AN EXPLICIT SCHILDER TYPE THEOREM FOR SUPER-BROWNIAN MOTIONS: INFINITE INITIAL MEASURES

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ABSTRACT. In this paper, we prove that an explicit Schilder type theorem for super-Brownian motions holds for infinite initial measures, which concludes a long-time attacking on a conjecture of Fleischmann, Gärtner and Kaj (1996).

1. INTRODUCTION

The aim of this paper is to conclude a long-time attacking on a conjecture of [17] by proving an explicit Schilder type (which is also a Cramér type) theorem for super-Brownian motions holds for infinite initial measures.

Super-Brownian motion (SBM) was introduced independently by S. Watanabe in 1968 and D. A. Dawson in 1975. The name “Super-Brownian motion (Super-process)” was coined by E. B. Dynkin in the late 1980s and has been studied by many authors (see [6], [23], [15], [14], [25], [24] and the references therein). SBM is the most fundamental branching measure-valued diffusion process (superprocess). Like ordinary Brownian motion, SBM is a universal object which arises in models from combinatorics (lattice tree and algebraic series), statistical mechanics (critical percolation), interacting particle systems (voter model and contact process), population theory and mathematical biology, and nonlinear partial differential equations ([23], [29], [3]).

To begin our introduction of SBM, fix a natural number d and a real number $r > d/2$. Put

$$\phi_r(x) = (1 + |x|^2)^{-r}, \quad \forall x \in \mathbb{R}^d.$$

Let $\mathcal{B}_b(\mathbb{R}^d)$ (resp. $C_b(\mathbb{R}^d)$) be the set of all bounded measurable (resp. continuous) functions on $\mathbb{R}^d$, and $C_c(\mathbb{R}^d)$ the set of all continuous functions with compact support on $\mathbb{R}^d$. Write

$$\langle \mu, f \rangle = \mu(f) = \int_{\mathbb{R}^d} f(x) \mu(dx)$$

for the integral of a function f against a measure μ on $\mathbb{R}^d$ if it exists. Denote by $\mathcal{M}_r(\mathbb{R}^d)$ the set of all Radon measures μ on $\mathbb{R}^d$ with $\langle \mu, \phi_r \rangle < \infty$ endowed with the following τ_r topology:

$$\mu_n \Longrightarrow \mu \text{ if and only if } \lim_{n \rightarrow \infty} \langle \mu_n, f \rangle = \langle \mu, f \rangle \text{ for any } f \in C_c(\mathbb{R}^d) \cup \{\phi_r\}.$$

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Let Δ be the Laplace operator on $\mathbb{R}^d$. Then SBM $X = (X_t)_{t \geq 0}$ is the unique $\mathcal{M}_r(\mathbb{R}^d)$ -valued diffusion process such that for any $\mu \in \mathcal{M}_r(\mathbb{R}^d)$ and any $0 \leq f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\mathbb{E}[\exp\{-\langle X_t, f \rangle\} \mid X_0 = \mu] = \exp\{-\langle \mu, u_t^{\sigma, \rho} f \rangle\}, \quad t \in \mathbb{R}_+ := [0, \infty).$$

Here $\sigma, \rho \in (0, \infty)$ are fixed but arbitrary, and $u_t^{\sigma, \rho} f$ is the unique nonnegative solution to the equation

$$u_t = S_t^\sigma f - \frac{\rho}{2} \int_0^t S_{t-s}^\sigma [(u_s)^2] \, ds, \quad t \in \mathbb{R}_+,$$

and $\{S_t^\sigma\}_{t \geq 0}$ is the semigroup of the d -dimensional Brownian motion with generator $\sigma \Delta$. Note X corresponds to the binary branching mechanism $z \in \mathbb{R}_+ \rightarrow \frac{\rho}{2} z^2 \in \mathbb{R}_+$. For any $\mu \in \mathcal{M}_r(\mathbb{R}^d)$, let

$$\Omega_\mu = C_\mu([0, 1], \mathcal{M}_r(\mathbb{R}^d))$$

be the space of all continuous $\mathcal{M}_r(\mathbb{R}^d)$ -valued paths on the interval $[0, 1]$ starting at μ , equipped with the compact-open topology. Write $P_\mu^{\sigma, \rho}$ for the distribution of $(X_t)_{0 \leq t \leq 1}$ with $X_0 = \mu$ on Ω_μ .

Let $\mathcal{M}_F(\mathbb{R}^d)$ be the space of all finite measures on $\mathbb{R}^d$ furnished with the weak convergence topology. For any $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, denote by

$$\Omega_{\mu, F} = C_\mu([0, 1], \mathcal{M}_F(\mathbb{R}^d))$$

the set of all continuous maps $\omega = (\omega_t)_{0 \leq t \leq 1}$ from $[0, 1]$ into $\mathcal{M}_F(\mathbb{R}^d)$ with $\omega_0 = \mu$. Equip $\Omega_{\mu, F}$ with the corresponding compact-open topology. If X_0 is in $\mathcal{M}_F(\mathbb{R}^d)$, then the SBM $(X_t)_{t \geq 0}$ is an $\mathcal{M}_F(\mathbb{R}^d)$ -valued diffusion process. Namely, for any $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, $P_\mu^{\sigma, \rho}$ is a probability on $\Omega_{\mu, F}$.

In this paper, we study the large deviation principle (LDP) for $\{P_\mu^{\sigma, \epsilon \rho}\}_{\epsilon > 0}$ on Ω_μ as $\epsilon \downarrow 0$ for any $\mu \neq 0$. For LDPs, refer to [8], [9], [19] and [31]. On one hand, this is a Freidlin-Wentzell type LDP ([17] called it a Schilder type LDP by comparing the form of its (possible) rate function with that of the Schilder theorem for Brownian motions). On the other hand, since

$$P_\mu^{\sigma, \epsilon \rho}[(X_t)_{0 \leq t \leq 1} \in \cdot] = P_{\mu/\epsilon}^{\sigma, \rho}[(\epsilon X_t)_{0 \leq t \leq 1} \in \cdot], \quad \forall \epsilon \in (0, \infty),$$

and for any natural number n , $P_{n\mu}^{\sigma, \rho}[(\frac{1}{n}X_t)_{0 \leq t \leq 1} \in \cdot]$ is the law of the empirical mean of n independent SBMs distributed as $P_\mu^{\sigma, \rho}$ because of the branching property. The mentioned LDP is an infinite-dimensional version of the well-known classical Cramér theorem with continuous parameters ([27], pp. 321-322). This is a fascinating feature of the LDP for $\{P_\mu^{\sigma, \epsilon \rho}\}_{\epsilon > 0}$. Notice this fascinating feature is precisely the case for the Schilder's theorem itself and generally stochastic processes possessing an appropriate "self-similarity" ([9]), and was told to the author by Professor O. Zeitouni. Without a doubt, SBM is one of the most important and fundamental processes with the mentioned fascinating feature.

Note Schilder type theorem and Cramér type theorem are two major topics for the large deviation theory. Recall the first statement of the Cramér theorem (on $\mathbb{R}^1$) was due to [4], and [12] first extended Cramér theorem to separable Banach spaces. While classical Schilder theorem was first derived by [28] and then based on Fernique's inequality, it was generalized to abstract Wiener space by [9]. A nontopological form of Schilder theorem for centered Gaussian processes based on

the isoperimetric inequality was established by [2]. For more information on these two theorems, see [8].

Let $\Phi := \Phi(\mathbb{R}^d)$ be the set of all continuous functions f on $\mathbb{R}^d$ satisfying that $\frac{f(x)}{\phi_r(x)}$ has a finite limit as x goes to infinity, and $\|\cdot\|$ the uniform norm on $\mathcal{B}_b(\mathbb{R}^d)$. Endow Φ with the norm

$$\|f\|_{\Phi} = \|f/\phi_r\|, \quad f \in \Phi.$$

Then $(\Phi, \|\cdot\|_{\Phi})$ is a separable Banach space. Note $\{S_t^{\sigma}\}_{t \geq 0}$ is a strongly continuous semigroup of bounded linear operators on $(\Phi, \|\cdot\|_{\Phi})$ (cf. [27] or Lemmas (3.2)-(3.3) in [5]). Let $(\sigma\Delta, \text{Dom}(\sigma\Delta))$ be its strong generator with domain $\text{Dom}(\sigma\Delta) \subset \Phi$. Then $\text{Dom}(\sigma\Delta) = \text{Dom}(\Delta)$.

Furnish the Schwartz space $\mathcal{D} := C_c^{\infty}(\mathbb{R}^d)$, the set of all smooth functions on $\mathbb{R}^d$ with compact supports, with its inductive topology via the subspaces

$$\mathcal{D}_K = \{\phi \in \mathcal{D} \mid \phi(x) = 0, \forall x \notin K\},$$

where sets K are compact in $\mathbb{R}^d$. Let $\mathcal{D}^*$ be the dual space of $\mathcal{D}$, namely the space of all Schwartz distributions on $\mathbb{R}^d$. Then $\mathcal{M}_r(\mathbb{R}^d) \subset \mathcal{D}^*$.

For any $\nu \in \mathcal{M}_r(\mathbb{R}^d)$, define $\Delta^*\nu \in \mathcal{D}^*$, by

$$(1.1) \quad \langle \Delta^*\nu, f \rangle = \langle \nu, \Delta f \rangle, \quad \forall f \in \mathcal{D}.$$

Then for any $\varphi \in \text{Dom}(\Delta)$, $\langle \Delta^*\nu, \varphi \rangle$ defined by $\langle \nu, \Delta\varphi \rangle$ is determined uniquely by (1.1).

We call a map

$$t \in [0, 1] \rightarrow \theta_t \in \mathcal{D}^*$$

absolutely continuous if for any compact set K in $\mathbb{R}^d$ there are a neighborhood V_K of 0 in $\mathcal{D}_K$ and an absolutely continuous real-valued function h_K on $[0, 1]$ satisfying

$$(1.2) \quad |\langle \theta_t, f \rangle - \langle \theta_s, f \rangle| \leq |h_K(t) - h_K(s)|, \quad \forall s, t \in [0, 1], \quad \forall f \in V_K.$$

For such a map, the derivatives $\dot{\theta}_t \in \mathcal{D}^*$ exist in the Schwartz distributional sense for almost all $t \in [0, 1]$ with respect to the Lebesgue measure ([7]).

Let H_{μ}^{σ} be the space of all absolutely continuous (in time t in the sense of Schwartz distributions) paths $\omega = (\omega_t)_{0 \leq t \leq 1} \in \Omega_{\mu}$ with derivatives $\dot{\omega}_t$ such that for almost every t with respect to the Lebesgue measure, the Schwartz distribution $\dot{\omega}_t - \sigma\Delta^*\omega_t$ is absolutely continuous with respect to ω_t , and

$$\int_0^1 \left\langle \omega_t, \left| \frac{d(\dot{\omega}_t - \sigma\Delta^*\omega_t)}{d\omega_t} \right|^2 \right\rangle dt < \infty.$$

Here $\nu \in \mathcal{D}^*$ is absolutely continuous with respect to $w \in \mathcal{M}_r(\mathbb{R}^d)$ if

$$\langle \nu, f \rangle = \langle w, hf \rangle, \quad \forall f \in \mathcal{D},$$

for some locally w -integrable measurable function h on $\mathbb{R}^d$, and write $h = \frac{d\nu}{dw}$. For any $\omega \in H_{\mu}^{\sigma}$, let

$$(1.3) \quad g_t = g_t^{\omega} := \frac{d(\dot{\omega}_t - \sigma\Delta^*\omega_t)}{d\omega_t}, \quad t \in [0, 1].$$

Put

$$I_{\mu}^{\sigma, \rho}(\omega) = \begin{cases} \frac{1}{2\rho} \int_0^1 \left\langle \omega_t, |g_t|^2 \right\rangle dt, & \text{if } \omega \in H_{\mu}^{\sigma}, \\ \infty, & \text{if } \omega \in \Omega_{\mu} \setminus H_{\mu}^{\sigma}. \end{cases}$$

Note the introduction of H_μ^σ and $I_\mu^{\sigma,\rho}$ was attributed to [17] (or perhaps [16]). In addition, for any $\mu \in \mathcal{M}_F(\mathbb{R}^d) \setminus \{0\}$, let

$$H_{\mu,F}^\sigma = H_\mu^\sigma \cap \Omega_{\mu,F}, \quad I_{\mu,F}^{\sigma,\rho}(\omega) = \infty I_{[\omega \in \Omega_{\mu,F} \setminus H_{\mu,F}^\sigma]} + I_\mu^{\sigma,\rho}(\omega) I_{[\omega \in H_{\mu,F}^\sigma]}, \quad \omega \in \Omega_{\mu,F}.$$

Now a long-standing conjecture is stated as follows:

Conjecture 1.1. *For any $\mu \in \mathcal{M}_r(\mathbb{R}^d) \setminus \mathcal{M}_F(\mathbb{R}^d)$ on Ω_μ , $\{P_\mu^{\sigma,\epsilon\rho}\}_{\epsilon>0}$ satisfies an LDP with good rate function $I_\mu^{\sigma,\rho}$ as $\epsilon \downarrow 0$. Whereas for any $\mu \in \mathcal{M}_F(\mathbb{R}^d) \setminus \{0\}$ on $\Omega_{\mu,F}$, $\{P_\mu^{\sigma,\epsilon\rho}\}_{\epsilon>0}$ satisfies an LDP with good rate function $I_{\mu,F}^{\sigma,\rho}$ as $\epsilon \downarrow 0$.*

Notice the above important and deep conjecture was (implicitly) stated in [17] and the difference between the mentioned LDP for finite initial measures and for infinite initial measures was first noticed by [33].

Assume $\mu \neq 0$. In Schied [27], LDP for $\{P_\mu^{\sigma,\epsilon\rho}\}_{\epsilon>0}$ as $\epsilon \downarrow 0$ was given, and the good rate function was represented by a variation formula, which was expected to be $I_\mu^{\sigma,\rho}$ (see also Schied [26]). Notice that although the two variation formulae in [27] and [17] representing the good rate functions for the mentioned LDP respectively seem different, they are identical due to the fact that the rate function for any LDP on a Polish space is unique ([8], pp. 117-118, Remarks).

Recall that along the lines of Fleischmann and Kaj [18], a weak LDP for $\{P_\mu^{\sigma,\epsilon\rho}\}_{\epsilon>0}$ as $\epsilon \downarrow 0$ on Ω_μ was established in a topology weaker than the compact-open topology by an original preprint of Fleischmann, Gärtner and Kaj [16]. Then Fleischmann, Gärtner and Kaj [17] also derived LDP for $\{P_\mu^{\sigma,\epsilon\rho}\}_{\epsilon>0}$ as $\epsilon \downarrow 0$. When replacing the underlying process, Brownian motion, by a killed or reflected Brownian motion in a bounded domain of $\mathbb{R}^d$, they got the corresponding explicit rate function $I_\mu^{\sigma,\rho}$. However, for the Brownian motion case, relying on an additional unproven ‘Hypothesis of local blow-up’ ([17], Hypothesis 1.2.4), [17] proved the variation formula representing the rate function in [27] and [17] to be $I_\mu^{\sigma,\rho}$.

To prove the conjecture, it suffices to calculate the variation formula for the good rate function. However, this variation formula is not covered by the standard theory of variational methods (e.g., [30]). As just mentioned, Hypothesis 1.2.4 (‘Hypothesis of local blow-up’) in [17] can be applied to calculate the mentioned variation. Though local blow-up properties for solutions to nonlinear partial differential equations (PDEs) have been studied extensively by many references (see [20], [1], [32] and the references therein), there is no statement on local blow-up properties in the PDE theory which implies the mentioned local blow-up hypothesis.

Recall that Djehiche and Kaj [10] took a Hamiltonian approach and a Girsanov transformation technique to prove an LDP result for some measure-valued jump processes, and they compared the rate function therein with that in [17] of the LDP for $\{P_\mu^{\sigma,\epsilon\rho}\}_{\epsilon>0}$ as $\epsilon \downarrow 0$ and hoped to extend the method in [10] to the LDP for $\{P_\mu^{\sigma,\epsilon\rho}\}_{\epsilon>0}$. See [10], pp. 1419 and 1437-1438. Then Djehiche and Kaj [11] took the just mentioned method to prove directly upper and lower bounds of LDP for $\{P_\mu^{\sigma,\epsilon\rho}\}_{\epsilon>0}$ as $\epsilon \downarrow 0$. In [11], though the rate function for the upper bound LDP was $I_\mu^{\sigma,\rho}$, it required a restriction to more regular paths to get a lower large deviation bound.

Thanks to previous results on the LDP for SBMs ([26], [27], [16], [17], [18], [10], [11]) and the Le Gall’s Brownian snake ([23]), an important breakthrough on Conjecture 1.1 was made by Xiang [33]. Recall [33] proved Conjecture 1.1 for any

nonzero finite initial measures μ and a rather surprising consequence: $H_{\mu,F}^\sigma = H_\mu^\sigma$. However, for infinite initial measures, [33] was only able to prove Conjecture 1.1 for the case $d = 1$ and $\frac{1}{2} < r \leq 1$.

Though the fact that [33] had to resort to certain restrictions (finite initial measures or a restricted class of infinite initial measures for the case $d = 1$) points to the subtlety and difficulty of Conjecture 1.1, fortunately, a short proof has been discovered to prove this conjecture holds for infinite initial measures.

Theorem 1.2. *For any $\mu \in \mathcal{M}_r(\mathbb{R}^d) \setminus \mathcal{M}_F(\mathbb{R}^d)$, on Ω_μ , $\{P_\mu^{\sigma,\epsilon\rho}\}_{\epsilon>0}$ satisfies an LDP with the good rate function $I_\mu^{\sigma,\rho}$ as $\epsilon \downarrow 0$. That is, for any closed subset $C \subseteq \Omega_\mu$,*

$$\limsup_{\epsilon \downarrow 0} \epsilon \log P_\mu^{\sigma,\epsilon\rho}[\omega \in C] \leq - \inf_{\omega \in C} I_\mu^{\sigma,\rho}(\omega),$$

for any open subset $O \subseteq \Omega_\mu$,

$$\liminf_{\epsilon \downarrow 0} \epsilon \log P_\mu^{\sigma,\epsilon\rho}[\omega \in O] \geq - \inf_{\omega \in O} I_\mu^{\sigma,\rho}(\omega),$$

$I_\mu^{\sigma,\rho}$ is lower semicontinuous, and $\{\omega \in \Omega_\mu \mid I_\mu^{\sigma,\rho}(\omega) \leq a\}$ is compact in Ω_μ for any $a \in [0, \infty)$.

As mentioned in [33] Remark 1.3(1), to prove Theorem 1.2, it suffices to prove that for any $\omega \in H_\mu^\sigma$ with $\mu \in \mathcal{M}_r(\mathbb{R}^d) \setminus \mathcal{M}_F(\mathbb{R}^d)$,

$$(1.4) \quad I_\mu^{\sigma,\rho}(\omega) = \lim_{\delta \downarrow 0} \inf_{\tilde{\omega} \in O_{\delta,F}(\omega)} I_{\tilde{\omega}_0,F}^{\sigma,\rho}(\tilde{\omega}),$$

where

$$O_{\delta,F}(\omega) = \{\tilde{\omega} \in \Omega_{\mathcal{M}_r(\mathbb{R}^d)} : q(\tilde{\omega}, \omega) < \delta, \tilde{\omega}_0 \in \mathcal{M}_F(\mathbb{R}^d), \tilde{\omega} \in H_{\tilde{\omega}_0,F}^\sigma\},$$

where

$$\Omega_{\mathcal{M}_r(\mathbb{R}^d)} = C([0, 1], \mathcal{M}_r(\mathbb{R}^d))$$

is the space of all continuous maps from $[0, 1]$ into $\mathcal{M}_r(\mathbb{R}^d)$ equipped with the compact-open topology, and where $q(\cdot, \cdot)$ is the corresponding compatible metric on $\Omega_{\mathcal{M}_r(\mathbb{R}^d)}$. We will prove (1.4) in Section 2.

Similarly, to prove Conjecture 1.1 in [33] and the present paper, one can prove an extended version of Conjecture 1.1 holds for super- α -stable processes and more general super-diffusions with the binary branching mechanism. Note unlike the local operator Δ , for super- α -stable processes with $\alpha \in (0, 2)$, the underlying generator Δ_α is not local, which will make the related proof slightly different.

2. PROOF OF THEOREM 1.2

Sketch of the proof. By Lemma 2.2, it suffices to prove (1.4) holds for any $\omega \in H_{\omega_0}^\sigma$ such that

$$(2.1) \quad (\omega_t - \eta dx)_{0 \leq t \leq 1} \in H_{\omega_0 - \eta dx}^\sigma, \quad \omega_0 - \eta dx \in \mathcal{M}_r(\mathbb{R}^d) \setminus \mathcal{M}_F(\mathbb{R}^d),$$

where dx is the Lebesgue measure on $\mathbb{R}^d$ and $\eta \in (0, 1)$. Note Lemma 2.1 is used to prove Lemma 2.2. By Lemma 2.3,

$$\omega^\epsilon \in H_{\omega_0^\epsilon}^\sigma, \quad \lim_{\epsilon \downarrow 0} I_{\omega_0^\epsilon}^{\sigma,\rho}(\omega^\epsilon) = I_{\omega_0}^{\sigma,\rho}(\omega), \quad \omega \in H_{\omega_0}^\sigma,$$

where for any $\epsilon \in (0, 1)$,

$$\omega^\epsilon = (\omega_t^\epsilon)_{0 \leq t \leq 1}, \quad \omega_t^\epsilon = (S_\epsilon^\sigma)^* \omega_t, \quad t \in [0, 1],$$

and for any $\nu \in \mathcal{M}_r(\mathbb{R}^d)$, $(S_\epsilon^\sigma)^* \nu \in \mathcal{M}_r(\mathbb{R}^d)$ is defined by $\langle (S_\epsilon^\sigma)^* \nu, f \rangle = \langle \nu, S_\epsilon^\sigma[f] \rangle$, $f \in \Phi$.

Fix an $\omega \in H_{\omega_0}^\sigma$ satisfying (2.1) and an $\epsilon \in (0, 1)$. Then the proof is boiled down to find $\omega^{\epsilon, n} \in \Omega_{\omega_0^{\epsilon, n}, F}$ for any natural number n such that

$$(2.2) \quad \omega^{\epsilon, n} \in H_{\omega_0^{\epsilon, n}, F}^\sigma, \quad \lim_{n \rightarrow \infty} I_{\omega_0^{\epsilon, n}}^{\sigma, \rho}(\omega^{\epsilon, n}) = I_{\omega_0^\epsilon}^{\sigma, \rho}(\omega^\epsilon).$$

Note that $\omega_t^\epsilon = q_t^\epsilon(y) \, dy$, $t \in [0, 1]$, where $q_t^\epsilon(y)$ is given by (2.6). By Lemma 2.3(i),

$$I_{\omega_0^\epsilon}^{\sigma, \rho}(\omega^\epsilon) = \frac{1}{2\rho} \int_0^1 \left\langle \omega_t^\epsilon, |g_t^\epsilon|^2 \right\rangle dt.$$

To prove (2.2), it suffices to let

$$\omega_t^{\epsilon, n} = f_n^\epsilon(t, y) \, \omega_t^\epsilon(dy) = f_n^\epsilon(t, y) q_t^\epsilon(y) \, dy, \quad t \in [0, 1],$$

where $0 \leq f_n^\epsilon \in \Phi_{[0, 1]}$ satisfies that for almost all t with respect to the Lebesgue measure,

$$(2.3) \quad \frac{d(\dot{\omega}_t^{\epsilon, n} - \sigma \Delta^* \omega_t^{\epsilon, n})}{d\omega_t^{\epsilon, n}} = g_t^\epsilon,$$

and $f_n^\epsilon \uparrow 1$ pointwise as $n \uparrow \infty$, and where $\Phi_{[0, 1]} := C([0, 1], \Phi)$ is the space of all Φ -valued continuous paths $\psi = (\psi_t)_{0 \leq t \leq 1}$ on the interval $[0, 1]$ equipped with the norm $\|\psi\|_{\Phi_{[0, 1]}} = \sup_{t \in [0, 1]} \{\|\psi_t\|_\Phi\}$. However, notice that (2.9), a nonrigorous

deduction, shows that if f_n^ϵ is smooth, then for almost all t with respect to the Lebesgue measure,

$$\begin{aligned} \frac{d(\dot{\omega}_t^{\epsilon, n} - \sigma \Delta^* \omega_t^{\epsilon, n})}{d\omega_t^{\epsilon, n}}(y) &= \frac{\partial_t(f_n^\epsilon(t, y) q_t^\epsilon(y)) - \sigma \Delta(f_n^\epsilon(t, y) q_t^\epsilon(y))}{f_n^\epsilon(t, y) q_t^\epsilon(y)} \\ &= g_t^\epsilon(y) + \frac{\partial_t f_n^\epsilon(t, y) - \sigma \Delta f_n^\epsilon(t, y)}{f_n^\epsilon(t, y)} - \frac{2\sigma \nabla f_n^\epsilon(t, y) \cdot \nabla q_t^\epsilon(y)}{f_n^\epsilon(t, y) q_t^\epsilon(y)}, \end{aligned}$$

which forces f_n^ϵ to satisfy

$$\partial_t f_n^\epsilon(t, y) = \sigma \Delta f_n^\epsilon(t, y) + \frac{2\sigma \nabla f_n^\epsilon(t, y) \cdot \nabla q_t^\epsilon(y)}{q_t^\epsilon(y)}.$$

Thus stochastic differential equation (SDE) (2.10) comes into picture, and naturally f_n^ϵ should be defined by the first formula in (2.12). **This is a key novel point of our proof.** Note by Lemma 2.3(ii) and Lemma 2.4, SDE (2.10) has a unique strong solution $(Y_t^\epsilon(x))_{0 \leq t \leq 1}$.

Though $Y_t^\epsilon(x)$ is smooth in $x \in \mathbb{R}^d$ for any fixed t , we cannot prove $f_n^\epsilon(t, x)$ is in $C^{1,2}$ -class in $(t, x) \in [0, 1] \times \mathbb{R}^d$. To obtain (2.3), we have to approximate SDE (2.10) by SDE (2.11). The function $f_n^{\epsilon, m}$ specified in the second formula in (2.12) likes f_n^ϵ and converges to f_n^ϵ as $m \uparrow \infty$, but the former has better properties than the latter. See Lemma 2.5. Since (2.16)-(2.18) hold for $f_n^{\epsilon, m}$ (let $m \uparrow \infty$), we can easily get (2.19), which implies that (2.3) holds. **This is another key novel aspect of our proof.** So far we have proven (2.2). $\square$

Note $(\Phi_{[0, 1]}, \|\cdot\|_{\Phi_{[0, 1]}})$ is a separable Banach space. Let $\mathcal{V} := \mathcal{V}^{\sigma, \rho}$ be the set of all $(\phi, \psi) \in \Phi \times \Phi_{[0, 1]}$ such that the following nonlinear integral equation has a unique solution $\Lambda[\phi, \psi] = (\Lambda[\phi, \psi](s, \cdot))_{0 \leq s \leq 1}$ in $\Phi_{[0, 1]}$:

$$(2.4) \quad v_s = S_{1-s}^\sigma \phi + \int_s^1 S_{t-s}^\sigma \psi_t \, dt + \frac{\rho}{2} \int_s^1 S_{t-s}^\sigma (v_t^2) \, dt, \quad s \in [0, 1].$$

Then $\mathcal{V}$ is open in $\Phi \times \Phi_{[0,1]}$, and

$$\{(\phi, \psi) \in \Phi \times \Phi_{[0,1]} \mid \phi(x) \leq 0, \psi_s(x) \leq 0, \forall (s, x) \in [0, 1] \times \mathbb{R}^d\} \subset \mathcal{V}.$$

From [17], for any $\omega_0 \in \mathcal{M}_r(\mathbb{R}^d) \setminus \{0\}$,

$$(2.5) \quad \sup_{(0, \psi) \in \mathcal{V}} \left\{ \int_0^1 \langle \omega_t, \psi_t \rangle dt - \langle \omega_0, \Lambda[0, \psi](0, \cdot) \rangle \right\} = I_{\omega_0}^{\sigma, \rho}(\omega), \quad \forall \omega \in \Omega_{\omega_0}.$$

In fact, the variation in (2.5) is equal to the good rate function of the LDP for $\{P_{\omega_0}^{\sigma, \epsilon \rho}\}_{\epsilon > 0}$ as $\epsilon \downarrow 0$ under the local blow-up hypothesis ([17]). By (2.5), we immediately obtain

Lemma 2.1. *For any sequence $\{\omega^{[n]}\}_{n \geq 1} \subset \Omega_{\mathcal{M}_r(\mathbb{R}^d)}$ converging to $\omega \in \Omega_{\mathcal{M}_r(\mathbb{R}^d)}$ with $\omega_0 \neq 0$,*

$$I_{\omega_0}^{\sigma, \rho}(\omega) \leq \liminf_{n \rightarrow \infty} I_{\omega_0^{[n]}}^{\sigma, \rho}(\omega^{[n]}).$$

For any $\omega \in \Omega_\mu$ and any $\eta \in (0, 1)$, let $\nu^{(\eta)} = \nu + \eta dx$ for any $\nu \in \mathcal{M}_r(\mathbb{R}^d)$, and let

$$\omega^{(\eta)} := (\omega_t^{(\eta)})_{0 \leq t \leq 1} = (\omega_t + \eta dx)_{0 \leq t \leq 1} \in \Omega_{\mu^{(\eta)}}.$$

Lemma 2.2. *For any $\omega \in H_\mu^\sigma$ with $\mu \neq 0$ and any $\eta \in (0, 1)$, $\omega^{(\eta)} \in H_{\mu^{(\eta)}}^\sigma$. Further,*

$$\lim_{\eta \downarrow 0} I_{\mu^{(\eta)}}^{\sigma, \rho}(\omega^{(\eta)}) = I_\mu^{\sigma, \rho}(\omega).$$

Proof. Let

$$h_t^{(\eta)} = \frac{d\omega_t}{d\omega_t^{(\eta)}}, \quad t \in [0, 1].$$

Clearly, $t \in [0, 1] \rightarrow \omega_t^{(\eta)} \in \mathcal{M}_r(\mathbb{R}^d) \subset \mathcal{D}^*$ is absolutely continuous, and for almost all $t \in [0, 1]$ with respect to the Lebesgue measure,

$$\dot{\omega}_t^{(\eta)} = \dot{\omega}_t = g_t d\omega_t + \sigma \Delta^* \omega_t.$$

Since for any $f \in \mathcal{D}$, $\int_{\mathbb{R}^d} \Delta f(x) dx = 0$, we see that $\Delta^*(\eta dx) = 0$. Thus for almost all $t \in [0, 1]$ with respect to the Lebesgue measure,

$$\begin{aligned} \dot{\omega}_t^{(\eta)} &= g_t d\omega_t + \sigma \Delta^* \omega_t^{(\eta)} = g_t h_t^{(\eta)} d\omega_t^{(\eta)} + \sigma \Delta^* \omega_t^{(\eta)} \\ \text{and } \dot{\omega}_t^{(\eta)} - \sigma \Delta^* \omega_t^{(\eta)} &\ll \omega_t^{(\eta)}, \quad \frac{d(\dot{\omega}_t^{(\eta)} - \sigma \Delta^* \omega_t^{(\eta)})}{d\omega_t^{(\eta)}} = g_t h_t^{(\eta)}. \end{aligned}$$

Notice that $h_t^{(\eta)}$ can be chosen satisfying $0 \leq h_t^{(\eta)} \leq 1$, $t \in [0, 1]$, and

$$\int_0^1 \left\langle \omega_t^{(\eta)}, |g_t h_t^{(\eta)}|^2 \right\rangle dt \leq \int_0^1 \left\langle \omega_t^{(\eta)}, |g_t|^2 h_t^{(\eta)} \right\rangle dt = \int_0^1 \langle \omega_t, |g_t|^2 \rangle dt < \infty.$$

So for any $\eta \in (0, 1)$, $\omega^{(\eta)} \in H_{\mu^{(\eta)}}^\sigma$, and $\limsup_{\eta \downarrow 0} I_{\mu^{(\eta)}}^{\sigma, \rho}(\omega^{(\eta)}) \leq I_\mu^{\sigma, \rho}(\omega)$. By Lemma 2.1, we have

$$\lim_{\eta \downarrow 0} I_{\mu^{(\eta)}}^{\sigma, \rho}(\omega^{(\eta)}) = I_\mu^{\sigma, \rho}(\omega).$$

□

As mentioned in the Introduction, to prove Theorem 1.2 it suffices to prove (1.4). By Lemma 2.2, it only needs to show that (1.4) holds for $\omega^{(\eta)}$, where $\omega \in H_{\omega_0}^\sigma$ and $\eta \in (0, 1)$.

Hence, throughout the rest of the paper, assume $\omega \in H_{\omega_0}^\sigma$ satisfies (2.1). Then there is a measurable subset $\mathcal{N}$ of $[0, 1]$ with zero Lebesgue measure such that for any $t \in [0, 1] \setminus \mathcal{N}$, $\dot{\omega}_t \in \mathcal{D}^*$ exists and $\dot{\omega}_t - \sigma \Delta^* \omega_t$ is absolutely continuous with respect to ω_t and

$$\left\langle \omega_t, \left| \frac{d(\dot{\omega}_t - \sigma \Delta^* \omega_t)}{d\omega_t} \right|^2 \right\rangle = \langle \omega_t, |g_t|^2 \rangle < \infty,$$

where g_t is defined by (1.3). For any $s \in (0, \infty)$, let

$$P_s^\sigma(x) = \frac{1}{(4\pi\sigma s)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4\sigma s}\right), \quad \forall x \in \mathbb{R}^d.$$

Put

$$(2.6) \quad q_t^\epsilon(y) := \int_{\mathbb{R}^d} P_\epsilon^\sigma(y-x) \omega_t(dx), \quad t \in [0, 1], \quad y \in \mathbb{R}^d.$$

Then $q_t^\epsilon(y) \in [\eta, \infty)$ is continuous in $(t, y) \in [0, 1] \times \mathbb{R}^d$, and $\omega_t^\epsilon(dy) = q_t^\epsilon(y) dy$. Let

$$(2.7) \quad g_t^\epsilon(\cdot) = \frac{\int_{\mathbb{R}^d} g_t(x) P_\epsilon^\sigma(\cdot-x) \omega_t(dx)}{q_t^\epsilon(\cdot)}, \quad t \in [0, 1],$$

where $g_t^\epsilon \equiv 0$ if $t \in \mathcal{N}$.

To state Lemma 2.3, introduce the following notation. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ of nonnegative integers, any smooth function f on $\mathbb{R}^d$ and any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, let

$$|\alpha| = \alpha_1 + \dots + \alpha_d, \quad \alpha! = \alpha_1! \dots \alpha_d!, \quad x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d},$$

$$D^\alpha f(x) = D_x^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

Lemma 2.3. (i) For any $\epsilon \in (0, 1)$, $\omega^\epsilon \in H_{\omega_0}^\sigma$. Namely, there is a measurable subset $\mathcal{N}_\epsilon$ of $[0, 1]$ with zero Lebesgue measure such that $\mathcal{N} \subseteq \mathcal{N}_\epsilon$, and for every $t \in [0, 1] \setminus \mathcal{N}_\epsilon$, $\dot{\omega}_t^\epsilon \in \mathcal{D}^*$ exists and $\dot{\omega}_t^\epsilon - \sigma \Delta^* \omega_t^\epsilon$ is absolutely continuous with respect to ω_t^ϵ , and $g_t^\epsilon = \frac{d(\dot{\omega}_t^\epsilon - \sigma \Delta^* \omega_t^\epsilon)}{d\omega_t^\epsilon}$. Further $\int_0^1 \langle \omega_t^\epsilon, |g_t^\epsilon|^2 \rangle dt < \infty$.

(ii) For any $\epsilon \in (0, 1)$, $q_t^\epsilon(y)$ is absolutely continuous in t and of the derivatives

$$\partial_t q_t^\epsilon(y) = \begin{cases} \langle \dot{\omega}_t, P_\epsilon^\sigma(y-\cdot) \rangle = \langle \omega_t, g_t P_\epsilon^\sigma(y-\cdot) \rangle + \langle \omega_t, \sigma \Delta P_\epsilon^\sigma(y-\cdot) \rangle, & t \in [0, 1] \setminus \mathcal{N}, \\ 0, & t \in \mathcal{N} \end{cases}$$

in the sense that $\partial_t q_t^\epsilon(y)$ is absolutely integrable in t with respect to the Lebesgue measure, and $q_t^\epsilon(y) = q_0^\epsilon(y) + \int_0^t \partial_s q_s^\epsilon(y) ds$, $t \in [0, 1]$. In addition, $q_t^\epsilon(y)$ is smooth in $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, and for any multi-index α of nonnegative integers,

$$D_y^\alpha q_t^\epsilon(y) = \int_{\mathbb{R}^d} D_y^\alpha P_\epsilon^\sigma(y-x) \omega_t(dx).$$

Particularly, for any $1 \leq i, j \leq d$,

$$(2.8) \quad \partial_{y_i} q_t^\epsilon(y) = \int_{\mathbb{R}^d} \partial_{y_i} P_\epsilon^\sigma(y-x) \omega_t(dx), \quad \partial_{y_i y_j}^2 q_t^\epsilon(y) = \int_{\mathbb{R}^d} \partial_{y_i y_j}^2 P_\epsilon^\sigma(y-x) \omega_t(dx),$$

and clearly each $D_y^\alpha q_t^\epsilon(y)$ is continuous in $(t, y) \in [0, 1] \times \mathbb{R}^d$. So for any $t \in [0, 1] \setminus \mathcal{N}_\epsilon$,

$$(2.9) \quad \frac{\partial_t q_t^\epsilon(y) - \sigma \Delta q_t^\epsilon(y)}{q_t^\epsilon(y)} = g_t^\epsilon(y), \quad y \in \mathbb{R}^d.$$

$$(iii) \quad \lim_{\epsilon \downarrow 0} I_{\omega_0^\epsilon}^{\sigma, \rho}(\omega^\epsilon) = I_{\omega_0}^{\sigma, \rho}(\omega).$$

Proof. By Lemmas 9.1-9.2 in [33] and Lemma 2.1, we have that Lemma 2.3(i) and Lemma 2.3(iii) hold. For the first part of Lemma 2.3(ii) and (2.8), see Lemma 9.3 and (9.2) in [33]. Note the idea of proving [33] Lemma 9.3 can be used to deduce that $q_t^\epsilon(y)$ is in fact smooth in y and $D_y^\alpha q_t^\epsilon(y) = \int_{\mathbb{R}^d} D_y^\alpha P_\epsilon^\sigma(y-x) \omega_t(dx)$. $\square$

Lemma 2.4. *For any $\epsilon \in (0, 1)$, there is a positive constant $C_0 = C_0(\epsilon)$ such that for any $1 \leq i \leq d$,*

$$\left| \frac{\partial_{y_i} q_t^\epsilon(y)}{q_t^\epsilon(y)} \right| \leq C_0(1 + |y|), \quad \forall y \in \mathbb{R}^d, \quad \forall t \in [0, 1].$$

Generally, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ of nonnegative integers, there exists a positive constant $C_\alpha = C_\alpha(\epsilon)$ satisfying that for any $1 \leq i \leq d$,

$$\left| D_y^\alpha \left(\frac{\partial_{y_i} q_t^\epsilon(y)}{q_t^\epsilon(y)} \right) \right| \leq C_\alpha \left(1 + |y|^{|\alpha|+1} \right), \quad \forall y \in \mathbb{R}^d, \quad \forall t \in [0, 1].$$

Proof. By (2.8), for any $y \in \mathbb{R}^d$ and $t \in [0, 1]$,

$$\partial_{y_i} q_t^\epsilon(y) = -\frac{1}{2\epsilon\sigma} \int_{\mathbb{R}^d} (y_i - x_i) P_\epsilon^\sigma(y-x) \omega_t(dx).$$

Thus for any $y \in \mathbb{R}^d$ and $t \in [0, 1]$,

$$\begin{aligned} \left| \frac{\partial_{y_i} q_t^\epsilon(y)}{q_t^\epsilon(y)} \right| &\leq \frac{1}{2\epsilon\sigma} \frac{\int_{\mathbb{R}^d} |y-x| P_\epsilon^\sigma(y-x) \omega_t(dx)}{q_t^\epsilon(y)} \\ &\leq \frac{1}{2\epsilon\sigma} \frac{\int_{\mathbb{R}^d} |y| P_\epsilon^\sigma(y-x) \omega_t(dx)}{q_t^\epsilon(y)} + \frac{1}{2\epsilon\sigma} \frac{\int_{\mathbb{R}^d} |x| P_\epsilon^\sigma(y-x) \omega_t(dx)}{q_t^\epsilon(y)} \\ &= \frac{|y|}{2\epsilon\sigma} + \frac{1}{2\epsilon\sigma} \frac{\int_{\mathbb{R}^d} |x| P_\epsilon^\sigma(y-x) \omega_t(dx)}{q_t^\epsilon(y)}. \end{aligned}$$

Note for any $y \in \mathbb{R}^d$ and $a \in (0, \infty)$,

$$\frac{\int_{\{|x| \leq a|y|\}} |x| P_\epsilon^\sigma(y-x) \omega_t(dx)}{q_t^\epsilon(y)} \leq a|y|, \quad q_t^\epsilon(y) \geq \eta, \quad \forall t \in [0, 1].$$

To prove the first part of the lemma, it suffices to verify that for any $c_1 \in [2, \infty)$, there is a positive constant $c_2 = c_2(\epsilon, c_1)$,

$$\int_{\{|x| > c_1|y|\}} |x| P_\epsilon^\sigma(y-x) \omega_t(dx) \leq c_2, \quad y \in \mathbb{R}^d \setminus \{0\}, \quad t \in [0, 1].$$

In fact, for any $y \in \mathbb{R}^d \setminus \{0\}$ and $t \in [0, 1]$,

$$\begin{aligned} & \int_{\{|x| > c_1|y|\}} |x| P_\epsilon^\sigma(y-x) \omega_t(dx) \\ & \leq \frac{1}{(4\pi\sigma\epsilon)^{\frac{d}{2}}} \int_{\{|x| > c_1|y|\}} |x| \exp\left\{-\frac{(1-1/c_1)^2|x|^2}{4\sigma\epsilon}\right\} \omega_t(dx) \\ & \leq \frac{1}{(4\pi\sigma\epsilon)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |x| \exp\left\{-\frac{(1-1/c_1)^2|x|^2}{4\sigma\epsilon}\right\} \omega_t(dx) \\ & \leq \frac{1}{(4\pi\sigma\epsilon)^{\frac{d}{2}}} \sup_{0 \leq s \leq 1} \left\{ \int_{\mathbb{R}^d} |x| \exp\left\{-\frac{(1-1/c_1)^2|x|^2}{4\sigma\epsilon}\right\} \omega_s(dx) \right\} < \infty. \end{aligned}$$

Inductively, similar to the first part of the lemma, it is easy to check that the second part of the lemma holds by some tedious calculuses. $\square$

For any $1 \leq m < \infty$, let $\zeta_m : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a smooth map such that $\zeta_m(x) = x$ if $|x| \leq m$ and $= 0$ if $|x| \geq 2m$. For any $s \in [0, 1]$ and $x \in \mathbb{R}^d$, put

$$q_s^{\epsilon,m}(x) = q_s^\epsilon(\zeta_m(x)), \quad b_s^{\epsilon,m}(x) = \frac{\nabla q_s^{\epsilon,m}(x)}{q_s^{\epsilon,m}(x)}, \quad b_s^\epsilon(x) = \frac{\nabla q_s^\epsilon(x)}{q_s^\epsilon(x)}.$$

Then by Lemma 2.3(ii), for any $(s, x) \in [0, 1] \times \mathbb{R}^d$,

$$\eta \leq q_s^{\epsilon,m}(x) \leq \max_{0 \leq t \leq 1, |y| \leq 2m} |q_t^\epsilon(\zeta_m(y))| < \infty,$$

$q_s^{\epsilon,m}(x)$ is continuous in $(s, x) \in [0, 1] \times \mathbb{R}^d$, for any fixed $s \in [0, 1]$, $q_s^{\epsilon,m}(x)$ is smooth in $x \in \mathbb{R}^d$, and $D^\alpha q_s^{\epsilon,m}(x)$ is bounded and continuous in $(s, x) \in [0, 1] \times \mathbb{R}^d$ for any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ of nonnegative integers.

By Lemma 2.3(ii) and Lemma 2.4, uniformly in time s , $b_s^\epsilon(x)$ is locally Lipschitzian and satisfies the linear growth condition in spatial variable x , and $b_s^{\epsilon,m}(x)$ is Lipschitzian in x . Also,

$$\max_{(s,x) \in [0,1] \times \mathbb{R}^d} |b_s^{\epsilon,m}(x)| < \infty.$$

Let $B = (B_t)_{t \geq 0}$ ($B_0 = 0$) be the standard d -dimensional Brownian motion. Then by the standard SDE theory,

$$(2.10) \quad Y_t^\epsilon(x) = x + \sqrt{2\sigma} B_t + \int_0^t 2\sigma b_s^\epsilon(Y_s^\epsilon(x)) \, ds, \quad t \in [0, 1], \quad x \in \mathbb{R}^d,$$

has a unique strong solution $((Y_t^\epsilon(x))_{0 \leq t \leq 1}, x \in \mathbb{R}^d)$,

$$(2.11) \quad Y_t^{\epsilon,m}(x) = x + \sqrt{2\sigma} B_t + \int_0^t 2\sigma b_s^{\epsilon,m}(Y_s^{\epsilon,m}(x)) \, ds, \quad t \in [0, 1], \quad x \in \mathbb{R}^d,$$

has a unique strong solution $((Y_t^{\epsilon,m}(x))_{0 \leq t \leq 1}, x \in \mathbb{R}^d)$, and for any $x \in \mathbb{R}^d$, almost surely,

$$\lambda_m(x) = \hat{\lambda}_m(x), \text{ and } Y_t^\epsilon(x) = Y_t^{\epsilon,m}(x) \text{ for all } t \leq \lambda_m(x).$$

Here

$$\begin{aligned}\lambda_m(x) &= \inf \{t \in [0, 1] : |Y_t^\epsilon(x)| \geq m\} \\ (\lambda_m(x) = 1 \text{ if } \{t \in [0, 1] : |Y_t^\epsilon(x)| \geq m\} = \emptyset), \\ \widehat{\lambda}_m(x) &= \inf \{t \in [0, 1] : |Y_t^{\epsilon, m}(x)| \geq m\} \\ (\widehat{\lambda}_m(x) = 1 \text{ if } \{t \in [0, 1] : |Y_t^{\epsilon, m}(x)| \geq m\} = \emptyset).\end{aligned}$$

Clearly, for any $x \in \mathbb{R}^d$,

$$\lim_{m \uparrow \infty} P[\lambda_m(x) = 1] = 1.$$

Following the same line for proving Proposition 2.2 in [21], Chapter V, Section 2, though the drift coefficient is time-inhomogeneous in the considered setting, one can prove that for any $t \in [0, 1]$, almost surely, $Y_t^{\epsilon, m}(x)$ is smooth in $x \in \mathbb{R}^d$. Therefore, for any $t \in [0, 1]$, almost surely, $Y_t^\epsilon(x)$ is also smooth in $x \in \mathbb{R}^d$.

Let $\{\hbar_n\}_{n \geq 1} \subset \mathcal{D}$ be a sequence such that

$$0 \leq \hbar_n \leq 1, \quad \hbar_n(x) = 1 \text{ if } |x| \leq n \text{ and } = 0 \text{ if } |x| \geq n+1, \quad \forall n \geq 1.$$

Clearly, $\lim_{k \uparrow \infty} \hbar_k(x) = 1$ for any $x \in \mathbb{R}^d$. Write $C_b^2(\mathbb{R}^d)$ for the family of all bounded continuous functions on $\mathbb{R}^d$ with bounded continuous derivatives of orders one and two. For any $n \geq 1$ and $m \geq 1$, and any $(t, x) \in [0, 1] \times \mathbb{R}^d$, let

$$\begin{aligned}(2.12) \quad f_n^\epsilon(t, x) &= \mathbb{E}[\hbar_n(Y_t^\epsilon(x))], \quad f_n^{\epsilon, m}(t, x) = \mathbb{E}[\hbar_n(Y_t^{\epsilon, m}(x))], \\ A_t^\epsilon f(x) &= \sigma \Delta f(x) + \frac{2\sigma \nabla f(x) \cdot \nabla q_t^\epsilon(x)}{q_t^\epsilon(x)}, \quad \forall f \in C_b^2(\mathbb{R}^d), \\ A_t^{\epsilon, m} f(x) &= \sigma \Delta f(x) + \frac{2\sigma \nabla f(x) \cdot \nabla q_t^{\epsilon, m}(x)}{q_t^{\epsilon, m}(x)}, \quad \forall f \in C_b^2(\mathbb{R}^d).\end{aligned}$$

Lemma 2.5. *Given any $\epsilon \in (0, 1)$ and any $n \geq 1$ and $m \geq 1$, then*

$$\begin{aligned}\lim_{k \uparrow \infty} f_n^{\epsilon, k}(t, x) &= f_n^\epsilon(t, x) \text{ for any } (t, x) \in [0, 1] \times \mathbb{R}^d, \quad (f_n^\epsilon(t, \cdot))_{0 \leq t \leq 1} \in \Phi_{[0, 1]}, \\ (f_n^{\epsilon, m}(t, \cdot))_{0 \leq t \leq 1} &\in \Phi_{[0, 1]}, \quad f_n^{\epsilon, m}(t, x) \text{ is in } C^{1, 2}\text{-class in } (t, x) \in [0, 1] \times \mathbb{R}^d, \\ \partial_t f_n^{\epsilon, m}(t, x) &= A_t^{\epsilon, m}[f_n^{\epsilon, m}(t, \cdot)](x), \quad t \in [0, 1], \quad x \in \mathbb{R}^d.\end{aligned}$$

Proof. Clearly, for any $(t, x) \in [0, 1] \times \mathbb{R}^d$, $\lim_{k \uparrow \infty} f_n^{\epsilon, k}(t, x) = f_n^\epsilon(t, x)$. For any $u > \frac{d}{2}$, put

$$\begin{aligned}\Phi_u &= \left\{ f \in C_b(\mathbb{R}^d) \mid \lim_{|x| \rightarrow \infty} \frac{f(x)}{\phi_u(x)} \text{ exists and is finite} \right\}, \\ \|f\|_{\Phi_u} &= \|f/\phi_u\|, \quad f \in \Phi_u,\end{aligned}$$

where $\phi_u(x) = (1 + |x|^2)^{-u}$, $x \in \mathbb{R}^d$. Notice for any $u > \frac{d}{2}$, $\left\| \frac{\Delta \phi_u}{\phi_u} \right\| < \infty$ and

$$\frac{|\nabla \phi_u(x)|}{\phi_u(x)} \leq \frac{2|x|}{1 + |x|^2}, \quad x \in \mathbb{R}^d.$$

By Lemma 2.4, for any $u > \frac{d}{2}$ and $t \in [0, 1]$,

$$\frac{|\nabla \phi_u(x)|}{\phi_u(x)} \frac{|\nabla q_t^\epsilon(x)|}{q_t^\epsilon(x)} \leq 2\sqrt{d} C_0 \frac{|x|(1 + |x|)}{1 + |x|^2} \leq 4\sqrt{d} C_0, \quad \forall x \in \mathbb{R}^d.$$

Thus, it is easy to see that for any $u > \frac{d}{2}$,

$$\widehat{c}_1 := \widehat{c}_1(u, \epsilon) := \sup_{0 \leq t \leq 1} \left\| \frac{A_t^\epsilon \phi_u}{\phi_u} \right\| < \infty.$$

Since for any $u > \frac{d}{2}$, $t \in [0, 1]$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} \mathbb{E} [\phi_u (Y_t^\epsilon(x))] &= \phi_u(x) + \int_0^t \mathbb{E} [A_s^\epsilon \phi_u (Y_s^\epsilon(x))] \, ds \\ &\leq \phi_u(x) + \widehat{c}_1 \int_0^t \mathbb{E} [\phi_u (Y_s^\epsilon(x))] \, ds, \end{aligned}$$

by the Gronwall inequality,

$$\mathbb{E} [\phi_u (Y_t^\epsilon(x))] \leq \phi_u(x) e^{t\widehat{c}_1}, \quad x \in \mathbb{R}^d, \quad t \in [0, 1].$$

Therefore, for any $u > \frac{d}{2}$ and $t \in [0, 1]$,

$$f_n^\epsilon(t, x) \leq \|h_n\|_{\Phi_u} \mathbb{E} [\phi_u (Y_t^\epsilon(x))] \leq \|h_n\|_{\Phi_u} e^{\widehat{c}_1} \phi_u(x), \quad x \in \mathbb{R}^d,$$

which implies $\sup_{0 \leq t \leq 1} \|f_n^\epsilon(t, \cdot)/\phi_u\| < \infty$. Particularly, by $\sup_{0 \leq t \leq 1} \|f_n^\epsilon(t, \cdot)/\phi_{r+1}\| < \infty$, we obtain that

$$(f_n^\epsilon(t, \cdot))_{0 \leq t \leq 1} \in \Phi_{[0,1]}.$$

Similarly, we can prove that

$$(f_n^{\epsilon,m}(t, \cdot))_{0 \leq t \leq 1} \in \Phi_{[0,1]}.$$

To prove the lemma, we need to prove that for any natural number N and any multi-index α of nonnegative integers,

$$(2.13) \quad \sup_{|x| \leq N} \mathbb{E} [|D_x^\alpha h_n (Y_t^{\epsilon,m}(x))|^p] < \infty, \quad \forall t \in [0, 1], \quad \forall p > 2.$$

Since the drift coefficient in SDE (2.11) is bounded, applying Itô's formula to SDE (2.11), by a routine argument, we can prove that for some positive constant $\widehat{c}_2 = \widehat{c}_2(\epsilon, p, m)$,

$$(2.14) \quad \mathbb{E} \left[\sup_{0 \leq t \leq 1} |Y_t^{\epsilon,m}(x)|^p \right] \leq \widehat{c}_2 (1 + |x|^p), \quad x \in \mathbb{R}^d.$$

Notice the derivatives of the drift coefficient $2\sigma b_s^{\epsilon,m}(\cdot)$ for SDE (2.11) with any order are bounded uniformly in $s \in [0, 1]$. Keeping (2.14) in mind, we can easily prove that for any natural number N and any multi-index α of nonnegative integers,

$$(2.15) \quad \sup_{|x| \leq N} \mathbb{E} [|D_x^\alpha Y_t^{\epsilon,m}(x)|^p] < \infty, \quad \forall t \in [0, 1], \quad \forall p > 2.$$

To see the idea for proving (2.15), which is standard (see [22], pp. 175-176, Corollary 4.6.7 for a stronger related result on general time-homogenous SDEs for instance; clearly in our case the related proof is simpler than that of the just-mentioned corollary), we prove $\sup_{|x| \leq N} \sum_{i=1}^d \mathbb{E} [|\partial_{x_i} Y_t^{\epsilon,m}(x)|^p] < \infty$ for example. For simplicity, assume $d = 1$ for the time being. Then

$$\partial_x Y_t^{\epsilon,m}(x) = 1 + 2\sigma \int_0^t \partial_x Y_s^{\epsilon,m}(x) (b_s^{\epsilon,m})' (Y_s^{\epsilon,m}(x)) \, ds, \quad t \in [0, 1].$$

By Itô's formula,

$$|\partial_x Y_t^{\epsilon,m}(x)|^p = 1 + 2\sigma p \int_0^t |\partial_x Y_s^{\epsilon,m}(x)|^p (b_s^{\epsilon,m})'(Y_s^{\epsilon,m}(x)) \, ds, \quad t \in [0, 1].$$

Since $\partial_x Y_t^{\epsilon,m}(x)$ is continuous in $t \in [0, 1]$, for any $k \geq 2$, define the following stopping time:

$$\tau_k = \inf \{s \in [0, 1] : |\partial_x Y_s^{\epsilon,m}(x)| \geq k\} \quad (\tau_k = 1 \text{ if } \{s \in [0, 1] : |\partial_x Y_s^{\epsilon,m}(x)| \geq k\} = \emptyset),$$

then almost surely, $\tau_k \uparrow 1$ as $k \uparrow \infty$. So

$$|\partial_x Y_{t \wedge \tau_k}^{\epsilon,m}(x)|^p = 1 + 2\sigma p \int_0^{t \wedge \tau_k} |\partial_x Y_{s \wedge \tau_k}^{\epsilon,m}(x)|^p (b_s^{\epsilon,m})'(Y_{s \wedge \tau_k}^{\epsilon,m}(x)) \, ds, \quad t \in [0, 1].$$

Write

$$C_1 := C_1(\epsilon, m) := \sup_{s \in [0, 1]} \|(b_s^{\epsilon,m})'(\cdot)\| < \infty.$$

Then

$$\mathbb{E} [|\partial_x Y_{t \wedge \tau_k}^{\epsilon,m}(x)|^p] \leq 1 + 2\sigma p C_1 \int_0^t \mathbb{E} [|\partial_x Y_{s \wedge \tau_k}^{\epsilon,m}(x)|^p] \, ds, \quad t \in [0, 1].$$

By Gronwall's inequality, for any $x \in \mathbb{R}^d$,

$$\mathbb{E} [|\partial_x Y_{t \wedge \tau_k}^{\epsilon,m}(x)|^p] \leq \exp \{2\sigma p C_1 t\}, \quad t \in [0, 1].$$

By Fatou's lemma, for any $x \in \mathbb{R}^d$,

$$\mathbb{E} [|\partial_x Y_t^{\epsilon,m}(x)|^p] \leq \liminf_{k \uparrow \infty} \mathbb{E} [|\partial_x Y_{t \wedge \tau_k}^{\epsilon,m}(x)|^p] \leq \exp \{2\sigma p C_1 t\}, \quad t \in [0, 1].$$

By (2.14)-(2.15), (2.13) holds immediately.

It is well known that the following Taylor formula holds ([13], p. 70, Theorem 2.8.4): for any smooth function f on $\mathbb{R}^d$, any nonnegative integer k , and any $x, y \in \mathbb{R}^d$,

$$f(x+y) = \sum_{|\alpha| \leq k} \frac{y^\alpha}{\alpha!} (D^\alpha f)(x) + (k+1) \sum_{|\alpha|=k+1} \frac{y^\alpha}{\alpha!} \int_0^1 (1-s)^k (D^\alpha f)(x+sy) \, ds.$$

Combining this with (2.13), it is easy to see that for any multi-index α of non-negative integers, we can interchange the order of the expectation $\mathbb{E}[\cdot]$ and the differentiation D^α when considering the differentiation of $\mathbb{E}[\hbar_n(Y_t^{\epsilon,m}(x))]$, namely, $D_x^\alpha f_n^{\epsilon,m}(t, x)$ exists and

$$D_x^\alpha f_n^{\epsilon,m}(t, x) = \mathbb{E}[D_x^\alpha \hbar_n(Y_t^{\epsilon,m}(x))].$$

Note that

$$f_n^{\epsilon,m}(t, x) = \hbar_n(x) + \int_0^t \mathbb{E}[A_s^{\epsilon,m} \hbar_n(Y_s^{\epsilon,m}(x))] \, ds, \quad t \in [0, 1], \quad x \in \mathbb{R}^d.$$

Now the lemma holds immediately. $\square$

Remark 2.6. By Lemma 2.4, the derivatives of the drift coefficient for SDE (2.10) with any order satisfy a polynomial growth condition. It is unknown whether (2.13) holds for $((Y_t^\epsilon(x))_{0 \leq t \leq 1}, x \in \mathbb{R}^d)$, and further whether $f_n^\epsilon(t, x)$ is in $C^{1,2}$ -class in $(t, x) \in [0, 1] \times \mathbb{R}^d$. This makes the proof of (2.19) somewhat indirect. In fact, we prove (2.19) by an approximation argument.

Let

$$\omega^{\epsilon,n} = (\omega_t^{\epsilon,n})_{0 \leq t \leq 1}, \quad \omega_t^{\epsilon,n}(\mathrm{d}y) = f_n^\epsilon(t, y) q_t^\epsilon(y) \, \mathrm{d}y.$$

By Lemma 2.5, we obtain

Corollary 2.7. $\omega_0^{\epsilon,n} \in \mathcal{M}_F(\mathbb{R}^d)$, $\omega^{\epsilon,n} \in \Omega_{\omega_0^{\epsilon,n}, F}$.

Lemma 2.8. *Given any $\epsilon \in (0, 1)$. Then (2.2) holds.*

Proof. By Lemma 2.3(i), $\omega^\epsilon \in H_{\omega_0^\epsilon}^\sigma$, and for almost every $t \in [0, 1]$ with respect to the Lebesgue measure, $\frac{\mathrm{d}(\omega_t^\epsilon - \sigma \Delta^* \omega_t^\epsilon)}{\mathrm{d}\omega_t^\epsilon} = g_t^\epsilon$. By Lemma 2.5, for any $h \in \mathcal{D}$,

$$\begin{aligned} f_n^{\epsilon,m}(t, \cdot)h &\in \text{Dom}(\Delta), \quad t \in [0, 1]; \\ (\Delta(f_n^{\epsilon,m}(t, \cdot)h))_{0 \leq t \leq 1} &\in \Phi_{[0,1]}, \quad (\partial_t(f_n^{\epsilon,m}(t, \cdot)h))_{0 \leq t \leq 1} \in \Phi_{[0,1]}. \end{aligned}$$

So by [33] Proposition 5.2, for any $t \in [0, 1]$ and $h \in \mathcal{D}$, and any $m \geq 1$,

$$\begin{aligned} \langle \omega_t^\epsilon, f_n^{\epsilon,m}(t, \cdot)h \rangle &= \langle \omega_0^\epsilon, f_n^{\epsilon,m}(0, \cdot)h \rangle + \int_0^t \langle \omega_s^\epsilon, g_s^\epsilon f_n^{\epsilon,m}(s, \cdot)h \rangle \, \mathrm{d}s \\ (2.16) \quad &+ \int_0^t \langle \omega_s^\epsilon, \sigma \Delta(f_n^{\epsilon,m}(s, \cdot)h) \rangle \, \mathrm{d}s + \int_0^t \langle \omega_s^\epsilon, \partial_s(f_n^{\epsilon,m}(s, \cdot)h) \rangle \, \mathrm{d}s. \end{aligned}$$

For any $s \in [0, 1]$ and $h \in \mathcal{D}$, by Lemma 2.3(ii), $q_s^\epsilon(\cdot)$ is a smooth function, and further

$$\begin{aligned} (2.17) \quad \langle \omega_s^\epsilon, \sigma \Delta(f_n^{\epsilon,m}(s, \cdot)h) \rangle &= \int_{\mathbb{R}^d} \sigma \Delta(f_n^{\epsilon,m}(s, \cdot)h)(y) q_s^\epsilon(y) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} f_n^{\epsilon,m}(s, y) h(y) \sigma \Delta q_s^\epsilon(y) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} h(y) \sigma \Delta(f_n^{\epsilon,m}(s, \cdot)q_s^\epsilon)(y) \, \mathrm{d}y - \int_{\mathbb{R}^d} h(y) q_s^\epsilon(y) \sigma \Delta f_n^{\epsilon,m}(s, y) \, \mathrm{d}y \\ &\quad - \int_{\mathbb{R}^d} 2\sigma h(y) \sum_{i=1}^d \partial_i f_n^{\epsilon,m}(s, y) \partial_i q_s^\epsilon(y) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} \sigma \Delta h(y) (f_n^{\epsilon,m}(s, \cdot)q_s^\epsilon)(y) \, \mathrm{d}y - \int_{\mathbb{R}^d} h(y) q_s^\epsilon(y) \sigma \Delta f_n^{\epsilon,m}(s, y) \, \mathrm{d}y \\ &\quad + \int_{\mathbb{R}^d} 2\sigma f_n^{\epsilon,m}(s, y) \sum_{i=1}^d \partial_i (h \partial_i q_s^\epsilon)(y) \, \mathrm{d}y. \end{aligned}$$

By Lemma 2.5, for any $s \in [0, 1]$ and $h \in \mathcal{D}$, we have

$$\begin{aligned} (2.18) \quad \partial_s(f_n^{\epsilon,m}(s, \cdot)h) &= h \sigma \Delta f_n^{\epsilon,m}(s, \cdot) + 2\sigma h \sum_{i=1}^d \partial_i f_n^{\epsilon,m}(s, \cdot) \frac{\partial_i q_s^{\epsilon,m}}{q_s^{\epsilon,m}}, \\ \langle \omega_s^\epsilon, \partial_s(f_n^{\epsilon,m}(s, \cdot)h) \rangle &= \int_{\mathbb{R}^d} h(y) q_s^\epsilon(y) \sigma \Delta f_n^{\epsilon,m}(s, y) \, \mathrm{d}y + \int_{\mathbb{R}^d} 2\sigma h(y) \frac{q_s^\epsilon(y)}{q_s^{\epsilon,m}(y)} \sum_{i=1}^d \partial_i f_n^{\epsilon,m}(s, y) \partial_i q_s^{\epsilon,m}(y) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} h(y) q_s^\epsilon(y) \sigma \Delta f_n^{\epsilon,m}(s, y) \, \mathrm{d}y - \int_{\mathbb{R}^d} 2\sigma f_n^{\epsilon,m}(s, y) \sum_{i=1}^d \partial_i \left(\frac{q_s^\epsilon}{q_s^{\epsilon,m}} h \partial_i q_s^{\epsilon,m} \right)(y) \, \mathrm{d}y. \end{aligned}$$

Since $h \in \mathcal{D}$, by the dominated convergence theorem, for any $t \in [0, 1]$, as $m \uparrow \infty$,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} f_n^{\epsilon, m}(s, y) \sum_{i=1}^d \partial_i (h \partial_i q_s^\epsilon)(y) \, dy ds \\ & \quad - \int_0^t \int_{\mathbb{R}^d} f_n^{\epsilon, m}(s, y) \sum_{i=1}^d \partial_i \left(\frac{q_s^\epsilon}{q_s^{\epsilon, m}} h \partial_i q_s^{\epsilon, m} \right) (y) \, dy ds \rightarrow 0, \\ & \int_0^t \langle \omega_s^\epsilon, f_n^{\epsilon, m}(s, \cdot) \sigma \Delta h \rangle \, ds \rightarrow \int_0^t \langle \omega_s^\epsilon, f_n^\epsilon(s, \cdot) \sigma \Delta h \rangle \, ds = \int_0^t \langle \omega_s^{\epsilon, n}, \sigma \Delta h \rangle \, ds. \end{aligned}$$

Thus for any $t \in [0, 1]$ and any $h \in \mathcal{D}$, as $m \uparrow \infty$,

$$\int_0^t \langle \omega_s^\epsilon, \sigma \Delta (f_n^{\epsilon, m}(s, \cdot) h) \rangle \, ds + \int_0^t \langle \omega_s^\epsilon, \partial_s (f_n^{\epsilon, m}(s, \cdot) h) \rangle \, ds \rightarrow \int_0^t \langle \omega_s^{\epsilon, n}, \sigma \Delta h \rangle \, ds.$$

Combining this with that for any $t \in [0, 1]$,

$$\begin{aligned} \lim_{m \uparrow \infty} \int_0^t \langle \omega_s^\epsilon, g_s^\epsilon f_n^{\epsilon, m}(s, \cdot) h \rangle \, ds &= \int_0^t \langle \omega_s^\epsilon, g_s^\epsilon f_n^\epsilon(s, \cdot) h \rangle \, ds = \int_0^t \langle \omega_s^{\epsilon, n}, g_s^\epsilon h \rangle \, ds, \\ \lim_{m \uparrow \infty} \langle \omega_t^\epsilon, f_n^{\epsilon, m}(t, \cdot) h \rangle &= \langle \omega_t^\epsilon, f_n^\epsilon(t, \cdot) h \rangle = \langle \omega_t^{\epsilon, n}, h \rangle, \end{aligned}$$

by (2.16)-(2.18), we obtain that for any $h \in \mathcal{D}$,

$$(2.19) \quad \langle \omega_t^{\epsilon, n}, h \rangle = \langle \omega_0^{\epsilon, n}, h \rangle + \int_0^t \langle \omega_s^{\epsilon, n}, g_s^\epsilon h \rangle \, ds + \int_0^t \langle \omega_s^{\epsilon, n}, \sigma \Delta h \rangle \, ds, \quad t \in [0, 1].$$

Therefore, for any compact set K in $\mathbb{R}^d$, let

$$\begin{aligned} V_K &= V_{K, n} = \left\{ h \in \mathcal{D}_K : \sup_{x \in K} \{ |h|(x) + |\Delta h|(x) \} < 1 \right\}, \\ h_K(t) &= \int_0^t \langle \omega_s^{\epsilon, n}, |g_s^\epsilon| I_K \rangle \, ds + \int_0^t \langle \omega_s^{\epsilon, n}, \sigma I_K \rangle \, ds, \quad t \in [0, 1]. \end{aligned}$$

Then V_K is a neighborhood of 0 in $\mathcal{D}_K$ and h_K is an absolutely continuous real-valued function, and

$$t \in [0, 1] \rightarrow \omega_t^{\epsilon, n} \in \mathcal{M}_r(\mathbb{R}^d) \subset \mathcal{D}^*$$

is absolutely continuous due to the fact that (1.2) holds for $\omega^{\epsilon, n}$. Furthermore, by (2.19), for almost all $t \in [0, 1]$ with respect to the Lebesgue measure,

$$\dot{\omega}_t^{\epsilon, n} = g_t^\epsilon d\omega_t^{\epsilon, n} + \sigma \Delta^* \omega_t^{\epsilon, n} \in \mathcal{D}^* \text{ exists}$$

and $\dot{\omega}_t^{\epsilon, n} - \sigma \Delta^* \omega_t^{\epsilon, n}$ is absolutely continuous with respect to $\omega_t^{\epsilon, n}$, and

$$\frac{d(\dot{\omega}_t^{\epsilon, n} - \sigma \Delta^* \omega_t^{\epsilon, n})}{d\omega_t^{\epsilon, n}} = g_t^\epsilon.$$

Since

$$\begin{aligned} & \int_0^1 \left\langle \omega_t^{\epsilon, n}, \left| \frac{d(\dot{\omega}_t^{\epsilon, n} - \sigma \Delta^* \omega_t^{\epsilon, n})}{d\omega_t^{\epsilon, n}} \right|^2 \right\rangle dt = \int_0^1 \left\langle \omega_t^\epsilon, f_n^\epsilon(t, \cdot) |g_t^\epsilon|^2 \right\rangle dt \\ & \leq \int_0^1 \left\langle \omega_t^\epsilon, |g_t^\epsilon|^2 \right\rangle dt < \infty, \end{aligned}$$

by Corollary 2.7, we obtain that $\omega^{\epsilon,n} \in H_{\omega_0^{\epsilon,n},F}^\sigma$, and by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 \left\langle \omega_t^{\epsilon,n}, \left| \frac{d(\dot{\omega}_t^{\epsilon,n} - \sigma \Delta^* \omega_t^{\epsilon,n})}{d\omega_t^{\epsilon,n}} \right|^2 \right\rangle dt = \int_0^1 \langle \omega_t^\epsilon, |g_t^\epsilon|^2 \rangle dt.$$

Lemma 2.8 holds. $\square$

Proof of Theorem 1.2. As mentioned before, to prove Theorem 1.2 it suffices to prove that (1.4) holds for any $\omega \in H_\mu^\sigma$ satisfying (2.1). Clearly, for any $\epsilon \in (0, 1)$, as $n \rightarrow \infty$, $\omega^{\epsilon,n} \rightarrow \omega^\epsilon$ in $\Omega_{\mathcal{M}_r}(\mathbb{R}^d)$. Also,

$$\text{as } \epsilon \downarrow 0, \omega^\epsilon \rightarrow \omega \text{ in } \Omega_{\mathcal{M}_r}(\mathbb{R}^d).$$

Therefore, by Lemma 2.3(iii), Lemma 2.8 and Lemma 2.1, we see that (1.4) holds for any $\omega \in H_\mu^\sigma$ satisfying (2.1). $\square$

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