THE RAHMAN POLYNOMIALS AND THE LIE ALGEBRA $\mathfrak{sl}_3(\mathbb{C})$

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ABSTRACT. We interpret the Rahman polynomials in terms of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Using the parameters of the polynomials we define two Cartan subalgebras for $\mathfrak{sl}_3(\mathbb{C})$, denoted H and $\tilde{H}$. We display an antiautomorphism $\dagger$ of $\mathfrak{sl}_3(\mathbb{C})$ that fixes each element of H and each element of $\tilde{H}$. We consider a certain finite-dimensional irreducible $\mathfrak{sl}_3(\mathbb{C})$ -module V consisting of homogeneous polynomials in three variables. We display a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V such that $\langle \beta\xi, \zeta \rangle = \langle \xi, \beta^\dagger \zeta \rangle$ for all $\beta \in \mathfrak{sl}_3(\mathbb{C})$ and $\xi, \zeta \in V$. We display two bases for V ; one diagonalizes H and the other diagonalizes $\tilde{H}$. Both bases are orthogonal with respect to $\langle \cdot, \cdot \rangle$. We show that when $\langle \cdot, \cdot \rangle$ is applied to a vector in each basis, the result is a trivial factor times a Rahman polynomial evaluated at an appropriate argument. Thus for both transition matrices between the bases each entry is described by a Rahman polynomial. From these results we recover the previously known orthogonality relation for the Rahman polynomials. We also obtain two seven-term recurrence relations satisfied by the Rahman polynomials, along with the corresponding relations satisfied by the dual polynomials. These recurrence relations show that the Rahman polynomials are bispectral. In our theory the roles of H and $\tilde{H}$ are interchangeable, and for us this explains the duality and bispectrality of the Rahman polynomials. We view the action of H and $\tilde{H}$ on V as a rank 2 generalization of a Leonard pair.

1. THE RAHMAN POLYNOMIALS

We begin by recalling the Rahman polynomials [4], [5]. In what follows $\{p_i\}_{i=1}^4$ denote complex numbers. They are essentially arbitrary, although certain combinations are forbidden in order to avoid dividing by zero. Define

$$\begin{aligned} t &= \frac{(p_1 + p_2)(p_1 + p_3)}{p_1(p_1 + p_2 + p_3 + p_4)}, & u &= \frac{(p_1 + p_3)(p_3 + p_4)}{p_3(p_1 + p_2 + p_3 + p_4)}, \\ v &= \frac{(p_1 + p_2)(p_2 + p_4)}{p_2(p_1 + p_2 + p_3 + p_4)}, & w &= \frac{(p_2 + p_4)(p_3 + p_4)}{p_4(p_1 + p_2 + p_3 + p_4)}. \end{aligned}$$

Fix an integer $N \geq 0$ and let a, b, c, d denote mutually commuting indeterminates. Define

$$P(a, b, c, d) = \sum_{\substack{0 \leq i, j, k, \ell \\ i+j+k+\ell \leq N}} \frac{(-a)_{i+j}(-b)_{k+\ell}(-c)_{i+k}(-d)_{j+\ell}}{i!j!k!\ell!(-N)_{i+j+k+\ell}} t^i u^j v^k w^\ell.$$

We are using the shifted factorial notation

$$(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1), \quad n = 0, 1, 2, \dots$$

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We will use the fact that for nonnegative integers m, n ,

$$(1) \quad (-m)_n = 0 \quad \text{if} \quad n > m.$$

For nonnegative integers m, n whose sum is at most N , the corresponding Rahman polynomial is $P(m, n, c, d)$ in the variables c, d , and the corresponding dual Rahman polynomial is $P(a, b, m, n)$ in the variables a, b [5, Section 2]. The Rahman polynomials and their duals satisfy an orthogonality relation which we now describe. Define

$$\nu = \frac{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)(p_3 + p_4)}{(p_1 p_4 - p_2 p_3)^2}.$$

Define $\eta_0 = \nu^{-1}$ and

$$\begin{aligned} \eta_1 &= \frac{p_1 p_2 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)}, \\ \eta_2 &= \frac{p_3 p_4 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_2 + p_4)(p_3 + p_4)}. \end{aligned}$$

Define $\tilde{\eta}_0 = \nu^{-1}$ and

$$\begin{aligned} \tilde{\eta}_1 &= \frac{p_1 p_3 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)}, \\ \tilde{\eta}_2 &= \frac{p_2 p_4 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)}. \end{aligned}$$

A short computation shows

$$\eta_0 + \eta_1 + \eta_2 = 1, \quad \tilde{\eta}_0 + \tilde{\eta}_1 + \tilde{\eta}_2 = 1.$$

For notational convenience define $k_i = \nu \tilde{\eta}_i$ and $\tilde{k}_i = \nu \eta_i$ for $0 \leq i \leq 2$ so that $k_0 = 1$ and $\tilde{k}_0 = 1$.

We credit the following result to Rahman and Hoare [5]. However, a similar and more general theorem was given earlier in [11, Theorem 1.1].

Theorem 1.1 (Rahman and Hoare [5]). *Fix nonnegative integers s, t whose sum is at most N , and nonnegative integers σ, τ whose sum is at most N . Then both*

$$(2) \quad \sum_{\substack{0 \leq i, j, k \\ i+j+k=N}} P(j, k, s, t) P(j, k, \sigma, \tau) \tilde{\eta}_0^i \tilde{\eta}_1^j \tilde{\eta}_2^k \begin{pmatrix} N \\ i \ j \ k \end{pmatrix} = \frac{\delta_{s\sigma} \delta_{t\tau}}{\tilde{k}_1^s \tilde{k}_2^t} \begin{pmatrix} N \\ r \ s \ t \end{pmatrix}^{-1},$$

$$(3) \quad \sum_{\substack{0 \leq i, j, k \\ i+j+k=N}} P(s, t, j, k) P(\sigma, \tau, j, k) \eta_0^i \eta_1^j \eta_2^k \begin{pmatrix} N \\ i \ j \ k \end{pmatrix} = \frac{\delta_{s\sigma} \delta_{t\tau}}{k_1^s k_2^t} \begin{pmatrix} N \\ r \ s \ t \end{pmatrix}^{-1},$$

where $r = N - s - t$.

In this paper we interpret the Rahman polynomials in terms of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Our results are summarized as follows. Using the parameters $\{p_i\}_{i=1}^4$ we define two Cartan subalgebras for $\mathfrak{sl}_3(\mathbb{C})$, denoted H and $\tilde{H}$. We display an anti-automorphism $\dagger$ of $\mathfrak{sl}_3(\mathbb{C})$ that fixes each element of H and each element of $\tilde{H}$. We consider an irreducible $\mathfrak{sl}_3(\mathbb{C})$ -module V consisting of the homogeneous polynomials in three variables that have total degree N . We display a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V such that $\langle \beta \xi, \zeta \rangle = \langle \xi, \beta^\dagger \zeta \rangle$ for all $\beta \in \mathfrak{sl}_3(\mathbb{C})$ and $\xi, \zeta \in V$. We display two bases for V ; one diagonalizes H and the other diagonalizes $\tilde{H}$. Both bases are orthogonal with respect to $\langle \cdot, \cdot \rangle$. We show that when $\langle \cdot, \cdot \rangle$ is applied to

a vector in each basis, the result is a trivial factor times a Rahman polynomial evaluated at an appropriate argument. Thus for both transition matrices between the bases each entry is described by a Rahman polynomial. From these results we obtain an elementary proof of Theorem 1.1. We also obtain two seven-term recurrence relations satisfied by the Rahman polynomials, along with the corresponding relations satisfied by their duals. These recurrence relations show that the Rahman polynomials are bispectral, a feature hinted at by Grünbaum [4]. We view the actions of H and $\tilde{H}$ on V as a rank 2 generalization of a Leonard pair [12]. This is discussed along with some open problems at the end of the paper.

2. THE LIE ALGEBRA $\mathfrak{sl}_3(\mathbb{C})$

We will be discussing the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$; for background information see [1]. Let $\text{Mat}_3(\mathbb{C})$ denote the $\mathbb{C}$ -algebra consisting of the 3×3 matrices that have all entries in $\mathbb{C}$. We index the rows and columns by $0, 1, 2$. For $0 \leq i, j \leq 2$ let e_{ij} denote the matrix in $\text{Mat}_3(\mathbb{C})$ that has (i, j) -entry 1 and all other entries 0. The Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is the set of matrices in $\text{Mat}_3(\mathbb{C})$ that have trace 0, together with the Lie bracket $[\beta, \gamma] = \beta\gamma - \gamma\beta$. We will consider two Cartan subalgebras of $\mathfrak{sl}_3(\mathbb{C})$, denoted H and $\tilde{H}$. The subalgebra H consists of the diagonal matrices in $\mathfrak{sl}_3(\mathbb{C})$. Define

$$\varphi = \text{diag}(-1/3, 2/3, -1/3), \quad \phi = \text{diag}(-1/3, -1/3, 2/3).$$

Then φ, ϕ form a basis for H . We will describe $\tilde{H}$ shortly. Define

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1-t & 1-v \\ 1 & 1-u & 1-w \end{pmatrix}$$

and

$$W = \text{diag}(\eta_0, \eta_1, \eta_2), \quad \tilde{W} = \text{diag}(\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2).$$

A brief calculation shows $\nu W U \tilde{W} U^t = I$, where I denotes the identity matrix and U^t denotes the transpose of U . Define

$$\vartheta = \frac{(p_1 + p_3)(p_2 + p_4)}{p_2 p_3 - p_1 p_4}, \quad \tilde{\vartheta} = \frac{(p_1 + p_2)(p_3 + p_4)}{p_2 p_3 - p_1 p_4}$$

and note that $\vartheta \tilde{\vartheta} = \nu$. Define $R = \tilde{\vartheta} \tilde{W} U^t$ and note that $R^{-1} = \vartheta W U$. We have

$$R = \begin{pmatrix} \frac{p_2 p_3 - p_1 p_4}{(p_1 + p_3)(p_2 + p_4)} & \frac{p_2 p_3 - p_1 p_4}{(p_1 + p_3)(p_2 + p_4)} & \frac{p_2 p_3 - p_1 p_4}{(p_1 + p_3)(p_2 + p_4)} \\ \frac{p_1 p_3 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_2 p_3 - p_1 p_4)} & \frac{-p_3}{p_1 + p_3} & \frac{p_1}{p_1 + p_3} \\ \frac{p_2 p_4 (p_1 + p_2 + p_3 + p_4)}{(p_2 + p_4)(p_2 p_3 - p_1 p_4)} & \frac{p_4}{p_2 + p_4} & \frac{-p_2}{p_2 + p_4} \end{pmatrix}$$

and

$$R^{-1} = \begin{pmatrix} \frac{p_2 p_3 - p_1 p_4}{(p_1 + p_2)(p_3 + p_4)} & \frac{p_2 p_3 - p_1 p_4}{(p_1 + p_2)(p_3 + p_4)} & \frac{p_2 p_3 - p_1 p_4}{(p_1 + p_2)(p_3 + p_4)} \\ \frac{p_1 p_2 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 p_3 - p_1 p_4)} & \frac{-p_2}{p_1 + p_2} & \frac{p_1}{p_1 + p_2} \\ \frac{p_3 p_4 (p_1 + p_2 + p_3 + p_4)}{(p_3 + p_4)(p_2 p_3 - p_1 p_4)} & \frac{p_4}{p_3 + p_4} & \frac{-p_3}{p_3 + p_4} \end{pmatrix}.$$

Note that $R W R^t = \tilde{\vartheta} \vartheta^{-1} \tilde{W}$. For $0 \leq i, j \leq 2$ define $\tilde{e}_{ij} = R e_{ij} R^{-1}$. Define $\tilde{H} = R H R^{-1}$ and note that $\tilde{H}$ is a Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{C})$. Define $\tilde{\varphi} = R \varphi R^{-1}$

and $\tilde{\phi} = R\phi R^{-1}$. Note that $\tilde{\varphi}, \tilde{\phi}$ is a basis for $\tilde{H}$. We have

$$\begin{aligned}
\tilde{\varphi} &= \frac{p_2(p_1p_4 - p_2p_3)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)}e_{01} + \frac{p_1(p_2p_3 - p_1p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)}e_{02} \\
&+ \frac{p_1p_2p_3(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_1p_4 - p_2p_3)}e_{10} - \frac{p_1p_3}{(p_1 + p_2)(p_1 + p_3)}e_{12} \\
&+ \frac{p_1p_2p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_2p_3 - p_1p_4)}e_{20} - \frac{p_2p_4}{(p_1 + p_2)(p_2 + p_4)}e_{21} \\
&+ \left(\frac{p_2p_3}{(p_1 + p_2)(p_1 + p_3)} - \frac{p_1p_2(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)} \right) \varphi \\
&+ \left(\frac{p_1p_4}{(p_1 + p_2)(p_2 + p_4)} - \frac{p_1p_2(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)} \right) \phi, \\
\tilde{\phi} &= \frac{p_4(p_2p_3 - p_1p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2 + p_4)}e_{01} + \frac{p_3(p_1p_4 - p_2p_3)}{(p_1 + p_3)(p_3 + p_4)(p_2 + p_4)}e_{02} \\
&+ \frac{p_1p_3p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2p_3 - p_1p_4)}e_{10} - \frac{p_1p_3}{(p_1 + p_3)(p_3 + p_4)}e_{12} \\
&+ \frac{p_2p_3p_4(p_1 + p_2 + p_3 + p_4)}{(p_2 + p_4)(p_3 + p_4)(p_1p_4 - p_2p_3)}e_{20} - \frac{p_2p_4}{(p_2 + p_4)(p_3 + p_4)}e_{21} \\
&+ \left(\frac{p_1p_4}{(p_1 + p_3)(p_3 + p_4)} - \frac{p_3p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2 + p_4)} \right) \varphi \\
&+ \left(\frac{p_2p_3}{(p_2 + p_4)(p_3 + p_4)} - \frac{p_3p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2 + p_4)} \right) \phi, \\
\varphi &= \frac{p_3(p_1p_4 - p_2p_3)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)}\tilde{e}_{01} + \frac{p_1(p_2p_3 - p_1p_4)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)}\tilde{e}_{02} \\
&+ \frac{p_1p_2p_3(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_1p_4 - p_2p_3)}\tilde{e}_{10} - \frac{p_1p_2}{(p_1 + p_2)(p_1 + p_3)}\tilde{e}_{12} \\
&+ \frac{p_1p_3p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2p_3 - p_1p_4)}\tilde{e}_{20} - \frac{p_3p_4}{(p_1 + p_3)(p_3 + p_4)}\tilde{e}_{21} \\
&+ \left(\frac{p_2p_3}{(p_1 + p_2)(p_1 + p_3)} - \frac{p_1p_3(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)} \right) \tilde{\varphi} \\
&+ \left(\frac{p_1p_4}{(p_1 + p_3)(p_3 + p_4)} - \frac{p_1p_3(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)} \right) \tilde{\phi}, \\
\phi &= \frac{p_4(p_2p_3 - p_1p_4)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)}\tilde{e}_{01} + \frac{p_2(p_1p_4 - p_2p_3)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)}\tilde{e}_{02} \\
&+ \frac{p_1p_2p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_2p_3 - p_1p_4)}\tilde{e}_{10} - \frac{p_1p_2}{(p_1 + p_2)(p_2 + p_4)}\tilde{e}_{12} \\
&+ \frac{p_2p_3p_4(p_1 + p_2 + p_3 + p_4)}{(p_2 + p_4)(p_3 + p_4)(p_1p_4 - p_2p_3)}\tilde{e}_{20} - \frac{p_3p_4}{(p_2 + p_4)(p_3 + p_4)}\tilde{e}_{21} \\
&+ \left(\frac{p_1p_4}{(p_1 + p_2)(p_2 + p_4)} - \frac{p_2p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)} \right) \tilde{\varphi} \\
&+ \left(\frac{p_2p_3}{(p_2 + p_4)(p_3 + p_4)} - \frac{p_2p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)} \right) \tilde{\phi}.
\end{aligned}$$

By an *antiautomorphism* of $\mathfrak{sl}_3(\mathbb{C})$ we mean a $\mathbb{C}$ -linear bijection $\theta : \mathfrak{sl}_3(\mathbb{C}) \rightarrow \mathfrak{sl}_3(\mathbb{C})$ such that $[\beta, \gamma]^\theta = -[\beta^\theta, \gamma^\theta]$ for all $\beta, \gamma \in \mathfrak{sl}_3(\mathbb{C})$. There exists an antiautomorphism

$\dagger$ of $\mathfrak{sl}_3(\mathbb{C})$ such that $\beta^\dagger = \tilde{W}\beta\tilde{W}^{-1}$ for all $\beta \in \mathfrak{sl}_3(\mathbb{C})$. Note that $(\beta^\dagger)^\dagger = \beta$ for all $\beta \in \mathfrak{sl}_3(\mathbb{C})$. We have

$$(4) \quad \begin{array}{c|ccc|ccc|cc} \beta & e_{01} & e_{12} & e_{02} & e_{10} & e_{21} & e_{20} & \varphi & \phi \\ \hline \beta^\dagger & e_{10}\tilde{\eta}_1/\tilde{\eta}_0 & e_{21}\tilde{\eta}_2/\tilde{\eta}_1 & e_{20}\tilde{\eta}_2/\tilde{\eta}_0 & e_{01}\tilde{\eta}_0/\tilde{\eta}_1 & e_{12}\tilde{\eta}_1/\tilde{\eta}_2 & e_{02}\tilde{\eta}_0/\tilde{\eta}_2 & \varphi & \phi \end{array}$$

Using $RWR^t = \tilde{\vartheta}\vartheta^{-1}\tilde{W}$ one finds

$$(5) \quad \begin{array}{c|ccc|ccc|cc} \beta & \tilde{e}_{01} & \tilde{e}_{12} & \tilde{e}_{02} & \tilde{e}_{10} & \tilde{e}_{21} & \tilde{e}_{20} & \tilde{\varphi} & \tilde{\phi} \\ \hline \beta^\dagger & \tilde{e}_{10}\eta_1/\eta_0 & \tilde{e}_{21}\eta_2/\eta_1 & \tilde{e}_{20}\eta_2/\eta_0 & \tilde{e}_{01}\eta_0/\eta_1 & \tilde{e}_{12}\eta_1/\eta_2 & \tilde{e}_{02}\eta_0/\eta_2 & \tilde{\varphi} & \tilde{\phi} \end{array}$$

Note that $\dagger$ fixes each element of H and each element of $\tilde{H}$. We now show that H and $\tilde{H}$ together generate $\mathfrak{sl}_3(\mathbb{C})$. Define $\psi = -\varphi - \phi$ and $\tilde{\psi} = -\tilde{\varphi} - \tilde{\phi}$. Then

$$\begin{aligned} e_{01} &= \frac{-[\varphi, [\psi, \tilde{\psi}]] + [\varphi, [\varphi, [\psi, \tilde{\psi}]]]}{2\tilde{\eta}_0}, & e_{10} &= \frac{-[\varphi, [\psi, \tilde{\psi}]] - [\varphi, [\varphi, [\psi, \tilde{\psi}]]]}{2\tilde{\eta}_1}, \\ e_{02} &= \frac{-[\phi, [\psi, \tilde{\psi}]] + [\phi, [\phi, [\psi, \tilde{\psi}]]]}{2\tilde{\eta}_0}, & e_{20} &= \frac{-[\phi, [\psi, \tilde{\psi}]] - [\phi, [\phi, [\psi, \tilde{\psi}]]]}{2\tilde{\eta}_2}, \\ e_{12} &= \frac{-[\varphi, [\phi, \tilde{\psi}]] - [\varphi, [\varphi, [\phi, \tilde{\psi}]]]}{2\tilde{\eta}_1}, & e_{21} &= \frac{-[\varphi, [\phi, \tilde{\psi}]] + [\varphi, [\varphi, [\phi, \tilde{\psi}]]]}{2\tilde{\eta}_2}. \end{aligned}$$

3. AN $\mathfrak{sl}_3(\mathbb{C})$ -MODULE

We now define a certain $\mathfrak{sl}_3(\mathbb{C})$ -module. Let x, y, z denote mutually commuting indeterminates. Let $\mathbb{C}[x, y, z]$ denote the $\mathbb{C}$ -algebra consisting of the polynomials in x, y, z that have all coefficients in $\mathbb{C}$. We abbreviate $A = \mathbb{C}[x, y, z]$. By a *derivation* of A we mean a $\mathbb{C}$ -linear map $\partial : A \rightarrow A$ such that $\partial(\xi\zeta) = \partial(\xi)\zeta + \xi\partial(\zeta)$ for all $\xi, \zeta \in A$. By [1, p. 207] the space A is an $\mathfrak{sl}_3(\mathbb{C})$ -module on which each element of $\mathfrak{sl}_3(\mathbb{C})$ acts as a derivation and

$$\begin{array}{c|ccc|ccc|cc} \xi & e_{01}.\xi & e_{12}.\xi & e_{02}.\xi & e_{10}.\xi & e_{21}.\xi & e_{20}.\xi & \varphi.\xi & \phi.\xi \\ \hline x & 0 & 0 & 0 & y & 0 & z & -x/3 & -x/3 \\ y & x & 0 & 0 & 0 & z & 0 & 2y/3 & -y/3 \\ z & 0 & y & x & 0 & 0 & 0 & -z/3 & 2z/3 \end{array}$$

Let V denote the subspace of A consisting of the homogeneous polynomials that have total degree N . The following is a basis for V :

$$(6) \quad x^r y^s z^t, \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.$$

The action of $\mathfrak{sl}_3(\mathbb{C})$ on this basis is described as follows:

$$\begin{array}{c|ccc} \xi & e_{01}.\xi & e_{12}.\xi & e_{02}.\xi \\ \hline x^r y^s z^t & s x^{r+1} y^{s-1} z^t & t x^r y^{s+1} z^{t-1} & t x^{r+1} y^s z^{t-1} \\ \hline \xi & e_{10}.\xi & e_{21}.\xi & e_{20}.\xi \\ \hline x^r y^s z^t & r x^{r-1} y^{s+1} z^t & s x^r y^{s-1} z^{t+1} & r x^{r-1} y^s z^{t+1} \\ \hline \xi & \varphi.\xi & \phi.\xi \\ \hline x^r y^s z^t & (s - N/3) x^r y^s z^t & (t - N/3) x^r y^s z^t \end{array}$$

By the above data V is an $\mathfrak{sl}_3(\mathbb{C})$ -submodule of A , and this submodule is irreducible [9, p. 97]. Let $\mathbb{I}$ denote the set consisting of the 3-tuples of nonnegative integers whose sum is N . For $\lambda = (r, s, t) \in \mathbb{I}$ let V_λ denote the subspace of V spanned by

$x^r y^s z^t$. Then $V = \sum_{\lambda \in \mathbb{I}} V_\lambda$ (direct sum), and this is the weight space decomposition of V with respect to $\tilde{H}$. By construction $\dim(V_\lambda) = 1$ for all $\lambda \in \mathbb{I}$.

We now consider the weight space decomposition of V with respect to $\tilde{H}$. To describe this decomposition we make a change of variables. Recall the matrix R and define

$$\begin{aligned}\tilde{x} &= R_{00}x + R_{10}y + R_{20}z, \\ \tilde{y} &= R_{01}x + R_{11}y + R_{21}z, \\ \tilde{z} &= R_{02}x + R_{12}y + R_{22}z.\end{aligned}$$

Thus R is the transition matrix from x, y, z to $\tilde{x}, \tilde{y}, \tilde{z}$. Using $R = \tilde{\vartheta}\tilde{W}U^t$ we obtain

$$(7) \quad \tilde{\vartheta}^{-1}\tilde{x} = \tilde{\eta}_0x + \tilde{\eta}_1y + \tilde{\eta}_2z,$$

$$(8) \quad \tilde{\vartheta}^{-1}\tilde{y} = \tilde{\vartheta}^{-1}\tilde{x} - t\tilde{\eta}_1y - v\tilde{\eta}_2z,$$

$$(9) \quad \tilde{\vartheta}^{-1}\tilde{z} = \tilde{\vartheta}^{-1}\tilde{x} - u\tilde{\eta}_1y - w\tilde{\eta}_2z.$$

Using $R^{-1} = \vartheta WU$ we obtain

$$\begin{aligned}\vartheta^{-1}x &= \eta_0\tilde{x} + \eta_1\tilde{y} + \eta_2\tilde{z}, \\ \vartheta^{-1}y &= \vartheta^{-1}x - t\eta_1\tilde{y} - u\eta_2\tilde{z}, \\ \vartheta^{-1}z &= \vartheta^{-1}x - v\eta_1\tilde{y} - w\eta_2\tilde{z}.\end{aligned}$$

The action of $\mathfrak{sl}_3(\mathbb{C})$ on $\tilde{x}, \tilde{y}, \tilde{z}$ is described as follows:

ξ	$\tilde{e}_{01} \cdot \xi$	$\tilde{e}_{12} \cdot \xi$	$\tilde{e}_{02} \cdot \xi$	$\tilde{e}_{10} \cdot \xi$	$\tilde{e}_{21} \cdot \xi$	$\tilde{e}_{20} \cdot \xi$	$\tilde{\varphi} \cdot \xi$	$\tilde{\phi} \cdot \xi$
$\tilde{x}$	0	0	0	$\tilde{y}$	0	$\tilde{z}$	$-\tilde{x}/3$	$-\tilde{x}/3$
$\tilde{y}$	$\tilde{x}$	0	0	0	$\tilde{z}$	0	$2\tilde{y}/3$	$-\tilde{y}/3$
$\tilde{z}$	0	$\tilde{y}$	$\tilde{x}$	0	0	0	$-\tilde{z}/3$	$2\tilde{z}/3$

The following is a basis for V :

$$(10) \quad \tilde{x}^r \tilde{y}^s \tilde{z}^t, \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.$$

The action of $\mathfrak{sl}_3(\mathbb{C})$ on this basis is described as follows:

ξ	$\tilde{e}_{01} \cdot \xi$	$\tilde{e}_{12} \cdot \xi$	$\tilde{e}_{02} \cdot \xi$
$\tilde{x}^r \tilde{y}^s \tilde{z}^t$	$s\tilde{x}^{r+1}\tilde{y}^{s-1}\tilde{z}^t$	$t\tilde{x}^r\tilde{y}^{s+1}\tilde{z}^{t-1}$	$t\tilde{x}^{r+1}\tilde{y}^s\tilde{z}^{t-1}$
ξ	$\tilde{e}_{10} \cdot \xi$	$\tilde{e}_{21} \cdot \xi$	$\tilde{e}_{20} \cdot \xi$
$\tilde{x}^r \tilde{y}^s \tilde{z}^t$	$r\tilde{x}^{r-1}\tilde{y}^{s+1}\tilde{z}^t$	$s\tilde{x}^r\tilde{y}^{s-1}\tilde{z}^{t+1}$	$r\tilde{x}^{r-1}\tilde{y}^s\tilde{z}^{t+1}$
ξ	$\tilde{\varphi} \cdot \xi$	$\tilde{\phi} \cdot \xi$	
$\tilde{x}^r \tilde{y}^s \tilde{z}^t$	$(s - N/3)\tilde{x}^r\tilde{y}^s\tilde{z}^t$	$(t - N/3)\tilde{x}^r\tilde{y}^s\tilde{z}^t$	

For each $\lambda = (r, s, t) \in \mathbb{I}$ let $\tilde{V}_\lambda$ denote the subspace of V spanned by $\tilde{x}^r \tilde{y}^s \tilde{z}^t$. Observe that $V = \sum_{\lambda \in \mathbb{I}} \tilde{V}_\lambda$ (direct sum), and this is the weight space decomposition of V with respect to $\tilde{H}$. By construction $\dim(\tilde{V}_\lambda) = 1$ for all $\lambda \in \mathbb{I}$.

We comment on how H and $\tilde{H}$ act on the weight spaces of each other. A pair of elements (r, s, t) and (r', s', t') in $\mathbb{I}$ will be called *adjacent* whenever $(r-r', s-s', t-t')$ is a permutation of $(1, -1, 0)$. Then H and $\tilde{H}$ act on the weight spaces of each other as follows. For all $\lambda \in \mathbb{I}$,

$$(11) \quad \tilde{H}V_\lambda \subseteq V_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \text{ adj } \lambda}} V_\mu, \quad H\tilde{V}_\lambda \subseteq \tilde{V}_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \text{ adj } \lambda}} \tilde{V}_\mu.$$

4. A BILINEAR FORM

In this section we introduce a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V . As we will see, both

$$(12) \quad \langle V_\lambda, V_\mu \rangle = 0 \quad \text{if} \quad \lambda \neq \mu, \quad \lambda, \mu \in \mathbb{I},$$

$$(13) \quad \langle \tilde{V}_\lambda, \tilde{V}_\mu \rangle = 0 \quad \text{if} \quad \lambda \neq \mu, \quad \lambda, \mu \in \mathbb{I}.$$

We define $\langle \cdot, \cdot \rangle$ as follows. With respect to $\langle \cdot, \cdot \rangle$ the vectors (6) are mutually orthogonal and

$$(14) \quad \|x^r y^s z^t\|^2 = \frac{r!s!t!}{\tilde{\eta}_0^r \tilde{\eta}_1^s \tilde{\eta}_2^t} \vartheta^N, \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.$$

We are using the notation $\|\xi\|^2 = \langle \xi, \xi \rangle$. The form $\langle \cdot, \cdot \rangle$ is symmetric, nondegenerate, and satisfies (12). Using (4) and the tables below (6) we obtain

$$(15) \quad \langle \beta \xi, \zeta \rangle = \langle \xi, \beta^\dagger \zeta \rangle \quad \forall \beta \in \mathfrak{sl}_3(\mathbb{C}), \quad \forall \xi, \zeta \in V.$$

Line (13) follows from (15) and since $\dagger$ fixes each element of $\tilde{H}$. The following is the basis for V that is dual to (6) with respect to $\langle \cdot, \cdot \rangle$:

$$(16) \quad \frac{\tilde{\eta}_0^r \tilde{\eta}_1^s \tilde{\eta}_2^t}{r!s!t!} \frac{x^r y^s z^t}{\vartheta^N}, \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.$$

The vector $\tilde{x}^N$ is equal to $N!\nu^N$ times the sum of the vectors (16). This is verified using (7) and the trinomial theorem.

Lemma 4.1. *With respect to $\langle \cdot, \cdot \rangle$ the vectors (10) are mutually orthogonal and*

$$(17) \quad \|\tilde{x}^r \tilde{y}^s \tilde{z}^t\|^2 = \frac{r!s!t!}{\tilde{\eta}_0^r \tilde{\eta}_1^s \tilde{\eta}_2^t} \tilde{\vartheta}^N, \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.$$

Proof. By (13) the vectors (10) are mutually orthogonal. We now verify (17). We do this by induction on $s + t$. First assume $s + t = 0$ so that $(r, s, t) = (N, 0, 0)$. We must show $\|\tilde{x}^N\|^2 = N! \eta_0^{-N} \tilde{\vartheta}^N$. The vector $\tilde{x}^N$ is equal to $N!\nu^N$ times the sum of the vectors (16), and these vectors are mutually orthogonal. Therefore $\|\tilde{x}^N\|^2$ is equal to $(N!)^2 \nu^{2N}$ times the sum of the square norms of the vectors (16). Computing this sum using (14) we obtain

$$\begin{aligned} \|\tilde{x}^N\|^2 &= (N!)^2 \nu^{2N} \vartheta^{-N} \sum_{\substack{0 \leq r, s, t \\ r+s+t=N}} \frac{\tilde{\eta}_0^r \tilde{\eta}_1^s \tilde{\eta}_2^t}{r!s!t!} \\ &= N! \nu^{2N} \vartheta^{-N} \sum_{\substack{0 \leq r, s, t \\ r+s+t=N}} \tilde{\eta}_0^r \tilde{\eta}_1^s \tilde{\eta}_2^t \binom{N}{r \ s \ t} \\ &= N! \nu^{2N} \vartheta^{-N} (\tilde{\eta}_0 + \tilde{\eta}_1 + \tilde{\eta}_2)^N \\ &= N! \nu^{2N} \vartheta^{-N} \\ &= N! \eta_0^{-N} \tilde{\vartheta}^N. \end{aligned}$$

Next assume $s + t > 0$ so that $s > 0$ or $t > 0$. We assume $s > 0$; the case $t > 0$ is similar. By (15),

$$(18) \quad \langle \tilde{e}_{01} \tilde{x}^r \tilde{y}^s \tilde{z}^t, \tilde{x}^{r+1} \tilde{y}^{s-1} \tilde{z}^t \rangle = \langle \tilde{x}^r \tilde{y}^s \tilde{z}^t, \tilde{e}_{01}^\dagger \tilde{x}^{r+1} \tilde{y}^{s-1} \tilde{z}^t \rangle.$$

Using the first table below (10) and then induction,

$$\begin{aligned}\langle \tilde{e}_{01} \tilde{x}^r \tilde{y}^s \tilde{z}^t, \tilde{x}^{r+1} \tilde{y}^{s-1} \tilde{z}^t \rangle &= \|\tilde{x}^{r+1} \tilde{y}^{s-1} \tilde{z}^t\|^2 s \\ &= \frac{(r+1)!s!t!}{\eta_0^{r+1} \eta_1^{s-1} \eta_2^t} \tilde{\vartheta}^N.\end{aligned}$$

Using (5) and the second table below (10),

$$\begin{aligned}\langle \tilde{x}^r \tilde{y}^s \tilde{z}^t, \tilde{e}_{01}^\dagger \tilde{x}^{r+1} \tilde{y}^{s-1} \tilde{z}^t \rangle &= \langle \tilde{x}^r \tilde{y}^s \tilde{z}^t, \tilde{e}_{10} \tilde{x}^{r+1} \tilde{y}^{s-1} \tilde{z}^t \rangle \eta_1 / \eta_0 \\ &= \|\tilde{x}^r \tilde{y}^s \tilde{z}^t\|^2 (r+1) \eta_1 / \eta_0.\end{aligned}$$

Evaluating (18) using these comments we obtain

$$\|\tilde{x}^r \tilde{y}^s \tilde{z}^t\|^2 = \frac{r!s!t!}{\eta_0^r \eta_1^s \eta_2^t} \tilde{\vartheta}^N.$$

We have verified (17), and the lemma is proved. $\square$

The following is the basis for V that is dual to (10) with respect to $\langle \cdot, \cdot \rangle$:

$$(19) \quad \frac{\eta_0^r \eta_1^s \eta_2^t}{r!s!t!} \frac{\tilde{x}^r \tilde{y}^s \tilde{z}^t}{\tilde{\vartheta}^N}, \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.$$

The vector $\tilde{x}^N$ is equal to $N!\nu^N$ times the sum of the vectors (19).

5. THE RAHMAN POLYNOMIALS AND $\mathfrak{sl}_3(\mathbb{C})$

In this section the Rahman polynomials are related to the $\mathfrak{sl}_3(\mathbb{C})$ -module V .

We would like to acknowledge that the following result is similar to [11, Section 2], although the setup is somewhat different.

Theorem 5.1. *For a basis vector $\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau$ from (10),*

$$(20) \quad \tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau = N!\nu^N \sum_{\substack{0 \leq r, s, t \\ r+s+t=N}} P(s, t, \sigma, \tau) \frac{\tilde{\eta}_0^r \tilde{\eta}_1^s \tilde{\eta}_2^t}{r!s!t!} \frac{x^r y^s z^t}{\vartheta^N}.$$

For a basis vector $x^\rho y^\sigma z^\tau$ from (6),

$$(21) \quad x^\rho y^\sigma z^\tau = N!\nu^N \sum_{\substack{0 \leq r, s, t \\ r+s+t=N}} P(\sigma, \tau, s, t) \frac{\eta_0^r \eta_1^s \eta_2^t}{r!s!t!} \frac{\tilde{x}^r \tilde{y}^s \tilde{z}^t}{\tilde{\vartheta}^N}.$$

Proof. We first prove (20). Since $\rho + \sigma + \tau = N$,

$$(22) \quad \tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau = \tilde{\vartheta}^N (\tilde{\vartheta}^{-1} \tilde{x})^\rho (\tilde{\vartheta}^{-1} \tilde{y})^\sigma (\tilde{\vartheta}^{-1} \tilde{z})^\tau.$$

Using (8) and the trinomial theorem,

$$\begin{aligned}(\tilde{\vartheta}^{-1} \tilde{y})^\sigma &= (\tilde{\vartheta}^{-1} \tilde{x} - t \tilde{\eta}_1 \tilde{y} - v \tilde{\eta}_2 \tilde{z})^\sigma \\ &= \sum_{\substack{0 \leq i, k \\ i+k \leq \sigma}} (\tilde{\vartheta}^{-1} \tilde{x})^{\sigma-i-k} (-t \tilde{\eta}_1 \tilde{y})^i (-v \tilde{\eta}_2 \tilde{z})^k \binom{\sigma}{i \quad k \quad \sigma-i-k} \\ &= \sum_{\substack{0 \leq i, k \\ i+k \leq \sigma}} (\tilde{\vartheta}^{-1} \tilde{x})^{\sigma-i-k} (t \tilde{\eta}_1 \tilde{y})^i (v \tilde{\eta}_2 \tilde{z})^k \frac{(-\sigma)_{i+k}}{i!k!}.\end{aligned}$$

Similarly, using (9),

$$\begin{aligned}
 (\tilde{\vartheta}^{-1}\tilde{z})^\tau &= (\tilde{\vartheta}^{-1}\tilde{x} - u\tilde{\eta}_1y - w\tilde{\eta}_2z)^\tau \\
 &= \sum_{\substack{0 \leq j, \ell \\ j+\ell \leq \tau}} (\tilde{\vartheta}^{-1}\tilde{x})^{\tau-j-\ell} (-u\tilde{\eta}_1y)^j (-w\tilde{\eta}_2z)^\ell \binom{\tau}{j \quad \ell \quad \tau-j-\ell} \\
 &= \sum_{\substack{0 \leq j, \ell \\ j+\ell \leq \tau}} (\tilde{\vartheta}^{-1}\tilde{x})^{\tau-j-\ell} (u\tilde{\eta}_1y)^j (w\tilde{\eta}_2z)^\ell \frac{(-\tau)_{j+\ell}}{j!\ell!}.
 \end{aligned}$$

Evaluating (22) using the above comments and (1), we obtain

$$\begin{aligned}
 \tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau &= \tilde{\vartheta}^N \sum_{\substack{0 \leq i, j, k, \ell \\ i+j+k+\ell \leq N}} t^i u^j v^k w^\ell \frac{(-\sigma)_{i+k} (-\tau)_{j+\ell}}{i!j!k!\ell!} \\
 (23) \quad &\times (\tilde{\vartheta}^{-1}\tilde{x})^{N-i-j-k-\ell} (\tilde{\eta}_1y)^{i+j} (\tilde{\eta}_2z)^{k+\ell}.
 \end{aligned}$$

In the expression (23) consider the first factor. Using (7) and the trinomial theorem,

$$\begin{aligned}
 (\tilde{\vartheta}^{-1}\tilde{x})^{N-i-j-k-\ell} &= (\tilde{\eta}_0x + \tilde{\eta}_1y + \tilde{\eta}_2z)^{N-i-j-k-\ell} \\
 &= \sum_{\substack{0 \leq r, b, c \\ r+b+c=N-i-j-k-\ell}} (\tilde{\eta}_0x)^r (\tilde{\eta}_1y)^b (\tilde{\eta}_2z)^c \binom{N-i-j-k-\ell}{r \quad b \quad c}.
 \end{aligned}$$

Therefore (23) is equal to

$$\sum_{\substack{0 \leq r, b, c \\ r+b+c=N-i-j-k-\ell}} (\tilde{\eta}_0x)^r (\tilde{\eta}_1y)^{b+i+j} (\tilde{\eta}_2z)^{c+k+\ell} \binom{N-i-j-k-\ell}{r \quad b \quad c},$$

which is equal to

$$\begin{aligned}
 &\sum_{\substack{0 \leq r, b, c \\ r+b+c=N-i-j-k-\ell}} (\tilde{\eta}_0x)^r (\tilde{\eta}_1y)^{b+i+j} (\tilde{\eta}_2z)^{c+k+\ell} \\
 &\times \frac{(-b-i-j)_{i+j} (-c-k-\ell)_{k+\ell}}{(-N)_{i+j+k+\ell}} \binom{N}{r \quad b+i+j \quad c+k+\ell}.
 \end{aligned}$$

After changing variables $s = b + i + j$, $t = c + k + \ell$ and using (1) the above sum becomes

$$(24) \quad \sum_{\substack{0 \leq r, s, t \\ r+s+t=N}} (\tilde{\eta}_0x)^r (\tilde{\eta}_1y)^s (\tilde{\eta}_2z)^t \frac{(-s)_{i+j} (-t)_{k+\ell}}{(-N)_{i+j+k+\ell}} \binom{N}{r \quad s \quad t}.$$

Therefore (23) is equal to (24). Upon replacing (23) by (24) the expression $\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau$ becomes a sum which is equal to the right-hand side of (20). This proves (20), and the proof of (21) is similar. $\square$

Theorem 5.2. For nonnegative integers s, t whose sum is at most N , both

$$(25) \quad P(s, t, \tilde{\varphi} + N/3 I, \tilde{\phi} + N/3 I) x^N = x^r y^s z^t,$$

$$(26) \quad P(\varphi + N/3 I, \phi + N/3 I, s, t) \tilde{x}^N = \tilde{x}^r \tilde{y}^s \tilde{z}^t,$$

where $r = N - s - t$.

Proof. Concerning (26), observe

$$\begin{aligned} \tilde{x}^r \tilde{y}^s \tilde{z}^t &= N! \nu^N \sum_{\substack{0 \leq i, j, k \\ i+j+k=N}} P(j, k, s, t) \frac{\tilde{\eta}_0^i \tilde{\eta}_1^j \tilde{\eta}_2^k}{i! j! k!} \frac{x^i y^j z^k}{\vartheta^N} \\ &= P(\varphi + N/3 I, \phi + N/3 I, s, t) N! \nu^N \sum_{\substack{0 \leq i, j, k \\ i+j+k=N}} \frac{\tilde{\eta}_0^i \tilde{\eta}_1^j \tilde{\eta}_2^k}{i! j! k!} \frac{x^i y^j z^k}{\vartheta^N} \\ &= P(\varphi + N/3 I, \phi + N/3 I, s, t) \tilde{x}^N. \end{aligned}$$

Line (25) is similarly obtained. $\square$

Theorem 5.3. *For a basis vector $x^r y^s z^t$ from (6) and a basis vector $\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau$ from (10),*

$$\langle x^r y^s z^t, \tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau \rangle = N! \nu^N P(s, t, \sigma, \tau).$$

Proof: By (20),

$$\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau = N! \nu^N \sum_{\substack{0 \leq i, j, k \\ i+j+k=N}} P(j, k, \sigma, \tau) \frac{\tilde{\eta}_0^i \tilde{\eta}_1^j \tilde{\eta}_2^k}{i! j! k!} \frac{x^i y^j z^k}{\vartheta^N}.$$

In the above equation take the inner product of each side with $x^r y^s z^t$ and use the fact that the bases (6), (16) are dual with respect to $\langle \cdot, \cdot \rangle$. $\square$

Proof of Theorem 1.1. We first verify (2). Abbreviate $\rho = N - \sigma - \tau$. By (20) we have

$$\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau = N! \nu^N \sum_{\substack{0 \leq i, j, k \\ i+j+k=N}} P(j, k, \sigma, \tau) \frac{\tilde{\eta}_0^i \tilde{\eta}_1^j \tilde{\eta}_2^k}{i! j! k!} \frac{x^i y^j z^k}{\vartheta^N}.$$

In the above equation take the inner product of each side with $\tilde{x}^r \tilde{y}^s \tilde{z}^t$. In the resulting equation simplify the left-hand side using Lemma 4.1 and simplify the right-hand side using Theorem 5.3. This gives (2). Line (3) is similarly verified. $\square$

6. SOME SEVEN-TERM RECURRENCE RELATIONS

In this section we display two seven-term recurrence relations satisfied by the Rahman polynomials, along with the corresponding relations satisfied by their duals. To obtain these relations we use the $\mathfrak{sl}_3(\mathbb{C})$ -module V from Section 3. Here we would like to acknowledge the work of Grünbaum [4, Section 6], which hints at the existence of these relations and contains a wealth of related data for the case $N = 5$.

Theorem 6.1. *Fix nonnegative integers s, t whose sum is at most N , and non-negative integers σ, τ whose sum is at most N . Then (i)–(iv) hold below.*

- (i) $(s - N/3)P(s, t, \sigma, \tau)$ is a weighted sum with the following terms and coefficients:

term	coefficient
$P(s, t, \sigma - 1, \tau)$	$\sigma \frac{p_3(p_1 p_4 - p_2 p_3)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)}$
$P(s, t, \sigma, \tau - 1)$	$\tau \frac{p_1(p_2 p_3 - p_1 p_4)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)}$
$P(s, t, \sigma + 1, \tau)$	$\rho \frac{p_1 p_2 p_3(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_1 p_4 - p_2 p_3)}$
$P(s, t, \sigma + 1, \tau - 1)$	$\tau \frac{-p_1 p_2}{(p_1 + p_2)(p_1 + p_3)}$
$P(s, t, \sigma, \tau + 1)$	$\rho \frac{p_1 p_3 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2 p_3 - p_1 p_4)}$
$P(s, t, \sigma - 1, \tau + 1)$	$\sigma \frac{-p_3 p_4}{(p_1 + p_3)(p_3 + p_4)}$
$P(s, t, \sigma, \tau)$	$(\sigma - N/3) \left(\frac{p_2 p_3}{(p_1 + p_2)(p_1 + p_3)} - \frac{p_1 p_3(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)} \right) + (\tau - N/3) \left(\frac{p_1 p_4}{(p_1 + p_3)(p_3 + p_4)} - \frac{p_1 p_3(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)} \right)$

- (ii) $(t - N/3)P(s, t, \sigma, \tau)$ is a weighted sum with the following terms and coefficients:

term	coefficient
$P(s, t, \sigma - 1, \tau)$	$\sigma \frac{p_4(p_2 p_3 - p_1 p_4)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)}$
$P(s, t, \sigma, \tau - 1)$	$\tau \frac{p_2(p_1 p_4 - p_2 p_3)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)}$
$P(s, t, \sigma + 1, \tau)$	$\rho \frac{p_1 p_2 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_2 p_3 - p_1 p_4)}$
$P(s, t, \sigma + 1, \tau - 1)$	$\tau \frac{-p_1 p_2}{(p_1 + p_2)(p_2 + p_4)}$
$P(s, t, \sigma, \tau + 1)$	$\rho \frac{p_2 p_3 p_4(p_1 + p_2 + p_3 + p_4)}{(p_2 + p_4)(p_3 + p_4)(p_1 p_4 - p_2 p_3)}$
$P(s, t, \sigma - 1, \tau + 1)$	$\sigma \frac{-p_3 p_4}{(p_2 + p_4)(p_3 + p_4)}$
$P(s, t, \sigma, \tau)$	$(\sigma - N/3) \left(\frac{p_1 p_4}{(p_1 + p_2)(p_2 + p_4)} - \frac{p_2 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)} \right) + (\tau - N/3) \left(\frac{p_2 p_3}{(p_2 + p_4)(p_3 + p_4)} - \frac{p_2 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)} \right)$

- (iii) $(\sigma - N/3)P(s, t, \sigma, \tau)$ is a weighted sum with the following terms and coefficients:

term	coefficient
$P(s - 1, t, \sigma, \tau)$	$s \frac{p_2(p_1 p_4 - p_2 p_3)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)}$
$P(s, t - 1, \sigma, \tau)$	$t \frac{p_1(p_2 p_3 - p_1 p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)}$
$P(s + 1, t, \sigma, \tau)$	$r \frac{p_1 p_2 p_3(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_1 p_4 - p_2 p_3)}$
$P(s + 1, t - 1, \sigma, \tau)$	$t \frac{-p_1 p_3}{(p_1 + p_2)(p_1 + p_3)}$
$P(s, t + 1, \sigma, \tau)$	$r \frac{p_1 p_2 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_2 p_3 - p_1 p_4)}$
$P(s - 1, t + 1, \sigma, \tau)$	$s \frac{-p_2 p_4}{(p_1 + p_2)(p_2 + p_4)}$
$P(s, t, \sigma, \tau)$	$(s - N/3) \left(\frac{p_2 p_3}{(p_1 + p_2)(p_1 + p_3)} - \frac{p_1 p_2(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)} \right) + (t - N/3) \left(\frac{p_1 p_4}{(p_1 + p_2)(p_2 + p_4)} - \frac{p_1 p_2(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)} \right)$

- (iv) $(\tau - N/3)P(s, t, \sigma, \tau)$ is a weighted sum with the following terms and coefficients:

term	coefficient
$P(s - 1, t, \sigma, \tau)$	$s \frac{p_4(p_2 p_3 - p_1 p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2 + p_4)}$
$P(s, t - 1, \sigma, \tau)$	$t \frac{p_3(p_1 p_4 - p_2 p_3)}{(p_1 + p_3)(p_3 + p_4)(p_2 + p_4)}$
$P(s + 1, t, \sigma, \tau)$	$r \frac{p_1 p_3 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2 p_3 - p_1 p_4)}$
$P(s + 1, t - 1, \sigma, \tau)$	$t \frac{-p_1 p_3}{(p_1 + p_3)(p_3 + p_4)}$
$P(s, t + 1, \sigma, \tau)$	$r \frac{p_2 p_3 p_4(p_1 + p_2 + p_3 + p_4)}{(p_2 + p_4)(p_3 + p_4)(p_1 p_4 - p_2 p_3)}$
$P(s - 1, t + 1, \sigma, \tau)$	$s \frac{-p_2 p_4}{(p_2 + p_4)(p_3 + p_4)}$
$P(s, t, \sigma, \tau)$	$(s - N/3) \left(\frac{p_1 p_4}{(p_1 + p_3)(p_3 + p_4)} - \frac{p_3 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2 + p_4)} \right) + (t - N/3) \left(\frac{p_2 p_3}{(p_2 + p_4)(p_3 + p_4)} - \frac{p_3 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_3 + p_4)(p_2 + p_4)} \right)$

In the above tables $r = N - s - t$ and $\rho = N - \sigma - \tau$.

Proof. (i) By (15) and since $\varphi^\dagger = \varphi$,

$$(27) \quad \langle \varphi.x^r y^s z^t, \tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau \rangle = \langle x^r y^s z^t, \varphi.\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau \rangle.$$

The left-hand side of (27) is equal to $N!\nu^N$ times $(s - N/3)P(s, t, \sigma, \tau)$. To see this, first observe $\varphi.x^r y^s z^t = (s - N/3)x^r y^s z^t$ from the tables below (6), and then use Theorem 5.3. The right-hand side of (27) is equal to $N!\nu^N$ times the weighted sum in the theorem statement. To obtain this fact, reduce the right-hand side of (27) using the following three steps: (a) eliminate φ using the long formula in Section 2; (b) evaluate the result using the tables below (10); (c) apply Theorem 5.3. The result follows from the above comments.

(ii)–(iv) Similar to the proof of (i) above. $\square$

We interpret Theorem 6.1 as follows. Parts (i), (ii) indicate that the Rahman polynomials are common eigenvectors for a pair of difference operators, while parts (iii), (iv) indicate that the dual Rahman polynomials are common eigenvectors for analogous difference operators. Following [2] we can say that the Rahman polynomials solve a bispectral problem. This feature of the Rahman polynomials was hinted at earlier by Grünbaum [4].

7. DIRECTIONS FOR FUTURE RESEARCH

Motivated by the results in this paper we now pose some problems. To state the problems we adopt a general point of view. Let $\mathbb{F}$ denote a field and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\text{End}(V)$ denote the $\mathbb{F}$ -algebra consisting of the $\mathbb{F}$ -linear maps from V to V . Fix integers $M \geq 1$ and $N \geq 0$. Let $\mathbb{I} = \mathbb{I}(M, N)$ denote the set consisting of the $(M+1)$ -tuples of nonnegative integers whose sum is N . A pair of elements $(r_0, r_1, \dots, r_M)$ and $(r'_0, r'_1, \dots, r'_M)$ in $\mathbb{I}$ will be called *adjacent* whenever $(r_0 - r'_0, r_1 - r'_1, \dots, r_M - r'_M)$ is a permutation of $(1, -1, 0, 0, \dots, 0)$.

Problem 7.1. Find all the pairs $H, \tilde{H}$ that satisfy the following conditions:

- (i) H is an M -dimensional subspace of $\text{End}(V)$ whose elements are diagonalizable and mutually commute.
- (ii) $\tilde{H}$ is an M -dimensional subspace of $\text{End}(V)$ whose elements are diagonalizable and mutually commute.
- (iii) There exists a bijection $\lambda \mapsto V_\lambda$ from $\mathbb{I}$ to the set of common eigenspaces of H such that for all $\lambda \in \mathbb{I}$,

$$\tilde{H}V_\lambda \subseteq V_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \text{ adj } \lambda}} V_\mu.$$

- (iv) There exists a bijection $\lambda \mapsto \tilde{V}_\lambda$ from $\mathbb{I}$ to the set of common eigenspaces of $\tilde{H}$ such that for all $\lambda \in \mathbb{I}$,

$$H\tilde{V}_\lambda \subseteq \tilde{V}_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \text{ adj } \lambda}} \tilde{V}_\mu.$$

- (v) There does not exist a subspace W of V such that $HW \subseteq W$, $\tilde{H}W \subseteq W$, $W \neq 0$, $W \neq V$.
- (vi) Each of $V_\lambda, \tilde{V}_\lambda$ has dimension 1 for $\lambda \in \mathbb{I}$.

- (vii) There exists a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V such that both

$$\begin{aligned}\langle V_\lambda, V_\mu \rangle &= 0 & \text{if } \lambda \neq \mu, & \lambda, \mu \in \mathbb{I}, \\ \langle \tilde{V}_\lambda, \tilde{V}_\mu \rangle &= 0 & \text{if } \lambda \neq \mu, & \lambda, \mu \in \mathbb{I}.\end{aligned}$$

Note 7.2. By our results earlier in this paper, the Rahman polynomials give a solution to Problem 7.1 with $M = 2$.

Note 7.3. For $M = 1$ a solution to Problem 7.1 is essentially the same as a *Leonard pair* [12]. The Leonard pairs have been studied extensively and are well understood; see [13] and the references therein. The Leonard pairs correspond to a class of orthogonal polynomials in one variable. This class coincides with the terminating branch of the Askey scheme [10] and consists of the q -Racah, q -Hahn, dual q -Hahn, q -Krawtchouk, dual q -Krawtchouk, quantum q -Krawtchouk, affine q -Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk, Bannai/Ito, and orphan polynomials [13, Section 35]. We expect that for general M the solutions to Problem 7.1 correspond to a class of M -variable orthogonal polynomials that resembles the above class. In particular, the M -variable polynomials from [3, Appendix A.3], [6], and [11] are likely members of this class.

Note 7.4. For $M = 1$ the conditions in Problem 7.1 are redundant. Indeed condition (vii) follows from (i)–(vi) [13, Lemma 15.2].

For the ambitious reader we now pose some harder problems.

Problem 7.5. Find all the pairs $H, \tilde{H}$ that satisfy conditions (i)–(v) in Problem 7.1.

Note 7.6. For $M = 1$ a solution to Problem 7.5 is essentially the same as a *tridiagonal pair* [8]; for a discussion of tridiagonal pairs see [7] and the references therein. The tridiagonal pairs over an algebraically closed field are classified [7, Corollary 18.1].

Problem 7.7. Above Problem 7.1 we defined an adjacency relation on a set $\mathbb{I}$. Given its features a Lie theorist will recognize that the solutions to Problems 7.1 and 7.5 belong to the root system A_M . Investigate the analogous objects that belong to other root systems.

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