A NONCONSTANT COEFFICIENTS DIFFERENTIAL OPERATOR ASSOCIATED TO SLICE MONOGENIC FUNCTIONS

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ABSTRACT. Slice monogenic functions have had a rapid development in the past few years. One of the main properties of such functions is that they allow the definition of a functional calculus, called S -functional calculus, for (bounded or unbounded) noncommuting operators. In the literature there exist two different definitions of slice monogenic functions that turn out to be equivalent under suitable conditions on the domains on which they are defined. Both the existing definitions are based on the validity of the Cauchy-Riemann equations in a suitable sense. The aim of this paper is to prove that slice monogenic functions belong to the kernel of the global operator defined by $G(x) := |\underline{x}|^2 \frac{\partial}{\partial x_0} + \underline{x} \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$, where $\underline{x}$ is the 1-vector part of the paravector $x = x_0 + \underline{x}$ and $n \in \mathbb{N}$. Despite the fact that G has nonconstant coefficients, we are able to prove that a subclass of functions in the kernel of G have a Cauchy formula. Moreover, we will study some relations among the three classes of functions and we show that the kernel of the operator G strictly contains the functions given by the other two definitions.

1. INTRODUCTION

In the past few years, the theory of slice monogenic functions has had a rapid development mainly stimulated by the application to a very elegant functional calculus for not necessarily commuting operators, either bounded or unbounded; see [2, 4, 7, 8, 9] and the recent monograph [10]. Slice monogenic functions are defined on paravectors in the Euclidean space and have values in a Clifford algebra, but there exists a parallel theory for functions of a quaternionic variable (see [14]) associated to a quaternionic functional calculus (see [3]). The most general case has been treated in [15] in which the authors consider functions with values in any real alternative algebra defined on open sets in the so-called quadratic cone of the algebra.

To better explain the problems we treat in this paper and to show what is known in the literature, we need some notation and definitions.

Let us denote by $\mathbb{R}_n$ the real Clifford algebra over n imaginary units $e_1, \dots, e_n$ satisfying the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$, where $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index, $e_A = e_{i_1} e_{i_2} \dots e_{i_r}$ and $e_\emptyset = 1$, $x_A \in \mathbb{R}$. Some specific elements in $\mathbb{R}_n$ can be identified with vectors in the Euclidean space $\mathbb{R}^n$: an element

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$(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ can be identified with a so-called 1-vector in the Clifford algebra through the map $(x_1, x_2, \dots, x_n) \mapsto \underline{x} = x_1 e_1 + \dots + x_n e_n$.

The element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ will be identified with $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_j e_j$, which is called a paravector. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \dots + x_n^2$. The real part x_0 of x will also be written as $\text{Re}[x]$. A function $f : U \subseteq \mathbb{R}^{n+1} \rightarrow \mathbb{R}_n$ is treated as a function $f(x)$ of the paravector x . The symbol $\mathbb{S}$ will denote the sphere of unit 1-vectors in $\mathbb{R}^n$, i.e.

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \dots + e_n x_n : x_1^2 + \dots + x_n^2 = 1\}.$$

Note that $\mathbb{S}$ is an $(n-1)$ -dimensional sphere in $\mathbb{R}^{n+1}$ lying on the hyperplane $x_0 = 0$ that can be identified with the Euclidean space $\mathbb{R}^n$. The vector space $\mathbb{R} + I\mathbb{R}$ passing through 1 and $I \in \mathbb{S}$ will be denoted by $\mathbb{C}_I$, while an element belonging to $\mathbb{C}_I$ will be denoted by $u + Iv$, for $u, v \in \mathbb{R}$. Observe that $\mathbb{C}_I$, for every $I \in \mathbb{S}$, is a 2-dimensional real subspace of $\mathbb{R}^{n+1}$ isomorphic to the complex plane. The isomorphism turns out to be an algebra isomorphism.

Given a paravector $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us set

$$(1.1) \quad I_x = \begin{cases} \frac{\underline{x}}{|\underline{x}|} & \text{if } \underline{x} \neq 0, \\ \text{any element of } \mathbb{S} & \text{otherwise,} \end{cases}$$

so, by definition, we have $x \in \mathbb{C}_{I_x}$. Given an element $x \in \mathbb{R}^{n+1}$, we can associate to it the sphere defined by $[x] = \{y \in \mathbb{R}^{n+1} : y = x_0 + I|\underline{x}|\}$, where $I \in \mathbb{S}$. The sphere $[x]$ is $(n-1)$ -dimensional if x is not real, while it reduces to a point if $x \in \mathbb{R}$.

We now introduce the following definition; see [7].

Definition 1.1 (Slice monogenic functions). Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f : U \rightarrow \mathbb{R}_n$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I$. We say that f is a (left) slice monogenic function, or s-monogenic function, if for every $I \in \mathbb{S}$, we have

$$(1.2) \quad \frac{1}{2} \left(\frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) f_I(u + Iv) = 0.$$

We denote by $\mathcal{SM}(U)$ the set of slice monogenic functions defined on U .

Observe that in Definition 1.1, we do not require any property on the open set U . However, in order that slice monogenic functions have good properties (such as the Cauchy formula), they must be defined on suitable open sets which are characterized in the following definitions:

Definition 1.2 (Slice domains). Let $U \subseteq \mathbb{R}^{n+1}$ be a domain. We say that U is a slice domain (s-domain for short) if $U \cap \mathbb{R}$ is nonempty and if $\mathbb{C}_I \cap U$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.

Definition 1.3 (Axially symmetric domains). Let $U \subseteq \mathbb{R}^{n+1}$. We say that U is axially symmetric if, for all $u + Iv \in U$, the whole $(n-1)$ -sphere $[u + Iv]$ is contained in U .

We now consider a second class of functions: the so-called holomorphic maps of a paravector variable; see Definition 1.4 below. The subclass of functions with α, β real-valued appear in the literature in several papers, in particular in the works related to the Fueter mapping theorem; see for example [12, 13, 16, 18, 20, 21, 22, 23, 24, 25]. In the formulation given below, this class of functions is used in [5].

This setting, in the case of functions of a paravector variable, has been foreseen in [23] and it has been further extended in [15] in which the authors treat the topic in a very general setting.

Definition 1.4 (Holomorphic maps of a paravector variable). Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and $\mathcal{U} \subseteq \mathbb{R} \times \mathbb{R}$ be such that $x = u + Iv \in U$ for all $(u, v) \in \mathcal{U}$. The (left) holomorphic maps of a paravector variable defined on U are of the form $f(x) = \alpha(u, v) + I\beta(u, v)$ where α, β are $\mathbb{R}_n$ -valued differentiable functions, satisfying the following even-odd conditions: $\alpha(u, v) = \alpha(u, -v)$, $\beta(u, v) = -\beta(u, -v)$ for all $(u, v) \in \mathcal{U}$ and moreover α and β satisfy the Cauchy-Riemann system

$$(1.3) \quad \begin{cases} \partial_u \alpha - \partial_v \beta = 0, \\ \partial_v \alpha + \partial_u \beta = 0. \end{cases}$$

We denote by $\mathcal{HM}(U)$ the set of holomorphic maps of a paravector variable on the open set U .

If we do not make any assumption on the open set U on which slice monogenic functions and holomorphic maps of a paravector variable are defined, then these two classes of functions are not related. However, there is a class of domains on which the two sets of functions coincide, as proven in [5], [15].

Theorem 1.5. Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s -domain and let $f : U \rightarrow \mathbb{R}^{n+1}$ be a real differentiable function. Then $f \in \mathcal{HM}(U)$ if and only if $f \in \mathcal{SM}(U)$.

As one may notice, the definition of slice monogenic functions suggests the existence of an operator G which is not defined slice by slice but “globally”, such as the Dirac operator for the classical monogenic functions. The operator G turns out to be a nonconstant coefficient differential operator (compare with Definition 7 in [15] and Definition 11.28 in [16]).

Definition 1.6. Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f : U \rightarrow \mathbb{R}_n$ be a real differentiable function. Let $G(x)$ be the operator defined by

$$G(x) := |\underline{x}|^2 \frac{\partial}{\partial x_0} + \underline{x} \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}.$$

The set of functions defined on the open set U and in the kernel of G will be denoted by $\mathcal{GM}(U)$. Moreover, we set $\mathcal{GM}_p(U) := \{f \in \mathcal{GM}(U) \mid \alpha \text{ and } \beta \text{ do not depend on } I \in \mathbb{S}\}$, where $\alpha := \frac{1}{2}(f(u + Iv) + f(u - Iv))$, $\beta := \frac{I}{2}(f(u - Iv) - f(u + Iv))$ and $x = u + Iv$, $u = \text{Re}(x)$, $v = |\underline{x}|$.

The main aim of this paper is to solve the following problem:

Problem 1.7. Determine the relations among the class of slice monogenic functions, the class of holomorphic maps of a paravector variable and the class of functions in the kernel of the differential operator G .

One may wonder if the null solutions of G coincide with slice monogenic functions or with holomorphic maps of a paravector variable. As we will show in Section 2, the kernel of G is a larger set. The answer to Problem 1.7 is summarized in the following results:

- (1) Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s -domain and let $f \in \mathcal{SM}(U)$. Then $f \in \mathcal{GM}(U)$.

- (2) Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain and let $f \in \mathcal{HM}(U)$. Then $f \in \mathcal{GM}(U)$.
- (3) Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric domain which does not intersect the real axis and let $f : U \rightarrow \mathbb{R}_n$ be a real differentiable function. Then $f \in \mathcal{HM}(U)$ if and only if $f \in \mathcal{GM}_p(U)$.
- (4) Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain that intersects the real axis. Then $\mathcal{HM}(U)$ is strictly contained in $\mathcal{GM}(U)$.

As a consequence of point (3), functions in $\mathcal{GM}_p(U)$, where U is an axially symmetric domain disjoint from the real axis, admit a Cauchy formula (see Theorem 4.1).

2. SLICE MONOGENIC FUNCTIONS AND NULL SOLUTIONS OF G

We begin this section by recalling some properties of holomorphic functions that will be useful in the sequel. By $Hol(\Lambda)$ we will denote the space of holomorphic functions $f : \Lambda \rightarrow \mathbb{C}$, where $\Lambda \subseteq \mathbb{C}$ is a domain. We will denote the complex conjugation as

$$Z_{\mathbb{C}}(z) = \bar{z}, \quad \forall z \in \mathbb{C},$$

and, in the sequel, we will consider domains Λ in the complex plane as in the following definition.

Definition 2.1 (*Z-invariant domain*). Let Λ be a domain in the complex plane $\mathbb{C}$. We say that Λ is a Z -invariant domain if $Z_{\mathbb{C}}(\Lambda) = \Lambda$.

Roughly speaking, Λ is a domain symmetric with respect to the real axis.

Remark 2.2. Let Λ be a Z -invariant domain in $\mathbb{C}$. The Cauchy-Riemann equations show that $f \in Hol(\Lambda)$ if and only if $Z_{\mathbb{C}} \circ f \circ Z_{\mathbb{C}} \in Hol(\Lambda)$.

Remark 2.3. Observe that if $U \subset \mathbb{R}^{n+1}$ is an axially symmetric s-domain, then the domain $U \cap \mathbb{C}_I$ is Z -invariant in the complex plane $\mathbb{C}_I$, for any $I \in \mathbb{S}$.

Definition 2.4. Let Λ be a Z -invariant domain in $\mathbb{C}$. We define the following function spaces:

$$\begin{aligned} Hol_c(\Lambda) &:= \{f \in Hol(\Lambda) \mid Z_{\mathbb{C}} \circ f \circ Z_{\mathbb{C}} = f\}, \\ Hol_{\bar{c}}(\Lambda) &:= \{f \in Hol(\Lambda) \mid -Z_{\mathbb{C}} \circ f \circ Z_{\mathbb{C}} = f\}. \end{aligned}$$

Remark 2.5. Note that $Hol_c(\Lambda)$ and $Hol_{\bar{c}}(\Lambda)$ are real linear spaces. Indeed, let $f, g \in Hol_c(\Lambda)$. Then

$$Z_{\mathbb{C}} \circ (f\lambda + g) \circ Z_{\mathbb{C}} = Z_{\mathbb{C}} \circ f \circ Z_{\mathbb{C}}\lambda + Z_{\mathbb{C}} \circ g \circ Z_{\mathbb{C}} = f\lambda + g,$$

for any $\lambda \in \mathbb{R}$. A similar result holds for $Hol_{\bar{c}}(\Lambda)$. Moreover observe that the mapping $f \mapsto if$ defined from $Hol_c(\Lambda)$ to $Hol_{\bar{c}}(\Lambda)$, is an isomorphism of real linear spaces.

We now state a result on holomorphic functions which is crucial in the sequel and, for the sake of completeness, we give a sketch of its proof.

Proposition 2.6. *Let Λ be a Z -invariant domain in $\mathbb{C}$ and let $f \in Hol(\Lambda)$. Then*

- (a) *there exists a unique pair of functions $f_1 \in Hol_c(\Lambda)$ and $f_2 \in Hol_{\bar{c}}(\Lambda)$ such that $f = f_1 + f_2$;*
- (b) *there exists a unique pair of functions $f_1, f_2 \in Hol_c(\Lambda)$ such that $f = f_1 + if_2$.*

Proof. To prove (a) we choose

$$f_1 = \frac{1}{2}(f + Z_{\mathbb{C}} \circ f \circ Z_{\mathbb{C}}) \quad \text{and} \quad f_2 = \frac{1}{2}(f - Z_{\mathbb{C}} \circ f \circ Z_{\mathbb{C}}).$$

Suppose that $h_1 \in \text{Hol}_c(\Lambda)$ and $h_2 \in \text{Hol}_{\bar{c}}(\Lambda)$ are such that $f = h_1 + h_2$. Then $f_1 + f_2 = h_1 + h_2$, or equivalently $f_1 - h_1 = h_2 - f_2$. Therefore we get

$$f_1 - h_1 = Z_{\mathbb{C}} \circ (h_2 - f_2) \circ Z_{\mathbb{C}} = -(h_2 - f_2) = -(f_1 - h_1).$$

This implies that $f_1 = h_1$ and $f_2 = h_2$. To prove (b) we observe that we can choose

$$f_1 = \frac{1}{2}(f + Z_{\mathbb{C}} \circ f \circ Z_{\mathbb{C}}), \quad f_2 = \frac{i}{2}(-f + Z_{\mathbb{C}} \circ f \circ Z_{\mathbb{C}}).$$

□

If we consider slice monogenic functions on axially symmetric s-domains we have a remarkable fact that is stated in the following result; see [2], [4].

Theorem 2.7 (Representation formula). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f \in \mathcal{SM}(U)$. For any vector $x = u + I_x v \in U$ the following formula holds:*

$$(2.1) \quad f(x) = \frac{1}{2} [1 - I_x I] f(u + Iv) + \frac{1}{2} [1 + I_x I] f(u - Iv), \quad I \in \mathbb{S}.$$

Remark 2.8. The meaning of the Representation formula is the following: if we know the values of the function f on $U \cap \mathbb{C}_I$ for a given $I \in \mathbb{S}$, then formula (2.1) allows us to reconstruct the values of f in all its domain U . If we do not make the assumption that $U \cap \mathbb{R} \neq \emptyset$, then the values of two holomorphic maps on two different complex planes $U \cap \mathbb{C}_I$ and $U \cap \mathbb{C}_J$ for $I \neq J$ are not, in general, related.

Lemma 2.9 (Splitting Lemma (see [7])). *Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $f : U \rightarrow \mathbb{R}_n$ be an s-monogenic function. For every $I = I_1 \in \mathbb{S}$, let $I_2, \dots, I_n$ be a completion to a basis of $\mathbb{R}_n$ satisfying the defining relations $I_r I_s + I_s I_r = -2\delta_{rs}$. Then there exist 2^{n-1} many holomorphic functions $F_A : U \cap \mathbb{C}_I \rightarrow \mathbb{C}_I$ such that for every $z = u + Iv$,*

$$f_I(z) = \sum_{|A|=0}^{n-1} F_A(z) I_A, \quad I_A = I_{i_1} \dots I_{i_s},$$

where $A = i_1 \dots i_s$ is a subset of $\{2, \dots, n\}$, with $i_1 < \dots < i_s$, or, when $|A| = 0$, $I_{\emptyset} = 1$.

Given a domain $U \subset \mathbb{R}^{n+1}$ we set $U_I := U \cap \mathbb{C}_I$ and we denote by $\text{Hol}(U_I)$ the complex linear space of holomorphic functions from U_I to $\mathbb{C}_I$.

Let $x \in \mathbb{R}^{n+1}$ be a nonzero paravector; we set $x = x_0 + \underline{x} = u + I_x v$, where $u = x_0$, $v = |\underline{x}|$ and $I_x = v^{-1} \underline{x}$, with $v \neq 0$ and for $v = 0$, see (1.1). Moreover if $I_x = (\zeta_1, \dots, \zeta_n)$, then $x_k = v \zeta_k$ for $k = 1, \dots, n$. The Splitting Lemma and the Representation formula allow us to define the following operators.

Definition 2.10. The restriction operator induced by the Splitting Lemma 2.9 will be denoted by

$$Q_I : \mathcal{SM}(U) \longrightarrow \sum_A \text{Hol}(U_I) I_A, \quad I_A = I_{i_1} \dots I_{i_s},$$

$$f \longmapsto f|_{U_I} = \sum_A F_A(z) I_A.$$

The extension operator induced by the Representation formula, Theorem 2.7, will be denoted by

$$P_I : \sum_A \text{Hol}(U_I) I_A \longrightarrow \mathcal{SM}(U),$$

$$P_I[f](x) = P_I[f](u + I_x v) = \frac{1}{2} \left[(1 + I_x I) f(u - Iv) + (1 - I_x I) f(u + Iv) \right].$$

Remark 2.11. Observe that $Q_I \circ P_I$ is the identity operator on $\sum_A \text{Hol}(U_I) I_A$ while $P_I \circ Q_I$ is the identity operator on $\mathcal{SM}(U)$.

Proposition 2.12. *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s -domain. Let α, β be real-valued functions and let $u = x_0$, $v = |\underline{x}|$.*

(a) *If $f = \alpha + I\beta \in \text{Hol}_c(U_I)$, then*

$$P_I[f](u + I_x v) = \alpha(u + Iv) + I_x \beta(u + Iv), \quad \forall x = u + I_x v \in U.$$

(b) *If $f = \alpha + I\beta \in \text{Hol}_{\bar{c}}(U_I)$, then*

$$P_I[f](u + I_x v) = (\beta(u + Iv) - I_x \alpha(u + Iv)) I, \quad \forall x = u + I_x v \in U.$$

Proof. Let $u + Iv \in U_I$. Since $f \in \text{Hol}_c(U_I)$ we have that

$$(2.2) \quad f(u - Iv) = \alpha(u + Iv) - I\beta(u + Iv).$$

By the definition of the operator P_I , and from (2.2) we have:

$$P_I[f](u + I_x v) = \frac{1}{2} \left[(1 + I_x I)(\alpha(u + Iv) - I\beta(u + Iv)) \right. \\ \left. + (1 - I_x I)(\alpha(u + Iv) + I\beta(u + Iv)) \right],$$

where $u + I_x v \in U$, from which point (a) follows since β is real-valued. Point (b) follows from the definition of P_I and from the fact that since $f \in \text{Hol}_{\bar{c}}(U_I)$ we have that

$$(2.3) \quad f(u - Iv) = -\alpha(u + Iv) + I\beta(u + Iv).$$

□

Definition 2.13. We denote by $\eta : \mathbb{R} \times (\mathbb{R}^+ \cup \{0\}) \times (-\pi, \pi] \times \dots \times (0, \pi] \times (0, \pi] \rightarrow \mathbb{R}^{n+1}$ the map:

$$\eta(u, v, \theta_1, \dots, \theta_{n-1}) = u + v \cos(\theta_1) e_1 + v \sin(\theta_1) \cos(\theta_2) e_2 + \dots \\ + v \sin(\theta_1) \dots \sin(\theta_{n-2}) \cos(\theta_{n-1}) e_{n-1} + v \sin(\theta_1) \dots \sin(\theta_{n-2}) \sin(\theta_{n-1}) e_n.$$

Proposition 2.14. *We have that*

$$\frac{\partial(I_x \circ \eta)}{\partial u} = \frac{\partial(I_x \circ \eta)}{\partial v} = 0.$$

Proof. If $x = x_0 + x_1 e_1 + \dots + x_n e_n$, then from the definition of η we have

$$I_x \circ \eta = \cos(\theta_1) e_1 + \sin(\theta_1) \cos(\theta_2) e_2 + \dots + \sin(\theta_1) \dots \sin(\theta_{n-2}) \cos(\theta_{n-1}) e_{n-1} \\ + \sin(\theta_1) \dots \sin(\theta_{n-2}) \sin(\theta_{n-1}) e_n.$$

So the statement follows from the fact that $I_x \circ \eta$ does not depend on u, v . □

Lemma 2.15. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s -domain and let $h \in \mathcal{SM}(U)$. Then we have*

$$(2.4) \quad \frac{\partial h \circ \eta}{\partial u} + (I_x \circ \eta) \frac{\partial h \circ \eta}{\partial v} = 0, \quad \text{on } \eta^{-1}(U).$$

Proof. We first consider the open set $U \setminus (U \cap \mathbb{R})$ and we split the proof into two steps.

Step 1. Let $I \in \mathbb{S}$ and let $f = \alpha + I\beta \in Hol_c(U_I)$, where α, β are real-valued functions. Using Propositions 2.12 and 2.14, one obtains that

$$\begin{aligned} & \frac{\partial P_I[f] \circ \eta}{\partial u}(u, v, \theta_1, \dots, \theta_{n-1}) + I_x \circ \eta \frac{\partial P_I[f] \circ \eta}{\partial v}(u, v, \theta_1, \dots, \theta_{n-1}) \\ &= \frac{\partial \alpha}{\partial u}(u + vI) + I_x \circ \eta \frac{\partial \beta}{\partial u}(u + vI) + I_x \circ \eta \frac{\partial \alpha}{\partial v}(u + vI) + (I_x \circ \eta)(I_x \circ \eta) \frac{\partial \beta}{\partial v}(u + vI) \\ &= \left(\frac{\partial \alpha}{\partial u}(u + vI) - \frac{\partial \beta}{\partial v}(u + vI) \right) + I_x \circ \eta \left(\frac{\partial \beta}{\partial v}(u + vI) + \frac{\partial \alpha}{\partial v}(u + vI) \right) = 0, \end{aligned}$$

where we used the fact that α, β satisfy the Cauchy-Riemann equations.

Step 2. Now, let $h \in \mathcal{SM}(U)$. Then $Q_I[h] \in \sum_A Hol(U_I)I_A$, and using (b) of Proposition 2.6, there exist $f_{0A}, f_{1A} \in Hol_c(U_I)$ such that

$$Q_I[h] = \sum_A (f_{0A} + If_{1A})I_A.$$

Since $P_I \circ Q_I$ is the identity operator on $\mathcal{SM}(U)$, for any $(u, v, \theta_1, \dots, \theta_{n-1}) \in \eta^{-1}(U \setminus (U \cap \mathbb{R}))$ we have that

$$\begin{aligned} & \frac{\partial h \circ \eta}{\partial u}(u, v, \theta_1, \dots, \theta_{n-1}) + I_x \circ \eta \frac{\partial h \circ \eta}{\partial v}(u, v, \theta_1, \dots, \theta_{n-1}) \\ &= \frac{\partial P_I \circ Q_I[h] \circ \eta}{\partial u} + I_x \circ \eta \frac{\partial P_I \circ Q_I[h] \circ \eta}{\partial v} \\ &= \frac{\partial P_I[\sum_A (f_{0A} + If_{1A})I_A] \circ \eta}{\partial u} + I_x \circ \eta \frac{\partial P_I[(f_{0A} + If_{1A})I_A] \circ \eta}{\partial v} \\ &= \sum_A \left(\frac{\partial P_I[f_{0A}] \circ \eta}{\partial u} + I_x \circ \eta \frac{\partial P_I[f_{0A}] \circ \eta}{\partial v} \right) I_A \\ &\quad + \sum_A \left(\frac{\partial P_I[f_{1A}] \circ \eta}{\partial u} + I_x \circ \eta \frac{\partial P_I[f_{1A}] \circ \eta}{\partial v} \right) II_A = 0, \end{aligned}$$

where we used Step 1. This concludes the proof for $U \setminus (U \cap \mathbb{R})$.

We finally consider the set $U \cap \mathbb{R}$. We show that h satisfies (2.4) also in a neighborhood of the real points. Let $\bigcup_{u_0 \in \mathbb{R}} B(u_0, r_{u_0})$ be a neighborhood on $U \cap \mathbb{R}$ consisting of open balls $B(u_0, r_{u_0}) \subset U$ with center at the real point u_0 and radius r_{u_0} . On each ball, $h|_{B(u_0, r_{u_0})}$ admits power series expansion because it is slice monogenic, so we have that it satisfies (2.4). This completes the proof. $\square$

Theorem 2.16. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s -domain and let $h \in \mathcal{SM}(U)$. Then $h \in \mathcal{GM}(U)$.*

Proof. Let $x = x_0 + \underline{x} = u + I_x v$, $I_x = \zeta_1 e_1 + \dots + \zeta_n e_n$; then $x_k = v \zeta_k$, $k = 1, \dots, n$ with $v = \sqrt{x_1^2 + \dots + x_n^2}$ and so $I_x = v^{-1}(x_1 e_1 + \dots + x_n e_n)$. The statement follows as a direct consequence of Lemma 2.15 and of the following identities:

$$\frac{\partial h \circ \eta}{\partial u} = \sum_{j=0}^n \frac{\partial h}{\partial x_j} \frac{\partial x_j}{\partial u} = \frac{\partial h}{\partial x_0}$$

and

$$\frac{\partial h \circ \eta}{\partial v} = \sum_{j=0}^n \frac{\partial h}{\partial x_j} \frac{\partial x_j}{\partial v} = \sum_{j=0}^n \frac{\partial h}{\partial x_j} \zeta_j = \sum_{j=0}^n \frac{\partial h}{\partial x_j} \frac{x_j}{v} = \frac{1}{v} \sum_{j=0}^n x_j \frac{\partial h}{\partial x_j}.$$

□

3. HOLOMORPHIC MAPS OF A PARAVECTOR VARIABLE AND NULL SOLUTIONS OF G

In [15], but see also [5], it is proved that for holomorphic maps of a paravector variable, defined on a domain which does not necessarily intersect the real axis, the Representation Formula still holds. In fact formula (2.1) is a consequence of the fact that f can be written in the form $f(x) = \alpha(u, v) + I\beta(u, v)$.

Theorem 3.1. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $f \in \mathcal{HM}(U)$. Then f satisfies the Representation Formula (2.1).*

We now state and prove the main results relating holomorphic maps of a paravector variable and the null solutions of G .

Theorem 3.2. *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric domain and let $f \in \mathcal{HM}(U)$. Then $f \in \mathcal{GM}(U)$.*

Proof. Suppose that $f \in \mathcal{HM}(U)$. Then by definition there exist $\alpha, \beta : U \rightarrow \mathbb{R}_n$ real differentiable functions such that

$$(3.1) \quad \alpha(u, v) = \alpha(u, -v), \quad \beta(u, v) = -\beta(u, -v),$$

and they satisfy the Cauchy-Riemann equations:

$$\frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial v}, \quad \frac{\partial \alpha}{\partial v} = -\frac{\partial \beta}{\partial u}.$$

Let $\mu : U \subseteq \mathbb{R}^{n+1} \mapsto \mathbb{R} \times (\mathbb{R}^+ \cup \{0\}) \times (-\pi, \pi] \times \dots \times (0, \pi] \times (0, \pi]$ be the map defined by

$$\mu(x) = (u, v, \theta_1, \dots, \theta_{n-1}).$$

Then μ is invertible for any $\underline{x} \neq \underline{0}$ and, as before, we denote by η its inverse:

$$\begin{aligned} \eta(u, v, \theta_1, \dots, \theta_{n-1}) = x &:= u + v \cos(\theta_1) e_1 + v \sin(\theta_1) \cos(\theta_2) e_2 + \dots \\ &\dots + v \sin(\theta_1) \dots \sin(\theta_{n-2}) \cos(\theta_{n-1}) e_{n-1}. \end{aligned}$$

Let us consider $U \setminus (U \cap \mathbb{R})$: we have $f \circ \eta : \mu(U) \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}_n$ and $f \circ \eta = \alpha + (I_x \circ \eta)\beta$. By a direct computation, thanks to the Cauchy-Riemann equations, one has

$$\frac{\partial f \circ \eta}{\partial u} + (I_x \circ \eta) \frac{\partial f \circ \eta}{\partial v} = \frac{\partial \alpha}{\partial u} + (I_x \circ \eta) \frac{\partial \beta}{\partial u} + (I_x \circ \eta) \frac{\partial \alpha}{\partial v} - \frac{\partial \beta}{\partial v} = 0.$$

By setting $x_k = v \zeta_k$, $k = 1, \dots, n$ and using the relations:

$$\begin{aligned} \frac{\partial f \circ \eta}{\partial u} &= \sum_{j=0}^n \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial u} = \frac{\partial f}{\partial x_0}, \\ \frac{\partial f \circ \eta}{\partial v} &= \sum_{j=0}^n \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial v} = \sum_{j=0}^n \frac{\partial f}{\partial x_j} \zeta_j = \sum_{j=0}^n \frac{\partial f}{\partial x_j} \frac{x_j}{v} = \frac{1}{v} \sum_{j=1}^n x_j \frac{\partial f}{\partial x_j}, \end{aligned}$$

one obtains that

$$(x_1^2 + \dots + x_n^2) \frac{\partial f}{\partial x_0} + \left(\sum_{j=1}^n x_j e_j \right) \sum_{m=1}^n x_m \frac{\partial f}{\partial x_m} = 0;$$

therefore $f \in \mathcal{GM}(U)$. If U intersects the real axis, then for any $u_0 \in U \cap \mathbb{R}$ there exists a ball $B = B(u_0, r_{u_0})$ containing u_0 and contained in U . On B the function f is slice monogenic, so, by Theorem 2.16, the statement follows. $\square$

Theorem 3.3. *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric domain which does not intersect the real axis and let $f : U \rightarrow \mathbb{R}_n$ be a real differentiable function. Then $f \in \mathcal{HM}(U)$ if and only if $f \in \mathcal{GM}_p(U)$.*

Proof. The fact that if $f \in \mathcal{HM}(U)$, then $f \in \mathcal{GM}_p(U)$ is a consequence of the previous theorem and of the fact that $\frac{1}{2}(f(u + Iv) + f(u - Iv))$, $\frac{I}{2}(f(u - Iv) - f(u + Iv))$ are equal to α and β in (3.1), so they do not depend on $I \in \mathbb{S}$. Let us show the converse. Define the map

$$\tilde{\mu} : U \subseteq \mathbb{R}^{n+1} \rightarrow \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}, \quad \tilde{\mu}(x) = (u, v, I),$$

where $u = \operatorname{Re}(x)$, $v = |\underline{x}|$ and $I = I_x$. Note that $\tilde{\mu}$ is well defined since $\underline{x} \neq 0$ because U does not intersect the real line. The map $\tilde{\mu}$ is invertible and its inverse will be denoted by $\tilde{\eta} : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S} \rightarrow \mathbb{R}^{n+1}$ and it is defined by $\tilde{\eta}(u, v, I) = u + Iv$. We denote by Z the conjugation operator on the set of paravectors. When a paravector is written as $x = u + Iv$ the conjugation acts as $Z(x) = u - Iv$, or equivalently if $x = x_0 + e_1 x_1 + \dots + e_n x_n$, then $w = Z(x) = x_0 - e_1 x_1 - \dots - e_n x_n$. We define the following functions:

$$\begin{aligned} \alpha &:= \frac{1}{2} (f \circ \tilde{\eta} + f \circ Z \circ \tilde{\eta}), \\ \beta &:= \frac{I_x}{2} (f \circ Z \circ \tilde{\eta} - f \circ \tilde{\eta}). \end{aligned}$$

Assume that $f \in \mathcal{GM}_p(U)$ for all $x \in U$, so we have that

$$(3.2) \quad \frac{\partial f}{\partial x_0} + \frac{1}{x_1^2 + \dots + x_n^2} \left(\sum_{j=1}^n x_j e_j \right) \sum_{m=1}^n x_m \frac{\partial f}{\partial x_m} = 0$$

and, by the symmetry of U with respect to the real axis, we also have

$$\left[\frac{\partial f}{\partial y_0} + \frac{1}{y_1^2 + \dots + y_n^2} \left(\sum_{j=1}^n y_j e_j \right) \sum_{m=1}^n y_m \frac{\partial f}{\partial y_m} \right] \circ Z = 0.$$

Then

$$\frac{\partial f}{\partial y_0} \circ Z + \frac{1}{(-y_1)^2 + \dots + (-y_n)^2} \left(\sum_{j=1}^n (-y_j) e_j \right) \sum_{m=1}^n (-y_m) \frac{\partial f}{\partial y_m} \circ Z = 0.$$

Denoting $y = Z(x)$ one has that $\frac{\partial f \circ Z}{\partial x_0} = \frac{\partial f}{\partial y_0} \circ Z$ and $\frac{\partial f \circ Z}{\partial x_m} = -\frac{\partial f}{\partial y_m} \circ Z$.

Therefore

$$(3.3) \quad \frac{\partial f \circ Z}{\partial x_0} - \frac{1}{x_1^2 + \dots + x_n^2} \left(\sum_{j=1}^n x_j e_j \right) \sum_{m=1}^n x_m \frac{\partial f \circ Z}{\partial x_m} = 0.$$

We can rewrite (3.2) and (3.3) in the form

$$\frac{\partial f \circ \tilde{\eta}}{\partial u} = -I_x \frac{\partial f \circ \tilde{\eta}}{\partial v}$$

and

$$\frac{\partial f \circ Z \circ \tilde{\eta}}{\partial u} = I_x \frac{\partial f \circ Z \circ \tilde{\eta}}{\partial v}.$$

We can now show that α and β defined above satisfy the Cauchy-Riemann system; in fact:

$$\begin{aligned} \frac{\partial \alpha}{\partial u} &= \frac{1}{2} \left(\frac{\partial f \circ \tilde{\eta}}{\partial u} + \frac{\partial f \circ Z \circ \tilde{\eta}}{\partial u} \right) \\ &= \frac{1}{2} \left(-I_x \frac{\partial f \circ \tilde{\eta}}{\partial v} + I_x \frac{\partial f \circ Z \circ \tilde{\eta}}{\partial v} \right) = \frac{\partial}{\partial v} \frac{I_x}{2} (f \circ Z \circ \tilde{\eta} - f \circ \tilde{\eta}) = \frac{\partial \beta}{\partial v} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \beta}{\partial u} &= \frac{I_x}{2} \left(\frac{\partial f \circ Z \circ \tilde{\eta}}{\partial u} - \frac{\partial f \circ \tilde{\eta}}{\partial u} \right) \\ &= \frac{I_x}{2} \left(I_x \frac{\partial f \circ Z \circ \tilde{\eta}}{\partial v} + I_x \frac{\partial f \circ \tilde{\eta}}{\partial v} \right) = -\frac{1}{2} \left(\frac{\partial f \circ Z \circ \tilde{\eta}}{\partial v} + \frac{\partial f \circ \tilde{\eta}}{\partial v} \right) = -\frac{\partial \alpha}{\partial v}. \end{aligned}$$

Since we assumed that $f \in \mathcal{GM}_p(U)$, then α and β do not depend on $I_x \in \mathbb{S}$ and the statement follows. $\square$

Remark 3.4. There are functions in $\mathcal{GM}(U)$ such that the corresponding α, β satisfy the Cauchy-Riemann system but depend on I_x . For example, one can consider the function $f(x) = f(u + I_x v) = xJI_x$, where J is a fixed element in $\mathbb{S}$. We have $\alpha = vI_xJI_x$, $\beta = -uI_xJI_x$.

We now consider functions in the kernel of operator G which are defined on an open set $U \subseteq \mathbb{R}^{n+1}$ intersecting the real axis. We show that there exist distribution solutions that cannot be interpreted as holomorphic maps of a paravector variable.

Theorem 3.5. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s -domain. Then $\mathcal{HM}(U)$ is strictly contained in $\mathcal{GM}(U)$.*

Proof. We consider the simplest case, i.e. $n = 1$, so the elements in $\mathcal{HM}(U)$ are simply holomorphic functions. As is well known, see [17], if P is an elliptic operator, and $\mathcal{A}, \mathcal{D}'$ are the sheaves of real analytic functions and of distributions, respectively, then $\mathcal{A}^P = \mathcal{D}'^P$, so to look for real analytic null solutions of P is the same as to look for distributional solutions. In particular, this result applies to the case in which P is the Cauchy-Riemann operator. Thus holomorphic functions can be considered as the distributional solutions of $\frac{\partial}{\partial \bar{z}} f = 0$; if U is an open set in $\mathbb{R}^2 \cong \mathbb{C}$ and Hol is the sheaf of holomorphic functions, we can write

$$\mathcal{HM}(U) = Hol(U) = \mathcal{D}'^P(U).$$

Let us now consider the distributional solutions to the equation

$$(3.4) \quad x_1^2 \left(\frac{\partial f}{\partial x_0} + e_1 \frac{\partial f}{\partial x_1} \right) = 0.$$

A holomorphic function f obviously satisfies the equation, but there are other solutions. In the spaces of distributions $\mathcal{D}'$ the equation

$$x_1^2 u(x_1) = 0$$

has solutions of the form $u(x_1) = c_1 \delta(x_1) + c_2 \delta'(x_1)$, where $c_1, c_2 \in \mathbb{R}$ and δ is the Dirac distribution. Assume that a solution f to (3.4) depends on x_1 only. Then the equation (3.4) reduces to

$$x_1^2 \frac{\partial f}{\partial x_1} = 0,$$

so it admits solutions of the form $f(x_1) = c_1 H(x_1) + c_2 \delta(x_1) + c_3$, $c_\ell \in \mathbb{R}$, $\ell = 1, 2, 3$, where $H(x_1)$ is the Heaviside function. The solution f is defined everywhere on $\mathbb{R}^2 \cong \mathbb{C}$. Thus there exist null solutions of G which are not holomorphic functions. More in general we have $\mathcal{HM}(U) \subsetneq \mathcal{GM}(U)$, for $U \subseteq \mathbb{R}^{n+1}$. $\square$

Remark 3.6. Note that if we consider (3.4) on an open set which does not intersect the real axis $x_1 = 0$, then we can disregard the factor x_1^2 in front of the equation and we end up with the Cauchy-Riemann equation.

4. THE CAUCHY FORMULA FOR NULL SOLUTIONS OF G

The Cauchy formula for slice monogenic functions or for holomorphic maps of a paravector variable is a consequence of the Representation formula. So for functions in $\mathcal{SM}(U)$, where U is an axially symmetric s-domain (see [4]), or for functions in $\mathcal{HM}(U)$, where U is an axially symmetric domain only, there is a Cauchy formula (see [15]):

Theorem 4.1 (Cauchy formula). *Let $W \subset \mathbb{R}^{n+1}$ be an open set and let $f \in \mathcal{HM}(W)$. Let U be a bounded axially symmetric open set such that $\overline{U} \subset W$. Suppose that the boundary of $U \cap \mathbb{C}_I$ consists of a finite number of rectifiable Jordan curves for any $I \in \mathbb{S}$. Then, if $x \in U$, we have*

$$(4.1) \quad f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S^{-1}(s, x) ds_I f(s),$$

where $ds_I = ds/I$ and

$$(4.2) \quad S^{-1}(s, x) = -(x^2 - 2x\operatorname{Re}[s] + |s|^2)^{-1}(x - \overline{s}),$$

and the integral does not depend on U nor on the imaginary unit $I \in \mathbb{S}$.

The proof of this result follows the same lines of the one given in [2] and [4] for slice monogenic functions. In [2] we have also proved the following result:

Proposition 4.2. *The function $S^{-1}(s, x)$ is left s-monogenic in the variable x and right s-monogenic in the variable s for $x \notin [s]$.*

Remark 4.3. The definition of a right s-monogenic function is analogous to the one in Definition 1.1: just replace (1.2) by

$$\frac{1}{2} \left(\frac{\partial}{\partial u} f_I(u + Iv) + \frac{\partial}{\partial v} f_I(u + Iv)I \right) = 0.$$

We introduce the definition of functions in the kernel of the G -operator acting on the right:

Definition 4.4. Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f : U \rightarrow \mathbb{R}_n$ be a real differentiable function. We say that the real differentiable function $f : U \rightarrow \mathbb{R}_n$ is in the kernel of the right G -operator if

$$G_R f := |\underline{x}|^2 \frac{\partial f}{\partial x_0} + \sum_{j=1}^n x_j \frac{\partial f}{\partial x_j} \underline{x} = 0, \quad \text{on } U.$$

We have the following analog of Proposition 4.2:

Theorem 4.5. Let $S^{-1}(s, x)$ be the function defined in (4.2). Then $S^{-1}(\cdot, x)$, for $x \notin [s]$, belongs to $\text{Ker } G$ and $S^{-1}(s, \cdot)$ is in the kernel of the right operator G_R .

Proof. This is a consequence of Proposition 4.2 and of Theorem 2.16. The fact that the function $S^{-1}(s, \cdot)$ is in the kernel of the right operator G_R can be proved by suitably modifying the proof of Theorem 2.16. $\square$

Remark 4.6. It is possible to prove Theorem 4.5 directly, but the computations are quite long. We only observe that

$$\frac{\partial S^{-1}}{\partial x_0} = 2(x^2 - 2xs_0 + |s|^2)^{-2}(x - s_0)(x - \bar{s}) - (x^2 - 2xs_0 + |s|^2)^{-1},$$

and, for $j = 1, \dots, n$, we have

$$\begin{aligned} \frac{\partial S^{-1}}{\partial x_j} &= 2(x^2 - 2xs_0 + |s|^2)^{-1}(x_0 e_j - x_j - e_j s_0)(x^2 - 2xs_0 + |s|^2)^{-1}(x - \bar{s}) \\ &\quad - (x^2 - 2xs_0 + |s|^2)^{-1} e_j. \end{aligned}$$

The previous calculations imply that

$$G(S^{-1}) = (x_1^2 + \dots + x_n^2) \frac{\partial S^{-1}}{\partial x_0} + \sum_{j=1}^n x_j e_j \left(\sum_{j=1}^n x_j \frac{\partial S^{-1}}{\partial x_j} \right) = 0.$$

Analogously, it follows that, for $x \notin [s]$, $S^{-1}(s, \cdot)$ is in the kernel of the right operator G_R .

Remark 4.7. Let $f(x) = \sum_{m \geq 0} x^m a_m$, where $a_m \in \mathbb{R}_n$ and $g(x) = \sum_{m \geq 0} a_m x^m$, where $a_m \in \mathbb{R}_n$ for every $m \in \mathbb{N}$. Suppose that the series converges in an open ball B_r with center at the origin and radius $r > 0$. It is easy to verify that f belongs to $\mathcal{GM}(B_r)$ and g is in the kernel of the right operator G_R on B_r .

Note that the powers x^m belong also to the set of the so-called holomorphic Cliffordian functions introduced, in odd dimension, in [19].

We now provide an important result that shows that for functions in $\mathcal{GM}_p(U)$ there is a Cauchy formula, at least on axially symmetric open sets that do not intersect the real axis.

Theorem 4.8. Let $W \subset \mathbb{R}^{n+1}$ be an open set and let $f \in \mathcal{GM}_p(U)$, where U is a bounded axially symmetric open set such that $\overline{U} \subset W$ that does not intersect the real line. Suppose that the boundary of $U \cap \mathbb{C}_I$ consists of a finite number of rectifiable Jordan curves for any $I \in \mathbb{S}$. Then, if $x \in U$, we have

$$(4.3) \quad f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S^{-1}(s, x) ds_I f(s),$$

where $ds_I = ds/I$ and $S^{-1}(s, x)$ is given by (4.2) and the integral does not depend on U nor on the imaginary unit $I \in \mathbb{S}$.

Proof. The previous theorem shows that on axially symmetric open sets not intersecting the real axis the functions in $\mathcal{HM}(U)$ coincide with the functions in $\mathcal{GM}_p(U)$; thus the Cauchy formula holds. $\square$

5. THE QUATERNIONIC SETTING

Denote by $q = x_0 + x_1i + x_2j + x_3k = x_0 + \underline{q}$ a quaternion and by $\mathbb{H}$ the vector space of quaternions. The sphere of purely imaginary quaternions will be denoted by

$$\mathbb{S} = \{\underline{q} = e_1x_1 + e_2x_2 + e_3x_3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

In this section we will consider functions defined on open sets of $\mathbb{H}$ with values in $\mathbb{H}$. Thus these functions do not coincide with slice monogenic functions with values in the quaternions, i.e. with the slice monogenic functions that are obtained by setting $n = 2$. In the case we consider holomorphic maps of a quaternionic variable with α, β real-valued, see Definition 5.2 below, we obtain the functions considered by Fueter in his papers [12, 13]. In this setting the three definitions given in the preceding sections become:

Definition 5.1 (Slice regular functions). Let $U \subseteq \mathbb{H}$ be an open set and let $f : U \rightarrow \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I$. We say that f is a (left) slice regular function, or s-regular function, if for every $I \in \mathbb{S}$, we have

$$\frac{1}{2} \left(\frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) f_I(u + Iv) = 0.$$

We denote by $\mathcal{SR}(U)$ the set of slice regular functions defined on U .

The notions of slice domains and of axially symmetric domains are similar to the ones already given.

Definition 5.2 (Holomorphic maps of a quaternionic variable). Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and $\mathcal{U} \subseteq \mathbb{R}^2$ be such that $x = u + Iv \in U$ for all $(u, v) \in \mathcal{U}$. The set of (left) holomorphic maps of a quaternionic variable defined on U are of the form $f(q) = \alpha(u, v) + I\beta(u, v)$, where α, β are $\mathbb{H}$ -valued differentiable functions, satisfying the following even-odd conditions, $\alpha(u, v) = \alpha(u, -v)$, $\beta(u, v) = -\beta(u, -v)$ for all $(u, v) \in \mathcal{U}$, and moreover α and β satisfy the Cauchy-Riemann system

$$(5.1) \quad \begin{cases} \partial_u \alpha - \partial_v \beta = 0, \\ \partial_v \alpha + \partial_u \beta = 0. \end{cases}$$

We denote by $\mathcal{HR}(U)$ the set of holomorphic maps of a quaternionic variable defined on the open set U .

Definition 5.3 (Functions in the kernel of G , the quaternionic case). Let $U \subseteq \mathbb{H}$ be an open set and let $f : U \rightarrow \mathbb{H}$ be a real differentiable function. Let $G(q)$ be the operator defined by

$$G(q) := |\underline{q}|^2 \frac{\partial f}{\partial x_0} + \underline{q} \sum_{j=1}^3 x_j \frac{\partial}{\partial x_j}.$$

The set of functions defined on the open set U and in the kernel of G will be denoted by $\mathcal{GR}(U)$. Moreover, we set $\mathcal{GR}_p(U) := \{f \in \mathcal{GR}(U) \mid \alpha \text{ and } \beta \text{ do not depend on } I \in \mathbb{S}\}$, where $\alpha := \frac{1}{2}(f(u + Iv) + f(u - Iv))$, $\beta := \frac{I}{2}(f(u - Iv) - f(u + Iv))$ and $u = \operatorname{Re}(q)$, $v = |q|$.

The following results can be proved by obvious modifications of the proofs in the monogenic setting.

Theorem 5.4. *Let $U \subseteq \mathbb{H}$ be an axially symmetric s -domain and let $f \in \mathcal{SR}(U)$. Then $f \in \mathcal{GR}(U)$.*

Theorem 5.5. *Let $U \subseteq \mathbb{H}$ be an axially symmetric domain and let $f \in \mathcal{HR}(U)$. Then $f \in \mathcal{GR}(U)$.*

Theorem 5.6. *Let $U \subset \mathbb{H}$ be an axially symmetric domain which does not intersect the real axis and let $f : U \rightarrow \mathbb{H}$ be a real differentiable function. Then $f \in \mathcal{HR}(U)$ if and only if $f \in \mathcal{GR}_p(U)$.*

Theorem 5.7. *Let $U \subseteq \mathbb{H}$ be an axially symmetric domain that intersects the real axis. Then $\mathcal{HR}(U)$ is strictly contained in $\mathcal{GR}(U)$.*

As a consequence of Theorem 5.6, functions in $\mathcal{GR}_p(U)$ defined on an axially symmetric domain which does not intersect the real axis have a Cauchy formula which is analogous to the one proven in Theorem 4.8. This formula corresponds to Corollary 28 in [15].

6. CONCLUDING REMARKS

The theory of monogenic functions of several paravector variables generalizes the case of holomorphic functions in several complex variables to the Clifford algebra setting. This generalization is highly nontrivial and can be carried out only under the additional assumption on the number of paravector variables with respect to the dimension of the Clifford algebra. The main results of such a theory can be found in the book [6] where Algebraic Analysis is the main tool to study this class of functions. In the case of slice monogenic functions there are additional difficulties due to the fact that the Algebraic Analysis techniques cannot be easily applied. Here we introduce a new class of functions in several paravector variables that is based on the operator G . We still denote by $\mathbb{R}_n$ the Clifford algebra over $n \in \mathbb{N}$ imaginary units and we take $m \in \mathbb{N}$; the paravectors $x_\ell \in \mathbb{R}_n$ for $\ell = 1, \dots, m$ will be denoted by

$$x_\ell = x_{\ell,0} + x_{\ell,1}e_1 + \dots + x_{\ell,n}e_n = x_{\ell,0} + \underline{x}_\ell.$$

Definition 6.1 (Functions in the kernel of several G -operators). Let U be an open set in $(\mathbb{R}^{n+1})^m$, for $n, m \in \mathbb{N}$, and consider a real differentiable function $f : U \rightarrow \mathbb{R}_n$, and the operators

$$(6.1) \quad G(x_\ell) := |\underline{x}_\ell|^2 \frac{\partial}{\partial x_{\ell,0}} + \underline{x}_\ell \sum_{j=1}^n x_{\ell,j} \frac{\partial}{\partial x_{\ell,j}}, \quad \ell = 1, \dots, m.$$

The set of functions defined on the open set U and in the kernel of $G(x_\ell)$, for $\ell = 1, \dots, m$, will be denoted by $\mathcal{GM}_m(U)$.

From the definition, it immediately follows that:

Proposition 6.2. *Let $f_\ell(x_\ell)$ be functions defined on the open sets $V_\ell \subseteq \mathbb{R}^{n+1}$ and suppose they are in the kernel of $G(x_\ell)$, for $\ell = 1, \dots, m$. Then given arbitrary constants $C_\ell \in \mathbb{R}_n$, $\ell = 1, \dots, m$, the function F defined by $F(x_1, \dots, x_m) = \sum_{\ell=1}^m f_\ell(x_\ell)C_\ell$ belongs to the kernel of (6.4) on $V_1 \times \dots \times V_m$.*

Let us consider the following example in which we consider two variables.

Example 6.3. Let U be an open set in $\mathbb{R}^3 \times \mathbb{R}^3$ that does not intersect the real line. Then the function

$$(6.2) \quad f(x_1, x_2) = -\operatorname{Im}(x_2) + \frac{x_2}{|\underline{x}_2|} \left(\frac{1}{2} \operatorname{Re}(x_1)^2 - \frac{1}{2} \operatorname{Im}(x_1)^2 + \operatorname{Re}(x_2) \right) + \frac{x_1}{|\underline{x}_1|} \frac{x_2}{|\underline{x}_2|} \operatorname{Re}(x_1) \operatorname{Im}(x_1),$$

satisfies the system

$$(6.3) \quad \begin{cases} |\underline{x}_1|^2 \frac{\partial}{\partial x_{1,0}} f(x_1, x_2) + \underline{x}_1 \sum_{j=1}^3 x_{1,j} \frac{\partial}{\partial x_{1,j}} f(x_1, x_2) = 0, \\ |\underline{x}_2|^2 \frac{\partial}{\partial x_{2,0}} f(x_1, x_2) + \underline{x}_2 \sum_{j=1}^3 x_{2,j} \frac{\partial}{\partial x_{2,j}} f(x_1, x_2) = 0. \end{cases}$$

The above example shows that this class of functions of several variables is nontrivial and deserves further investigations. Finally, we observe that the case of several quaternionic variables based on the G operator can be naturally introduced as follows.

We denote the components of $(q_1, \dots, q_m) \in \mathbb{H}^m$ by

$$q_\ell = x_{\ell,0} + x_{\ell,1}i + x_{\ell,2}j + x_{\ell,3}k = x_{\ell,0} + \underline{x}_\ell$$

for $\ell = 1, \dots, m$, where $m \in \mathbb{N}$. We have:

Definition 6.4 (Functions in the kernel of several G -operators: the quaternionic case). Let U be an open set in $\mathbb{H}^m$, for $m \in \mathbb{N}$, and consider a real differentiable function $f : U \rightarrow \mathbb{H}$, and the operators

$$(6.4) \quad G(q_\ell) := |\underline{x}_\ell|^2 \frac{\partial}{\partial x_{\ell,0}} + \underline{x}_\ell \sum_{j=1}^3 x_{\ell,j} \frac{\partial}{\partial x_{\ell,j}}, \quad \ell = 1, \dots, m.$$

The set of functions defined on the open set U and in the kernel of $G(q_\ell)$, for $\ell = 1, \dots, m$, will be denoted by $\mathcal{GR}_m(U)$.

Remark 6.5. A different approach to define a theory of several quaternionic variables, but using the notion of holomorphic functions of a paravector variable, has been introduced in [11].

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