$ is not a congruence relation.

Proof. Since $T_1 \wedge T_2 = T_1 \cap T_2$ and $T_1 \vee T_2 \supseteq T_1 \cup T_2$, we immediately have suitable conditions on $n(T_1 \wedge T_2)$, $n(T_1 \vee T_2)$, $J(T_1 \wedge T_2)$ and $J(T_1 \vee T_2)$. It is also obvious that $m(T_1 \cap T_2) = \max(m_1, m_2)$, $r(T_1 \cap T_2) = \text{lcm}(r_1, r_2)$. Since $m(T_1 \cup T_2) = \min(m_1, m_2)$, we have $m(T_1 \vee T_2) \leq \min(m_1, m_2)$. Assume that $e \in T_1 \vee T_2$ satisfies $m(e) = m(T_1 \vee T_2)$. Then e has a form $v_1 = v_k$ and is obtained by transitivity from the equations $v_i = v_{i+1}; i \in \{1, \dots, k-1\}$. All the equations $v_i = v_{i+1}$ belong either to T_1 or to T_2 . Let x be a variable that belongs to $\text{supp}(v_1) \cup \text{supp}(v_k)$, and let $|v_1|_x \neq |v_k|_x$. Then there is $i \in \{1, \dots, k-1\}$ such that $|v_i|_x \neq |v_{i+1}|_x$. This implies that $\min(m_1, m_2) \leq m(T_1 \vee T_2)$, and we

have the equality $m(T_1 \vee T_2) = \min(m_1, m_2)$. Similarly, one can prove the equality $r(T_1 \vee T_2) = \text{lcm}(r_1, r_2)$. $\square$

Corollary 4.10. *Every equivalence class $S(n, J, m, r)$ is an interval in $\mathcal{L}(SP)$.*

5. THE LATTICE $\mathcal{L}(SP)$

Now we supplement the objects n, J, m, r by an equivalence relation π on a set $\mathcal{W} \setminus J$. Using these five objects, we describe every single theory in $\mathcal{L}(SP)$. The equivalence relation π will play the same role as the similar equivalence relation that we define in section 2.2. The equivalence relation π will be called a remainder of the theory T .

5.1. Remainders. Before we describe the structure of the lattice $\mathcal{L}(SP)$, we give some description of the structure of remainders. First, we prove the following.

Lemma 5.1. *Let $v = w$ be the equation that belongs to T . If either*

- (a) $\text{supp}(v) \neq \text{supp}(w)$ (irregular equations)
- or
- (b) $v <_n w$, (S -equations),

then both v and w belong to $J(T)$.

Proof. We start with case (a). We assume, without loss of generality, that $x \in \text{supp}(v) \setminus \text{supp}(w)$. Since $m(T) = 0$, we know that $J(T)$ is not empty. Hence, there is a number t such that $x^t \in J(T)$. Let v' denote the word obtained from v by substituting in v the variable x by z^t , where $z \notin \text{supp}(v) \cup \text{supp}(w)$. It is obvious that $v' \in J(T)$ and the equation $v' = w$ belongs to T . It follows that $w \in J(T)$. Consequently, $v \in J(T)$. This completes the proof in case (a).

Now consider case (b). In view of (a), we may assume that $\text{supp}(v) = \text{supp}(w) = \{x_1, \dots, x_k\}$. Hence, we have, immediately, that $|v| < |w|$. Moreover, there is an operation $f \in \mathcal{O}$ such that either $f(v) = w$ or $(|f(v)|_{x_1}, \dots, |f(v)|_{x_k}) \equiv_K (|w|_{x_1}, \dots, |w|_{x_k})$. It follows that $|f^t(v)| < |f^{t+1}(v)|$, for every natural number t . Consequently, $|f^n(v)| \geq n$. Thus, the equation $f^n(v) = f^{n+1}(v)$ belongs to T_n . By Remark 2.9, $f^n(v) \in J_n(T) \subseteq J(T)$. By transitivity of equality, the equation $v = f^n(v)$ belongs to T . This implies that $v \in J(T)$ and $w \in J(T)$, and completes the proof of the lemma. $\square$

Let us define $\pi(T)$ to be the set

$$(T \setminus T(n(T), J(T), m(T), r(T))) \cup \{v = v : v \in \mathcal{W} \setminus J(T)\}.$$

Thus, $T = T(n(T), J(T), m(T), r(T)) \cup \pi(T)$. As in the commutative case, π is a relatively small part of T , and similarly, we call it a *remainder of the type (n, J, m, r)* . For an equation $e = (v, w) \in T$, either $v, w \in J(T)$ and e satisfies the conditions (J1) and (J2) or $v, w \notin J(T)$ and $e \in \pi(J)$. We establish properties that are satisfied by an equation $(v, w) \in \pi(T)$.

By Lemma 5.1, $\pi(T)$ can be viewed as an equivalence relation on the set $\mathcal{W} \setminus J(T)$.

Lemma 5.2. *Let $T \in \mathcal{L}(SP)$ and $(v, w) \in \pi(T)$. Then the following conditions hold:*

- (II1) $\text{supp}(v) = \text{supp}(w)$,
- (II2) conditions (J1) and (J2) hold,
- (II3) for every $f \in \mathcal{O}$, either $(f(v), f(w)) \in \pi(T)$ or $f(v), f(w) \in J(T)$.

Proof. The condition (II1) is an immediate consequence of Lemma 5.1. The condition (II2) holds for every equation $e \in T$. The condition (II3) is an immediate consequence of the fact that $T = T(n(T), J(T), m(T), r(T)) \cup \pi(T)$. $\square$

As a consequence of (II3) we have additional conditions.

Corollary 5.3. *For T, v, w as above, we have:*

- $(\sigma(v), \sigma(w)) \in \pi(T)$, for every $\sigma \in P$,
- if for some $x \in \text{supp}(v)$ we have $|v|_x \neq |w|_x$, $2|v|_x \geq n(T)$ and $2|w|_x \geq n$, then $\{vt^s, t^sv, wt^s, t^sw\} \subset J(T)$, where $s = \min(|v|_x, |w|_x)$.

Proof. Since $v, w \notin J$, $\sigma(v) \equiv_n v$, and $\sigma(w) \equiv_n w$, we have $\sigma(v), \sigma(w) \notin J$. Therefore, by (II3), $(\sigma(v), \sigma(w)) \in \pi$. Moreover, since $(v, w) \in T$, for substitutions we have $(t_{xz}(v), t_{xz}(w)) \in T$. Since $2|v|_x \geq n(T)$ and $2|w|_x \geq n(T)$, the following equations hold: $t_{xz}(v) = vz^{|v|_x} = z^{|v|_x}v$ and $t_{xv}(w) = wz^{|w|_x} = z^{|w|_x}w$. Using those properties, the transitivity of the equality, and the fact that $(vz^s, wz^s) \in T$ and $(z^sv, z^sw) \in T$, we have the second statement. $\square$

One may observe that, in fact, (II1) is also a consequence of (II3). However, (II1) is so important that it is worth mentioning it as a separate condition.

As in [4], we describe more precisely the structure of equivalence blocks of equivalence relations π . By Lemma 5.1, two words that belong to a block B have to be either incomparable or equivalent in the bqo $(\mathcal{W}, \leq_n)$. By Corollary 5.3, the block B is also the block for the symmetric group S_k under its action on the words with support of cardinality k . Consequently, we have an induced action of S_k on the set of blocks that consists of the words with the support of cardinality k . By G_B , as in [4], we denote the setwise stabilizer of B in S_k under this action. Since the block B consists of the words that are either incomparable or equivalent, we may choose a maximal antichain $A \subseteq B$ such that, for every $v \in B$, there is $w \in A$ and $\sigma \in S_k$ with $\sigma(w) = v$.

It is clear that the block B is determined by the antichain A and the group G_B as follows.

Lemma 5.4. *For every block B of a remainder π , if A is a maximal antichain contained in B , then $B = G_B A = \{g(v) : g \in G_B, v \in A\}$.*

It is clear that if elements of A have the support of cardinality k , then the group G_B contains all point stabilizers of elements of A in a group S_k . In fact, G_B contains point stabilizers of all elements of B . We prove additional conditions that every remainder has to satisfy.

Lemma 5.5. *Let π be a remainder of type (n, J, m, r) and B be a block of π that contains a word $x_1 \cdots x_k$, with $k < n$. Then the following holds:*

- (1) B consists of the words of the form $x_{\sigma(1)} \cdots x_{\sigma(k)}$, where $\sigma \in S_k$.
- (2) $G_B \neq S_k$.

Proof. We prove (2). Since, $\pi = T \setminus T(n(T), J(T), m(T), r(T))$ for some T with $n(T) = n$, $J(T) = J$, $m(T) = m$ and $r(T) = r$, there is $\sigma \in S_n$ such that $(x_1 \cdots x_k, x_{\sigma(1)} \cdots x_{\sigma(k)}) \notin T \supseteq \pi$.

Now we prove (1). We assume, to the contrary, that there exists a word w in B that is not in the form $x_{\sigma(1)} \cdots x_{\sigma(k)}$. Then, by (II1), $|w| > k$. Using substitutions, identification variables and transitivity, it is easy to find a word $v \in B$ such that

$|v| \geq n$. By (II1), $\text{supp}(v) = \text{supp}(w)$. Thus, we may choose v of the form $x_1^{i_1} \cdots x_k^{i_k}$, where at least one i_j is greater than 1. But, obviously, $x_1 \cdots x_k < v$. This is in contradiction with Lemma 5.1(b). $\square$

For a fixed remainder π , we may define a quasi-ordering on a set of equivalence blocks of π . We define $B_1 \leq_\pi B_2$ if and only if there is $f \in \mathcal{O}$ such that $f(B_1) \subseteq B_2$. By (II3), this definition is equivalent to the condition that there is $b \in B_1$ and $f \in \mathcal{O}$ such that $f(b) \in B_2$.

Since $(W, \leq_n)$ is bqo, using Fact 2.4, we have the following.

Corollary 5.6. *The quasi-ordering $((W \setminus J)_{/\pi}, \leq_\pi)$ is bqo.*

In the same way as in [4, Lemma 1.2], we obtain the following.

Lemma 5.7. *Let $B, C \in \pi$. Then B and C are equivalent if and only if $p(B) = C$ for some $p \in P$.*

Fix a positive integer n . Since there are only finitely many words w such that $|w| < n$ and $\text{supp}(w) = \{x_1, \dots, x_k\}$, we have only finite many blocks $B \in \pi$ such that there is $w \in B$ with $|w| < n$. Hence, by Remark 2.10, we have the following.

Corollary 5.8. *For every remainder π there is a finite set $\bar{\pi}$ of blocks of π such that every block $C \in \pi$ has a form $pl^*r^*(B)$ for some $B \in \bar{\pi}$, $p \in P$ and operations l^*, r^* of the form $r^* = (r_{y_1})^{t_1} \cdots (r_{y_k})^{t_k}$ and $l^* = (l_{z_1})^{s_1} \cdots (l_{z_j})^{s_j}$.*

5.2. Theories of $\mathcal{L}(SP)$. The following theorem describes every theory of $\mathcal{L}(SP)$ using the objects n, J, m, r, π .

Theorem 5.9. *Given integers $m \geq 0$, $r > 0$, $n \geq 2$, and an order filter J such that $m(J) = m$ and $x_1 \cdots x_{n-1} \notin J$, let π be an equivalence relation on the set $W \setminus J$ satisfying conditions (II1), (II2) and (II3) and such that for the block B with $x_1 \cdots x_{n-1} \in B$, we have $G_B \neq S_{n-1}$. Let $T(n, J, m, r, \pi) = T(n, J, m, r) \cup \pi$. Then the following hold:*

- (i) $T(n, J, m, r, \pi) \in \mathcal{L}(SP)$,
- (ii) $T(n, J, m, r, \pi) \in S(n, J, m, r)$,
- (iii) $(v, w) \in T(n, J, m, r, \pi)$ if and only if either both $v, w \in J$ and satisfy conditions (J1) and (J2) or both $v, w \notin J$ and $(v, w) \in \pi$.

Conversely, every $T \in \mathcal{L}(SP)$ other than $\mathcal{I}$ is of this form.

Proof. We prove (i). We have to show that the consequences of the equations that belong to T , themselves, belong to T . For $(v, w) \in T$ and $(w, u) \in T$ we have to show that

- (a) $(r(v), r(w)) \in T$,
- (b) $(l(v), l(w)) \in T$,
- (c) $(g_{x_i x_j}(v), g_{x_i x_j}(w)) \in T$,
- (d) $(t_{x_i y}(v), t_{x_i y}(w)) \in T$,
- (e) $(p(v), p(w)) \in T$, and
- (f) $(v, u) \in T$.

Since $T(n, J, m, r) \in \mathcal{L}(SP)$ and by the definition, if $(v', w') \in T$, then either $(v', w') \in T(n, J, m, r)$ and $v', w' \in J$ or $(v', w') \in \pi$ and $v', w' \notin J$, it is enough to check these conditions with assumptions $(v, w) \in \pi$ and $(w, u) \in \pi$.

By Corollary 5.3, (e) holds. To check (a)–(d), let f be one of the operations $r, l, g_{x_i x_j}, t_{x_i y}$. By (II3) either $(f(v), f(w)) \in \pi$ and we are done or $f(v), f(w) \in J$.

Moreover, by (II2) the conditions (J1) and (J2) hold for (v, w) . We show that they hold for $(f(v), f(w))$. This is clear for $f \in \{r, l, t_{x_i y}\}$. Let $f = g_{x_i, x_j}$. Assume $|f(v)|_{x_h} \neq |f(w)|_{x_h}$. If $h \neq i$, then $|f(v)|_{x_h} = |v|_{x_h}$ and $|f(w)|_{x_h} = |w|_{x_h}$. We have $|v|_{x_h} \geq m$, $|w|_{x_h} \geq m$, and r divides $|v|_{x_h} - |w|_{x_h}$. If $h = i$, then $|f(v)|_{x_h} = |v|_{x_i} + |v|_{x_j}$ and $|f(w)|_{x_h} = |w|_{x_i} + |w|_{x_j}$. Obviously, r divides $|f(v)|_{x_h} - |f(w)|_{x_h}$. Moreover, either $|v|_{x_i} \neq |w|_{x_i}$, $|v|_{x_i} \geq m$, and $|w|_{x_i} \geq m$ or $|v|_{x_j} \neq |w|_{x_j}$, $|v|_{x_j} \geq m$, and $|w|_{x_j} \geq m$. Therefore, (J1) and (J2) hold for $(f(v), f(w))$. Thus, $(f(v), f(w)) \in T$. This proves (i).

Since $T(n, J, m, r) \subseteq T$ and for the block B with $x_1 \cdots x_n \in B$ we have $G_B \neq S_n$, it is clear that $n(T) = n$. Moreover, $J(T) = J$, since in addition we have (II3). Finally, $m(T) = m$ and $r(T) = r$, since in addition (J1) and (J2) hold for every equation in T . This proves (ii). Now (iii) is obvious.

The converse holds by Lemma 5.2. $\square$

Since every order filter in the set $(\mathcal{W}, \leq_n)$ is finitely generated, using Corollary 5.8, we obtain the following well-known fact that we will need later.

Theorem 5.10. *Every equational theory of strongly permutative semigroups is finitely generated.*

The above is an easy consequence of the results of Perkins [18] and Putcha-Yaqub [20].

6. STRUCTURE OF $\mathcal{L}(SP)$

In this section, we describe the structure of $\mathcal{L}(SP)$ as a lattice and as an ordering.

6.1. Inclusion and the operations $\vee, \wedge$. For two theories $T_1, T_2 \in \mathcal{L}(SP)$, we describe conditions under which $T_1 \subseteq T_2$. Moreover, we determine theories $T_1 \wedge T_2$ and $T_1 \vee T_2$.

Theorem 6.1. *Let $T_1 = T(n_1, J_1, m_1, r_1, \pi_1)$ and $T_2 = T(n_2, J_2, m_2, r_2, \pi_2)$. Then $T_1 \supseteq T_2$ if and only if $n_1 \leq n_2$, $J_1 \supseteq J_2$, $m_1 \leq m_2$, r_1 divides r_2 , and $\pi_1 \cup J_1^2 \supseteq \pi_2$.*

Proof. The “only if” part is obvious. For the “if” part, since $n_1 \leq n_2$, it is enough to show that if $(v, w) \in \pi_2 \subseteq \pi_1 \cup J_1^2$, then $(v, w) \in E_1$. If $(v, w) \in \pi_1$, then this is obvious. If $(v, w) \in J_1^2$, then, since the equation (v, w) satisfies condition Π_2 with m_2 and r_2 , it satisfies Π_2 with m_1 and r_1 . Therefore, $(v, w) \in T(n_1, J_1, m_1, r_1) \subseteq T_1$, as required. $\square$

For an order filter J , $m \geq 0$, and $r > 0$, by $T(J, m, r)$ we denote a set of those (v, w) with $v, w \in J$ that satisfy conditions (J1) and (J2) with parameters m and r .

Theorem 6.2. *Let $T_1 = T(n_1, J_1, m_1, r_1, \pi_1)$ and $T_2 = T(n_2, J_2, m_2, r_2, \pi_2)$. Then, $T_1 \wedge T_2 = T_1 \cap T_2 = T(\max(n_1, n_2), J_1 \cap J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2), \pi)$, where $\pi = (\pi_1 \cap \pi_2) \cup (\pi_1 \cap T(J_2, m_2, r_2)) \cup (\pi_2 \cap T(J_1, m_1, r_1))$.*

Proof. The equality $n(T_1 \cap T_2) = \max(n_1, n_2)$ is obvious. The rest is a consequence of the fact that $(T(J_1, m_1, r_1) \cup \pi_1) \cap (T(J_2, m_2, r_2) \cup \pi_2) = (T(J_1, m_1, r_1) \cap T(J_2, m_2, r_2)) \cup (T(J_1, m_1, r_1) \cap \pi_2) \cup (T(J_2, m_2, r_2) \cap \pi_1) \cup (\pi_1 \cap \pi_2) = T(J_1 \cap J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2)) \cup (T(J_1, m_1, r_1) \cap \pi_2) \cup (T(J_2, m_2, r_2) \cap \pi_1) \cup (\pi_1 \cap \pi_2)$. $\square$

Before we formulate a similar theorem for the operation $\vee$, we give some definitions. Let $T_1 = T(n_1, J_1, m_1, r_1, \pi_1)$ and $T_2 = T(n_2, J_2, m_2, r_2, \pi_2)$. We denote $G_n(T_i) = S_n$ for $n \geq n_i$ and $G_n(T_i) = G_B$ if $n < n_i$, and $x_1 \cdots x_n$ belongs to the class B in π_i . Now, we define, inductively, $G_n(T_1 \cup T_2)$:

$$\begin{aligned} G_2(T_1 \cup T_2) &= \langle G_2(T_1), G_2(T_2) \rangle, \\ G_{k+1}(T_1 \cup T_2) &= \langle G_{k+1}(T_1), G_{k+1}(T_2), G_k(T_1) \times S_1, G_k(T_2) \times S_1, \\ &\quad S_1 \times G_k(T_1), S_1 \times G_k(T_2), G_k^i(T_1), G_k^i(T_2) \rangle, \end{aligned}$$

where $i \in \{1, \dots, k\}$ and $G_k^i(T_j)$ is a subgroup of S_{k+1} generated by permutations $\sigma^i : \sigma \in G_k(T_j)$, and σ^i rearranges the set $\{1, \dots, k+1\}$ just as σ rearranges the set $\{1, \dots, k\}$ treating the pair $i, i+1$ as a single element. See [15] for details. In fact $G_k(T_1 \cup T_2)$ is a minimal subgroup of S_k generated by all permutations of this form arising from the groups $\langle G_l(T_1), G_l(T_2) \rangle$ for all $l \leq k$.

By $n(T_1 \cup T_2)$, we denote a minimal n such that $S_n = G_n(T_1 \cup T_2)$. Obviously, $n(T_1 \vee T_2) \leq \min(n_1, n_2)$. We also consider the equivalence relation R which is the equivalence closure of the relation $J_1^2 \cup J_2^2 \cup \pi_1 \cup \pi_2$. By $J(T_1 \cup T_2)$ we denote the equivalence block in R which contains the set $J_1 \cup J_2$. Moreover, by $\pi(T_1 \cup T_2)$ we denote the equivalence relation $R \setminus (J(T_1 \cup T_2))^2$ on the set $\mathcal{W} \setminus J(T_1 \cup T_2)$.

Theorem 6.3. *Let $T_1 = T(n_1, J_1, m_1, r_1, \pi_1)$ and $T_2 = T(n_2, J_2, m_2, r_2, \pi_2)$. Then $T_1 \vee T_2 = T(n(T_1 \cup T_2), J(T_1 \cup T_2), \min(m_1, m_2), \gcd(r_1, r_2), \pi(T_1 \cup T_2))$.*

Proof. The equality $n(T_1 \vee T_2) = n(T_1 \cup T_2)$ is an immediate consequence of the definition of $n(T_1 \cup T_2)$, Lemma 5.5(1), and the fact that if the equation $x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)}$ with $k < \min(n_1, n_2)$ is a consequence of the set of equations $\{e_1, \dots, e_j\}$, then every equation e_l has a form $p_1(x_1 \cdots x_i) = p_2(x_1 \cdots x_i)$ for some $i \leq k$, and permutations $p_1, p_2 \in P$. (Otherwise, there is $x_1 \cdots x_i \in J_l$ for some $i \leq k$ and $l \in \{1, 2\}$. In this situation $n(T_l) \leq i$, which contradicts the fact that $i \leq k < n(T_l)$.)

The equalities $J(T_1 \vee T_2) = J(T_1 \cup T_2)$ and $\pi(T_1 \vee T_2) = \pi(T_1 \cup T_2)$ are consequences of the fact that taking transitivity of the equations is the same as taking the transitive closure of the relation on equations and the fact that the relation $J_1^2 \cup J_2^2 \cup \pi_1 \cup \pi_2$ is symmetric and reflective and that these properties are preserved by transitive closure. The equality $m(T_1 \vee T_2) = \min(m_1, m_2)$ holds by the fact that we have $m(w, v) \geq \min(m(w, u), m(u, v))$ for every $w, v, u \in W$. Finally, we prove the equality $r(T_1 \vee T_2) = \gcd(r_1, r_2)$.

The inequality $r(T_1 \vee T_2) \geq \gcd(r_1, r_2)$ holds by the fact that we have $m(w, v) \geq \gcd(m(w, u), m(u, v))$ for every $w, v, u \in W$. If either $r_1 = \infty$ or $r_2 = \infty$, then the opposite inequality is obvious. Assume that both $r_1 < \infty$ and $r_2 < \infty$ hold. Then, there is k such that the equation $x^k = x^{k+r_1}$ belongs to T_1 and the equation $x_k = x^{k+r_1}$ belongs to T_2 . It is easy to see that the equation $x^k = x^{k+\gcd(r_1, r_2)}$ belongs to $T_1 \vee T_2$. This proves the second inequality and completes the proof. $\square$

6.2. Covering relation. In this section we describe, in terms of the objects n, J, m, r, π , the covering relation in the lattice $\mathcal{L}(SP)$. Let $T \in \mathcal{L}(SP)$. We say that T_1 is a *cover* of T in $\mathcal{L}(SP)$ if $T_1 > T$ and there is no T_2 such that $T_1 > T_2 > T$. In this case, we also say that T is a *dual cover* of T_1 .

For an antichain A of minimal words in J that have the same n variables, we say that A is an (n, m, r) -*separable* form J if for every $v, w \in A$ there is $p \in S_n$ such that the conditions (J1) and (J2) are satisfied for the pair $(v, p(w))$. Then, by π_A , we

denote the set of all those pairs (v, w) such that $p(v), q(w) \in A$ for some $p, q \in P$, and the conditions (J1) and (J2) are satisfied. Let $\{x_1 \cdots x_n\}$ be a one-element antichain.

Theorem 6.4. *Let $T_1 = T(n_1, J_1, m_1, r_1, \pi_1)$ and $T_2 = T(n_2, J_2, m_2, r_2, \pi_2)$ be equational theories of strongly permutative semigroups. Then T_2 is a dual cover of T_1 if and only if one of the following holds:*

- (1) $n_2 = n_1 + 1$, $J_2 = J_1$, $m_2 = m_1$, $r_2 = r_1$, and

$$\pi_2 = (\pi_1 \setminus \{pS_n\{(x_1 \cdots x_n)\} : p \in P\}) \cup \{pG\{(x_1 \cdots x_n)\} : p \in P\},$$
 where G is a co-atom in S_n .
- (2) $n_2 = n_1$, $J_2 = J_1 \setminus \bigcup_{p \in P} pA$, $m_2 = m_1$, $r_2 = r_1$, and $\pi_2 = \pi_1 \cup \pi_A$, where $A \subset J_1$ is an (n_1, m_1, r_1) -separable antichain.
- (3) $n_2 = n_1$, $J_2 = J_1$, $m_2 = m_1 + 1$, $r_2 = r_1$, and $\pi_2 = \pi_1$.
- (4) $n_2 = n_1$, $J_2 = J_1$, $m_2 = m_1$, $r_2 = r_1 q$, and $\pi_2 = \pi_1$, where q is a prime.
- (5) $n_2 = n_1$, $J_2 = J_1$, $m_2 = m_1$, $r_2 = r_1$, and either
 - (a) $\pi_2 = (\pi_1 \setminus \{p(GA) : p \in P\}) \cup \{p(HA) : p \in P\}$ or
 - (b) $\pi_2 = (\pi_1 \setminus \{p(GA) : p \in P\}) \cup \{p(GA_1) \cup p(GA_2) : p \in P\},$
 where $G, H \subseteq S_n$, H is a dual cover of G , A, A_1, A_2 are antichains such that $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, and n is a number of variables of elements of A .

Proof. We prove the “if” part. It is obvious that in every case $T_2 < T_1$. Let $T = T(n, J, m, r, \pi)$ be such that $T_2 \leq T \leq T_1$. We must show that either $T = T_1$ or $T = T_2$. This is obvious in cases (3) and (4).

In case (1), we have $J = J_1 = J_2$, $m = m_1 = m_2$, $r = r_1 = r_2$. If $n = n_1$, then the set $\{p(x_1 \cdots x_n) : p \in S_n\}$ has to be a subset of a block of π_1 . Therefore, $T = T_1$. If $n = n_2$, then the set $\{p(x_1 \cdots x_n) : p \in S_n\}$ cannot be a subset of a block of π_1 . This is possible only if $T = T_2$.

In case (2), we have $n = n_1 = n_2$, $m = m_1 = m_2$, $r = r_1 = r_2$. Assume first that there exists $u \in A \setminus J$. Since A is (n_1, m_1, r_1) -separable from J_1 , for every pair $v, w \in A$ there is $p \in S_n$ such that $(v, p(w)) \in \pi_A \subseteq \pi_2$. Since $T \supseteq T_2$ and $u \notin J$, we have $A \cap J = \emptyset$. Therefore $J = J_2$. Then it is easy to see that $\pi = \pi_2$. Hence, $T = T_2$. If $J = J_1$, then it is clear that $\pi = \pi_1$, and hence $T = T_1$.

In case (5), we have $n = n_1 = n_2$, $J = J_1 = J_2$, $m = m_1 = m_2$, $r = r_1 = r_2$. In both subcases, it is clear that if $\pi > \pi_2$, then $\pi = \pi_1$, and $T = T_1$. Otherwise, $T = T_2$.

It remains to show the “only if” part. First we show that if either $m_2 \neq m_1$ or $r_2 \neq r_1$, then we have cases (3) or (4), respectively.

Indeed, if $m_2 > m_1$, then $T = T(n_2, J_2, m_2 - 1, r_2, \pi_2)$ is a well-defined theory and $E_2 < E \leq E_1$. Hence, $E = E_1$, and we have case (3).

If $r_2 < r_1$, then, similar to the case above, a set $T = T(n_2, J_2, m_2, r_2/q, \pi_2)$ is a well-defined theory such that $T_1 < T \leq T_2$ (where q is a prime dividing r_1/r_2). Thus, $T = T_1$ and case (4) holds.

Now, suppose that $m_2 = m_1$, $r_2 = r_1$, and $n_2 > n_1$. We show that this must be case (1). But before we do this, we need the following lemma.

Lemma 6.5. *Under the condition above, $J_2 = J_1$.*

Proof. Assume to the contrary that $J_1 \neq J_2$. First, we show that the word $x_1 \cdots x_{n_2-1}$ does not belong to J_1 . Otherwise, let n be the smallest such that

the word $x_1 \cdots x_n$ belongs to J_1 . The one-element set $\{x_1 \cdots x_n\}$ is (n_1, m_1, r_1) -separable from J_1 . Moreover, if $v \leq x_1 \cdots x_n$, then $v = y_1 \cdots y_k$, where $k \leq n$ and $y_i \neq y_j$ for every i and j . Therefore, $T_3 = T(n_1, J_3, m_1, r_1, \pi_3)$, where $J_3 = J_1 \setminus \{p\{x_1 \cdots x_n\} : p \in P\}$ and $\pi_3 = \pi_1 \cup \{pS_n\{x_1 \cdots x_n\} : p \in P\}$ is a well-defined theory and is a dual cover of T_1 . Since $x_1 \cdots x_n \notin J_2$, we have $T_3 \geq T_2$. Hence, $T_3 = T_2$. This contradicts the assumption $n_1 \neq n_2$.

Now we consider a set $T_4 = T(n_2, J_1, m_2, r_2, \pi_4)$, where $\pi_4 = \pi_2 \setminus J_1^2$. We show that $T_4 \in \mathcal{L}(SP)$. Since $m_2 = m_1$ and $x_1 \cdots x_{n_2} \notin J_1$, conditions on J_1 are satisfied. We check conditions (II1) – (II3). (II1) is obvious. Since $\pi_4 \subseteq \pi_2$ and the parameters m and r are the same, (II2) holds for π_4 . We check (II3). Let $f \in \mathcal{O}$. It is enough to show that if $(v, w) \in \pi_4$ and $f(v) \in J_1$, then $f(w) \in J_1$. It is clear by the fact that $T_1 > T_2$ and $\pi_4 \subseteq \pi_2$. Thus, $T_4 \in \mathcal{L}(SP)$. It is clear that $T_1 \geq T_4 \geq T_2$. Since $n_1 > n_2$, $T_1 > T_4$. Hence, $T_4 = T_2$. Thus, $J_1 = J_2$. This completes the proof of the lemma. $\square$

Let $B \in \pi_2$ be a block containing the word $x_1 \cdots x_{n_1}$. We consider an equivalence relation π_5 defined as follows: $\pi_5 = \pi_2 \cup \{C : C \in \pi_1, C >_\pi B\}$. It is easy to verify that π_5 is a remainder of the type $(n_1 + 1, J_2, m_2, r_2)$ and $\pi_1 \supseteq \pi_5 \supseteq \pi_2$. Let $T_5 = T(n_1 + 1, J_2, m_2, r_2, \pi_5)$. Obviously, $T_1 \geq T_5 \geq T_2$ and $T_5 \neq T_1$. Thus, $T_5 = T_2$. Consequently, $n_2 = n_1 + 1$. Moreover, π_2 and π_1 coincide on all blocks $\leq_\pi$ greater than B .

We consider an equivalence relation π_6 defined as follows:

$$\pi_6 = (\pi_2 \setminus \{pB : p \in P\}) \cup \{pS_{n_1}\{x_1 \cdots x_{n_1}\} : p \in P\}.$$

It is easy to verify that π_6 is a remainder of the type (n_1, J_1, m_1, r_1) and $\pi_1 \supseteq \pi_6 \supseteq \pi_2$. Let $T_6 = T(n_1, J_1, m_1, r_1, \pi_6)$. Obviously, $T_1 \geq T_6 \geq T_2$ and $T_6 \neq T_2$. Thus, $T_6 = T_1$. Consequently, π_2 coincides with π_1 on all blocks that are not $\leq_\pi$ -equivalent to B .

Let G be a maximal subgroup of S_{n_1} such that $G_B \subseteq G$. We consider an equivalence relation π_7 defined as follows:

$$\pi_7 = (\pi_2 \setminus \{pB : p \in P\}) \cup \{pG\{x_1 \cdots x_{n_1}\} : p \in P\}.$$

It is easy to verify that π_7 is a remainder of the type (n_2, J_2, m_2, r_2) and $\pi_1 \supseteq \pi_7 \supseteq \pi_2$. Let $T_7 = T(n_2, J_2, m_2, r_2, \pi_7)$. Obviously, $T_1 > T_7 \geq T_2$. Thus, $T_7 = T_2$. Consequently, $G = G_B$, and we have case (1).

Now suppose that $n_2 = n_1$, $m_2 = m_1$, $r_2 = r_1$, and $J_2 \neq J_1$. We show that this must be case (2). Let $\pi_8 = \pi_2 \setminus J_1^2$. It is easy to see that π_8 is a remainder of the type (n_1, J_1, m_1, r_1) . Let $T_8 = T(n_1, J_1, m_1, r_1, \pi_8)$ and then, $T_1 \geq T_8 > T_2$. Thus, $T_8 = T_1$. Therefore, $\pi_1 = \pi_2 \setminus J_1^2$.

Let $B \in \pi_2$ be a minimal block that is not a block in π_1 . We consider an equivalence relation $\pi_9 = \pi_1 \cup \{pB : p \in P\}$. Let $J_9 = J_1 \setminus \{v : \text{there is } p \in P \text{ such that } p(v) \in B\}$. Since $J_2 \subseteq J_9 \subseteq J_1$, $m_1 = m_2$, and $n_1 = n_2$, the conditions on J_9 are satisfied. We prove that π_9 is a remainder of the type (n_1, J_1, m_1, r_1) . Conditions (II1) and (II2) are obvious. We check (II3). Assume $(v, w) \in \pi_9$ and $f \in \mathcal{O}$. Since $\pi_9 \supseteq \pi_2 \setminus J_9^2$, it is enough to show that if $f(v) \in J_9$, then $f(w) \in J_9$. If $f(v) \in J_9$, then $f(v) \in J_1$. Hence, $f(w) \in J_1$. Moreover, $f(v) \notin pB$ for any $p \in P$. Therefore, $f(w) \notin pB$ for any $p \in P$. Hence, $f(w) \in J_9$. Let $T_9 = T(n_1, J_9, m_1, r_1, \pi_9)$. We have $T_1 > T_2 \geq T_2$. Hence, $T_9 = T_1$.

Let A be an antichain in B . It is clear that all elements of A are minimal in J_1 . Since $n_1 = n_2$, $m_1 = m_2$, $r_1 = r_2$, and A is the subset of a block in π_2 , A is (n_1, m_1, r_1) -separable from J_1 . Let $T_{10} = T(n_2, J_2, m_2, r_2, \pi_{10})$, where $\pi_{10} = \pi_1 \cup \pi_A$. Then T_{10} is a dual cover of T_1 . It is easy to see that $T_{10} \supseteq T_2$. Thus, $T_{10} = T_2$, and we have case (2).

We show that the remaining cases ($n_1 = n_2$, $J_1 = J_2$, $m_1 = m_2$, and $r_1 = r_2$) lead to case (5). The idea of the proof is the same as in [4, Section 3].

Let $B \in \pi_1$ be minimal such that B is not a block in π_2 . Let

$$\pi_{11} = (\pi_1 \setminus \{pB : p \in P\}) \cup \{pC : C \in \pi_2, C \subseteq B, p \in P\}.$$

We leave it to the reader to check that π_{11} is a remainder of the form (n_1, J_1, m_1, r_1) (see [4, Lemma 3.2]). Since $\pi_1 \supseteq \pi_{11} \supseteq \pi_2$ and $\pi_{11} \neq \pi_1$, we have $\pi_{11} = \pi_2$. Consequently, π_2 differs from π_1 only within the blocks that are equivalent to B . We show that this difference can be only of the kind given in (5)(a) or (5)(b).

Let $C \in \pi_2$ such that $C \subseteq B$. We define $C^B = G_B C$. Let

$$\pi_{12} = (\pi_2 \setminus \{pC : C \in \pi_2, C \subseteq B, p \in P\}) \cup \{pC^B : C \subseteq B, p \in P\}.$$

Again, we leave it to the reader to check that π_{12} is a remainder of the form (n_1, J_1, m_1, r_1) (see [4, Lemma 3.4]). Since $\pi_1 \supseteq \pi_{12} \supseteq \pi_2$, we have either $\pi_{12} = \pi_1$ or $\pi_{12} = \pi_2$. We show that these two possibilities lead to the possibilities (5)(a) and (5)(b), respectively.

First, assume that $\pi_{12} = \pi_1$. Let $C \in \pi_2$ be such that $C \subseteq B$. Let G be a maximal subgroup of G_B that contains G_C . We define

$$\pi_{13} = (\pi_1 \setminus \{pB : p \in P\}) \cup \{pGC : p \in P\}.$$

It is easy to see that π_{13} is a remainder of the form (n_1, J_1, m_1, r_1) and $\pi_1 \supseteq \pi_{13} \supseteq \pi_2$. Since $\pi_{13} \neq \pi_1$, we have $\pi_{13} = \pi_2$, and we have case (5)(a).

Now we assume that $\pi_{12} = \pi_2$. Let $C \in \pi_2$ such that $C \subseteq B$. We have $C^B = C$. Hence, $G_C = G_B$ and $B = C_1 \cup C_n$, where $n > 1$. Let A^i be a maximal antichain in C_i . It is easy to see that $\bigcup_{i=1}^n A_i$ is a maximal antichain in B . Define $A_1 = A^1$ and $A_2 = \bigcup_{i=2}^n A^i$. We define π_{14} as follows:

$$\pi_{14} = (\pi_1 \setminus \{pB : p \in P\}) \cup \{pG_B A_1 : p \in P\} \cup \{pG_B A_2 : p \in P\}.$$

It is not difficult to see that π_{14} is a remainder of the form (n_1, J_1, m_1, r_1) . Moreover, $\pi_1 \supseteq \pi_{14} \supseteq \pi_2$ and $\pi_{14} \neq \pi_1$. Therefore, $\pi_{14} = \pi_2$, and we have cases (5)(b). This completes the proof of the theorem. $\square$

7. FIRST-ORDER DEFINABILITY

The second part of the paper is devoted to showing one possible application of the description of the lattice $L(SP)$ developed in the previous sections. We prove that every equational theory of strongly permutative semigroups is individually definable (Theorem 10.9). We start with the following.

Theorem 7.1. *The set $\mathcal{L}(Com)$ is definable in $\mathcal{L}(SP)$.*

Proof. To this end, we have a look at the upper part of the lattice $L(SP)$. The greatest element in $\mathcal{L}(SP)$ is theory $T(2, [(1)], 0, 1, \emptyset)$ of all equations. There are the following co-atoms in $\mathcal{L}(SP)$:

- $S = T(2, [(x)], 1, 1, \emptyset)$, the theory of semilattices;

- $A_p = T(2, [(x)], 0, p, \emptyset)$, the theory of Abelian groups of prime exponents, for all primes $p \geq 2$;
- $N = T(2, [(xy)], 0, 1, \Theta)$ of semigroups with zero multiplication, defined by $xy = uz$.

Obviously the set of co-atoms is definable.

Let M denote an equational theory of strongly permutative semigroups generated by the equation $xy = yx$. This is the smallest theory that belongs to $L(Com)$. It is well known (see [3], for instance) that the theory M is a meeting of all theories A_p . Since $M \subset S \cap N$, M is a meeting of all atoms of $\mathcal{L}(SP)$. Thus, M is definable.

$L(Com)$ is the set of all theories of $\mathcal{L}(SP)$ that are greater than or equal to M . $\square$

An immediate consequence is

Corollary 7.2. *Every set of theories and every individual theory that is definable in $\mathcal{L}(Com)$ is definable in $\mathcal{L}(SP)$, too.* $\square$

A theory T is what we call a *balanced* theory if it consists of the equations of the form $y_1 \cdots y_k = y_{\sigma(1)} \cdots y_{\sigma(k)}$, where y_i are not necessarily different. In other words an equation is balanced if every letter occurs in both its sides the same number of times.

Corollary 7.3. *The set $\mathcal{L}(Bal)$ of strongly permutative balanced theories is definable in $\mathcal{L}(SP)$.*

Proof. M is the greatest balanced theory. Since M is definable, the set of balanced theories $\mathcal{L}(Bal)$ is definable as a set of theories that are less than or equal to M . $\square$

In addition, we have

Corollary 7.4. *The parameters m and r are definable.*

Proof. These parameters are definable in $L(Com)$ (see [13]). Now, theories that have given m and r are those theories T where the theory $T \vee M$ has parameters m and r . $\square$

8. DEFINABILITY OF THE PARAMETER n

In this section, we show that, for a given $n \geq 2$, the set of the theories $T \in \mathcal{L}(SP)$, such that $n(T) = n$ is definable.

Note that if T is a balanced theory, then we have $J(T) = \emptyset$, $m(T) = \infty$, and $r(T) = \infty$. This means, in particular, that every dual-cover of a given balanced theory has the form as in Theorem 6.4(1) or (5). We recall that for $G < S_n$ and variables $y_1, \dots, y_n$ (not necessarily different), by $G\{y_1 \cdots y_n\}$ we denote the block $\{y_{g(1)} \cdots y_{g(n)} : g \in G\}$. We say that the block K generates a theory T if T is generated by all the equations (v, w) , where $v, w \in K$. In the sequel, we assume that the variables x_i and x_j , where $i \neq j$, are different unless we say otherwise. By $Bal(n)$, we denote the theory generated by the block $S_n\{x_1 \cdots x_n\}$ (when, recall, $\{x_1 \cdots x_n\}$ is a one-element antichain).

The greatest balanced theory is $M = Bal(2)$. It consists of all the balanced equations and is generated by the equation $xy = yx$. It has the only one co-atom $Bal(3)$, which is obviously definable.

We wish to show that all the theories $Bal(n)$ are definable. We do it inductively. We know that theories $Bal(2)$ and $Bal(3)$ are definable. Assume that $Bal(n)$, $n \geq 3$, is definable. We show that $Bal(n+1)$ is definable. Dual covers of $Bal(n)$ are theories B_G generated by the blocks:

- $G\{x_1 \cdots x_n\}$,
- $S_n\{w\}$, where w is $\leq_n$ -equivalent with $x_1^2 x_2 \cdots x_{n-1}$,
- $S_{n+1}\{x_1 \cdots x_{n+1}\}$,

where G is a maximal subgroup of S_n . Obviously, the set of dual covers of $Bal(n)$ is definable.

It is well known (see [2], for instance) that S_n has the following maximal subgroups:

- $S_I \times S_J$, where I, J is a partition of the set $\{1, \dots, n\}$ such that $|I| \neq |J|$,
- $S_k Wr S_t$, for a partition $I_1, \dots, I_k$ of the set $\{1, \dots, n\}$ such that $|I_i| = t$ for every $1 \leq i \leq k$ and $kt = n$,
- a maximal primitive subgroup of S_n .

As usual, by A_n we denote the alternating group consisting of even permutations of the set $\{1, \dots, n\}$. We say that a group $G \subseteq S_n$ has an *inner fixed point interval* if there is $1 < i < n$ such that G preserves the sets $\{1, \dots, i-1\}$ and $\{i+1, \dots, n\}$, and moreover, the action of G is nontrivial on each of these sets.

The following result was proved in [19] and also in [15] with another proof. We prefer the formulation from [15]. Unfortunately, it contains a small gap. The correct formulation is the following.

Theorem 8.1 ([19, 15]). *Let T be a balanced theory such that $G\{x_1 \cdots x_n\}$ is a block containing $x_1 \cdots x_n$ and $H\{x_1 \cdots x_{n+1}\}$ is a block containing $x_1 \cdots x_{n+1}$. Then $H = S_{n+1}$ or $H = A_{n+1}$ if and only if G has no inner fixed point intervals and is not a subgroup of S_5 generated by a permutation $(1, 4)(2, 5)$. In such a case $H = A_{n+1}$ if and only if the set of odd numbers is fixed by G and $G \subseteq A_n$.*

Observe that the subgroup of S_5 generated by the permutation $(1, 4)(2, 5)$ is not maximal. Moreover, the permutation group A_n contains a permutation σ such that $\sigma(k) - k$ is odd for some k . In addition, every other maximal subgroup of S_n contains an odd permutation. Therefore, by Theorem 8.1, we obtain that, except for the permutation groups $S_{\{1\}} \times S_{\{2, \dots, n\}}$ and $S_{\{1, \dots, n-1\}} \times S_{\{n\}}$, the equations generated by the block $S_{n+1}\{x_1 \cdots x_{n+1}\}$ are consequences of the equations generated by the block $G\{x_1 \cdots x_n\}$ for a maximal subgroup G of S_n . In the two exceptional cases the consequences are equations generated by the blocks $S_{\{1\}} \times S_{\{2, \dots, n+1\}}\{x_1 \cdots x_{n+1}\}$ and $S_{\{1, \dots, n\}} \times S_{\{n+1\}}\{x_1 \cdots x_{n+1}\}$, respectively.

By the remark above, if G is neither $S_{\{1\}} \times S_{\{2, \dots, n\}}$ nor $S_{\{1, \dots, n-1\}} \times S_{\{n\}}$, then a theory B_G has at most one dual cover that is not contained in another theory of the form B_H . If there is such a dual cover, then it is generated by the blocks listed below:

- $G\{x_1 \cdots x_n\}$,
- $S_{n+1}\{x_1 \cdots x_{n+1}\}$,
and for $n > 3$
- $S_n\{w\}$, where w is $\leq_n$ -equivalent either to the word $x_1^3 x_2 \cdots x_{n-2}$ or to the word $x_1^2 x_2^2 x_3 \cdots x_{n-2}$.

We denote this dual cover C_G .

For $G \in \{S_{\{1\}} \times S_{\{2, \dots, n\}}, S_{\{1, \dots, n-1\}} \times S_{\{n\}}\}$ there are two dual covers of B_G that are not contained in another theory of the form B_H . One is as in the case above, and the second is generated by the blocks:

- $G\{x_1 \cdots x_n\}$,
- $S_n\{w\}$, where w is $\leq_n$ -equivalent to the word $x_1^2 x_2 \cdots x_{n-1}$,
- $S_{n+2}\{x_1 \cdots x_{n+2}\}$.

We denote it D_G .

This implies that the sets

$$\{B_G : G \in \{S_{\{1\}} \times S_{\{2, \dots, n\}}, S_{\{1, \dots, n-1\}} \times S_{\{n\}}\}\}$$

and

$$\{B_G : G \notin \{S_{\{1\}} \times S_{\{2, \dots, n\}}, S_{\{1, \dots, n-1\}} \times S_{\{n\}}\}\}$$

are definable.

We show that, in both cases where $G \in \{S_{\{1\}} \times S_{\{2, \dots, n\}}, S_{\{1, \dots, n-1\}} \times S_{\{n\}}\}$, we can distinguish between C_G and D_G . Indeed, in both of these cases D_G has two dual covers that are not under B_H for $H \neq G$. We list them for the group $S_{\{1\}} \times S_{\{2, \dots, n\}}$ (the second case is analogous):

- (1) the theory generated by the blocks:
 - $G\{x_1 \cdots x_n\}$,
 - $S_n\{w\}$, where w is $\leq_n$ -equivalent to the word $x_1^2 x_2 \cdots x_{n-1}$,
 - $S_{n+3}\{x_1 \cdots x_{n+3}\}$;
- (2) the theory $F_{S_{\{1\}} \times S_{\{2, \dots, n\}}}$ generated by the blocks:
 - $G\{x_1 \cdots x_n\}$,
 - $S_{n+2}\{x_1 \cdots x_{n+2}\}$,
 - $S_{\{1\}} \times S_{\{2, \dots, n+1\}}\{w\}$, where w is $\leq_n$ -equivalent to the word $x_1 x_2^2 \cdots x_n$, and for $n > 3$,
 - $S_n\{w\}$, where w is $\leq_n$ -equivalent either to the word $x_1^3 x_2 \cdots x_{n-2}$ or with the word $x_1^2 x_2^2 x_3 \cdots x_{n-2}$.

For theories C_G , we have two cases. For $n = 3$, C_G has the only one dual cover that is not under B_H for $H \neq G$. This is F_G . For $n > 3$, C_G has three of them. Again, we list them for the group $S_{\{1\}} \times S_{\{2, \dots, n\}}$:

- (1) the theory F_G ,
- (2) the theory generated by the blocks:
 - $G\{x_1 \cdots x_n\}$,
 - $S_{n+1}\{x_1 \cdots x_{n+1}\}$,
and for $n > 3$
 - $S_n\{w\}$, where w is $\leq_n$ -equivalent either to the word $x_1^3 x_2 \cdots x_{n-2}$ or (for $n \geq 6$) to the word $x_1^2 x_2^2 x_3^2 x_4 \cdots x_{n-3}$,
- (3) the theory generated by the blocks:
 - $G\{x_1 \cdots x_n\}$,
 - $S_{n+1}\{x_1 \cdots x_{n+1}\}$,
and for $n > 3$
 - $S_n\{w\}$, where w is $\leq_n$ -equivalent to the word $x_1^2 x_2^2 x_3 \cdots x_{n-2}$.

Consequently, the set $\{C_G : G \in \{S_{\{1\}} \times S_{\{2, \dots, n\}}, S_{\{1, \dots, n-1\}} \times S_{\{n\}}\}\}$ is definable.

For $G \notin \{S_{\{1\}} \times S_{\{2, \dots, n\}}, S_{\{1, \dots, n-1\}} \times S_{\{n\}}\}$, by E_G we denote the minimal theory which is under B_G and is not under B_H for $H \neq G$. For $G \in \{S_{\{1\}} \times$

$S_{\{2,\dots,n\}}, S_{\{1,\dots,n-1\}} \times S_{\{n\}}$, by E_G we denote the minimal theory which is under C_G and is not under B_H for $H \neq G$. It is clear that E_G is generated by the blocks:

- $G\{x_1 \cdots x_n\}$,
- $S_{n+1}\{x_1 \cdots x_{n+1}\}$.

Obviously, the set consisting of all the theories E_G is definable. Therefore, the meet of E_G is definable. We show that this meet is equal to the theory $Bal(n)$. In fact, this is enough to take the meet of those G that have a form $S_{\{i\}} \times S_{\{1,\dots,n\} \setminus \{i\}}$.

Fix a word $w = y_1 \cdots y_n$. Then a block containing w in E_G , for $G = S_{\{1,\dots,n\} \setminus \{i\}}$, has the form $G(w)$. This means that if a word $v = z_1 \cdots z_n$ belongs to the block $G(w)$, then v is $\leq_n$ -equivalent with w and $v = \sigma(w)$ for some $\sigma \in G$. In particular, $z_i = y_i$. Therefore, the block of w in a theory $\bigwedge E_G$ contains only one element, w . This means that $\bigwedge E_G \subseteq Bal(n+1)$. Obviously, the opposite inclusion also holds. Consequently, $Bal(n+1)$ is definable. Thus, by induction, $Bal(n)$ is definable for every $n \geq 2$.

Corollary 8.2. *For given $n \geq 2$, the set of theories of $\mathcal{L}(SP)$ of the form $T(n, J, m, r, \pi)$ is definable (the parameter n is definable).*

Proof. This is the set of those theories which contain the theory $Bal(n)$ and does not contain the theory $Bal(n-1)$. $\square$

For a given n, m , and r , we denote $S(n, m, r)$ to be the set of all theories of the form $T(n, J, m, r, \pi)$.

Corollary 8.3. *Every set $S(n, m, r)$ is definable.*

9. DEFINABILITY OF THE PARAMETER J

Let Θ denote the trivial remainder (i.e., consisting of the trivial equations only). As a next step, in this section we show that for a given n, J, m , and r , the theory $T(n, J, m, r, \Theta)$ is definable, up to duality. Such theories are called Schwabauer theories. In view of earlier results, it is the same as showing that every order filter J is definable, up to duality. We start with those filters of the bqo $\leq_n$ that are generated by a single word.

Fix $n \geq 2$. Let $Q(n, 0, 1)$ be a set of those theories $T \in S(n, 0, 1)$ for which there is exactly one dual cover of T that belongs to the set $S(n, 0, 1)$. Obviously, the set $Q(n, 0, 1)$ is definable. It is clear that if $T \in Q(n, 0, 1)$, then T is generated by the block $S_n\{x_1 \cdots x_n\}$, where $x_i \neq x_j$ for $i \neq j$, and one additional equation $e = (w, v)$, which is not balanced (otherwise $m = r = \infty$ and $T \notin Q(n, 0, 1)$). We consider a few cases.

Since $m(e') > 0$ for every regular equation e' , we may consider only irregular equations. Then assume that e is irregular and moreover that v and w are incomparable. If it happens that $r(e) = 1$, then, by Lemma 5.1, we have that $T = T(n, J, 0, 1, \Theta)$, where $J = [v, w]$. However, this theory has two dual covers that belong to $S(n, 0, 1)$. They have the forms $T(n, J_v, 0, 1, \pi_1)$ and $T(n, J_w, 0, 1, \pi_2)$ for some π_1 and π_2 , where $J_u = J \setminus \{u' : u' \text{ is } \leq_n\text{-equivalent with } u\}$. Therefore, $T \notin Q(n, 0, 1)$.

Now let v and w be comparable. We may assume $v \leq_n w$. Let $J = [v]$. If $r(e) = 1$, then by Lemma 5.1, $T = T(n, J, 0, 1, \Theta)$. Moreover, T has exactly one dual cover that belongs to $S(n, 0, 1)$. It has the form $T(n, J_v, 0, 1, \pi)$ for some π . It remains to show that, for every word v , there exists a word $w \geq_n v$ such that

$e = (v, w)$ is irregular and $r(e) = 1$. Take $w = vy$, where y is a variable that does not occur in $\text{supp}(v)$. Then, w is as required. This implies that the set $S(n, 0, 1)$ is a set of those theories of the form $T(n, J, 0, 1, \Theta)$ for which J is generated by a single word.

For a fixed $n \geq 2$, $m \geq 0$, and $r > 0$ such that either $m > 0$ or $r > 1$, we define $Q(n, m, r)$ to be the set of all theories of the form $T(n, J, m, r, \Theta)$, where J is generated by a single word.

Corollary 9.1. *All the sets $Q(n, m, r)$ are definable.*

Proof. Assume that either $m > 0$ or $r > 1$. Let $T(n, J_m, m, r, \Theta)$ be a theory, where J_m is a filter in a bqo $\leq_n$ consisting of all the words w such that $m(w) \geq m$. Then, $T(n, J_m, m, r, \Theta)$ is the largest element in $S(n, m, r)$, and therefore, $T(n, J_m, m, r, \Theta)$ is definable. The set $Q(n, m, r)$ consists of all those theories T that belong to the set $S(n, m, r)$ and for which there exists a theory $T' \in Q(n, 0, 1)$ such that $T = T' \wedge T(n, J_m, m, r, \Theta)$. Therefore, the set $Q(n, m, r)$ is definable. $\square$

Now we have a natural order preserving “one to one” correspondence between the set of the classes of equivalent words in a bqo $(W, \leq_n)$ and the set $Q(n, 0, 1)$. Proving definability of an element (set) in one of these bqo’s is the same as proving definability of the corresponding element (set) in the other. To simplify the notation, we will write w , also, for the equivalence class containing the word w . Moreover, it is clear that if a class w is definable in $(W, \leq_n)$, for some n , then w is definable in $(W, \leq_{n'})$, for every $n' \leq n$. ($T(n', [w], 0, 1, \Theta)$ is a join of $T(n, [w], 0, 1, \Theta)$ and $Bal(n')$.) The same is true for definable sets of classes. Therefore, we may always assume that n is suitably large. We will do it without further mention. We also assume, without further mention, that the letters with different names are different, unless we say otherwise.

Now we show that in every bqo $(W, \leq_n)$, each class is definable up to duality (the reversing of letters). For a class w , by $[w]$, we denote the set of those classes v such that $\text{supp}(v) = \text{supp}(w)$ and $|v|_x = |w|_x$ for every $x \in \text{supp}(w)$. First, we use the fact (proved in [14, Theorem 4.3]) that each element in the bqo $(W, \leq_2)$ is definable. This implies that, for every n , each set $[w]$ is definable in $(W, \leq_n)$ (it consists of a single class w for $n = 2$). Moreover, if $w \in \{x^k, x_1 \cdots x_k : \text{where } x_i \text{ are all different and } k \in \mathbb{N}_+\}$, then $[w] = \{w\}$. Thus, in these cases w is definable.

Consider the set $[x^2y]$. It consists of three classes: x^2y , yx^2 , and xyx . The first two are $\leq_n$ greater than x^2 . The other is not. Therefore xyx is definable. The elements x^2y and yx^2 are the smallest undefinable elements. However, they are definable, up to duality. We show that, if we could distinguish between those two elements, we would define every class in $(W, \leq_n)$. This will show that each element is definable up to duality. We assume now that in $(W, \leq_n)$ we are able to single out element x^2y . We show that we are able to define every element.

For a fixed $k \geq 2$, the element $x^2y_1 \cdots y_k$ is the only element belonging to the set $[x^2y_1 \cdots y_k]$ that is a cover of the class $x^2y_1 \cdots y_{k-1}$ and is not greater than the class yx^2 . Then, by induction on k , the class $x^2y_1 \cdots y_k$ is definable for every k . Similarly, the elements $y_1 \cdots y_k x^2$, $k \geq 1$, are definable.

Fix $k \geq 3$. The element $x^k y$ is the only member of the set $[x^k y]$ that is greater than the classes x^k , $x^2y_1 \cdots y_{k-1}$ but not greater than the class $y_1 \cdots y_{k-1} x^2$. Hence, $x^k y$ is definable and, similarly, yx^k is definable. For $l \geq 2$, an element $x^k y_1 \cdots y_l$ is the only member of $[x^k y_1 \cdots y_l]$ that is greater than $x^k y_1 \cdots y_{l-1}$ but not greater

than yx^k . By induction, all elements $x^ky_1 \cdots y_l$ are definable. Similarly, the same is true for elements $y_1 \cdots y_l x^k$. For fixed $k \geq 2$, $l \geq 1$, and $j \geq 1$, the element $y_1 \cdots y_l x^k z_1 \cdots z_j$ is the only member of the set $[y_1 \cdots y_l x^k z_1 \cdots z_j]$ that is greater than both $x^k z_1 \cdots z_j$ and $y_1 \cdots y_l x^k$.

For $j, k, l \geq 0$, we denote $w(j, k, l)$ to be the word $y_1 \cdots y_j x z_1 \cdots z_k x t_1 \cdots t_l$. Then the class $w(j, k, l)$ is the only member of the set $[w(j, k, l)]$ that is smaller than $y_1 \cdots y_j x^{k+2} t_1 \cdots t_l$ and is not smaller than $y_1 \cdots y_{j+1} x^{k+1} t_1 \cdots t_l$ or than $y_1 \cdots y_j x^{k+1} t_1 \cdots t_{l+1}$. Thus, the classes $w(j, k, l)$ are definable.

Finally, let $w = x_1 \cdots x_k$ be an arbitrary word where not all x_i are different. Then the class w is the only member of the set $[w]$ that is greater than the classes $w(i-1, j-i-1, k-j)$ for all $1 \leq i < j \leq k$ such that $x_i = x_j$. Thus, we have the following.

Lemma 9.2. *For every $n \geq 2$, each class is definable in the bqo $(W, \leq_n)$ with the element x^2y singled out.*

This immediately implies the following.

Theorem 9.3. *For every $n \geq 2$, each class is definable in the bqo $(W, \leq_n)$, up to duality. Moreover, a class w is definable if and only if the word w is symmetric.*

Since for every $n \geq 2$ each filter J of the bqo $(Q, \leq_n)$ is finitely generated, we have the following.

Corollary 9.4. *For a given n, J, m , and r , the theory $T(n, J, m, r, \Theta)$ is definable, up to duality. Moreover, the set of theories of the form $T(n, J, m, r, \pi)$ is definable, up to duality (the parameter J is definable, up to duality).*

10. DEFINABILITY OF REGULAR EQUATIONS

In the previous section, we have proved that every Schwabauer theory is definable. Since every irregular equation generates a Schwabauer theory, we can restrict our investigation to the theories generated by regular equations. Moreover, if a regular equation $e = (v, w)$ is such that $v <_n w$, then e also generates a Schwabauer theory. Therefore, we may restrict ourselves to the theories generated by a regular equation $e = (v, w)$, where either $w = p(v)$ for some permutation p (a *substitutive equation*) or v and w are $\leq_n$ incomparable (a *parallel equation*).

Generally, this part is harder than the previous one. Yet, in this case we can follow exactly some arguments applied in [14] and [6]. Therefore, we only point out corresponding proofs and give some additional comments, if necessary. First, we observe that we can recognize e up to permutation.

Theorem 10.1. *Given $\leq_n$ incomparable words v and w , the following properties are definable in $\mathcal{L}(SP)$ (up to duality):*

- (i) *an equation $(v, p(v)) \in T$ for some permutation p ,*
- (ii) *T is generated by a set of equations of the form $(v, p(v))$ and the block $S_n\{x_1 \cdots x_n\}$,*
- (iii) *an equation $(v, p(w)) \in T$ for some permutation p ,*
- (iv) *T is generated by a set of equations of the form $(v, p(w))$ and the block $S_n\{x_1 \cdots x_n\}$.*

Proof. The proof is exactly the same as the proofs of [14, Theorem 5.1 and Theorem 5.4]. $\square$

For every k , any subgroup of S_k which is not cyclic of prime power order, has at least two (inclusion) incomparable subgroups. Hence, we may improve Theorem 10.1 by taking the theories of the form (ii) that do not have two incomparable smallest theories generated by the sets of equations of the form as in (ii). In case (iv), we simply take the smallest theories of this form.

Corollary 10.2. *Given $\leq_n$ incomparable words v and w , the following properties are definable in $\mathcal{L}(SP)$ (up to duality):*

- (i) *T is generated by an equation of the form $(v, p(v))$ and the block $S_n\{x_1 \cdots x_n\}$, where the order of the permutation p is a prime power,*
- (ii) *T is generated by an equation of the form $(v, p(w))$ and the block $S_n\{x_1 \cdots x_n\}$.*

10.1. Theories generated by substitutive equations. We proceed to prove that every theory in $\mathcal{L}(SP)$ is definable (up to duality). We consider these theories that are generated by an equation $(v, p(v))$ and the block $S_n\{x_1 \cdots x_n\}$. We may assume that n is large enough (i.e., $n > |v| + 1$, for instance). For a fixed n , a word v and a permutation p , we define $T(n, (v, p(v)))$ to be the theory generated by an equation $(v, p(v))$ and the block $S_n\{x_1 \cdots x_n\}$.

Lemma 10.3. *Let $v = v(x_1, \dots, x_k)$, $p \in S_k$, and $T = T(n, (v, p(v)))$. Let $i \neq j$, $s \neq t$ be fixed. Then there exists $\sigma = p^l$ for some l such that $\{\sigma(i), \sigma(j)\} = \{s, t\}$ if and only if there is $q \in S_{k-1}$ such that $(g_{x_i x_j}(v), q(g_{x_s x_t}(v))) \in T$.*

Proof. Exactly the same as the proof of [6, Lemma 5.1]. □

Now we may prove the following.

Theorem 10.4. *Every theory of the form $T(n, (v, p(v)))$, where the order of p is a prime power, is definable (up to duality).*

Proof. Using Lemma 10.3, exactly the same as the proof of [6, Theorem 5.3]. Note however that, since n is large enough, the word v has no inner symmetries. Therefore, the case is similar to the case in [6], where terms of the chain are different from each other. This offers the possibility to simplify the proof considerably. □

As a consequence, we immediately have the following.

Theorem 10.5. *A theory of the form $T(n, (v, p(v)))$ is definable (up to duality) for every $n \geq 2$, a word v , and a permutation p of $\text{supp}(v)$.*

Proof. It is enough to observe that $T(n, (v, p(v))) = \text{Bal}(n) \vee \bigvee T(m, (v, p_i(v)))$, where m is large enough, $p = p_1 \cdots p_k$, and an order of each p_i is a prime power. □

10.2. Theories generated by parallel equations. The last case to consider is that of the theories generated by an equation (v, w) and the block $S_n\{x_1 \cdots x_n\}$, where the equation (v, w) is regular and v and w are incomparable. We may assume again that n is large enough (i.e., $n > \max(|v|, |w|) + 1$, for instance). For a fixed n and words v and w such that the equation (v, w) is parallel, we define $T(n, (v, w))$ to be a theory generated by an equation (v, w) and the block $S_n\{x_1 \cdots x_n\}$.

We are going to prove that every theory of the form $T(n, (v, w))$ is definable (up to duality). We start with the case when $|\text{supp}(v)| \geq 3$.

Lemma 10.6. *Let (v, w) be a parallel equation such that $|\text{supp}(v)| \geq 3$. Then $T(n, (v, w))$ is definable (up to duality).*

Proof. The proof is exactly the same as [12, Lemma 4.7]. Note that the proof of [12, Lemma 4.7] uses only [12, Lemma 4.6], which we do not need, since we have Theorem 10.5. In particular, the proof does not use [12, Lemma 4.4].

Alternatively, it is also possible to use the proof of [6, Lemma 6.2] in the easier version, since for n large enough, the words v and w have no inner symmetries. $\square$

It remains to prove the similar lemma with assumption $|\text{supp}(v)| = 2$. This is different from the commutative case of [6]. We are not able to follow [12, Lemma 4.4], since it uses the fact that the theory generated by the equation (x, xy) is definable (up to duality). In the present case, we have an additional commutativity which begins from some level n , and therefore, the equation (x, xy) generates the theory of all equations. Yet the following lemma settles the problem.

Lemma 10.7. *Let (v, w) be a parallel equation such that $|\text{supp}(v)| = 2$. Then $T(n, (v, w))$ is definable (up to duality).*

Proof. Let $v = v(x, y)$, $w = w(x, y)$, and $z \notin \{x, y\}$. We consider an equation $e = (vz, wz)$. It is clear that the equation e is parallel and $\text{supp}(vz) = 3$. Hence, by Lemma 10.6, the theory $T(n, e)$ is definable. Obviously, e is a consequence of the equation $(v(x, y), w(x, y))$ and therefore, $T(n, (v(x, y), w(x, y))) \supset T(n, e)$. Since n is large enough, $vz, wz \notin J(T(n, (v(x, y), w(y, x))))$. This implies that $T(n, e) \not\subseteq T(n, (v(x, y), w(y, x)))$. Hence, we can distinguish between theories $T(n, (v(x, y), w(x, y)))$ and $T(n, (v(x, y), w(y, x)))$. Thus, both those theories are definable (up to duality). $\square$

Using Lemma 10.6 and Lemma 10.7, we can prove the following.

Corollary 10.8. *A theory $T(n, (v, w))$ is definable (up to duality) for every $n \geq 2$ and each parallel equation (v, w) .*

Proof. It is enough to observe that $T(n, (v, w)) = \text{Bal}(n) \vee T(m, (v, w))$, where m is large enough. $\square$

Combining the results from sections 7 – 10 and using Theorem 5.10, we have the result announced at the beginning of section 7.

Theorem 10.9. *Every equational theory of strongly permutative semigroups is individually first-order definable, up to duality, in the lattice of equational theories of strongly permutative semigroups.*

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