POINTWISE DIMENSION, ENTROPY AND LYAPUNOV EXPONENTS FOR C^1 MAPS

WEN HUANG AND PENGFEI ZHANG

ABSTRACT. Let f be a C^1 self-map on a smooth Riemannian manifold M , μ be an f -invariant ergodic Borel probability measure with a compact support Λ and $\chi_\mu^1 > \cdots > \chi_\mu^s$ be the Lyapunov exponents of μ with respect to f . If $\chi_\mu^1 > 0$, then we give a lower bound of the lower pointwise dimension of μ in terms of χ_μ^1 and of the entropy $h_\mu(f)$. Moreover, if $Df_{\{\cdot\}}$ is non-degenerate on Λ and $\chi_\mu^s > 0$, then we give an upper bound of the upper pointwise dimension of μ in terms of χ_μ^s and of the entropy $h_\mu(f)$. Furthermore, if f is $C^{1+\alpha}$ for some $\alpha > 0$, then the non-degeneracy condition can be removed.

As direct applications of the above results, we also give the corresponding lower and upper bounds of some classical characteristics of dimensional type of μ in terms of the Lyapunov exponents and of the entropy $h_\mu(f)$.

1. INTRODUCTION

In the study of the interrelation between dimension, entropy and Lyapunov exponents, Young introduced in [19] the notation of pointwise dimension. Generally, for a Borel measure m on a complete metric space X , the *lower* and *upper pointwise dimensions* of $x \in X$ with respect to m are defined by

$$\underline{d}_m(x) = \liminf_{r \rightarrow 0} \frac{\log m(B(x, r))}{\log r} \text{ and } \bar{d}_m(x) = \limsup_{r \rightarrow 0} \frac{\log m(B(x, r))}{\log r},$$

where $B(x, r)$ is the ball centered at x of radius r . If $\underline{d}_m(x) = \bar{d}_m(x)$ at $x \in X$, we will use $d_m(x)$ to denote this common value as the *pointwise dimension* at x with respect to m . The lower and upper pointwise dimensions characterize the local geometrical structure of m with respect to the metric on X , and are important tools in the dimension theory of dynamical systems (see Framer, Ott and Yorke [5], Eckmann and Ruelle [4], Ledrappier and Young [11], Young [19, 20, 21], Barreira, Pesin and Schmeling [2] and Barreira and Wolf [3] for example).

In this paper we will investigate the interrelations between pointwise dimensions, entropy and Lyapunov exponents of ergodic Borel probability measures for C^1 self-maps on smooth Riemannian manifolds. One of our main results can be stated as follows.

Theorem 1. *Let f be a C^1 self-map on a smooth Riemannian manifold M , and μ be an f -invariant ergodic Borel probability measure with a compact support Λ .*

Received by the editors August 30, 2009 and, in revised form, November 30, 2010.

2010 *Mathematics Subject Classification.* Primary 37C45; Secondary 37D25.

Key words and phrases. Pointwise dimension, local entropy, Lyapunov exponents, ergodic measures.

The first author was supported by the NSFC, the Fok Ying Tung Education Foundation, FANEDD (Grant 200520) and the Fundamental Research Funds for the Central Universities.

- (1) If $\chi_\mu^1 > 0$, then $\underline{d}_\mu(x) \geq \frac{h_\mu(f)}{\chi_\mu^1}$ for μ -a.e. $x \in M$.
- (2) Suppose f is non-degenerate on Λ . If $\chi_\mu^s > 0$, then $\frac{h_\mu(f)}{\chi_\mu^1} \leq \underline{d}_\mu(x) \leq \overline{d}_\mu(x) \leq \frac{h_\mu(f)}{\chi_\mu^s}$ for μ -a.e. $x \in M$.

Note that for the ergodic measure μ , if $\chi_\mu^1 = 0$, then by Ruelle's inequality [17] we have $h_\mu(f) = 0$. Hence the conclusion $\underline{d}_\mu(x) \cdot \chi_\mu^1 \geq h_\mu(f)$ holds trivially.

The pointwise dimensions are closely related to some other characteristics of dimensional type of the Borel measure m on a complete metric space X . More precisely, let $Z \subset X$ be a subset of X , $\text{HD}(Z)$ be the Hausdorff dimension of Z , and $\underline{\text{BD}}(Z)$ and $\overline{\text{BD}}(Z)$ be the lower and upper Box dimensions of Z . Then the Hausdorff dimension and the lower and upper Box dimensions of the measure m are defined respectively by

$$\begin{aligned}\text{HD}(m) &= \inf\{\text{HD}(Z) : Z \subset M \text{ and } m(M \setminus Z) = 0\}, \\ \underline{\text{BD}}(m) &= \liminf_{\delta \rightarrow 0} \{\underline{\text{BD}}(Z) : Z \subset M \text{ and } m(M \setminus Z) < \delta\} \text{ and} \\ \overline{\text{BD}}(m) &= \liminf_{\delta \rightarrow 0} \{\overline{\text{BD}}(Z) : Z \subset M \text{ and } m(M \setminus Z) < \delta\}.\end{aligned}$$

To introduce some other characteristics we need the notion of entropy. That is, for any finite measurable partition β of X , the entropy of β with respect to m is defined by

$$H_m(\beta) = \sum_{B \in \beta} -m(B) \log m(B).$$

For $\epsilon > 0$, let

$$H_m(\epsilon) = \inf\{H_m(\beta) : \beta \text{ is a finite measurable partition of } X \text{ and } \max_{B \in \beta} \text{diam}(B) \leq \epsilon\}.$$

Then the lower and upper information dimensions of m are defined respectively by

$$\underline{\text{ID}}(m) = \liminf_{\epsilon \rightarrow 0} \frac{H_m(\epsilon)}{-\log \epsilon} \text{ and } \overline{\text{ID}}(m) = \limsup_{\epsilon \rightarrow 0} \frac{H_m(\epsilon)}{-\log \epsilon}.$$

In [9] Ledrappier considered another characteristic of dimensional type of m as

$$\overline{\text{LD}}(m) = \lim_{\delta \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \frac{\log N(\epsilon, \delta, m)}{-\log \epsilon},$$

where $N(\epsilon, \delta, m)$ is the minimum number of balls of diameter ϵ covering a subset in X of measure greater than $1 - \delta$. The corresponding lower dimension can be defined similarly by $\underline{\text{LD}}(m) = \lim_{\delta \rightarrow 0} \liminf_{\epsilon \rightarrow 0} \frac{\log N(\epsilon, \delta, m)}{-\log \epsilon}$. All above characteristics of dimensional type of m are closely related to pointwise dimensions. In fact, Young established a useful criterion in [19] which says that if $\underline{d}_m(x) \geq C_1$ and $\overline{d}_m(x) \leq C_2$ for m -a.e. x , then $C_1 \leq D(m) \leq C_2$, where $D(m)$ stands for each above characteristic.

There is an extensive literature discussing the relations of these characteristics of invariant measures and some other dynamical characteristics, for example, the measure-theoretic entropy and Lyapunov exponents of the measures if the measures are preserved by some self-maps. In [9] Ledrappier proved that for a C^1 self-map f on M and an f -invariant ergodic Borel probability measure μ with a compact support, if $\chi_\mu^1 > 0$, then $\overline{\text{LD}}(\mu) \geq \frac{h_\mu(f)}{\chi_\mu^1}$. In [6] Hu showed, among other results, that

for a C^2 map f on M with a compact invariant set Λ on which f is expanding, and an f -invariant ergodic Borel probability measure μ on Λ , $\frac{h_\mu(f)}{\chi_\mu^1} \leq D(\mu) \leq \frac{h_\mu(f)}{\chi_\mu^s}$, where $D(\mu)$ stands for $\text{HD}(\mu)$, $\underline{\text{LD}}(\mu)$, $\overline{\text{LD}}(\mu)$, $\underline{\text{BD}}(\mu)$ or $\overline{\text{BD}}(\mu)$. We also get similar results to these characteristics. In fact, the following result is a consequence of Theorem 1 by Young's criterion.

Theorem 2. *Let f be a C^1 self-map on a smooth Riemannian manifold M , and μ be an f -invariant ergodic Borel probability measure with a compact support Λ .*

(1) *If $\chi_\mu^1 > 0$, then $D(\mu) \geq \frac{h_\mu(f)}{\chi_\mu^1}$.*

(2) *Suppose f is non-degenerate on Λ . If $\chi_\mu^s > 0$, then $\frac{h_\mu(f)}{\chi_\mu^1} \leq D(\mu) \leq \frac{h_\mu(f)}{\chi_\mu^s}$,*

where $D(\mu)$ stands for $\text{HD}(\mu)$, $\underline{\text{LD}}(\mu)$, $\overline{\text{LD}}(\mu)$, $\underline{\text{ID}}(\mu)$, $\overline{\text{ID}}(\mu)$, $\underline{\text{BD}}(\mu)$ or $\overline{\text{BD}}(\mu)$.

Young proved a sharp result (see [19, Corollary 5.1]) that if $f : M \rightarrow M$ is a $C^{1+\alpha}$ diffeomorphism and μ is a hyperbolic ergodic Borel probability measure, then

$$(1.1) \quad h_\mu(f) \cdot \left(\frac{1}{\chi_\mu^1} - \frac{1}{\chi_\mu^s} \right) \leq \text{HD}(\mu) \leq h_\mu(f) \cdot \left(\frac{1}{\chi_\mu^k} - \frac{1}{\chi_\mu^{k+1}} \right),$$

where $k = \max\{i \geq 1 : \chi_\mu^i > 0\}$. Young's approaches use some typical $C^{1+\alpha}$ tools such as *Lyapunov charts* and *Pesin theory*. We do not know whether (1.1) holds for C^1 maps.

In order to obtain the upper bound of the upper pointwise dimension of μ in terms of χ_μ^s and the entropy $h_\mu(f)$ in Theorem 1, we require the non-degeneracy condition for only the C^1 self-map f . In this case, the non-degeneracy condition is used to give some estimates of the distortion of the differential Df_x (see Lemma 2.2). This is crucial for an estimate of the upper bound of the sizes of Bowen balls in the proof of Theorem 1, which is no longer meaningful if there exist some degenerate points of the C^1 self-map f .

When f is a $C^{1+\alpha}$ self-map on M for some $\alpha > 0$, we can give a better estimate of the distortion of the differential Df_x (see Lemma 4.1). This gives rise to a positive function $\hat{\epsilon}$ such that for a non-degenerate point x , $\hat{\epsilon}(x)$ decreases *mildly* to 0 as x gets close to some degenerate point. Then rescaling Bowen balls by the function $\hat{\epsilon}$ and applying the *rescaled version* of Brin-Katok local entropy formula [12, 19, 1], we can remove the non-degeneracy condition of f and obtain the following result.

Theorem 3. *Let f be a $C^{1+\alpha}$ self-map on a smooth Riemannian manifold M , and μ be an f -invariant ergodic Borel probability measure with a compact support Λ . If $\chi_\mu^s > 0$, then*

(1) *$\frac{h_\mu(f)}{\chi_\mu^1} \leq \underline{d}_\mu(x) \leq \overline{d}_\mu(x) \leq \frac{h_\mu(f)}{\chi_\mu^s}$ for μ -a.e. $x \in M$, and*

(2) *$\frac{h_\mu(f)}{\chi_\mu^1} \leq D(\mu) \leq \frac{h_\mu(f)}{\chi_\mu^s}$, where $D(\mu)$ stands for $\text{HD}(\mu)$, $\underline{\text{LD}}(\mu)$, $\overline{\text{LD}}(\mu)$, $\underline{\text{ID}}(\mu)$, $\overline{\text{ID}}(\mu)$, $\underline{\text{BD}}(\mu)$ or $\overline{\text{BD}}(\mu)$.*

Recall that a measure μ is *exact dimensional* if $d_\mu(x) = d$ for μ -a.e. $x \in X$. In this case we denote $\dim(\mu) = d$. Also, μ is called *conformal* if all the Lyapunov exponents of μ coincide. In this case we denote the common value by χ_μ . In [14] Manning proved that $\text{HD}(\mu) = \frac{h_\mu(f)}{\chi_\mu}$, where f is an analytic self-map on the complex plane $\mathbb{C}$ with a compact invariant Julia set $J(f)$ on which $f'(z) \neq 0$, and μ is an f -invariant ergodic Borel probability measure on $J(f)$ with the Lyapunov

exponent $\chi_\mu > 0$. In [13] Mañé showed that if f is a rational function on the Riemann sphere $\widehat{\mathbb{C}}$ and μ is an f -invariant ergodic Borel probability measure of f with $\chi_\mu > 0$, then $\dim(\mu) = \frac{h_\mu(f)}{\chi_\mu}$. In [10] Ledrappier and Misiurewicz proved that, for any C^1 map $f : [0, 1] \rightarrow [0, 1]$, if f and f' are piecewise monotonic and μ is an f -invariant ergodic Borel probability measure with $\chi_\mu > 0$, then $\dim(\mu) = \frac{h_\mu(f)}{\chi_\mu}$. Here we get similar results which are direct applications of Theorems 1, 2 and 3.

Corollary 4. *Let f be a C^1 self-map on a smooth Riemannian manifold M and μ be an f -invariant ergodic Borel probability measure with a compact support Λ on which f is non-degenerated. If μ is conformal and $\chi_\mu > 0$, then μ is an exact dimensional measure and $\dim(\mu) = \frac{h_\mu(f)}{\chi_\mu}$. Moreover, $D(\mu) = \frac{h_\mu(f)}{\chi_\mu}$, where $D(\mu)$ stands for $\text{HD}(\mu)$, $\underline{\text{LD}}(\mu)$, $\overline{\text{LD}}(\mu)$, $\underline{\text{ID}}(\mu)$, $\overline{\text{ID}}(\mu)$, $\underline{\text{BD}}(\mu)$ or $\overline{\text{BD}}(\mu)$.*

Corollary 5. *Let f be a $C^{1+\alpha}$ self-map on a smooth Riemannian manifold M , and μ be an f -invariant ergodic Borel probability measure with a compact support Λ . If μ is conformal and $\chi_\mu > 0$, then μ is an exact dimensional measure and $\dim(\mu) = \frac{h_\mu(f)}{\chi_\mu}$. Moreover, $D(\mu) = \frac{h_\mu(f)}{\chi_\mu}$, where $D(\mu)$ stands for $\text{HD}(\mu)$, $\underline{\text{LD}}(\mu)$, $\overline{\text{LD}}(\mu)$, $\underline{\text{ID}}(\mu)$, $\overline{\text{ID}}(\mu)$, $\underline{\text{BD}}(\mu)$ or $\overline{\text{BD}}(\mu)$.*

We remark that the condition $\chi_\mu > 0$ is indispensable in Corollary 4, since Kalinin and Sadovshaya proved in [8] that for a generic circle diffeomorphism $f : S^1 \rightarrow S^1$ with an irrational rotation number, there is a unique f -invariant Borel probability measure μ that satisfies $\underline{d}_\mu(x) = 0 < \overline{d}_\mu(x) = 1$ for μ -a.e. $x \in S^1$ (hence μ is not exact dimensional).

The rest of the paper is organized as follows. In Section 2 we review some basic notation and preliminary results which will be used in later sections. In Section 3 we give the proofs of Theorem 1. We first prove a weak version of Theorem 1 (see Theorem 3.4) using the Brin–Katok local entropy formula. Then applying Theorem 3.4 to the iterations f^k for each $k \in \mathbb{N}$, we prove Theorem 1. Theorem 2 is an immediate consequence of Theorem 1 by Young’s criterion. In Section 4 we first refine the estimates of the distortions and get a rescaling function $\hat{e}$. Then we use the rescaled local entropy formula to prove the statements in Theorem 3.

2. PRELIMINARIES

In this section we give some basic notation and results which will be used in later sections. Let M be a smooth Riemannian manifold with a Riemannian metric. This Riemannian metric naturally induces a norm $\|\cdot\|$ on each fiber $T_x M$ and a geodesic distance ρ on M such that the manifold M is also a metric space. For each $x \in M$ and $r > 0$, let $B(x, r)$ denote the set $\{y \in M : \rho(x, y) < r\}$.

Let g be a C^1 self-map on M . The *operator norm* $\|\cdot\|$ and the *co-norm* $[\cdot]$ of $Dg_x : T_x M \rightarrow T_{gx} M$ are defined respectively by

$$\|Dg_x\| = \max_{u \in T_x M : \|u\|=1} \|Dg_x(u)\| \text{ and } [Dg_x] = \min_{u \in T_x M : \|u\|=1} \|Dg_x(u)\|.$$

Some authors also use the notation $m(Dg_x) = [Dg_x]$.

Definition 2.1. Let $g : M \rightarrow M$ be a C^1 self-map and Λ be a g -invariant subset. The map g is said to be *non-degenerate* at $x \in M$ if $[Dg_x] > 0$, and g is said to be *non-degenerate* on Λ if g is non-degenerate at each $x \in \Lambda$.

Generally, the case that $Dg_x = 0$ is not excluded. To deal with these *bad* points we define the *positive part* ϕ^+ of a real function $\phi : M \rightarrow \mathbb{R}$ on M as $\phi^+(x) = \max\{0, \phi(x)\}$. In the same spirit we define $\|Dg_x\|' = \max\{1, \|Dg_x\|\}$. Evidently we have $\|Dg_x\|' \geq 1$ and $\log(\|Dg_x\|') = (\log \|Dg_x\|)^+$.

The following lemma is fairly straightforward.

Lemma 2.2. *Let g be a C^1 self-map on M , and Λ be a non-empty compact g -invariant subset of M . Then the following statements hold.*

- (1) *For any $\delta > 0$ there exists $\epsilon = \epsilon(\delta, g, \Lambda) > 0$ such that $\rho(gx, gy) \leq \rho(x, y) \cdot \|Dg_x\|' \cdot e^\delta$ for each $x \in \Lambda$ and $y \in B(x, \epsilon)$.*
- (2) *If g is non-degenerate on Λ , then for any $\delta \in (0, 1)$ there exists $\epsilon = \epsilon(\delta, g, \Lambda) > 0$ such that $\rho(gx, gy) \geq \rho(x, y) \cdot \lfloor Dg_x \rfloor \cdot e^{-\delta}$ for each $x \in \Lambda$ and $y \in B(x, \epsilon)$.*

The proof uses only elementary tools in Riemannian geometry. For the first conclusion it suffices to note that $\|Dg_x\|' \cdot e^\delta \geq \|Dg_x\|' \cdot (1 + \delta) \geq \|Dg_x\|' + \delta \geq \|Dg_x\| + \delta$ for all $x \in M$.

Remark 2.3. The non-degeneracy assumption guarantees the existence of a uniform constant ϵ in the second conclusion. In general the constant ϵ will also depend on the choice of points in Λ , and might decay *wildly* to 0 as x gets close to some degenerate point. We will show the decay is *mild* if the map is $C^{1+\alpha}$. See Lemma 4.1 and Theorem 4.3 for more details.

The following lemma says that in the definition of pointwise dimensions we can use some proper sequential limits instead of the limit $r \rightarrow 0$. See [19] for details.

Lemma 2.4. *Let μ be a Borel probability measure on M . If a sequence of positive real numbers $\{r_n\}_{n=1}^\infty$ satisfies $\lim_{n \rightarrow +\infty} r_n = 0$ and $\lim_{n \rightarrow +\infty} \frac{\log r_{n+1}}{\log r_n} = 1$, then for any $x \in M$,*

$$\underline{d}_\mu(x) = \liminf_{n \rightarrow +\infty} \frac{\log \mu(B(x, r_n))}{\log r_n} \quad \text{and} \quad \bar{d}_\mu(x) = \limsup_{n \rightarrow +\infty} \frac{\log \mu(B(x, r_n))}{\log r_n}.$$

Remark 2.5. By Lemma 2.4, it is clear that both $\underline{d}_\mu(\cdot)$ and $\bar{d}_\mu(\cdot)$ are Borel functions on M .

Now assume μ is a g -invariant ergodic Borel probability measure on M with a compact support Λ_μ . By Lemma 2.2 (1) for $\delta = 1$, we see that there exists $\epsilon = \epsilon(1, g, \Lambda_\mu) > 0$ such that $\rho(gx, gy) \leq \rho(x, y) \cdot e \cdot \|Dg_x\|'$ for any $x \in \Lambda_\mu$ and $y \in B(x, \epsilon)$. This implies that $g(B(x, \frac{r}{e\|Dg_x\|'})) \subset B(gx, r)$ for any $x \in \Lambda_\mu$ and $0 < r < \epsilon$. Therefore $B(x, \frac{r}{e\|Dg_x\|'}) \subset g^{-1}B(gx, r)$, and hence $\mu(B(x, \frac{r}{e\|Dg_x\|'})) \leq \mu(g^{-1}B(gx, r)) = \mu(B(gx, r))$ for any $x \in \Lambda_\mu$ and $0 < r < \epsilon$. Passing $r \rightarrow 0$, we get that

$$(2.1) \quad \underline{d}_\mu(x) \geq \underline{d}_\mu(gx) \quad \text{and} \quad \bar{d}_\mu(x) \geq \bar{d}_\mu(gx), \quad \text{for any } x \in \Lambda_\mu.$$

Moreover, since μ is ergodic and the functions $\underline{d}_\mu(\cdot)$ and $\bar{d}_\mu(\cdot)$ are Borel measurable, we know that both $\underline{d}_\mu(x)$ and $\bar{d}_\mu(x)$ are constant for μ -a.e. $x \in \Lambda_\mu$ by (2.1). Denote these two constants by $\underline{d}_\mu$ and $\bar{d}_\mu$ respectively. Let

$$E_\mu(\underline{d}) = \{x \in \Lambda_\mu : \underline{d}_\mu(x) = \underline{d}_\mu\} \quad \text{and} \quad E_\mu(\bar{d}) = \{x \in \Lambda_\mu : \bar{d}_\mu(x) = \bar{d}_\mu\}.$$

For convention we restate the above observations in a lemma.

Lemma 2.6. *Let μ be a g -invariant ergodic Borel probability measure with a compact support. Then $\mu(E_\mu(\bar{d})) = \mu(E_\mu(\underline{d})) = 1$.*

Finally let's recall two basic properties relating the lower pointwise dimension with the Hausdorff dimension of Borel measures. Let μ be a Borel probability measure supported on a Borel subset $X \subseteq \mathbb{R}^\ell$.

- (1) If $\underline{d}_\mu(x) \geq \alpha$ for μ -a.e. $x \in X$, then $\text{HD}(\mu) \geq \alpha$.
- (2) If $\underline{d}_\mu(x) \leq \alpha$ for μ -a.e. $x \in X$, then $\text{HD}(\mu) \leq \alpha$.

See [16, Theorem 7.1 and 7.2] for details. By Whitney's *Embedding Theorem*, the above statements also hold for Borel probability measures on the manifold M . Combining these two properties and Lemma 2.6, we have the following result.

Proposition 2.7. *Let g be a C^1 self-map on M and μ be a g -invariant ergodic Borel probability measure on M with a compact support Λ_μ . Then $\text{HD}(\mu) = \underline{d}_\mu$.*

3. PROOF OF THEOREMS 1 AND 2

In this section, our aim is to prove Theorem 1 and Theorem 2. Throughout this section, we let M be a smooth Riemannian manifold, f be a C^1 self-map on M and μ be an f -invariant ergodic Borel probability measure with a compact support Λ_μ .

Now we introduce the notation of Lyapunov exponents of f with respect to μ . By Oseledets' *Multiplicative Ergodic Theorem* (see [15] or [18, Theorem 10.2]), there are an integer $s \geq 1$, numbers $\chi_\mu^1 > \chi_\mu^2 > \cdots > \chi_\mu^s$ (χ_μ^s may be $-\infty$), and a filtration $T_x M = V_x^1 \supset \cdots \supset V_x^s \supset V_x^{s+1} = \{0\}$ of $T_x M$ for μ -a.e. $x \in M$ such that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \chi_\mu^i, \quad \text{if } v \in V_x^i \setminus V_x^{i+1}, 1 \leq i \leq s.$$

The numbers χ_μ^i , counting with their multiplicities $n_i = \dim V_x^i - \dim V_x^{i+1}$, are called the *Lyapunov exponents* of f with respect to μ . Next we will give another description of the largest exponent χ_μ^1 and the smallest exponent χ_μ^s that fits our needs in the rest of this paper.

Let $\varphi_n(x) = \log \|Df_x^n\|$ for each point $x \in M$ and each integer $n \in \mathbb{N}$. Then $\{\varphi_n\}_{n=1}^\infty$ is a family of continuous functions on M and $\varphi_{n'+n}(x) \leq \varphi_{n'}(f^n x) + \varphi_n(x)$ for any $x \in M$ and $n', n \in \mathbb{N}$. Since Λ_μ is compact, $D = \{\|Df_x\| : x \in \Lambda_\mu\} < +\infty$. Then the integral $\int_M \log \|Df_x\| d\mu(x)$ is well defined and $\int_M \log \|Df_x\| d\mu(x) \in [-\infty, \log D]$. By the Kingman *Sub-additive Ergodic Theorem* (see [7] or [18, Theorem 10.1]), the limit $\chi_\mu^1 = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Df_x^n\|$ exists for μ -a.e. $x \in M$. Clearly the limit function $\chi_{\{\cdot\}}^1$ is f -invariant. Since μ is ergodic, $\chi_{\{\cdot\}}^1$ is μ -a.e. constant and $\chi_x^1 = \chi_\mu^1$ for μ -a.e. $x \in M$. Consequently we have

$$(3.1) \quad \chi_\mu^1 = \lim_{n \rightarrow +\infty} \frac{1}{n} \int_M \log \|Df_x^n\| d\mu(x) = \inf_{n \geq 1} \frac{1}{n} \int_M \log \|Df_x^n\| d\mu(x).$$

Note that $\chi_\mu^1 = -\infty$ is not excluded.

Similarly, let's consider another family of functions $\{\phi_n\}_{n=1}^\infty$ on M by setting $\phi_n(x) = \log \lfloor Df_x^n \rfloor$ for each $x \in M$ and $n \in \mathbb{N}$. It is not hard to show $\phi_{n'+n}(x) \geq \phi_{n'}(f^n x) + \phi_n(x)$ for any $x \in M$ and $n', n \in \mathbb{N}$. Clearly,

$$\int_M \log \lfloor Df_x^n \rfloor d\mu(x) \leq \int_M \log \|Df_x^n\| d\mu(x) \leq n \log D.$$

Again the limit $\chi_x^s = \lim_{n \rightarrow +\infty} \frac{1}{n} \log [Df_x^n]$ exists for μ -a.e. $x \in M$ and $\chi_x^s = \chi_\mu^s$ for μ -a.e. $x \in M$. We also have

$$(3.2) \quad \chi_\mu^s = \lim_{n \rightarrow +\infty} \frac{1}{n} \int_M \log [Df_x^n] d\mu(x) = \sup_{n \geq 1} \frac{1}{n} \int_M \log [Df_x^n] d\mu(x).$$

Also $\chi_\mu^s = -\infty$ is not excluded. If f is non-degenerated on Λ_μ , then $[Df_x] > 0$ on Λ_μ , and hence $d := \inf\{[Df_x] : x \in \Lambda_\mu\} > 0$. Then $\chi_\mu^s \in [\log d, \log D]$.

Our next effort in this section is to prove the following theorem (it is equivalent to Theorem 1).

Theorem 3.1. (1) $\underline{d}_\mu(x) \max\{\chi_\mu^1, 0\} \geq h_\mu(f)$ for μ -a.e. $x \in M$.
 (2) If f is non-degenerate on Λ_μ , then $\bar{d}_\mu(x) \max\{\chi_\mu^s, 0\} \leq h_\mu(f)$ for μ -a.e. $x \in M$.

In order to prove this theorem we need some more preparations. Let $g : M \rightarrow M$ be a C^1 self-map and ν be a g -invariant ergodic Borel probability measure. For $n \in \mathbb{N}$, $x \in M$ and $\delta > 0$, let

$$B_n(x, \delta) = \{y \in M : \max_{0 \leq i \leq n-1} \rho(g^i x, g^i y) < \delta\},$$

$$h_\nu^+(g, x, \delta) = \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \nu(B_n(x, \delta)) \text{ and } h_\nu^-(g, x, \delta) = \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \nu(B_n(x, \delta)).$$

The following results are the Brin–Katok *local entropy formula*. See [1].

Lemma 3.2. Let g be a C^1 self-map on M and ν be a g -invariant ergodic Borel probability measure on M with a compact support Λ_ν . Then the following statements hold.

- (1) The limits $h_\nu^+(g, x) = \lim_{\delta \rightarrow 0} h_\nu^+(g, x, \delta)$ and $h_\nu^-(g, x) = \lim_{\delta \rightarrow 0} h_\nu^-(g, x, \delta)$ coincide for ν -a.e. $x \in M$.
- (2) $h_\nu^+(g, x) = h_\nu^-(g, x) = h_\nu(g)$ for ν -a.e. $x \in M$.

Recall that the set of *generic points* of ν is defined by

$$G_\nu = \{x \in \Lambda_\nu : \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(g^i x) = \int_M \phi d\nu \text{ for each } \phi \in C(M, \mathbb{R})\}.$$

By the Birkhoff *Pointwise Ergodic Theorem*, we know $\nu(G_\nu) = 1$ since ν is ergodic.

We denote $\log^+ \|Dg_x\| = (\log \|Dg_x\|)^+$ for short. Let $\chi_\nu^+ = \int_M \log^+ \|Dg_x\| d\nu(x)$ and $\chi_\nu^- = \int_M \log [Dg_x] d\nu(x)$. Clearly $\chi_\nu^+ \geq 0$. Note that χ_ν^- may be non-positive even if g is non-degenerate on Λ_ν .

Lemma 3.3. Let g be a C^1 self-map on M and ν be a g -invariant ergodic Borel probability measure on M with a compact support Λ_ν . Then

- (1) $\chi_\nu^+ \cdot \underline{d}_\nu(x) \geq h_\nu^-(g, x)$ for any $x \in G_\nu$.
- (2) If g is non-degenerate on Λ_ν , then $\max\{\chi_\nu^-, 0\} \cdot \bar{d}_\nu(x) \leq h_\nu^+(g, x)$.

Proof. (1) To show (1), it is sufficient to show $\underline{d}_\nu(x) \cdot (2\delta + \chi_\nu^+) \geq h_\nu^-(g, x)$ for any $\delta \in (0, 1)$ and $x \in G_\nu$.

Given $\delta \in (0, 1)$, for each $\ell \in \mathbb{N}$, let's consider the set

$$Y_\ell = \{p \in G_\nu : \frac{1}{n} \sum_{k=0}^{n-1} \log^+ \|Dg_{g^k p}\| \leq \chi_\nu^+ + \delta \text{ for all } n > \ell\}.$$

Since $\log^+ \|Dg_{\{\cdot\}}\|$ is continuous on M , $\bigcup_{\ell \in \mathbb{N}} Y_\ell = G_\nu$. Hence for each $x \in G_\nu$ there exists an integer $\ell_x \in \mathbb{N}$ such that $x \in Y_{\ell_x}$. Then for any $n > \ell_x$,

$$(3.3) \quad 0 \leq \sum_{k=0}^{n-1} \log^+ \|Dg_{g^k x}\| \leq n(\chi_\nu^+ + \delta).$$

Let $\epsilon = \epsilon(\delta, g, \Lambda_\nu) > 0$ be given as in Lemma 2.2 (1) such that $\rho(gy, gz) \leq \rho(y, z)e^\delta \|Dg_y\|'$ for any $y \in \Lambda_\nu$ and $z \in B(y, \epsilon)$. Now fix $\zeta \in (0, \epsilon)$.

Claim. Let $n \geq 1$. For each $y \in M$ with $\rho(x, y) < \zeta e^{-n\delta - \sum_{k=0}^{n-1} \log^+ \|Dg_{g^k x}\|}$ we have that $\rho(g^i x, g^i y) \leq \rho(x, y) e^{i\delta + \sum_{k=0}^{i-1} \log^+ \|Dg_{g^k x}\|} < \zeta$ for $1 \leq i \leq n$.

Proof of the Claim. We prove the Claim by induction on n . For $n = 1$, it is just Lemma 2.2 (1). Now we assume the Claim holds for all $n \leq \ell$. Then for $n = \ell + 1$, let $y \in M$ with $\rho(x, y) < \zeta e^{-(\ell+1)\delta - \sum_{k=0}^{\ell} \log^+ \|Dg_{g^k x}\|}$. It is clear that $\rho(x, y) < \zeta e^{-\ell\delta - \sum_{k=0}^{\ell-1} \log^+ \|Dg_{g^k x}\|}$ since $\log^+ \|Dg_{g^\ell x}\| \geq 0$. Using the assumption for $n = \ell$, we know $\rho(g^i x, g^i y) \leq \rho(x, y) e^{i\delta + \sum_{k=0}^{i-1} \log^+ \|Dg_{g^k x}\|} < \zeta$ for $1 \leq i \leq \ell - 1$. In particular, for $i = \ell - 1$ we have

$$\rho(g^{\ell-1} x, g^{\ell-1} y) \leq \rho(x, y) e^{(\ell-1)\delta + \sum_{k=0}^{\ell-2} \log^+ \|Dg_{g^k x}\|} < \zeta e^{-\delta - \log^+ \|Dg_{g^{\ell-1} x}\|}.$$

Moreover, since $g^\ell x \in \Lambda_\nu$ and $\zeta \in (0, \epsilon)$, we have, by the choice of ϵ ,

$$\rho(g^\ell x, g^\ell y) \leq \rho(g^{\ell-1} x, g^{\ell-1} y) e^\delta \|Dg_{g^{\ell-1} x}\|' \leq \rho(x, y) e^{\ell\delta + \sum_{k=0}^{\ell-1} \log^+ \|Dg_{g^k x}\|} < \zeta.$$

So the Claim also holds for $n = \ell + 1$. This ends the proof of the Claim. $\square$

Given any $x \in G_\nu$, let ℓ_x be given so that $x \in Y_{\ell_x}$. Then by (3.3) and the Claim we have, for $n > \ell_x$,

$$B(x, \zeta e^{-n(2\delta + \chi_\nu^+)}) \subseteq B(x, \zeta e^{-n\delta - \sum_{k=0}^{n-1} \log^+ \|Dg_{g^k x}\|}) \subseteq B_n(x, \zeta).$$

Let $r_n = \zeta e^{-n(2\delta + \chi_\nu^+)}$. Clearly $\lim_{n \rightarrow +\infty} r_n = 0$ and $\lim_{n \rightarrow +\infty} \frac{\log r_{n+1}}{\log r_n} = 1$. Then applying Lemma 2.4 we get

$$\begin{aligned} \underline{d}_\nu(x) &= \liminf_{n \rightarrow \infty} \frac{\log \nu(B(x, r_n))}{\log r_n} = \liminf_{n \rightarrow +\infty} \frac{\log \nu(B(x, \zeta e^{-n(2\delta + \chi_\nu^+)})}{\log(\zeta e^{-n(2\delta + \chi_\nu^+)})} \\ &\geq \liminf_{n \rightarrow +\infty} \frac{\log \nu(B_n(x, \zeta))}{-n(2\delta + \chi_\nu^+) + \log \zeta} = \frac{h_\nu^-(g, x, \zeta)}{2\delta + \chi_\nu^+}. \end{aligned}$$

This is equal to saying $\underline{d}_\nu(x)(2\delta + \chi_\nu^+) \geq h_\nu^-(g, x, \zeta)$ for each $\zeta \in (0, \epsilon)$. Letting $\zeta \rightarrow 0$ in this inequality, we get $\underline{d}_\nu(x)(2\delta + \chi_\nu^+) \geq h_\nu^-(g, x)$. This ends the proof of (1).

(2) Assume that g is non-degenerate on Λ_ν . If $\chi_\nu^- \leq 0$, then the result is trivial since $h_\nu^+(g, x) \geq 0$. In the following we assume $\chi_\nu^- > 0$. To show (2), it is sufficient to show that for any $\delta \in (0, \min\{1, \frac{\chi_\nu^-}{3}\})$, $\bar{d}_\nu(x) \leq \frac{h_\nu^+(f, x)}{\chi_\nu^- - 2\delta}$ for all $x \in G_\nu$.

Given $\delta \in (0, \min\{1, \frac{\chi_\nu^-}{3}\})$, for each $\ell \in \mathbb{N}$ let's consider the set

$$Z_\ell = \{p \in G_\nu : \frac{1}{n} \sum_{i=0}^{n-1} \log [Dg_{g^i p}] \geq \chi_\nu^- - \delta \text{ for all } n > \ell\}.$$

Since $\log \lfloor Dg_{\{\cdot\}} \rfloor$ is continuous on M , $\bigcup_{\ell \in \mathbb{N}} Z_\ell = G_\nu$. Hence for each $x \in G_\nu$ there exists an integer $\ell_x \in \mathbb{N}$ such that $x \in Z_{\ell_x}$. Then for any $n > \ell_x$,

$$(3.4) \quad \sum_{i=0}^{n-1} \log \lfloor Dg_{g^i x} \rfloor \geq n(\chi_\nu^- - \delta) > 0.$$

Let $\epsilon = \epsilon(\delta, g, \Lambda_\nu) > 0$ be given as in Lemma 2.2 (2) such that $\rho(gz, gy) \geq \rho(z, y)e^{-\delta} \lfloor Dg_y \rfloor$ for any $y \in \Lambda_\nu$ and $z \in B(y, \epsilon)$.

Fix any $\zeta \in (0, \epsilon)$. Thus for any $y \in M$ and $n \geq 1$ with $\max_{0 \leq j \leq n} \rho(g^j x, g^j y) < \zeta$, we have $\rho(g^{\ell-1} x, g^{\ell-1} y) \leq \rho(g^\ell x, g^\ell y)e^\delta (\lfloor Dg_{g^{\ell-1} x} \rfloor)^{-1}$ for each $1 \leq \ell \leq n$. Combining these inequalities from $\ell = n$ to 1, we get

$$(3.5) \quad \rho(x, y) \leq \rho(g^n x, g^n y)e^{n\delta} \left(\prod_{i=0}^{n-1} \lfloor Dg_{g^i x} \rfloor \right)^{-1} < \zeta e^{n\delta - \sum_{i=0}^{n-1} \log \lfloor Dg_{g^i x} \rfloor}.$$

Combining (3.4) and the above inequalities we have that

$$B_{n+1}(x, \zeta) \subseteq B(x, \zeta e^{n\delta - \sum_{i=0}^{n-1} \log \lfloor Dg_{g^i x} \rfloor}) \subseteq B(x, \zeta e^{n(2\delta - \chi_\nu^-)}),$$

for any $n > \ell_x$. Using Lemma 2.4 for $r_n = \zeta e^{n(2\delta - \chi_\nu^-)}$, we have

$$\begin{aligned} \bar{d}_\nu(x) &= \limsup_{n \rightarrow \infty} \frac{\log \nu(B(x, r_n))}{\log r_n} = \limsup_{n \rightarrow +\infty} \frac{\log \nu(B(x, \zeta e^{n(2\delta - \chi_\nu^-)}))}{\log(\zeta e^{n(2\delta - \chi_\nu^-)})} \\ &\leq \limsup_{n \rightarrow +\infty} \frac{\log \nu(B_{n+1}(x, \zeta))}{n(2\delta - \chi_\nu^-) + \log \zeta} = \frac{h_\nu^+(g, x, \zeta)}{\chi_\nu^- - 2\delta}. \end{aligned}$$

Letting $\zeta \rightarrow 0$ in the above inequalities, we get $\bar{d}_\nu(x) \leq \frac{h_\nu^+(g, x)}{\chi_\nu^- - 2\delta}$. This finishes the proof of (2). $\square$

As an immediate consequence of Lemma 2.6, Lemma 3.2 and Lemma 3.3, we have the following theorem which can be viewed as a weak version of Theorem 3.1.

Theorem 3.4. *Let g be a C^1 self-map on M and ν be a g -invariant ergodic Borel probability measure on M with a compact support Λ_ν . Then the following statements hold.*

- (1) $\underline{d}_\nu \cdot \chi_\nu^+ \geq h_\nu(g)$.
- (2) *If g is non-degenerated on Λ_ν , then $\bar{d}_\nu \cdot \max\{\chi_\nu^-, 0\} \leq h_\nu(g)$.*

Now we come back to the initial map f and consider its iteration f^k for each $k \in \mathbb{N}$. Since (Λ_μ, μ, f) is ergodic, it is well known that for each $k \in \mathbb{N}$, there exist a factor p_k of k and an f^k -invariant ergodic Borel probability measure ν_k with $\text{supp}(\nu_k) \subseteq \Lambda_\mu$ such that $f^{p_k} \nu_k = \nu_k$, $\mu = \frac{1}{p_k} \sum_{j=0}^{p_k-1} f^j \nu_k$, where $f^j \nu_k(A) = \nu_k(f^{-j} A)$ for any Borel subset A of M . Since Λ_μ is f -invariant and ν_k is an f^k -invariant ergodic measure supported on Λ_μ , $f^j \nu_k$ is also an f^k -invariant ergodic measure and $\text{supp}(f^j \nu_k) \subseteq \Lambda_\mu$ for $j = 0, 1, \dots, p_k - 1$. To finish the proof of Theorem 3.1, we need the following lemma.

Lemma 3.5. *The following statements hold.*

- (1) $h_{\nu_k}(f^k) = h_{f\nu_k}(f^k) = \dots = h_{f^{p_k-1}\nu_k}(f^k) = kh_\mu(f)$.
- (2) $\bar{d}_{\nu_k} = \bar{d}_{f\nu_k} = \dots = \bar{d}_{f^{p_k-1}\nu_k} \geq \bar{d}_\mu$
- (3) $\underline{d}_{\nu_k} = \underline{d}_{f\nu_k} = \dots = \underline{d}_{f^{p_k-1}\nu_k} = \underline{d}_\mu$.

Proof. (1) It is a well-known result.

(2) By Lemma 2.6, we have $f\nu_k(E_{f\nu_k}(\underline{d})) = 1$, i.e., $\nu_k(f^{-1}E_{f\nu_k}(\bar{d})) = 1$. Hence $\nu_k(E_{\nu_k}(\bar{d}) \cap f^{-1}E_{f\nu_k}(\bar{d})) = 1$. Let $x \in E_{\nu_k}(\bar{d}) \cap f^{-1}E_{f\nu_k}(\bar{d})$. Then $\bar{d}_{\nu_k}(x) = \bar{d}_{\nu_k}$ and $\bar{d}_{f\nu_k}(fx) = \bar{d}_{f\nu_k}$.

Let $\delta = 1$. By Lemma 2.2 (2), there exists $\epsilon = \epsilon(1, f, \Lambda_\mu) > 0$ such that $\rho(fx, fy) \leq \rho(x, y) \cdot e \cdot \|Df_x\|'$ for any $y \in B(x, \epsilon)$. This implies that $fB(x, \frac{r}{e \cdot \|Df_x\|'}) \subset B(fx, r)$ for any $0 < r < \epsilon$. Therefore $B(x, \frac{r}{e \cdot \|Df_x\|'}) \subset f^{-1}B(fx, r)$, and hence

$$\nu_k(B(x, \frac{r}{e \cdot \|Df_x\|'})) \leq \nu_k(f^{-1}B(fx, r)) = f\nu_k(B(fx, r))$$

for any $0 < r < \epsilon$. Then we have

$$\begin{aligned} \bar{d}_{\nu_k} &= \bar{d}_{\nu_k}(x) = \limsup_{r \rightarrow 0} \frac{\log \nu_k(B(x, \frac{r}{e \cdot \|Df_x\|'}))}{\log(\frac{r}{e \cdot \|Df_x\|'})} \geq \limsup_{r \rightarrow 0} \frac{\log f\nu_k(B(fx, r))}{\log(\frac{r}{e \cdot \|Df_x\|'})} \\ &= \limsup_{r \rightarrow 0} \frac{\log f\nu_k(B(fx, r))}{\log r} = \bar{d}_{f\nu_k}(fx) = \bar{d}_{f\nu_k}. \end{aligned}$$

In the same way we can show $\bar{d}_{f\nu_k} \geq \bar{d}_{f^2\nu_k} \geq \dots \geq \bar{d}_{f^{p_k}\nu_k}$. Noting that $f^{p_k}\nu_k = \nu_k$, we get $\bar{d}_{\nu_k} = \bar{d}_{f\nu_k} = \dots = \bar{d}_{f^{p_k-1}\nu_k}$. We are left to compare $\bar{d}_{\nu_k}$ and $\bar{d}_\mu$.

Since $1 = \mu(E_\mu(\bar{d})) = \frac{1}{p_k} \sum_{i=0}^{p_k-1} f^i\nu_k(E_\mu(\bar{d}))$, one has $\nu_k(E_\mu(\bar{d})) = 1$. So $\nu_k(E_{\nu_k}(\bar{d}) \cap E_\mu(\bar{d})) = 1$. Let $y \in E_{\nu_k}(\bar{d}) \cap E_\mu(\bar{d})$. Then $\bar{d}_{\nu_k}(y) = \bar{d}_{\nu_k}$ and $\bar{d}_\mu(y) = \bar{d}_\mu$. Since $\mu = \frac{1}{p_k} \sum_{j=0}^{p_k-1} f^j\nu_k$, $\frac{1}{p_k}\nu_k(B(y, r)) \leq \mu(B(y, r))$ for any $r > 0$. Thus

$$\begin{aligned} \bar{d}_{\nu_k} &= \bar{d}_{\nu_k}(y) = \limsup_{r \rightarrow 0} \frac{\log \nu_k(B(y, r))}{\log r} = \limsup_{r \rightarrow 0} \frac{\log \frac{\nu_k(B(y, r))}{p_k}}{\log r} \\ &\geq \limsup_{r \rightarrow 0} \frac{\log \mu(B(y, r))}{\log r} = \bar{d}_\mu(y) = \bar{d}_\mu. \end{aligned}$$

(3) Following completely from the proof of (2), we have

$$\underline{d}_{\nu_k} = \underline{d}_{f\nu_k} = \dots = \underline{d}_{f^{p_k-1}\nu_k} \geq \underline{d}_\mu.$$

Note that if $\mu(E) = 0$, then $\nu_k(E) = 0$. So $\text{HD}(\nu_k) \leq \text{HD}(\mu)$. Moreover, by Proposition 2.7, we have $\text{HD}(\nu_k) = \underline{d}_{\nu_k}$ and $\text{HD}(\mu) = \underline{d}_\mu$. Combining these relations we have $\underline{d}_{\nu_k} = \underline{d}_{f\nu_k} = \dots = \underline{d}_{f^{p_k-1}\nu_k} = \underline{d}_\mu$. $\square$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. (1) By the Kingman Sub-additive Ergodic Theorem we have

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \log \|Df_x^k\| = \chi_\mu^1 \text{ for } \mu\text{-a.e. } x \in \Lambda_\mu.$$

Moreover, it is not hard to see that

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \log^+ \|Df_x^k\| = \max\{\chi_\mu^1, 0\} \text{ for } \mu\text{-a.e. } x \in \Lambda_\mu.$$

Thus by the *Dominated Convergent Theorem* we have

$$(3.6) \quad \lim_{k \rightarrow +\infty} \frac{1}{k} \int_M \log^+ \|Df_x^k\| d\mu(x) = \max\{\chi_\mu^1, 0\}.$$

For each $k \in \mathbb{N}$, we let p_k and ν_k be given such that ν_k is an f^k -invariant ergodic Borel probability measure and $\mu = \frac{1}{p_k} \sum_{j=0}^{p_k-1} f^j \nu_k$. For any $j \in \{0, 1, \dots, p_k - 1\}$, we know $\underline{d}_{f^j \nu_k} \cdot \chi_{f^j \nu_k}^+ \geq h_{f^j \nu_k}(f^k) = kh_\mu(f)$ by Theorem 3.4 (1) and Lemma 3.5 (1). Combining this with the fact $\underline{d}_\mu = \underline{d}_{f^j \nu_k}$ (see Lemma 3.5 (3)), we have $kh_\mu(f) \leq \underline{d}_\mu \cdot \chi_{f^j \nu_k}^+$. Summing over j from 0 to $p_k - 1$ in this inequality and then dividing both sides by $k \cdot p_k$, we get

$$\begin{aligned} h_\mu(f) &\leq \underline{d}_\mu \cdot \left(\frac{1}{k \cdot p_k} \sum_{j=0}^{p_k-1} \chi_{f^j \nu_k}^+ \right) = \underline{d}_\mu \cdot \frac{1}{k} \int_M \log^+ \|Df_x^k\| d \left(\frac{1}{p_k} \sum_{j=0}^{p_k-1} f^j \nu_k \right) (x) \\ &= \underline{d}_\mu \cdot \frac{1}{k} \int_M \log^+ \|Df_x^k\| d\mu(x). \end{aligned}$$

Letting $k \rightarrow +\infty$ in this inequality, we get $h_\mu(f) \leq \underline{d}_\mu \cdot \max\{\chi_\mu^1, 0\}$ by (3.6).

(2) Assume that f is non-degenerate on Λ_μ . For each $k \in \mathbb{N}$, let p_k and ν_k be given such that ν_k is an f^k -invariant ergodic Borel probability measure with $\mu = \frac{1}{p_k} \sum_{j=0}^{p_k-1} f^j \nu_k$. Clearly f^k is also non-degenerate on Λ_μ . Let $j \in \{0, 1, \dots, p_k - 1\}$. Then by Theorem 3.4 (2) we have $h_{f^j \nu_k}(f^k) \geq \bar{d}_{f^j \nu_k} \cdot \max\{0, \chi_{f^j \nu_k}^-\}$. Combining with Lemma 3.5 (1) and (2) we indeed have $kh_\mu(f) \geq \bar{d}_\mu \cdot \max\{0, \chi_{f^j \nu_k}^-\}$. Summing over j in this inequality from 0 to $p_k - 1$ and then dividing both sides by $k \cdot p_k$, we get

$$\begin{aligned} h_\mu(f) &\geq \bar{d}_\mu \cdot \left(\frac{1}{k \cdot p_k} \sum_{j=0}^{p_k-1} \chi_{f^j \nu_k}^- \right) = \bar{d}_\mu \cdot \frac{1}{k} \int_M \log \lfloor Df_x^k \rfloor d \left(\frac{1}{p_k} \sum_{j=0}^{p_k-1} f^j \nu_k \right) (x) \\ &= \bar{d}_\mu \cdot \frac{1}{k} \int_M \log \lfloor Df_x^k \rfloor d\mu(x). \end{aligned}$$

Letting $k \rightarrow +\infty$ in this inequality, we get $h_\mu(f) \geq \bar{d}_\mu \cdot \chi_\mu^s$ by (3.2). Since $h_\mu(f) \geq 0$ we indeed get $h_\mu(f) \geq \bar{d}_\mu \cdot \max\{\chi_\mu^s, 0\}$. $\square$

Following is Young's criterion in [19] (also see [6, Fact 2]).

Proposition 3.6. *Let μ be a Borel probability measure on a compact set $\Lambda_\mu \subset M$. If there are constants C_1 and C_2 such that $C_1 \leq \underline{d}_\mu(x) \leq \bar{d}_\mu(x) \leq C_2$ for μ -a.e. $x \in M$, then $C_1 \leq D(\mu) \leq C_2$, where $D(\mu)$ stands for $\text{HD}(\mu)$, $\underline{\text{LD}}(\mu)$, $\overline{\text{LD}}(\mu)$, $\underline{\text{ID}}(\mu)$, $\overline{\text{ID}}(\mu)$, $\underline{\text{BD}}(\mu)$ or $\overline{\text{BD}}(\mu)$.*

Proof of Theorem 2. It immediately follows from Theorem 3.1 and Proposition 3.6. $\square$

4. PROOF OF THEOREM 3

In this section our aim is to prove Theorem 3. For this purpose we need some related preparations. The following lemma is similar to Lemma 2.2, but with a more involved proof. We give a proof for the lack of precise references.

Lemma 4.1. *Let g be a $C^{1+\alpha}$ self-map on M for some $\alpha \in (0, 1)$, and Λ be a non-empty compact g -invariant subset of M . Then there exist $r_0 = r_0(g, \Lambda) > 0$*

and $C = C(g, \Lambda) > 0$ such that

$$\rho(gx, gy) \geq \rho(x, y) \lfloor Dg_x \rfloor - C\rho(x, y)^{1+\alpha},$$

for any $x \in \Lambda$ and $y \in B(x, r_0)$.

Proof. Recall that the exponential map $\exp_x : T_x M \rightarrow M$ at $x \in M$ is defined by $\exp_x(v) = \gamma_v(1)$, where $\gamma_v(t)$ is the geodesic with $\gamma_v(0) = x, \dot{\gamma}_v(0) = v$. The following properties are well known.

- (i) For $x \in M$, $\exp_x(0_x) = x$ and $D(\exp_x)_{0_x} = \text{Id}_{T_x M}$.
- (ii) There exist a positive number $R = R(\Lambda) > 0$ and an open neighborhood U of Λ whose closure $\overline{U}$ is compact, such that $\exp_x : T_x M(R) \rightarrow M$ is embedding and

$$\rho(\exp_x(v), x) = \|v\| \text{ for any } v \in T_x M(R) \text{ and } x \in U,$$

where $T_x M(s) = \{v \in T_x M : \|v\| < s\}$ for $s > 0$. Moreover, $\exp_x(T_x M(r)) = B(x, r) \subset M$ for any $x \in U$ and $r \in (0, R)$.

Since Λ is compact and g -invariant, there exist a positive number $r_0 = r_0(g, \Lambda) \in (0, R)$ and an open neighborhood $W \subseteq U$ of Λ such that for any $x \in W, y \in M$ with $\rho(x, y) < r_0$ we have $gx \in U$ and $\rho(gx, gy) < R$.

Consider a map $\tilde{g} : T_W M(r_0) \rightarrow T_U M(R)$ corresponding to the map g on W as

$$\tilde{g}(v) = \exp_{gx}^{-1} \circ g \circ \exp_x(v), \text{ for } v \in T_x M(r_0) \text{ and } x \in W,$$

where $T_W M(r_0) = \{v \in T_W M : \|v\| < r_0\}$ and $T_U M(R) = \{u \in T_U M : \|u\| < R\}$. Since the exponential map is always smooth enough, $\tilde{g}$ is as smooth as g . Define the partial derivative D_2 of $\tilde{g}$ with respect to the fiber variable, also named as the partial derivative with respect to the second variable, as

$$D_2 \tilde{g}_v = D(\tilde{g}|_{T_x M(r_0)})_v : T_v(T_x M) \rightarrow T_{\tilde{g}(v)}(T_{gx} M)$$

for any $x \in W$ and $v \in T_x M(r_0)$. In the following we use the fact that $T_x M$ is a linear space and $T_v(T_x M)$ is isometric to $T_x M$; so is $T_{\tilde{g}(v)}(T_{gx} M)$ to $T_{gx} M$. Hence it is reasonable to express $D_2 \tilde{g}_v : T_x M \rightarrow T_{gx} M$ for any $v \in T_x M(r_0)$ and $x \in W$. At $v = 0_x$ we have $\tilde{g}(0_x) = 0_{gx}$, $D(\exp_{gx}^{-1})_{gx} = \text{Id}_{T_{gx} M}$ and

$$D_2 \tilde{g}_{0_x} = D(\exp_{gx}^{-1} \circ g \circ \exp_x)_{0_x} = \text{Id}_{T_{gx} M} \circ Dg_x \circ \text{Id}_{T_x M} = Dg_x.$$

Claim. There exists $C = C(g, \Lambda) > 0$ such that for any $x \in \Lambda$ and $y \in B(x, r_0)$,

$$\|\exp_{gx}^{-1}(gy) - Dg_x(\exp_x^{-1}(y))\| \leq C\rho(x, y)^{1+\alpha}.$$

Proof of the Claim. Since the exponential map is always smooth enough, $\tilde{g}$ is as smooth as g . So for $x \in W$, $\tilde{g} : T_x M(r_0) \rightarrow T_{gx} M(R)$ is uniformly $C^{1+\alpha}$. Hence $D_2 \tilde{g}_v$ is uniformly C^α for $v \in T_x M(r_0)$ and $x \in W$. Thus there exists a positive constant $C = C(g, \Lambda) > 0$ such that

$$(4.1) \quad \|D_2 \tilde{g}_u - D_2 \tilde{g}_v\| \leq C\|u - v\|^\alpha \text{ for any } u, v \in T_x M(r_0) \text{ and } x \in W.$$

Let $x \in \Lambda$ and $y \in B(x, r_0)$ be given. Let $\gamma(t) = t \exp_x^{-1}(y) \in T_x M$ for $t \in [0, 1]$. Then

$$\exp_{g_x}^{-1}(gy) = \tilde{g}(\exp_x^{-1}(y)) = \int_0^1 \frac{d(\tilde{g} \circ \gamma(t))}{dt} dt = \int_0^1 D_2 \tilde{g}_{\gamma(t)}(\exp_x^{-1}(y)) dt.$$

By the choice of C we have

$$\begin{aligned} \|\exp_{g_x}^{-1}(gy) - D_2 \tilde{g}_{0_x}(\exp_x^{-1}(y))\| &= \left\| \int_0^1 (D_2 \tilde{g}_{\gamma(t)} - D_2 \tilde{g}_{0_x})(\exp_x^{-1}(y)) dt \right\| \\ &\leq \int_0^1 \|D_2 \tilde{g}_{\gamma(t)} - D_2 \tilde{g}_{0_x}\| \cdot \|\exp_x^{-1}(y)\| dt = \rho(x, y) \int_0^1 \|D_2 \tilde{g}_{\gamma(t)} - D_2 \tilde{g}_{0_x}\| dt \\ &\stackrel{(4.1)}{\leq} \rho(x, y) \int_0^1 C \|\gamma(t) - 0_x\|^\alpha dt \leq C \|\rho(x, y)\|^{1+\alpha}. \end{aligned}$$

Note that $D_2 \tilde{g}_{0_x} = Dg_x$. This completes the proof of the Claim. $\square$

Let C be given as in the Claim. For $x \in \Lambda$ and $y \in B(x, r_0)$, by the Claim we have

$$\begin{aligned} \rho(gx, gy) = \|\exp_{g_x}^{-1}(gy)\| &\geq \|Dg_x(\exp_x^{-1}(y))\| - C\rho(x, y)^{1+\alpha} \\ &\geq \lfloor Dg_x \rfloor \|\exp_x^{-1}(y)\| - C\rho(x, y)^{1+\alpha} = \rho(x, y) \lfloor Dg_x \rfloor - C\rho(x, y)^{1+\alpha}. \end{aligned}$$

This ends the proof of the lemma. $\square$

The following *rescaled* local entropy formula is due to Mañé [12]. See also [1, 21].

Lemma 4.2. *Let ν be a g -invariant ergodic Borel probability measure with a compact support Λ_ν , $\kappa > 0$ be a positive constant, and $\hat{e}$ be a measurable function on Λ_ν satisfying $0 < \hat{e}(x) \leq \kappa$ for $x \in \Lambda_\nu$ and $\int_{\Lambda_\nu} -\log \hat{e}(x) d\nu(x) < +\infty$. Then for ν -a.e. $x \in M$,*

$$h_\nu(g) = \lim_{t \rightarrow 0} \limsup_{k \rightarrow +\infty} -\frac{1}{k} \log \nu(B_k(x, t\hat{e})),$$

where $B_k(x, t\hat{e}) \stackrel{\Delta}{=} \{y \in \Lambda_\nu : \rho(g^i x, g^i y) < t \cdot \hat{e}(g^i x), \forall 0 \leq i < k\}$.

The following theorem is a weak version of Theorem 3.

Theorem 4.3. *Let g be a $C^{1+\alpha}$ self-map on M for some $\alpha \in (0, 1)$, and ν be a g -invariant ergodic Borel probability measure on M with a compact support Λ_ν . Then for ν -a.e. $x \in M$, $\max\{\chi_\nu^-, 0\} \cdot \bar{d}_\nu(x) \leq h_\nu(g)$.*

Proof. If $\chi_\nu^- \leq 0$, then the result is clear since $h_\nu(g) \geq 0$. Then we assume $\chi_\nu^- > 0$. It is sufficient to show that for any $\delta \in (0, \min\{1, \frac{\chi_\nu^-}{3}\})$, $\bar{d}_\nu(x) \leq \frac{h_\nu(g)}{\chi_\nu^- - 2\delta}$ for ν -a.e. $x \in \Lambda_\nu$.

Now let $\delta \in (0, \min\{1, \frac{\chi_\nu^-}{3}\})$ be fixed. By Lemma 4.1, there exist $r_0 = r_0(g, \Lambda_\nu) > 0$ and $C = C(g, \Lambda_\nu) > 0$ such that

$$(4.2) \quad \rho(gx, gy) \geq \rho(x, y) \lfloor Dg_x \rfloor - C\rho(x, y)^{1+\alpha},$$

for any $x \in \Lambda_\nu$ and $y \in B(x, r_0)$. Define a rescaling function $\hat{e} : \Lambda_\nu \rightarrow (0, r_0]$ by

$$(4.3) \quad \hat{e}(x) = \begin{cases} \min\{r_0, \left(\frac{1-e^{-\delta}}{C} \lfloor Dg_x \rfloor\right)^{\frac{1}{\alpha}}\}, & \text{if } \lfloor Dg_x \rfloor > 0; \\ r_0, & \text{if } \lfloor Dg_x \rfloor = 0. \end{cases}$$

Clearly, $\hat{\epsilon}$ is measurable and $0 < \hat{\epsilon}(x) \leq r_0$ on Λ_ν . Given any $x \in \Lambda_\nu$ and $y \in B(x, \hat{\epsilon}(x))$, if $\lfloor Dg_x \rfloor = 0$, then it is obvious that $\rho(gx, gy) \geq 0 = \rho(x, y) \lfloor Dg_x \rfloor e^{-\delta}$. If $\lfloor Dg_x \rfloor > 0$, then by (4.2) and (4.3) we have

$$\begin{aligned} \rho(gx, gy) &\geq \rho(x, y) \cdot (\lfloor Dg_x \rfloor - C\rho(x, y)^\alpha) \\ &\geq \rho(x, y) \cdot (\lfloor Dg_x \rfloor - C\hat{\epsilon}(x)^\alpha) \geq \rho(x, y) \lfloor Dg_x \rfloor e^{-\delta}. \end{aligned}$$

That is, for any $x \in \Lambda_\nu$ and $y \in B(x, \hat{\epsilon}(x))$,

$$(4.4) \quad \rho(gx, gy) \geq \rho(x, y) \lfloor Dg_x \rfloor e^{-\delta}.$$

Since $\chi_m^- > 0$, $\log \lfloor Dg_{\{\cdot\}} \rfloor$ is integrable with respect to ν . Then from the definition of $\hat{\epsilon}$ we see that $\log \hat{\epsilon}$ is also integrable with respect to ν (this is what we mean by $\hat{\epsilon}$ decreases mildly). By Lemma 4.2, there exists a measurable subset $Y \subset \Lambda_\nu$ with $\nu(Y) = 1$ such that for any $x \in Y$,

$$h_\nu(g) = \lim_{t \rightarrow 0} \limsup_{k \rightarrow +\infty} -\frac{\log \nu(B_k(x, t\hat{\epsilon}))}{k}.$$

Since $\log \hat{\epsilon}$ is integrable with respect to ν , there exists a subset $W \subseteq \Lambda_\nu$ with $\nu(W) = 1$ such that $\lim_{k \rightarrow +\infty} \frac{\log \hat{\epsilon}(g^k x)}{k} = 0$ for any $x \in W$. Then it is not hard to see

$$\lim_{k \rightarrow +\infty} \frac{\log t\hat{\epsilon}(g^k x)}{k} = 0 \text{ for any } x \in W \text{ and } t \in (0, 1).$$

For each $\ell \in \mathbb{N}$ let's consider the set

$$Z_\ell = \{p \in \Lambda_\nu : \frac{1}{n} \sum_{i=0}^{n-1} \log \lfloor Dg_{g^i p} \rfloor \geq \chi_m^- - \delta \text{ for all } n > \ell\}.$$

Let $Z = \bigcup_{\ell \in \mathbb{N}} Z_\ell$. The ergodicity of ν implies that $\nu(Z) = 1$. So $\nu(Z \cap W \cap Y) = 1$.

Once again it is sufficient to show that $\bar{d}_\nu(x) \leq \frac{h_m(g)}{\chi_\nu^- - 2\delta}$ for each point $x \in Z \cap W \cap Y$. Pick such a point $x \in Z \cap W \cap Y$. There is an integer $\ell_x \in \mathbb{N}$ such that $x \in Z_{\ell_x}$. For any $k > \ell_x$ and $t \in (0, 1)$, if $y \in B_{k+1}(x, t\hat{\epsilon})$, then for each $0 \leq i \leq k$, $g^i y \in B(g^i x, t\hat{\epsilon}(g^i x))$ and $\rho(g^{i+1}x, g^{i+1}y) \geq \rho(g^i x, g^i y) \lfloor Dg_{g^i x} \rfloor e^{-\delta}$ by (4.4). Combining these inequalities from $i = k$ to 0 we get

$$\rho(x, y) \leq \rho(g^k x, g^k y) e^{k\delta - \sum_{i=0}^{k-1} \log \lfloor Dg_{g^i x} \rfloor} < t \cdot \hat{\epsilon}(g^k x) \cdot e^{k(2\delta - \chi_\nu^-)}.$$

Let $r_k(t) = t\hat{\epsilon}(g^k x) e^{k(2\delta - \chi_\nu^-)}$. Then we have

$$(4.5) \quad B_{k+1}(x, t\hat{\epsilon}) \subset B(x, t\hat{\epsilon}(g^k x) e^{k(2\delta - \chi_\nu^-)}) = B(x, r_k(t)),$$

for each $k > \ell_x$ and $t \in (0, 1)$. We also have $\lim_{k \rightarrow +\infty} r_k(t) = 0$ and

$$\begin{aligned} \lim_{k \rightarrow +\infty} \frac{\log r_{k+1}(t)}{\log r_k(t)} &= \lim_{k \rightarrow +\infty} \frac{\log t\hat{\epsilon}(g^{k+1}x) + (k+1)(2\delta - \chi_\nu^-)}{\log t\hat{\epsilon}(g^k x) + k(2\delta - \chi_\nu^-)} \\ &= \lim_{k \rightarrow +\infty} \frac{k+1}{k} \cdot \frac{\frac{\log t\hat{\epsilon}(g^{k+1}x)}{k+1} + (2\delta - \chi_\nu^-)}{\frac{\log t\hat{\epsilon}(g^k x)}{k} + (2\delta - \chi_\nu^-)} = 1, \text{ since } x \in W. \end{aligned}$$

Moreover, applying Lemma 2.4 to $r_k = r_k(t)$, we have

$$\begin{aligned}\bar{d}_\nu(x) &= \limsup_{k \rightarrow \infty} \frac{\log \nu(B(x, r_k))}{\log r_k} = \limsup_{k \rightarrow +\infty} \frac{\log \nu(B(x, r_k(t)))}{\log r_k(t)} \\ &= \limsup_{k \rightarrow +\infty} \frac{\log \nu(B(x, r_k(t)))}{k \left(\frac{\log t \hat{\epsilon}(g^k x)}{k} + (2\delta - \chi_\nu^-) \right)} = \limsup_{k \rightarrow +\infty} \frac{\log \nu(B(x, r_k(t)))}{k(2\delta - \chi_\nu^-)} \\ &\leq \limsup_{k \rightarrow +\infty} \frac{\log \nu(B_{k+1}(x, t\hat{\epsilon}))}{k(2\delta - \chi_\nu^-)} \quad (\text{by (4.5)}) \\ &= \frac{1}{\chi_\nu^- - 2\delta} \cdot \limsup_{k \rightarrow +\infty} -\frac{1}{k} \log \nu(B_k(x, t\hat{\epsilon})).\end{aligned}$$

Finally passing $t \rightarrow 0$ in the above inequality we get, by the choice of Y ,

$$\bar{d}_\nu(x) \leq \frac{1}{\chi_\nu^- - 2\delta} \cdot \lim_{t \rightarrow 0} \limsup_{k \rightarrow +\infty} -\frac{1}{k} \log \nu(B_k(x, t\hat{\epsilon})) = \frac{1}{\chi_\nu^- - 2\delta} h_\nu(g),$$

since $x \in Y$. This ends the proof of the theorem. $\square$

We are ready to prove Theorem 3.

Proof of Theorem 3. (1) Let $f : M \rightarrow M$ be of class $C^{1+\alpha}$ for $\alpha \in (0, 1)$ and μ be an ergodic probability measure with $\chi_\mu^s > 0$. Given $k \in \mathbb{N}$, let $p_k \in \mathbb{N}$ and ν_k be given such that ν_k is an f^k -invariant ergodic measure on Λ with $p_k | k$, $f^{p_k} \nu_k = \nu_k$ and $\mu = \frac{1}{p_k} \sum_{j=0}^{p_k-1} f^j \nu_k$. For each $j \in \{0, \dots, p_k-1\}$, by Theorem 4.3 we have $h_{f^j \nu_k}(f^k) \geq \bar{d}_{f^j \nu_k} \cdot \max\{0, \chi_{f^j \nu_k}^-\}$ since the support of $f^j \nu_k$ is contained in Λ . Combining this and Lemma 3.5 (1) and (2) we indeed have $kh_\mu(f) \geq \bar{d}_\mu \cdot \max\{0, \chi_{f^j \nu_k}^-\}$ for each $j \in \{0, \dots, p_k-1\}$. Summing over j from 0 to p_k-1 in this inequality and dividing both sides by $k \cdot p_k$, we get

$$h_\mu(f) \geq \bar{d}_\mu \cdot \left(\frac{1}{k \cdot p_k} \sum_{j=0}^{p_k-1} \chi_{f^j \nu_k}^- \right) = \bar{d}_\mu \cdot \frac{1}{k} \int_M \log [Df_x^k] d\mu(x).$$

Since this inequality holds for each $k \in \mathbb{N}$, we get $h_\mu(f) \geq \bar{d}_\mu \cdot \chi_\mu^s$ by passing k to $+\infty$. This ends the proof of Theorem 3 (1).

(2) It immediately follows from Proposition 3.6 and Theorem 3 (1). $\square$

ACKNOWLEDGEMENTS

The authors are grateful to the referees for their careful readings, useful comments and constructive suggestions which helped them to significantly improve the organization and the readability of this paper.

REFERENCES

1. M. Brin and A. Katok, *On local entropy*, Geometric dynamics (Rio de Janeiro, 1981), 30–38, Lecture Notes in Math., **1007**, Springer, Berlin, 1983. MR730261 (85c:58063)
2. L. Barreira, Ya. Pesin and J. Schmeling, *Dimension and product structure of hyperbolic measures*, Ann. of Math. (2) **149** (1999), 755–783. MR1709302 (2000f:37027)
3. L. Barreira and C. Wolf, *Pointwise dimension and ergodic decompositions*, Ergod. Th. and Dynam. Sys. **26** (2006), 653–671. MR2237462 (2007b:37040)
4. J.P. Eckmann and D. Ruelle, *Ergodic theory of chaos and strange attractors*, Rev. Modern Phys. **57** (1985), no. 3, 617–656. MR800052 (87d:58083a)
5. J. Farmer, E. Ott and J. Yorke, *The dimension of chaotic attractors*, Physica **7D** (1983), 153–180. MR719051 (84m:58022)

6. H. Hu, *Dimensions of invariant sets of expanding maps*, Comm. Math. Phys. **176** (1996), 307–320. MR1374415 (96k:58135)
7. J.F.C. Kingman, *Subadditive Processes*, Lecture Notes in Mathematics, Vol. **539**, Springer, 1976. MR0438477 (55:11388)
8. B. Kalinin and V. Sadovshaya, *On pointwise dimension of non-hyperbolic measures*, Ergod. Th. and Dynam. Sys. **6** (2002), 1783–1801. MR1944404 (2003j:37034)
9. F. Ledrappier, *Some relations between dimension and Lyapunov exponents*, Comm. Math. Phys. **81** (1981), 229–238. MR632758 (83b:58044)
10. F. Ledrappier and M. Misiurewicz, *Dimension of invariant measures of maps with exponent zero*, Ergod. Th. and Dynam. Sys. **5** (1985), 595–610. MR829860 (87j:58058)
11. F. Ledrappier and L.S. Young, *The metric entropy of diffeomorphisms. Part 2; Relation between entropy, exponents and dimension*, Ann. Math. **122** (1985), 540–574. MR819557 (87i:58101b)
12. R. Mañé, *A proof of Pesin's formula*, Ergod. Th. and Dynam. Sys. **1** (1981), 95–102. MR627789 (83b:58042)
13. R. Mañé, *The Hausdorff dimension of invariant probabilities of rational maps*, Dynamical Systems, Valparaíso (1986) 86–117, Lecture Notes in Math., Vol. **1331**, Springer, Berlin, 1988. MR961095 (90j:58073)
14. A. Manning, *The dimensions of the maximal measure for a polynomial map*, Ann. Math. **119** (1984), 425–430. MR740898 (85i:58068)
15. V.I. Oseledets, *A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems*, Tran. Moscow Math. Soc. **19** (1968), 197–221.
16. Ya. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Mathematics, Chicago University Press, 1997. MR1489237 (99b:58003)
17. D. Ruelle, *An inequality for the entropy of differentiable maps*, Bol. Soc. Bras. Mat. **9** (1978), 83–87. MR516310 (80f:58026)
18. P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, Vol. **79**, Springer-Verlag, 1981. MR648108 (84e:28017)
19. L.S. Young, *Dimension, entropy and Lyapunov exponents*, Ergod. Th. and Dynam. Sys. **2** (1982), 109–124. MR684248 (84h:58087)
20. L.S. Young, *Ergodic theory of attractors*, Proc. ICM (Zurich, 1994), Birkhauser, Basel, 1995, 1230–1237. MR1404024 (98b:58101)
21. L.S. Young, *Ergodic theory of differentiable dynamical systems*, Real and Complex Dynamical Systems (B. Branner and P. Hjorth, eds.), Kluwer Acad. Publ., 1995, 293–336. MR1351527 (96h:58101)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI
 ANHUI 230026, PEOPLE'S REPUBLIC OF CHINA
E-mail address: wenh@mail.ustc.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI
 ANHUI 230026, PEOPLE'S REPUBLIC OF CHINA
E-mail address: pfzhang5@mail.ustc.edu.cn