RATIONALITY OF RATIONALLY CONNECTED THREEFOLDS ADMITTING NON-ISOMORPHIC ENDOMORPHISMS

DE-QI ZHANG

ABSTRACT. We prove a structure theorem for non-isomorphic endomorphisms of weak $\mathbb{Q}$ -Fano threefolds or, more generally, for threefolds with a big anti-canonical divisor. Also provided is a criterion for a fibred rationally connected threefold to be rational. As a consequence, we show (without using the classification) that every smooth Fano threefold having a non-isomorphic surjective endomorphism is rational.

1. INTRODUCTION

We work over the field $\mathbb{C}$ of complex numbers. We shall prove Theorems 1.1, 1.2 and 1.3.

Theorem 1.1. *Let X be a $\mathbb{Q}$ -factorial projective threefold having only terminal singularities, a surjective endomorphism $f : X \rightarrow X$ of degree > 1 , and a K_X -negative extremal contraction $X \rightarrow Y$ with $\dim Y \in \{1, 2\}$. Suppose either one of the following three conditions:*

- (1) *X is Gorenstein.*
- (2) *$K_X^3 \neq 0$.*
- (3) *The ramification divisor R_f is non-zero.*

Suppose further that X is rationally connected. Then X is rational.

Theorem 1.2. *Let X be a smooth Fano threefold with a surjective endomorphism $f : X \rightarrow X$ of degree > 1 . Then X is rational.*

Here is a sketch of the proof of Theorem 1.2 using Theorem 1.1. Let $X \rightarrow X_1$ be the composite of blowdowns between smooth threefolds such that X_1 is a primitive smooth Fano threefold in the sense of [15]. The morphism f , replaced by some power, descends to a surjective morphism $f_1 : X_1 \rightarrow X_1$, using the finiteness of extremal rays in the Mori cone $\overline{NE}(X)$. If $\rho(X_1) = 1$, then $X_1 \cong \mathbb{P}^3$ by [1] or [6], and we are done! Suppose that $\rho(X_1) \geq 2$. By [15, Theorem 5], X_1 has an extremal contraction of conic bundle type, so X_1 is rational by Theorem 1.1. Alternatively, as a referee remarked, such an X_1 fits one of the four families (or else is known to be rational; cf. [15]) whose explicit description may be used to give another probably simpler proof of Theorem 1.2.

In [11], Kollár-Xu constructed many examples of endomorphisms $f : X \rightarrow X$ of degree > 1 on quotients X of projective spaces, which are Fano varieties of Picard

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number one with at worst terminal singularities; some X might be irrational by invoking David Saltman's famous counterexamples to Noether's problem.

As another type of example, one takes an elliptic curve E of period $\sqrt{-1}$ and the abelian variety $A = E^n$ with $n \geq 1$. Let $\mu_4 \cong \langle \sqrt{-1} \rangle$ act diagonally on A . Then $X = A/\mu_4$ has only log terminal singularities and is rationally connected when $n \leq 3$. For $m \neq 0, \pm 1$, the multiplication map $m_A : A \rightarrow A$ ($a \mapsto ma$) descends to an endomorphism $f : X \rightarrow X$ of degree $m^{2n} > 1$. When $n = 3$, it is not known whether X is rational.

These examples suggest that the Gorenstein requirement might be necessary for Theorem 1.1 or 1.2.

For a uniruled threefold X with $-K_X$ big and an endomorphism, we have the following equivariant minimal model program (MMP); see Corollaries 4.1, 4.2 and Remarks 4.3, 4.4 for applications. As a referee remarked, the assertion (4) is well known and also follows from [21, Lemma 2.8].

Theorem 1.3. *Let X be a $\mathbb{Q}$ -factorial projective threefold with only terminal singularities, a big $-K_X$ and a surjective endomorphism $f : X \rightarrow X$ of degree $q^3 > 1$. Let $X = X_0 \cdots \rightarrow X_1 \cdots \rightarrow X_t$ be a composition of K_{X_i} -negative flips and divisorial contractions such that there is no K_{X_t} -negative extremal contraction of birational type. Replacing f by some power, we have (cf. also 4.4):*

- (1) *For every $1 \leq j \leq t$, the dominant rational map $f_j : X_j \cdots \rightarrow X_j$ induced from f is holomorphic. Set $f_0 := f$.*
- (2) *There is a K_{X_t} -negative extremal contraction $X_t \rightarrow Y$ with $\dim Y \leq 2$. The morphism f_t descends to a surjective endomorphism $h : Y \rightarrow Y$.*
- (3) *Either $\dim Y \in \{1, 2\}$, or X_t is a $\mathbb{Q}$ -Fano threefold of Picard number one (a consequence of MMP). In the latter case, all f_i ($0 \leq i \leq t$) are polarized (cf. 2.1(1)), and further*

$$f_i^* | N^1(X_i) = q \text{ id}.$$

- (4) *Suppose further that (X, Θ) is klt weak $\mathbb{Q}$ -Fano for some $\Theta \geq 0$. Then for all $0 \leq i \leq t$, there are some $\Theta_i \geq 0$ such that the pairs (X_i, Θ_i) are klt $\mathbb{Q}$ -Fano.*

Remark 1.4. Theorem 1.2 was proved in [25] when the surjective morphism $f : X \rightarrow X$ is assumed further to be polarized (cf. 2.1(1) below for its definition). The author does not know whether Theorem 1.2 follows from the result in [25], but being polarized is a strong condition (nevertheless, see also Theorem 1.3(3)). For instance, taking any endomorphism $g : Y \rightarrow Y$, then $f = g \times h : Y \times \mathbb{P}^1 \rightarrow Y \times \mathbb{P}^1$ is not polarized if $(\deg h)^{\dim Y} \neq \deg g$ (cf. [20, Lemma 2.1]).

For an arbitrary uniruled variety X with an endomorphism f , we would like to have an equivariant MMP: $X \cdots \rightarrow X_m$ to reduce to the Fano fibration case. To do so, we need to prove that the K -negative extremal rays appearing in the composition $X \cdots \rightarrow X_m$ of birational contractions are stabilized by f and its descents. This is not easy to prove because the Mori cone $\overline{NE}(X)$ may have infinitely many extremal rays and $X \cdots \rightarrow X_m$ may involve flips and hence may not be holomorphic. Fortunately, for Theorem 1.3, these difficulties are overcome by Theorem 2.16.

We refer to [4], [5], [12], [18], [19], [20], [24], and [25] for the recent development in the study of the dynamics of endomorphisms of complex varieties.

2. PRELIMINARY RESULTS

2.1. Conventions are as in Hartshorne's book, [9] and [10]. Some more:

(1) Every endomorphism f of a projective variety in this paper is assumed to be surjective; so it is finite by the projection formula.

An endomorphism $f : X \rightarrow X$ is *polarized* if there is an ample Cartier integral divisor H such that $f^*H \sim qH$ for some $q > 0$. So $\deg(f) = q^{\dim X}$.

(2) Let $f : X \rightarrow X$ be an endomorphism and $\sigma_V : V \rightarrow X$ and $\sigma_Y : X \rightarrow Y$ morphisms. We say that f *lifts* to an endomorphism $f_V : V \rightarrow V$ if $f \circ \sigma_V = \sigma_V \circ f_V$; f *descends* to an endomorphism $f_Y : Y \rightarrow Y$ if $\sigma_Y \circ f = f_Y \circ \sigma_Y$.

(3) Every boundary divisor in a pair is assumed to be an effective $\mathbb{Q}$ -divisor.

(4) A pair (X, Θ) is called a *klt weak $\mathbb{Q}$ -Fano variety* (resp. *klt $\mathbb{Q}$ -Fano variety*) if X is a $\mathbb{Q}$ -factorial normal projective variety, the pair (X, Θ) has at worst klt singularities as in [10, Definition 2.34] and $-(K_X + \Theta)$ is nef and big (resp. ample). When $\Theta = 0$, the pair (X, Θ) is simply denoted as X . Thus X is a *klt weak $\mathbb{Q}$ -Fano variety*, or a *klt $\mathbb{Q}$ -Fano variety* if so is $(X, 0)$. A klt weak $\mathbb{Q}$ -Fano variety (resp. a klt $\mathbb{Q}$ -Fano variety) with (X, Θ) terminal (strengthened condition) is called a *weak $\mathbb{Q}$ -Fano variety* (resp. a *$\mathbb{Q}$ -Fano variety*).

(5) Suppose that X is a normal projective variety and (X, Θ) is terminal or klt. An *extremal ray* in the Mori cone $\overline{\text{NE}}(X)$, the nef cone $\text{Nef}(X)$ or some other convex cones is in the sense of a locally polyhedral cone. A $(K_X + \Theta)$ -*negative extremal ray* in $\overline{\text{NE}}(X)$ is an extremal ray whose intersection with $K_X + \Theta$ is negative. By a $(K_X + \Theta)$ -*negative extremal contraction* $X \rightarrow Y$, we mean the contraction of a $(K_X + \Theta)$ -negative extremal ray in $\overline{\text{NE}}(X)$. In particular, the Picard number $\rho(X) = \rho(Y) + 1$. When $\Theta = 0$, such a contraction is simply called an *extremal contraction*, or *divisorial contraction*, or *flip*, accordingly.

(6) By a *conic bundle* $X \rightarrow Y$, we mean an extremal contraction of relative dimension 1, where X is normal projective with only terminal singularities (so a general fibre is $\mathbb{P}^1$).

(7) For a surjective morphism $f : X \rightarrow Y$, the *discriminant* of f is defined as $D(X/Y) := \{y \in Y; f \text{ is not smooth over } y\}$. Let $D_s(X/Y)$ be the s -dimensional part of $D(X/Y)$, which is also regarded as a reduced scheme.

(8) For an endomorphism $\varphi : V \rightarrow V$ of a finite-dimensional real vector space V , $\rho(\varphi) := \max\{|\lambda|; \lambda \in \mathbb{C} \text{ is an eigenvalue of } \varphi\}$ is the *spectral radius*.

(9) For a $\mathbb{Q}$ -factorial normal projective variety X , denote $S(X) := \{\text{prime divisor } M; M|_M \text{ is not pseudo-effective}\}$. When $\dim X = 2$, our $S(X)$ is the set of negative curves on X .

(10) For a normal projective surface X , denote by $\text{Weil}(X)$ the finite-dimensional $\mathbb{R}$ -vector space of Weil divisors modulo Mumford numerical equivalence: two Weil divisors are *Mumford numerically equivalent* if their Mumford pullbacks to a resolution are numerically equivalent.

Lemma 2.2(1) is from [16, Theorem 1.2.7]. For Lemma 2.2(2), see [14, Theorem (3.5)] (for the smooth case) and [16, Corollary 2.4.2] (originally due to Cutkosky).

Lemma 2.2. *Let $X \rightarrow Y$ be a conic bundle with $(\dim X, \dim Y) = (3, 2)$. Then we have:*

- (1) *Y has at worst Du Val singularities of type A_n .*
- (2) *If X is Gorenstein, then Y is smooth.*

The following are known results about the rationality of threefolds; see [8, §§2.2 - 2.3], [13, §4.7] and the references therein.

Theorem 2.3. *Let X be a $\mathbb{Q}$ -factorial projective threefold with at worst terminal singularities. Then X is rational if either one of the following three conditions is satisfied.*

- (1) *There is an extremal contraction $X \rightarrow B \cong \mathbb{P}^1$ such that a general fibre F is a del Pezzo surface with $K_F^2 \geq 5$.*
- (2) *X is smooth. There is an extremal contraction $X \rightarrow Y$ onto a rational surface Y , which is a non-standard conic bundle (cf. Remark 2.4).*
- (3) *X is smooth. There is an extremal contraction $X \rightarrow Y$ onto a rational surface Y , which is a standard conic bundle. Either $Y = \mathbb{P}^2$ and the discriminant $D := D(X/Y)$ has $\deg(D) \leq 4$, or there is a $\mathbb{P}^1$ -fibration on Y such that $F \cdot D \leq 3$ for a general fibre F .*

Remark 2.4. (i) In (2) and (3) of Theorem 2.3, Y is smooth (cf. Lemma 2.2). According to [13, 4.7], a conic bundle $\pi : X \rightarrow Y$ is *standard* if $\text{Pic}(X) = \pi^* \text{Pic}(Y) \oplus \mathbb{Z}K_X$.

- (ii) In Theorem 2.3(2), by [13, §4.7], ‘ $X \rightarrow Y$ is non-standard’ if and only if ‘ $X \rightarrow Y$ is a $\mathbb{P}^1$ -bundle in the Zariski-topology’; such a $\mathbb{P}^1$ -bundle is locally trivial and hence X is rational, due to the triviality of the Brauer group $\text{Br}(Y)$ for smooth rational surfaces.
- (iii) Theorem 2.3(3) was proved by Iskovskikh [7, Theorem 1]; see his survey [8, §2.3, Theorem 8] for its variation and references.

For the first part of Lemma 2.5 below, see [13, Lemma 4.1, Remark 4.2]; the last part follows from the first part and the Riemann-Roch theorem as proved in [3, Lemma 2.3].

Lemma 2.5. *Let $X \rightarrow Y$ be a conic bundle with X smooth and $(\dim X, \dim Y) = (3, 2)$. Then the discriminant $D = D(X/Y)$ is a divisor of normal crossings, and every smooth rational component of D (if it exists) meets at least two points of the other components. In particular, we have $|K_Y + D_i| \neq \emptyset$, when Y is rational and D_i is a connected component of D .*

Lemma 2.6. *Let $\pi : X \rightarrow Y$ be a conic bundle with $(\dim X, \dim Y) = (3, 2)$ and Y a rational surface. Then either the generic fibre is rational over $\mathbb{C}(Y)$ and hence X is rational, or there is a conic bundle $\pi' : X' \rightarrow Y'$ such that:*

- (1) *X' (and hence Y') are smooth, and there are birational morphisms $\sigma_x : X' \rightarrow X$ and $\sigma_y : Y' \rightarrow Y$ satisfying $\pi \circ \sigma_x = \sigma_y \circ \pi'$.*
- (2) *No connected component C'_i of $D(X'/Y')$ is contracted to a point by σ_y .*
- (3) *(The 1-dimensional part) $D_1(X/Y) = \sigma_{y*} D(X'/Y') \neq \emptyset$.*
- (4) *$D_1(X/Y)$ and $D(X'/Y')$ have equal numbers of connected components.*

Proof. (1) is actually proved in [13, Theorem 4.8]. If (2) is not true, C'_i is contracted to a point on Y which is at worst of Du Val singularity of type A_n (cf. Lemma 2.2) and hence C'_i is a rational tree, which contradicts Lemma 2.5. Part (3) is true because every reducible fibre over some $d \in D(X/Y)$ should be underneath only reducible fibres over some $d' \in D(X'/Y')$ and note that $\sigma_y : Y' \rightarrow Y$ is the blowup along $D(X/Y)$; see the construction in [13, Theorem 4.8]; note also that $(\pi')^* E$ is irreducible for every prime divisor $E \subset Y'$ (and especially for those $E \subset D(X'/Y')$). Part (4) is from Parts (2) and (3). $\square$

Proposition 2.7. *Let $X \rightarrow Y$ be a conic bundle with Y a rational surface. Suppose that either $D_1(X/Y) = \emptyset$, or there is a smooth rational curve F on $\tilde{Y}$, with $\sigma : \tilde{Y} \rightarrow Y$ a minimal resolution, such that $Bs|F| = \emptyset$ and $F \cdot \sigma^* D_1(X/Y) \leq 3$. Then X is rational.*

Proof. We may assume that $D_1(X/Y) \neq \emptyset$ (cf. Lemma 2.6) and F is a general member in $|F|$. Let $Y'' \rightarrow \tilde{Y}$ be the blowup of $m = F^2$ points on F away from $F \cap \sigma^{-1}D(X/Y)$. Take a blowup $X'' \rightarrow X$ so that the composite $X'' \rightarrow X \rightarrow Y \cdots \rightarrow Y''$ extends to a morphism with general fibre $\mathbb{P}^1$. By the proof of [13, Theorem 4.8], we have a lifting $\pi' : X' \rightarrow Y'$ of π as in Lemma 2.6 so that $Y' \rightarrow Y$ factors as $Y' \rightarrow Y'' \rightarrow \tilde{Y} \rightarrow Y$. Denote by $D' = D(X'/Y')$ and by $D'', \tilde{D}, D$ its pushforwards (as cycles) on $Y'', \tilde{Y}$ and Y . Note that $D = D_1(X/Y)$ by Lemma 2.6.

Claim 2.8. $\sigma^* D \geq \tilde{D}$.

We prove Claim 2.8. Write $\sigma^* D - \tilde{D} = \sum_i e_i E_i$ with E_i irreducible and σ -exceptional. Lemma 2.2 implies that each E_i is a (-2) -curve. By Zariski's lemma, we have only to show the assertion that $0 \geq E_j \cdot (\sigma^* D - \tilde{D}) = -E_j \cdot \tilde{D}$ for every E_j . We need to consider the case where E_j is a component of $\tilde{D}$. Applying Lemma 2.5 to D' and using Lemma 2.6, we have $E_j \cdot \tilde{D} \geq 0$. This proves Claim 2.8.

We continue the proof of Proposition 2.7. The proper transform $F'' \subset Y''$ of F is a fibre of a $\mathbb{P}^1$ -fibration. Let $F' \subset Y'$ be the total transform of F'' . Then $F' \cdot D' = F'' \cdot D'' = F \cdot \tilde{D} \leq F \cdot \sigma^* D \leq 3$. Now Proposition 2.7 follows from Theorem 2.3. $\square$

Lemma 2.9. *Let $X \rightarrow Y$ be a conic bundle with $\dim Y = 2$, and $f : X \rightarrow X$ a surjective endomorphism descending to an endomorphism $h : Y \rightarrow Y$. Then $h^{-1}(D) = D$ for the (1-dimensional part) $D := D_1(X/Y)$, so after replacing f by some power, we have $h^{-1}(D_i) = D_i$ for every irreducible component D_i of D .*

Proof. We have $h^{-1}(D) \subseteq D$ since the reducibility of a fibre X_d over $d \in D$ implies that of $X_{d'}$ for every $d' \in h^{-1}(d)$. So $D \supseteq h^{-1}(D) \supseteq h^{-2}(D) \supseteq \cdots$. Considering the number of irreducible components, we have $h^{-s}(D) = h^{-s-1}(D)$ for some $s > 1$. Since h is surjective and applying h^s and h^{s+1} we get $h^{\pm}(D) = D$. This proves the lemma. $\square$

The result below can be proved as in [17, Lemma 8 and Proposition 10]. The assumption on X_i can be weakened to being normal Moishezon by using the Mumford intersection on surfaces.

Lemma 2.10. *Let $\varphi : X_1 \rightarrow X_2$ be a finite morphism between two $\mathbb{Q}$ -factorial normal projective surfaces with the same Picard number $\rho(X_1) = \rho(X_2)$. Then we have:*

- (1) φ induces a bijection $\varphi : S(X_1) \rightarrow S(X_2)$ ($C \mapsto \varphi(C)$) with a well-defined inverse mapping C' to $\varphi^{-1}(C')$. Further, $\varphi^* C_2 = \alpha C_1$, whenever $C_2 \in S(X_2)$, $C_1 = \varphi^{-1}(C_2)$ and $\alpha = \sqrt{\deg(\varphi) C_2^2 / C_1^2}$.
- (2) Suppose that $\deg(\varphi) > 1$. Then $|S(X_i)| < \infty$; we may also assume that $\varphi|_{S(X_1)} = \text{id}$, when $\varphi : X_1 \rightarrow X_2 = X_1$ is an endomorphism and φ is replaced by some power.

As a referee remarked, Lemma 2.11 and Lemma 2.12(2) are well known and the latter also follows from [21, Lemma 2.8]. We give proofs for them for the convenience of readers.

Lemma 2.11. *Let (X, Θ) be a klt weak $\mathbb{Q}$ -Fano variety. Then we have:*

- (1) (X, Θ_0) is klt $\mathbb{Q}$ -Fano for some $\Theta_0 \geq \Theta$.
- (2) The cone $\overline{\text{NE}}(X)$ has only finitely many extremal rays (K_X negative or non-negative).
- (3) X is rationally connected.

Proof. (3) is proved in [26, Theorem 1]. For (1), by [10, Proposition 2.61], $-(K_X + \Theta) \sim_{\mathbb{Q}} A_k + \frac{1}{k}E$ for some ample $\mathbb{Q}$ -divisor A_k and effective $\mathbb{Q}$ -divisor E . Choose $k \gg 0$ so that $(X, \Theta + \frac{1}{k}E)$ has at worst klt singularities; see [10, Corollary 2.35]. Now $-(K_X + \Theta + \frac{1}{k}E) \sim_{\mathbb{Q}} A_k$ is ample. (1) is proved. (2) is true by the cone theorem as in [10, Theorem 3.7]. $\square$

Lemma 2.12. (1) *Let $\varphi : V \rightarrow W$ be a generically finite surjective morphism between projective varieties. Suppose that the cone $\overline{\text{NE}}(V)$ has only finitely many extremal rays $\mathbb{R}_{\geq 0}[C_i]$ (K_V negative or non-negative). Then $\overline{\text{NE}}(W)$ has only finitely many extremal rays, each of which is generated by some $\varphi(C_i)$.*

(2) *Suppose that (V, Θ) is a klt weak $\mathbb{Q}$ -Fano variety. Let $\sigma : V \dashrightarrow W$ be a K_V -flip or a K_V -divisorial contraction. Then (W, Θ_W) is klt $\mathbb{Q}$ -Fano for some $\Theta_W \geq 0$. Hence both cones $\overline{\text{NE}}(V)$ and $\overline{\text{NE}}(W)$ have only finitely many extremal rays. (We remark that if V is terminal, then so is W ; cf. [10, Corollary 3.42]).*

Proof. (1) is true, since φ induces a surjective additive map $\varphi_* : \overline{\text{NE}}(V) \rightarrow \overline{\text{NE}}(W)$.

For (2), consider the case where σ is a flip. Let $\pi : V \rightarrow Y$ and $\pi^+ : W = V^+ \rightarrow Y$ be the flipping contractions. Let H be an ample $\mathbb{Q}$ -divisor on Y . By Lemma 2.11, we may assume that $-(K_V + \Theta)$ is ample after replacing Θ . Thus $-m(K_V + \Theta) - \pi^*H$ is very ample for $m \gg 0$ and hence linearly equivalent to an irreducible divisor G_m . Then $(V, \Theta + \frac{1}{m}G_m)$ is klt by [10, Corollary 2.31]. Let $\Theta'_W \subset W$ be the proper transform of $\Theta + \frac{1}{m}G_m$. Then W is $\mathbb{Q}$ -factorial and (W, Θ'_W) is klt by [10, Proposition 3.37, Corollary 3.42]. Also $-(K_W + \Theta'_W)$ is $\mathbb{Q}$ -linearly equivalent to the nef and big divisor $(\pi^+)^*H/m$. Now (2) follows from Lemma 2.11.

Consider the case where σ is divisorial. By Lemma 2.11, we may assume that (V, Θ) is klt $\mathbb{Q}$ -Fano. Take a small ample $\mathbb{Q}$ -divisor $H_W \subset W$ such that $-(K_V + \Theta) - \sigma^*H_W$ is still ample and hence it is $\mathbb{Q}$ -linearly equivalent to some $\Delta > 0$ with $(V, \Theta + \Delta)$ klt by the argument in the previous case. Then $-(K_V + \Theta + \Delta) \sim_{\mathbb{Q}} \sigma^*H_W$. Set $\Theta_W := \sigma_*(\Theta + \Delta)$. Now $-(K_W + \Theta_W) \sim_{\mathbb{Q}} \sigma_*\sigma^*H_W = H_W$, which is ample, and $K_V + \Theta + \Delta \sim_{\mathbb{Q}} \sigma^*(K_W + \Theta_W)$. Thus (W, Θ_W) is klt (and $\mathbb{Q}$ -Fano) because so is $(V, \Theta + \Delta)$. This proves the lemma. $\square$

Below is Birkhoff's generalization of the Perron-Frobenius theorem (cf. [2]).

Theorem 2.13. *Let C be a strictly convex closed cone of a finite-dimensional real vector space V such that C generates V as a vector space. Let $\varphi : V \rightarrow V$ be an endomorphism such that $\varphi(C) \subseteq C$. Then the spectral radius $\rho(\varphi)$ is an eigenvalue of φ and corresponds to an eigenvector in $C \setminus \{0\}$.*

Lemma 2.14. *Let V be a normal projective surface and $f : V \rightarrow V$ a surjective endomorphism. Then $\deg(f) \leq \rho(f^*)^2$, where $\rho(f^*)$ is the spectral radius of $f^*|_{\text{Weil}(V)}$.*

Proof. By Theorem 2.13, there is a non-zero $\mathbb{R}$ -Cartier nef divisor $L \in \text{Weil}(V)$ such that $f^*L \equiv \rho L$, where $\rho := \rho(f^*)$. Take a Jordan canonical basis $\{L_i\}_{i \geq 1}$ for $f^*|_{\text{Weil}(V)}$. Write $f^*L_i = \lambda_i L_i + L_{i-1}$ (set $L_0 := 0$). Note that $|\lambda_i| \leq \rho$. We may assume that $L.L_i = 0$ for all $i < s$ and $L.L_s \neq 0$. Now $\deg(f)L.L_s = f^*L.f^*L_s = \rho L.(\lambda_s L_s + L_{s-1}) = \rho \lambda_s L.L_s$. So $\deg(f) = \rho \lambda_s \leq \rho^2$. $\square$

Lemma 2.15. *Let $X = X_0 \cdots \rightarrow X_1 \cdots \rightarrow X_r$ be a composition of flips and divisorial contractions corresponding to $(K_{X_i} + \Theta_i)$ -negative extremal rays on some $\mathbb{Q}$ -factorial klt pairs (X_i, Θ_i) . Suppose that $f = f_0 : X \rightarrow X$ is a surjective endomorphism and descends to surjective endomorphisms $f_i : X_i \rightarrow X_i$ ($1 \leq i \leq r$). Suppose further that for some $0 \leq k \leq r$ we have $(f_k^{s_k})^*|_{N^1(X_k)} = q^{s_k} \text{id}$ for some $q > 1$ and integer $s_k > 0$. Then all f_j ($0 \leq j \leq r$) are polarized, and $(f_j^s)^*|_{N^1(X_j)} = q^s \text{id}$ for some integer $s > 0$.*

Proof. Let $H' \subset X_k$ be an ample $\mathbb{Q}$ -divisor and set $H := \sum_{i=0}^{s_k-1} (f_k^i)^* H' / q^i$. Then $f_k^* H \equiv qH$. Thus f_k is polarized by [20, Lemma 2.2].

By the remark above, we only need to show the second assertion, and for that we will prove by the ascending induction on $|k - i|$. We may assume that $s_k = 1$, after replacing f by some power. It suffices to show that $f_i^*|_{N^1(X_i)} = q \text{id}$ for $i = k - 1$ and $k + 1$. So we may assume that $(k, r) = (0, 1)$ or $(1, 1)$.

Case 1. $\sigma : X \rightarrow X_1$ is a divisorial contraction with E the exceptional divisor. E is irreducible by [9, Proposition 5-1-6], and $f^*E = eE$ for some $e > 0$ because both f and its descent f_1 are necessarily finite morphisms. If $k = 0$ so that $f_i^*|_{N^1(X_i)} = q \text{id}$ for $i = 0$, then the same holds for $i = 1$ because $\sigma^*N^1(X_1)$ is a subspace of $N^1(X)$ and f_1 is the descent of f . Suppose that $k = 1$ so that $f_1^*|_{N^1(X_1)} = q \text{id}$. Then $f^*|_{N^1(X)} = \text{diag}[e, q, \dots, q]$ with respect to a basis consisting of E and a basis of $\sigma^*N^1(X_1)$. For an ample divisor $H' \subset X_1$, the divisor $H := \sigma^*H'$ is nef and big and $f^*H \equiv qH$, so $e = q$ by [20, Lemma 2.1]. Thus $f^*|_{N^1(X)} = q \text{id}$. We are done.

Case 2. $\sigma : X \cdots \rightarrow X_1$ is a flip. Then σ induces an isomorphic linear map between $N^1(X)$ and $N^1(X_1)$ compatible with the two actions $f_i^*|_{N^1(X_i)}$ with $i = 0, 1$; see the proof of [10, Proposition 3.37]. So the scalarity of one action implies that of the other. This proves the lemma. $\square$

The theorem below is the key in proving Theorem 1.3.

Theorem 2.16. *Let X be a $\mathbb{Q}$ -factorial projective threefold with only terminal singularities, big $-K_X$ and a surjective endomorphism $f : X \rightarrow X$ of degree $d > 1$. Let $\mathbb{R}_{\geq 0}[\ell]$ be a K_X -negative extremal ray. Then, replacing f by some power, we have:*

- (1) $f(\ell)$ is parallel to ℓ , i.e., $f(\ell) \in \mathbb{R}_{> 0}[\ell]$.
- (2) Suppose that $\mathbb{R}_{\geq 0}[\ell]$ gives rise to a divisorial contraction $\sigma : X \rightarrow X_1$, or a flip $\sigma : X \cdots \rightarrow X^+ =: X_1$, or a Fano fibration $X \rightarrow X_1$ with $\dim X_1 \leq 2$. Then f descends to a surjective endomorphism of X_1 . Further, $-K_{X_1}$ is also big when σ is divisorial or a flip.

Proof. We only need to show the first assertion of Theorem 2.16 because the second follows from the first; see [25, Lemmas 2.12 and 3.6]. Indeed, the bigness of $-K_X$ implies that of $-K_{X_1} = \sigma_*(-K_X)$ when σ is divisorial; $-K_{X_1}$ is also big when σ is a flip, because σ is then isomorphic in codimension one.

We now prove Theorem 2.16(1). We may assume that ℓ is an irreducible curve. Write $-K_X = A + D$ with A an ample $\mathbb{Q}$ -divisor and D an effective $\mathbb{Q}$ -divisor. By [10, Theorem 3.7], for every $1 > \varepsilon > 0$, the first assertion below is true.

- Claim 2.17.* (1) The subcone $R_\varepsilon := \overline{\text{NE}}(X)_{K_X + \varepsilon A < 0}$ of $\overline{\text{NE}}(X)$ contains only finitely many extremal rays.
- (2) Suppose that $\mathbb{R}_{\geq 0}[\ell]$ gives rise to a divisorial contraction $\sigma : X \rightarrow X_1$ with E the exceptional (necessarily prime) divisor and ℓ taken to be a fibre of σ . Then the subspace $N^1(X).E \subset N_1(X)$ has rank ≤ 2 , and the ray $\mathbb{R}_{\geq 0}[\ell]$ is extremal in (the closure of) the cone $\text{Nef}(X).E \subset N_1(X)$.

For Claim 2.17(2), if $\sigma(E)$ is a point, then $N^1(X).E = \mathbb{R}[\ell]$, and we are done! Suppose that $\sigma(E)$ is a curve. For any divisor M on X we have $(M - aA).\ell = 0$ for some $a \in \mathbb{R}$ and hence $M - aA = \sigma^*L$ for some divisor L on Y_1 ; see [9, Lemma 3-2-5]. Then $M.E = aA.E + b\ell$ with $b = L.\sigma(E)$. Thus $N^1(X).E$ is spanned by $A.E$ and ℓ . Claim 2.17(2) is true because the ray there, is extremal in the bigger cone $\overline{\text{NE}}(X)$. This proves Claim 2.17.

We return to the proof of Theorem 2.16.

Note that every $\mathbb{R}_{\geq 0}[f^s(\ell)]$ ($s \in \mathbb{Z}$) is an extremal ray in $\overline{\text{NE}}(X)$ (K_X negative or non-negative); see [25, Lemma 2.11]. If $f^s(\ell)$ is contained in R_ε for infinitely many $s = s_1, s_2, \dots$, then by Claim 2.17, we may assume that $f^{s_1}(\ell)$ is parallel to $f^{s_2}(\ell)$ for some $s_2 > s_1$. Thus $f^{s_2-s_1}(\ell)$ is parallel to ℓ because f_* is an automorphism of the vector space $N_1(X)$. We are done.

Therefore, we may assume that $f^s(\ell)$ is not contained in R_ε for all $s \in \mathbb{Z} \setminus \{0\}$ after f is replaced by some power. Note that $\bigcup_{s>0} f^s(\ell) \subset D$ because:

$$\begin{aligned} -(K_X + \varepsilon A) &= (1 - \varepsilon)A + D, \\ 0 &\geq -f^s(\ell).(K_X + \varepsilon A) = (1 - \varepsilon)f^s(\ell).A + f^s(\ell).D > f^s(\ell).D. \end{aligned}$$

Case 1. $\mathbb{R}_{\geq 0}[\ell]$ gives rise to a Fano fibration $\sigma : X \rightarrow X_1$ with $\dim X_1 \leq 2$ and ℓ chosen to be in a general fibre of σ . Then $f(\ell)$ is not contained in D . This is a contradiction.

Case 2. $\mathbb{R}_{\geq 0}[\ell]$ gives rise to a divisorial contraction $\sigma : X \rightarrow X_1$ with E the exceptional (prime) divisor and ℓ taken as a fibre of σ . Since $f^s(\ell).D < 0$ ($s \neq 0$), we have $f(\ell).D_1 < 0$ for an irreducible component D_1 of D . Every fibre ℓ' of σ is parallel to ℓ , so $f(\ell').D_1 < 0$. Thus $f(\ell') \subset D_1$ and hence $f(E) = D_1$. Now $f^2(\ell).D < 0$ implies that $f^2(\ell).D_j < 0$ for some irreducible component D_j of D . As argued above, $f^2(E) = D_j$. Since D has only finitely many components, we have $f^{r_2}(E) = f^{r_1}(E)$ for some $r_2 > r_1$. Thus for fibres ℓ_t of σ , we have $f^{r_2}(\ell_t) = f^{r_1}(\ell_t)$ for some irreducible curve $m_t \subset E$. So $f^r(\ell_t)$ (and hence $f^r(\ell)$) is parallel to m_t , where $r := r_2 - r_1 > 0$. For an extremal curve ℓ'' , denote

$$\Sigma_{\ell''} := \{C \subset X; [C] \in \mathbb{R}_{\geq 0}[\ell'']\}, \quad U_{\ell''} := \bigcup_{C \in \Sigma_{\ell''}} C,$$

where these C are irreducible curves. Then $E = U_\ell$. By [25, Lemma 2.11],

$$f^r(E) = f^r(U_\ell) = U_{f^r(\ell)} \supset \bigcup_t m_t \subset E.$$

The map $\{\ell_t\} \rightarrow \{m_t\}$ is a finite-to-finite map. So $\{m_t\}$ is an infinite set and hence

$$f^r(E) = \overline{\bigcup_t m_t} = E.$$

By Claim 2.17, (the closure of) $\text{Nef}(X).E$ is generated by one or two extremal rays (one being $\mathbb{R}_{\geq 0}[\ell]$) each of which is preserved by f_*^{2r} . Thus $f^{2r}(\ell)$ is parallel to ℓ . We are done.

Case 3. $\mathbb{R}_{\geq 0}[\ell]$ gives rise to a flip $X \cdots \rightarrow X^+$ with flipping contractions $\pi : X \rightarrow Y_1$ and $\pi^+ : X^+ \rightarrow Y_1$. We may assume that $f^i(\ell)$ and $f^j(\ell)$ are not parallel for all $i \neq j$. In particular, $f^i(\ell)$'s are pairwise distinct. Replacing f by some power, we may assume that

$$\bigcup_{s>0} f^s(\ell) \subset \bigcup_{i=1}^r D_i \subseteq \text{Supp } D$$

and $I_j := \{s > 0 \mid f^s(\ell) \subset D_j\}$ is an infinite set for all $1 \leq j \leq r$. Thus

$$D_j = \overline{\bigcup_{s \in I_j} f^s(\ell)}.$$

Each $\Sigma_{f^s(\ell)} = f^s(\Sigma_\ell)$ is a union of finitely many (and, when $s \gg 0$, the same number of) curves. Since $f^{-s}(\Sigma_{f^s(\ell)}) = \Sigma_\ell$ (cf. [25, Lemma 2.11]), replacing ℓ by some $f^s(\ell)$, we may assume that for $\ell_i := f^i(\ell)$ ($i \geq 0$) we have $f^{-1}(\ell_{i+1}) = \ell_i$ and hence $f^{-i}f^i(\ell_0) = \ell_0$. Now $f^\pm$ permutes the D_j 's. Indeed,

$$f(D_1) = f\left(\overline{\bigcup_{s \in I_1} f^s(\ell)}\right) = \overline{\bigcup_{s \in I_1} f^{s+1}(\ell)} \subset \bigcup_{i=1}^r D_i.$$

Hence $f(D_1) = D_{j_0}$ for some j_0 . Also, by the choice of j_0 and ℓ_0 ,

$$f^{-1}(D_{j_0}) = f^{-1}\overline{\bigcup_{s \in I_1} f^{s+1}(\ell)} = \overline{\bigcup_{s \in I_1} f^{-1}f^{s+1}(\ell)} = \overline{\bigcup_{s \in I_1} f^s(\ell)} = D_1.$$

Replacing f by some power, we may assume that $f^\pm(D_j) = D_j$ for all $1 \leq j \leq r$, and also $f(\ell).D_1 < 0$ after relabelling, noting that only finitely many $f^s(\ell)$ are in $D \setminus \sum_{i=1}^r D_i$. In particular, $f(\ell) \subset D_1$ and hence $D_1|D_1 \neq 0$. Write $f^*D_1 = e_1D_1$ for some integer $e_1 > 0$. Then $f^*(D_1|D_1) = e_1D_1|D_1$.

Consider the case $\deg(f|D_1 : D_1 \rightarrow D_1) = 1$. Then $e_1 = d > 1$. We claim that $f|D_1$ induces an automorphism h of positive entropy on the normalization $\widetilde{D}_1$, the pullback of $-D_1|D_1$ is a nef eigenvector of h^* and e_1 equals the spectral radius $\rho(h^*)$ of $h^*|NS(\widetilde{D}_1)$. Indeed, applying Theorem 2.13 to the action of h^* on the nef cone of $\widetilde{D}_1$, we have $h^*L = \rho(h^*)L$ for some non-zero $\mathbb{R}$ -Cartier nef divisor L ; thus the pullback of $-D_1|D_1$ is a positive multiple of L by the uniqueness of the eigenvalue of h^* of modulus > 1 as in [12, Theorem 3.2] applied to an equivariant desingularization of D_1 . The claim is proved.

On the other hand, e_1 (being the spectral radius) must be a Salem number (and an algebraic integer of degree ≥ 2 over $\mathbb{Q}$) and e_1 and e_1^{-1} are roots of their common minimal polynomial (over $\mathbb{Q}$) dividing the characteristic polynomial of $h^*|NS(\widetilde{D}_1)$;

see the proof of [12, Theorem 3.2] or [23, Proposition 2.11]. Indeed, $e_1 \in \mathbb{N}$ and hence $(x - e_1)$ is the minimal polynomial of e_1 over $\mathbb{Q}$, which is of degree one, which is absurd!

Therefore, $f|D_1 : D_1 \rightarrow D_1$ has degree $d_1 > 1$. Then $d = d_1 e_1$. Our $f|D_1$ induces a degree- d_1 endomorphism h of the normalization $\widetilde{D}_1$ of D_1 . After relabeling, we may assume that no $\ell_i = f^i(\ell)$ ($i \geq 0$) is included in the non-normal locus of D_1 or the image of the ramification divisor R_h . Denote by $\widetilde{\ell}_i \subset \widetilde{D}_1$ the pullback of ℓ_i . Then we have $h^* \widetilde{\ell}_{j+1} = \widetilde{\ell}_j$ for all $j \geq 0$. Let N be the denominator of the intersection matrix of the exceptional curves of a resolution of D_1 . Then the Mumford intersection satisfies $C_1.C_2 \in \frac{1}{N^2} \mathbb{Z}$ for all curves C_i on $\widetilde{D}_1$. Now

$$d_1 \widetilde{\ell}_i . \widetilde{\ell}_j = h^* \widetilde{\ell}_i . h^* \widetilde{\ell}_j = \widetilde{\ell}_{i-1} . \widetilde{\ell}_{j-1}, \quad \frac{1}{N^2} \mathbb{Z} \ni \widetilde{\ell}_i . \widetilde{\ell}_j = \frac{1}{d_1^a} \widetilde{\ell}_{i-a} . \widetilde{\ell}_{j-a}.$$

Applying the above to $i = j = a \gg 0$ or $i = j - 1 = a \gg 0$, we get

$$\widetilde{\ell}_i^2 = \widetilde{\ell}_{i+1}^2 = \widetilde{\ell}_i . \widetilde{\ell}_{i+1} = 0.$$

Thus $\widetilde{\ell}_{i+1}$ is parallel to $\widetilde{\ell}_i$, by pulling back to a resolution of D_1 and applying the Hodge index theorem. So $\ell_{i+1} = f^{i+1}(\ell)$ is parallel to $\ell_i = f^i(\ell)$. Hence $f(\ell)$ is parallel to ℓ . This proves Theorem 2.16. $\square$

Lemma 2.18. *Let $X = X_0$ and X_1 be $\mathbb{Q}$ -factorial projective threefolds with only terminal singularities and $f = f_0 : X \rightarrow X$ and $f_1 : X_1 \rightarrow X_1$ surjective endomorphisms of the same degree $d > 1$. Suppose that either $\sigma : X \rightarrow X_1$ is a K_X -divisorial contraction or $\sigma : X \cdots \rightarrow X_1$ is a K_X -flip, and that f_1 is the descent of f in either case. Then the ramification divisor $R_f = 0$, if and only if $R_{f_1} = 0$, if and only if f_i is étale over $X_i \setminus \text{Sing } X_i$ for both i .*

Proof. Note that the second part follows from the first and the purity of the branch loci. If $\sigma : X \cdots \rightarrow X_1$ is a flip, then σ switches R_f and R_{f_1} because σ is an isomorphism in codimension 1 and f_1 is the descent of f , so $R_f = 0$ if and only if $R_{f_1} = 0$.

Consider the case where $\sigma : X \rightarrow X_1$ is divisorial with E the exceptional (prime) divisor. Then $\sigma_* R_f = R_{f_1}$, so $R_f = 0$ implies $R_{f_1} = 0$.

Suppose the contrary that $R_{f_1} = 0$ and $R_f \neq 0$. Then $R_f = (e - 1)E$, where $f^* E = eE$ with $e \geq 2$. Take a fibre ℓ of σ . Then $\ell.K_X < 0$. It is well known that $\ell.E < 0$. Since f and f_1 are compatible, $f^* \ell \equiv c\ell$ for some $c > 0$. Now $d\ell.E = f^* \ell . f^* E = c\ell.E$ implies that $c = d/e$. Thus $f_* \ell = (d/c)\ell = e\ell$. Now multiplying the equality $K_X = f^* K_X + (e - 1)E$ by ℓ and by the projection formula, we obtain

$$\ell.K_X = \ell.f^* K_X + (e - 1)\ell.E = e\ell.K_X + (e - 1)\ell.E.$$

Thus $0 < -\ell.K_X = \ell.E < 0$, which is absurd. So $R_f = 0$ if and only if $R_{f_1} = 0$. $\square$

3. FIBRATION-PRESERVING ENDOMORPHISMS

In this section, we shall prove three lemmas used in the proof of the main theorems.

Remark 3.1. Every normal projective variety Y of dimension ≤ 2 and dominated by a rationally connected normal projective variety X is rational.

Lemma 3.2. *Let X be a rationally connected projective threefold with only terminal singularities, and $\pi : X \rightarrow Y$ a conic bundle. Suppose that $f : X \rightarrow X$ is an endomorphism of degree $d > 1$ and it descends to an automorphism $h : Y \rightarrow Y$. Then X is rational.*

Proof. By Proposition 2.7, it suffices to show the claim below.

Claim 3.3. The (1-dimensional part) $D := D_1(X/Y)$ is an empty set.

We prove the claim. Suppose to the contrary that $D \neq \emptyset$. Replacing f by some power, we may assume, as in Lemma 2.9, that $h^{-1}(D_i) = D_i$ for every irreducible component D_i of D . Thus $h^*D_i = D_i$ and $h^*D = D$ because $\deg(h) = 1$.

Let F be a very ample divisor on Y away from $\Sigma := D_0(X/Y) \cup \text{Sing}(D) \cup \pi(\text{Sing } X) \cup \text{Sing}(Y)$ and $h(\Sigma)$. Set $s := F \cdot D > 0$. Since the Picard number $\rho(X/Y) = 1$, $X_F := \pi^*F$ is irreducible and is a smooth ruled surface (admitting a ruling $X_F \rightarrow F$) with s singular fibres S_i lying over the s points P_i in $D \cap F$. Further, each S_i consists of two intersecting lines $S_i(1) + S_i(2)$ (see [14], or [13, §4.1, §4.8]).

Note that the Picard number $\rho(X_F) = 2 + s$. Denote $F' = h^*F$. We have $f^*X_F = X_{F'} := \pi^*F'$, which is also a smooth ruled surface with a ruling $X_{F'} \rightarrow F'$. Our f restricts to a finite morphism $f|_{X_{F'}} : X_{F'} \rightarrow X_F$ of degree d while the latter descends to the isomorphism $h|_{F'} : F' \rightarrow F$.

Now $D \cdot F' = h^*D \cdot h^*F = D \cdot F = s$. So $X_{F'}$ is a smooth ruled surface with the Picard number $\rho(X_{F'}) = 2 + s = \rho(X_F)$. By Lemma 2.10, we have $(f|_{X_{F'}})^*(S_i(j)) = d_{ij}S_i(j)'$ for some $d_{ij} \in \mathbb{N}$, where $S_i(1)' + S_i(2)' = (f|_{X_{F'}})^*S_i$ are the s singular fibres of the ruling $X_{F'} \rightarrow F'$ lying over the s points $P'_i = D \cap F'$. Now $-d = dS_i(j)^2 = ((f|_{X_{F'}})^*S_i(j))^2 = (d_{ij}S_i(j)')^2 = -d_{ij}^2$, so $d_{ij} = \sqrt{d} > 1$. Thus $h^*P_i = \sqrt{d}P'_i$ as 0-cycles, with $P'_i = h^{-1}(P_i)$, contradicting $\deg(h) = 1$. This proves the claim and Lemma 3.2. $\square$

Lemma 3.4. *Let X be a rationally connected projective threefold with only terminal singularities, and $\pi : X \rightarrow Y$ a conic bundle. Suppose that $f : X \rightarrow X$ is an endomorphism of degree $d > 1$ and it descends to an endomorphism $h : Y \rightarrow Y$ of degree $e > 1$. Then either X is rational, or $d = e$, X is non-Gorenstein, f is étale over $X \setminus \text{Sing } X$, and $K_X^3 = 0$.*

Proof. By Lemma 2.2, Y is a rational surface with at worst Du Val singularities. So K_Y is Cartier and $K_{\tilde{Y}} = \sigma^*K_Y$ if $\sigma : \tilde{Y} \rightarrow Y$ is a minimal resolution.

By Proposition 2.7, we may assume that the (1-dimensional part) $D := D_1(X/Y) \neq \emptyset$. Replacing f by some power, we may assume that $h^{-1}(D_i) = D_i$ for every irreducible component D_i of D , as in Lemma 2.9. So $h^*D_i = d_iD_i$ for some $d_i \in \mathbb{N}$. Thus $K_Y + D = h^*(K_Y + D) + G$, where G is an effective divisor having no common component with D . Inductively, for all $s > 0$, we have

$$K_Y + D = h^*(K_Y + D) + G = (h^s)^*(K_Y + D) + \sum_{i=0}^{s-1} (h^i)^*G.$$

In view of Proposition 2.7, we may impose an extra assumption that $\sigma^*D \cdot F' \geq 4$ where F' is a general fibre of a $\mathbb{P}^1$ -fibration on $\tilde{Y}$ when $\tilde{Y} \neq \mathbb{P}^2$ and F' is a line when $\tilde{Y} = \mathbb{P}^2$. We apply Lemma 2.6 and use the notation $X' \rightarrow Y'$ there. We

may assume that $Y' \rightarrow Y$ factors as $Y' \rightarrow \tilde{Y} \rightarrow Y$. Let $D' := D(X'/Y')$ and let $\tilde{D} \subset \tilde{Y}$ be the image of D' . By Lemma 2.5, $|K_{Y'} + D'| \neq \emptyset$, so $|K_{\tilde{Y}} + \tilde{D}| \neq \emptyset$ and $|K_Y + D| \neq \emptyset$.

Take an ample divisor H on Y . Then for all $s > 0$,

$$H.(K_Y + D) = H.(h^s)^*(K_Y + D) + \sum_{i=0}^{s-1} H.(h^i)^*G \geq \sum_{i=0}^{s-1} H.(h^i)^*G.$$

Thus $G = 0$ and $K_Y + D = h^*(K_Y + D)$. If $K_Y + D \equiv 0$, then for the F' above we have $\sigma^*D.F' = \sigma^*(-K_Y).F' = -K_{\tilde{Y}}.F' \leq 3$, a contradiction to the extra assumption above. So $K_Y + D$ is not numerically trivial. By [18, Theorem 7.1.1; cf. also Remark 7.1.2, Theorem 2.6.5], there is a fibration $Y \rightarrow B \cong \mathbb{P}^1$ (for $g(B) \leq q(Y) = 0$) with a general fibre F such that

$$K_Y + D \sim_{\mathbb{Q}} aF$$

for some rational number $a > 0$. Since $h^*(K_Y + D) = K_Y + D$, we have $h^*F \sim_{\mathbb{Q}} F$, so h descends to an automorphism $h_B : B \rightarrow B$. Now $0 = F.aF = F.(K_Y + D) \geq F.K_Y$. Hence either $F \cong \mathbb{P}^1$ or F is an elliptic curve. If $F \cong \mathbb{P}^1$, then for $F' := \sigma^*F$, we have $F'.\sigma^*D = F.(-K_Y) = 2$. This contradicts the extra assumption above.

We still have to consider the case where F is elliptic and $Y \rightarrow B$ is an elliptic fibration.

Claim 3.5. (1) h is étale over $Y \setminus \text{Sing } Y$.

(2) Every fibre of the elliptic fibration $Y \rightarrow B$ is irreducible.

(3) We have $S(Y) = \emptyset$, so $\text{Sing } Y \neq \emptyset$.

(4) The Picard number $\rho(Y) = 2$.

(5) If $\deg(f) = d = e = \deg(h)$, then X is non-Gorenstein and f is étale over $X \setminus \text{Sing } X$, so $dK_X^3 = (f^*K_X)^3 = K_X^3$ and $K_X^3 = 0$.

We now prove Claim 3.5. Replacing f by some power, we may assume that both h and h^{-1} stabilize every negative curve on Y as in Lemma 2.10. (1) is proved in [18, Lemma 6.1.4].

(2) If C is a curve in a reducible fibre, then C is a negative curve and hence $h^*C = bC$ for some $b \in \mathbb{N}$, where $b^2C^2 = (h^*C)^2 = eC^2$ and $b = \sqrt{e} > 1$. Thus $h_B^*P = bP$ for the point P over which lies the curve C . This is impossible because $\deg(h_B) = 1$.

Here is a note before continuing the proof of Claim 3.5: Note that $0 = F.aF = F.(K_Y + D) = F.D$ and hence $D = \sum_{i=1}^s D_i$ with D_i irreducible and the support of a fibre F_i . So $D \equiv wF$ for some $w > 0$. Since $K_{\tilde{Y}}^2 = K_Y^2 = (aF - D)^2 = 0$, the induced elliptic fibration $\tilde{Y} \rightarrow \mathbb{P}^1$ (with a general fibre $\tilde{F}$ say) is relatively minimal, and $\sigma : \tilde{Y} \rightarrow Y$ is just the contraction of (-2) -curves in fibres of $\tilde{Y} \rightarrow B$ into Du Val singularities.

(3) Note that $\tilde{Y} = Y$ implies that there is a (-1) -curve on Y (noting that $\tilde{Y}$ is rational and $K_{\tilde{Y}}^2 = 0$). So we have only to consider the case where there is a negative curve E on Y . Then $h^*E = \sqrt{e}E$ as in (2). Also E is horizontal to the elliptic fibration $Y \rightarrow B$ by (2), so $E.F > 0$. Now $e(E.F) = h^*E.h^*F = \sqrt{e}E.F$ and hence $e = \sqrt{e}$. This is impossible for $e = \deg(h) > 1$ by the assumption.

(4) Since $\deg(h) = e > 1$ and $K_{\tilde{Y}} = \sigma^*K_Y$ (and hence K_Y) are not pseudo-effective, $\rho(Y) \neq 2$ would imply that h is polarized by some ample divisor H so that $h^*H \sim \sqrt{e}H$ as in [18, Theorem 4.4.6]; this leads to $e(H.F) = h^*H.h^*F = \sqrt{e}H.F$, which is absurd.

(5) Note that $d/e = \deg(\ell \rightarrow f(\ell))$ for a general fibre ℓ of $\pi : X \rightarrow Y$. To distinguish, we write $f : X_1 = X \rightarrow X_2 = X$, $h : Y_1 = Y \rightarrow Y_2 = Y$ and $\pi_i : X_i \rightarrow Y_i$ (identical to π) so that $\pi_2 \circ f = h \circ \pi_1$.

Suppose that $d = e$. We assert then that X_1 is isomorphic to the normalization Z of $X_2 \times_{Y_2} Y_1$. Indeed, $\pi_1 : X_1 \rightarrow Y_1$ and $f : X_1 \rightarrow X_2$ factor as $X_1 \rightarrow Z \rightarrow Y_1$ and $X_1 \rightarrow Z \rightarrow X_2$, where $Z \rightarrow Y_1$ and $Z \rightarrow X_2$ are the natural projections. Since $\pi_2 : X_2 \rightarrow Y_2$ has connected fibres and h is finite, Z is irreducible. Note that the map $X_1 \rightarrow Z$ is finite because so is $f : X_1 \rightarrow X_2$. The map $X_1 \rightarrow Z$ is also birational because $\deg(X_1/X_2) = \deg(Y_1/Y_2) = \deg(Z/X_2)$. Thus the map $X_1 \rightarrow Z$ is an isomorphism and we can and will identify $X_1 = Z$.

By (1), $f : X \rightarrow X$ is étale in codimension 1, so we have (*): f is étale over $X \setminus \text{Sing } X$ by the purity of the branch loci. Thus X is non-Gorenstein. Indeed, a Gorenstein 3-dimensional terminal singularity is an isolated hypersurface singularity and hence has trivial local π_1 by a result of Milnor, so that $f^s : X \rightarrow X$ is étale of degree d^s for any $s > 0$ (by (*)), contradicting the fact that the rationally connected variety X has finite π_1 . This proves Claim 3.5.

We continue the proof of Lemma 3.4. Let ρ be the spectral radius of $h^*|N^1(Y)$ and L a non-zero $\mathbb{R}$ -Cartier nef divisor such that $h^*L \equiv \rho L$. Note that $\rho \geq \sqrt{e} > 1$ by Lemma 2.14. So L is not parallel to F and hence $L.F > 0$ by the Hodge index theorem applied to the pullbacks on $\tilde{Y}$. Thus $e(L.F) = h^*L.h^*F = \rho L.F$ and $\rho = e$. Since $eL^2 = (h^*L)^2 = (eL)^2$, we have $L^2 = 0$.

Claim 3.6. (1) We have $\overline{\text{NE}}(Y) = \mathbb{R}_{\geq 0}L + \mathbb{R}_{\geq 0}F$ with L and F extremal rays.

(2) L (replaced by its positive multiple) is an integral Cartier divisor with $h^0(Y, L) \geq 2$ and the Iitaka D -dimension $\kappa(Y, L) = 1$.

(3) A multiple of L is linearly equivalent to a general fibre of a $\mathbb{P}^1$ -fibration $\psi : Y \rightarrow C \cong \mathbb{P}^1$.

We prove Claim 3.6.

(1) Note that $S(Y) = \emptyset$ and $L^2 = 0 = F^2$ imply that L and F are extremal and hence (1) follows because the Picard number $\rho(Y) = 2$. Indeed, for $G = L$ or F , decompose $G \equiv G_1 + G_2$ as a sum of pseudo-effective divisors. Let $G_i \equiv P_i + N_i$ be the Zariski decomposition. Since $S(Y) = \emptyset$, we have $N_i = 0$ and hence $G_i = P_i$ is nef. Now $0 = G^2 = G_1^2 + G_2^2 + 2G_1.G_2$ implies that $G_i.G_j = 0$. Then, by the Hodge index theorem, G_1 is parallel to G_2 and hence both G_i are parallel to G . So both L and F are extremal.

(2) We can choose F and an ample divisor H to be generators of $N^1(Y)$. Replacing L by its multiple, we may write $L = \varepsilon F + uH$ with $\varepsilon = \pm 1$ and $u \in \mathbb{R}$. Now $0 = L^2 = 2\varepsilon uF.H + u^2H^2$ implies that $u = (-2\varepsilon F.H)/H^2 \in \mathbb{Q}$. Thus we may assume that L is integral and Cartier.

By the Riemann-Roch theorem and noting that $L.F > 0$ and $\sigma^*K_Y = K_{\tilde{Y}} = -v\sigma^*F$ for some $v > 0$ due to Kodaira's canonical divisor formula,

$$h^0(Y, L) = h^0(\tilde{Y}, \sigma^*L) \geq \frac{L(L - K_Y)}{2} + 1 > 1.$$

So $\kappa(L) = 1$ because $L^2 = 0$.

(3) For the L in (2), write $|L| = |M| + \text{Fix}$ with Fix the fixed part. Now $0 = L^2 \geq L.M \geq 0$ and $0 = L^2 = L.M + L.\text{Fix} \geq M^2 + L.\text{Fix} \geq M^2 \geq 0$. So $L.M = 0$ and hence L is parallel to M by the Hodge index theorem. Also $M^2 = 0$ and hence $\text{Bs}|M| = \emptyset$. Replacing L by a multiple of M we may assume that L is a fibre of the fibration $\psi : Y \rightarrow C \cong \mathbb{P}^1$ derived from $\Phi_{|M|}$. Note that $K_Y = -vF$. Since $\sigma^*L.K_{\bar{Y}} = L.K_Y = L.(-vF) < 0$, our (3) is true. This proves Claim 3.6.

We continue the proof of Lemma 3.4. By Claim 3.6(3), we may assume that L is a general fibre of ψ . Since the arithmetic genus $p_a(D_i) = 1$, we have $L.D > 0$. Since $h^*L \equiv eL$, our $h : Y \rightarrow Y$ descends to an endomorphism $h_C : C \rightarrow C$ of degree $e > 1$. By [4, Theorem 5.1] and replacing f by some power, we may assume that $h_C(c) = c$ for a general point $c \in C$. Our h induces an endomorphism $h|_{Y_c} : Y_c \rightarrow Y_c$ of $Y_c := \psi^{-1}(c)$, and this endomorphism is an automorphism for $\deg(h) = e = \deg(h_C)$. Let $X_c = \pi^*Y_c$. Since $Y_c \equiv L$ by our choice of L , we have $h^*Y_c \equiv eY_c$. So $f : X \rightarrow X$ induces an endomorphism $f|_{X_c} : X_c \rightarrow X_c$ of degree d/e .

By Claim 3.5, we may assume that $d > e$. Since $r := Y_c.D = L.wF > 0$, our X_c is a smooth ruled surface with a ruling $X_c \rightarrow Y_c \cong \mathbb{P}^1$ and r singular fibres $S_i(1)+S_i(2)$ (two intersecting (-1) -curves) lying over the s points $P_i \in Y_c \cap D$. By Lemma 2.10 and relabelling, we have $(f|_{X_c})^*S_i(1) = \sqrt{d/e}S_j(1)$. Thus $(h|_{Y_c})^*P_i = \sqrt{d/e}P_j$ as 0-cycles. This is impossible for $\deg(h|_{Y_c}) = 1$. Lemma 3.4 is proved. $\square$

Lemma 3.7. *Let X be a $\mathbb{Q}$ -factorial rationally connected projective threefold with only terminal singularities, and $\pi : X \rightarrow Y$ an extremal contraction with $\dim Y = 1$. Suppose that $f : X \rightarrow X$ is an endomorphism of degree $d > 1$. Then either X is rational; or X is non-Gorenstein, f is étale over $X \setminus \text{Sing } X$, and $K_X^3 = 0$.*

We now prove Lemma 3.7. Since the Picard number $\rho(X) = \rho(Y) + 1 = 2$, we may assume that f preserves each of the two extremal rays of $\overline{\text{NE}}(X)$, so it descends to a surjective endomorphism $h : Y \rightarrow Y$ after replacing f by its square.

Claim 3.8. Suppose that h is an automorphism. Then X is rational.

Proof. Choose a general $y = y_0 \in Y$ such that $y_i := h^i(y_0)$ is not in $D(X/Y)$ or $\pi(\text{Sing } X)$ for all $i \geq 0$. Set $X_i := \pi^*(y_i)$ which is a (smooth) del Pezzo surface. Then $f^*X_{i+1} = X_i$ and the restriction $f_i = f|_{X_i} : X_i \rightarrow X_{i+1}$ is a finite surjective morphism of degree d . By Theorem 2.3, we may assume that $K_{X_i}^2 = K_{X_0}^2 \leq 4$ for all $i \geq 0$. Note that the Picard number $\rho(X_i) = \rho(X_0)$. We shall reach a contradiction later.

Let N_i be the union of all negative curves (i.e., (-1) -curves) on X_i . By Lemma 2.10, both f and f^{-1} induce natural bijections (inverse to each other) between N_i and N_{i+1} . Indeed, $f_i^*N_{i+1}(j) = \sqrt{d}N_i(j)$ with $\deg(f_i) = d$ if we label $N_i = \sum_j N_i(j)$ such that $f_i^{-1}N_{i+1}(j) = N_i(j)$. Set $K_i = K_{X_i}$. Then $K_i + N_i = f_i^*(K_{i+1} + N_{i+1}) + G_i$, where G_i is an effective divisor having no common components with N_i . By iterating, for all $s > 0$, we have:

$$K_0 + N_0 = (f_s \circ f_{s-1} \circ \cdots \circ f_0)^*(K_{s+1} + N_{s+1}) + G_0 + \sum_{i=1}^s (f_{i-1} \circ \cdots \circ f_0)^*G_i.$$

Since $K_i^2 \leq 4$, our N_i is a loop N'_i plus a positive divisor N''_i , so both $K_i + N'_i$ and $K_i + N_i$ are pseudo-effective (see [3, Lemma 2.3]); indeed, a del Pezzo surface of degree 6 contains a loop of (-1) -curves. Multiplying the above displayed equality by

an ample divisor and letting $s \rightarrow \infty$, we see that $G_i = 0$ for almost all i . Relabelling X_i , we may assume that $G_i = 0$ for all $i \geq 0$. Thus, denoting $g_s := f_s \circ f_{s-1} \circ \cdots \circ f_0$, we have

$$\begin{aligned} K_0 + N_0 &= g_s^*(K_{s+1} + N_{s+1}) = g_s^*(K_{s+1} + N'_{s+1}) + g_s^*N''_{s+1} \\ &= g_s^*(K_{s+1} + N'_{s+1}) + d^{(s+1)/2} g_s^{-1} N''_{s+1}. \end{aligned}$$

Multiplying the equality by an ample divisor and letting $s \rightarrow \infty$, we get a contradiction. $\square$

By the above claim, we may assume that $\deg(h) > 1$. Let $y_0 \in Y$ be a general h -periodic point which is not contained in $\pi(\text{Sing } X) \cup D(X/Y)$ or the branch locus of h (see [4, Theorem 5.1]). We may assume that $h(y_0) = y_0$ after replacing f by some power. Set $X_0 := \pi^*(y_0)$ which is a (smooth) del Pezzo surface. Then the restriction $f_0 = f|_{X_0} : X_0 \rightarrow X_0$ is a finite morphism of degree $d_0 := d/\deg(h)$. If $d_0 > 1$, then $K_{X_0}^2 \geq 6$ by [17] or [22, Theorem 3], so X is rational by Theorem 2.3.

Therefore, we may assume that $d_0 = 1$ and $\deg(h) = \deg(f) = d$. So $f^*F \equiv dF$ for a general fibre F of π . To distinguish, we write $f : X_1 = X \rightarrow X_2 = X$, $h : Y_1 = Y \rightarrow Y_2 = Y$ and $\pi_i : X_i \rightarrow Y_i$ (identical to π) so that $\pi_2 \circ f = h \circ \pi_1$. As in the proof of Claim 3.5, X_1 is isomorphic to the normalization Z of $X_2 \times_{Y_2} Y_1$.

In particular, the ramification divisor R_f is supported on fibres and we can write $R_f \equiv bF$, noting that every fibre is irreducible for $\rho(X/Y) = 1$. If $b = 0$, i.e. $R_f = 0$, then $K_X^3 = 0$ and X is non-Gorenstein, as in the proof of Claim 3.5. Thus we may assume that $b > 0$.

Since the Picard number $\rho(X) = 2$, we have $\text{Nef}(X) = \mathbb{R}_{\geq 0}L_1 + \mathbb{R}_{\geq 0}L_2$ with L_i extremal rays. We assert that a general fibre F of π is an extremal ray in $\text{Nef}(X)$. Indeed, if $F = G_1 + G_2$ is the decomposition into nef divisors, then $0 = \ell.F = \ell.G_1 + \ell.G_2$ for a curve ℓ in F and hence $\ell.G_i = 0$, so $G_i \in \pi^*N^1(Y)$ and G_i is parallel to F (see [9, Lemma 3-2-5]). The assertion is proved. So we may assume that $L_1 = F$ and hence $f^*L_1 \equiv dL_1$. Also $L_1^2 = F^2 = 0$ (for later use). We can also write $f^*L_2 \equiv \lambda L_2$ for some $\lambda > 0$.

Write $K_X \equiv a_1L_1 + a_2L_2$. Since $-K_X|_F \equiv -a_2L_2|_F$ is ample we have $a_2 < 0$. Now

$$K_X = f^*K_X + R_f \equiv (a_1dL_1 + a_2\lambda L_2) + bL_1.$$

Comparing coefficients of L_i , we get

$$a_1 = \frac{b}{1-d} < 0, \quad \lambda = 1.$$

So $-K_X$ is an interior point in $\text{Nef}(X)$ and hence ample. Also $f^*[N^1(X)]$ can be diagonalized as $\text{diag}[d, 1]$ with respect to the basis $\{L_1, L_2\}$. Write $\overline{\text{NE}}(X) = \mathbb{R}_{\geq 0}\ell_1 + \mathbb{R}_{\geq 0}\ell_2 \subset N_1(X)$. Since $N_1(X)$ is dual to $N^1(X)$, we have $B = \deg(f)(A^T)^{-1}$ if A (resp. B) is the matrix representation of $f^*[N^1(X)]$ (resp. $f^*[N_1(X)]$) with respect to some basis. Thus $f^*\ell_i \equiv r_i\ell_i$, where $\{r_1, r_2\} = \{1, d\}$. We may assume that ℓ_1 is contained in F so that π is the contraction of the extremal ray ℓ_1 . Then $\ell_1.L_1 = 0$, so $\ell_1.L_2 > 0$. Thus $d(\ell_1.L_2) = f^*\ell_1.f^*L_2 = r_1\ell_1.L_2$, whence $r_1 = d$. So $r_2 = 1$. Therefore, we have

$$f^*L_1 \equiv dL_1, \quad f^*L_2 \equiv L_2, \quad f^*\ell_1 \equiv d\ell_1, \quad f^*\ell_2 \equiv \ell_2.$$

$d(\ell_2.L_2) = f^*\ell_2.f^*L_2 = \ell_2.L_2$ implies that $\ell_2.L_2 = 0$. Thus we have

$$\ell_i.L_i = 0 \quad (i = 1, 2), \quad \ell_i.L_j > 0 \quad (i \neq j).$$

Let $\pi_2 : X \rightarrow Y_2$ be the contraction of the (necessarily K_X -negative) extremal ray generated by ℓ_2 . Then f descends to an endomorphism $h_2 : Y_2 \rightarrow Y_2$ and the Picard number $\rho(Y_2) = \rho(X) - 1 = 1$. Hence $\dim Y_2 \neq 0$. If $\dim Y_2 = 2$, the result follows from Lemmas 3.2 and 3.4.

Consider the case $\dim Y_2 = 1$. Since $\ell_2.L_2 = 0$, L_2 is parallel to a fibre of π_2 . Hence $L_2^2 = 0$. Take an ample H and write $H = u_1 L_1 + u_2 L_2$ with $u_i > 0$. Then $H^3 = 0$ for $L_i^2 = 0$, which is absurd!

Consider the case $\dim Y_2 = 3$ so that π_2 is birational. Let H be an ample generator of $\text{Pic}(Y_2)$. Then $\pi_2^* H$ is extremal in $\text{Nef}(X)$ as in the proof for the extremality of $L_1 = F$ above and since $\rho(Y_2) = \rho(X) - 1 = 1$. Hence we may assume that $L_2 = \pi_2^* H$ because $\ell_2.\pi_2^* H = 0$ and $\ell_2.L_1 > 0$ imply that $\pi_2^* H$ is not parallel to L_1 . Set $h_2^* H = \alpha H$. Then $\alpha = 1$ because $L_2 \equiv f^* L_2 = f^* \pi_2^* H = \pi_2^* h_2^* H = \alpha L_2$. Thus $\deg(h_2) = (h_2^* H)^3 / H^3 = 1$, whence $\deg(f) = 1$ for h_2 is the descent of f . This is a contradiction. This proves Lemma 3.7.

4. PROOFS OF THEOREMS AND COROLLARIES

In this section we shall prove Theorems 1.1, 1.2 and 1.3 and their corollaries below. In fact, Theorem 1.1 is a consequence of Lemmas 3.2, 3.4 and 3.7.

Corollary 4.1. *Let X be a $\mathbb{Q}$ -factorial projective threefold with only terminal singularities and a surjective endomorphism $f : X \rightarrow X$ of degree $q^3 > 1$. Assume either one of the following three conditions:*

- (1) X is Gorenstein.
- (2) $K_X^3 \neq 0$.
- (3) The ramification divisor R_f is non-zero.

Assume further that $() : X$ is rationally connected and $-K_X$ is big. Then X is rational, unless $(f^s)^*|N^1(X) = q^s \text{id}$ for some integer $s \geq 1$ (and hence f is polarized).*

The hypothesis $(*)$ in Corollary 4.1 above is satisfied if (X, Θ) is klt weak $\mathbb{Q}$ -Fano for some Θ (cf. Lemma 2.11).

Corollary 4.2. *With the situation as in Theorem 1.3(3), suppose that either $\dim Y = 0$ and X_t is smooth, or $\dim Y \in \{1, 2\}$ and at least one X_i ($0 \leq i \leq t$) satisfies either one of the following three conditions:*

- (1) X_i is Gorenstein.
- (2) $K_{X_i}^3 \neq 0$.
- (3) The ramification divisor R_{f_i} is non-zero.

Suppose further that X is rationally connected. Then X is rational.

Remark 4.3. (1) In the situation of Theorem 1.3, the ramification divisor $R_{f_k} = 0$ for some $k \in \{0, 1, \dots, t\}$, if and only if $R_{f_i} = 0$ for all $i \in \{0, 1, \dots, t\}$, if and only if f_i is étale over $X_i \setminus \text{Sing } X_i$; see Lemma 2.18.

- (2) By the proof of Claim 3.5(5), in Theorem 1.1 and Corollaries 4.1 and 4.2, we have:

condition(1) $\Rightarrow$ condition(3); condition(2) $\Rightarrow$ condition(3).

Remark 4.4. Some remarks on Theorem 1.3 related to the building blocks of endomorphisms:

- (1) Every X_i ($0 \leq i \leq t$) is $\mathbb{Q}$ -factorial and has only terminal singularities and big $-K_{X_i}$; see [9] or [10, Proposition 3.37, Corollary 3.42], and Theorem 2.16(2).
- (2) Note that $-K_X$ is big and X is rationally connected, whenever (X, Θ) is klt weak $\mathbb{Q}$ -Fano for some Θ ; see Lemma 2.11.
Now consider the case $\dim Y \in \{1, 2\}$ in Theorem 1.3(3).
- (3) Every general fibre of $X_t \rightarrow Y$ is $\mathbb{P}^1$ (when $\dim Y = 2$) or a smooth Fano surface = del Pezzo surface (when $\dim Y = 1$), since X_t has only terminal singularities.
- (4) Suppose further that X is rationally connected. If $\deg(h : Y \rightarrow Y) \geq 2$ and either the Picard number $\rho(Y) \neq 2$ or $f_t^*|N^1(X_t) = q \text{ id}$, then h is polarized (cf. [18, Theorem 4.4.6] and [25, Lemma 2.2]), and hence the set of h -periodic points is dense in Y by [4, Theorem 5.1], so there is a surjective endomorphism $h|_{X_y} : X_y \rightarrow X_y$ for some general $y \in Y$, after h is replaced by some power. Here $X_y \subset X$ is the fibre over y and is $\mathbb{P}^1$ or del Pezzo.

The remark below gives sufficient conditions to descend f to a surjective endomorphism $h : Y \rightarrow Y$ as in Theorems 1.1 and 1.3.

Remark 4.5. Let $\pi : X \rightarrow Y$ be a K_X -negative extremal contraction and $f : X \rightarrow X$ a surjective endomorphism. Then f , replaced by some power, descends to a surjective endomorphism $h : Y \rightarrow Y$ if any one of the following seven conditions is satisfied.

- (1) X is a klt weak $\mathbb{Q}$ -Fano variety.
- (2) (X, Θ) is a klt weak $\mathbb{Q}$ -Fano variety for some Θ .
- (3) The cone $\overline{\text{NE}}(X)$ has only finitely many extremal rays (K_X negative or non-negative).
- (4) $\dim Y \leq 1$.
- (5) The Picard number $\rho(X) \leq 2$.
- (6) X is a $\mathbb{Q}$ -factorial projective threefold with only terminal singularities and big $-K_X$; the degree $\deg(f) \geq 2$.
- (7) $\dim Y < \dim X$; see [25, Theorem 2.13, or Appendix].

Proof of Remark 4.5. The situation in Remark 4.5(6) is done by Theorem 2.16. If $\dim Y \leq 1$, then $\rho(X) = \rho(Y) + 1 \leq 2$. If $\rho(X) \leq 2$, then $\overline{\text{NE}}(X)$ has at most two extremal rays. Thus by Lemma 2.11, we have only to consider the situation in Remark 4.5(3). Our $f^\pm$ permutes extremal rays of $\overline{\text{NE}}(X)$; see [25, Lemma 2.11]. Hence $f^{\pm s}$ stabilizes every such extremal ray for some $s > 1$. So Remark 4.5 is true (cf. [25, Lemma 3.6] for the flip case). $\square$

Proof of Theorem 1.2. Since X is Fano, it is rationally connected and has only finitely many extremal rays $\mathbb{R}_{\geq 0}[C_i]$ all of which are K_X -negative; see Lemma 2.11. Let $X \rightarrow X_1$ be the composite of blowdowns between smooth threefolds such that X_1 is a primitive smooth Fano threefold in the sense of [15]. Replacing f by some power, we may assume that f descends to a surjective morphism $f_1 : X_1 \rightarrow X_1$. This is because every smooth X' appearing in between X and X_1 is obtained by contracting a K -negative extremal ray, the cone $\overline{\text{NE}}(X')$ is generated by finitely

many extremal rays (the images of $\mathbb{R}_{\geq 0}[C_i]$) and a finite morphism permutes these rays; see [25, Lemma 2.11]. If $\rho(X_1) = 1$, then X_1 is a smooth Fano threefold of Picard number one having an endomorphism $f_1 : X_1 \rightarrow X_1$ with $\deg(f_1) = \deg(f) > 1$, so $X_1 \cong \mathbb{P}^3$ by [1] or [6], and we are done! If $\rho(X_1) \geq 2$, by [15, Theorem 5], X_1 has an extremal contraction of conic bundle type, so X_1 is rational by Theorem 1.1. This proves Theorem 1.2. $\square$

Proof of Theorem 1.3. Let $X = X_0 \cdots \rightarrow X_1 \cdots \rightarrow X_t$ be a composition of K_{X_i} flips and divisorial contractions so that there is no K_{X_t} -negative extremal contraction of birational type.

Claim 4.6. For every $0 \leq i \leq t$, the X_i is $\mathbb{Q}$ -factorial uniruled with only terminal singularities, $-K_{X_i}$ is big, K_{X_i} is not nef, and there is a K_{X_i} -negative extremal contraction.

We prove Claim 4.6. The bigness of $-K_X$ implies that of $-K_{X_i}$; see Theorem 2.16. So we can write $-K_{X_i} = A_i + G_i$ with A_i an ample $\mathbb{Q}$ -divisor and G_i an effective $\mathbb{Q}$ -divisor. Thus X_i is uniruled by [15, Theorem 1]. For the other parts, see [10, Proposition 3.37, Corollary 3.42, Theorem 3.7]. This proves Claim 4.6.

We return to the proof of Theorem 1.3. Let $X_t \rightarrow X_{t+1} = Y$ be a K_{X_t} -negative extremal contraction. Then $\dim Y \leq 2$ by the choice of X_t . Theorem 1.3 (1) and (2) follow from Theorem 2.16. For Theorem 1.3 (3), if $\dim Y = 0$, then $\rho(X_t) = 1$ (so $f_t^*[N^1(X_t)]$ is a scalar) and hence $-K_{X_t}$ is ample because K_{X_t} is not nef. Thus (3) follows from Lemma 2.15. (4) is proved in Lemma 2.12. This proves Theorem 1.3. $\square$

Proof of Corollary 4.2. Note that $\deg(f_t) = \deg(f) > 1$. If $\dim Y = 0$, by the assumption, X_t is a smooth Fano threefold with a surjective endomorphism f_t of degree > 1 , so $X_t \cong \mathbb{P}^3$ by [1] or [6], and we are done! Consider the case $\dim Y \in \{1, 2\}$. By the proof of Theorem 1.1 (in three lemmas), we may assume that $R_{f_t} = 0$. Then $R_{f_i} = 0$ for all $i \in \{0, 1, \dots, t\}$ by Lemma 2.18. This proves Corollary 4.2 (cf. Remark 4.3). $\square$

Proof of Corollary 4.1. In the notation of Theorem 1.3, we may assume that $\dim Y \in \{1, 2\}$; see also the first paragraph in the proof of Lemma 2.15. Now Corollary 4.1 follows from Corollary 4.2. $\square$

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REFERENCES

- [1] E. Amerik, M. Rovinsky and A. Van de Ven, A boundedness theorem for morphisms between threefolds. *Ann. Inst. Fourier (Grenoble)* **49** (2) (1999) 405–415. MR1697369 (2000f:14056)
- [2] G. Birkhoff, Linear transformations with invariant cones, *Amer. Math. Monthly* **74** (1967) 274–276. MR0214605 (35:5454)
- [3] J. A. Chen, M. Chen and D.-Q. Zhang, A non-vanishing theorem for $\mathbb{Q}$ -divisors on surfaces, *J. Algebra* **293** (2) (2005) 363–384. MR2172344 (2006g:14014)

- [4] N. Fakhruddin, Questions on self maps of algebraic varieties, *J. Ramanujan Math. Soc.* **18** (2) (2003) 109–122. MR1995861 (2004f:14038)
- [5] C. Favre, Holomorphic self-maps of singular rational surfaces, *Publ. Mat.* **54** (2010), no. 2, 389–432. MR2675930
- [6] J. -M. Hwang and N. Mok, Finite morphisms onto Fano manifolds of Picard number 1 which have rational curves with trivial normal bundles, *J. Alg. Geom.* **12** (2003) 627–651. MR1993759 (2005f:14086)
- [7] V. A. Iskovskikh, On the rationality problem for conic bundles, *Duke Math. J.* **54** (1987), no. 2, 271–294. MR899398 (88i:14038)
- [8] V. A. Iskovskikh, On the rationality problem for algebraic threefolds, *Proc. Steklov Inst. Math.* **218** (3) (1997) 186–227. MR1642381 (99g:14002)
- [9] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, *Algebraic geometry, Sendai, 1985* (T. Oda, ed.), *Adv. Stud. Pure Math.*, **10**, Kinokuniya and North-Holland, 1987, pp. 283–360. MR946243 (89e:14015)
- [10] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., **134**, Cambridge Univ. Press, 1998. MR1658959 (2000b:14018)
- [11] J. Kollár and C. Xu, Fano Varieties with Large Degree Endomorphisms, arXiv:0901.1692.
- [12] C. McMullen, Dynamics on $K3$ surfaces: Salem numbers and Siegel disks, *J. Reine Angew. Math.* **545** (2002) 201–233. MR1896103 (2003a:37057)
- [13] M. Miyanishi, Algebraic methods in the theory of algebraic threefolds—surrounding the works of Iskovskikh, Mori and Sarkisov, *Algebraic varieties and analytic varieties (Tokyo, 1981)*, pp. 69–99, *Adv. Stud. Pure Math.*, Vol. **1**, North-Holland, Amsterdam, 1983. MR715647 (85b:14054)
- [14] S. Mori, Threefolds whose canonical bundles are not numerically effective, *Ann. of Math. (2)* **116** (1982), no. 1, 133–176. MR662120 (84e:14032)
- [15] S. Mori and S. Mukai, Classification of Fano 3-folds with $B_2 \geq 2$, *Manuscripta Math.* **36** (1981/82), no. 2, 147–162; Erratum, *Manuscripta Math.* **110** (2003), no. 3, 407. MR641971 (83f:14032); MR1969009
- [16] S. Mori and Y. Prokhorov, On $\mathbb{Q}$ -conic bundles, *Publ. Res. Inst. Math. Sci.* **44** (2008), no. 2, 315–369. MR2426350 (2009e:14062)
- [17] N. Nakayama, Ruled surfaces with non-trivial surjective endomorphisms, *Kyushu J. Math.* **56** (2002) 433–446. MR1934136 (2003j:14050)
- [18] N. Nakayama, On complex normal projective surfaces admitting non-isomorphic surjective endomorphisms, Preprint 2 September 2008.
- [19] N. Nakayama and D. -Q. Zhang, Building blocks of étale endomorphisms of complex projective manifolds, *Proc. Lond. Math. Soc. (3)* **99** (2009), no. 3, 725–756. MR2551469
- [20] N. Nakayama and D. -Q. Zhang, Polarized endomorphisms of complex normal varieties, *Math. Ann.* **346** (2010), no. 4, 991–1018. MR2587100
- [21] Yu. G. Prokhorov and V. V. Shokurov, Towards the second main theorem on complements, *J. Algebraic Geom.* **18** (2009) 151–199. MR2448282 (2009i:14007)
- [22] D. -Q. Zhang, On endomorphisms of algebraic surfaces. *Topology and geometry: commemorating SISTAG*, pp. 249–263, *Contemp. Math.*, Vol. **314**, Amer. Math. Soc., Providence, RI 2002. MR1941637 (2004c:14070)
- [23] D. -Q. Zhang, Automorphism groups and anti-pluricanonical curves, *Math. Res. Lett.* **15** (2008), no. 1, 163–183. MR2367182 (2009f:14072)
- [24] D. -Q. Zhang, Dynamics of automorphisms of compact complex manifolds, *Proceedings of The Fourth International Congress of Chinese Mathematicians (ICCM2007)*, Vol. **II**, pp. 678–689; also: arXiv:0801.0843.
- [25] D. -Q. Zhang, Polarized endomorphisms of uniruled varieties, With an appendix by Y. Fujimoto and N. Nakayama, *Compos. Math.* **146** (2010), no. 1, 145–168. MR2581245
- [26] Q. Zhang, Rational connectedness of log $\mathbb{Q}$ -Fano varieties, *J. Reine Angew. Math.* **590** (2006) 131–142. MR2208131 (2006m:14021)

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076, SINGAPORE
E-mail address: matzdzq@nus.edu.sg