

CHARACTER VARIETIES

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ABSTRACT. We study properties of irreducible and completely reducible representations of finitely generated groups Γ into reductive algebraic groups G . In particular, we study the geometric invariant theory of the G action on the space of G -representations of Γ by conjugation.

Let $X_G(\Gamma)$ be the G -character variety of Γ . We prove that for every completely reducible, scheme smooth $\rho : \Gamma \rightarrow G$

$$T_{[\rho]} X_G(\Gamma) \simeq T_0(H^1(\Gamma, Ad\rho) // S_\Gamma),$$

where $H^1(\Gamma, Ad\rho)$ is the first cohomology group of Γ with coefficients in the Lie algebra \mathfrak{g} of G twisted by $\Gamma \xrightarrow{\rho} G \xrightarrow{Ad} GL(\mathfrak{g})$ and S_Γ is the centralizer of $\rho(\Gamma)$ in G . The condition of ρ being scheme smooth is very important as there are groups Γ such that

$$\dim T_{[\rho]} X_G(\Gamma) < T_0 H^1(\Gamma, Ad\rho),$$

for a Zariski open subset of points in $X_G(\Gamma)$. We prove, however, that all irreducible representations of surface groups are scheme smooth.

Let M be an orientable 3-manifold with a connected boundary F of genus $g \geq 2$. Let $X_G^g(F)$ be the subset of the G -character variety of $\pi_1(F)$ composed of conjugacy classes of good representations $\rho : \Gamma \rightarrow G$, i.e., irreducible representations such that the centralizer of $\rho(\Gamma)$ is the center of G . By a theorem of Goldman, $X_G^g(F)$ is a holomorphic symplectic manifold. We prove that the set of good G -representations of $\pi_1(F)$ which extend to representations of $\pi_1(M)$ is a complex isotropic subspace of $X_G^g(F)$. It is Lagrangian, if these representations correspond to reduced points of the G -character variety of M . It is an open problem whether it is always the case.

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1. SUMMARY OF RESULTS

Let G be a complex reductive algebraic group, for example a classical group of matrices, $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $O(n, \mathbb{C})$, $Sp(n, \mathbb{C})$, or its quotient. Let Γ be a finitely generated group. We say that a representation $\rho : \Gamma \rightarrow G$ is *irreducible* if $\rho(\Gamma)$ is not contained in any proper parabolic subgroup of G . Additionally, we say that ρ is *completely reducible* if for every parabolic subgroup $P \subset G$ containing $\rho(\Gamma)$, there is a Levi subgroup $L \subset P$ containing $\rho(\Gamma)$. In particular, $\rho : \Gamma \rightarrow GL(n, \mathbb{C})$ is irreducible if and only if \mathbb{C}^n is a simple Γ -module (via ρ), and it is completely reducible if \mathbb{C}^n is a semi-simple Γ -module.

We discuss properties of irreducible and completely reducible representations in Sections 3-4. For example, we prove that $\rho : \Gamma \rightarrow G$ is completely reducible if and only if the algebraic closure of $\rho(\Gamma)$ in G is a linearly reductive group. Furthermore, a completely reducible representation $\rho : \Gamma \rightarrow G$ is irreducible if and only if the centralizer of $\rho(\Gamma)$ is a finite extension of $C(G)$.

The space, $Hom(\Gamma, G)$, of all group homomorphisms from Γ to G is an algebraic set on which G acts by conjugation. In Section 7, we study properties of this action from the point of view of the Geometric Invariant Theory. In particular, we observe that ρ is a poly-stable point under that action if and only if ρ is completely reducible. If ρ is irreducible, then it is a stable point. Finally, ρ is properly stable if and only if ρ is irreducible and $C(G)$ is finite.

The categorical quotient, $X_G(\Gamma) = Hom(\Gamma, G)//G$, is called the G -character variety of Γ , cf. Section 11. Although it is a coarser quotient than the set theory one, it has the advantage of having a natural structure of an affine algebraic set. Every element of $X_G(\Gamma)$ is represented by a unique completely reducible representation. With $Hom(\Gamma, G)$ and $X_G(\Gamma)$, there are naturally associated algebraic schemes $\mathcal{H}om(\Gamma, G)$ and $\mathcal{X}_G(\Gamma) = \mathcal{H}om(\Gamma, G)//G$ such that $\mathbb{C}[Hom(\Gamma, G)]$ and $\mathbb{C}[X_G(\Gamma)]$ are nil-radical quotients of $\mathcal{O}(\mathcal{H}om(\Gamma, G))$ and $\mathcal{O}(\mathcal{X}_G(\Gamma))$, cf. Sections 5 and 12.

We formulate cohomological descriptions of the tangent spaces to $\mathcal{X}_G(\Gamma)$ and to $X_G(\Gamma)$ in Section 13. More specifically, we prove the following: Let $H^*(\Gamma, Ad, \rho)$ denote the group cohomology of Γ with coefficients in the Lie algebra \mathfrak{g} of G twisted by the homomorphism

$$\Gamma \xrightarrow{\rho} G \xrightarrow{Ad} End(\mathfrak{g}),$$

where Ad is the adjoint representation. Denote the stabilizer of ρ under the G action (by conjugation) on $Hom(\Gamma, G)$ by S_ρ . It is the centralizer of $\rho(\Gamma)$ in G . There is a natural action of S_ρ on $H^1(\Gamma, Ad, \rho)$, cf. Section 13.

We call an irreducible $\rho : \Gamma \rightarrow G$ *good* if the stabilizer of its image, S_ρ , is the center of G . We say that ρ is *scheme smooth* if $\mathcal{H}om(\Gamma, G)$ is non-singular at ρ .

Theorem 1 (cf. Theorem 53). (1) For every good $\rho : \Gamma \rightarrow G$ there exists a natural linear isomorphism

$$\phi : H^1(\Gamma, Ad \rho) \xrightarrow{\cong} T_{[\rho]} \mathcal{X}_G(\Gamma).$$

(2) If $\rho : \Gamma \rightarrow G$ is completely reducible and scheme smooth, then

$$T_0 (H^1(\Gamma, Ad \rho) // S_\rho) \simeq T_{[\rho]} \mathcal{X}_G(\Gamma) = T_{[\rho]} X_G(\Gamma).$$

The quotient on the left side in the statement of Theorem 1(2) may be non-trivial even if ρ is irreducible, cf. discussion in Section 4 and Example 20 in particular. We discuss the question of the existence of a natural isomorphism in part (2) of the above theorem in Section 13.

Versions of the above theorem for $G = PSL(2, \mathbb{C})$ appear in [Po, Proposition 3.5] and [HP2, Prop. 5.2]. Although statements similar to Theorem 1(1) appear in the literature for general G , there is widespread confusion about the necessary assumptions and all proofs known to us are incomplete.

The following result illuminates the importance of the requirement of ρ being scheme smooth in Theorem 1(2). Denote the set of equivalence classes of good representations in $\mathcal{X}_G(\Gamma)$ by $\mathcal{X}_G^g(\Gamma)$.

Theorem 2 (cf. Theorem 57). $\mathcal{X}_G^g(\Gamma)$ is reduced iff $T_{[\rho]} X_G(\Gamma) = H^1(\Gamma, Ad \rho)$ for a non-empty Zariski open set of $[\rho]$'s in $X_G^g(\Gamma)$.

It is easy to see that all representations of free groups are scheme smooth. We prove an analogous statement for surface groups, i.e., fundamental groups of closed, orientable surfaces of genus ≥ 2 .

Theorem 3. For every reductive G , all irreducible G -representations of surface groups are scheme smooth.

This theorem has some important consequences to the theory of skein modules, cf. Corollary 52:

Theorem 4. For every orientable surface F , there is an isomorphism between the Kauffman bracket skein algebra $\mathcal{S}_{2,\infty}(F, \mathbb{C}, -1)$ and $\mathbb{C}[X_{SL(2,\mathbb{C})}(\pi_1(F))]$ sending a link $K_1 \cup \dots \cup K_n$ in $F \times [0, 1]$ to $(-1)^n \tau_{K_1} \cdot \dots \cdot \tau_{K_n}$, where $\tau_K \in \mathbb{C}[X_{SL(2,\mathbb{C})}(\pi_1(F))]$ sends the equivalence class of $\rho : \pi_1(F) \rightarrow SL(2, \mathbb{C})$ to $tr \rho(\gamma)$, and $\gamma \in \pi_1(F)$ is any element representing K (with some orientation).

Although this result was announced in [PS2, Thm. 7.3], its proof relied on [PS2, Thm. 7.4] whose proof was not provided. An alternative proof of the above statement was provided independently by L. Charles and J. Marché in [CM].

Denote the set of irreducible G -representations of Γ by $Hom^i(\Gamma, G)$. It is a Zariski open subset of $Hom(\Gamma, G)$, cf. Proposition 27. Since each equivalence class in $Hom(\Gamma, G) // G$ contains a unique closed orbit and the orbit of every irreducible representation is closed (Theorem 30), the categorical quotient of $Hom(\Gamma, G)$ restricted to $Hom^i(\Gamma, G)$ is the set theoretic quotient. Denote $Hom^i(\Gamma, G) // G$ by $X_G^i(\Gamma)$.

Proposition 5. Let Γ be a free group or a surface group. Then

- (1) $X_G^i(\Gamma)$ is an orbifold.
- (2) If $G = GL(n, \mathbb{C})$ or $SL(n, \mathbb{C})$, then $X_G^i(\Gamma)$ is a manifold. (See also [FL2].)

We do not know if Proposition 5(2) holds for any reductive groups other than $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$, cf. Question 19 and Proposition 49.

We say that ρ is *good* if ρ is irreducible and S_ρ is the center of G . Denote the set of all such representations by $Hom^g(\Gamma, G)$. It is a Zariski open subset of $Hom(\Gamma, G)$, cf. Proposition 33. Let

$$X_G^g(\Gamma) = Hom^g(\Gamma, G)//G = Hom^g(\Gamma, G)/G.$$

By the above discussion $X_G^g(\Gamma)$ is an open subset of $X_G^i(\Gamma)$. Furthermore, it is a smooth manifold for free groups and surface groups Γ .

For a topological space Y , we abbreviate $X_G(\pi_1(Y))$ by $X_G(Y)$. Let F be a closed orientable surface of genus ≥ 2 . Goldman proved that every non-degenerate symmetric bilinear Ad -invariant form B on \mathfrak{g} gives rise to a holomorphic symplectic 2-form ω_B on $X_G^g(F)$, [Go2], cf. Section 14.

According to folk knowledge, if M is a compact orientable 3-manifold with a connected boundary F , then the image of the map $r^* : X_G(M) \rightarrow X_G(F)$ induced by the embedding $r : F = \partial M \hookrightarrow M$ is “roughly speaking” a Lagrangian subspace of the G -character variety of F . In Section 15, we formulate this claim precisely, and we prove it in detail. Let $Y_G(M)$ be the non-singular part of

$$X_G^g(F) \cap r^*(X_G(M)).$$

Theorem 6. (1) $Y_G(M)$ is an isotropic submanifold of $X_G^g(F)$ with respect to ω_B .

In particular, every connected component of $Y_G(M)$ of dimension $\frac{1}{2} \dim X_G(F)$ is Lagrangian.

(2) If a connected component C of $Y_G(M)$ contains the conjugacy class of a representation which is a reduced point of $\mathcal{H}om(\pi_1(M), G)$, then C is a Lagrangian submanifold of $Y_G(M)$.

Although we do not know any example of a 3-manifold M such that $Y_G(M)$ is not Lagrangian, in light of the above theorem and the results of M. Kapovich mentioned in Section 12, we believe that such examples exist.

Here is a different version of (2) above:

Theorem 7. If $X_G^s(M)$ denotes the set of equivalence classes of scheme smooth representations in $X_G(M)$, then $X_G^g(F) \cap r^*(X_G^s(M))$ is an immersed Lagrangian submanifold of $X_G^g(F)$.

Note however that $X_G^s(M)$ may be empty even if $\pi_1(M)$ has good G -representations.

Theorems 6 and 7 are relevant to Chern-Simons theory, cf. Section 15.

In this paper we assume familiarity with basic algebraic geometry and the theory of algebraic groups. The standard references for these topics are [Ha, Shf, Bo, Hu].

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2. REDUCTIVE GROUPS

Every algebraic group G contains a unique maximal normal connected solvable subgroup called its *radical* and is denoted by $Rad G$. A connected group G is *semi-simple* if and only if $Rad G$ is trivial. A connected group G is *reductive* if and only if $Rad G$ is an algebraic torus, $(\mathbb{C}^*)^n$. In particular, \mathbb{C}^* and all classical matrix

groups, $SL(n, \mathbb{C})$, $O(n, \mathbb{C})$, $Sp(n, \mathbb{C})$, are reductive. Furthermore, Cartesian products, quotients and finite connected covers of reductive groups are reductive. In fact, all reductive groups can be obtained in this way from simple algebraic groups.

Denote the center of G by $C(G)$ and the connected component of the identity in $C(G)$ by $C^0(G)$. For every reductive G , $C^0(G) = (\mathbb{C}^*)^n$. A reductive group G is semi-simple if and only if $C^0(G)$ is trivial.

Let $[G, G]$ be the commutator of G . If G is reductive, then $[G, G]$ is semi-simple. Furthermore, by [Bo, Proposition IV.14.2], the epimorphism

$$(1) \quad \nu : C^0(G) \times [G, G] \rightarrow G, \quad \nu(g, h) = g \cdot h$$

has a finite kernel isomorphic to

$$K = C^0(G) \cap [G, G].$$

Therefore, there is a finite quotient

$$(2) \quad \pi : G \rightarrow C^0(G)/K \times [G, G]/K.$$

An algebraic group G is *linearly reductive* if all its $GL(n, \mathbb{C})$ -representations are completely reducible. G is linearly reductive if and only if the connected component, G^0 , of its identity is reductive. (This property does not hold for groups over fields of non-zero characteristic.) Therefore, linearly reductive groups are “virtually reductive”.

A maximal connected solvable subgroup of G is called a *Borel subgroup*. A closed subgroup $P \subset G$ is *parabolic* if one of the following equivalent conditions holds:

- (a) G/P is a complete variety,
- (b) G/P is a projective variety,
- (c) P contains a Borel subgroup of G , cf. [Bo].

A Levi subgroup of an algebraic group H is a connected subgroup $L \subset H$ such that H is a semi-direct product of L and the unipotent radical of H . Since L is isomorphic to the quotient of H by its unipotent radical, it is always reductive. By a result of Mostow, every algebraic group contains a Levi subgroup, cf. [Bo, IV.11.22]

3. IRREDUCIBLE AND COMPLETELY REDUCIBLE SUBGROUPS

We say that a subgroup H (closed or not) of G is *irreducible* if it is not contained in any proper parabolic subgroup of G . We also say that H is *completely reducible* if for every parabolic subgroup $P \subset G$ containing H , there is a Levi subgroup of P containing H as well, [Se, BMR]. In particular, every irreducible subgroup is completely reducible.

The following is an important characterization of completely reducible subgroups:

Proposition 8. *For every reductive G , a subgroup $H \subset G$ is completely reducible if and only if the algebraic closure of H in G is a linearly reductive group.*

Proof. \Rightarrow (1) Assume first that H is irreducible, i.e., not contained in any proper parabolic subgroup of G . Let \overline{H} be the Zariski closure of H in G and let $Rad_u(\overline{H})$ be the unipotent radical of \overline{H} . Let $P = \mathcal{P}(Rad_u(\overline{H}))$ be the parabolic subgroup defined in [Hu, 30.3]. Then $\overline{H} \subset N_G(Rad_u(\overline{H}))$ and $N_G(Rad_u(\overline{H})) \subset P$ by [Hu, 30.3 Corollary A]. If \overline{H} is not linearly reductive, then $Rad_u(\overline{H})$ is non-trivial and

$Rad_u(\overline{H}) \subset Rad_u(P)$ by [Hu, 30.3 Corollary A]. Therefore, P is a proper subgroup of G .

(2) Now we carry the proof in full generality by induction with respect to $dim G$: If $dim G = 1$, then $G = \mathbb{C}^*$ and the statement holds. Assume now that it holds for all reductive algebraic groups G of dimension less than n . Let $dim G = n$. If H lies in a proper parabolic subgroup of G , then it also lies in a Levi subgroup of P and the statement follows from inductive hypothesis. If H does not lie in a proper parabolic subgroup of G , then H is irreducible in G and the statement follows from part (1).

\Leftarrow Suppose \overline{H} is linearly reductive and $H \subset P$. Since P is closed, $\overline{H} \subset P$. Now the statement follows from the fact that every closed linearly reductive subgroup of P lies in a Levi subgroup of P . Since we do not know a good reference to this fact, we enclose its proof here: There is an exact sequence

$$\{e\} \rightarrow Rad_U P \rightarrow P \xrightarrow{\tau} L \rightarrow \{e\},$$

where $Rad_U P$ is the unipotent radical of P . Since \overline{H}^0 is reductive, it has no connected unipotent subgroups and, therefore, τ is an embedding of \overline{H}^0 into L . Therefore, the kernel K of τ restricted \overline{H} is finite. By [Bo, Corollary 4.8], $Rad_U P$ is a subgroup of upper triangular matrices and, therefore, it has no elements of finite order. Hence, K is trivial. \square

A representation $\phi : \Gamma \rightarrow G$ is *irreducible* or *completely reducible* if $\phi(\Gamma) \subset G$ is. In particular, a representation $\rho : \Gamma \rightarrow GL(n, \mathbb{C})$ is irreducible if and only if \mathbb{C}^n does not have any Γ -invariant subspaces other than $\{0\}$ and \mathbb{C}^n . Additionally, $\rho : \Gamma \rightarrow GL(n, \mathbb{C})$ is completely reducible if and only if \mathbb{C}^n decomposes into a sum of irreducible Γ -modules.

Since a quotient of a reductive group is reductive, Proposition 8 implies:

Corollary 9. *For every homomorphism $\phi : G_1 \rightarrow G_2$ of reductive groups, the image of a completely reducible subgroup of G_1 is completely reducible in G_2 .*

Similarly, we have:

Lemma 10. *For every epimorphism $\phi : G_1 \rightarrow G_2$ of reductive groups, the image of an irreducible subgroup of G_1 is irreducible in G_2 .*

Proof. Suppose that $\phi(H)$ lies inside a proper parabolic subgroup $P \subset G_2$. Since ϕ induces an isomorphism $G_1/\phi^{-1}(P) \rightarrow G_2/P$ and G_2/P is complete, $G_1/\phi^{-1}(P)$ is complete as well, implying that $\phi^{-1}(P)$ is a proper parabolic subgroup of G_1 containing H . \square

The following example shows that irreducibility of $H \subset G_1$ does not imply irreducibility of $\phi(H) \subset G_2$ if $\phi : G_1 \rightarrow G_2$ is not an epimorphism, even if ϕ is irreducible itself.

Proposition 11. *Let $H = \{A : A \cdot A^T = \pm I\} \subset SL(2, \mathbb{C})$.*

- (1) *H is isomorphic to $O(2, \mathbb{C})$.*
- (2) *$H \subset SL(2, \mathbb{C})$ is irreducible.*
- (3) *The image of H under the adjoint representation $Ad : SL(2, \mathbb{C}) \rightarrow SL(3, \mathbb{C})$ is completely reducible but not irreducible in $SL(3, \mathbb{C})$.*

Proof. (1) H is a non-abelian split $\mathbb{Z}/2$ extension of $SO(2) \simeq \mathbb{C}^*$. However, since $H^2(\mathbb{Z}/2, \mathbb{C}^*) = 0$ (with twisted coefficients), $O(2, \mathbb{C})$ is the unique non-abelian split extension of $SO(2, \mathbb{C})$ by $\mathbb{Z}/2$.

(2) Since H is reductive, it is completely reducible in $SL(2, \mathbb{C})$ by Proposition 8. If it was reducible, it would be a subgroup of diagonal matrices, \mathbb{C}^* . Since H is non-abelian, it is irreducible in $SL(2, \mathbb{C})$.

(3) Complete reducibility follows from Corollary 9. We claim that the group $Ad(H)$ lies in the parabolic subgroup of $SL(3, \mathbb{C})$ composed of transformations of $sl(2, \mathbb{C}) \simeq \mathbb{C}^3$ which preserve $Span(M) \subset sl(2, \mathbb{C})$, where $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Indeed, since $SL(2, \mathbb{C}) = Sp(2, \mathbb{C})$, $AMA^T = M$ for every $A \in SL(2, \mathbb{C})$. If $A \in H$, then $A^T = \pm A^{-1}$, and the claim follows. \square

We say that $H \subset G$ is *Ad-irreducible* if $Ad(H) \subset GL(\mathfrak{g})$ is irreducible. The $H \subset SL(2, \mathbb{C})$ above is irreducible but not *Ad-irreducible*. By Proposition 8, every irreducible subgroup $H \subset G$ is *completely Ad-reducible*, i.e., $Ad(H) \subset GL(\mathfrak{g})$ is completely reducible. We are going to show that *Ad-irreducibility* implies irreducibility.

Lemma 12. *Let $\phi : G \rightarrow GL(n, \mathbb{C})$ be an irreducible representation of a reductive group G . If H a subgroup of G such that $\phi(H)$ is irreducible, then either*

- (a) H is irreducible, or
- (b) $Ker \phi$ contains the unipotent radical (i.e., the maximal connected unipotent subgroup) of a Borel subgroup of G .

Proof. Suppose that $H \subset P \subsetneq G$. Then ϕ restricted to P is irreducible as well. Let U be the unipotent radical of P . Denote the space of vectors in $V = \mathbb{C}^n$ invariant under the U -action by V^U . Since P is a semi-direct product of U and its Levi subgroup L , [Bo], $P = L \rtimes U$, $l^{-1}ul \in U$, for every $u \in U$ and $l \in L$, and

$$u \cdot l \cdot v = l \cdot l^{-1}ul \cdot v = l \cdot v \quad \text{for every } v \in V^U.$$

Therefore $l \cdot v \in V^U$ and, consequently, V^U is preserved by P . Since ϕ restricted to P is irreducible, by Shur's lemma V^U is either 0 or V . However, U is a connected solvable group and, therefore, $V^U \neq 0$, by the Lie-Kolchin theorem, [Bo, Cor. 10.5]. Hence $V^U = V$ and, consequently, $U \subset Ker \phi$. If B is a Borel subgroup of G contained in P , then the unipotent radical of B is contained in U . \square

Since the kernel of the adjoint representation is the center of G , [Bo, I.3.15], and its unipotent radical is trivial, Lemma 12 implies:

Corollary 13. *Every Ad-irreducible subgroup of a reductive group is irreducible.*

4. STABILIZERS OF IRREDUCIBLE REPRESENTATIONS

Proposition 14. *The centralizer of an Ad-irreducible subgroup of a reductive group G is the center of G .*

Proof. Let $H \subset G$ be *Ad-irreducible*. By Shur's lemma the centralizer of $Ad(H)$ is the group of scalar matrices in $GL(\mathfrak{g})$. Hence,

$$Ad(C_G(H)) \subset C_{GL(\mathfrak{g})}(Ad(H)) \subset \{c \cdot I : c \in \mathbb{C}^*\}.$$

On the other hand, since the center of $Ad(G) = G/C(G)$ is trivial, cf. [FH, Thm. 23.16], $Ad(G) \cap \{c \cdot I : c \in \mathbb{C}^*\} = \{I\}$. Hence, $Ad(C_G(H))$ is trivial, implying that $C_G(H) \subset C(G)$. \square

Proposition 15. *The centralizer of an irreducible subgroup of a reductive group G is a finite extension of the center of G .*

Proof. Suppose that the centralizer, $C_G(H)$, of H is an infinite extension of the center. Let T be a maximal torus in $C_G(H)$. Then $rank T > rank C(G)$ and $H \subset C_G(T)$. Recall that T is either regular, semi-regular, or singular, [Bo, §13.1]. If T is regular, then $C_G(T)$ is a maximal torus. If T is semi-regular, then $C_G(T)$ is contained in a Borel subgroup, cf. proof of [Bo, IV.13.1 Proposition]. In either case $H \subset C_G(T)$ is reducible. Therefore, T is singular. In that case T is the connected component of identity of $\bigcap_{\alpha \in I} Ker \alpha$, where the intersection is over a certain proper, non-empty subset I of positive roots. By [Bo, IV.14.17], T lies inside of a proper parabolic subgroup of G (denoted by Borel by P_I). \square

The following lemma will be useful later.

Lemma 16. *For every Levi subgroup L of every proper parabolic subgroup of a reductive group G , $dim C(L) > dim C(G)$.*

Proof. Suppose that L is a Levi subgroup of a parabolic group P in G . As before, let $C^0(G)$ be the connected component of the identity in the center of G . Then $C^0(G) \subset L$ and $L' = L/C^0(G)$ is a Levi subgroup of the parabolic subgroup $P' = P/C^0(G)$ of $G' = G/C^0(G)$. Therefore, without loss of generality we can assume that $dim C(G) = 0$.

By [Bo, Prop. 11.23], it is enough to prove that the radical of P , $\mathcal{R}P$, has a positive dimension. Fix a root system for G . We are going to use the notation of [Bo]. By classification of parabolic subgroups in [Bo, §14.17], $P = P_I$ for some subset I of positive roots Δ of G . Let T_I be the identity component of $\bigcap_{\alpha \in I} Ker \alpha$. By [Bo, Prop. 14.18], $T_I \subset \mathcal{R}P$. Since T_I is an algebraic torus of dimension $rank G - |I|$, $dim \mathcal{R}P > 0$ unless $I = \Delta$. In this case, $P = G$. \square

Proposition 15 and Lemma 16 imply:

Corollary 17. *A completely reducible subgroup $H \subset G$ is irreducible if and only if $dim C_G(H) = dim C(G)$.*

We will say that a reductive group G has property CI if the centralizer of every irreducible subgroup of G coincides with the center of G .

Example 18. $G = GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ are CI. Indeed, $H \subset G$ is irreducible if and only if elements of H linearly span $M(n, \mathbb{C})$. Consequently, the centralizer of every irreducible subgroup $H \subset G$ is the center of G .

Question 19. Are $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ the only CI-groups?

Example 20. $PSL(2, \mathbb{C})$ is not CI. To see that consider the subgroup $H \subset PSL(2, \mathbb{C})$, generated by $\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $|H| = 4$, H is the Klein group. One can easily see that H is its own centralizer in $PSL(2, \mathbb{C})$ (while the center of $PSL(2, \mathbb{C})$ is trivial). Being finite, H is linearly reductive and completely reducible by Proposition 8. By Corollary 17, H is irreducible in $PSL(2, \mathbb{C})$.

Example 21. $SO(n, \mathbb{C})$ is not CI: Let DM_n be the group of diagonal matrices in

$$SO(n, \mathbb{C}) = \{A : A \cdot A^T = I\} \subset SL(n, \mathbb{C}).$$

Then $DM_n \simeq (\mathbb{Z}/2)^{n-1}$ and it is easy to see that DM_n is its own centralizer in $SO(n, \mathbb{C})$. Being finite, DM_n is linearly reductive and completely reducible by Proposition 8. By Corollary 17, DM_n is irreducible in $SO(n, \mathbb{C})$.

Proposition 22. $Sp(2n, \mathbb{C})$ is not CI.

Proof. (Based on the idea of S. Lawton, cf. [FL2]) Denote by $D(\alpha_1, \dots, \alpha_n)$ the diagonal matrix with entries $\alpha_1, \dots, \alpha_n$, and by $AD(\alpha_1, \dots, \alpha_n)$ the anti-diagonal matrix

$$\begin{pmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 \\ \alpha_n & 0 & 0 & 0 \end{pmatrix}.$$

The matrices $D(\alpha_1, \dots, \alpha_n, \alpha_n^{-1}, \dots, \alpha_1^{-1})$ and $AD(\beta_1, \dots, \beta_n, -\beta_n^{-1}, \dots, -\beta_1^{-1})$, for $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{C}^*$ form a subgroup of $Sp(2n, \mathbb{C}) = \{A : AJA^T = J\}$, where $J = AD(1, \dots, 1, -1, \dots, -1)$. Denote that subgroup by H_n . An elementary computation shows that the center of H_n is composed of matrices $D(\alpha_1, \dots, \alpha_n, \alpha_n^{-1}, \dots, \alpha_1^{-1})$, where $\alpha_1, \dots, \alpha_n \in \{\pm 1\}$. Since H_n is a finite extension of $(\mathbb{C}^*)^n$, it is linearly reductive and, hence, by Proposition 8, it is completely reducible in $Sp(2n, \mathbb{C})$. Since $C(\Gamma_n)$ is a finite extension of $C(Sp(2n, \mathbb{C})) = \{\pm 1\}$, Γ_n is irreducible by Corollary 17. \square

By the following result, $PSO(n, \mathbb{C}), PSp(2n, \mathbb{C})$ are not CI either.

Proposition 23. A quotient of a non-CI group by a finite subgroup is non-CI.

Proof. Let $\Gamma \subset G$ be irreducible and such that $C_G(\Gamma)$ is a proper extension of $C(G)$. If $\pi : G \rightarrow G'$ is a quotient with finite kernel, then $\text{Ker } \pi \subset C(G)$ and, consequently, the centralizer of $\pi(\Gamma)$ in G' is a proper extension of $C(G')$. Now the statement follows from Lemma 10. \square

5. REPRESENTATION VARIETIES

If Γ is a finitely generated group and G an affine complex algebraic group, then the space of all G -representations of Γ , $\text{Hom}(\Gamma, G)$, is an algebraic set.

Remark 24. $\text{Hom}(\Gamma_1 * \Gamma_2, G) = \text{Hom}(\Gamma_1, G) \times \text{Hom}(\Gamma_2, G)$. Hence, for the free group on n generators, $\text{Hom}(F_n, G) = G^n$.

Example 25. Each point of $\text{Hom}(\mathbb{Z}^2, SL(2, \mathbb{C}))$ is represented by $\rho : \mathbb{Z}^2 \rightarrow SL(2, \mathbb{C})$ defined by

$$\rho(1, 0) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad \rho(0, 1) = \begin{pmatrix} x_5 & x_6 \\ x_7 & x_8 \end{pmatrix},$$

satisfying relations

$$\begin{aligned} x_1x_4 - x_2x_3 - 1 &= x_5x_8 - x_6x_7 - 1 = x_2x_7 - x_3x_6 = 0, \\ -x_2x_5 + x_1x_6 - x_4x_6 + x_2x_8 &= x_3x_5 - x_1x_7 + x_4x_7 - x_3x_8 = 0. \end{aligned}$$

The algebraic set $\text{Hom}(\mathbb{Z}^2, SL(2, \mathbb{C})) \subset \mathbb{C}^8$ is irreducible by [Ri1, Thm. C].

For a more through study of representation varieties, it is useful to associate with each Γ and G as above an affine algebraic scheme, also called the representation variety, whose set of closed points coincides with $Hom(\Gamma, G)$. That scheme, containing sometimes more subtle information about G -representations of Γ than $Hom(\Gamma, G)$, is constructed below.

If G is an affine complex algebraic group, then $\mathbb{C}[G]$ is a Hopf algebra with the coproduct

$$\Delta : \mathbb{C}[G] \rightarrow \mathbb{C}[G] \otimes \mathbb{C}[G] = \mathbb{C}[G \times G]$$

being the dual to the group product $G \times G \rightarrow G$ and the antipode

$$S : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$$

being the dual to the inverse map $g \rightarrow g^{-1}$. Consequently, for any commutative \mathbb{C} -algebra A with product $m : A \times A \rightarrow A$, the space of algebra homomorphisms, $Hom(\mathbb{C}[G], A)$, is a group with the multiplication

$$Hom(\mathbb{C}[G], A) \times Hom(\mathbb{C}[G], A) \ni (f, g) \rightarrow m(f \otimes g)\Delta \in Hom(\mathbb{C}[G], A)$$

and the inverse

$$Hom(\mathbb{C}[G], A) \ni f \rightarrow fS \in Hom(\mathbb{C}[G], A).$$

We will denote $Hom(\mathbb{C}[G], A)$ with that group structure by $G(A)$. The functor $G(\cdot)$ is an affine group scheme, [Wa]. For example, $G(A) = SL(n, A)$ for $G = SL(n, \mathbb{C})$.

We say that a commutative \mathbb{C} -algebra $R(\Gamma, G)$ is a *universal representation algebra of Γ into G* and $\rho_U : \Gamma \rightarrow G(R(\Gamma, G))$ is a *universal representation* if for every commutative \mathbb{C} -algebra A and every representation $\rho : \Gamma \rightarrow G(A)$, there is a \mathbb{C} -algebra homomorphism $f : R(\Gamma, G) \rightarrow A$ inducing a representation $G(f) : G(R(\Gamma, G)) \rightarrow G(A)$ such that $\rho = G(f)\rho_U$, [BH, Si1].

Lemma 26. *For every Γ and every G as above,*

- (1) $R(\Gamma, G)$ and ρ_U exist.
- (2) $R(\Gamma, G)$ is well defined up to an isomorphism of \mathbb{C} -algebras.
- (3) $\rho_U : \Gamma \rightarrow G(R(\Gamma, G))$ is unique up to a composition with $G(f)$ where f is a \mathbb{C} -algebra automorphism of $R(\Gamma, G)$.

Proof. (1) Since each affine algebraic group G is a closed subgroup of $GL(n, \mathbb{C})$, the coordinate ring of G is a quotient of

$$\mathbb{C}[GL(n, \mathbb{C})] = \mathbb{C}[d, x_{ij}, 1 \leq i, j \leq n] / (d \cdot \det(x_{ij}) - 1).$$

Let

$$\mathbb{C}[G] = \mathbb{C}[d, x_{ij}, 1 \leq i, j \leq n] / I_G$$

for an appropriate ideal I_G . For the free group, $F_N = \langle \gamma_1, \dots, \gamma_N \rangle$,

$$R(F_N, G) = \mathbb{C}[d_1, x_{1ij}, 1 \leq i, j \leq n] / I_G \otimes \dots \otimes \mathbb{C}[d_N, x_{Nij}, 1 \leq i, j \leq n] / I_G$$

and

$$(3) \quad \rho_U(\gamma_t) = (x_{tij}) \in G(R(\Gamma, G)), \quad \text{for } t = 1, \dots, N$$

satisfy the required universal properties.

If

$$\Gamma = \langle \gamma_1, \dots, \gamma_N \rangle / H,$$

then we define $R(\Gamma, G)$ as the quotient of $R(\langle \gamma_1, \dots, \gamma_N \rangle, G)$ by an ideal I generated by all relations necessary for (3) to be a well defined group homomorphism. Therefore, each normal generator of $H \triangleleft \langle \gamma_1, \dots, \gamma_N \rangle$ introduces n^2 relations to I

(although some of them may be redundant). It is easy to see that, in this way, (3) descends to a universal representation $\rho_U : \Gamma \rightarrow G(R(\Gamma, G))$.

(2) and (3) follow immediately from the definition. □

Every $\rho \in \text{Hom}(\Gamma, G)$ defines a \mathbb{C} -algebra homomorphism $h_\rho : R(\Gamma, G) \rightarrow \mathbb{C}$ (unique up to an automorphism of $R(\Gamma, G)$) such that

$$\rho = G(h_\rho)\rho_U.$$

$\text{Ker } h_\rho$ is a maximal ideal in $R(\Gamma, G)$ and, hence, a closed point in the affine scheme $\text{Hom}(\Gamma, G) = \text{Spec } R(\Gamma, G)$. Conversely, every closed point in $\text{Hom}(\Gamma, G)$ defines a representation $\rho : \Gamma \rightarrow G$. Therefore, $\text{Hom}(\Gamma, G)$ is the set of closed points of $\text{Hom}(\Gamma, G)$ and there is a natural map

$$\text{Hom}(\Gamma, G) \rightarrow \mathcal{H}om(\Gamma, G)$$

dual to

$$(4) \quad R(\Gamma, G) \rightarrow R(\Gamma, G)/\sqrt{0} = \mathbb{C}[\text{Hom}(\Gamma, G)].$$

By [KM, Thm. 1.2], $R(\Gamma, PSL(2, \mathbb{C}))$ contains non-zero nilpotent elements for some Artin groups Γ . Furthermore, M. Kapovich proves that $R(\pi_1(M), SL(2, \mathbb{C}))$ and $R(\pi_1(M), PSL(2, \mathbb{C}))$ contain non-zero nilpotents for some 3-dimensional manifolds M , [Ka1, Ka2]. See further comments in Section 12.

6. SPACES OF IRREDUCIBLE REPRESENTATIONS

Proposition 27. *For every Γ and every reductive group G , the set of irreducible G -representations of Γ is Zariski open in $\text{Hom}(\Gamma, G)$.*

Proof. The proposition follows from Corollary 17 and from [Ne, Prop. 3.8]. Since the proof of this referenced result is non-elementary, we enclose a complete simple proof here.

(1) First, a simple proof for $G = GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$: If $\rho : \Gamma \rightarrow G$ is irreducible, then, by Shur's lemma, the elements of $\rho(\Gamma)$ linearly span the space of $n \times n$ matrices, $M(n, \mathbb{C})$. Conversely, if $\rho(\Gamma)$ lies in a parabolic subgroup of G , then the elements of $\rho(\Gamma)$ do not span $M(n, \mathbb{C})$.

Enumerate all elements of Γ in a sequence $\gamma_1, \gamma_2, \dots$. Let U_s be the space of all ρ 's such that $\rho(\gamma_1), \dots, \rho(\gamma_s)$ span $M(n, \mathbb{C})$. Since the space of all irreducible ρ 's is the union of all U_s 's, it is enough to prove that each U_s is open. This condition is equivalent to an existence of a sequence i_1, \dots, i_{n^2} such that the $n^2 \times n^2$ matrix whose columns are $\rho(\gamma_{i_1}), \dots, \rho(\gamma_{i_{n^2}})$ considered as vectors in $M(n, \mathbb{C}) = \mathbb{C}^{n^2}$ has a non-zero determinant. This is a Zariski open condition.

(2) Here is a fairly elementary proof for all G :
The set of irreducible representations $\Gamma \rightarrow G$ is the complement of

$$\bigcup_P \text{Hom}(\Gamma, P) \subset \text{Hom}(\Gamma, G),$$

where the union on the left is over all proper parabolic subgroups of G . By [Bo, Thm. 14.18], there are only finitely many parabolic subgroups of G up to conjugation. Therefore it is enough to prove that for a given P

$$X_P = \bigcup_{g \in G} \text{Hom}(\Gamma, gPg^{-1}) \subset \text{Hom}(\Gamma, G)$$

is closed. X_P is the union of closed sets parameterized by a complete variety G/P . By the Projective Extension Theorem, [CLO, Ch. 8 §5 Thm. 6], such a union is closed. \square

The adjoint representation induces a map $Ad_* : Hom(\Gamma, G) \rightarrow Hom(\Gamma, GL(\mathfrak{g}))$. $\rho : \Gamma \rightarrow G$ is *Ad-irreducible* if $Ad\rho$ is irreducible. Since the set of *Ad-irreducible* representations $\Gamma \rightarrow G$ is the Ad_* -preimage of the irreducible representations in $Hom(\Gamma, GL(\mathfrak{g}))$, we conclude with

Corollary 28. *The set of Ad-irreducible representations is Zariski open in $Hom(\Gamma, G)$.*

Proposition 29. *Let G be a reductive group.*

(1) *For a free group, F_N , of rank $N \geq 2$, the irreducible representations form a dense subset of $Hom(F_N, G)$ (in complex topology).*

(2) *For every surface group¹ Γ , the irreducible representations are dense in a non-empty set of irreducible components of $Hom(\Gamma, G)$.*

Proof. (1) Since $Hom(F_N, G)$ is an irreducible algebraic set and the set of irreducibles is Zariski open in it, it is enough to show that the set of irreducibles is non-empty. Since every free group F_N of rank $N \geq 2$ maps onto F_2 , it is enough to prove that statement for F_2 . The set of irreducible G -representations of F_2 is the complement of

$$\bigcup_P Hom(F_2, P) \subset Hom(F_2, G) = G \times G,$$

where the union of sets on the left is over all proper parabolic subgroups of G . By [Bo, Thm. 14.18], there are only finitely many parabolic subgroups of G up to conjugation. Since for each P

$$\bigcup_{g \in G} Hom(F_2, gPg^{-1})$$

is the image of the G action on $Hom(F_2, P)$ with stabilizer P , its dimension is at most

$$2 \cdot \dim P + \dim G - \dim P < 2 \cdot \dim G = \dim Hom(F_2, G).$$

Therefore, there exists an irreducible representation.

(2) Again, it is enough to prove that an irreducible representation exists. This follows from the fact that Γ maps onto the free group of rank 2. \square

Note that Proposition 29 does not apply to $F = \text{torus}$. For example, all representations of $\mathbb{Z} \times \mathbb{Z}$ to $GL(n, \mathbb{C})$ are reducible for $n > 1$.

7. STABLE AND PROPERLY STABLE REPRESENTATIONS IN THE SENSE OF GIT

Let O_ρ be the orbit of $\rho \in Hom(\Gamma, G)$ under the G action on $Hom(\Gamma, G)$ by conjugation. In the language of geometric invariant theory, ρ is *poly-stable* if O_ρ is closed.

Theorem 30. *For any reductive algebraic group G , $O_\rho \subset Hom(\Gamma, G)$ is closed if and only if ρ is completely reducible.*

¹A surface group is the fundamental group of a closed, orientable surface of genus ≥ 2 .

Proof. (The proof for $G = GL(n, \mathbb{C})$ can be found in [LM, Thm. 1.27]) \Rightarrow We follow an argument of the proof of [JM, Thm. 1.1]: Assume that O_ρ is closed. If ρ is contained in a proper parabolic subgroup P , then by conjugating ρ with a one parameter group in the center of a Levi subgroup L of P one can obtain a representation $\rho' \in \overline{O_\rho}$ whose image lies in L . Since O_ρ is closed, $\rho' = g^{-1}\rho g$, for some $g \in P$. Hence ρ lies in the Levi subgroup gLg^{-1} .

\Leftarrow Any finitely generated group Γ is a quotient of a free group F of finite rank. Denote the epimorphism $F \rightarrow \Gamma$ by π . Since $Hom(\Gamma, G)$ is a closed subset of $Hom(F, G)$ and $O_\rho = O_{\pi\rho} \cap Hom(\Gamma, G)$, it is enough to prove that $O_{\pi\rho} \subset Hom(F, G)$ is closed. This statement follows from [Ri2, Thm. 3.6]. \square

According to the geometric invariant theory, a point x of an affine set X is *stable* with respect to a G action on X (and the trivial line bundle on X) if there is a Zariski open neighborhood of x preserved by G on which the G action is closed [MFK, Do].

Corollary 31. (1) *Every irreducible representation is a stable point of $Hom(\Gamma, G)$ under the G action by conjugation.*

(2) *$\rho \in Hom(F_n, G)$ is stable if and only if ρ is irreducible.*

Proof. (1) This follows from Proposition 27.

(2) Every stable ρ is completely reducible by Theorem 30. Every completely reducible representation of a free group which is not irreducible can be deformed by an arbitrarily small deformation to a representation which is not completely reducible. \square

A point $x \in X$ is *properly stable* if it is stable and its stabilizer is finite.

Corollary 32. *For every reductive group G ,*

(1) *ρ is a properly stable point of $Hom(\Gamma, G)$ under conjugation action of G if and only if ρ is irreducible and $C(G)$ is finite.*

(2) *ρ is a properly stable point of $Hom(\Gamma, G)$ under conjugation action of $G/C(G)$ if and only if ρ is irreducible.*

Proof. (1) \Rightarrow : ρ is completely reducible by Theorem 30, and it is irreducible by Lemma 16.

\Leftarrow : By Theorem 30 and Propositions 27 and 15.

The same argument shows (2) \square

Following [JM], we say that a representation ρ is *good* if O_ρ is closed and S_ρ is the center of G . By Theorem 30 and Corollary 17, every good representation is irreducible. By Proposition 14, every *Ad*-irreducible representation is good.

Proposition 33. *For every Γ the space of good G -representations is open in the space of all G -representations of Γ .*

Proof. By [JM, Proposition 1.1], the G action on the space of all irreducible G -representations of Γ is proper. The good representations, if they exist, form a set which is the union of the principal orbits of that action. For every proper action, the union of principal orbits is an open subset, cf. [GO, Thm. 1.5]. \square

8. TANGENT SPACES

Let A be a commutative \mathbb{C} -algebra, let m be a closed point of $\text{Spec } A$, i.e., a maximal ideal $m \triangleleft A$, and let r_m be the projection $A \rightarrow A/m = \mathbb{C}$. The tangent space to $\text{Spec } A$ at m is the dual vector space to m/m^2 ,

$$T_m \text{Spec } A = (m/m^2)^*.$$

Here is an equivalent definition of the tangent space which will be useful later: Let $\pi : \mathbb{C}[\varepsilon]/(\varepsilon^2) \rightarrow \mathbb{C}$ be the homomorphism sending ε to 0, and let $\mathcal{T}_m \text{Spec } A$ be the complex vector space of \mathbb{C} -algebra homomorphisms $A \rightarrow \mathbb{C}[\varepsilon]/(\varepsilon^2)$ which descend to r_m when composed with π . Since every such homomorphism is of the form $r_m + \tau\varepsilon$, where $\tau : A \rightarrow \mathbb{C}$ is a derivation,

$$\mathcal{T}_m \text{Spec } A = \{\tau : A \rightarrow \mathbb{C} : \tau(ab) = r_m(a)\tau(b) + r_m(b)\tau(a)\}.$$

A straightforward calculation shows that for every $v \in T_m \text{Spec } A$,

$$\lambda_v(a) = v(a - r_m(a))$$

is a derivation in $\mathcal{T}_m \text{Spec } A$. A direct computation shows that

$$(5) \quad \lambda : T_m \text{Spec } A \rightarrow \mathcal{T}_m \text{Spec } A,$$

sending v to λ_v is an isomorphism of vector spaces, [EH, VI.1.3]. From now on we will identify these two spaces and call them the *Zariski tangent space* to $\text{Spec } A$ at m .

The above discussion applies to $A = \text{Hom}(\Gamma, G) = \text{Spec } R(\Gamma, G)$. Each $\rho \in \text{Hom}(\Gamma, G)$ defines a projection $r_\rho : R(\Gamma, G) \rightarrow \mathbb{C}$ and a closed point $m_\rho = \text{Ker } \rho$ in $\text{Hom}(\Gamma, G)$. We will abbreviate $T_{m_\rho} \text{Hom}(\Gamma, G)$ to $T_\rho \text{Hom}(\Gamma, G)$. Each tangent vector $\tau \in T_\rho \text{Hom}(\Gamma, G)$ defines a group homomorphism

$$(6) \quad \Gamma \xrightarrow{\rho U} G(R(\Gamma, G)) \xrightarrow{G(r_\rho + \tau\varepsilon)} G(\mathbb{C}[\varepsilon]/(\varepsilon^2)).$$

By abuse of notation, we denote by π the extension of the homomorphism

$$\pi : \mathbb{C}[\varepsilon]/(\varepsilon^2) \rightarrow \mathbb{C}, \quad \pi(\varepsilon) = 0,$$

to the induced group homomorphism

$$\pi : G(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \rightarrow G(\mathbb{C}) = G.$$

Proposition 34. *Consider a closed embedding $G \subset GL(n, \mathbb{C})$. (Such an embedding exists for every affine algebraic group.)*

(1) *For every $g \in G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$,*

$$\sigma(g) = \frac{g \cdot \pi(g)^{-1} - I}{\varepsilon} \in M_n(\mathbb{C}[\varepsilon]/(\varepsilon^2)),$$

and $\sigma(g)$ belongs to the Lie algebra $\mathfrak{g} \subset M_n(\mathbb{C})$ of G .

(2) *For every $g_1, g_2 \in G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$,*

$$(7) \quad \sigma(g_1 g_2) = \sigma(g_1) + \text{Ad } \pi(g_1) \cdot \sigma(g_2),$$

where $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ is the adjoint representation.

Proof. (1) If $h \in G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ is such that $\pi(h) = I$, then $\frac{h-I}{\varepsilon}$ belongs to the Zariski tangent space to G at the identity, that is, the Lie algebra of G . Now the statement follows by substitution $h = g\pi(g)^{-1}$.

(2) This follows by a direct computation. □

For every $\tau \in T_\rho \mathcal{H}om(\Gamma, G)$, $G(r_\rho + \tau\varepsilon)\rho_U(\gamma) \in G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$, cf. (6). Therefore, by Proposition 34, we have a function $\sigma : \Gamma \rightarrow \mathfrak{g}$

$$(8) \quad \sigma(\gamma) = \frac{(G(r_\rho + \tau\varepsilon)\rho_U(\gamma))(G(r_\rho)\rho_U(\gamma))^{-1} - I}{\varepsilon}$$

satisfying (7), which is the cocycle condition for the first cohomology group of Γ with coefficients in \mathfrak{g} twisted by $Ad \rho$. Hence, (8) defines a linear map

$$\Psi_\rho : T_\rho \mathcal{H}om(\Gamma, G) \rightarrow Z^1(\Gamma, Ad \rho)$$

sending τ to σ .

The adjoint action of the centralizer of $\rho(\Gamma)$, S_ρ , on \mathfrak{g} induces a S_ρ -action on $Z^1(\Gamma, Ad \rho)$. Additionally, every $g \in S_\rho$ acts on $T_\rho \mathcal{H}om(\Gamma, G)$ by sending $\tau \in T_\rho \mathcal{H}om(\Gamma, G)$ to $g\tau$ such that

$$G(r_\rho + g\tau\varepsilon) = gG(r_\rho + \tau\varepsilon)g^{-1}.$$

The homomorphism Ψ_ρ is S_ρ -equivariant.

We are going to prove that Ψ_ρ is an isomorphism by constructing its inverse. An easy calculation shows that for every $\sigma \in Z^1(\Gamma, Ad \rho)$,

$$\gamma \rightarrow (I + \sigma(\gamma)\varepsilon) \cdot \rho(\gamma)$$

is a group homomorphism from Γ to $G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ (cf. [LM, Prop. 2.2] for $G = GL(n, \mathbb{C})$). Therefore, σ defines a homomorphism $\Phi_\rho(\sigma) : R(\Gamma, G) \rightarrow \mathbb{C}[\varepsilon]/(\varepsilon^2)$ such that $\pi\Phi_\rho(\sigma) = r_\rho$. Hence, $\Phi_\rho(\sigma) \in T_\rho \mathcal{H}om(\Gamma, G)$. In other words, we have defined a linear map

$$\Phi_\rho : Z^1(\Gamma, Ad \rho) \rightarrow T_\rho \mathcal{H}om(\Gamma, G).$$

A straightforward computation shows (cf. [JM, Lem. 2.2], [LM, Prop. 2.2] for $G = GL(n, \mathbb{C})$ and [Be, Prop. 1.2] for $G = PSL(2, \mathbb{C})$):

Theorem 35. Ψ_ρ and Φ_ρ are inverses of each other, and therefore, they are S_ρ -equivariant isomorphisms between $Z^1(\Gamma, Ad \rho)$ and $T_\rho \mathcal{H}om(\Gamma, G)$.

9. SMOOTH AND REDUCED REPRESENTATIONS

A closed point x of an algebraic scheme X is *reduced* if the local ring $O_{X,x}$ has no non-zero nilpotent elements. By this definition, reduced points form a Zariski open subset of X .

Recall that a point x of an affine algebraic set or of an algebraic scheme X is simple if x belongs to a unique irreducible component of X and the dimension of that component coincides with $\dim T_x X$. Every simple point of an algebraic scheme is reduced.

We say that $\rho : \Gamma \rightarrow G$ is *smooth* (respectively, *scheme smooth*) if ρ is a simple point of $\mathcal{H}om(\Gamma, G)$ (respectively, of $\mathcal{H}om(\Gamma, G)$). ρ is *reduced* if ρ is a reduced point of $\mathcal{H}om(\Gamma, G)$. Note that ρ is scheme smooth iff it is reduced and smooth.

Remark 36. (1) The set of all smooth representations,
 (2) the set of all reduced representations,
 (3) the set of all scheme smooth representations,
 are Zariski open subsets $\mathcal{H}om(\Gamma, G)$.

Proof. (1) This follows from the fact that the set of simple points are Zariski open.

(2) $Hom^r(\Gamma, G)$ is a preimage of the Zariski open set of reduced points in $Hom(\Gamma, G)$ under the map $Hom(\Gamma, G) \rightarrow Hom(\Gamma, G)$.

(3) This set is the intersection of the first two. □

All G -representations of a free group are scheme smooth, since $R(F_n, G)$ is the coordinate ring of the n -th cartesian power of G , which is a non-singular algebraic set.

Proposition 37. *For every reductive group G and every surface group Γ , all irreducible representations $\rho : \Gamma \rightarrow G$ are scheme smooth and, hence, reduced.*

Proof. By Proposition 15, the centralizer of $\rho(\Gamma)$ is a finite extension of the center of G . Hence, by [Go1, Prop. 1.2] and Theorem 35,

$$(9) \quad \dim T_\rho \mathcal{H}om(\Gamma, G) = \dim Z^1(\Gamma, Ad \rho) = (2g - 1)\dim G + \dim C(G).$$

(1) Assume first that G is semi-simple. Then $\dim C(G) = 0$. Since Γ has a presentation with $2g$ generators and one relation,

$$\dim C \geq (2g - 1)\dim G$$

for all irreducible components $C \subset Hom(\Gamma, G)$. Therefore,

$$(10) \quad \dim C \geq \dim T_\rho \mathcal{H}om(\Gamma, G).$$

Since (10) must be an equality, ρ is scheme smooth.

(2) For an arbitrary reductive group G , consider epimorphism (1),

$$\nu : C^0(G) \times [G, G] \rightarrow G.$$

Since it has a finite kernel, the induced map

$$Hom(\Gamma, C^0(G)) \times Hom(\Gamma, [G, G]) \rightarrow Hom(\Gamma, G)$$

is finite. Since $Hom(\Gamma, C^0(G)) = (C^0(G))^{2g}$ and, by (1), $Hom(\Gamma, [G, G])$ is composed of irreducible components of dimension at least $(2g - 1) \dim [G, G]$, the set $Hom(\Gamma, G)$ is composed of irreducible components of dimension at least

$$2g \cdot \dim C^0(G) + (2g - 1) \dim [G, G] = (2g - 1)\dim G + \dim C^0(G).$$

Therefore,

$$\dim C \geq \dim T_\rho \mathcal{H}om(\Gamma, G).$$

Now the argument goes exactly as in (1). □

By [Shf, II §2 Thm. 6], we obtain:

Corollary 38. *For every reductive group G , every irreducible representation of every surface group Γ belongs to a unique irreducible component of $Hom(\Gamma, G)$.*

Proposition 39. *If G is a reductive group and Γ is a surface group such that $Hom(\Gamma, G)$ is irreducible, then $R(\Gamma, G)$ has no non-zero nilpotents. Consequently, map (4), $R(\Gamma, G) \rightarrow \mathbb{C}[Hom(\Gamma, G)]$, is an isomorphism.*

Proof. Recall from Section 8 that for every $\rho : \Gamma \rightarrow G$ there is a corresponding maximal ideal $m_\rho \triangleleft R(\Gamma, G)$. If we assume that $R(\Gamma, G)$ has a non-zero nilpotent, then that nilpotent projects onto a non-zero nilpotent in the localization $R(\Gamma, G)_{m_\rho}$ for an open set of ρ 's in $Hom(\Gamma, G)$. These ρ 's are not reduced.

Since $Hom(\Gamma, G)$ is irreducible, by Proposition 29(2), the set of irreducible representations is dense in $Hom(\Gamma, G)$. Therefore, Γ has an irreducible but not reduced G -representation, contradicting Proposition 37. \square

A version of this theorem for F torus appears in [Si5].

Corollary 40. *For $G = GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ and every surface group Γ the ring $R(\Gamma, G)$ is an integral domain, i.e., it has no proper zero divisors.*

Proof. By Theorems 1 and 3 in [RBC], $Hom(\Gamma, GL(n, \mathbb{C}))$ and $Hom(\Gamma, SL(n, \mathbb{C}))$ are irreducible. Therefore, the statement follows by Proposition 39. \square

This corollary has some important consequences to the theory of skein modules, cf. Corollary 52, and it is needed to complete Goldman’s construction of symplectic forms on character varieties of surfaces, cf. Section 14.

The following remains open:

Question 41. Does $R(\Gamma, G)$ have non-zero nilpotents for some algebraic group G and a surface group Γ ?

Finally, we would like to remark that Proposition 37 does not hold for non-surface groups. In fact, there appears to be no easy characterization of simple points of $Hom(\Gamma, G)$ in general.

Example 42. Let $\rho_1 : \mathbb{Z}^2 \rightarrow SL(2, \mathbb{C})$ be the trivial representation, and let $\rho_2 : \Gamma \rightarrow SL(2, \mathbb{C})$ be an irreducible representation. These representations define a representation $\rho_1 * \rho_2 : \mathbb{Z}^2 * \Gamma \rightarrow SL(2, \mathbb{C})$ which is irreducible. On the other hand, the Jacobian matrix $(\partial r_i / \partial x_j)_{i=1, \dots, 5, j=1, \dots, 8}$ of the five relations in Example 25 has rank 2 at $\rho_1 = (x_1, \dots, x_8) = (1, 0, 0, 1, 1, 0, 0, 1)$. Therefore, $Hom(\mathbb{Z}^2, SL(2, \mathbb{C}))$ is singular at ρ_1 and, by Remark 24, $\rho_1 * \rho_2$ is a singular point of $Hom(\mathbb{Z}^2 * \Gamma, G)$.

10. ORBITS

As before, let O_ρ be the orbit of ρ in $Hom(\Gamma, G)$ under the G action by conjugation. Since O_ρ is homogeneous and (as every algebraic set) it has a simple point, all its points are simple, i.e., O_ρ is a non-singular algebraic set.

The following theorem generalizes [JM, Lemma 2.2] and [LM, Cor. 2.4].

Theorem 43. *For every ρ the inclusion*

$$T_\rho O_\rho \subset T_\rho Hom(\Gamma, G)$$

corresponds to

$$B^1(\Gamma, Ad \rho) \subset Z^1(\Gamma, Ad \rho)$$

under the isomorphism Ψ_ρ .

Proof. Since O_ρ is the image of the map $f_\rho : G \rightarrow Hom(\Gamma, G)$, $f_\rho(g) = g\rho g^{-1}$, O_ρ is the left quotient of G by the stabilizer of ρ , S_ρ , cf. [Bo, II.6.1]. Furthermore, the quotient map $G \rightarrow G/S_\rho$ induces an epimorphism $T_{[\rho]} G \rightarrow T_\rho(G/S_\rho)$. (Proof: S_ρ acts on G by left translation, with the trivial stabilizer. By Luna’s Étale Slice Theorem, [Lu], $G \rightarrow G//S_\rho$ is étale, inducing an isomorphism of tangent spaces. Note that $G//S_\rho = G/S_\rho$ since all orbits are closed.) Therefore every tangent vector v to O_ρ at ρ is of the form $d(f_\rho)(I + A\varepsilon)$. By the discussion of Section 8,

every vector $v \in T_\rho \mathcal{H}om(\Gamma, G)$ corresponds to a group homomorphism (6). By this correspondence, v corresponds to

$$(11) \quad \gamma \rightarrow (I + A\varepsilon)\rho(\gamma)(I + A\varepsilon)^{-1}.$$

Since

$$(I + A\varepsilon)^{-1} = I - A\varepsilon \pmod{\varepsilon^2},$$

(11) can be written as

$$\gamma \rightarrow (I + A\varepsilon)\rho(\gamma)(I - A\varepsilon) = (I + (A - \rho(\gamma)A\rho(\gamma)^{-1})\varepsilon)\rho(\gamma) \pmod{\varepsilon^2}.$$

The right side is $(I + \tau\varepsilon)\rho(\gamma)$, where

$$(12) \quad \tau = A - \rho(\gamma)A\rho(\gamma)^{-1} \in B^1(\Gamma, Ad \rho) \subset Z^1(\Gamma, Ad \rho).$$

Therefore, Ψ_ρ of Theorem 35 maps $T_\rho O_\rho$ to $B^1(\Gamma, Ad \rho)$. Since Ψ_ρ is an embedding, in order to show Ψ_ρ is an isomorphism from $T_\rho O_\rho$ to $B^1(\Gamma, Ad \rho)$, it is enough to prove that

$$(13) \quad \dim T_\rho O_\rho \geq \dim B^1(\Gamma, Ad \rho).$$

Since $Ad\rho(\gamma) : \mathfrak{g} \rightarrow \mathfrak{g}$ is the differential of the map

$$G \rightarrow G, \quad g \rightarrow \rho(\gamma)g\rho(\gamma)^{-1}$$

and that map is constant on S_ρ , $Ad\rho(\Gamma)$ acts trivially on the Lie algebra $L(S_\rho) \subset \mathfrak{g}$. Therefore (12) vanishes for $A \in L(S_\rho)$. Consequently,

$$\dim B^1(\Gamma, Ad \rho) \leq \dim \mathfrak{g} - \dim S_\rho.$$

Since the left side of (13) is $\dim G - \dim S_\rho$, inequality (13) follows. □

Remark 44. The action of S_ρ on $T_\rho \mathcal{H}om(\Gamma, G)$ and on $Z^1(\Gamma, Ad \rho)$ preserves $T_\rho O_\rho$ and $B^1(\Gamma, Ad \rho)$.

Theorems 35 and 43 imply Weil’s infinitesimal rigidity theorem, cf. [Wei].

Corollary 45. *If $H^1(\Gamma, Ad \rho) = 0$, then*

- (1) $\rho : \Gamma \rightarrow G$ is scheme smooth (and, hence, reduced).
- (2) O_ρ contains an open neighborhood of ρ (in complex topology) in $\mathcal{H}om(\Gamma, G)$.

Proof. (1) Since O_ρ is smooth, we have

$$\begin{aligned} \dim B^1(\Gamma, Ad \rho) &= \dim O_\rho \leq \dim \mathcal{H}om(\Gamma, G) \leq \dim T_\rho \mathcal{H}om(\Gamma, G) \\ &\leq \dim T_\rho \mathcal{H}om(\Gamma, G) = \dim Z^1(\Gamma, Ad \rho), \end{aligned}$$

by Theorems 35 and 43. Since all relations above are equalities, ρ is scheme smooth.

(2) This follows from the fact that $\mathcal{H}om(\Gamma, G)$ is smooth at ρ and has the same dimension as O_ρ . □

Since $H^1(\Gamma, Ad \rho) = 0$, for all finite groups Γ and all ρ , cf. [Wb, Thm. 6.5.8], we also have:

Corollary 46. *All G -representations of every finite group are reduced.*

11. CHARACTER VARIETIES

The categorical quotient of $\text{Hom}(\Gamma, G)$ by the G action by conjugation,

$$X_G(\Gamma) = \text{Hom}(\Gamma, G)//G,$$

is called the G -character variety of Γ . By definition, it is an affine algebraic set together with the map $\text{Hom}(\Gamma, G) \rightarrow \text{Hom}(\Gamma, G)//G$ which is constant on all G -orbits, with the property that every morphism from $\text{Hom}(\Gamma, G)$ into an affine algebraic set Y which is constant on all G -orbits factors through $\text{Hom}(\Gamma, G) \rightarrow \text{Hom}(\Gamma, G)//G$, cf. [Do, Fo, MFK]. If G is reductive, then the categorical quotient exists. The reason for considering the categorical quotient rather than the set theory quotient is that the quotient topology on $\text{Hom}(\Gamma, G)/G$ is often not a Zariski topology of any algebraic set. In particular, it often contains points which are not closed. Character varieties are often reducible, despite being called “varieties”.

Every equivalence class in $X_G(\Gamma)$ contains a unique closed orbit. Therefore, by Theorem 30, each element of the G -character variety of Γ is represented by a unique completely reducible representation.

Example 47. Let \mathbb{T} be a maximal torus of G . The map $\mathbb{T} \rightarrow \text{Hom}(\mathbb{Z}, G)$ sending g to the G -representation $1 \rightarrow g$ of \mathbb{Z} factors through an isomorphism

$$\mathbb{T}/W \rightarrow X_G(\mathbb{Z}),$$

where W is the Weyl group of G , cf. [St, 6.4].

Example 48. (1) The $SL(2, \mathbb{C})$ -character variety of the free group on two generators is isomorphic to \mathbb{C}^3 .

(2) The $SL(3, \mathbb{C})$ -character variety of the free group on two generators is a hypersurface in 9-dimensional affine space, [La1, Thm. 8], [Si1].

Algebraic properties of character varieties are discussed in [AP, BH, BK1, BK2, BK3, BKCh, GM, Ho1, Ho2, La1, La2, La3, La4, La5, LP, LM, Na, PBK, RBK, RBC, Si1, Si4, Si5, Wh]. Character varieties appear in many ways in mathematics and physics. Of particular importance are character varieties of surface groups, discussed in Section 14. They appear as moduli spaces of hyperbolic, projective, and other geometric structures on surfaces, cf. [BIW, Go1, Go3, Go9, GM1, GM2, GW, JM, KM, Li, Sa, Wa], as well as on the moduli spaces of flat connections, holomorphic bundles, and of Higgs bundles, [AB, AMW, CHM, Da, DDW, DWWW, DWW, FL1, FL2, GGM, Hi1, Hi2, HLR, HT, Je2, Je3, JK, Ki, NS, Me, MW, Ol, Rac, Sim1, Sim2, Th1, Th2, Th3, Wi2, Wi3, Za]. These connections inspired an investigation of topology (and, more specifically, cohomology) of character varieties in many of the papers cited here.

In a related fashion, character varieties of surfaces also appear as examples of symplectic reduction (and, furthermore, Kähler and hyper-Kähler reductions) as well as in the context of Hamiltonian actions, [AB, Au, Go2, Je1]. The action of the mapping class groups on character varieties is discussed in [Go6, Go7, Go8, PX, SS, We, Wi].

In mathematical physics, character varieties appear in the context of Yang-Mills and Chern-Simons quantum field theories [At, AB, Ba, Fr, Gu, JW, KK, We1, We2, We3, Wi1, Wi2, Wi3] as well as in the related skein theory of quantum invariants of 3-manifolds [Bu, CM, FG, FGL, Ga, Ge, Le2, PS1, PS2, Si2, Si3].

In low-dimensional topology, character varieties appear in the Bass-Culler-Shalen theory, in the context of A -polynomial and in other related areas [Be, BB, BLZ, BZ1, BZ2, BZ3, Du, DG, CCGLS, CL, CS, HS, MS, Mo, Shl] as well as in the context of Casson-Walker-Lescop invariant, [AM, BC1, BC2, BHe, BN, Cu]. Varieties of representations of 3-manifold groups have been studied for a long time also in relation to the Alexander polynomial, hyperbolic geometry, and for other independent reasons [BF, Ril1, HLM1, HLM2, HP1, HP2, Le1, LR1, LR2, Po, Ti]. Finally, character varieties appear in the discussion of local rigidity of discrete subgroups of Lie groups, cf. Corollary 45, [Wei, Ra, Rag].

These are just sample references to the above topics, as there are hundreds of papers devoted to every one of them.

Denote the set of irreducible representations in $\text{Hom}(\Gamma, G)$ by $\text{Hom}^i(\Gamma, G)$. The G action by conjugation preserves this set. Since all orbits in $\text{Hom}^i(\Gamma, G)$ are closed (cf. Theorem 30) and each equivalence class in a categorical quotient contains a unique closed orbit, the categorical quotient, $\text{Hom}^i(\Gamma, G)/G$, is the set-theoretic quotient as well. Denote it by $X_G^i(\Gamma)$.

Proposition 49. *Let G be a reductive group.*

(1) *For the free group F_n , $X_G^i(F_n)$ is a complex orbifold of complex dimension*

$$(n-1)\dim G + \dim C(G).$$

(2) *For every closed orientable surface S_g of genus $g > 1$, $X_G^i(\pi_1(S_g))$ is a complex orbifold of complex dimension*

$$(2g-2)\dim G + 2\dim C(G).$$

(3) *If G is CI (e.g., $G = GL(n, \mathbb{C}), SL(n, \mathbb{C})$), then $X_G^i(F_n)$ and $X_G^i(\pi_1(S_g))$ are manifolds for all $n, g > 1$. (See also [FL2].)*

Proof. (1) Since $\text{Hom}^i(F_n, G)$ is an open subset of G^n , it is smooth. By [JM, Prop. 1.1], the $G/C(G)$ action on $\text{Hom}^i(F_n, G)$ is properly discontinuous. By Proposition 15, $G/C(G)$ acts on $\text{Hom}^i(F_n, G)$ with finite centralizers. The quotient of a manifold by a properly discontinuous group action with finite stabilizers is an orbifold.

(2) By Proposition 37, all irreducible representations of $\pi_1(S_g)$ are scheme smooth. Therefore, by Theorem 35 and by [Go2, Prop. 1.2], $\text{Hom}^i(\pi_1(S_g))$ is a complex manifold of dimension $(2g-1)\dim G + \dim C(G)$. Now the proof follows as in (1).

(3) By definition of a CI group, the $G/C(G)$ action on $X_G^i(F_n)$ is free. By [JM, Prop. 1.1], this action is also properly discontinuous. The quotient of a manifold by a free properly discontinuous group action is a manifold [Th, Proposition 3.5.7]. \square

Recall that a representation $\rho : \Gamma \rightarrow G$ is good if and only if it is irreducible and the centralizer of $\rho(\Gamma)$ coincides with the center of G . By Proposition 33, the space of good representations, $\text{Hom}^g(\Gamma, G)$, is an open subset of the irreducible ones. Since $G/C(G)$ acts freely and properly discontinuously on $\text{Hom}^g(\Gamma, G)$, we have:

Corollary 50. *For every reductive group G and every surface group or a free group Γ , $X_G^g(\Gamma) = \text{Hom}^g(\Gamma, G)/G$ is an open subset of $X_G^i(\Gamma)$ and a smooth complex manifold.*

For a topological space Y , we will abbreviate $X_G(\pi_1(Y))$ by $X_G(Y)$.

Proposition 51. (1) If M is a compact 3-manifold with a connected boundary of genus g , then

$$\dim X_G(M) \geq \dim G \cdot (g - 1) + \dim C(G).$$

(2) For a given non-abelian reductive group G and a positive integer g , there is no upper bound on $\dim X_G(M)$ over compact 3-manifolds M with connected boundary of fixed genus g .

Proof. (1) If M is a compact manifold with connected boundary of genus g , then $\pi_1(M)$ has a presentation with n generators and p relations such that

$$1 - n + p = \chi(M) = 1 - g.$$

Hence $\dim \text{Hom}(\pi_1(M), G) \geq \dim G \cdot (n - p) = \dim G \cdot g$. Since $X_G(M)$ is the quotient of $\text{Hom}(\pi_1(M), G)$ by $G/C(G)$, the statement follows.

(2) It is enough to prove that there is no upper bound on $\dim \text{Hom}(\pi_1(M), G)$ over compact manifolds M with connected boundary of fixed genus g . Since every non-abelian reductive group contains either $SL(2, \mathbb{C})$ or $PSL(2, \mathbb{C})$,

$$\begin{aligned} \dim \text{Hom}(\pi_1(M), G) \\ \geq \min(\dim \text{Hom}(\pi_1(M), SL(2, \mathbb{C})), \dim \text{Hom}(\pi_1(M), PSL(2, \mathbb{C}))). \end{aligned}$$

Since the quotient map $SL(2, \mathbb{C}) \rightarrow PSL(2, \mathbb{C})$ induces a finite map

$$\begin{aligned} \text{Hom}(\pi_1(M), SL(2, \mathbb{C})) &\rightarrow \text{Hom}(\pi_1(M), PSL(2, \mathbb{C})), \\ \dim \text{Hom}(\pi_1(M), SL(2, \mathbb{C})) &\leq \dim \text{Hom}(\pi_1(M), PSL(2, \mathbb{C})). \end{aligned}$$

(The inequality stems from the fact that this map does not have to be onto.) Therefore, it is enough to prove that there is no upper bound on

$$\dim \text{Hom}(\pi_1(M), SL(2, \mathbb{C})).$$

Let K_n be the connected sum of n copies of a knot K . Cooper and Long [CL] proved that $\dim \text{Hom}(\pi_1(S^3 \setminus K_n), SL(2, \mathbb{C})) \geq n + 3$. (Although their argument is made for hyperbolic knots K only, it generalizes to all knots by the result of [KrM], cf. [DG].) Let $K_{n,g}$ be a graph obtained by connecting g unlinked copies of K_n in S^3 by $g - 1$ tunnels, and let $M_{n,g}$ be the complement of an open tubular neighborhood of $K_{n,g}$ in S^3 . Then $\pi_1(M_{n,g})$ is the free product of g copies of $\pi_1(S^3 \setminus K_n)$ and

$$\dim \text{Hom}(\pi_1(M_{n,g}), SL(2, \mathbb{C})) \geq g \cdot (n + 3),$$

by Remark 24. Since $\partial M_{n,g}$ is a surface of genus g , the statement follows. □

12. CHARACTER VARIETIES AS ALGEBRAIC SCHEMES

The algebraic scheme $\mathcal{X}_G(\Gamma) = \text{Hom}(\Gamma, G)/G$ is often also called the G -character variety of Γ . By the definition of categorical quotient, $\mathcal{X}_G(\Gamma) = \text{Spec } R(\Gamma, G)^G$. The epimorphism $R(\Gamma, G) \rightarrow \mathbb{C}[\text{Hom}(\Gamma, G)]$ induces the map

$$(14) \quad R(\Gamma, G)^G \rightarrow \mathbb{C}[\text{Hom}(\Gamma, G)]^G,$$

which is an epimorphism by the properties of the Reynolds operator. Hence, we have an embedding of schemes

$$(15) \quad i : X_G(\Gamma) \hookrightarrow \mathcal{X}_G(\Gamma).$$

Since the kernel of (14) is the nil-radical of $R(\Gamma, G)^G$, (15) is a bijection between the points of $X_G(\Gamma)$ and the closed points of $\mathcal{X}_G(\Gamma)$.

In [Si1], we have described the coordinate ring of $\mathcal{X}_{SL(n,\mathbb{C})}(\Gamma)$ as a space of n -valent graphs reminiscent of Feynman diagrams in an arbitrary path connected topological space X with $\pi_1(X) = \Gamma$.

By Corollary 40, $R(\Gamma, G)^G$ has no zero divisors for surface groups and $G = GL(n, \mathbb{C}), SL(n, \mathbb{C})$. Hence, by [RBC], $\mathcal{X}_G(\Gamma)$ is reduced and irreducible for such groups. For the F torus and $G = SL(2, \mathbb{C})$, $R(\pi_1(F), G)^G$ is reduced as well by [PS1, Thm. 3.3]. These facts have an important consequence for the theory of skein modules.

Corollary 52. *For every closed orientable surface F , the map ϕ of [PS2, Thm. 7.1] is an isomorphism between the skein algebra of $\pi_1(F)$ and $\mathbb{C}[X_{SL(2,\mathbb{C})}(\pi_1(F))]$. Consequently, ϕ composed with the isomorphism of [PS2, Thm. 2.8] is an isomorphism from the skein algebra of a surface, $S_{2,\infty}(F, \mathbb{C}, -1)$, to $\mathbb{C}[X_{SL(2,\mathbb{C})}(\pi_1(F))]$.*

Although this result was announced in [PS2, Thm. 7.3], its proof required [PS2, Thm. 4.7] whose proof was not provided. An alternative proof of the above statement was provided independently by L. Charles and J. Marché in [CM].

$\mathcal{X}_G(\Gamma)$ is not always reduced. Kapovich and Millson proved that for every affine (possibly unreduced) variety X over \mathbb{Q} there is an Artin group Γ such that a Zariski open subset of $\mathcal{X}_{PSL(2,\mathbb{C})}(\Gamma)$ is isomorphic to a Zariski open subset of X [KM]. Additionally, for every $x \in X$ there is a representation ρ of an Artin group Γ into $PSL(2, \mathbb{C})$ such that the analytic germ of $\mathcal{X}_{PSL(2,\mathbb{C})}(\Gamma)$ at $[\rho]$ coincides with the analytic germ of X at x [KM].

Kapovich proved that the same is true for 3-manifold groups. That is, for every $x \in X$ as above there is a closed 3-manifold M and a $PSL(2, \mathbb{C})$ -representation ρ of $\Gamma = \pi_1(M)$ such that the analytic germ of $\mathcal{X}_{PSL(2,\mathbb{C})}(\Gamma)$ at $[\rho]$ coincides with the analytic germ of X at x [Ka1, Ka2]. In particular $\mathcal{X}_{PSL(2,\mathbb{C})}(\Gamma)$ contains non-zero nilpotent elements for some Artin groups and some 3-manifold groups Γ .

13. TANGENT SPACES TO CHARACTER VARIETIES

For every ρ the map (15) induces an embedding

$$(16) \quad T_{[\rho]} X_G(\Gamma) \hookrightarrow T_{[\rho]} \mathcal{X}_G(\Gamma).$$

It is an isomorphism if ρ is reduced. We are going to give a cohomological description of these tangent spaces.

For every $\rho : \Gamma \rightarrow G$, the S_ρ action on $Z^1(\Gamma, Ad \rho)$ descends to an action on $H^1(\Gamma, Ad \rho)$.

Theorem 53. *(1) If $\rho : \Gamma \rightarrow G$ is completely reducible, then the isomorphism Φ_ρ of Theorem 35 combined with the natural projection $\mathcal{H}om(\Gamma, G) \rightarrow \mathcal{X}_G(\Gamma)$ induces a natural linear map*

$$\phi : H^1(\Gamma, Ad \rho) \rightarrow T_{[\rho]} \mathcal{X}_G(\Gamma).$$

(2) If ρ is good, then ϕ is an isomorphism.

(3) If ρ is completely reducible and scheme smooth (but not necessarily good), then

$$\dim T_0 (H^1(\Gamma, Ad \rho) // S_\rho) = \dim T_{[\rho]} \mathcal{X}_G(\Gamma) = \dim T_{[\rho]} X_G(\Gamma).$$

The naturality of the morphism ϕ means that for every $\alpha : \Gamma' \rightarrow \Gamma$ and $\beta : G \rightarrow G'$ such that $\rho : \Gamma \rightarrow G$ and $\beta\rho\alpha : \Gamma' \rightarrow G'$ are completely reducible, the following

diagram commutes:

$$\begin{array}{ccc} H^1(\Gamma, Ad \rho) & \xrightarrow{\phi} & T_{[\rho]} \mathcal{X}_G(\Gamma) \\ \downarrow & & \downarrow \\ H^1(\Gamma', Ad \beta \rho \alpha) & \xrightarrow{\phi} & T_{[\beta \rho \alpha]} \mathcal{X}_{G'}(\Gamma'), \end{array}$$

where the vertical homomorphisms are induced by α and β .

We discuss the existence of a natural isomorphism in part (3) of the above theorem at the end of this section.

Versions of the above theorem for $G = PSL(2, \mathbb{C})$ appear in [Po, Prop. 3.5] and [HP2, Prop. 5.2]. (Note that every irreducible $PSL(2, \mathbb{C})$ -representation is Ad -irreducible and, hence, good, by Proposition 14.) A related discussion (for abelian representations) appears in [Be].

Although statements similar to Theorem 53(2) appear in the literature for general G , there is a lot of confusion about the necessary assumptions and there is a lack of discussion of the naturality of the isomorphism. Furthermore, all proofs known to us are incomplete.

Proof of Theorem 53. The proof of the above theorem is based on Luna’s Étale Slice Theorem, [Lu], cf. [MFK], [PV, Thm. 6.1], which applies to closed orbits of group actions on affine (not necessarily reduced) schemes.

(1) Since the orbit of ρ is closed by Theorem 30, by Luna’s Slice Theorem there exists a subscheme $S \subset \mathcal{H}om(\Gamma, G)$, called an étale slice, such that

- (i) $\rho \in S$.
- (ii) S is preserved by the S_ρ action.
- (iii) If we denote the categorical quotient of $G \times S$ by the diagonal S_ρ -action (which is right on G and left on S) by $G \times_{S_\rho} S$, then the natural map $G \times_{S_\rho} S \rightarrow \mathcal{H}om(\Gamma, G)$ sending (g, s) to gs is étale.
- (iv) $S//S_\rho \rightarrow \mathcal{X}_G(\Gamma)$ is étale.

In particular, it means that the differentials of the above maps

$$(17) \quad T_{(e, \rho)} (G \times_{S_\rho} S) \rightarrow T_\rho \mathcal{H}om(\Gamma, G)$$

and

$$(18) \quad T_{[\rho]} S//S_\rho \rightarrow T_{[\rho]} \mathcal{X}_G(\Gamma)$$

are isomorphisms.

The natural projection,

$$\mathcal{H}om(\Gamma, G) \rightarrow \mathcal{X}_G(\Gamma)$$

induces

$$(19) \quad T_\rho \mathcal{H}om(\Gamma, G) \rightarrow T_{[\rho]} \mathcal{X}_G(\Gamma),$$

which can be written as

$$(20) \quad T_\rho \mathcal{H}om(\Gamma, G) \xrightarrow{\cong} T_{(e, \rho)} (G \times_{S_\rho} S) \rightarrow T_{[\rho]}(S//S_\rho) \xrightarrow{\cong} T_{[\rho]} \mathcal{X}_G(\Gamma),$$

where the first map is the inverse of (17), the second is the differential of

$$(21) \quad G \times_{S_\rho} S \rightarrow S//S_\rho,$$

and the third one is (18).

Since the map $G \rightarrow G \times_{S_\rho} S$ composed with (21) is constant, the image of $T_e G$ in $T_{(e,\rho)}G \times_{S_\rho} S$ belongs to the kernel of $T_\rho \mathcal{H}om(\Gamma, G) \rightarrow T_\rho(S//S_\rho)$. Since this image corresponds to $T_\rho O_\rho \subset T_\rho \mathcal{H}om(\Gamma, G)$, (20) factors through

$$(22) \quad T_\rho \mathcal{H}om(\Gamma, G)/T_\rho O_\rho \rightarrow T_\rho(S//S_\rho) \xrightarrow{\simeq} T_\rho \mathcal{X}_G(\Gamma).$$

Now the statement follows from Theorems 35 and 43.

(2) If ρ is good, then $S_\rho = C(G)$ acts trivially on S and (21) can be written as the projection

$$G/C(G) \times S \rightarrow S.$$

Its differential is the projection $T_e G/C(G) \times T_\rho S \rightarrow T_\rho S$ whose kernel is $T_\rho O_\rho$. Therefore, the left morphism of (22) is an isomorphism.

(3) Since the stabilizer of the S_ρ -action on $G \times S$ at (e, ρ) is trivial, by Luna's Slice Theorem we have an étale morphism $(G \times S)/\{e\} \rightarrow G \times_{S_\rho} S$ yielding an isomorphism of the tangent spaces. Composing it with (17), we obtain an isomorphism

$$(23) \quad T_e G \times T_\rho S \rightarrow T_\rho \mathcal{H}om(\Gamma, G).$$

The image of $T_e G$ in $T_\rho \mathcal{H}om(\Gamma, G)$ is $T_\rho O_\rho$. Therefore,

$$(24) \quad T_\rho S \simeq \mathcal{H}om(\Gamma, G)/T_\rho O_\rho \simeq H^1(\Gamma, Ad \rho).$$

Since ρ is a simple point of $\mathcal{H}om(\Gamma, G)$, we can assume that S is non-singular at ρ by [Lu, Remark III.1.1]. Therefore, there exists an isomorphism

$$(25) \quad T_\rho(S//S_\rho) \simeq T_0(T_\rho S//S_\rho)$$

by [Lu, Lemma III.1] (cf. [PV, Thm. 6.4]). This isomorphism combined with (24) and (18) implies the statement. \square

Since O_ρ is always reduced, it is a subvariety of $\mathcal{H}om(\Gamma, G)$ in $\mathcal{H}om(\Gamma, G)$. A version of the above argument applied to the G action on $\mathcal{H}om(\Gamma, G)$ yields the following:

Theorem 54. (1) *If ρ is completely reducible, then the projection $\mathcal{H}om(\Gamma, G) \rightarrow X_G(\Gamma)$ induces a linear map*

$$T_\rho \mathcal{H}om(\Gamma, G)/B^1(\Gamma, Ad \rho) \rightarrow T_{[\rho]} X_G(\Gamma).$$

(2) *If ρ is good, then this is an isomorphism.*

(3) *If ρ is smooth, then*

$$T_0\left(\left(T_\rho \mathcal{H}om(\Gamma, G)/B^1(\Gamma, Ad \rho)\right)//S_\rho\right) \simeq T_{[\rho]} X_G(\Gamma).$$

Corollary 55. *If ρ is good, then $\mathcal{X}_G(\Gamma)$ is reduced at $[\rho]$ if and only if ρ is reduced.*

Proof. $\mathcal{X}_G(\Gamma)$ is reduced if and only if

$$T_{[\rho]} X_G(\Gamma) = T_{[\rho]} \mathcal{X}_G(\Gamma),$$

which holds if and only if

$$T_\rho \mathcal{H}om(\Gamma, G) = T_\rho \mathcal{H}om(\Gamma, G)$$

by Theorems 53(2) and 54(2). \square

Question 56. (1) Is there a natural linear map

$$T_0 (H^1(\Gamma, Ad \rho) // S_\rho) \rightarrow T_{[\rho]} \mathcal{X}_G(\Gamma)$$

for all completely reducible ρ , which is an isomorphism for scheme smooth ρ ?

(2) Is this map an injection or surjection for ρ which are not scheme smooth?

The difficulty in answering this question comes from the following three factors:

- (1) Étale slices are not unique.
- (2) The isomorphism (25) is non-canonical for a given slice S .
- (3) Map ϕ does not descend to an isomorphism

$$T_0 (H^1(\Gamma, Ad \rho) // S_\rho) \rightarrow T_{[\rho]} \mathcal{X}_G(\Gamma)$$

if $H^1(\Gamma, Ad \rho) \rightarrow T_0 (H^1(\Gamma, Ad \rho) // S_\rho)$ is not onto.

The following result illuminates the importance of the requirement of ρ being scheme smooth in Theorem 53(2).

Theorem 57. $\mathcal{X}_G^g(\Gamma)$ is reduced iff $T_{[\rho]} X_G(\Gamma) = H^1(\Gamma, Ad \rho)$ for a non-empty Zariski open set of $[\rho]$'s in $X_G^g(\Gamma)$.

Proof. \Rightarrow By Theorem 53(2), the equality holds for all $[\rho] \in X_G^g(\Gamma)$.

\Leftarrow If $\mathcal{X}_G^g(\Gamma)$ is not reduced, then $\mathcal{X}_G^g(\Gamma)$ is not reduced at $[\rho]$ for an open set Ω of $[\rho]$'s in $X_G^g(\Gamma)$. Since the set of smooth and good representations, $Hom^{sg}(\Gamma, G)$, is the non-singular part of the set of good representations, it is an open dense subset of $Hom^g(\Gamma, G)$. Hence $\Omega \cap Hom^{sg}(\Gamma, G)$ is non-empty and open.

By Corollary 55, ρ 's in $\Omega \cap Hom^{sg}(\Gamma, G)$ are not reduced. Therefore,

$$\dim T_\rho Hom(\Gamma, G) < \dim T_\rho Hom(\Gamma, G) = \dim Z^1(\Gamma, Ad \rho)$$

by Theorem 35. Since

$$\dim T_{[\rho]} X_G(\Gamma) = \dim T_\rho Hom(\Gamma, G) - \dim B^1(\Gamma, Ad \rho)$$

for $\rho \in Hom^{sg}(\Gamma, G)$, by Theorem 54(3), we get

$$\dim T_{[\rho]} X_G(\Gamma) < \dim H^1(\Gamma, Ad \rho)$$

for $\rho \in \Omega \cap Hom^{sg}(\Gamma, G)$. □

14. SYMPLECTICITY OF CHARACTER VARIETIES OF SURFACES

Let G be a reductive group, and let \mathfrak{g} be its Lie algebra. A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ is Ad -invariant if $B(Ad(g)x, Ad(g)y) = B(x, y)$.

Let F be a closed orientable surface. For every representation $\rho : \pi_1(F) \rightarrow G$ and every Ad -invariant bilinear form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$, the cup product defines a pairing

$$(26) \quad \omega_B : H^1(F, Ad \rho) \times H^1(F, Ad \rho) \xrightarrow{\cup} H^2(F, Ad \rho \otimes Ad \rho) \xrightarrow{B} H^2(F, \mathbb{C}) = \mathbb{C},$$

which can also be identified with the pairing

$$(27) \quad H^1(F, Ad \rho) \times H_1(F, Ad \rho) \xrightarrow{\cap} H_0(F, Ad \rho \otimes Ad \rho) \xrightarrow{B} H_0(F, \mathbb{C}) = \mathbb{C}$$

via the Poincaré duality with twisted coefficients, [Sp],

$$\cap[F] : H^n(F, Ad \rho) \rightarrow H_{2-n}(F, Ad \rho),$$

where $[F] \in H_2(F, \mathbb{C})$ is a fundamental class of F .

Let Γ be a group, and let (C_*, ∂) be a chain complex of left $\mathbb{Z}\Gamma$ -modules. Let M_1, M_2 be left $\mathbb{Z}\Gamma$ -modules. If $B : M_1 \times M_2 \rightarrow \mathbb{C}$ is a $\mathbb{Z}\Gamma$ -invariant pairing, i.e., $B(rm_1, rm_2) = B(m_1, m_2)$ for every $r \in \mathbb{Z}\Gamma$, $m_1 \in M_1$, $m_2 \in M_2$, then, by [Br, V §3], the cap product induces a pairing

$$(28) \quad H^n(\text{Hom}_{\mathbb{Z}\Gamma}(C_*, M_2), \delta) \times H_n(M_1 \otimes_{\mathbb{Z}\Gamma} C_*, \partial) \rightarrow M_1 \otimes_{\mathbb{Z}\Gamma} M_2 \xrightarrow{B} \mathbb{C}.$$

In the above formula M_1 is considered as a right $\mathbb{Z}\Gamma$ -module via $m \cdot \gamma = \gamma^{-1} \cdot m$.

Lemma 58. *If B is a duality pairing, i.e., if B induces an isomorphism $M_1 \simeq \text{Hom}(M_2, \mathbb{C})$, then the pairing (28) is non-degenerate.*

Proof. The cochain complex $(\text{Hom}_{\mathbb{Z}\Gamma}(C_*, M_2), \delta)$ can be written as

$$\begin{aligned} (\text{Hom}_{\mathbb{Z}\Gamma}(C_*, \text{Hom}(M_1, \mathbb{C})), \delta) &= (\text{Hom}_{\mathbb{Z}\Gamma}(C_* \otimes M_1, \mathbb{C}), \delta) \\ &= (\text{Hom}(C_* \otimes_{\mathbb{Z}\Gamma} M_1, \mathbb{C}), \delta) = \text{Hom}((C_* \otimes_{\mathbb{Z}\Gamma} M_1, \partial), \mathbb{C}). \end{aligned}$$

Hence

$$(29) \quad H_n(\text{Hom}_{\mathbb{Z}\Gamma}(C_*, M_2), \delta) = H_n(\text{Hom}((C_* \otimes_{\mathbb{Z}\Gamma} M_1, \partial), \mathbb{C})).$$

Since \mathbb{C} is a divisible group, $\text{Hom}(\cdot, \mathbb{C})$ is an exact functor in the category of abelian groups. Hence, (29) becomes

$$H_n(\text{Hom}_{\mathbb{Z}\Gamma}(C_*, M_2), \delta) = \text{Hom}(H_n(C_* \otimes_{\mathbb{Z}\Gamma} M_1, \partial), \mathbb{C}).$$

It is easy to verify that this isomorphism is induced by (28). □

If B is symmetric, then (26) is skew-symmetric. Therefore, Lemma 58 implies:

Corollary 59 (cf. [Go1]). *If $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ is symmetric, Ad -invariant, and non-degenerate, then (26) is a symplectic form on $H^1(F, Ad \rho)$.*

If \mathfrak{g} is simple, then the Killing form is unique among symmetric, G -invariant, non-degenerate forms on \mathfrak{g} , up to a constant multiple.

Let F be a closed orientable surface of genus ≥ 2 now. By Corollary 50, the space of conjugacy classes of good representations,

$$X_G^g(F) = \text{Hom}^g(\pi_1(F), G) // G = \text{Hom}^g(\pi_1(F), G) / G$$

is a complex manifold and, by Proposition 37 and Theorem 53(2),

$$T_{[\rho]} X_G^g(F) = H^1(\pi_1(F), Ad \rho).$$

Remark 60. Goldman gave an “algebraic” description of the form ω_B on $X_G^g(F)$, [Go2]. His argument implies that ω_B is holomorphic. However, we do not know if ω_B can be written as $\sum_{i=1}^n f_i dg_i \wedge dh_i$, for some regular functions $f_1, \dots, f_n, g_1, \dots, g_n, h_1, \dots, h_n$ on $X_G(F)$.

Goldman proves by an argument from gauge theory that for every non-degenerate, symmetric, Ad -invariant B , ω_B is closed [Go1]. Therefore, $(X_G^g(F), \omega_B)$ is a holomorphic symplectic manifold. (Note that our Proposition 37 is needed to complete Goldman’s construction of ω_B .)

Although $X_G^g(\text{torus})$ is empty for most G , $X_G(\text{torus})$ is a singular symplectic manifold as well [Si5].

15. 3-MANIFOLDS AND LAGRANGIAN SUBSPACES

Let M be an orientable compact 3-manifold with a connected boundary F . The embedding $\partial M \hookrightarrow M$ induces a homomorphism $r : \pi_1(F) \rightarrow \pi_1(M)$ and a map $r^* : X_G(M) \rightarrow X_G(F)$.

It is often believed that the non-singular part of the image of $X_G(M)$ in $X_G^g(F)$ is Lagrangian. We investigate this claim in this section.

Theorem 61. *Consider the symplectic form ω_B induced by a symmetric non-degenerate bilinear Ad -invariant form B on \mathfrak{g} .*

(1) *Let $\rho : \pi_1(M) \rightarrow G$ be such that ρr is good. Then the image of $T_{[\rho]} \mathcal{X}_G(M)$ in $T_{[\rho]} X_G(F)$ is Lagrangian.*

(2) *The non-singular part of $r^*(X_G(M))$ in $X_G^g(F)$, denoted by us by $Y_G(M)$, is an isotropic submanifold of $X_G^g(F)$. (In particular, every connected component of $Y_G(M)$ of dimension $\frac{1}{2} \dim X_G(F)$ is Lagrangian.)*

(3) *If a connected component of $Y_G(M)$ contains an equivalence class of a reduced G -representation of $\pi_1(M)$, then it is a Lagrangian submanifold of $X_G(F)$.*

Although we do not know any example of a 3-manifold M such that $Y_G(M)$ is not Lagrangian, in light of the above theorem and the results of M. Kapovich mentioned in Section 12 we believe that such examples exist.

Here is an alternative version of Theorem 6.1(3):

Theorem 62. *Let $X_G^s(M)$ be the set of equivalence classes of scheme smooth G -representations of $\pi_1(M)$. Then $X_G^g(F) \cap r^*(X_G^s(M))$ is an immersed Lagrangian submanifold of $X_G^g(F)$.*

Note that $X_G^s(M)$ may be empty, even if there exist good G -representations of $\pi_1(M)$. Note also that $r^* : X_G^s(M) \rightarrow X_G^g(F)$ does not have to be an immersion. Indeed, it is easy to choose an example of a 3-manifold M satisfying the statement of Proposition 51(2) all of whose G -representations are reduced and, hence, $X_G^s(M)$ has an arbitrarily large dimension. We prove versions of Theorems 61 and 62 for F torus in [Si5].

Theorems 61 and 62 have applications to Chern-Simons quantum field theory, [Ba, Fr, Gu, JW, We1, We2, Si3]. It is a $(2 + 1)$ -topological quantum field theory, associating a Hilbert space $H(F)$ to every closed orientable surface F and a vector $I(M) \in H(F)$ to every 3-manifold M with $\partial M = F$. Although a mathematically rigorous version of this theory exists for compact groups G , [RT, BHMV], its constructions are combinatorial and algebraic in nature, yielding little information about the relations between $I(M)$ and the topology of M . Furthermore, a rigorous construction of this theory is still missing for complex algebraic groups.

One hopes to achieve a geometric construction of Chern-Simons TQFT for all groups through a procedure of geometric quantization which associates Hilbert spaces H to symplectic manifolds X and vectors in H to Lagrangian subspaces of X [Sn, Wo]. It is natural to expect that Witten's $I(M) \in H(F)$ is the vector associated with $Y_G(M)$. However, several obstacles exist in this approach, the first one being the question of whether $Y_G(M)$ is Lagrangian.

Theorems 61 and 62 are relevant also to Floer homology theory for 3-manifolds, cf. [Cu, BC1, BC2]. If $F \subset M$ is a surface separating M into M_1 and M_2 and $Y_G(M_1), Y_G(M_2)$ are Lagrangian submanifolds of $X_G(F)$, then one may consider Floer symplectic homology groups for such a splitting. (A difficulty with this approach lies in the non-compactness of $X_G(F)$.)

For every representation $\rho : \pi_1(M) \rightarrow G$, the homomorphism $r : \pi_1(F) \rightarrow \pi_1(M)$ induces $r^* : H^1(M, Ad \rho) \rightarrow H^1(F, Ad \rho r)$. The proofs of Theorems 61 and 62 are based on the following:

Theorem 63. *For every $\rho : \pi_1(M) \rightarrow G$, $r^*H^1(M, Ad \rho)$ is a Lagrangian subspace of the symplectic space $(H^1(F, Ad \rho r), \omega_B)$ with respect to every non-degenerate, Ad-invariant, symmetric, bilinear form B on \mathfrak{g} .*

In particular, for the trivial representation $\rho : \pi_1(M) \rightarrow \mathbb{C}^*$, Theorem 63 implies the following classical result:

Corollary 64. *For every compact, orientable 3-manifold with a connected boundary F , the image of the map $r^* : H^1(M, \mathbb{C}) \rightarrow H^1(F, \mathbb{C})$ induced by the embedding $r : F \hookrightarrow M$ is a Lagrangian subspace of $H^1(F, \mathbb{C})$ with the symplectic form being the cup product.*

Proof of Theorem 63. (1) We prove that

$$\dim r^*H^1(M, Ad \rho) = \frac{1}{2} \dim H^1(F, Ad \rho r)$$

first, by filling in the details of the approach of [Fr]. (This approach was communicated to us by Charlie Frohman.) By Poincaré-Lefschetz duality we have

$$(30) \quad \begin{array}{ccccc} H_2(M, F, Ad \rho) & \xrightarrow{\partial} & H_1(F, Ad \rho r) & \xrightarrow{r_*} & H_1(M, Ad \rho) \\ & \downarrow & \downarrow \eta & & \downarrow \\ H^1(M, Ad \rho) & \xrightarrow{r^*} & H^1(F, Ad \rho r) & \xrightarrow{\delta} & H^2(M, F, Ad \rho), \end{array}$$

where all vertical maps are isomorphisms induced by Poincaré duality. By Corollary 59, the cap product

$$H^1(F, Ad \rho r) \times H_1(F, Ad \rho r) \xrightarrow{\cap} H_0(F, Ad \rho r \otimes Ad \rho r) \xrightarrow{B} H_0(F, \mathbb{C}) = \mathbb{C}$$

is non-degenerate. Similarly,

$$\begin{aligned} H^1(M, Ad \rho) \times H_1(M, Ad \rho) &\xrightarrow{\cap} H_0(M, Ad \rho \otimes Ad \rho) \xrightarrow{B} \mathbb{C}, \\ H^2(M, F, Ad \rho) \times H_2(M, F, Ad \rho) &\xrightarrow{\cap} H_0(M, F, Ad \rho \otimes Ad \rho) \xrightarrow{B} \mathbb{C} \end{aligned}$$

are non-degenerate by Lemma 58. Consider the isomorphisms

$$\begin{aligned} H^1(M, Ad \rho) &\simeq (H_1(M, Ad \rho))^*, & H^1(F, Ad \rho r) &\simeq (H_1(F, Ad \rho r))^*, \\ H^2(M, F, Ad \rho) &\simeq (H_2(M, F, Ad \rho))^* \end{aligned}$$

defined by these pairings. Under these identifications, r^* and r_* and ∂ and δ are the duals of each other. Hence

$$\begin{aligned} \text{rank } r^* &= \text{rank } r_* = \text{rank } \delta = \dim H^1(F, Ad \rho r) - \dim \text{Ker } \delta \\ &= \dim H^1(F, Ad \rho r) - \text{rank } r^*. \end{aligned}$$

(2) It remains to be proven that $r^*H^1(M, Ad \rho)$ is an isotropic subspace of $H^1(F, Ad \rho)$.

The pairing (27) identifies $H_1(F, Ad \rho r)$ with $H^1(F, Ad \rho r)^*$. The isomorphism η^{-1} of (30) sends $\alpha \in H^1(F, Ad \rho r)$ to $\eta^{-1}(\alpha) \in H_1(F, Ad \rho r)$, which under the above identification is the functional $f_\alpha : H^1(F, Ad \rho r) \rightarrow \mathbb{C}$, $f_\alpha(\beta) = \omega_B(\alpha, \beta)$.

By Lemma 58, the pairing

$$(31) \quad H^1(M, Ad \rho) \times H_1(M, Ad \rho) \xrightarrow{\cap} H_0(M, Ad \rho \otimes Ad \rho) \xrightarrow{B} H_0(M, \mathbb{C}) = \mathbb{C}$$

is non-degenerate. If we use it to identify $H_1(M, Ad \rho)$ with $H^1(M, Ad \rho)^*$, then r_* in (30) sends f_α to $f_\alpha r_* : H^1(M, Ad \rho) \rightarrow \mathbb{C}$. By commutativity and exactness of (30), $f_\alpha r_* = 0$ for every $\alpha \in r^*(H^1(M, Ad \rho))$. In other words, $f_\alpha(\beta) = 0$ for every $\alpha, \beta \in r^*(H^1(M, Ad \rho))$. \square

Remark 65. Let G be reductive and $\rho : \pi_1(M) \rightarrow G$ be such that $\rho r : \pi_1(F) \rightarrow G$ is good. Then ρ is good as well and, by Theorem 53(1) and (2), the following diagram commutes:

$$\begin{array}{ccc} H^1(M, Ad \rho) & \xrightarrow{r^*} & H^1(F, Ad \rho r) \\ \downarrow \phi & & \downarrow \phi \\ T_{[\rho]} \mathcal{X}_G(M) & \xrightarrow{dr^*} & T_{[\rho r]} \mathcal{X}_G(F), \end{array}$$

where ϕ is the isomorphism of Theorem 53(2).

Proof of Theorem 61. (1) is a direct consequence of Theorems 53, 63 and Remark 65.

(2) Since $Y_G(M)$ is smooth, it is enough to show the statement

$$(32) \quad T_{[\rho r]} Y_G(M) \subset T_{[\rho r]} X_G(F) \quad \text{is isotropic}$$

for a dense subset of points $[\rho r] \in Y_G(M)$ (in the complex topology). Let $X'_G(M) \subset X_G(M)$ be the non-singular part of $(r^*)^{-1} Y_G(M)$. Since $r^*(X'_G(M))$ is dense in $Y_G(F)$, it is enough to show that (32) holds for a dense subset of points in $r^*(X'_G(M))$. By Sard's theorem there is a dense set of points in $[\rho r] \in r^*(X'_G(M))$ for which

$$T_{[\rho r]} Y_G(M) = dr^* T_{[\rho]} X_G(M).$$

Therefore, it is enough to show that $dr^* T_{[\rho]} X_G(M)$ is isotropic in $T_{[\rho r]} X_G(F)$. This follows from (1) and the fact that $T_{[\rho]} X_G(M)$ is a subspace of $T_{[\rho]} \mathcal{X}_G(M)$.

(3) Let $\rho : \pi_1(M) \rightarrow G$ be a reduced, irreducible representation whose conjugacy class belongs to $Y_G(M)$. By Theorem 53(2), both ϕ 's in the diagram of Remark 65 are isomorphisms. By Theorem 63, $\dim T_{[\rho]} C = \frac{1}{2} \dim T_{[\rho]} X_G(F)$. Since C and $X_G^g(F)$ are smooth, that equality holds for all points of C . Now the statement follows from (2). \square

Proof of Theorem 62. By Proposition 33, $X_G^g(F) \subset X_G(F)$ is open and, hence, $U = (r^*)^{-1}(X_G^g(F)) \cap X_G^s(M)$ is an open subset of $X_G^s(M)$. Since ϕ 's in the diagram of Remark 65 are isomorphisms, by Theorem 63,

$$dr^* : T_{[\rho]} U \rightarrow T_{[\rho]} X_G^g(F)$$

has constant rank for all $[\rho]$ in U . By the Constant Rank Theorem, for every $[\rho] \in U$ there is a neighborhood V of $[\rho]$ in U and a neighborhood W of $r^*([\rho])$ in $X_G^g(G)$ such that $r^*(V) \cap W$ a submanifold of W . Consequently, $r^*(U) = r^*(X_G^s(M)) \cap X_G^g(F)$ is an immersed submanifold of $X_G^g(F)$. \square

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