HARMONIC MAP HEAT FLOW WITH ROUGH BOUNDARY DATA

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ABSTRACT. Let B_1 be the unit open disk in $\mathbb{R}^2$ and M a closed Riemannian manifold. In this note, we first prove the uniqueness for weak solutions of the harmonic map heat flow in $H^1([0, T] \times B_1, M)$ whose energy is non-increasing in time, given initial data $u_0 \in H^1(B_1, M)$ and boundary data $\gamma = u_0|_{\partial B_1}$. Previously, this uniqueness result was obtained by Rivière (when M is the round sphere and the energy of initial data is small) and Freire (when M is an arbitrary closed Riemannian manifold), given that $u_0 \in H^1(B_1, M)$ and $\gamma = u_0|_{\partial B_1} \in H^{3/2}(\partial B_1)$. The point of our uniqueness result is that no boundary regularity assumption is needed. Second, we prove the exponential convergence of the harmonic map heat flow, assuming that energy is small at all times.

1. INTRODUCTION

Let B_1 be the unit open disk in $\mathbb{R}^2$ and M a closed Riemannian manifold. Suppose that $u \in H^1([0, T] \times B_1, M)$ is the weak solution of the initial-boundary value problem for the harmonic map heat flow, given initial data $u_0 \in H^1(B_1, M)$ and boundary data $\gamma = u_0|_{\partial B_1}$.¹ In this note, we study the uniqueness and the rate of convergence of the weak solution u .

Under the additional assumption that $\gamma \in H^{3/2}(\partial B_1, M)$, the initial-boundary value problem for the harmonic map heat flow has been investigated intensively by several mathematicians, such as Chang [2], Rivière [14] and Freire [7], [6] and [8]. Define

$$(1.1) \quad V^T = H^1([0, T] \times B_1, M) \cap L^\infty([0, T], H^1(B_1, M)) \cap L^2([0, T], H^2(B_1, M)).$$

Space V^T plays a crucial role in the papers referred to above. However, without $\gamma \in H^{3/2}(\partial B_1)$, we are not able to show that u with non-increasing energy is in $V^{T'}$ for some $T' > 0$, as Freire did in [6]. Because, otherwise, it would imply that $\gamma \in H^{3/2}(\partial B_1, M)$ by the Sobolev trace theorem. But the image of trace operator on $H^1(B_1)$ is exactly $H^{1/2}(\partial B_1)$, and $H^{3/2}(\partial B_1)$ is a proper subset of $H^{1/2}(\partial B_1)$.

To get around this, we make use of the interior gradient estimate for u and Hardy's inequality. The main difficulty is to deal with the L^2 inner product of $|\nabla u|^2$ and h^2 for $\forall h \in H_0^1(B_1)$, which arises from the non-linear term in the harmonic map heat flow equation. First, using Hélein's existence result of the Coulomb frame, we derive the interior gradient estimate for u with small energy. Second, assuming the energy is non-increasing in time, we conclude that $u(t, \cdot) \rightarrow u_0$ in the $H^1(B_1)$ topology, and thus we can uniformly bound the energy on small disks

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¹The restriction here is the trace operator acting on u_0 .

for short time. Thus, applying the gradient estimate for u restricted on small disks, we bound $|\nabla u|^2(t, x)$ by t^{-1} and $(1 - |x|)^{-2}$ for short time. Finally, by Hardy's inequality, we bound the L^2 inner product of $|\nabla u|^2$ and h^2 for small time $t > 0$.

In Section 5, we prove the existence and uniqueness theorem for weak solutions of the harmonic map heat flow in $\bigcap_{T>0} H^1([0, T] \times B_1, M)$ whose energy is non-increasing in time, given small energy initial data $u_0 \in H^1 \cap C^0(\bar{B}_1)$ and boundary data $\gamma = u_0|_{\partial B_1}$. By the Sobolev trace theorem, such weak solutions are not in V^T in general.

Throughout, we use the subscripts t , x_1 , x_2 and r to denote taking derivatives with respect to t , x_1 , x_2 and r ; $\nabla \cdot$ and $\nabla^2 \cdot$ denote the gradient and the Hessian operator, respectively; “sup” in this note is “esssup” in the usual literature; constants in proofs are not preserved when crossing lemmas, propositions, and theorems.

By the Nash embedding theorem, M can be isometrically embedded in some Euclidean space $(\mathbb{R}^N, \langle \cdot, \cdot \rangle)$. Given $w \in H^1(B_1, M)$, we define the energy functional $E(w) = \frac{1}{2} \int_{B_1} |\nabla w|^2$. The harmonic map heat flow is the negative L^2 gradient flow of the energy functional. Thus, given $u_0 \in H^1(B_1, M)$ and $\gamma = u_0|_{\partial B_1}$, $u \in H^1([0, T] \times B_1, M)^2$ is the weak solution of the initial-boundary value problem for the harmonic map heat flow, if

$$(1.2) \quad \begin{cases} u_t - \Delta u = -A_u(\nabla u, \nabla u) & \text{on } (0, T) \times B_1, \\ u(t, x) = \gamma(x) & \text{for } t \geq 0, x \in \partial B_1, \\ \lim_{t \rightarrow 0+} u(t, \cdot) = u_0 & \text{in } L^2(B_1, M) \text{ topology,} \end{cases}$$

where A is the second fundamental form of M in $\mathbb{R}^N$ at the point u . We recall that

$$(1.3) \quad A_u(\nabla u, \nabla u) \text{ is perpendicular to } M \text{ at } u.$$

We define the weak solution $u \in H^1([0, T] \times B_1, M)$ of the first equation in (1.2) by

$$(1.4) \quad \int_0^T \int_{B_1} \langle u_t, \xi \rangle + \langle \nabla u, \nabla \xi \rangle + \langle A_u(\nabla u, \nabla u), \xi \rangle dx dt = 0,$$

for $\forall \xi \in C_c^\infty((0, T) \times B_1, \mathbb{R}^N)$. In [17], it is shown that this definition is equivalent to that, for a.e. $t \in [0, T]$,

$$(1.5) \quad \int_{\{t\} \times B_1} \langle u_t, \zeta \rangle + \langle \nabla u, \nabla \zeta \rangle + \langle A_u(\nabla u, \nabla u), \zeta \rangle dx = 0, \quad \forall \zeta \in H_0^1 \cap L^\infty(B_1, \mathbb{R}^N).$$

Note that the definition given by equation (1.5) allows us to freeze the time and is more convenient for our proofs of Theorems 1.1 and 1.3.

The main results of this note are twofold. First, we show the uniqueness for weak solutions of (1.2) whose energy is non-increasing.

Theorem 1.1. *If u and v are weak solutions of (1.2) in $H^1([0, T] \times B_1, M)$ satisfying $E(u(t_2, \cdot)) \leq E(u(t_1, \cdot))$, $E(v(t_2, \cdot)) \leq E(v(t_1, \cdot))$ for $t_1 \leq t_2$, and having the same initial data $u_0 \in H^1(B_1, M)$ and boundary value $\gamma = u_0|_{\partial B_1}$, then $u = v$ on $[0, T] \times B_1$.*

²By Theorem 3 on page 287 of [5], if $w \in H^1([0, T] \times B_1)$, then there exists $\tilde{w} \in C^0([0, T], L^2(B_1))$ and $\tilde{w}(t, \cdot) = w(t, \cdot)$ for a.e. t . Thus, in this note, we always choose $\tilde{w}$ representing w in $H^1([0, T] \times B_1)$. In other words, we always assume that functions in $H^1([0, T] \times B_1)$ are actually also in $C^0([0, T], L^2(B_1))$.

Remark 1.2. In [1], Bertsch, Dal Passo and van der Hout proved that there exist initial data $u_0 \in H^1(B_1, S^2)$ and boundary data $\gamma = u_0|_{\partial B_1}$ such that (1.2) has infinitely many weak solutions which do not satisfy the non-increasing energy condition. Thus, Theorem 1.1 appears to be the optimal uniqueness statement for weak solutions of the harmonic map heat flow with time independent boundary data.

Second, we study the rate of convergence of small energy weak solutions of (1.2), and we conclude that

Theorem 1.3. *There exists $\epsilon_0 > 0$, depending only on M , so that if u is the weak solution of (1.2) in $\bigcap_{T>0} H^1([0, T] \times B_1, M)$ satisfying that $E(u(t, \cdot)) < \epsilon_0$ for a.e. t , then there exist $T_0 > 0$, $\alpha_0 > 0$ and $C_0 > 0$ such that, for a.e. $t > T_0$,*

$$(1.6) \quad \|u(t, \cdot) - u_\infty\|_{H^1} \leq C_0 e^{-\alpha_0 t},$$

where u_∞ is some harmonic map from B_1 to M with the same boundary value γ .

Remark 1.4. It follows from Corollary 3.3 in [3] that u_∞ is the unique harmonic map in the class of

$$\{w \in H^1(B_1, M) \mid w|_{\partial B_1} = \gamma \text{ and } E(w) \leq E_1\},$$

where $E_1 > 0$ is a constant depending only on M .

Remark 1.5. Although Theorems 1.1 and 1.3 are stated for the unit open disk in this note, the proofs could be modified to apply to any bounded open set in $\mathbb{R}^2$ and even general two dimensional Riemannian manifolds.

2. INTERIOR GRADIENT ESTIMATE FOR THE HARMONIC MAP HEAT FLOW

In this section, we derive the interior gradient estimate for small energy solutions of the harmonic map heat flow. This is one of the key ingredients in the proofs of Theorems 1.1 and 1.3. First, using Hélein's existence result of the Coulomb frame, we show that $u(t, \cdot) \in H^2(B_{1/2}, M)$ for a.e. t . Next, we follow Struwe's method in [15] to conclude that $u \in L^2([0, \bar{T}], H^2(B_{1/4}, M))$ for $1 < \bar{T} < 2$ and obtain the gradient estimate for u at $(1, 0, 0)$.

The following elementary geometric fact is obtained in [4] and will be used frequently in this note. For self-containedness, we include the proof in Appendix A.

Lemma 2.1 (Lemma A.1 in [4]). *There exists $C > 0$, depending on M , so that if $x, y \in M$, then $|(x - y)^\perp| \leq C|x - y|^2$, where $(x - y)^\perp$ is the normal component to M at y .*

First, we derive the local integral bounds for $|\nabla^2 u|$ and $|\nabla u|$:

Lemma 2.2. *Suppose that $1 < \bar{T} < 2$ and $u \in H^1([0, \bar{T}] \times B_1, M)$ satisfies*

$$(2.1) \quad u_t - \Delta u = -A_u(\nabla u, \nabla u)$$

on $(0, \bar{T}) \times B_1$. Then there exists $\epsilon > 0$, depending only on M , so that if $E(u(t, \cdot)) \leq \epsilon$ for a.e. $t \in [0, \bar{T}]$, then

$$(2.2) \quad \int_0^{\bar{T}} \int_{B_{1/4}} |\nabla^2 u|^2 dx dt \leq 10^5 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot)),$$

$$(2.3) \quad \int_0^{\bar{T}} \int_{B_{1/4}} |\nabla u|^4 dx dt \leq 10^8 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot))^2.$$

Proof. First, note that $|u_t| \in L^2(B_1)$ for a.e. t . Fix such t . Following the proof of Theorem 4.1.1 in [11], there exists $\delta_1 > 0$, depending only on M , so that if $E(u(t, \cdot)) \leq \delta_1$, then there is a finite energy harmonic section (i.e., “Coulomb frame”) $e(t) = (e_1(t), \dots, e_n(t))$ of the bundle of orthonormal frames for $u(t, \cdot)^*(TM)$, and one can construct $\beta(t) \in L^\infty(B_1, GL(n, \mathbb{C}))$ satisfying that $|\beta(t)| \leq \lambda_1$, $|\beta(t)^{-1}| \leq \lambda_1$, and

$$(2.4) \quad \partial_{\bar{z}}(\beta^{-1}\alpha(t)) = \frac{1}{4}\beta(t)^{-1}f,$$

where λ_1 depends only on M and the upper bound of the energy of $u(t, \cdot)$, $z = x_1 + ix_2$, $\alpha = (\langle \partial_z u, e_1 \rangle, \dots, \langle \partial_z u, e_n \rangle)$ and $f = (\langle u_t, e_1 \rangle, \dots, \langle u_t, e_n \rangle)$. Thus, by the elliptic regularity for $\partial_{\bar{z}}$ operator (see the theorem on page 80 of [10]), $\beta^{-1}\alpha(t) \in H^1(B_{3/4})$. It follows from the Sobolev embedding theorem (see Theorem 2 on page 265 of [5]) and $|\beta(t)| \leq \lambda_1$ that $\alpha(t) \in L^p(B_{3/4})$ for $1 < p < \infty$. In particular, $|\nabla u|(t, \cdot) \in L^4(B_{3/4})$. Therefore, by Theorem 8.8 in [9], $u(t, \cdot) \in H^2(B_{1/2})$.

Next, let ϕ be a smooth cut-off function, which is one in $B_{1/4}$, compactly supported in $B_{1/2}$, $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq 8$. Thus, by Lemma 6.7 in Chapter III of [16] and equation (2.1),

$$\begin{aligned} \int_{\{t\} \times B_1} |\Delta u|^2 \phi^2 dx &\leq \int_{\{t\} \times B_1} |u_t|^2 \phi^2 dx + \sup_M |A|^2 \int_{\{t\} \times B_1} |\nabla u|^4 \phi^2 \\ &\leq \int_{\{t\} \times B_1} |u_t|^2 \phi^2 dx + \lambda_2 E(u(t, \cdot)) \left(\int_{\{t\} \times B_1} |\nabla^2 u|^2 \phi^2 dx + \int_{\{t\} \times B_1} |\nabla u|^2 dx \right), \end{aligned}$$

where $\lambda_2 = 512 \sup_M |A|^2$. On the other hand, approximating $u(t, \cdot)$ by smooth functions in $H^2(B_{1/2})$ and integration by parts, give

$$(2.5) \quad \int_{\{t\} \times B_1} |\Delta u|^2 \phi^2 dx \geq \frac{1}{2} \int_{\{t\} \times B_1} |\nabla^2 u|^2 \phi^2 dx - 8 \int_{\{t\} \times B_1} |\nabla u|^2 |\nabla \phi|^2 dx.$$

If $4\lambda_2 E(u(t, \cdot)) \leq 1$, then

$$(2.6) \quad \int_{\{t\} \times B_1} |\nabla^2 u|^2 \phi^2 dx \leq 32 \int_{\{t\} \times B_1} [|u_t|^2 \phi^2 + |\nabla u|^2 (1 + |\nabla \phi|^2)] dx.$$

Thus, integrating over $[0, \bar{T}]$, we have

$$(2.7) \quad \int_0^{\bar{T}} \int_{B_1} |\nabla^2 u|^2 \phi^2 dx dt \leq 32 \int_0^{\bar{T}} \int_{B_1} [|u_t|^2 \phi^2 + |\nabla u|^2 (1 + |\nabla \phi|^2)] dx dt,$$

if $E(u(t, \cdot)) \leq \min\{\delta_1, \lambda_2^{-1}/4\}$ for a.e. t . And it follows from the proof of Lemma 3.4 in [15] (replacing the test function u by $u\phi^2$) that

$$(2.8) \quad \int_0^{\bar{T}} \int_{B_1} |u_t|^2 \phi^2 dx dt \leq 1026 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot)).$$

Therefore,

$$(2.9) \quad \int_0^{\bar{T}} \int_{B_1} |\nabla^2 u|^2 \phi^2 dx dt \leq 10^5 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot)),$$

and it follows from Lemma 6.7 in Chapter III of [16] that

$$\begin{aligned} \int_0^{\bar{T}} \int_{B_1} |\nabla u|^4 \phi^4 dx dt &\leq 8 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot)) \int_0^{\bar{T}} \int_{B_1} (|\nabla \phi|^2 |\nabla u|^2 + |\nabla^2 u|^2 \phi^2) dx dt \\ &\leq 10^8 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot))^2. \end{aligned} \quad \square$$

Now we are ready to prove the interior gradient estimate:

Lemma 2.3. *Under the assumption of Lemma 2.2, there exist $\epsilon_1 \in (0, \epsilon]$ and $C_1 > 0$, depending only on M , so that if $E(u(t, \cdot)) \leq \epsilon_1$ for a.e. $t \in [0, \bar{T}]$, then*

$$(2.10) \quad |\nabla u|^2(1, 0, 0) \leq C_1 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot)).$$

Proof. We will follow the suggestion in the remark after Lemma 3.10 in [15] to obtain the interior gradient estimate for u . Let ϕ be a smooth cut-off function, which is one in $B_{1/8}$, compactly supported in $B_{1/4}$, $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq 16$. Also, we define $D^h w(t, x) = (w(t+h, x) - w(t, x))/h$ for $0 < h < h_0 \ll 1$, where w takes value in $\mathbb{R}$ or $\mathbb{R}^N$. Thus, for $0 < t_1 \leq t_2 \leq \bar{T} - h_0$, using equation (2.1) and integration by parts, we get

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{B_1} \partial_t |D^h u|^2 \phi^2 dx dt + 2 \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 \phi^2 dx dt \\ &\leq 4 \int_{t_1}^{t_2} \int_{B_1} |D^h u| |\phi| |\nabla D^h u| |\nabla \phi| dx dt + 2h^{-1} \int_{t_1}^{t_2} \int_{B_1} \langle A_u(\nabla u, \nabla u), D^h u \rangle \phi^2 dx dt \\ &\quad - 2h^{-1} \int_{t_1}^{t_2} \int_{B_1} \langle A_{u(t+h, x)}(\nabla u, \nabla u), D^h u \rangle \phi^2 dx dt \\ &\leq 4 \int_{t_1}^{t_2} \int_{B_1} |D^h u| |\phi| |\nabla D^h u| |\nabla \phi| dx dt + \lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 |\nabla u|^2(t, x) \phi^2 dx dt \\ &\quad + \lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 |\nabla u|^2(t+h, x) \phi^2 dx dt \\ &\leq \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 \phi^2 dx dt + 4 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 |\nabla \phi|^2 dx dt \\ &\quad + \lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 (|\nabla u|^2(t, x) + |\nabla u|^2(t+h, x)) \phi^2 dx dt, \end{aligned}$$

where $\lambda_1 = C \sup_M |A|$, and we use (1.3) and Lemma 2.1 in the second inequality. Thus, absorbing the first term on the right-hand side in the left, gives

$$\begin{aligned} &\int_{\{t_2\} \times B_1} |D^h u|^2 \phi^2 dx + \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 \phi^2 dx dt \\ (2.11) \quad &\leq \int_{\{t_1\} \times B_1} |D^h u|^2 \phi^2 dx + 4 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 |\nabla \phi|^2 dx dt \\ &\quad + \lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 (|\nabla u|^2(t, x) + |\nabla u|^2(t+h, x)) \phi^2 dx dt. \end{aligned}$$

By Lemma 6.7 in Chapter III of [16], we get

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{B_1} |D^h u|^4 \phi^4 dx dt \\ & \leq 8 \sup_{t_1 \leq t \leq t_2} \int_{\{t\} \times B_1} |D^h u|^2 \phi^2 dx \cdot \int_{t_1}^{t_2} \int_{B_1} (|D^h u|^2 |\nabla \phi|^2 + |\nabla D^h u|^2 \phi^2) dx dt. \end{aligned}$$

If $8\lambda_1 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot)) < 10^{-4}$, then, by Lemma 2.2 and Hölder's inequality, we have

$$\begin{aligned} & \lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 (|\nabla u|^2(t, x) + |\nabla u|^2(t+h, x)) \phi^2 dx dt \\ & \leq 2\lambda_1 \left(\int_{t_1}^{t_2} \int_{B_1} |D^h u|^4 \phi^4 dx dt \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \int_{B_{1/4}} (|\nabla u|^4(t, x) + |\nabla u|^4(t+h, x)) dx dt \right)^{\frac{1}{2}} \\ & \leq \left(\sup_{t_1 \leq t \leq t_2} \int_{\{t\} \times B_1} |D^h u|^2 \phi^2 dx \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \int_{B_1} (|D^h u|^2 |\nabla \phi|^2 + |\nabla D^h u|^2 \phi^2) dx dt \right)^{\frac{1}{2}} \\ & \leq \frac{1}{2} \sup_{t_1 \leq t \leq t_2} \int_{\{t\} \times B_1} |D^h u|^2 \phi^2 dx + \frac{1}{2} \int_{t_1}^{t_2} \int_{B_1} (|D^h u|^2 |\nabla \phi|^2 + |\nabla D^h u|^2 \phi^2) dx dt. \end{aligned}$$

Hence, (2.11) gives

$$\begin{aligned} (2.12) \quad & \int_{\{t_2\} \times B_1} |D^h u|^2 \phi^2 dx - \int_{\{t_1\} \times B_1} |D^h u|^2 \phi^2 dx \\ & \leq \frac{9}{2} \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 |\nabla \phi|^2 dx dt + \frac{1}{2} \sup_{t_1 \leq t \leq t_2} \int_{\{t\} \times B_1} |D^h u|^2 \phi^2 dx. \end{aligned}$$

We conclude from (2.12) that

$$\int_{\{t_2\} \times B_1} |D^h u|^2 \phi^2 dx \leq 2 \inf_{0 \leq t \leq t_2} \int_{\{t\} \times B_1} |D^h u|^2 \phi^2 dx + \lambda_2 \int_0^{t_2} \int_{B_{1/4}} |D^h u|^2 dx dt,$$

where $\lambda_2 > 0$ is a universal constant. Therefore,

$$(2.13) \quad \int_{\{t_2\} \times B_1} |D^h u|^2 \phi^2 dx \leq 2(t_2^{-1} + \lambda_2) \int_0^{t_2} \int_{B_{1/4}} |D^h u|^2 dx dt.$$

Note that u is in $H^1([0, \bar{T}] \times B_1)$ and thus

$$(2.14) \quad \int_0^{\bar{T}-h_0} \int_{B_1} |D^h u - u_t|^2 dx dt \leq \int_0^1 \int_0^{\bar{T}-h_0} \int_{B_1} |u_t(t+sh, x) - u_t(t, x)|^2 dx dt ds.$$

Since elements in L^2 are continuous in the mean,

$$(2.15) \quad \lim_{h \rightarrow 0} \int_0^{\bar{T}-h_0} \int_{B_1} |D^h u - u_t|^2 dx dt = 0.$$

Therefore, letting $h \rightarrow 0$, by the arbitrariness of h_0 , (2.13) and (2.8), we conclude that for a.e. t ,

$$(2.16) \quad \int_{\{t\} \times B_1} |u_t|^2 \phi^2 dx \leq 10^4 (t^{-1} + \lambda_2) \sup_{0 \leq s \leq \bar{T}} E(u(s, \cdot)).$$

By the same argument used to derive (2.6), for $\frac{1}{2} < t < \bar{T}$,

$$(2.17) \quad \int_{\{t\} \times B_1} |\nabla^2 u|^2 \phi^2 dx \leq \lambda_3 \sup_{0 \leq s \leq \bar{T}} E(u(s, \cdot)),$$

where $\lambda_3 > 0$ is a universal constant. Hence, by the Sobolev embedding theorem, for $1 < p < \infty$,

$$(2.18) \quad \int_{\frac{1}{2}}^{\bar{T}} \int_{B_{1/8}} |\nabla u|^p dx dt \leq \lambda_4 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot))^{\frac{p}{2}},$$

where λ_4 depends only on p . Thus, by inserting cut-off functions and the theorem on page 72 of [10], $|u_t|$ and $|\nabla^2 u|$ are in $L^p([1/4, \bar{T}_1] \times B_{1/16})$ for $1 < \bar{T}_1 < \bar{T}$. Furthermore, using the Bochner formula and the Gauss equation, one can derive the evolution equation for $g = |\nabla u|^2$ (see page 128 of [10]), that is,

$$(2.19) \quad g_t - \Delta g = -2|\text{Hess}_u|^2 + 2\langle A_u(u_{x_1}, u_{x_1}), A_u(u_{x_2}, u_{x_2}) \rangle - 2|A_u(u_{x_1}, u_{x_2})|^2.$$

Thus, $g \in W^{1,p}([1/8, \bar{T}_2] \times B_{1/32})$ and $|\nabla^2 g| \in L^p([1/8, \bar{T}_2] \times B_{1/32})$ for $1 < \bar{T}_2 < \bar{T}_1$. Therefore, by the local maximum principle (see Theorem 7.36 in [12]),

$$(2.20) \quad g(1, 0, 0) = |\nabla u|^2(1, 0, 0) \leq \lambda_5 \sup_{0 \leq t \leq T} E(u(t, \cdot)),$$

where $\lambda_5 > 0$ depends only on M , assuming that $E(u(t, \cdot)) < \epsilon_1$ for *a.e.* t and $\epsilon_1 = \min\{\epsilon, 10^{-4}\lambda_1^{-1}/8\}$. $\square$

3. PROOF OF THEOREM 1.1

In [6] and [8], Freire first constructed the optimal tangent frames for each fixed time and rewrote equation (2.1) under these frames. Next, he used the parabolic perturbation argument to show that any weak solution $u \in H^1([0, T] \times B_1, M)$ of (1.2), satisfying $E(u(t, \cdot)) \leq E(u_0)$ for *a.e.* $t \in [0, T]$, is in $V^{T'}$ for some $T' \in (0, T)$, given initial data $u_0 \in H^1(B_1, M)$ and boundary data $\gamma = u_0|_{\partial B_1} \in H^{3/2}(B_1, M)$; see Theorem 1.1 in [6]. Finally, combining with the results of Struwe in [15] and Chang in [2], he concluded the uniqueness for energy non-increasing weak solutions of (1.2) on $[0, T] \times B_1$ by iteration. However, without the assumption on boundary regularity, the proof by Freire does not apply to our case (as explained in the second paragraph of the Introduction). Instead, we make use of the interior gradient estimate and Hardy's inequality to show the uniqueness for energy non-increasing weak solutions of (1.2) in $H^1([0, T] \times B_1, M)$ with initial data $u_0 \in H^1(B_1, M)$ and boundary data $\gamma = u_0|_{\partial B_1}$.

We start with showing Hardy's inequality for the unit open disk. This turns out to be the other key ingredient. Such a Hardy's inequality also holds for general domains in $\mathbb{R}^2$; see [13].

Lemma 3.1. *For $h \in H_0^1(B_1, \mathbb{R})$,*

$$(3.1) \quad \int_{B_1} \frac{h^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx \leq 4 \int_{B_1} |\nabla h|^2 dx.$$

Proof. First, we prove the lemma for $h \in C_c^\infty(B_1, \mathbb{R})$. Rewriting the left-hand side of inequality (3.1) in polar coordinates and using integration by parts, we get

$$\begin{aligned} \int_{B_1} \frac{h^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx &= \int_0^1 \int_0^{2\pi} \frac{h^2 r}{(1-r)^2} d\theta dr \\ &= - \int_0^1 \int_0^{2\pi} \frac{h^2}{1-r} d\theta dr - \int_0^1 \int_0^{2\pi} \frac{2hh_r r}{1-r} d\theta dr \\ &\leq 2 \left(\int_0^1 \int_0^{2\pi} \frac{h^2 r}{(1-r)^2} d\theta dr \right)^{\frac{1}{2}} \left(\int_0^1 \int_0^{2\pi} h_r^2 r d\theta dr \right)^{\frac{1}{2}} \\ &\leq 2 \left(\int_{B_1} \frac{h^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx \right)^{\frac{1}{2}} \left(\int_{B_1} |\nabla h|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, inequality (3.1) follows by absorbing the second term of the product on the right-hand side in the left.

Since $C_c^\infty(B_1)$ is dense in $H_0^1(B_1)$, there exists a sequence of $h_n \in C_c^\infty(B_1, \mathbb{R})$ such that $h_n \rightarrow h$ in $H^1(B_1)$ topology and $h_n \rightarrow h$ a.e. in B_1 . By Fatou's lemma (see Theorem 3 on page 648 of [5]),

$$\begin{aligned} \int_{B_1} \frac{h^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx &\leq \liminf_{n \rightarrow \infty} \int_{B_1} \frac{h_n^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx \\ &\leq \liminf_{n \rightarrow \infty} 4 \int_{B_1} |\nabla h_n|^2 dx = 4 \int_{B_1} |\nabla h|^2 dx. \quad \square \end{aligned}$$

Next, to avoid repeating computation in Section 5, we will prove a general stability lemma below, i.e., Lemma 3.2. Suppose that u and v are weak solutions of (1.2) in $H^1([0, T] \times B_1, M)$ satisfying that the energy is non-increasing and with initial data u_0 and v_0 , respectively. For this moment, u_0 may not be equal to v_0 . And let $\epsilon_2 = \min\{\epsilon_1, C^{-1}C_1^{-1} \sup_M |A|/32\}$.

The key to the proof of Lemma 3.2 is to bound the L^2 inner products $\langle |\nabla u|^2, h^2 \rangle_{L^2}$ and $\langle |\nabla v|^2, h^2 \rangle_{L^2}$ on B_1 , for $\forall h \in H_0^1(B_1)$. Such integrals arise from the non-linear terms $A_u(\nabla u, \nabla u)$ and $A_v(\nabla v, \nabla v)$ in equation (2.1). First, by the energy non-increasing assumption and Lemma 2.3, we can bound $|\nabla u|$ and $|\nabla v|$ for $x_0 \in B_1$ and small time $t_0 > 0$. Namely, since the energy of $u(t, \cdot)$ is non-increasing in time, $u(t, \cdot) \rightarrow u_0$ weakly in $H^1(B_1)$ and strongly in $L^2(B_1)$, as $t \rightarrow 0$. Then,

$$\begin{aligned} &\lim_{t \rightarrow 0} \int_{B_1} |\nabla u(t, x) - \nabla u_0|^2 dx \\ &= \lim_{t \rightarrow 0} \int_{B_1} |\nabla u(t, x)|^2 dx - \int_{B_1} 2\langle \nabla u(t, x), \nabla u_0 \rangle dx + \int_{B_1} |\nabla u_0|^2 dx \\ &\leq 0. \end{aligned}$$

Thus, $u(t, \cdot) \rightarrow u_0$ strongly in $H^1(B_1)$, and by the same argument, $v(t, \cdot) \rightarrow v_0$ strongly in $H^1(B_1)$, as $t \rightarrow 0$. Hence, by the absolute continuity of integration, there exist $R_0 > 0$ and $T' \in (0, \min\{R_0^2, T\}]$ so that for $x_0 \in B_1$ and $t \in [0, T']$,

$$(3.2) \quad \frac{1}{2} \int_{\{t\} \times (B_{R_0}(x_0) \cap B_1)} |\nabla u|^2 dx < \epsilon_2 \quad \text{and} \quad \frac{1}{2} \int_{\{t\} \times (B_{R_0}(x_0) \cap B_1)} |\nabla v|^2 dx < \epsilon_2.$$

Note that equation (2.1) is invariant under the transformation $(t, x) \rightarrow (\lambda^2 t, \lambda x)$ for $\lambda > 0$, and the energy is invariant under conformal transformations of domains

in $\mathbb{R}^2$. Fix $(t_0, x_0) \in (0, T') \times B_1$. Let $\lambda = \min\{\sqrt{t_0}, 1 - |x_0|\}$. Define $u_\lambda(s, y) = u(\lambda^2 s, x_0 + \lambda y)$ and $v_\lambda(s, y) = v(\lambda^2 s, x_0 + \lambda y)$. Thus, u_λ and v_λ satisfy equation (2.1) on $(0, \lambda^{-2} T') \times B_1$, and for $s \in [0, \lambda^{-2} T']$, $E(u_\lambda(s, \cdot)) < \epsilon_2$ and $E(v_\lambda(s, \cdot)) < \epsilon_2$. Hence, by Lemma 2.3, for $(t_0, x_0) \in (0, T') \times B_1$,

$$(3.3) \quad |\nabla u_\lambda|^2(\lambda^{-2} t_0, 0, 0) \leq C_1 \epsilon_2 \quad \text{and} \quad |\nabla v_\lambda|^2(\lambda^{-2} t_0, 0, 0) \leq C_1 \epsilon_2.$$

Therefore,

$$(3.4) \quad |\nabla u|^2(t_0, x_0) \leq C_1 [t_0^{-1} + (1 - |x_0|)^{-2}] \epsilon_2,$$

$$(3.5) \quad |\nabla v|^2(t_0, x_0) \leq C_1 [t_0^{-1} + (1 - |x_0|)^{-2}] \epsilon_2.$$

Then, combining inequalities (3.4) and (3.5) with Lemma 3.1, we can bound the L^2 inner products $\langle |\nabla u|^2, h^2 \rangle_{L^2}$ and $\langle |\nabla v|^2, h^2 \rangle_{L^2}$ on B_1 , for $\forall h \in H_0^1(B_1)$ and $t \in (0, T')$.

Lemma 3.2. *There exists $C_2 > 0$, depending only on M , so that*

$$(3.6) \quad \int_0^{T'} \int_{B_1} |\nabla u - \nabla v|^2 t^{-\frac{1}{2}} dx dt + \frac{1}{2\sqrt{T'}} \int_{\{T'\} \times B_1} |u - v|^2 dx \leq N,$$

where

$$(3.7) \quad \begin{aligned} N = & \left(\frac{1}{\sqrt{T'}} + 2\sqrt{T'} \right) \int_{B_1} (|w_0|^2 + |\nabla w_0|^2) dx \\ & + 8\sqrt{2T'(E(u_0) + E(v_0))} \left(\int_{B_1} |\nabla w_0|^2 dx \right)^{\frac{1}{2}} \\ & + 4C_2 \int_0^{T'} \int_{B_1} (|\nabla u|^2 + |\nabla v|^2)(|w_0|^2 + |w_0|) t^{-\frac{1}{2}} dx dt. \end{aligned}$$

Proof. Define $w = u - v$. It is clear that

$$(3.8) \quad \begin{aligned} & \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt \\ &= \int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w \rangle t^{-\frac{1}{2}} dx dt - \int_0^{T'} \int_{B_1} \langle \nabla v, \nabla w \rangle t^{-\frac{1}{2}} dx dt. \end{aligned}$$

We will estimate the first term of (3.8) and the second term can be estimated similarly. First, by footnote 2, $w - w_0 \in C^0([0, T], L^2(B_1))$ and the map $t \rightarrow \|w(t, \cdot) - w_0\|_{L^2(B_1)}^2$ is absolutely continuous, with

$$\begin{aligned} \frac{d}{dt} \|w(t, \cdot) - w_0\|_{L^2(B_1)}^2 &= 2 \int_{\{t\} \times B_1} \langle w_t, w - w_0 \rangle dx \\ &\leq 2 \left(\int_{\{t\} \times B_1} |w_t|^2 dx \right)^{\frac{1}{2}} \left(\int_{\{t\} \times B_1} |w - w_0|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

for *a.e.* $t \in [0, T]$. Thus, for $\forall t_0 \geq 0$, integrating over $[0, t_0]$ and by Hölder's inequality, we conclude that

$$\begin{aligned} \|w(t_0, \cdot) - w_0\|_{L^2(B_1)} &\leq \int_0^{t_0} \left(\int_{\{t\} \times B_1} |w_t|^2 dx \right)^{\frac{1}{2}} dt \\ &\leq \sqrt{t_0} \left(\int_0^{t_0} \int_{B_1} |w_t|^2 dx dt \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$(3.9) \quad \int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt \leq 2\sqrt{T'} \int_0^{T'} \int_{B_1} |w_t|^2 dx dt < +\infty,$$

$$(3.10) \quad \lim_{t \rightarrow 0} t^{-\frac{1}{2}} \int_{\{t\} \times B_1} |w - w_0|^2 dx = 0.$$

Next, (1.5) gives,

$$\begin{aligned} &\int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w \rangle t^{-\frac{1}{2}} dx dt \\ &= \int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w - \nabla w_0 \rangle t^{-\frac{1}{2}} dx dt + \int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w_0 \rangle t^{-\frac{1}{2}} dx dt \\ &= \int_0^{T'} \int_{B_1} \langle -u_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + \int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w_0 \rangle t^{-\frac{1}{2}} dx dt \\ &\quad - \int_0^{T'} \int_{B_1} \langle A_u(\nabla u, \nabla u), w - w_0 \rangle t^{-\frac{1}{2}} dx dt \\ &\leq \int_0^{T'} \int_{B_1} \langle -u_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + 2\sqrt{2T'E(u_0)} \left(\int_{B_1} |\nabla w_0|^2 \right)^{\frac{1}{2}} \\ &\quad - \int_0^{T'} \int_{B_1} \langle A_u(\nabla u, \nabla u), w - w_0 \rangle t^{-\frac{1}{2}} dx dt. \end{aligned}$$

We will bound the third term from above. In the following calculation, energy non-increasing condition and (3.9) guarantee that each quantity below is finite. Since $A_u(\nabla u, \nabla u)$ is perpendicular to M at u and $w = u - v$, we can apply Lemma 2.1 to the L^2 inner product of $A_u(\nabla u, \nabla u)$ and w on B_1 . Note that $w - w_0 = (u - u_0) - (v - v_0) \in H_0^1(B_1)$ for *a.e.* t fixed. Thus, we can apply the interior gradient estimate (3.4) and Lemma 3.1 to the L^2 inner product of $|\nabla u|^2$ and $|w - w_0|^2$ on B_1 . Hence,

$$\begin{aligned} &-\int_0^{T'} \int_{B_1} \langle A_u(\nabla u, \nabla u), w - w_0 \rangle t^{-\frac{1}{2}} dx dt \\ &\leq \lambda_1 \int_0^{T'} \int_{B_1} |\nabla u|^2 |w|^2 t^{-\frac{1}{2}} dx dt + \lambda_2 \int_0^{T'} \int_{B_1} |\nabla u|^2 |w_0| t^{-\frac{1}{2}} dx dt \\ &\leq \lambda_1 \int_0^{T'} \int_{B_1} |\nabla u|^2 |w - w_0|^2 t^{-\frac{1}{2}} dx dt + \lambda_2 \int_0^{T'} \int_{B_1} |\nabla u|^2 (|w_0|^2 + |w_0|) t^{-\frac{1}{2}} dx dt \end{aligned}$$

$$\begin{aligned}
&\leq \lambda_1 C_1 \epsilon_2 \int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt + 4\lambda_1 C_1 \epsilon_2 \int_0^{T'} \int_{B_1} |\nabla w - \nabla w_0|^2 t^{-\frac{1}{2}} dx dt \\
&\quad + \lambda_2 \int_0^{T'} \int_{B_1} |\nabla u|^2 (|w_0|^2 + |w_0|) t^{-\frac{1}{2}} dx dt,
\end{aligned}$$

where $\lambda_1 = C \sup_M |A|$ and $\lambda_2 > 0$ (changing from line to line in the computation above) depends only on M . The previous two inequalities and $32\lambda_1 C_1 \epsilon_2 < 1$ give

$$\begin{aligned}
(3.11) \quad &\int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w \rangle t^{-\frac{1}{2}} dx dt \\
&\leq \int_0^{T'} \int_{B_1} \langle -u_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + \frac{1}{32} \int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt \\
&\quad + \frac{1}{4} \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt + N_1,
\end{aligned}$$

where

$$\begin{aligned}
N_1 = &\lambda_2 \int_0^{T'} \int_{B_1} (|\nabla u|^2 + |\nabla v|^2) (|w_0|^2 + |w_0|) t^{-\frac{1}{2}} dx dt \\
&+ 2\sqrt{2T'(E(u_0) + E(v_0))} \left(\int_{B_1} |\nabla w_0|^2 \right)^{\frac{1}{2}} + \frac{\sqrt{T'}}{2} \int_{B_1} |\nabla w_0|^2 dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(3.12) \quad &-\int_0^{T'} \int_{B_1} \langle \nabla v, \nabla w \rangle t^{-\frac{1}{2}} dx dt \\
&\leq \int_0^{T'} \int_{B_1} \langle v_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + \frac{1}{32} \int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt \\
&\quad + \frac{1}{4} \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt + N_1.
\end{aligned}$$

Thus, combining (3.11) and (3.12), we get

$$\begin{aligned}
&\int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt \\
&\leq -\int_0^{T'} \int_{B_1} \langle w_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + \frac{1}{16} \int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt \\
&\quad + \frac{1}{2} \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt + 2N_1 \\
&\leq -\frac{1}{2\sqrt{T'}} \int_{\{T'\} \times B_1} |w - w_0|^2 dx + \frac{1}{2} \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt + 2N_1,
\end{aligned}$$

where the last inequality follows from integration by parts and (3.10). Therefore,

$$(3.13) \quad \begin{aligned} & \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt + \frac{1}{2\sqrt{T'}} \int_{\{T'\} \times B_1} |w|^2 dx \\ & \leq \frac{1}{\sqrt{T'}} \int_{B_1} |w_0|^2 dx + 4N_1, \end{aligned}$$

and $C_2 = \lambda_2$ in the lemma. $\square$

Now, under the condition of Theorem 1.1, that is, $u_0 = v_0$, we have $N \equiv 0$ and thus $u(t, \cdot) = v(t, \cdot)$ in $L^2(B_1)$ for each $t \in [0, T']$. Therefore, it follows from an open and closed argument that $u = v$ a.e. on $[0, T] \times B_1$.

4. PROOF OF THEOREM 1.3

Let u be the weak solution of (1.2) in $\bigcap_{T>0} H^1([0, T] \times B_1, M)$ with initial data $u_0 \in H^1(B_1, M)$ and boundary data $\gamma = u_0|_{\partial B_1}$. First, define $E = \sup_{t \geq 0} E(u(t, \cdot))$. If $E < \epsilon_1$, then, by a similar argument used to obtain (3.4), we get

$$(4.1) \quad |\nabla u|^2(t_0, x_0) \leq C_1 E \max\{t_0^{-1}, (1 - |x_0|)^{-2}\} \leq C_1 E [t_0^{-1} + (1 - |x_0|)^{-2}],$$

for $t_0 > 0$ and $x_0 \in B_1$.

Second, we derive two estimates of the kinetic energy: one holds for a.e. $t_0 > 0$; the other holds for t_0 large enough.

Lemma 4.1. *There exists $\epsilon_3 \in (0, \epsilon_1)$, depending only on M , so that if $E \leq \epsilon_3$, then for a.e. $t_0 > 0$,*

$$(4.2) \quad \int_{\{t_0\} \times B_1} |u_t|^2 dx \leq \frac{4}{t_0} \int_0^{t_0} \int_{B_1} |u_t|^2 dx dt,$$

and there exist $T_1 > 0$, $\alpha_1 > 0$ and $C_3 > 0$ such that for a.e. $t_0 \geq T_1$,

$$(4.3) \quad \int_{\{t_0\} \times B_1} |u_t|^2 dx \leq C_3 \exp[-\alpha_1(t - T_1)].$$

Proof. Assume that $E < \epsilon_1$. Let $0 < h < h_0 \ll 1$. We define the difference quotient $D^h u = (u(t + h, x) - u(t, x)) / h$. Note that

$$(4.4) \quad D^h u(t, \cdot) \in H_0^1 \cap L^\infty(B_1) \text{ for a.e. } t \text{ fixed.}$$

Thus, by equation (2.1), we get

$$\begin{aligned} & \frac{d}{dt} \int_{\{t\} \times B_1} |D^h u|^2 dx \\ & = -2 \int_{\{t\} \times B_1} |\nabla D^h u|^2 dx + 2h^{-1} \int_{\{t\} \times B_1} \langle A_u(\nabla u, \nabla u), D^h u \rangle dx \\ & \quad - 2h^{-1} \int_{\{t\} \times B_1} \langle A_{u(t+h, x)}(\nabla u, \nabla u), D^h u \rangle dx \\ & \leq -2 \int_{\{t\} \times B_1} |\nabla D^h u|^2 dx + \lambda_1 \int_{\{t\} \times B_1} |D^h u|^2 (|\nabla u|^2(t, x) + |\nabla u|^2(t + h, x)) dx, \end{aligned}$$

where $\lambda_1 = C \sup_M |A|$, and we use (1.3) and Lemma 2.1 in the last inequality. For $t > 0$, by (4.1), (4.4), and Lemma 3.1, we have

$$\begin{aligned} & \int_{\{t\} \times B_1} |D^h u|^2 |\nabla u|^2 dx \\ & \leq C_1 E \int_{\{t\} \times B_1} |D^h u|^2 [t^{-1} + (1 - |x|^2)^{-2}] dx \\ & \leq 4C_1 E \int_{\{t\} \times B_1} |\nabla D^h u|^2 dx + C_1 E t^{-1} \int_{\{t\} \times B_1} |D^h u|^2 dx. \end{aligned}$$

If $8C_1 \lambda_1 E \leq 1$, then

$$\begin{aligned} & \frac{d}{dt} \int_{\{t\} \times B_1} |D^h u|^2 dx \\ (4.5) \quad & \leq - \int_{\{t\} \times B_1} |\nabla D^h u|^2 dx + t^{-1} \int_{\{t\} \times B_1} |D^h u|^2 dx \\ & \leq -(C_s^{-1} - t^{-1}) \cdot \int_{\{t\} \times B_1} |D^h u|^2 dx, \end{aligned}$$

where we apply the Sobolev inequality to $D^h u(t, \cdot) \in H_0^1(B_1)$ in the first inequality and $C_s > 0$ is the Sobolev constant of B_1 .

Thus, integrating over $[t_0/2, t_0]$, (4.5) gives

$$(4.6) \quad \int_{\{t_0\} \times B_1} |D^h u|^2 dx \leq \frac{4}{t_0} \int_0^{t_0} \int_{B_1} |D^h u|^2 dx dt.$$

On the other hand, let $I = [2C_s, 2C_s + 1]$, and it is obvious that

$$(4.7) \quad \inf_{t \in I} \int_{\{t\} \times B_1} |D^h u|^2 dx \leq \int_I \int_{B_1} |D^h u|^2 dx dt.$$

Thus, for $t > T_1 = 2C_s + 2$, (4.5) also implies that

$$(4.8) \quad \int_{\{t\} \times B_1} |D^h u|^2 dx \leq \int_I \int_{B_1} |D^h u|^2 dx dt \cdot \exp[-C_s^{-1}(t - T_1)/2].$$

Letting $h \rightarrow 0$ and by (2.15), Lemma 4.1 follows with $\epsilon_3 = \min\{\epsilon_1, C_1^{-1} \lambda_1^{-1}/8\}$, $C_3 = \int_I \int_{B_1} |u_t|^2 dx dt$ and $\alpha_1 = C_s^{-1}/2$. $\square$

Finally, assume that $E < \epsilon_3$ and $32C_1 E \sup_M |A| < 1$. In the calculation below, we first apply Hölder's inequality to the L^2 inner product of u_t and $u(t_2, \cdot) - u(t_1, \cdot)$ on B_1 . Meanwhile, we use (1.3), Lemma 2.1, and Lemma 3.1 to bound the L^2 inner product of $A_{u(t_2, x)}(\nabla u, \nabla u)$ and $u(t_2, \cdot) - u(t_1, \cdot)$ on B_1 . Next, we deduce the last inequality from Cauchy's inequality and the assumption on the upper bound of E .

That is, for a.e. $T_1 < t_1 < t_2$,

$$\begin{aligned}
& \int_{B_1} \langle \nabla u(t_2, x), \nabla u(t_2, x) - \nabla u(t_1, x) \rangle dx \\
&= \int_{B_1} \langle -u_t(t_2, x) - A_{u(t_2, x)}(\nabla u, \nabla u), u(t_2, x) - u(t_1, x) \rangle dx \\
&\leq \sqrt{C_s} \left(\int_{B_1} |u_t(t_2, x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx \right)^{\frac{1}{2}} \\
&\quad + 4CC_1 E \sup_M |A| \cdot \int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx \\
&\leq 2C_s \int_{B_1} |u_t(t_2, x)|^2 dx + \frac{1}{4} \int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_{B_1} \langle \nabla u(t_1, x), \nabla u(t_1, x) - \nabla u(t_2, x) \rangle dx \\
&\leq 2C_s \int_{B_1} |u_t(t_1, x)|^2 dx + \frac{1}{4} \int_{B_1} |\nabla u(t_1, x) - \nabla u(t_2, x)|^2 dx.
\end{aligned}$$

Summing the two inequalities above, we get

$$(4.9) \quad \int_{B_1} |\nabla u(t_1, x) - \nabla u(t_2, x)|^2 dx \leq 4C_s \int_{B_1} (|u_t(t_1, x)|^2 + |u_t(t_2, x)|^2) dx.$$

Therefore, Theorem 1.3 follows immediately from Lemma 4.1 by choosing $\epsilon_0 = \min\{\epsilon_3, C^{-1}C_1^{-1} \inf_M |A|^{-1}/32\}$.

5. EXAMPLE OF HARMONIC MAP HEAT FLOW NOT IN V^T

In this section, we construct the unique weak solution $u \in \bigcap_{T>0} H^1([0, T] \times B_1, M)$ of (1.2) starting with small energy initial data $u_0 \in H^1 \cap C^0(\bar{B}_1, M)$ and boundary data $\gamma = u_0|_{\partial B_1}$. In general, the weak solution u is not in V^T .

Proposition 5.1. *There exists $\epsilon_4 > 0$, depending only on M , so that given $u_0 \in H^1 \cap C^0(\bar{B}_1, M)$ with $E(u_0) < \epsilon_4$, there exists a unique weak solution in $\bigcap_{T>0} H^1([0, T] \times B_1, M)$ of (1.2) whose energy is non-increasing. Moreover, for $0 \leq t_1 < t_2$,*

$$(5.1) \quad \frac{1}{7} \int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx \leq \int_{B_1} |\nabla u(t_1, x)|^2 dx - \int_{B_1} |\nabla u(t_2, x)|^2 dx.$$

Remark 5.2. Recently, Colding and Minicozzi showed that the H^1 distance between a harmonic map and an H^1 map with the same boundary value can be controlled by their gap in energy, assuming energy is small; see Theorem 3.1 in [3]. This is a key ingredient in the proof of finite time extinction of the Ricci flow. Combined with Theorem 1.3 and Remark 1.4, our estimate (5.1) can be viewed as a parabolic version of their theorem.

Proof. First, we may approximate u_0 by a sequence of maps $u_{m0} \in C^\infty(\bar{B}_1, M)$ in $H^1 \cap C^0$ topology.³ By Theorem 1.1 in [2], there exists $\delta_1 \in (0, \epsilon_2)$, depending

³Let $v_{m0} \in C^\infty(\bar{B}_1, \mathbb{R}^N)$ be the global approximations of u_0 , constructed in Theorem 3 on page 252 of [5]. Then u_{m0} could be the nearest point projection (onto M) of v_{m0} .

only on M , so that if $E(u_{m0}) < \delta_1$, then the weak solution $u_m \in \bigcap_{T>0} W_p^{1,2} \cap C^{1+(\mu/2), 2+\mu}((0, T) \times \bar{B}_1, M)^4$ of (1.2) exists, $\forall 0 < \mu < 1, p > 4/(1 - \mu)$, and for $0 \leq t_1 < t_2$,

$$(5.2) \quad \int_{t_1}^{t_2} \int_{B_1} |\partial_t u_m|^2 dx dt = E(u_m(t_1, \cdot)) - E(u_m(t_2, \cdot)).$$

If $E(u_0) < \delta_1/2$, then, by Lemma 3.2 and a diagonalization argument, there exists a subsequence (relabelled) of u_m satisfying that for $\forall T > 0$, $\partial_t u_m \rightharpoonup \partial_t u$ weakly in $L^2([0, T] \times B_1)$, $u_m \rightarrow u$ and $\nabla u_m \rightarrow \nabla u$ strongly in $L^2([0, T] \times B_1)$. Note that the boundary data $\gamma_m \rightarrow \gamma$ in $H^{1/2} \cap C^0(\partial B_1, M)$. Therefore, $u \in \bigcap_{T>0} H^1([0, T] \times B_1)$ is the weak solution of (1.2) with initial data u_0 and boundary data $\gamma = u_0|_{\partial B_1}$. Moreover, $E(u(t, \cdot)) \leq E(u_0)$ for *a.e.* t and there exists a zero measure set $I_1 \subseteq (0, \infty)$ so that if $t_1, t_2 \in I_1^c$ and $t_1 < t_2$, then

$$(5.3) \quad \int_{t_1}^{t_2} \int_{B_1} |u_t|^2 dx dt \leq E(u(t_1, \cdot)) - E(u(t_2, \cdot)).$$

Second, we will show that inequality (5.3) is actually equality for *a.e.* $0 < t_1 < t_2$. Let $0 < h < h_0 \ll 1$. Define $D^h u(t, x) = (u(t + h, x) - u(t, x))/h$. Thus, for $0 < t_1 < t_2$,

$$\begin{aligned} \int_{t_1}^{t_2} \int_{B_1} |u_t|^2 dx dt &= \lim_{h \rightarrow 0} \int_{t_1}^{t_2} \int_{B_1} \langle u_t, D^h u \rangle dx dt \\ &= \lim_{h \rightarrow 0} \int_{t_1}^{t_2} \int_{B_1} -\langle \nabla u, \nabla D^h u \rangle dx dt - \int_{t_1}^{t_2} \int_{B_1} \langle A_u(\nabla u, \nabla u), D^h u \rangle dx dt. \end{aligned}$$

We will bound the second term. In the following calculation, we use (1.3) and Lemma 2.1 in the first inequality, apply the gradient estimate (4.1) in the second inequality, and use (4.4) and Lemma 3.1 in the last inequality. Thus,

$$\begin{aligned} & \left| \int_{t_1}^{t_2} \int_{B_1} \langle A_u(\nabla u, \nabla u), D^h u \rangle dx dt \right| \\ & \leq hC \sup_M |A| \int_{t_1}^{t_2} \int_{B_1} |\nabla u|^2 |D^h u|^2 dx dt \\ & \leq hCC_1 E(u_0) \sup_M |A| \int_{t_1}^{t_2} \int_{B_1} [t^{-1} + (1 - |x|)^{-2}] |D^h u|^2 dx dt \\ & \leq t_1^{-1} hCC_1 E(u_0) \sup_M |A| \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 dx dt \\ & \quad + 4hCC_1 E(u_0) \sup_M |A| \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 dx dt. \end{aligned}$$

⁴ $w \in W_p^{1,2}((0, T) \times \bar{B}_1, M)$ means that u , $|\nabla u|$, $|\nabla^2 u|$ and $|u_t|$ are in $L^p((0, T) \times \bar{B}_1)$, and $u \in M$ for *a.e.* $(t, x) \in (0, T) \times \bar{B}_1$.

For the first term,

$$\begin{aligned}
 & \int_{t_1}^{t_2} \int_{B_1} -\langle \nabla u, \nabla D^h u \rangle dx dt \\
 &= \frac{1}{2h} \int_{t_1}^{t_2} \int_{B_1} (|\nabla u(t, x)|^2 - |\nabla u(t+h, x)|^2) dx dt + \frac{h}{2} \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 dx dt \\
 &= \frac{1}{2h} \int_{t_1}^{t_1+h} \int_{B_1} |\nabla u|^2 dx dt - \frac{1}{2h} \int_{t_2}^{t_2+h} \int_{B_1} |\nabla u|^2 dx dt + \frac{h}{2} \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 dx dt.
 \end{aligned}$$

If $E(u_0) < C^{-1}C_1^{-1} \inf_M |A|^{-1}/8$, then

$$\begin{aligned}
 & \int_{t_1}^{t_2} \int_{B_1} |u_t|^2 dx dt \\
 & \geq \lim_{h \rightarrow 0} \frac{1}{2h} \int_{t_1}^{t_1+h} \int_{B_1} |\nabla u|^2 dx dt \\
 & \quad - \frac{1}{2h} \int_{t_2}^{t_2+h} \int_{B_1} |\nabla u|^2 dx dt - \frac{h}{8t_1} \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 dx dt \\
 & \geq \lim_{h \rightarrow 0} \frac{1}{2h} \int_{t_1}^{t_1+h} \int_{B_1} |\nabla u|^2 dx dt - \frac{1}{2h} \int_{t_2}^{t_2+h} \int_{B_1} |\nabla u|^2 dx dt.
 \end{aligned}$$

Define $f(t) = \int_{B_1} |\nabla u(t, x)|^2 dx$ and $F(t) = \int_0^t f(s) ds$. Since $f \in L^1([0, T])$ for $\forall T \in (0, \infty)$, $F'(t) = f(t)$ for *a.e.* $t \in [0, T]$. Hence, there exists a zero measure set $I_2 \subseteq (0, \infty)$ so that if $t \in I_2^c$, then

$$(5.4) \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} \int_{B_1} |\nabla u|^2 dx dt = \int_{B_1} |\nabla u(t, x)|^2 dx.$$

Therefore, if $t_1, t_2 \in I_2^c$ and $0 < t_1 < t_2$, then

$$(5.5) \quad \int_{t_1}^{t_2} \int_{B_1} |u_t|^2 dx dt \geq E(u(t_1, \cdot)) - E(u(t_2, \cdot)).$$

Combining with inequality (5.3), we get

$$(5.6) \quad \int_{t_1}^{t_2} \int_{B_1} |u_t|^2 dx dt = E(u(t_1, \cdot)) - E(u(t_2, \cdot)),$$

if $t_1, t_2 \in (I_1 \cup I_2)^c$ and $0 < t_1 < t_2$.

Third, assume that $E(u_0) < \min\{\epsilon_3/2, \delta_1/2, C^{-1}C_1^{-1} \inf_M |A|^{-1}/32\}$. Since $u_{m0} \rightarrow u_0$ in $H^1(B_1, M)$ topology, we may also assume that $E(u_{m0}) \leq 2E(u_0)$.

Thus, for $0 \leq t_1 < t_2$, using (4.1) and Lemmas 2.1, 3.1 and 4.1,⁵ we estimate

$$\begin{aligned}
& \int_{B_1} \langle \nabla u_m(t_2, x), \nabla u_m(t_2, x) - \nabla u_m(t_1, x) \rangle dx \\
&= \int_{B_1} \langle -\partial_t u_m(t_2, x) - A_{u_m(t_2, x)}(\nabla u_m, \nabla u_m), u_m(t_2, x) - u_m(t_1, x) \rangle dx \\
&\leq \left(\int_{B_1} |\partial_t u_m(t_2, x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_1} |u_m(t_1, x) - u_m(t_2, x)|^2 dx \right)^{\frac{1}{2}} \\
&\quad + C \sup_M |A| \int_{B_1} |\nabla u_m(t_2, x)|^2 |u_m(t_1, x) - u_m(t_2, x)|^2 dx \\
&\leq \sqrt{t_2 - t_1} \cdot \left(\int_{B_1} |\partial_t u_m(t_2, x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \int_{B_1} |\partial_t u_m|^2 dx dt \right)^{\frac{1}{2}} \\
&\quad + CC_1 E(u_m(t_1, \cdot)) \sup_M |A| \int_{B_1} (t_2 - t_1)^{-1} |u_m(t_1, x) - u_m(t_2, x)|^2 dx \\
&\quad + CC_1 E(u_m(t_1, \cdot)) \sup_M |A| \int_{B_1} \frac{|u_m(t_1, x) - u_m(t_2, x)|^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx \\
&\leq \frac{33}{16} \int_{t_1}^{t_2} \int_{B_1} |\partial_t u_m|^2 dx dt + \frac{1}{4} \int_{B_1} |\nabla u_m(t_2, x) - \nabla u_m(t_1, x)|^2 dx,
\end{aligned}$$

where we use the fact that the energy of u_m is non-increasing in the last inequality. Thus,

$$\begin{aligned}
& \int_{B_1} |\nabla u_m(t_1, x)|^2 dx - \int_{B_1} |\nabla u_m(t_2, x)|^2 dx \\
&= \int_{B_1} |\nabla u_m(t_1, x) - \nabla u_m(t_2, x)|^2 dx \\
&\quad + 2 \int_{B_1} \langle \nabla u_m(t_2, x), \nabla u_m(t_1, x) - \nabla u_m(t_2, x) \rangle dx \\
&\geq \frac{1}{2} \int_{B_1} |\nabla u_m(t_1, x) - \nabla u_m(t_2, x)|^2 dx - 5 \int_{t_1}^{t_2} \int_{B_1} |\partial_t u_m|^2 dx dt.
\end{aligned}$$

Hence, it follows from (5.2) that

$$(5.7) \quad \frac{1}{7} \int_{B_1} |\nabla u_m(t_1, x) - \nabla u_m(t_2, x)|^2 dx \leq \int_{B_1} |\nabla u_m(t_1, x)|^2 dx - \int_{B_1} |\nabla u_m(t_2, x)|^2 dx.$$

By Lemma 3.2, there exists a zero measure set $I_3 \subseteq (0, \infty)$ so that if $t \in I_3^c$, then there exists a subsequence (relabelled) of u_m such that $u_m(t, \cdot) \rightarrow u(t, \cdot)$ in $H^1(B_1, M)$ topology. Hence, if $0 < t_1 < t_2$ and $t_1, t_2 \in (I_1 \cup I_2 \cup I_3)^c$, then

$$(5.8) \quad \frac{1}{7} \int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx \leq \int_{B_1} |\nabla u(t_1, x)|^2 dx - \int_{B_1} |\nabla u(t_2, x)|^2 dx,$$

$$(5.9) \quad \frac{1}{2} \int_{B_1} |\nabla u(t_1, x)|^2 dx - \frac{1}{2} \int_{B_1} |\nabla u(t_2, x)|^2 dx = \int_{t_1}^{t_2} \int_{B_1} |u_t|^2 dx dt.$$

⁵Note that Lemma 4.1 holds for every $t_0 > 0$ for u_m , since $u_m \in C^{1+(\mu/2), 2+\mu}((0, \infty) \times \bar{B}_1, M)$ and $\|u_t(t, \cdot)\|_{L^2(B_1)}$ is continuous in t .

We can modify the definition of u on $(I_1 \cup I_2 \cup I_3) \times B_1$ by taking limits. Therefore, the modified map solves (1.2) satisfying that the energy is non-increasing and the uniqueness for weak solutions of the harmonic map heat flow follows from Theorem 1.1. $\square$

APPENDIX A

Proposition A.1. *There exists $\epsilon_5 > 0$, depending on M , so that if $|x - y| < \epsilon_5$, then $\text{dist}_M(x, y) < 2|x - y|$, where $\text{dist}_M(x, y)$ is the intrinsic distance between x and y on M .*

Proof. If not, then there exists a sequence of $(x_j, y_j) \in M \times M$ such that $|x_j - y_j| \rightarrow 0$ but $\text{dist}_M(x_j, y_j) \geq 2|x_j - y_j|$. Since M is compact, there exist $x_0 \in M$ and a subsequence (relabelled) of (x_j, y_j) satisfying that $x_j \rightarrow x_0$ and $y_j \rightarrow x_0$. There exists $0 < \delta_1 < \sup_M |A|/4$ such that the geodesic ball $B_{\delta_1}^M(x_0)$ centered at x_0 with radius δ_1 is strictly geodesically convex. If j is sufficiently large, then x_j and y_j are in $B_{\delta_1}^M(x_0)$. Let l_j be the geodesic distance between x_j and y_j , and $\gamma_j : [0, l_j] \rightarrow B_{\delta_1}^M(x_0)$ be the unit speed minimizing geodesic joining x_j and y_j . Thus,

$$\begin{aligned} |y_j - x_j|^2 &= \int_0^{l_j} 2\langle \gamma_j(s) - x_j, \gamma_j'(s) \rangle ds \\ &= \int_0^{l_j} \int_0^s (2|\gamma_j'(\tau)|^2 + 2\langle \gamma_j(\tau) - x_j, \gamma_j''(\tau) \rangle) d\tau ds \\ &\geq \int_0^{l_j} \int_0^s 2\left(1 - 2\delta_1 \sup_M |A|\right) d\tau ds \\ &\geq \frac{l_j^2}{2}. \end{aligned}$$

Therefore, $\text{dist}_M(x_j, y_j) \leq \sqrt{2}|x_j - y_j|$, and this is a contradiction. $\square$

Proof of Lemma 2.1. If $|x - y| \geq \epsilon_5$, then $|(x - y)^\perp|/|x - y|^2 \leq \epsilon_5^{-1}$. Otherwise, let $\gamma : [0, l] \rightarrow M$ be the minimizing geodesic joining y to x with length $l \leq 2|x - y|$. Then

$$\begin{aligned} |(x - y)^\perp| &= \int_0^l \langle \gamma'(s), V \rangle ds = \int_0^l \int_0^s \langle \gamma''(\tau), V \rangle d\tau ds \\ &\leq \sup_M |A| \cdot \frac{l^2}{2} \leq 2 \sup_M |A| \cdot |x - y|^2, \end{aligned}$$

where $V = (x - y)^\perp/|x - y|$. Therefore, Lemma 2.1 follows immediately with $C = \max\{\epsilon_5^{-1}, 2 \sup_M |A|\}$. $\square$

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REFERENCES

- [1] M. Bertsch, R. Dal Passo, and R. van der Hout, *Nonuniqueness for the heat flow of harmonic maps on the disk*, Arch. Ration. Mech. Anal. 161 (2002), no. 2, 93–112. MR1870959 (2003a:35093)
- [2] K.C. Chang, *Heat flow and boundary value problem for harmonic maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1989), no. 5, 363–395. MR1030856 (90i:58037)
- [3] T.H. Colding and W.P. Minicozzi II, *Width and finite extinction time of Ricci flow*, Geom. Topol. 12 (2008), no. 5, 2537–2586. MR2460871 (2009k:53166)
- [4] T.H. Colding and W.P. Minicozzi II, *Width and mean curvature flow*, Geom. Topol. 12 (2008), no. 5, 2517–2535. MR2460870 (2009k:53165)
- [5] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics 19, American Mathematical Society, Providence, RI, 1998 (reprinted with corrections 2002). MR1625845 (99e:35001)
- [6] A. Freire, *Uniqueness for the harmonic map flow from surfaces to general targets*, Comment. Math. Helv. 70 (1995), no. 2, 310–338. MR1324632 (96f:58045)
- [7] A. Freire, *Uniqueness for the harmonic map flow in two dimensions*, Calc. Var. Partial Differential Equations 3 (1995), no. 1, 95–105. MR1384838 (97d:58058)
- [8] A. Freire, *Correction to: “Uniqueness for the harmonic map flow from surfaces to general targets”* [Comment. Math. Helv. 70 (1995), no. 2, 310–338; MR1324632 (96f:58045)], Comment. Math. Helv. 71 (1996), no. 2, 330–337. MR1396678 (97c:58032)
- [9] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001. MR1814364 (2001k:35004)
- [10] R.S. Hamilton, *Harmonic maps of manifolds with boundary*, Lecture Notes in Mathematics 471, Springer-Verlag, Berlin-New York, 1975. MR0482822 (58:2872)
- [11] F. Hélein, *Harmonic maps, conservation laws and moving frames*, second edition, Cambridge Tracts in Mathematics 150, Cambridge University Press, Cambridge, 2002. Translated from the 1996 French original. With a foreword by James Eells. MR1913803 (2003g:58024)
- [12] G. Lieberman, *Second order parabolic differential equations*, revised version 2005, World Scientific Publishing Co., Inc., River Edge, NJ, 1996. MR1465184 (98k:35003)
- [13] J. Nečas, *Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle*, Ann. Scuola Norm. Sup. Pisa (3) 16 (1962), 305–326. French. MR0163054 (29:357)
- [14] T. Rivière, *Flot des applications harmoniques en dimension deux*, preprint ENS-Cachan (1993). In “Applications harmoniques entre variétés”: Thèse de l’Université Paris 6.
- [15] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. 60 (1985), no. 4, 558–581. MR826871 (87e:58056)
- [16] M. Struwe, *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*, second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 34. Springer-Verlag, Berlin, 1996. MR1411681 (98f:49002)
- [17] L. Wang, *Remark on weak solutions*, preprint. Available at <http://math.mit.edu/~luwang/>.

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