ON THE CLASSIFICATION OF INDUCTIVE LIMITS OF II_1 FACTORS WITH SPECTRAL GAP

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ABSTRACT. We consider II_1 factors M which can be realized as inductive limits of subfactors, $N_n \nearrow M$, having spectral gap in M and satisfying the bi-commutant condition $(N'_n \cap M)' \cap M = N_n$. Examples are the enveloping algebras associated to non-Gamma subfactors of finite depth, as well as certain crossed products of McDuff factors by amenable groups. We use deformation/rigidity theory to obtain classification results for such factors.

1. INTRODUCTION

A von Neumann subalgebra Q of a II_1 factor M has *spectral gap in M* if any element $x \in M$ that almost commutes with Q must be close (in the Hilbert norm given by the trace) to an element that actually commutes with Q . This condition holds true, for instance, whenever Q has the property (T). But it is also satisfied under much weaker conditions on Q , e.g. if Q does not have amenable direct summand, provided this is “compensated” by certain requirements on how Q sits inside M .

In [P06a, P06b], spectral gap was used in combination with malleability properties of the ambient factor M , to derive rigidity results, using the deformation-rigidity theory developed in ([P01, P03, P04a]; see [P06c] for a survey). In this paper we obtain several more applications of spectral gap rigidity, by combining it instead with inductive limits deformation.

Our starting point is the observation that if Q has spectral gap in M and M is an inductive limit of subfactors, $N_n \nearrow M$, then $N'_n \cap M$ is approximately contained in $Q' \cap M$, for large n . Furthermore, if N_n and Q satisfy the bicommutant condition in M , i.e. $(Q' \cap M)' \cap M = Q$ and $(N'_n \cap M)' \cap M = N_n$, then it follows that Q is almost contained in N_n as well. In particular, by using the *intertwining subalgebras* techniques and the notation in [P03], one gets $Q \prec_M N_n$; i.e., Q can be conjugated into N_n by a unitary element of M (roughly). This shows that in order for the deformation-rigidity arguments to work, one needs the II_1 factors M to be inductive limits of subfactors, $N_n \nearrow M$, having spectral gap in M and satisfying the bi-commutant condition $(N'_n \cap M)' \cap M = N_n$. As it turns out, the additional condition $[N_n : N_m] < \infty$, $\forall n > m$, is also needed.

An important class of factors, which arise as inductive limits of subfactors satisfying all these conditions, are the enveloping algebras N_∞ associated to non-Gamma subfactors of finite depth $N_{-1} \subset N_0$, defined by $N_\infty = \overline{\bigcup_n N_n}$, where $N_n \nearrow N_\infty$ denotes the Jones tower for $N_{-1} \subset N_0$. The above argument then shows that any

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isomorphism between such enveloping factors is implemented by a *weak equivalence* of the finite depth subfactors they come from (see Definition 3.1.4° and Corollary 3.6). In other words, the enveloping algebra N_∞ of a non-Gamma subfactor with finite depth $N_{-1} \subset N_0$ “roughly” remembers $N_{-1} \subset N_0$.

Another class of factors satisfying the above conditions are the crossed product factors of the form $T \rtimes \Gamma$, where T is an *s-McDuff factor* (i.e., a factor of the form $N \overline{\otimes} R$, with N a non-Gamma II_1 factor and R the hyperfinite II_1 factor), Γ is a countable amenable group and $\Gamma \curvearrowright T$ is a *proper* action of Γ on T , where by definition this means that Γ acts outerly on both N and R (the unique, by [P06b], non-Gamma and hyperfinite components of T) and its image in $\text{Out}(T)$ is closed.

A concrete class of such examples is obtained as follows. Let N be a non-Gamma II_1 factor (e.g. $N = L(\mathbb{F}_n)$, for some $2 \leq n \leq \infty$), Γ a countable group and $\theta : \Gamma \hookrightarrow \text{Out}(N)$, $\rho : \Gamma \hookrightarrow \text{Out}(R)$ faithful Γ -kernels, such that their corresponding $H^3(\Gamma, \mathbb{T})$ -obstructions satisfy $\text{Ob}(\theta) = \overline{\text{Ob}(\rho)}$ and such that the image of $\theta(\Gamma)$ in $\text{Out}(N)$ is closed. Then $\sigma = \theta \otimes \rho$ implements a free cocycle action of Γ on $T = N \overline{\otimes} R$, and all the above conditions are satisfied. Note that by [P91], [PS00], if $N = L(\mathbb{F}_\infty)$, then, given any countable amenable group Γ and any element $\alpha \in H^3(\Gamma, \mathbb{T})$, there exists $\theta : \Gamma \hookrightarrow \text{Out}(L(\mathbb{F}_\infty))$ such that $\text{Ob}(\theta) = \alpha$ and such that $\theta(\Gamma)$ is closed in $\text{Out}(L(\mathbb{F}_\infty))$. Also, by [J80], there exists a faithful Γ -kernel $\rho : \Gamma \hookrightarrow \text{Out}(R)$ such that $\text{Ob}(\rho) = \overline{\alpha}$, which by [Oc83] is in fact unique, up to cocycle conjugacy.

Our results show that any isomorphism $\Delta : M_0 \simeq M_1$ between crossed product factors $M_i = T_i \rtimes_{\sigma_i} \Gamma_i$ in this class, “virtually” takes the s-McDuff factors T_0, T_1 onto each other, and cocycle conjugates σ_0, σ_1 , as well as their restrictions θ_0, θ_1 to the non-Gamma parts of T_0, T_1 . Moreover, if the Γ_i are torsion free, then Δ comes from an actual cocycle conjugacy of θ_0, θ_1 . In particular, the group Γ_i and the element $\alpha_i = \text{Ob}(\theta) \in H^3(\Gamma_i, \mathbb{T})$ are isomorphism invariants for M_i . For instance, since $H^3(\mathbb{Z}^3, \mathbb{T}) = \mathbb{Z}$, if we take $\Gamma = \mathbb{Z}^3$ and $\alpha \in \mathbb{T} \setminus \{\pm 1\}$, then the corresponding factor $M = L(\mathbb{F}_\infty) \overline{\otimes} R \rtimes_\sigma \mathbb{Z}^3$ satisfies $M \not\cong M^{\text{op}}$ and $M^{\otimes n}$ are nonisomorphic, for $n = 1, 2, 3, \dots$. This construction should be compared with Connes original examples satisfying such properties, obtained by using his $\chi(M)$ invariant (see [C75]). We mention that in a paper that we circulate in parallel ([P09]), we use similar deformation by inductive limits and spectral gap rigidity to calculate $\chi(M)$ for II_1 factors satisfying these assumptions.

2. SOME GENERALITIES ON SPECTRAL GAP

2.1. Definition. Let M be a II_1 factor and $Q \subset M$ a von Neumann subalgebra. We say that Q has *spectral gap* in M if $\forall \varepsilon > 0$, $\exists u_1, \dots, u_n \in \mathcal{U}(Q)$ and $\delta > 0$ such that if $x \in M$ satisfies $\| [x, u_i] \|_2 \leq \delta \|x\|_2$, $\forall i$, then $\|x - E_{Q' \cap M}(x)\|_2 \leq \varepsilon \|x\|_2$. Note that this is equivalent to the condition that the representation of $\mathcal{U}(Q)$ on the Hilbert space $L^2(M \ominus Q' \cap M) = L^2(M) \ominus L^2(Q' \cap M)$, given by $u \mapsto \text{Adu}$, has spectral gap, i.e. it does not weakly contain the trivial representation of $\mathcal{U}(Q)$.

2.2. Remarks. 1° Note that if Q is a II_1 factor, then Definition 2.1 automatically implies that Q is a non-Gamma II_1 factor.

2° A weaker version of spectral gap is the following: $\forall \varepsilon > 0$, $\exists u_1, \dots, u_n \in \mathcal{U}(Q)$ and $\delta > 0$ such that if $x \in (M)_1$ satisfies $\| [x, u_i] \|_2 \leq \delta$, $\forall i$, then $\|x - E_{Q' \cap M}(x)\|_2 \leq \varepsilon$. This latter condition, which we refer to as “ Q has *w-spectral gap* in M ”, is trivially equivalent to the condition $Q' \cap M^\omega = (Q' \cap M)^\omega$. In turn, spectral gap is equivalent to $Q' \cap L^2(M)^\omega = L^2(Q' \cap M)^\omega$, where for a Hilbert space $\mathcal{H}$ and

a free ultrafilter ω on $\mathbb{N}$, $\mathcal{H}^\omega$ denotes the ω -ultrapower Hilbert space. It is the stronger form of spectral gap that one usually checks in concrete examples, but the weaker version is what one typically needs in deformation-rigidity arguments. As also pointed out in (Definition 3.4 in [P06a]), the difference between w-spectral gap and spectral gap is in the same vein as the difference between strong ergodicity and spectral gap of a measure-preserving action of a group on a probability space $\Gamma \curvearrowright X$, emphasized by K. Schmidt in ([S81]). It is also analogous to the difference between the noninner amenability of a group Γ (as defined by Effros in [E75]) and the property non-Gamma of its associated II_1 factor $L(\Gamma)$, discovered recently by Vaes ([Va09]).

However, in many interesting cases, spectral gap is equivalent to w-spectral gap, notably in the case $Q \subset Q \overline{\otimes} S = M$. Indeed, by a result of Connes in [C76], both conditions are equivalent to Q being non-Gamma. More generally, if $Q_0 \subset Q$ is an inclusion of non-Gamma factors with finite index, then $Q_0 \subset Q \subset Q \overline{\otimes} S$ has spectral gap.

But one can easily “cook up” examples where they are not. Thus, let θ_k be the automorphism of $L(\mathbb{F}_\infty)$ that takes the k th generator to its negative, leaving all others fixed. Note that $\theta_k \rightarrow \text{id}$. For each $n \geq 1$, let $N_n \subset M_2(\mathbb{C}) \otimes L(\mathbb{F}_\infty)$ be the “diagonal” subfactor given by $\{xe_{11} + \theta_n(x)e_{22} \mid x \in L(\mathbb{F}_\infty)\}$, where $\{e_{ij}\}_{1 \leq i,j \leq 2}$ are the matrix units in $M_2(\mathbb{C}) = M_2(\mathbb{C}) \otimes 1$. Then let $M \simeq L(\mathbb{F}_\infty)$ and choose a partition of 1 with projections in M , $\{p_n\}_{n \geq 0} \in M$. For each n we identify $p_n M p_n$ with $M_2(L(\mathbb{F}_\infty))$, via some isomorphism σ_n , by using Voiculescu’s theorem ([V89]), and let $N \subset M$ be the “diagonal” subfactor $\{\Sigma_n \sigma_n^{-1}(xe_{11} + \theta_n(x)e_{22}) \mid x \in L(\mathbb{F}_\infty)\}$. Then $\xi_m = \sigma_m^{-1}(e_{12})/\tau(p_m)^{1/2} \in L^2(M)$ satisfy $\|\xi_m\|_2^2 = 1/2$ and $\|[\xi_m, x]\|_2 \rightarrow 0$, $\forall x \in N$. On the other hand, it is trivial to see that if N is a non-Gamma II_1 factor and $N \subset M$ is so that $N' \cap M$ is atomic with $pMp = Np$ for any minimal projection $p \in N' \cap M$, then N has w-spectral gap in M .

3° Several examples of subfactors $Q \subset M$ with spectral gap are emphasized in [P09]. For instance, if $N_{-1} \subset N_0$ is an inclusion of II_1 factors with finite index, $N_n \nearrow N_\infty$ is its associated Jones tower and enveloping algebra, and we assume that either N_0 has the property (T), or N_0 is merely non-Gamma but $N_{-1} \subset N_0$ has finite depth, then N_n has spectral gap in N_∞ , $\forall n$ (see 2° and 3° in Proposition 3.2 of [P09]). Also, if $\mathcal{G}$ is a standard λ -lattice ([P94]), Q is a II_1 factor and one denotes $N_{-1} = M_{-1}^{\mathcal{G}}(Q) \subset M_0^{\mathcal{G}}(Q) = N_0$ the inclusion of II_1 factors with standard invariant $\mathcal{G}$, as constructed in [P94], [P98], with $N_n \nearrow N_\infty$ the associated tower and enveloping algebra, then N_n has spectral gap in N_∞ , $\forall n$ (see 3.2.4° in [P09]).

Another class of examples is obtained below:

2.3. Proposition. *Let N be a non-Gamma II_1 factor and $T = N \overline{\otimes} R$. Let $\Gamma \curvearrowright^\sigma T$ be a cocycle action of a countable group Γ and denote $M = N \overline{\otimes} R \rtimes \Gamma$.*

1° *For any g , there exists $t > 0$ such that, modulo perturbation by an inner automorphism of T , one has $\sigma(g)(N) = N^t$, $\sigma(g)(R) = R^{1/t}$, where $N^t \overline{\otimes} R^{1/t}$ is the unique decomposition of $N \overline{\otimes} R$ in [OP03].*

2° *Denote $\theta(g) = \sigma(g)|_N$ (cf. part 1°). Then $\theta(g)$ is outer, $\forall g \neq e$, iff $\sigma(\Gamma) \cap \overline{\text{Int}(T)} = 1$ and iff $N' \cap M = R$.*

3° *Assume that the equivalent conditions in 2° are satisfied. Then $\rho(g) = \sigma(g)|_R$ is outer, $\forall g \neq e$, iff N satisfies the bicommutant condition $(N' \cap M)' \cap M = N$ and iff σ is centrally free.*

4° Assume that the equivalent conditions in 2° are satisfied. Then N has spectral gap in M iff $\sigma(\Gamma)$ is closed in $\text{Out}(T)$ and iff $\theta(\Gamma)$ is closed in $\text{Out}(N^\infty)$.

Proof. Part 1° is a consequence of (Theorem 5.1 in [P06b]). We leave the proof of 2° – 4° as an exercise. $\square$

We use the following convention throughout the paper: If M is a finite factor and $A, B \subset M$ are von Neumann subalgebras, then the notation $A \subset_\varepsilon B$, for some $\varepsilon > 0$, means that $\|x - E_B(x)\|_2 \leq \varepsilon$, $\forall x \in A$, $\|x\| \leq 1$.

2.4. Lemma. Let $Q \subset M$ be an inclusion of factors and assume that Q has spectral gap in M . If $N_n \subset M$ are von Neumann subalgebras such that $\lim_n \|E_{N_n}(x) - x\|_2 = 0$, $\forall x \in M$, then for any $\varepsilon > 0$, there exists n such that $N'_n \cap M \subset_\varepsilon Q' \cap M$. Moreover, for this n we have $(Q' \cap M)' \cap M \subset_{2\varepsilon} (N'_n \cap M)' \cap M$.

Proof. Since Q has spectral gap in M , by Definition 2.1, there exist $\delta > 0$ and $u_1, \dots, u_m \in \mathcal{U}(Q)$ such that if $x \in (M)_1$ satisfies $\|[u_i, x]\|_2 < \delta$, $\forall i$, then $x \in_\varepsilon Q' \cap M$.

By the hypothesis, there exists n such that $\|E_{N_n}(u_i) - u_i\|_2 \leq \delta/2$, $\forall i$. Thus, if $x \in N'_n \cap M$, then

$$\|[x, u_i]\|_2 \leq 2\|x\|\|E_{N_n}(u_i) - u_i\|_2 \leq \delta, \forall i,$$

implying that $x \in_\varepsilon Q' \cap M$. The other approximate inclusion is now automatic, by the triangle inequality. $\square$

3. VIRTUAL STRONG RIGIDITY RESULTS

3.1. Definition. 1° Recall from [P97], [P01] that a von Neumann subalgebra B of a II_1 factor M is *quasi-regular* in M if the von Neumann algebra generated by its *quasi-normalizer*, defined as the set $q\mathcal{N}(B)$ of all $x \in M$ with the property that $\text{sp} Bx\bar{B} \subset L^2(M)$ is finitely generated both as a left and as a right B -module, is equal to M . Note that this is equivalent to $B' \cap \langle M, e_B \rangle$ being generated by projections p with the property that both p and $J_M p J_M$ are finite in $\langle M, e_B \rangle$. Moreover, by (1.4.2 in [P01]), B is in fact quasi-regular in M iff it is *discrete* in M , in the sense of [ILP96], i.e. iff $B' \cap \langle M, e_B \rangle$ is generated by projections that are finite in $\langle M, e_B \rangle$.

2° If N, Q are von Neumann subalgebras of the II_1 factor M , then Q is *discrete over N* (inside M) if $L^2(M)$ is generated by $Q - N$ Hilbert-bimodules which are finitely generated over N . Arguing as in the proof of (1.4.2 in [P01]), this is easily seen to be equivalent to $Q' \cap \langle M, e_N \rangle$ being generated by projections that are finite in $\langle M, e_N \rangle$.

3° If $N \subset M$, $Q \subset P$ are quasi-regular von Neumann subalgebras of the II_1 factors M, P , then an isomorphism $\theta : M \simeq P$ is a *virtual isomorphism* of the inclusions $N \subset M$ and $Q \subset P$ if $L^2(P)$ is a direct sum of $\theta(N) - Q$ bimodules of finite index, in other words if Q is discrete over $\theta(N)$ and $\theta(N)$ is discrete over Q , inside P . Note that this notion becomes relevant only in the case that N, Q have infinite index in their respective ambient algebras. Indeed, if $N \subset M$, $Q \subset P$ are subfactors of finite Jones index, then any isomorphism of M, P satisfies the condition. In turn, for finite index inclusions of factors we have the following notion of “virtual isomorphism”:

4° Let $N_{-1} \subset N_0$, $Q_{-1} \subset Q_0$ be inclusions of II_1 factors with finite Jones index and N_∞, Q_∞ their enveloping algebras. An isomorphism $\Delta : N_\infty \simeq Q_\infty$ with the property that $L^2(Q_\infty)$ is generated by $\theta(N_{-1}) - Q_{-1}$ Hilbert bimodules which

are finite over both $\theta(N_{-1})$ and Q_{-1} , is called a *weak equivalence* of $N_{-1} \subset N_0$, $Q_{-1} \subset Q_0$.

5° Finally, if $N, Q \subset M$ are von Neumann subalgebras of the II_1 factor M , then a (nonzero) Hilbert $N - Q$ bimodule $\mathcal{H} \subset L^2(M)$, which is finitely generated both over N and over Q , is called a *virtual conjugacy* of (corners of) N, Q inside M . If such a bimodule exists, we say that N, Q are *virtually conjugate inside M* and write $N \sim_M Q$. As with the virtual isomorphism of inclusions defined in 3°, this relative version “doesn’t recognize” between finite index inclusions: if Q, N are any subfactors of finite index of a II_1 factor M , then $N \sim_M Q$.

3.2. Lemma. *Let N, Q be quasi-regular subfactors of the II_1 factor M . Then the following conditions are equivalent:*

- (a) *N is discrete over Q and Q is discrete over N .*
- (b) *$L^2(M)$ is generated by $N - Q$ bimodules that are finitely generated both over N and over Q .*
- (c) *$N \sim_M Q$.*

Proof. Let $\tilde{M} = M_2(M)$ and $\tilde{N} = N \oplus Q \subset \tilde{M}$. It is immediate to see that condition (b) is equivalent to $\tilde{N}$ being quasi-regular in $\tilde{M}$, while the condition (a) is equivalent to $\tilde{N}$ being discrete in $\tilde{M}$. Thus, (a) $\Leftrightarrow$ (b) by (part (iii) of 1.4.2 in [P01]). On the other hand, we trivially have (b) $\Rightarrow$ (c) and for the converse, all we need to prove is that $\tilde{N}$ is quasi-regular in $\tilde{M}$. But since the quasi-normalizers of Qe_{11} and Ne_{22} in $\tilde{M}$ are included into the quasi-normalizer of $\tilde{N}$ and they generate the factors $e_{11}Me_{11}, e_{22}Me_{22}$ respectively, and since $N \sim_M Q$ means there exists a nonzero element $x = e_{11}xe_{22} \in q\mathcal{N}_{\tilde{M}}(\tilde{N})$, it follows that the von Neumann algebra generated by $q\mathcal{N}_{\tilde{M}}(\tilde{N})$ must be all $\tilde{M}$. $\square$

3.3. Lemma. *Let $N = N_0 \subset N_1 \subset N_2 \subset \dots \nearrow M$ be an inductive limit of II_1 factors such that $[N_n : N] < \infty, \forall n$. Then N is quasi-regular in M . If in addition $N'_n \cap M$ is a factor, $\forall n \geq 0$, then $N'_{n+1} \cap M \subset N'_n \cap M$ has finite index, $N_n \vee N'_n \cap M$ is quasi-regular in M and N_n is quasi-regular in $(N'_n \cap M)' \cap M, \forall n \geq 0$.*

Proof. Since $e_{N_n} \in \langle M, e_N \rangle$ are finite projections and tend to 1, the first part follows trivially from (1.4.2 in [P01]).

To prove the second part, it is clearly sufficient to consider the case $n = 0$. The fact that $N'_1 \cap M$ has finite index in $N' \cap M$ was shown in [P92], but we include a full proof for the sake of completeness. Thus, denote by $e \in N_1$ the Jones projection for $N \subset N_1$ and let $P = \{e\}' \cap N \subset N$ be the corresponding downward basic construction (cf. [J82]; see also [PP83]). Let also $\{m_j\}_j$ be an orthonormal basis of N over P (cf. [PP83]). Thus, $\sum_j m_j e m_j^* = 1$ and $p_j = E_N(m_j^* m_j) \in \mathcal{P}(P), \forall j$. Note that $N' \cap M \ni x \mapsto \mathcal{E}(x) = \sum_j m_j x e m_j^*$ defines a conditional expectation of $N' \cap M$ onto $N'_1 \cap M$, satisfying $x e x = \mathcal{E}(x) e$ and

(3.3.1)

$$\begin{aligned} \tau(\mathcal{E}(x)y) &= \sum_j \tau(x e m_j^* m_j e y) = \sum_j \tau(p_j x e y) \\ &= \sum_j \tau(E_{P' \cap M}(p_j) x e y) = [N_1 : N] \tau(x e y) = \tau(b x y), \forall y \in N'_1 \cap M, \end{aligned}$$

where $b = [N_1 : N] E_{N' \cap N_1}(e)$. Notice that $[N_1 : N]^{-1} 1 \leq b \leq [N_1 : N] 1$ and denote $E(x) = \mathcal{E}(b^{-1/2} x b^{-1/2})$. By (3.3.1), it follows that E defines the trace-preserving conditional expectation of $N' \cap M$ onto $N'_1 \cap M$. Moreover, $e_0 = b^{-1/2} e b^{-1/2}$ is a

projection and $e_0 x e_0 = E(x) e_0, \forall x \in N' \cap M$. Thus, $[N' \cap M : N'_1 \cap M] \leq \tau(e_0)^{-1} = \tau(e)^{-1} = [N_1 : N]$.

The fact that $N \vee N' \cap M$ is quasi-regular in M follows as in [P97], by noticing that if $e_n \in N_n$ is a Jones projection for $N \subset N_n$, then $\mathcal{H}_n = L^2(N \vee N' \cap M e_n N \vee N' \cap M)$ is finitely generated both as a left and as a right Hilbert $N \vee N' \cap M$ -module. Since $\mathcal{H}_n \nearrow L^2(M)$, we are done.

Finally, to show that N is quasi-regular in $\tilde{N} = (N' \cap M)' \cap M$, note that one has the commuting square

$$\begin{array}{ccc} N \vee N' \cap M & \subset & M \\ \cup & & \cup \\ N & \subset & \tilde{N}. \end{array}$$

Thus, if we denote $e = e_{N \vee N' \cap M}^M$, then e implements the trace-preserving conditional expectation of $\tilde{N}$ onto N and if $f = \bigvee \{ueu^* \mid u \in \mathcal{U}(\tilde{N})\}$, then $Nf \subset \tilde{N}f \subset {}^e\mathcal{N} = \vee N(\tilde{N}f, e)$ is the basic construction for $N \subset \tilde{N}$. Since for any projection $p \in (N \vee N' \cap M)' \cap \langle M, e \rangle$ of finite trace, $fpf \in f\langle M, e \rangle f$ has finite trace, commutes with N and its Tr -preserving expectation onto $\mathcal{N}$ still has finite trace and commutes with N , it follows that $N' \cap \mathcal{N}$ is generated by finite projections. Thus, $N \subset \tilde{N}$ is quasi-regular. $\square$

3.4. Definition. An inclusion of II_1 factors $N \subset M$ is *proper* if there exists an increasing sequence of factors $N \subset N_1 \subset N_2 \subset \dots \subset M$ such that:

- (3.4.1) $\bigcup_n N_n = M$;
- (3.4.2) $[N_n : N] < \infty, \forall n$;
- (3.4.3) N_n has spectral gap in $M, \forall n$;
- (3.4.4) $(N'_n \cap M)' \cap M = N_n, \forall n$.

3.5. Theorem. Let $N \subset M, Q \subset P$ be proper inclusions of factors. Any $\theta : M \simeq P$ implements a virtual isomorphism between the inclusions $N \subset M, Q \subset P$ and between the inclusions $N \vee N' \cap M \subset M, Q \vee Q' \cap P \subset P$.

Proof. We identify P with M , via θ . Let $N \subset N_1 \subset \dots \nearrow M$ and $Q \subset Q_1 \subset \dots \nearrow P$ be inductive limits of subfactors satisfying conditions 3.4. By Lemma 2.4, the spectral gap and bicommutant properties for N_n and Q_m in M , given any $\varepsilon > 0$ there exist m, n such that $N'_m \cap M \subset_\varepsilon Q'_n \cap M \subset_\varepsilon N' \cap M$ and $N \subset_{2\varepsilon} Q_n \subset_{2\varepsilon} N_m$. Taking $\varepsilon < 1/4$, it follows that if we average the projection $e_{Q_n \vee (Q'_n \cap M)} \in \langle M, e_{Q_n \vee (Q'_n \cap M)} \rangle$ by unitaries in $\mathcal{U}(N)\mathcal{U}(N'_m \cap M)$ then we get a finite nonzero projection. Thus, by [P03] we get $N \vee (N'_m \cap M) \prec_M Q_n \vee (Q'_n \cap M)$. Since $N \vee (N'_m \cap M)$ has finite index in $N \vee (N' \cap M)$, this implies that $N \vee (N' \cap M) \prec_M Q_n \vee (Q'_n \cap M)$. But $Q \vee (Q' \cap M)$ and $Q_n \vee (Q'_n \cap M)$ have finite index over their common subfactor $Q \vee (Q' \cap M)$; thus $N \vee (N' \cap M) \prec_M Q_n \vee (Q'_n \cap M)$ implies $N \vee (N' \cap M) \prec_M Q \vee (Q' \cap M)$, which in turn implies $N \vee (N' \cap M) \prec_M Q \vee (Q' \cap M)$, with the latter being quasi-regular in M . Similarly, $Q \vee (Q' \cap M) \prec_M N \vee (N' \cap M)$ and the latter is quasi-regular in M . By the proof of (Lemma 2 in [P04b]), it then follows that $L^2(M)$ is a direct sum of finite-dimensional $N \vee (N' \cap M) - Q \vee (Q' \cap M)$ bimodules.

The proof that ${}_N L^2(M)_Q$ is generated by finite $N - Q$ bimodules is similar and is left as an exercise. $\square$

3.6. Corollary. *Let $N_{-1} \subset N_0$, $Q_{-1} \subset Q_0$ be non-Gamma subfactors with finite depth. Then any isomorphism of their enveloping algebras $\Delta : N_\infty \simeq Q_\infty$ implements a weak equivalence of $N_{-1} \subset N_0$, $Q_{-1} \subset Q_0$.*

Proof. By [P09], both $N_0 \subset N_\infty$ and $Q_0 \subset Q_\infty$ are proper inclusions, so Theorem 3.5 applies.

Let us mention that a “virtual strong rigidity result” similar to Theorem 3.5 was already shown in [P04b]:

3.7. Theorem. *Let $N \subset P$, $Q \subset P$ be irreducible, quasi-regular, rigid inclusions of II_1 factors such that M (resp. P) has Haagerup property relative to N (resp. Q). Then any $\theta : M \simeq P$ implements a virtual isomorphism between the inclusions $N \subset M$, $Q \subset P$.*

Proof. The proof of (Lemma 2 and the Theorem in [P04b]) actually shows this. $\square$

4. PROPER INCLUSIONS COMING FROM PROPER ACTIONS

We have already pointed out in Proposition 2.3 that if a group Γ acts on an s-McDuff factor $T = N \overline{\otimes} R$ such that its restrictions to the non-Gamma and hyperfinite parts N, R of T are both outer and its image in $\text{Out}(T)$ is closed, then N has spectral gap in $M = T \rtimes \Gamma$ and satisfies the bicommutant condition. We will consider here the following more specific case:

4.1. Definition. Let Γ be a discrete group and $N \overline{\otimes} R$ an s-McDuff factor, with N non-Gamma. A *proper* action of Γ on $N \overline{\otimes} R$ is a free, cocycle action with the property that σ normalizes N and R , its restrictions to N and R are both properly outer and $\sigma|_N$ is closed in $\text{Out}(N)$. Note that any such σ is of the form $\sigma = \theta \otimes \rho$, where $\theta : \Gamma \hookrightarrow \text{Out}(N)$, $\rho : \Gamma \hookrightarrow \text{Out}(R)$ are faithful Γ -kernels with $H^3(\Gamma, \mathbb{T})$ -obstructions satisfying $\text{Ob}(\rho) = \overline{\text{Ob}(\theta)}$. It is useful to keep in mind throughout the rest of the paper that by a well-known result of Ocneanu [Oc83] there exists a unique (up to conjugacy) Γ -kernel into $\text{Out}(R)$ with given $H^3(\Gamma, \mathbb{T})$ -obstruction.

4.2. Theorem. *If Γ is a countable amenable group, $N \overline{\otimes} R$ an s-McDuff factor and $\Gamma \curvearrowright^\sigma N \overline{\otimes} R$ is a proper action, then $N \subset M = N \overline{\otimes} R \rtimes \Gamma$ is a proper inclusion.*

Proof. It is easy to see that it is sufficient to prove the case when N is separable. But then the statement follows immediately from the following: $\square$

4.3. Lemma. *Let N be a separable II_1 factor, Γ a countable amenable group and $\theta : \Gamma \hookrightarrow \text{Out}(N)$ a faithful Γ -kernel.*

1° *There exists a hyperfinite subfactor $R \subset N$, with $R' \cap N = \mathbb{C}$, and a lifting $\theta' : \Gamma \rightarrow \text{Aut}(N)$ of θ such that $\theta'(g)(R) = R$, $\forall g \in \Gamma$, $\theta'(g)\theta'(h) = \text{Ad}(u_{g,h})\theta'(gh)$, for some $u_{g,h} \in \mathcal{U}(R)$, $\forall g, h \in \Gamma$, and θ'_R implements a faithful Γ -kernel in $\text{Out}(R)$.*

2° *Let $\rho : \Gamma \hookrightarrow \text{Out}(R)$ be a faithful Γ -kernel on the hyperfinite II_1 factor with $\text{Ob}(\rho) = \overline{\text{Ob}(\theta)}$ and denote by $\sigma = \theta \otimes \rho$ the corresponding cocycle action of Γ on $N \overline{\otimes} R$. Then there exists an increasing sequence of subfactors with finite index, $N \subset N_1 \subset \dots \nearrow N_\infty = N \overline{\otimes} R \rtimes_{\theta \otimes \rho} \Gamma$, such that $(N'_n \cap N_\infty)' \cap N_\infty = N_n$, $\forall n$.*

Proof. Note that $\theta \otimes \theta^{\text{op}}$ implements a cocycle action on $N \overline{\otimes} N^{\text{op}}$. Denote by $\tilde{M} = N \overline{\otimes} N^{\text{op}} \rtimes_{\theta \otimes \theta^{\text{op}}} \Gamma$ the corresponding crossed product II_1 factor. All work will be done in this “ambient” algebra. We will prove simultaneously 1° and 2°.

We first treat the case when Γ is generated by finitely many elements $g_0 = e, g_1, \dots, g_n \in \Gamma$. Note that by perturbing if necessary each θ_g by an inner automorphism implemented by a unitary in N , we may assume that σ fixes a subalgebra $M_{(n+1) \times (n+1)}(\mathbb{C})$ generated by matrix units $\{e_{ij}\}_{i,j}$. Thus, we can “split” N as $N = M_{(n+1) \times (n+1)}(\mathbb{C}) \otimes P$, while at the same time realizing P as a locally trivial “diagonal” subfactor of N , $P = \{\sum_{j=0}^n \theta(g_j)(x)e_{jj} \mid x \in P\}$.

By Theorem 3.3 in [P97], we can identify $N \overline{\otimes} N^{\text{op}} \subset \tilde{M}$ with the *symmetric enveloping inclusion* $N \vee N^{\text{op}} \subset N \boxtimes_{e_P} N^{\text{op}}$ of $P \subset N$, as defined in Section 2 of [P97]. By [P97] again, one can embed the Jones tower of factors $P \subset N \subset N_1 \subset \dots$, with its enveloping algebra N_∞ , as subfactors in the symmetric enveloping algebra, $N \otimes 1 = N \subset N_1 \subset \dots \nearrow N_\infty \subset \tilde{M}$. Moreover, any such embedding of the tower of factors comes from a tunnel of factors $N \supset P \supset P_1 \supset \dots$, by taking $N_n = P_{n-1}^{\text{op}} \cap \tilde{M}$, where $^{\text{op}}$ is the canonical symmetry of $\tilde{M}$, which leaves e_P fixed and sends N onto N^{op} (see Sections 1 and 2 in [P97]).

Other observations concerning the symmetric enveloping algebra $\tilde{M}$ that will be useful are the following (cf. Sections 1 and 2 in [P97]): $N' \cap \tilde{M} = N^{\text{op}}$ and $(N^{\text{op}})' \cap \tilde{M} = N$; more generally, $(N'_n \cap \tilde{M})' \cap \tilde{M} = N_n$, $\forall n \in \mathbb{Z}$, where $N_0 = N$, $N_{-1} = P$ and $N_k = P_{-k-1}$ for $k \leq -2$; given any other locally trivial “diagonal” subfactor $P^0 \subset N$, corresponding to another set of generators (with possible multiplicities) $h_0 = e, h_1, \dots, h_m \in \Gamma$ and appropriate inner perturbations of $\theta(h_i)$, one can alternatively view $\tilde{M}$ as the symmetric enveloping algebra $N \boxtimes_{e_{P^0}} N^{\text{op}}$; given any “level” $k \in \mathbb{Z}$ of the tower-tunnel N_k , one can identify $N \vee N^{\text{op}} \subset N \boxtimes_{e_P} N^{\text{op}} = N \otimes N^{\text{op}} \rtimes_{\theta \otimes \theta^{\text{op}}} \Gamma$ with $N^t \otimes N^{1/t^{\text{op}}} \subset N^t \otimes N^{1/t^{\text{op}}} \rtimes_{\theta^t \otimes \theta^{1/t^{\text{op}}}} \Gamma$, which we view also as $N_k \vee N'_k \cap \tilde{M} \subset \tilde{M}$ and as the amplification by $1/t^2$ of the symmetric enveloping inclusion $N_k \vee N_k^{\text{op}} \subset N_k \boxtimes_{e_{N_{k-1}}} N_k^{\text{op}}$, where $t = (n+1)^k$.

For clarity, we'll first give a short argument in the case when Γ is strongly amenable with respect to the set of generators $g_0, g_1, \dots, g_n$ (in the sense of [P89a]). Thus, by (proof of 2.1 in [P89b]), the tunnel $N \supset P \supset P_1 \supset P_2 \dots$ can be chosen in this case such that if we denote $R = \overline{\bigcup_n P'_n \cap N}$, $S_k = \overline{\bigcup_n P'_n \cap P_k}$, then, $R' \cap \tilde{M} = N' \cap \tilde{M} = N^{\text{op}}$ and $S'_k \cap \tilde{M} = P'_k \cap \tilde{M}$, $k \geq 0$. Equivalently, this means that after perturbing if necessary each $\theta(g_j)$ by an inner automorphism of N , there exists a θ -invariant hyperfinite subfactor Q of N , with relative commutant in $\tilde{M}$ equal to N^{op} , such that $\theta(g)|_Q \in \text{Aut}(Q)$, $g \in \Gamma$, implements a Γ -kernel, $\Gamma \hookrightarrow \text{Out}(Q)$, and such that $R = M_{(n+1) \times (n+1)}(\mathbb{C}) \otimes Q$, $S = \{\sum_j \theta(g_j)(x)e_{jj} \mid x \in Q\}$ (see proof of 2.1 in [P89b]). Note that if we let $N_n = P_{n-1}^{\text{op}} \cap \tilde{M}$, $N_\infty = \overline{\bigcup_n N_n} \subset \tilde{M}$ the representations inside $\tilde{M}$ of the tower and enveloping algebra of $P \subset N$ corresponding to the choice of tunnel P_n , then $(N'_1 \cap N_\infty \subset N' \cap N_\infty)$ coincides with $(S \subset R)^{\text{op}}$ (see [P92] or [P97]). Also, if we denote by $\rho : \Gamma \hookrightarrow \text{Out}(R^{\text{op}})$ the corresponding Γ -kernel, then by Section 3 in [P97], it follows that $\theta \otimes \theta^{\text{op}}$ implements a cocycle action of Γ on $N \vee N^{\text{op}}$ which leaves $N \vee R^{\text{op}}$ invariant, and such that $N \vee (N' \cap N_\infty) \subset N_\infty$ is isomorphic $N \otimes R^{\text{op}} \rtimes_{\theta \otimes \rho} \Gamma$. Moreover, from the way we chose the tunnel, we have

$$\begin{aligned} (N'_n \cap N_\infty)' \cap N_\infty &= (S_{n-1}^{\text{op}})' \cap N_\infty \subset (S_{n-1}^{\text{op}})' \cap \tilde{M} \\ &= (S'_{n-1} \cap \tilde{M})^{\text{op}} = (P'_{n-1} \cap \tilde{M})^{\text{op}} = N_n, \end{aligned}$$

which together with the trivial inclusion $(N'_n \cap N_\infty)' \cap N_\infty \supset N_n$ gives $(N'_n \cap N_\infty)' \cap N_\infty = N_n$.

The proof of the lemma in its full generality is very similar, but instead of a Jones tunnel of factors P_n , we now use (the proof of 7.1 and Remark 7.2.1° in [P97]) in combination with (the proof of 2.1 in [P89b]) to construct recursively a decreasing sequence of subfactors P_n , with each P_n obtained as a downward basic construction of a suitable local inclusion $P_{n-1}p \subset pN_m^0p$, for some m and $p \in P'_{n-1} \cap N_m^0$, where $P \subset N \subset N_1^0 \subset N_2^0 \subset \cdots \subset \tilde{M}$ is a representation of the Jones tower inside $\tilde{M}$, which we choose once and for all, from the beginning.

Let us consider first the case Γ generated by a finite set $g_0 = e, g_1, \dots, g_n$ and let $P \subset N \subset \tilde{M}$ be defined as before. Let $\{x_n\}_n \subset \tilde{M}$ be dense in the norm $\|\cdot\|_2$ in the unit ball of $\tilde{M}$. Assume we have constructed finite index subfactors $N = P_{-1} \supset P = P_0 \supset P_1 \supset \cdots \supset P_{n-1}$ such that $E_{(P'_j \cap P_k)' \cap \tilde{M}}(x_i) \in_{2^{-j}} P'_k \cap \tilde{M}$, $\forall 1 \leq i \leq j \leq n-1$, $-1 \leq k \leq j \leq n-1$, and such that each $P_j \subset N$ is a locally trivial subfactor, corresponding to some finite subset $e \in K_j \subset \Gamma$ and multiplicities $k_{g,j} \geq 1, g \in K_j$.

By arguing as in the proof of 4.10 in [P97], or as in the proof of 2.1 in [P89b], it follows that the inclusion $N \supset P_{n-1} = P_0^{-1}$ admits a tunnel $N \supset P_0^{-1} \supset P_1^{-1} \supset \cdots \supset P_k^{-1}$ such that $E_{(P_k^{-1'} \cap P_{n-1})' \cap \tilde{M}}(x_j) \in_{2^{-n-1}} P_k^{-1'} \cap \tilde{M}$. On the other hand, since $P_k^{-1} \subset N$ is locally trivial, given by automorphisms in Γ , there exists a factor $N \subset N_1^{-1} \subset \tilde{M}$ such that $(P_k^{-1})^{\text{op}} \subset N^{\text{op}} \subset P_k^{-1'} \cap \tilde{M}$ is a basic construction (from properties mentioned before); let $e \in P_k^{-1'} \cap (P_k^{-1})^{\text{op}'}$ be the corresponding Jones projection. Thus, $P_k^{-1'} \cap \tilde{M} = \text{sp} N^{\text{op}} e N^{\text{op}}$. By (6.1 in [P97]), for any $\delta > 0$ there exist m , a projection $p \in P_k^{-1'} \cap N_m$ and a downward basic construction $P_n \subset P_k^{-1} \simeq P_k^{-1}p \subset pN_m p$ such that $\|E_{(P_n' \cap P_k)' \cap \tilde{M}}(e) - E_{P_k' \cap \tilde{M}}(e)\|_2 \leq \delta$. Since

$$\|E_{(P_k^{-1'} \cap P_j)' \cap \tilde{M}}(x_i) - E_{P_k^{-1'} \cap \tilde{M}}(x_i)\|_2 \leq 2^{-n-1}$$

and $P_k^{-1'} \cap \tilde{M} = \text{sp} N^{\text{op}} e N^{\text{op}}$, $P'_j \cap \tilde{M} = \text{sp} N^{\text{op}} E_{P'_j \cap \tilde{M}}(e) N^{\text{op}}$, it follows that for $n-1 \geq j \geq -1$, we have

$$\begin{aligned} & \|E_{(P_n' \cap P_j)' \cap \tilde{M}}(x_i) - E_{P'_j \cap \tilde{M}}(x_i)\|_2 \\ & \leq \|E_{(P_n' \cap P_j)' \cap \tilde{M}}(x_i) - E_{(P_n' \cap N)' \cap (P_k^{-1'} \cap \tilde{M})}(x_i)\|_2 \\ & \quad + \|E_{(P_n' \cap P_j)' \cap \tilde{M}}(E_{P_k^{-1'} \cap \tilde{M}}(x_i) - E_{P'_j \cap \tilde{M}}(x_i))\|_2 \\ & \leq 2^{-n-1} + \|E_{(P_n' \cap P_j)' \cap \tilde{M}}(E_{P_k^{-1'} \cap \tilde{M}}(x_i)) - E_{P'_j \cap \tilde{M}}(x_i)\|_2, \end{aligned}$$

which for small enough δ gives $\|E_{(P_n' \cap P_j)' \cap \tilde{M}}(x_i) - E_{P'_j \cap \tilde{M}}(x_i)\|_2 \leq 2^{-n}$, $\forall i \leq n$.

We now define $N_n = P_n^{\text{op}'} \cap \tilde{M}$, $N_\infty = \overline{\bigcup_n N_n}$, $R = \overline{\bigcup_n P'_n \cap N}$, $S_j = \overline{\bigcup_n P'_n \cap P_j}$. Then N_∞ is generated by N, e_P, R^{op} . Moreover, by the way the P_n were chosen, we also have $R' \cap \tilde{M} = N^{\text{op}}$, $S'_j \cap \tilde{M} = P'_j \cap \tilde{M}$ and a nondegenerate commuting square of factors

$$\begin{array}{ccc} P & \subset & N \\ \cup & & \cup \\ S & \subset & R. \end{array}$$

As in the proof of (2.1 in [P89b]), this implies that there exist unitary elements $u_i \in N$ such that the $\text{Ad}(u_i)\theta_{g_i}$ normalize R . Thus, after some inner perturbations,

θ normalizes R . Moreover, the condition $R' \cap \tilde{M} = N^{\text{op}}$ implies that the restriction of θ to R is a faithful Γ -kernel (i.e. $\theta(g)|_R$ is inner iff $g = e$). Altogether, if we denote by ρ the restriction of θ^{op} to R^{op} , then $\theta \otimes \theta^{\text{op}}$ normalizes N_∞ , the inclusion $N \vee (N' \cap N_\infty) \subset N_\infty$ is isomorphic to $N \overline{\otimes} R^{\text{op}} \rtimes_{\theta \otimes \rho} \Gamma$ and if $S_n = P_n \cap S = \overline{\bigcup_k P'_k \cap P_n}$, then

$$\begin{aligned} (N'_n \cap N_\infty)' \cap N_\infty &= (S_{n-1}^{\text{op}})' \cap N_\infty \\ &= (P_{n-1}^{\text{op}})' \cap \tilde{M} \cap N_\infty = (P'_{n-1} \cap \tilde{M})^{\text{op}} \cap N_\infty = N_n, \end{aligned}$$

thus proving the statement in the case when Γ is finitely generated.

Finally, let's settle the case when $\Gamma = \{g_n\}_n$ is infinitely generated. Let $\Gamma_m \subset \Gamma$ be the subgroup generated by $g_0 = e, g_1, \dots, g_m$ and denote $\tilde{M}_m = N \vee N^{\text{op}} \rtimes_{\sigma} \Gamma_m \subset \tilde{M}$. We then construct $N \supset P \supset P_1 \dots$ recursively so as to satisfy $E_{(P'_j \cap P_i)' \cap \tilde{M}_j}(x_i) \in_{2^{-j}} P'_i \cap \tilde{M}, \forall 1 \leq i \leq j \leq n$. If we let N_n, N_∞, R, S_n be defined as before, then the statement follows in its full generality. $\square$

Let us also mention that by combining Lemma 4.3 with results in [Oc83], one obtains a generalization of the vanishing 2-cohomology (Theorem 2.1 in [P89b]):

4.4. Corollary. *Let N be a II_1 factor, Γ a countable amenable group and $\theta : \Gamma \hookrightarrow \text{Out}(N)$ a faithful Γ -kernel with trivial obstruction, $\text{Ob}(\theta) = 1$. Then there exists a lifting $\theta'(g) \in \text{Aut}(N), g \in \Gamma$, such that $\theta'(g)\theta'(h) = \theta'(gh), \forall g, h \in \Gamma$.*

Proof. It is clearly sufficient to prove the result in the case when N is separable. By 4.3.1°, there exists a hyperfinite subfactor $R \subset N$ and a lifting of the Γ -kernel θ to automorphisms $\theta(g)$ that leave R invariant and satisfy $\theta(g)\theta(h) = \text{Ad}(u_{g,h})\theta(gh)$, for some $u_{g,h} \in \mathcal{U}(R), \forall g, h \in \Gamma$. By [Oc83], there exist $w_g \in \mathcal{U}(R)$ such that $u_{g,h} = \theta(g)(w_h^*)w_g^*w_{gh}$, modulo scalars, $\forall g, h \in \Gamma$. But then, $\theta'(g) = \text{Ad}_{w_g} \circ \theta(g)$ satisfy the desired conditions. $\square$

5. STRONG RIGIDITY FOR FACTORS ARISING FROM PROPER ACTIONS

We now apply the results in the previous sections to derive a strong rigidity result for crossed product II_1 factors arising from proper actions.

5.1. Theorem. *Let Γ_i be a countable amenable group, $\sigma_i : \Gamma_i \rightarrow \text{Aut}(N_i \overline{\otimes} R_i)$ a proper cocycle action on the s -McDuff factor $N_i \overline{\otimes} R_i$, leaving the non-Gamma factor N_i invariant, and denote by $M_i = N_i \overline{\otimes} R_i \rtimes_{\sigma_i} \Gamma_i$ the corresponding crossed product factor, $i = 0, 1$. Let $\Delta : M_0 \simeq M_1$. Then Δ implements a virtual cocycle conjugacy of σ_0, σ_1 , i.e. a virtual isomorphism of the inclusions $N_0 \overline{\otimes} R_0 \subset M_0, N_1 \overline{\otimes} R_1 \subset M_1$.*

If in addition Γ_0, Γ_1 are torsion free, then Δ implements a stable cocycle conjugacy of σ_0, σ_1 . More precisely, there exists $u \in \mathcal{U}(M_1), \{v_g\}_g \subset \mathcal{U}(N_1 \overline{\otimes} R_1)$, an isomorphism $\delta : \Gamma_0 \simeq \Gamma_1$ and a splitting $N_1^t \overline{\otimes} R_1^{1/t}$ of $N_1 \overline{\otimes} R_1$, for some $t > 0$, such that if we denote $\Delta' = \text{Adu} \circ \Delta$ and by $\{u_g^i\}_g$ the canonical unitaries in $M_i = N_i \overline{\otimes} R_i \rtimes \Gamma_i$, then we have:

- (a) $\Delta'(N_0) = N_1^t, \Delta'(R_0) = R_1^{1/t};$
- (b) $\Delta'(x_g u_g^0) = \Delta_0(x_g) v_{\delta(g)} u_{\delta(g)}^1, \forall x_g \in N_0 \overline{\otimes} R_0$, where $\Delta_0 : N_0 \overline{\otimes} R_0 \simeq N_1 \overline{\otimes} R_1 = N_1^t \overline{\otimes} R_1^{1/t}$ is the isomorphism implemented by the restriction of Δ' to $N_0 \overline{\otimes} R_0$.

In particular, Δ implements an outer conjugacy between the kernels (Γ_0, σ_0) and (Γ_1, σ_1^t) .

Proof. By Theorem 4.2, the inclusions $N_i \subset M_i$ are proper, so by Theorem 3.5, Δ implements a virtual isomorphism of $N_0 \overline{\otimes} R_0 \subset M_0$ and $N_1 \overline{\otimes} R_1 \subset M_1$. If, in addition, the Γ_i are torsion free, then by (Lemma 2 in [P04b]) it follows that $N_0 \overline{\otimes} R_0 = N_0 \vee (N'_0 \cap M)$ and $N_1 \overline{\otimes} R_1 = N_1 \vee (N'_1 \cap M)$ are conjugate by a unitary in M , where we identified $M_0 = M_1 = M$ via Δ . The rest of the statement follows from (Lemma 6.5.1 in [P06a]). $\square$

5.2. Remark. Let Γ be strongly amenable with respect to the set of generators $g_0 = 1, g_1, \dots, g_n \in \Gamma$ and let $\alpha \in H^3(\Gamma, \mathbb{T})$. If Q is a II_1 factor, then the proof of Lemma 4.3 shows that the universal construction of subfactors in ([P94], [P98]) gives rise to a canonical faithful Γ -kernel $\theta : \Gamma \hookrightarrow \text{Out}(N)$ with $\text{Ob}(\theta) = \alpha$, where $N = R' \cap (Q \overline{\otimes} S *_S R)$, $S \subset R$ being the diagonal subfactor corresponding to a faithful Γ -kernel on the hyperfinite II_1 factor, with obstruction $\overline{\alpha}$. Moreover, [PS00] shows that if $Q = L(\mathbb{F}_\infty)$, then $N \simeq L(\mathbb{F}_\infty)$. Thus, the considerations in 4.3 and 5.1 give canonical faithful Γ -kernels with given obstruction on the free group factor $L(\mathbb{F}_\infty)$, with discrete closure in $\text{Out}(L(\mathbb{F}_\infty))$. Moreover, by 5.1, if Γ is torsion free, then two such Γ -kernels $\theta : \Gamma \hookrightarrow \text{Out}(L(\mathbb{F}_\infty))$ are cocycle conjugate iff the crossed product factors $L(\mathbb{F}_\infty) \overline{\otimes} R \rtimes_{\theta \otimes \rho} \Gamma$ are isomorphic, where $\rho : \Gamma \hookrightarrow \text{Out}(R)$ is a model Γ -kernel on the hyperfinite II_1 factor with obstruction $\text{Ob}(\rho)$ equal to $\overline{\alpha}$. Noticing that given any Γ -kernel $\theta : \Gamma \hookrightarrow \text{Out}(N)$ we have a natural identification $(N \overline{\otimes} R \rtimes_{\theta \otimes \rho} \Gamma)^{\text{op}} = N^{\text{op}} \overline{\otimes} R^{\text{op}} \rtimes_{\theta^{\text{op}} \otimes \rho^{\text{op}}} \Gamma$ and that $\text{Ob}(\theta^{\text{op}}) = \overline{\text{Ob}(\theta)}$, by 5.1 it follows that if $\alpha \neq \overline{\alpha}$, then the factor $\tilde{M} = L(\mathbb{F}_\infty) \overline{\otimes} R \rtimes_{\theta \otimes \rho} \Gamma$ is not anti-isomorphic to itself. This situation occurs if for instance $\Gamma = \mathbb{Z}^3$ and one takes $\alpha \in H^3(\mathbb{Z}^3, \mathbb{T}) = \mathbb{T}$, with $\alpha \neq \pm 1$. Similarly, if α^n are distinct for $n = 1, 2, \dots$ (which in this case amounts to $\alpha = \exp(2\pi i t)$ with t irrational), then the tensor products $\tilde{M}^{\otimes n}$ are mutually nonisomorphic. Note that these examples of II_1 factors are very similar in spirit with the ones in [C75].

5.3. Remark. Note that if $P \subset N$ is an inclusion of non-Gamma factors with finite index, then, while the condition that N has spectral gap in the enveloping algebra N_∞ is automatic under the finite depth assumption, this is no longer true in the infinite depth case, not even under the “strong amenability” condition on the standard graph $\mathcal{G}_{P,N}$. This is already clear from 2.3, 4.2, 5.1, where we see (implicitly) that if $P \subset N$ is a locally trivial subfactor given by a Γ -kernel of a finitely generated group, then N has spectral gap in N_∞ iff Γ is closed in $\text{Out}(N)$. If $N = L(\mathbb{F}_n)$, for some $2 \leq n \leq \infty$, then given any finitely generated group Γ that can be embedded faithfully into a unitary group $\mathcal{U}(m)$ (with no scalar values other than 1) it is easy to provide two examples of faithful Γ -kernels σ_0, σ_1 on $L(\mathbb{F}_n)$, one with compact closure in $\text{Out}(L(\mathbb{F}_n))$, the other one discrete. For instance, one can take $\sigma_0 : \Gamma \rightarrow \text{Aut}(R)$ to be a compact product action, obtained from the restriction to Γ of the diagonal action (via Ad) of $O(n)/\mathbb{T}$ on $R = \overline{\otimes}_k (M_{m \times m}(\mathbb{C}))_k$, extended to $L(\mathbb{F}_n) = R * L(\mathbb{F}_{n-1})$ by $\sigma_0 = \sigma_0 * 1$, while σ_1 can be taken to be a Bernoulli action $\Gamma \curvearrowright R = \overline{\otimes}_{g \in \Gamma} (M_{m \times m}(\mathbb{C}))_g$, extended to $L(\mathbb{F}_n) = R * L(\mathbb{F}_{n-1})$ by $\sigma_1 = \sigma_1 * 1$.

In particular, one can construct two subfactors $P^i \subset N, i = 0, 1$, of the free group factor $N = L(\mathbb{F}_n)$, both with index 4 and graph $A_\infty^{(1)} = A_{-\infty, \infty}$, the first one with the property that N has spectral gap in the enveloping factor N_∞^0 of $P^0 \subset N$, the second one so that N doesn't have spectral gap in the enveloping factor N_∞^1 of $P^1 \subset N$.

Similarly, by using a construction in ([P89a]), one has two subfactors $P^i \subset N^i \simeq L(\mathbb{F}_{2n-1})$, $i = 0, 1$, of index 4 and graph D_∞ , such that N^0 has spectral gap in N_∞^0 while N^1 doesn't have spectral gap in N_∞^1 , as follows: Let θ_0 be a properly outer action of the infinite dihedral group $\Gamma = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ on $L(\mathbb{F}_n)$ such that $\theta_0(\Gamma)$ has compact closure in $\text{Out}(L(\mathbb{F}_n))$. Let θ_1 be a properly outer action of Γ with discrete closure in $\text{Out}(L(\mathbb{F}_n))$. In each case, denote by σ_i, ρ_i the period 2 automorphisms of $L(\mathbb{F}_n)$ generating $\theta_i(\Gamma)$. Then denote $(P^i \subset N^i) = (L(\mathbb{F}_n)^{\sigma_i} \subset L(\mathbb{F}_n) \rtimes \rho_i)$. By [P89a], both subfactors have graph D_∞ . On the other hand, it is easy to check that N^0 does not have spectral gap in N_∞^0 , while N^1 does have spectral gap in N_∞^1 . Also, by [R92] or [PS00], it follows that $N^0, N^1 \simeq L(\mathbb{F}_{2n-1})$.

In conclusion, we see that for non-Gamma subfactors of finite index $P \subset N$, the condition that N has spectral gap in the enveloping algebra N_∞ is an isomorphism invariant for $P \subset N$. The spectral gap property means that the semigroup of outer symmetries of N (i.e. correspondences, or endomorphisms on $N^\infty = N \overline{\otimes} \mathcal{B}(\ell^2 \mathbb{N})$), generated by the irreducible direct summands of the Hilbert bimodule ${}_N L^2(N_\infty)_N$, is closed in $\text{End}(N^\infty)/\text{Int}(N^\infty)$. At the opposite end, we have the case when this semigroup is precompact.

5.4. Remark. The notion of proper action on an s-McDuff factor, $\Gamma \curvearrowright^\sigma N \overline{\otimes} R$, doesn't in fact need the condition that σ normalizes N, R . Indeed, by 2.3.1°, this is automatic if one considers instead the cocycle action implemented by σ on the II_∞ factors $N^\infty \overline{\otimes} R^\infty$. This, of course, entails a possible appearance of a trace-scaling restriction $\theta = \sigma|_{N^\infty}, \rho = \sigma|_{R^\infty}$, with $\text{mod} \theta(g) = \text{mod}(\rho(g))^{-1}, \forall g \in \Gamma$. Theorem 4.2 holds true for these general proper actions of countable amenable groups on II_∞ s-McDuff factors. The proof relies on the appropriate generalization of Lemma 4.3 to Γ -kernels in $\text{Out}(N^\infty)$, whose proof is very similar in spirit to the proof of 4.3, but more “painful” to formalize. Consequently, Theorem 5.1 holds true in this generality as well.

5.5. Remark. Finally, let us point out that a generalization of Lemma 4.3 in the abstract framework of subfactors holds true, as follows: Given an inclusion of separable factors of type II_1 , $N_{-1} \subset N_0$, with finite index and amenable graph (not necessarily extremal!), there exists a decreasing sequence of subfactors of finite index $N_0 \supset N_{-1} \supset N_{-2} \supset \dots$ such that if we denote $R_0 = \overline{\bigcup_n N'_n} \cap N_0$, $R_{-1} = \overline{\bigcup_n N'_{-n}} \cap N_{-1}$, then $R'_i \cap N_n = N'_i \cap N_n$, for any $i = 0, -1$ and $n \geq 0$, where $N_{-1} \subset N_0 \subset N_1 \subset \dots$ is the Jones tower. The proof is similar to the proof of 4.3.2°. This can be used to show that if N_0 has spectral gap in N_∞ , then $N_0 \subset N_\infty$ is a proper inclusion, i.e. satisfies conditions 3.3.

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