THE MODULI SPACE OF THICKENINGS

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ABSTRACT. Fix K a finite connected CW complex of dimension $\leq k$. An n -thickening of K is a pair

$$(M, f),$$

in which M is a compact n -dimensional manifold and $f: K \rightarrow M$ is a simple homotopy equivalence. This concept was first introduced by C.T.C. Wall approximately 40 years ago. Most of the known results about thickenings are in a range of dimensions depending on k , n and the connectivity of K .

In this paper we remove the connectivity hypothesis on K . We define *moduli space* of n -thickenings $T_n(K)$. We also define a *suspension map* $E: T_n(K) \rightarrow T_{n+1}(K)$ and compute its homotopy fibers in a range depending only on n and k . We will show that these homotopy fibers can be approximated by certain section spaces whose definition depends only on the choice of a certain stable vector bundle over K .

1. INTRODUCTION

Let K be a finite connected cell complex of dimension $\leq k$. An n -thickening of K is a pair

$$(N, h)$$

in which N is a compact smooth manifold and

$$h: K \rightarrow N$$

is a simple homotopy equivalence. In addition, one assumes

- (*codimension* ≥ 3). $k \leq n-3$.
- (π - π -*condition*). The inclusion $\partial N \rightarrow N$ induces an isomorphism of fundamental groups.
- (*Surgery dimensions*). $n \geq 6$.

The above definition is due to C.T.C. Wall [W1]; it generalizes the notion of a “non-stable neighborhood” due to B. Mazur [Ma].

In any case, the n -thickenings of K form the zero simplices of a Δ -set¹ denoted by

$$T_n(K).$$

A 1-simplex of $T_n(K)$ can be thought of as an s -cobordism between n -thickenings. More generally, the j -simplices of $T_n(K)$ are “ Δ^j -blocked” families of thickenings. The actual definition is too technical to give in this introduction; we defer it to §3.

The *moduli space* of n -thickenings

$$\mathbf{T}_n(K)$$

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¹I.e., a simplicial set without the degeneracies.

is the geometric realization of $T_n(K)$. The main purpose of this work is to obtain results about the homotopy type of the moduli space in a certain range of dimensions. It will turn out that this range depends only on n and k . In [W1], Wall computed the set of path components of the moduli space in a range that depends on n , k and the *connectivity* of K ; our results enable one to dispense with Wall's connectivity hypothesis.

We remark that the study of the path components of $\mathbf{T}_n(K)$ addresses a fundamental issue in differential topology:

Question. How does one enumerate the set of compact n -manifolds up to diffeomorphism within a fixed homotopy type?

In classical surgery theory, one imposes an additional constraint: the homotopy type of the boundary of the manifold is held fixed. In the theory of thickenings, this constraint is dropped.

To study the thickening problem, Wall defined a *suspension map*

$$E \mathbf{T}_n(K) \rightarrow \mathbf{T}_{n+1}(K)$$

and studied the deviation from it being an isomorphism on path components:

Theorem (Wall's Suspension Theorem [W1, §5]). *Assume*

- $2n \geq 3k + 3$ and
- K is a $(2k - n + 1)$ -connected based CW complex of dimension $\leq k$.

Then there is an exact sequence of pointed sets

$$\{K \wedge K, S^n\} \xrightarrow{P} \pi_0(\mathbf{T}_n(K)) \xrightarrow{E} \pi_0(\mathbf{T}_{n+1}(K)) \xrightarrow{H} \{K \wedge K, S^{n+1}\}.$$

Here $\{X, Y\}$ denotes the abelian group of stable homotopy classes of stable maps $X \rightarrow Y$. The names of the maps in the theorem are intended to remind one of the classical homotopy theory EHP sequence (cf. [J]). By an exact sequence of pointed sets, we mean that the inverse image of a base point by a map is the image of the previous map.

In our initial study of $\mathbf{T}_n(K)$, we discovered a gap in Wall's proof of the above theorem. The gap has to do with exactness at the term $\pi_0(\mathbf{T}_{n+1}(K))$. In a joint paper with J.R. Klein [A-K], the gap was repaired. The method we introduced in that paper ultimately led to the generalization which appears in the present work.

We now wish to formulate our main results. This will require some preparation. Let $E \rightarrow B$ be a (Hurewicz) fibration equipped with preferred section $B \rightarrow E$. Denote the fiber over $b \in B$ by F_b . Then one can form a new fibration

$$Q_\bullet E \rightarrow B$$

given by applying stable homotopy fiberwise. Explicitly, the fiber $b \in B$ is QF_b , where $Q = \Omega^\infty \Sigma^\infty$ is the stable homotopy functor. We denote by

$$\mathrm{sec}^{\mathrm{st}}(E \rightarrow B)$$

the space of sections of $Q_\bullet E \rightarrow B$. This is the space of *stable sections* of $E \rightarrow B$. It has the structure of an infinite loop space; in particular, its set of path components forms an abelian group.

We will apply this in the following situation: let ξ be a rank $n + 1$ vector bundle over K , and let

$$\mathcal{E}_\xi \rightarrow K \times K$$

be the fibration whose fiber over $(x, y) \in K \times K$ is given by the space

$$S_x^\xi \wedge (\Omega_x^y K)_+$$

where

- S_x^ξ is the based $(n+1)$ -sphere given by taking the one point compactification of the fiber of ξ at $x \in K$;
- $\Omega_x^y K$ is the space of paths from x to y in K ;
- $(\Omega_x^y K)_+$ is the effect of adding a basepoint to $\Omega_x^y K$.

Let us say that an $(n+1)$ -thickening $\alpha = (N, h)$ *compresses* if there exists an n -thickening $\beta = (M, g)$ such that $E(\beta)$ is in the same path component as α in $\mathbf{T}_{n+1}(K)$.

To each $\alpha = (N, h)$ that compresses we can associate a vector bundle ξ over K , which is given by pulling back the tangent bundle of N along the simple homotopy equivalence $h: K \rightarrow N$. We call ξ the *tangential data* associated with α .

Our first result determines the complete obstruction to compressing an $(n+1)$ -thickening in the metastable range.

Theorem A. *To an $(n+1)$ -thickening $\alpha = (N, f)$ with tangential data ξ , one can assign an element*

$$e(\alpha) \in \pi_0(\sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K))$$

*called the **Euler class**. This invariant vanishes when α compresses.*

Conversely, if $3k+2 \leq 2n$, and the Euler class vanishes, then α compresses.

Our second and final main result concerns the determination of the homotopy fibers of the suspension map E .

Theorem B. *Assume α compresses. Let $\mathbf{F}_n(\alpha)$ denote the homotopy fiber of $E: \mathbf{T}_n(K) \rightarrow \mathbf{T}_{n+1}(K)$ at α . Then there is a $(2n - 3k - 2)$ -connected map*

$$\mathcal{H} \mathbf{F}_n(\alpha) \rightarrow \Omega \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K).$$

With the assumptions of Theorem B, the *Hopf invariant* is the homotopy class represented by the composite

$$H: \Omega \mathbf{T}_{n+1}(K) \xrightarrow{\partial} \mathbf{F}_n(\alpha) \xrightarrow{\mathcal{H}} \Omega \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K)$$

in which ∂ represents the boundary map in the long exact sequence of the homotopy fibration, and $\Omega \mathbf{T}_{n+1}(K)$ is the based loop space of $\mathbf{T}_{n+1}(K)$.

Corollary C. *After fixing a basepoint of $\mathbf{T}_n(K)$ with tangential data ξ , there is a long exact homotopy sequence*

$$\begin{array}{ccccccc} \pi_{2n-3k-1}(\sec^{\text{st}}(\mathcal{E}_{\xi \oplus \epsilon} \rightarrow K \times K)) & \xrightarrow{P} & \pi_{2n-3k-2}(\mathbf{T}_n(K)) & \xrightarrow{E} & \pi_{2n-3k-2}(\mathbf{T}_{n+1}(K)) & & \\ & & \searrow H & & & & \\ \pi_{2n-3k-2}(\sec^{\text{st}}(\mathcal{E}_{\xi \oplus \epsilon} \rightarrow K \times K)) & \xrightarrow{P} & \cdots & \xrightarrow{H} & \pi_1(\sec^{\text{st}}(\mathcal{E}_{\xi \oplus \epsilon} \rightarrow K \times K)) & & \\ & & & \searrow P & & & \\ & & \pi_0(\mathbf{T}_n(K)) & \xrightarrow{E} & \pi_0(\mathbf{T}_{n+1}(K)). & & \end{array}$$

Remarks 1.1. (1) The last displayed H in the exact sequence is not asserted to be a homomorphism. This is because the Hopf invariant may fail to deloop.

(2) The above is an exact sequence of sets. In degrees > 1 it is an exact sequence of abelian groups.

(3) If in addition Wall's connectivity condition on K is assumed, then the above results reduce to Wall's suspension theorem. The main observation is that, in Wall's range, the spaces

$$\sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K) \quad \text{and} \quad \text{maps}^{\text{st}}(K \wedge K, S^{n+1})$$

have the same set of path components (where the right side denotes the function space of stable maps from $K \wedge K$ to S^{n+1}). We postpone the proof of the statement to the appendix.

Outline. §2 is largely preliminary material on Δ -sets and manifold ads. In §3 we define the moduli space of thickenings. In §4 we define the stabilization map. §5, §6 and §7 study the extent to which the stabilization map is surjective on path components. In §8 we identify the homotopy fibers of the stabilization map. In §9 we compute some examples. In §10 we show how our results generalize Wall's suspension theorem.

2. PRELIMINARIES

Spaces. We work in the category of compactly generated spaces. A map $X \rightarrow Y$ of non-empty spaces is r -connected if its homotopy fibers are $(r-1)$ -connected (by convention, every non-empty space is (-1) -connected, the empty space is (-2) -connected, and a non-empty space X is s -connected for $s \geq 0$ if $\pi_j(X) = 0$ for $j \leq s$ for all choices of basepoint). A *weak (homotopy) equivalence* is a map which is ∞ -connected. When a map $f: X \rightarrow Y$ is a weak equivalence, we often indicate this by $f: X \xrightarrow{\sim} Y$.

A *weak map* of spaces from X to Y consists of a finite chain of maps

$$X := X_0 \rightarrow X_1 \xleftarrow{\sim} X_2 \rightarrow X_3 \xleftarrow{\sim} \cdots \xleftarrow{\sim} X_{j-1} \rightarrow X_j =: Y$$

in which each left pointing arrow is a weak equivalence. A weak map induces an actual map in the homotopy category of spaces. A weak map is r -connected if all of the right pointing maps are r -connected.

There are set theoretic difficulties in defining the various moduli spaces appearing in this work. The problem is related to the fact that the category of spaces is not small. To get around this, we fix a suitably large Grothendieck universe $\mathcal{U}$ and assume that our spaces are subsets of that universe. It will also be convenient to assume that all manifolds are actually embedded in $\mathbb{R}^\infty$.

Review of Δ -sets. Introduced by Rourke and Sanderson [R-S], Δ -sets are simplicial sets without the degeneracy operators. They arise naturally in classification problems in differential topology, and enjoy many of the properties satisfied by simplicial sets.

Let Δ be the category whose objects are *finite ordered sets* and whose morphisms are the *order preserving injections*. Note that each object of Δ with cardinality $n+1$ is uniquely isomorphic to the ordered set

$$[n] := \{0 < 1 < \cdots < n\}.$$

For this reason, a functor from Δ to any category is determined by specifying a functor on the full subcategory of Δ consisting of the objects $[n]$.

A Δ -set is a functor

$$X: \Delta^{\text{op}} \rightarrow \mathbf{Set},$$

where $\mathbf{Set}$ denotes the category of sets.

In Δ , there are particular morphisms $\delta_i: [n-1] \rightarrow [n]$, for $0 \leq i \leq n$, given by

$$\delta_i(j) = \begin{cases} j, & \text{if } j < i, \\ j+1, & \text{otherwise.} \end{cases}$$

Let X_n denote the value of this functor at the object $[n]$, and let $d_i: X_n \rightarrow X_{n-1}$ be the map induced by δ_i . Then X_n is called the set of n -simplices, and the function d_i is called the i -th face map. A 0-simplex is also called a vertex. The face maps satisfy the identities

$$d_i d_j = d_{j-1} d_i \quad \text{if } i < j.$$

Conversely, if we are given sets $\{X_n\}_{n \in \mathbb{N}}$ and maps d_i satisfying the above identities, then these define a unique Δ -set.

Note that a simplicial set determines a Δ -set by forgetting its degeneracies.

Example 2.1. Let $\Delta[n]$ be the standard n -simplex. This is a Δ -set whose set of m -simplices $\Delta[n]_m$ is the set of all monotone injective functions $f: [m] \rightarrow [n]$ and whose face maps $d_i: \Delta[n]_m \rightarrow \Delta[n]_{m-1}$ are given by $d_i(f) = f \circ \delta_i$ for $0 \leq i \leq m$.

Definition 2.2. 1. A morphism $f: X \rightarrow Y$ of Δ -sets is just a natural transformation of functors.
2. A subfunctor $A \rightarrow X$ of Δ -sets is called a *sub Δ -set*; this is sometimes written as a pair (X, A) .

Example 2.3. Let $\iota_n := id_{[n]}$. The following are sub Δ -sets of $\Delta[n]$:

1. Let $\partial\Delta[n]$, the *boundary* of $\Delta[n]$, be the Δ -set whose j -simplices are specified by

$$(\partial\Delta[n])_j = \begin{cases} \Delta[n]_j, & \text{if } 0 \leq j \leq n-1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus $\partial\Delta[n]$ is the smallest sub Δ -set of $\Delta[n]$ containing the faces δ_j , $0 \leq j \leq n$ of the n -simplex ι_n .

2. Let $\Lambda^k[n]$, called the k -th *horn*, $0 \leq k \leq n$, be the sub Δ -set of $\Delta[n]$ whose j -simplices are given by

$$(\Lambda^k[n])_j = \begin{cases} \Delta[n]_j, & \text{if } 0 \leq j \leq n-2, \\ \Delta[n]_{n-1} - \{\delta_k\}, & \text{if } j = n-1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus $\Lambda^k[n]$ is a sub Δ -set of $\Delta[n]$ which is generated by all faces δ_j of the n -simplex ι_n except its k -th face.

We use

$\Delta\text{-Set}$

to denote the functor category of Δ -sets.

Let Δ^n be the standard affine topological n -simplex, given by the convex hull of $[n]$ thought of as an orthonormal basis of $\mathbb{R}^{n+1}$.

A map $[m] \rightarrow [n]$ then induces an evident (linear) map of convex hulls $a_*: \Delta^m \rightarrow \Delta^n$.

Definition 2.4. The (*geometric*) realization of a Δ -set X is the quotient space

$$|X| = \frac{\coprod X_n \times \Delta^n}{\sim},$$

in which (a^*x, s) is identified with (x, a_*t) . Alternatively, we can describe $|X|$ as the co-end of the functor $\Delta^{\text{op}} \times \Delta \rightarrow \mathbf{Set}$ given by $S \mapsto X_S \times \Delta^S$.

Note that $|X|$ has the structure of a CW complex ([R-S]).

A map $X \rightarrow Y$ is a *weak equivalence* if it induces a weak homotopy equivalence of spaces $|X| \rightarrow |Y|$.

The extension condition. Let $g: X \rightarrow Y$ be a map of Δ -sets. Consider a lifting problem

$$\begin{array}{ccc} \Lambda^i[n] & \longrightarrow & X \\ \cap \downarrow & \nearrow & \downarrow g \\ \Delta[n] & \longrightarrow & Y \end{array}$$

We say that g is a *Kan fibration* if the lifting problem has a solution for any k and n making the above diagram commute.

We say that X is a *Kan Δ -set* if $X \rightarrow *$ is a Kan fibration.

The Yoneda lemma. The j -simplices of a Δ -set X are in bijection with the Δ -maps $\Delta[j] \rightarrow X$. The correspondence is given by

$$(f: \Delta[j] \rightarrow X) \mapsto f(\text{id}_{[j]}),$$

where $\text{id}_{[j]}: [j] \rightarrow [j]$ is the identity.

Homotopy groups. Let X be a Δ -set equipped with basepoint $* \in X_0$. The unique map $a: [j] \rightarrow [0]$ induces a function $a: X_0 \rightarrow X_j$ which in turn describes a basepoint of X_j . For this reason, we can think of X as a *based Δ -set*.

Consider maps of Δ -pairs:

$$f: (\Delta[j], \partial\Delta[j]) \rightarrow (X, *).$$

Two such maps of f and g are said to be *homotopic* if there is a map

$$F: \Delta[j] \times \Delta[1] \rightarrow X$$

such that F maps $\partial\Delta[j] \times \Delta[1]$ to the basepoint, F restricted to $\Delta[j] \times 0$ coincides with f and F restricted to $\Delta[j] \times 1$ coincides with g .

If X is a Kan Δ -set, then homotopy is an equivalence relation. In this case, we define

$$\pi_j(X, *)$$

to be the resulting set of equivalence classes. Then, as usual, $\pi_0(X, *)$ is a based set, $\pi_1(X, *)$ is group and $\pi_j(X, *)$ is an abelian group when $j > 1$.

The function complex. For Δ -sets X and Y define

$$F(X, Y)$$

to be the Δ -set whose j -simplices are maps

$$X \times \Delta[j] \rightarrow Y.$$

The face map $d_i: F(X, Y)_j \rightarrow F(X, Y)_{j-1}$ is given as follows: by the Yoneda lemma an element of $F(X, Y)_j$ can be described as a certain kind of function $X \rightarrow Y_j$ which we can post compose with the i -th face map of Y_j to get a function $X \rightarrow Y_{j-1}$. The latter then corresponds to an element of $F(X, Y)_{j-1}$ (we omit the details).

The definition of the function complex generalizes to function spaces of pairs, etc., in the evident way.

The function complex has the “correct” homotopy type when Y is Kan in the sense that the evident map

$$|F(X, Y)| \rightarrow \text{maps}(|X|, |Y|)$$

is a weak equivalence.

The homotopy fiber. We will only need to describe the homotopy fiber of an inclusion.

Let (X, A) be a Δ -pair, and assume that X is based. The *homotopy fiber* F of $A \rightarrow X$ is the Δ -set whose j -simplices are given by the function complex of maps of triads

$$(\Delta[j] \times \Delta[1], \Delta[j] \times 1, \Delta[j] \times 0) \rightarrow (X, A, *).$$

If X is Kan, then F has the correct homotopy type in the sense that $|F|$ is identified with the homotopy fiber at $*$ of the map $|A| \rightarrow |X|$.

(Very special) manifold ads. Let $n \geq 2$ be an integer. Let M^m be a (smooth) manifold equipped with submanifolds

$$M(0), M(1), \dots, M(n-1) \subset \partial M$$

which are codimension zero inside ∂M . Set $[n-1] = \{0, 1, \dots, n-1\}$.

We will assume:

- The total intersection of the M_i is empty.
- If $i \neq j$, then $(M(i), \partial M(i))$ is transverse to $(M(j), \partial M(j))$ and $M(i) \cap M(j)$ is a codimension zero submanifold of $\partial M(j)$.
- For each non-trivial subset $S \subset [n-1]$, let $M(S)$ denote the intersection of the $M(j)$ for $j \in S$. Then $M(S)$ is a manifold, and if $i \notin S$, then $M(i)$ transversally intersects $M(S)$ in a codimension zero submanifold of $\partial M(S)$.

Here is some more notation: for $S \subset [n-1]$ a non-empty subset, we define

$$M_S := M([n-1] - S).$$

Our convention will be to set $M_{[n-1]} := M$.

The above definition is a special case of what is called an m -manifold $(n+1)$ -ad. This terminology, introduced by Wall [W2, §0], arises from the fact that one can specify the data as an $(n+1)$ -tuple

$$(M; M(0), M(1), \dots, M(n-1)).$$

Example 2.5. A 3-ad in the above sense consists of

$$(M; M_1, M_0)$$

in which $M_1 \amalg M_0$ is a codimension zero compact submanifold of ∂M . Define

$$\partial_1 M := \text{closure}(\partial M - (M_0 \amalg M_1)).$$

Then $(M, \partial_1 M)$ is a cobordism between $(M_0, \partial M_0)$ and $(M_1, \partial M_1)$.

We can also specify ads in the language of functors. Let $\mathbf{2}^n$ be the poset of subsets of $[n-1]$. Let $\mathcal{M}$ denote the category whose objects are smooth compact manifolds with boundary and whose morphisms are inclusions of one smooth manifold in the boundary of another. Then the above determines an intersection preserving functor

$$M_\bullet : \mathbf{2}^n \rightarrow \mathcal{M}$$

defined by

$$S \mapsto M_S.$$

Here is an example of such an $(n+1)$ -ad: let N be a compact manifold, possibly with boundary. For each non-empty $S \subset [n-1]$, form

$$M_S := N \times \Delta^S.$$

(We use the standard method of “rounding” corners of Δ^{n-1} to consider M_S as a smooth manifold with boundary.)

For an $(n+1)$ -ad $M_\bullet$, set

$$\partial_0 M_S := \coprod_{T \subsetneq S} M_T$$

and set

$$\partial_1 M_S := \text{closure}(\partial M_S - \partial_0 M_S).$$

Then we have a decomposition

$$\partial M_S = \partial_0 M_S \cup \partial_1 M_S.$$

3. THICKENINGS

Let K be a finite connected cell complex of dimension $\leq k$. An n -thickening of K is a pair

$$(M, f)$$

in which M is an n -dimensional compact manifold with boundary ∂M , and $f: K \rightarrow M$ is a simple homotopy equivalence. We additionally assume

- $k \leq n-3$;
- M satisfies the π - π -condition, i.e., ∂M is connected and the inclusion $\partial M \rightarrow M$ induces an isomorphism of fundamental groups.

A few words are in order. The first condition avoids knotting phenomena. The second condition ensures that the map $\partial M \rightarrow M$ is $(n-k-1)$ -connected, which is a homotopy theoretic way of thinking of the spine of M as being K .

As promised in the introduction, we will now describe the *moduli space* of n -thickenings of K . This will be the geometric realization of a Δ -set whose j -simplices are a “ Δ^j -blocked” version of a thickening.

Definition 3.1. Let K^k be a connected finite cell complex of dimension k . We denote by

$$T_n(K)$$

the Δ -set in which a 0-simplex is given by an n -thickening (M, f) of K .

If $j \geq 1$, then a j -simplex is specified by a pair

$$(M_\bullet, f_\bullet)$$

in which $M_\bullet$ is an $(n+j)$ -dimensional manifold $(j+2)$ -ad and

$$f_\bullet: K \times \Delta^j \rightarrow M_\bullet$$

is a simple j -blocked homotopy equivalence. This means that $f_\bullet$ respects faces and for each non-empty subset $S \subset [j]$ the map

$$f_S: K \times \Delta^S \rightarrow M_S$$

is a simple homotopy equivalence. We also require

- The inclusion

$$\partial_1 M_S \rightarrow M_S$$

is an isomorphism of path components and also of fundamental groups.

- For each $V \subset U$, the map

$$(M_V, \partial_1 M_V) \subset (M_U, \partial_1 M_U)$$

is a simple homotopy equivalence of pairs.

Example 3.2. A 1-simplex of $T_n(K)$ amounts to an s -cobordism $(M_{01}, \partial_1 M_{01})$ between $(M_0, \partial M_0)$ and $(M_1, \partial M_1)$, together with a simple homotopy equivalence

$$(K \times [0, 1], K \times 0, K \times 1) \xrightarrow{\sim} (M_{01}, M_0, M_1).$$

Proposition 3.3. $T_n(K)$ is a Kan Δ -set.

Proof. A map $\Lambda^k[j] \rightarrow T_n(K)$ amounts to specifying j simplices of $T_n(K)_{j-1}$:

$$f_i: K \times \Delta_i^{j-1} \rightarrow M_i, \quad i = 0, \dots, k-1, k+1, \dots, j,$$

such that

$$d_i f_\ell = d_{\ell-1} f_i \text{ if } i < \ell, i \neq k, \ell \neq k.$$

Then we have to find a j -simplex

$$F: K \times \Delta^m \rightarrow N$$

in $T_n(K)$ such that

$$d_i F = f_i.$$

Let us think of Δ^j as a subpolyhedron of

$$|\Lambda^k[j]| \times [0, 1].$$

Then there is an embedded isotopy from the identity map of this space to an embedding with image Δ^j (see, e.g., [Ho, p. 295]). Use this end of this isotopy to identify $|\Lambda^k[j]| \times [0, 1]$ with Δ^j . Next, set

$$N := \left(\bigcup_i M_i \right) \times [0, 1].$$

Then there we can define an extension

$$F: K \times |\Lambda^k[j]| \times [0, 1] \rightarrow N$$

by the formula

$$F(x, (y, t)) = (f_i(x, y), t) \text{ if } (x, y) \in K \times \Delta_i^{j-1}.$$

□

4. STABILIZATION

The suspension map. We consider the map

$$E: T_n(K) \rightarrow T_{n+1}(K)$$

defined by

$$(M_\bullet, f_\bullet) \mapsto (M_\bullet \times [0, 1], f'_\bullet)$$

in which, for each S , the map f'_S is the composite

$$K \times \Delta^S \xrightarrow{f_S} M_S = \frac{1}{2} \times M_S \xrightarrow{\subset} [0, 1] \times M_S.$$

Let

$$T_\infty(K) := \lim_{n \rightarrow \infty} T_n(K)$$

be the result of taking the colimit of the sequence defined by E .

Proposition 4.1. $T_\infty(K)$ is a Kan Δ -set.

Proof. Follows from that fact that $T_n(K)$ is Kan and from the small object argument (i.e., the compactness of $\Delta[j]$). $\square$

The tangent map. A j -simplex $(N_\bullet, h_\bullet)$ of $T_n(K)$ has an underlying $(n+j)$ -dimensional manifold $N_{[j]}$ (which, by our conventions, is embedded in $\mathbb{R}^\infty$). Let BO_k be the Grassmannian of k -planes in $\mathbb{R}^\infty$. The usual Gauss map for the tangent bundle of $N_{[j]}$ gives a

$$N_{[j]} \rightarrow BO_{n+j}.$$

We precompose this map with $h_{[j]}: K \times \Delta^j \rightarrow N_{[j]}$ to get a map

$$K \times \Delta^j \rightarrow BO_{n+j}.$$

Let BO be the direct limit of BO_k with respect to the inclusions

$$BO_k \subset BO_{k+1}$$

defined by mapping a k -plane V to the k -plane $\mathbb{R} \oplus V$.

Let

$$\text{maps}(K, BO)$$

denote the total singular complex of the space of maps $K \rightarrow BO$, considered as a Δ -set. This satisfies the Kan condition.

Now let j vary. Then the above construction assembles to a map of Δ -sets

$$\tau: T_n(K) \rightarrow \text{maps}(K, BO)$$

called the *tangent map*.

Note that with respect to τ , the stabilization map E corresponds to the self-equivalence of BO given by adding a copy of the real line.

Theorem 4.2. The tangent map τ defines a weak equivalence of Δ -sets,

$$T_\infty(K) \simeq \text{map}(K, BO).$$

Proof. This is basically a result of Wall [W1, Prop. 5.1] adapted to block families. We will sketch the argument in degree zero first, and then explain the changes which are needed to adapt the argument to block families.

Observe that if $L \xrightarrow{\sim} K$ is a homotopy equivalence of cell complexes, then composition gives an induced weak equivalence

$$T_n(K) \xrightarrow{\sim} T_n(L).$$

This allows us to replace K by a suitable compact framed manifold within its homotopy type. Hence, without loss in generality, it is enough to prove the result when K itself is a compact smooth manifold.

The next point is that a map $K \rightarrow BO$ is represented by a *smooth* vector bundle $\xi E(\xi) \rightarrow K$ equipped with a fiberwise inner product. The unit disk bundle $D(\xi)$ is then a smooth manifold with boundary $D(\xi|_{\partial K}) \cup S(\xi)$, where $\xi|_{\partial K}$ is the restriction of ξ to ∂K and $S(\xi)$ is the unit sphere bundle of ξ . The zero section $z: K \rightarrow D(\xi)$ then defines a thickening

$$(D(\xi), z).$$

The tangent bundle of $D(\xi)$ pulled back along z is canonically identified with ξ . Hence, $\tau(D(\xi), z) \cong \xi$. This shows that the map τ is surjective on π_0 .

To obtain surjectivity in higher degrees, the argument is modified as follows: a homotopy class in $x \in \pi_j(\text{map}(K, BO))$ is represented by a j -simplex $K \times \Delta^j \rightarrow BO$ such that the restriction to $K \times \partial\Delta^j$ is constant in the second factor, i.e., it is a composite

$$K \times \partial\Delta^j \xrightarrow{\text{project}} K \xrightarrow{\alpha} BO.$$

Furthermore, by the degree zero case, we can assume that $\alpha = \tau(N, h)$ for some thickening (N, h) .

Applying the same argument as in degree zero, but now working relatively, we can construct a j -simplex of $T_\infty(K)$ which is constant along the boundary (i.e., it has the form $(N \times \sigma, h \times \text{id})$ for each face σ of Δ^j), and whose tangential data coincides with x . This j -simplex yields the desired lift of x to $\pi_j(T_\infty(K))$. This completes the proof of surjectivity.

We now discuss injectivity in degree zero. Fix two n -thickenings (N_0, h_0) and (N_1, h_1) . Let ξ_i be the effect of applying τ to (N_i, h_i) , and assume that there is a one simplex

$$\xi K \times \Delta^1 \rightarrow BO$$

which restricts to ξ_i on $K \times i$.

Then there is a simple homotopy equivalence

$$f N_0 \xrightarrow{\sim} N_1$$

and an isomorphism of stable tangent bundles

$$\xi_0 \cong f^* \xi_1.$$

Since we are stabilizing, we can assume that $n \gg k$. Then by transversality and Smale-Hirsch theory ([H], [Sm]), $f N_0 \rightarrow N_1$ is homotopic to an embedding into the interior of N_1 , which also has the property that $f^*(\tau_{N_1})$ is bundle isomorphic to τ_{N_0} . Call this embedding

$$e: N_0 \hookrightarrow N_1.$$

Then

$$W = \text{closure}(N_1 - e(N_0))$$

is an s -cobordism between ∂N_1 and $e(\partial N_0)$. By the s -cobordism theorem, W is diffeomorphic rel $e(\partial N_0)$ to a collar manifold $e(\partial N_0) \times [0, 1]$.

Then the manifold

$$N_0 \cup W \cup N_1$$

is diffeomorphic to the boundary of $N_{01} := N_1 \times \Delta^1$ in which N_0 is identified with $N_1 \times 0$ and N_1 is identified with $N_1 \times 1$. This gives a 3-ad $N_\bullet$ whose faces are N_0 and N_1 .

Furthermore, there is an evident simple block equivalence

$$h_\bullet K \times \Delta^1 \rightarrow N_\bullet$$

which extends the maps h_i . The pair $(N_\bullet, h_\bullet)$ gives a 1-simplex of $T_\infty(K)$ whose faces are (N_i, h_i) . This shows injectivity in degree zero.

The case of injectivity in higher degrees is similar, and will be left to the reader. $\square$

5. COMPRESSION

In the previous section we defined the suspension map

$$E: T_n(K) \rightarrow T_{n+1}(K).$$

In this section we measure the deviation of E from being surjective on path components.

Compression and sectioning.

Definition 5.1. An $(n+1)$ -thickening $\alpha = (N, f)$ *compresses* if there is an n -thickening β and a 1-simplex σ of $T_{n+1}(K)$ such that

$$d_0\sigma = \alpha \quad \text{and} \quad d_1\sigma = E(\beta).$$

Definition 5.2. The *fiberwise suspension* of a map of spaces $E \rightarrow B$ is the map $S_BE \rightarrow B$ in which

$$S_BE = B \times 0 \cup E \times [0, 1] \cup B \times 1$$

is the evident double mapping cylinder.

When $E \rightarrow B$ is a (Hurewicz) fibration, then so is $S_BE \rightarrow B$ (Strøm [Str, p. 436]). There are also two preferred sections

$$s_-, s_+: B \rightarrow S_BE.$$

Although the following result is classical (cf. Larmore [L, Th. 4.3]; see also Becker[B]), we include a proof for the sake of completeness.

Proposition 5.3. *Assume that $E \rightarrow B$ is a fibration. If $E \rightarrow B$ admits a section, then s_- and s_+ are section homotopic.*

Conversely, let $E \rightarrow B$ be $(r+1)$ -connected and assume that B has the homotopy type of a complex of dimension $\leq 2r+1$. Furthermore, assume that s_- and s_+ are section homotopic. Then $E \rightarrow B$ admits a section.

Proof. A section $B \rightarrow E$ can be fiberwise suspended over B to give a map

$$B \times [0, 1] = S_BB \rightarrow S_BE.$$

This map is the desired homotopy from s_- to s_+ .

Conversely, the commutative square

$$\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow & & \downarrow s_+ \\ B & \xrightarrow{s_-} & S_BE \end{array}$$

is homotopy cocartesian. By the Blakers-Massey theorem (see, e.g., [G, p. 8]), it is also $(2r+1)$ -cartesian. This implies that the map from E into the homotopy pullback of $s_{\pm}$ (given by choosing a homotopy from s_- to s_+) is $(2r+1)$ -connected. The conclusion follows. $\square$

The next lemma shows that homotopy and section homotopy coincide. Although it will not be needed, we include it for the sake of clarification.

Lemma 5.4 (Crabb-James [C-J, p. 29]). *Let $E \rightarrow B$ be a Hurewicz fibration and suppose that $s, t: B \rightarrow E$ are sections. Then s and t are homotopic if and only if they are section homotopic.*

Returning to the original problem of compressing α , we let E_{α} be the homotopy pullback of the diagram

$$K \xrightarrow[\sim]{f} N \xleftarrow{\supset} \partial N.$$

Then $E_{\alpha} \rightarrow K$ is a fibration.

Theorem 5.5. *If α compresses, then the preferred sections*

$$s_{\pm}: K \rightarrow S_K E_{\alpha}$$

are homotopic.

Conversely, assume $3k+2 \leq 2n$, where K has the homotopy type of a complex of dimension $\leq k$. Then the existence of a homotopy between s_- and s_+ implies that α compresses.

Proof. To prove the first part, first assume that $\alpha = E(\beta)$, where $\beta = (M, g)$. Then $N = M \times [0, 1]$ and $M \times 0 \subset \partial N$. The map $g: K \rightarrow M \times 0$ followed by that inclusion into ∂N then gives rise to the section of $E_{\alpha} \rightarrow K$ in this instance. By the first part of Proposition 5.3, we get a homotopy between s_- and s_+ in this instance.

The more general case when α compresses uses the s -cobordism between ∂N and $\partial(M \times [0, 1])$, which is provided by the 1-simplex σ .

We now prove the converse. Our range assumptions imply that the map $E_{\alpha} \rightarrow K$ is $(n-k)$ -connected. Hence, by the second part of Proposition 5.3, the existence of a homotopy between s_+ and s_- enables us to conclude that the simple homotopy equivalence $f: K \rightarrow N$ factors through ∂N up to homotopy. Let $f_1: K \rightarrow \partial N$ be a choice of such a factorization.

The rest of the proof appeals to Wall's original argument which we will now sketch. The map $K \rightarrow \partial N$ is $(n-k-1)$ -connected (since $\partial N \rightarrow N$ is $(n-k)$ -connected). Our assumptions guarantee that

$$n-k-1 \geq 2k-n+1,$$

so by the Stallings-Wall embedding theorem (see 5.6 below) one can find an *embedding up to homotopy* of the map, i.e., there is a codimension zero compact submanifold $M \subset \partial N$ (satisfying the π - π -condition) and a simple homotopy equivalence $g: K \xrightarrow{\sim} M$ such that the composite

$$K \xrightarrow{g} M \subset \partial N$$

is homotopic to the given map. Then (M, g) is an n -thickening such that (M, g) and α are the faces of a one simplex of $T_{n+1}(K)$ where the underlying manifold of

the one simplex is

$$N \cup_{M \times 0} M \times [0, 1].$$

□

Theorem 5.6 (Stallings [Sta, p. 5], Wall [W1, p. 76]). *Let K be a connected finite complex. Let M be a manifold of dimension n with boundary. Assume $\dim K \leq k \leq n-3$. If $f: K \rightarrow M$ is $(2k-n+1)$ -connected, then f embeds up to homotopy.*

The Euler class. For an $(n+1)$ -thickening $\alpha = (N, f)$, set

$$\xi := f^*(\tau_N).$$

For $x \in K$ let

$$S_x^\xi$$

denote the one point compactification of the fiber of ξ at x . For $(x, y) \in K \times K$, let

$$\Omega_x^y K$$

denote the space of paths in K which begin at x and end at y .

There is a fibration

$$\mathcal{E}_\xi \rightarrow K \times K$$

whose fiber at $(x, y) \in K \times K$ is the space

$$S_x^\xi \wedge (\Omega_x^y K)_+.$$

Explicitly, if we equip ξ with a fiberwise inner product, then $\mathcal{E}_\xi$ can be described by the space of pairs (v, λ) in which λ is a path in K and v is a point in the unit disk of ξ at $\lambda(0)$, subject to the identification $(v, \lambda) \sim (w, \beta)$ for all v and w of length one. With this description the map $\mathcal{E}_\xi \rightarrow K \times K$ is given by $(v, \lambda) \mapsto (\lambda(0), \lambda(1))$. This fibration comes equipped with a preferred section $s: K \times K \rightarrow \mathcal{E}_\xi$.

Definition 5.7. Let $E \rightarrow B$ be a (Hurewicz) fibration equipped with a preferred section $s: B \rightarrow E$. Let

$$Q_\bullet E \rightarrow B$$

be the effect of applying the stable homotopy functor $Q = \Omega^\infty \Sigma^\infty$ fiberwise to $E \rightarrow B$ ([C-J]).

The *stable section space* of $E \rightarrow B$ is the space of sections of $Q_\bullet E \rightarrow B$. It will be denoted by

$$\sec^{\text{st}}(E \rightarrow B).$$

Remark 5.8. The stable section space has the structure of an infinite loop space. In particular, its set of path components is an abelian group.

Recall that $E_\alpha \rightarrow K$ is the fibration given by taking the homotopy pullback of $\partial N \rightarrow N$ along $f: K \rightarrow N$. The fiberwise suspension $S_K E_\alpha \rightarrow K$ has two preferred sections $s_\pm$. Our convention will be to use s_- to form its stable section space.

Theorem 5.9. *There is a weak equivalence of stable section spaces*

$$\sec^{\text{st}}(S_K E_\alpha \rightarrow K) \simeq \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K)$$

which can be made canonical up to contractible choice.

The proof of the theorem is deferred until the next section.

Remark 5.10. The right hand side only depends on the *stable isomorphism type* of the bundle ξ , not on the choice of the thickening $\alpha = (N, f)$.

Definition 5.11. The *Euler class*

$$e(\alpha) \in \pi_0(\sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K))$$

is the element which corresponds to the section $s_+ \in \sec^{\text{st}}(S_K E_\alpha \rightarrow K)$ via the weak equivalence of Theorem 5.9.

Lemma 5.12. *Suppose $p: E \rightarrow B$ is a fibration equipped with section $B \rightarrow E$. Assume that p is $(r+1)$ -connected and that B has the homotopy type of a cell complex of dimension $\leq b$. Then the evident map*

$$\sec(E \rightarrow B) \rightarrow \sec^{\text{st}}(E \rightarrow B)$$

is $(2r+1-b)$ -connected.

Proof. Let F denote any fiber of p . Then F is an r -connected based space and the map

$$F \rightarrow QF$$

is $(2r+1)$ -connected by the Freudenthal suspension theorem. The long exact homotopy sequence and the 5-lemma then imply that the map

$$E \rightarrow Q_\bullet E$$

is also $(2r+1)$ -connected. The induced map of section spaces is then $(2r+1-b)$ -connected by obstruction theory. $\square$

Corollary 5.13. *If α compresses, then $e(\alpha)$ is trivial. Conversely, if $3k+2 \leq 2n$ and $e(\alpha)$ is trivial, then α compresses.*

We note that if $n \geq 2k$, then the section space in Lemma 5.12 is path connected (by obstruction theory). Then $H(\alpha) = 0$ for any vertex α in $T_{n+1}(K)$, and it follows that E is surjective in this case.

6. EQUIVARIANT HOMOTOPY THEORY

Let B be a connected space. Define

$$\mathbf{Top}_{/B}$$

to be the category of spaces “over B ”. Specifically, an *object* is a space X and a choice of (structure) map $X \rightarrow B$. A *morphism* is a map of spaces which is compatible with the structure maps. Call a morphism a *weak equivalence* if it is a weak homotopy equivalence of underlying spaces. It can be shown that this notion of weak equivalence arises from a Quillen model structure on $\mathbf{Top}_{/B}$, but the full strength of this fact will not be needed.

Next, let G be a topological group which is cofibrant when considered as a topological space. Define

$$\mathbf{Top}^G$$

to be the category of (left) G -spaces and G -equivariant maps. A *weak equivalence* in this instance is deemed to be a morphism whose underlying map of (unequivariant) spaces is a weak homotopy equivalence. Again, this arises from a Quillen model structure.

In each of these cases, let the homotopy category be formed in the usual way by formally inverting the weak equivalences.

The correspondence. The prototype idea is that any connected based space is naturally the weak homotopy type of a classifying space of a topological group, the latter being a suitable model for the loop space.

Specifically, if B is a connected based space, then a construction of Kan gives a natural weak equivalence

$$B \simeq BG,$$

where G is a topological group object in the category of compactly generated spaces. The definition of G is as follows. Let $G.(B)$ denote the Kan loop group of the total singular complex of B . Then G is taken to be the geometric realization of $G.(B)$.

Define a functor

$$\mathbf{Top}^G \rightarrow \mathbf{Top}_{/BG}$$

by $X \rightarrow X \times_G EG$ (the Borel construction).

Proposition 6.1 (cf. [D-D-K, Th. 2.1 and Cor. 2.5]). *This functor can be derived to give an equivalence of homotopy categories*

$$ho\mathbf{Top}_{/BG} \simeq ho\mathbf{Top}^G.$$

It will be helpful to have a functor which gives rise to the inverse equivalence.

To this end, let $X \rightarrow BG$ be a map of spaces. Define its *thick homotopy fiber* by

$$F := \text{pullback}(EG \rightarrow BG \leftarrow X)$$

in which $EG \rightarrow BG$ is the universal principal G -bundle. Then $F \subset EG \times X$ is a subspace, where G acts (freely and on the left) on the first factor EG and therefore on $EG \times X$ using the trivial action on X . It is readily verified that this restricts to an action of G on F . The assignment $X \mapsto F$ defines a functor $\mathbf{Top}_{/BG} \rightarrow \mathbf{Top}^G$ which induces the inverse equivalence on homotopy categories.

Here is a little more detail which contains the crux of the statement: as EG is a contractible space, we see that F is a model for the (usual) homotopy fiber of $X \rightarrow BG$ at the preferred basepoint. The following shows how to recover $X \rightarrow BG$ up to homotopy from its thick homotopy fiber:

Lemma 6.2. *Assume that X is a cofibrant space. Then in the homotopy category of spaces over BG there is a canonical weak equivalence between the Borel construction*

$$EG \times_G F \rightarrow BG$$

and $X \rightarrow BG$.

Proof. The displayed map factors as

$$EG \times_G F \rightarrow X \rightarrow BG,$$

in which the first map is obtained from the equivariant weak equivalence

$$EG \times F \rightarrow \text{pt} \times F$$

by taking G -orbits (we are using the observation that X is obtained from F by taking G -orbits). Since F has the structure of a free G -CW complex (by covering homotopy property), both the source and the target of the above equivariant weak equivalence are fibrant and cofibrant. This implies that the map is an equivariant homotopy equivalence. Hence the map of orbit spaces is a homotopy equivalence. $\square$

Dictionary. The following table gives the correspondence between various constructions in the categories $\mathbf{Top}_B$ and $G\text{-}\mathbf{Top}$:

Fibration $E \rightarrow B = BG$	$F =$ thick homotopy fiber
section $B \rightarrow E$	$F =$ based G -space
$S_BE \rightarrow B$	$SF =$ unreduced suspension
$\sec(E \rightarrow B)$	$F^{hG} =$ homotopy fixed points
$\sec^{\text{st}}(S_BE \rightarrow B)$	$(QSF)^{hG}$

Naive G -spectra. A (*naive*) G -spectrum E consists of based (left) G -spaces E_i for $i \geq 0$ and equivariant based maps $\Sigma E_i \rightarrow E_{i+1}$ (by convention, G acts trivially on the suspension coordinate of ΣE_i). A *morphism* $E \rightarrow E'$ of G -spectra consists of equivariant based maps $E_i \rightarrow E'_i$ which are compatible with the structure maps. A *weak equivalence* of G -spectra is a morphism that yields an isomorphism on homotopy groups. A G -spectrum E is said to be *fibrant* if the adjoint maps $E_i \rightarrow \Omega E_{i+1}$ are weak homotopy equivalences. Up to a natural weak equivalence, one can always approximate a G -spectrum by one which is fibrant (this procedure is called *fibrant replacement*). The *zero-th space* E_0 of a fibrant spectrum E is often denoted $\Omega^\infty E$. If E isn't fibrant, we take $\Omega^\infty E$ to be the zero-th space of its fibrant replacement.

If X is a based G -space, then its *suspension spectrum* $\Sigma^\infty X$ is a G -spectrum with j -th space $Q(S^j \wedge X)$, where $Q = \Omega^\infty \Sigma^\infty$ is the stable homotopy functor (here G acts trivially on the suspension coordinates). We use the notation

$$S[G]$$

for the suspension spectrum of G_+ considered as a $(G \times G)$ -spectrum (the action on G_+ is given by left multiplication with respect to the first factor of $G \times G$ and right multiplication composed with the involution $g \mapsto g^{-1}$ on the second factor).

Let E be a fibrant G -spectrum. Then its *homotopy fixed point spectrum* E^{hG} is

$$\text{map}_G(EG_+, E),$$

where the j -th space of the latter is given by the mapping space of equivariant maps $EG_+ \rightarrow E_j$. Here EG is the free contractible G -space (arising from the bar construction), and EG_+ is the effect of adding a basepoint to EG .

The dualizing spectrum. Using the left factor action G on $S[G]$ we can form the homotopy fixed point spectrum

$$D_G := S[G]^{hG}$$

(cf. Klein [K]). This is called the *dualizing spectrum of G* . The right factor action of G on $S[G]$ gives D_G the structure of a G -spectrum.

7. PROOF OF THEOREM 5.9

The fibration $\mathcal{E}_\xi \rightarrow K \times K$. Recall that the fiber of $\mathcal{E}_\xi \rightarrow K \times K$ at (x, y) is identified with

$$S_x^\xi \wedge (\Omega_x^y K)_+.$$

Let $*$ $\in K$ denote the basepoint. In particular, the fiber at $(*, *)$ is given by

$$S_*^\xi \wedge (\Omega K)_+,$$

in which ΩK is the based loop space of K . We will give an equivariant model for this space.

In what follows with slight abuse of notation, we identify K with BG . Then we think of ξ as a vector bundle over BG . Let

$$S^\xi := \tilde{D}(\xi)/\tilde{S}(\xi),$$

where $\tilde{D}(\xi)$ (resp. $\tilde{S}(\xi)$) denotes the thick homotopy fiber of $D(\xi) \rightarrow BG$ (resp. $S(\xi) \rightarrow BG$). Then S^ξ is a G -equivariant model for S_*^ξ . Extend this to an action of $G \times G$ by letting the second factor act trivially. Let G^{ad} denote G with the action of $G \times G$ given by

$$(g, h) \cdot x := gxh^{-1}.$$

Let G_+^{ad} be the effect of adding a disjoint basepoint to G^{ad} .

Lastly, set

$$S^\xi[G] := S^\xi \wedge G_+^{\text{ad}},$$

and give this the diagonal $G \times G$ action.

With respect to these conventions we obtain

Lemma 7.1. *In the category of spaces over $BG \times BG$, there is a weak equivalence*

$$\mathcal{E}_\xi \simeq (EG \times EG) \times_{G \times G} S^\xi[G]$$

where $G \times G$ acts diagonally on $S^\xi \wedge G_+^{\text{ad}}$. In particular, there is a weak equivalence of infinite loop spaces

$$\text{sec}^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K) \simeq (QS^\xi[G])^{hG \times G}.$$

Proof. The fibration

$$\mathcal{E}_\xi \rightarrow K \times K$$

is the *fiberwise smash product* of two fibrations over $K \times K$. The first of these fibrations is given as follows: consider the vector bundle ξ over K , apply fiberwise one point compactification and take the resulting pullback along first factor projection $K \times K \rightarrow K$. This results in the fibration $E_1 \rightarrow K \times K$ whose fiber at (x, y) is S_x^ξ .

The second fibration over $K \times K$ can be described as follows: let $K^{[0,1]}$ be the space of paths in K , and let

$$K^{[0,1]} \rightarrow K \times K$$

be the fibration which evaluates a path at its endpoints. The fiber of this fibration at (x, y) is then precisely $\Omega_x^y K$. Consequently, if we add a basepoint to each fiber, we obtain a fibration $E_2 \rightarrow K \times K$ whose fiber is $(\Omega_x^y K)_+$.

Clearly, we have a homeomorphism over $K \times K$,

$$\mathcal{E}_\xi \cong E_1 \wedge_{K \times K} E_2,$$

where the right side denotes the fiberwise smash product of E_1 and E_2 .

The map $K^{[0,1]} \rightarrow K \times K$ is the effect of converting the diagonal $K \rightarrow K \times K$ into a fibration. In terms of the identification $K \simeq BG$, this diagonal map can be identified with the Borel construction of $G \times G$ acting on G^{ad} . The reason for this

is that one can consider

$$(EG \times EG) \times_{G \times G} G^{\text{ad}}$$

as being formed in 2 steps:

- (1) take the Borel construction on G^{ad} with respect to the subgroup $G \times 1 \subset G \times G$, and then
- (2) the Borel construction with respect to $1 \times G$.

The result of the first step gives the G -space EG equipped with the usual map to BG . Consequently, the application of the second step gives the evident map

$$EG \times_G EG \rightarrow BG \times BG.$$

The total space of this fibration is identified with BG by taking the G -orbits of the diagonal map $EG \rightarrow EG \times EG$. With respect to this identification, the displayed map coincides with the diagonal of BG .

The proof is completed by observing that the fiberwise smash product corresponds via the dictionary to the smash product of equivariant spaces. $\square$

The exponential law. Let E be a (fibrant) $(G \times H)$ -spectrum. Then we have an equivalence of homotopy fixed point spectrum in two steps:

$$E^{hG \times H} \simeq (E^{hH})^{hG}.$$

If we apply this reasoning to $G \times G$ acting on the spectrum $(\Sigma^\infty S^\xi[G])$, we obtain

Lemma 7.2. *There is a weak equivalence of spectra*

$$(\Sigma^\infty S^\xi[G])^{hG \times G} \simeq (S^\xi \wedge D_G)^{hG}.$$

Proof.

$$\begin{aligned} (\Sigma^\infty S^\xi[G])^{hG \times G} &\simeq ((\Sigma^\infty S^\xi \wedge G_+^{\text{ad}})^{hG})^{hG} && \text{by the exponential law,} \\ &\simeq ((S^\xi \wedge (\Sigma^\infty G_+^{\text{ad}})^{hG})^{hG})^{hG} && S^\xi \text{ has a trivial } 1 \times G\text{-action,} \\ &= (S^\xi \wedge D_G)^{hG}. \end{aligned}$$

$\square$

Combining this with Lemma 7.1 then yields

Corollary 7.3. *There is a weak equivalence of infinite loop spaces*

$$\text{sec}^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K) \simeq \Omega^\infty(S^\xi \wedge D_G)^{hG}.$$

Identification of $E_\alpha \rightarrow K$. Now consider the fibration

$$E_\alpha \rightarrow K.$$

Make the identification $K \simeq BG$ and let F denote the thick homotopy fiber. (Recall that $E_\alpha \rightarrow K$ is identified with the inclusion $\partial N \rightarrow N$ up to homotopy.) The following is essentially [K, Prop. 10.5].

Proposition 7.4 (Klein). *There is an equivariant weak equivalence*

$$S^\xi \wedge D_G \simeq \Sigma^\infty SF.$$

Remark 7.5. The case in which ξ is the trivial fibration is one of the main theorems of [K] which relates the Spivak fiber (the right side) to equivariant homotopy theory (the left side).

The above weak equivalence is defined by means of a certain kind of equivariant duality map. In particular, the equivalence is given up to contractible choice.

Proof of Theorem 5.9.

$$\begin{aligned} \sec^{\text{st}}(S_K(E_\alpha \rightarrow K)) &\simeq (QSF)^{hG} && \text{by the dictionary} \\ &\simeq \Omega^\infty(S^\xi \wedge D_G)^{hG} && \text{by Proposition 7.4} \\ &\simeq \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K) && \text{by Corollary 7.3.} \end{aligned}$$

□

8. THE HOMOTOPY FIBERS OF E

In this section we identify the homotopy fibers of the suspension map in a stable range.

Abstract description. Let $\alpha = (N, f)$ be a 0-simplex of $T_{n+1}(K)$. We give the description of the homotopy fiber of the suspension map E at the basepoint α (cf. the first four sections):

Definition 8.1. Let $F_n(\alpha)$ be the Δ -set in which a j -simplex is specified by a pair

$$(\beta, \sigma)$$

in which $\beta \in T_n(K)$ is a j -simplex and $\sigma: \Delta[j] \times \Delta[1] \rightarrow T_{n+1}(K)$ is a map of Δ -sets such that σ restricted to $\Delta[j] \times 0$ represents $\alpha \times id_{\Delta[j]}$ and σ restricted to $\Delta[j] \times 1$ represents $E(\beta)$.

Formulation. The last chapter gave criteria for deciding when $F_n(\alpha)$ is non-empty in the metastable range. The goal of this chapter is to identify $F_n(\alpha)$ in the metastable range. We will define a map

$$\mathcal{H}: |F_n(\alpha)| \rightarrow \Omega \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K),$$

where the space on the right hand side is the loop space of the stable section space which appeared in the previous chapter. We will then show that this map is highly connected:

Theorem 8.2. *Assume that K has the homotopy type of a cell complex of dimension $\leq k$ and that α admits a compression.*

Then there is a $(2n-3k-2)$ -connected map

$$\mathcal{H}: |F_n(\alpha)| \rightarrow \Omega \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K).$$

This will be the main result of the section.

Section spaces revisited. Let $p: E \rightarrow B$ be a (Hurewicz) fibration. Consider as before the fiberwise suspended fibration

$$S_B E \rightarrow B.$$

Denote its fiber at $b \in B$ by SF_b . This space comes equipped with two points given by the south and north poles of the suspension.

Definition 8.3. Let

$$\Omega_B^\pm S_B E \rightarrow B$$

be the fibration in which the fiber at $b \in B$ consists of a path from the south to the north pole of SF_b .

Explicitly, $\Omega_B^\pm S_B E$ is the homotopy pullback of the diagram

$$B \xrightarrow{s_-} S_B E \xleftarrow{s_+} B.$$

There is an evident map

$$E \rightarrow \Omega_B^\pm S_B E$$

which is $(2r+1)$ -connected whenever $E \rightarrow B$ is $(r+1)$ -connected (this follows from the Blakers-Massey theorem). It follows that the map of section spaces

$$\sec(E \rightarrow B) \rightarrow \sec(\Omega_B^\pm S_B E \rightarrow B)$$

is $(2r+1-b)$ -connected whenever B has the homotopy type of a cell complex of dimension $\leq b$.

Now if the section space on the right happens to come equipped with a basepoint, then any other section may be combined with the basepoint section to form a *based loop of sections* (the reason is that two paths between two points can be regarded as a based loop at one of the points). Summarizing,

Lemma 8.4. *Given a choice of basepoint in $\sec(\Omega_B^\pm S_B E \rightarrow B)$, there is a weak equivalence of spaces*

$$\sec(\Omega_B^\pm S_B E \rightarrow B) \simeq \Omega \sec(S_B E \rightarrow B).$$

Corollary 8.5. *Once a basepoint in $\sec(\Omega_B^\pm S_B E \rightarrow B)$ is chosen, there is a $(2r+1-b)$ -connected map*

$$\sec(E \rightarrow B) \rightarrow \Omega \sec(S_B E \rightarrow B).$$

We now give a convenient model for section spaces.

Definition 8.6. Let $p: E \rightarrow B$ be a fibration. Then

$$\sec.(E \rightarrow B)$$

is defined to be the Δ -set in which a j -simplex consists of

$$(p', s),$$

in which

•

$$p' E' \rightarrow B \times \Delta^j \times [0, 1]$$

is a fibration,

- the restriction of p' to $B \times \Delta^j \times 0$ coincides with

$$p \times \text{id } E \times \Delta^j \rightarrow B \times \Delta^j$$

(where $p: E \rightarrow B$ is the given fibration), and

- s is a section of p' along the subspace $B \times \Delta^j \times 1$.

The face operators are induced in the evident way from the faces of Δ^j .

Lemma 8.7. *Assume that B is a cell complex. Then $\sec.(E \rightarrow B)$ satisfies the Kan condition.*

Proof (Sketch). To keep notation in the proof consistent, we rename $\sec.(E \rightarrow B)$ by

$$\sec.(p).$$

Let $\mathcal{F}(E \rightarrow B)$ be the Δ -set whose j -simplices are fibrations over $B \times \Delta^j$. It is straightforward to check that $\mathcal{F}(E \rightarrow B)$ satisfies the Kan condition.

Let $P_p(E \rightarrow B)$ be the Δ -set of paths in $\mathcal{F}(E \rightarrow B)$ with initial point $p \in E \rightarrow B$. Then $P_p(E \rightarrow B) \rightarrow \mathcal{F}(E \rightarrow B)$ is a Kan fibration. Hence $P_p(E \rightarrow B)$ is also a Kan Δ -set.

The projection $(p', s) \rightarrow p'$ defines a map

$$\text{sec.}(p) \rightarrow P_p(E \rightarrow B).$$

As $P_p(E \rightarrow B)$ is Kan, it is sufficient to verify that this map satisfies the Kan condition. But this follows directly from the homotopy extension principle. $\square$

Lemma 8.8. *Assume that B is a cell complex. Then there is a preferred weak equivalence of spaces*

$$|\text{sec.}(E \rightarrow B)| \simeq \text{sec}(E \rightarrow B),$$

where the right side denotes the space of sections of $p \in E \rightarrow B$.

Proof. Let the notation be as in the previous proof. Then as above we have a Kan fibration

$$\text{sec.}(p) \rightarrow P_p(E \rightarrow B).$$

The fiber over the basepoint is just

$$S.(\text{sections}(p))$$

= the total singular complex of the space of sections of p . This has the homotopy type of the space of sections of p .

As $P_p(E \rightarrow B)$ is contractible, it follows that the inclusion from fiber to total space is a weak equivalence. The conclusion follows. $\square$

Combining the above with Lemma 5.12, we obtain

Corollary 8.9. *Assume $E \rightarrow B$ is an $(r+1)$ -connected fibration and B is a cell complex of dimension $\leq b$. Assume further that the fibration $\Omega_B^\pm S_B E \rightarrow B$ comes equipped with a section.*

Then there is a $(2r-b)$ -connected (weak) map

$$|\text{sec.}(E \rightarrow B)| \rightarrow \Omega \text{ sec}^{\text{st}}(S_B E \rightarrow B).$$

Defining the map $\mathcal{H}$. The first step is to define a map of Δ -sets

$$\Gamma F_n(\alpha) \rightarrow \text{sec.}(E_\alpha \rightarrow K),$$

where we recall that $E_\alpha \rightarrow K$ is the fibration defined by taking the homotopy pullback of

$$K \xrightarrow{f} N \leftarrow \partial N.$$

We first describe the map on 0-simplices. Recall that a zero simplex of $F_n(\alpha)$ consists of a 1-simplex

$$\sigma = (N_\bullet, f_\bullet)$$

of $T_{n+1}(K)$ such that $d_1 \sigma$ is of the form $E(\beta)$ for some $\beta = (M, g)$ a 0-simplex of $T_n(K)$ and $d_0 \sigma = \alpha$. Recall that $E(\beta)$ is the thickening given by

$$K \xrightarrow{g} M = M \times \frac{1}{2} \subset M \times [0, 1].$$

Then we have a fibration

$$E_\sigma \rightarrow K \times \Delta^1$$

given by taking the homotopy pullback of

$$K \times \Delta^1 \longrightarrow N_{01} \xleftarrow{\supset} \partial_1 N_{01}.$$

The fibration has the following properties:

- its restriction to $K \times 0$ is identified with $E_\alpha \rightarrow K$, and
- its restriction to $K \times 1$ comes equipped with a preferred section.

(The second property uses the preferred homotopy from $M \times \frac{1}{2} \subset M \times [0, 1]$ into $M \times 0 \subset \partial(M \times [0, 1])$.)

Consequently, these data describe a zero simplex of $\sec.(E_\alpha \rightarrow K)$. The map on higher dimensional simplices is given by essentially the same description (we omit the details).

In summary, what we have constructed is a map of Δ -sets

$$\Gamma F_n(\alpha) \rightarrow \sec.(E_\alpha \rightarrow K).$$

The fibration $E_\alpha \rightarrow K$ admits a non-trivial section, and so does $\Omega_K^\pm S_K E_\alpha \rightarrow K$.

Then by Corollary 8.9, there is a weak map of spaces

$$|\sec.(E_\alpha \rightarrow K)| \rightarrow \Omega \sec^{\text{st}}(S_K E_\alpha \rightarrow K).$$

By Theorem 5.9, we have a weak equivalence

$$\Omega \sec^{\text{st}}(S_K E_\alpha \rightarrow K) \simeq \Omega \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K).$$

Assembling the above, and applying Corollary 8.9, we get

Corollary 8.10. *There is a (weak) map of spaces*

$$|\sec.(E_\alpha \rightarrow K)| \rightarrow \Omega \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K).$$

If K is a cell complex of dimension $\leq k$ and α is an $(n+1)$ -thickening, then this map is $(2n - 3k - 2)$ -connected.

Definition 8.11. Fix a compression of α . Then the map

$$\mathcal{H}: |F_n(\alpha)| \rightarrow \Omega \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K)$$

is given by precomposing the map of Corollary 8.10 with the realization of the map

$$\Gamma F_n(\alpha) \rightarrow \sec.(E_\alpha \rightarrow K).$$

Connectivity of $\mathcal{H}$. We are now ready to prove the main result of this chapter. By Corollary 8.10 and the given assumptions, it will be sufficient to prove

Proposition 8.12. *The map*

$$\Gamma: F_n(\alpha) \rightarrow \sec.(E_\alpha \rightarrow K)$$

is $(2n - 3k - 2)$ -connected (after realization).

Digression: Hodgson's embedding theorem. Let (K, L) be a cofibration pair such that L is a finite complex and K is obtained from L by attaching cells of dimension $\leq k$. In this instance, we write

$$\dim(K, L) \leq k.$$

Let N be a compact n -manifold.

Definition 8.13. Fix a map of pairs

$$f := (f_K, f_L): (K, L) \rightarrow (N, \partial N).$$

Then a (relative) embedding up to homotopy of f in $(N, \partial N)$ consists of a pair

$$(W, h)$$

such that:

- $W \subset N$ is a compact n -manifold equipped with boundary decomposition

$$\partial W = \partial_0 W \cup_{\partial_0 W} \partial_1 W$$

such that W intersects ∂N transversely and $W \cap \partial N = \partial_0 W$.

•

$$h := (h_K, h_L): (K, L) \xrightarrow{\sim} (W, \partial_0 W)$$

is a simple homotopy equivalence.

- The composite of h with the inclusion $i(W, \partial_0 W) \subset (N, \partial N)$ is homotopic to f .
- The pairs $(W, \partial_1 W)$ and $(\partial_0 W, \partial_{01} W)$ satisfy the π - π condition.

Observe that the data $(\partial_0 W, h_L)$ define an embedding up to homotopy in the classical Stallings-Wall sense.

Theorem 8.14 (Hodgson [Ho, Th. 2.3]). *Assume f_K is r -connected, $r \geq 2k - n + 1$, and $k \leq n - 3$. Fix an embedding up to homotopy*

$$(W_L, h_L)$$

of the map $f_L: L \rightarrow \partial N$. Then there exists a relative embedding up to homotopy of f ,

$$(W, h),$$

such that $(\partial_0 W, h_L) = (W_L, h_L)$. Furthermore, the homotopy between $i \circ h$ and f can be taken constant along L .

Proof of Proposition 8.12. By Theorem 5.5, the map Γ is surjective on the path components if $3k + 2 \leq 2n$. We will show that Γ is surjective on homotopy in degree $j > 0$ when $j \leq 2n - 3k - 2$. The proof of injectivity is merely a relativized version of the the proof of surjectivity; to avoid clutter we will omit it.

To prove surjectivity, we can without loss of generality assume

$$\alpha = E(\beta),$$

where $\beta = (M, g)$ is an n -thickening. Hence, $N = M \times [0, 1]$.

An element of $\pi_j(\text{sec.}(E_\alpha \rightarrow K))$ is represented by a j -parameter family of sections of the fibration

$$E_\alpha \rightarrow K$$

(by Lemma 8.8). By definition, this is the same thing as a block map

$$F: K \times \Delta^j \rightarrow \partial N \times \Delta^j$$

over Δ^j such that:

- the restriction of F to $K \times \partial \Delta^j$ has the form $F_0 \times \text{id}$;
- the composition

$$K \times \Delta^j \xrightarrow{F} \partial N \times \Delta^j \xrightarrow{\subset} N \times \Delta^j$$

is homotopic over Δ^j to $f \times \text{id}$, where the homotopy is held constant along $K \times \partial \Delta^j$.

Identify M with the codimension zero submanifold $M \times 0 \subset \partial N$. Then we have a codimension zero submanifold

$$M \times \partial \Delta^j \subset \partial N \times \partial \Delta^j,$$

a simple homotopy equivalence

$$K \times \partial \Delta^j \xrightarrow[\simeq]{g \times \text{id}} M \times \partial \Delta^j,$$

and a map of pairs

$$(K \times \Delta^j, K \times \partial \Delta^j) \xrightarrow{(F, g \times \text{id})} (\partial N \times \Delta^j, M \times \partial \Delta^j).$$

By Theorem 8.14, there is

- a codimension zero submanifold

$$W \subset \partial N \times \Delta^j$$

whose boundary ∂W , when intersected with $\partial N \times \partial \Delta^j$, coincides with $M \times \partial \Delta^j$, and

- a simple homotopy equivalence

$$K \times \Delta^j \xrightarrow[\simeq]{h} W$$

extending $g \times \text{id}$, which, when followed by the inclusion into $\partial N \times \Delta^j$, is homotopic to F . The homotopy can be assumed to be fixed along $K \times \partial \Delta^j$.

Consider the manifold

$$V := N \times \Delta^j \cup W \times [0, 1],$$

where the gluing is along $W \times 0 \subset \partial N \times \Delta^j$. This can be thought of in two ways:

- as diffeomorphic to $N \times \Delta^j$ (since $W \times [0, 1]$ is a collar of $W \times 0$) or
- as an s -cobordism between $W \times 1$ and $\text{closure}(\partial N \times \Delta^j - W \times 0)$.

The s -cobordism theorem gives a diffeomorphism

$$V \cong W \times [0, 1]$$

relative to $W \times 0$. If we take the mapping cylinder of these identifications and glue them together, we can regard the result as an s -cobordism $U \subset N \times \Delta^j \times \Delta^1$ between $W \times [0, 1]$ and $N \times \Delta^j$. This s -cobordism can be regarded as an element of $\pi_j(F_n(\alpha))$ which lifts the given homotopy class of the family of sections. This completes the proof of surjectivity. $\square$

9. EXAMPLES

The problem with making computations of $\pi_0(T_n(K))$ is that it generally fails to have the structure of a group. However, we can obtain *upper bounds for the number of elements* in this set in certain fringe cases.

A lemma about section spaces. Suppose

$$K = L \cup_{\beta} D^k,$$

where L is a based CW complex of dimension $\leq k-2$, and $k \geq 2$.

Let

$$p E \rightarrow K$$

be a fibration equipped with section $s: K \rightarrow E$ and suppose that p is k -connected (the same k as before). Let F be the fiber of p at the basepoint.

Lemma 9.1. *With respect to these assumptions there is an isomorphism*

$$\pi_0(\sec(E \rightarrow K)) \cong \pi_k(F).$$

Proof. Obstruction theory says the obstructions to making a section $t: K \rightarrow E$ homotopic to the given section s live in the cohomology groups

$$H^*(K; \pi_*(F)), \quad * = 1, 2, \dots$$

(the coefficients are possibly twisted). By the assumptions, the only non-trivial group occurs in degree $j = k$, so this implies that we have a bijection

$$\pi_0(\sec(E \rightarrow K)) \cong H^k(K; \pi_k(F)).$$

To compute this, let $C_*(K)$ be the cellular chains on the universal cover of K ; this is a complex of free $\mathbb{Z}[\pi]$ -module, where $\pi = \pi_1(K)$. Note that $C_{k-1}(K)$ is trivial and $C_k(K) = \mathbb{Z}[\pi]$. The cohomology group is then easily computed from this, and we get $H^k(K; \pi_k(F)) = \pi_k(F)$. $\square$

Corollary 9.2 (Stable version). *With the same assumptions as above we get*

$$\pi_0(\sec^{\text{st}}(E \rightarrow K)) = \pi_k^{\text{st}}(F),$$

where the right side is the stable homotopy of F in degree k .

Proof. The stable section space in question is the same as the section space of

$$Q_{\bullet} E \rightarrow K.$$

Apply the previous lemma and use the fact that $\pi_k^{\text{st}}(F) = \pi_k(QF)$. $\square$

With K as above, we consider $K \times K$. This has the form

$$K \times K = (L \times K \cup K \times L) \cup D^{2k},$$

where the space in parenthesis has dimension $\leq 2k-2$.

Consider the case $2k = n$. By Wall, the tangent map gives an isomorphism $\pi_0(\mathbf{T}_{n+1}(K)) = [K, BO]$.

By Corollary C, we have a short exact sequence of sets,

$$\pi_1(\sec^{\text{st}}(\mathcal{E}_{\xi \oplus \epsilon} \rightarrow K \times K)) \rightarrow \pi_0(\mathbf{T}_n(K)) \xrightarrow{\tau} [K, BO] \rightarrow 1.$$

We will compute the first term of this sequence.

The main point is that there is an identification

$$\Omega \sec^{\text{st}}(\mathcal{E}_{\xi \oplus \epsilon} \rightarrow K \times K) \simeq \sec^{\text{st}}(\mathcal{E}_{\xi} \rightarrow K \times K).$$

The fiber of the fibration appearing on the right side is of the form $S^n \wedge (\Omega K)_+$, which is $(2k-1)$ -connected. Hence the projection map for that fibration is $(2k)$ -connected.

The corollary above then says

$$\pi_1(\sec^{\text{st}}(\mathcal{E}_{\xi \oplus \epsilon} \rightarrow K \times K)) \cong \pi_{2k}^{\text{st}}(S^n \wedge (\Omega K)_+) = \pi_0^{\text{st}}((\Omega K)_+) = \mathbb{Z}[\pi],$$

where $\pi = \pi_1(K)$ and the last identification made use of the Hurewicz theorem.

Corollary 9.3. *With $\pi = \pi_1(K)$, there is a short exact sequence of sets*

$$\mathbb{Z}[\pi] \rightarrow \pi_0(\mathbf{T}_{2k}(K)) \xrightarrow{\tau} [K, BO] \rightarrow 1.$$

In particular, $\pi_0(\mathbf{T}_{2k}(K))$ is countable.

Example 1. In the above, assume further that L is a CW complex of dimension $\leq k-2$ with the property that its homology vanishes in all degrees. Then K has the singular homology of a k -sphere in this case.

Furthermore,

$$[K, BO] \cong [S^k, BO] \cong \pi_{k-1}(O)$$

(the first isomorphism uses the fact that BO is a loop space and that $\Sigma K \simeq S^{k+1}$).

In particular, if

$$k \equiv 3, 5, 6, 7 \pmod{8},$$

then $[K, BO] = 0$.

For example, if $k = 11$, then there is a surjective function

$$\mathbb{Z}[\pi] \rightarrow \pi_0(\mathbf{T}_{22}(K)).$$

Example 2. In this example

$$K = S^1 \vee S^{11}.$$

Then there is again a short exact sequence of sets

$$\mathbb{Z}[t, t^{-1}] \rightarrow \pi_0(\mathbf{T}_{22}(S^1 \vee S^{11})) \rightarrow [S^1 \vee S^{11}, BO] \rightarrow 1,$$

and we have

$$[S^1 \vee S^{11}, BO] \cong \mathbb{Z}_2.$$

Consequently, the sequence of sets

$$\mathbb{Z}[t, t^{-1}] \rightarrow \pi_0(\mathbf{T}_{22}(S^1 \vee S^{11})) \rightarrow \mathbb{Z}_2 \rightarrow 1$$

is exact.

10. APPENDIX

In this section, we are going to prove that Wall's suspension theorem is retrieved once the connectivity assumption on K is added.

Theorem. *Let K be a finite connected cell complex of dimension $\leq k$, and let $\alpha = (N, f)$ be an $(n+1)$ -thickening of K with tangential data ξ . If K is $(2k-n+1)$ -connected, then there is a bijection*

$$\pi_0(\sec^{\text{st}}(\mathcal{E}_{\xi} \rightarrow K \times K)) \rightarrow \{K \wedge K, S^{n+1}\}.$$

Proof. Consider the projection map

$$S_x^{\xi} \wedge (\Omega_x^y K)_+ \rightarrow S_x^{\xi}$$

which is $(n+r+1)$ -connected, where r is the connectivity of K . We recall that S_x^{ξ} is the one point compactification of the fiber of ξ at $x \in K$ which we identify with S^{n+1} the $(n+1)$ -sphere. In Wall's range $r = 2k - n + 1$, so the map is $(2k+2)$ -connected.

The latter map induces a $(2k+2)$ -connected map of fibrations $\mathcal{E}_\xi \rightarrow E(\xi)$, where the fibration $E(\xi) \rightarrow K \times K$ represents the trivial fibration

$$K \times K \times S^{n+1} \rightarrow K \times K.$$

Using obstruction theory, we see that the map

$$\sec(\mathcal{E}_\xi \rightarrow K \times K) \rightarrow \sec(E(\xi) \rightarrow K \times K)$$

induced by the map of fibrations is 2-connected. We note that there is a commutative diagram

$$\begin{array}{ccc} \sec(\mathcal{E}_\xi \rightarrow K \times K) & \longrightarrow & \sec(E(\xi) \rightarrow K \times K) \\ \downarrow & & \downarrow \\ \sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K) & \longrightarrow & \sec^{\text{st}}(E(\xi) \rightarrow K \times K) \end{array}$$

where $\sec^{\text{st}}(E(\xi) \rightarrow K \times K)$ means the space of sections of the fibration whose fiber at (x, y) is $Q(S^{n+1})$, Q is the stable homotopy functor, and the vertical maps are inclusions. The bottom map is the stable version of the upper map defined above.

Since $S_x^\xi \wedge (\Omega_x^y K)_+$ and S^{n+1} are n -connected, then applying the Freudenthal theorem and the obstruction theory, we infer that both vertical maps are $2n-2k+1$ -connected. Now observe that $2n-2k+1 > 2$; then we see that the lower horizontal map is 2-connected. But

$$\pi_0(\sec^{\text{st}}(E(\xi) \rightarrow K \times K)) = \{K \times K, S^{n+1}\},$$

and then it follows that the bottom map induces a bijection

$$(1) \quad \pi_0(\sec^{\text{st}}(\mathcal{E}_\xi \rightarrow K \times K)) \rightarrow \{K \times K, S^{n+1}\}.$$

On the other hand, Wall notices that $[K \times K, S^{n+1}] = [K \wedge K, S^{n+1}]$ in his range. This observation can be extended to every positive integer l such that

$$[\Sigma^l(K \times K), S^{n+1+l}] = [\Sigma^l(K \wedge K), S^{n+1+l}].$$

Next we take the limit of the right hand side sequence and the left hand side sequence of abelian groups with respect to the group homomorphisms induced by the suspension maps, and we obtain

$$(2) \quad \{K \times K, S^{n+1}\} = \{K \wedge K, S^{n+1}\}.$$

Combine (1) and (2); then the desired result follows. $\square$

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