A NEW LITTLEWOOD-RICHARDSON RULE FOR SCHUR P -FUNCTIONS

SOOJIN CHO

ABSTRACT. A new description of shifted Littlewood-Richardson coefficients is given in terms of semistandard decomposition tableaux which were recently introduced by L. Serrano. We also show that the set of semistandard decomposition tableaux is invariant under the action of Lascoux-Schützenberger involution, providing a combinatorial proof of the symmetry of Schur P -functions. We find counterexamples to the conjecture made by L. Serrano on skew Schur P -functions, proving the falsity of the conjecture. Many combinatorial properties of semistandard decomposition tableaux are also shown.

0. INTRODUCTION

Schur Q -functions were introduced by I. Schur in relation to the projective representations of the symmetric and alternating groups [15]. Schur Q -functions are scaled to define Schur P -functions: For a *strict* partition λ with $\ell(\lambda)$ parts,

$$P_\lambda(x_1, \dots, x_n) = 2^{-\ell(\lambda)} Q_\lambda(x_1, \dots, x_n).$$

Shifted Littlewood-Richardson coefficients (shifted LR-coefficients) $f_{\lambda\mu}^\nu$ are the structural constants that appear in the expansion of the product of two Schur P -functions:

$$P_\lambda \cdot P_\mu = \sum_{\nu} f_{\lambda\mu}^\nu P_\nu.$$

There are many ways to prove that shifted LR-coefficients $f_{\lambda\mu}^\nu$ are nonnegative integers. To find explicit combinatorial rules to describe the $f_{\lambda\mu}^\nu$'s has been as important and interesting a problem in combinatorics of symmetric functions as it was to find a rule to describe the $c_{\lambda\mu}^\nu$'s, the (classical) Littlewood-Richardson coefficients (LR-coefficients), where $s_\lambda \cdot s_\mu = \sum_{\nu} c_{\lambda\mu}^\nu s_\nu$ for Schur S -functions s_λ , s_μ and s_ν . There are many combinatorial models to describe $c_{\lambda\mu}^\nu$: Littlewood-Richardson tableaux (LR-tableaux) [12], some special lattice words [19], and hives and puzzles [9, 10]. They are based on the theory of (ordinary) semistandard tableaux, plactic monoids, and Schubert calculus of Grassmannians, respectively, all of which are closely related to the theory of Schur functions.

Received by the editors March 29, 2011 and, in revised form June 13, 2011.

2010 *Mathematics Subject Classification.* Primary 05E05; Secondary 05E10.

Key words and phrases. Schur P -functions, shifted tableaux, semistandard decomposition tableaux, shifted Littlewood-Richardson coefficients.

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0012398).

This work was done while the author was visiting the Korea Institute for Advanced Study.

©2012 American Mathematical Society
 Reverts to public domain 28 years from publication

The theory of *shifted* tableaux was developed by D. Worley [3] and B. Sagan [14], and thanks to the notion of shifted jeu de taquin, it was shown that $f_{\lambda\mu}^\nu$ is the number of certain skew shifted tableaux that *rectifies* to a fixed shifted tableau (see Proposition 7.3). Littlewood-Richardson-Stembridge shifted tableaux (LRS-tableaux) are the ones that rectify to a special shifted tableau, and explicit combinatorial conditions for LRS-tableaux are given by J. Stembridge [17]. The shifted LR-coefficients $f_{\lambda\mu}^\nu$ also appear in the Schubert calculus of Grassmannians, of types B and D : $\tau_\lambda \cdot \tau_\mu = \sum_\nu f_{\lambda\mu}^\nu \tau_\nu$, where τ_λ is the Schubert class in the cohomology ring of an orthogonal Grassmannian. Recently, a shifted plactic monoid structure was constructed for Schur P -functions by L. Serrano in [16], which makes it possible to work with words to understand Schur P -functions: Semistandard decomposition tableaux (SSDT) are presented as a new combinatorial model for Schur P -functions in [16]. Our main concern is to find a combinatorial rule for the shifted LR-coefficients in terms of words (or SSDT's). We follow the same line of the calculation as Stembridge did in [19], where he used the bi-alternant formula along with the semistandard tableaux model for Schur functions and Bender-Knuth involution on the set of semistandard tableaux. For the calculation, we use Nimmo type formula (quotient of Pfaffians), shifted semistandard tableaux model and SSDT's along with the definition of Hall-Littlewood functions specialized at $t = -1$ for Schur P -functions. The shifted plactic monoid theory developed by Serrano plays an important role throughout our work to derive a combinatorial rule for $f_{\lambda\mu}^\nu$ in terms of *words*.

In more detail, we show that the set of SSDT's is invariant under the symmetric group action defined by Lascoux and Schützenberger on the set of words of $\{1, 2, \dots, n\}$, providing a combinatorial proof of the symmetry of Schur P -functions. Finally, various expressions of Schur P -functions are used for different purposes to define some selective SSDT's called *essential* SSDT's. The involution σ_i that is the action of the transposition $(i, i+1) \in S_n$ is used to define a sign reversing involution on the set of essential SSDT's which are among the candidates of special SSDT's that really contribute to $f_{\lambda\mu}^\nu$. The fixed SSDT's of the involution turn out to be equinumerous to the shifted LR-coefficient $f_{\lambda\mu}^\nu$. (See Theorem 5.12.)

The theory of *skew* Schur P -functions $P_{\lambda/\mu}$ is parallel to that of ordinary Schur P -functions:

$$P_{\lambda/\mu} = \sum_{\rho \leq \lambda} f_{\mu\rho}^\lambda 2^{\ell(\mu)+\ell(\rho)-\ell(\lambda)} P_\rho.$$

In [16], Serrano made a conjecture on plactic skew Schur P -functions. We find counterexamples of the conjecture and make an observation on the reason why the conjecture is not true. For the case of Schur P -functions, shifted semistandard tableaux and their plactic counterparts do not interact well enough, differently from the case of Schur functions.

The present paper is organized as follows. After we introduce some background work in Section 1, we prove that the Lascoux-Schützenberger involution is indeed a Bender-Knuth type involution on the set of SSDT's in Section 2. Section 3 and Section 4 are devoted to some technical calculations on the product of two Schur P -functions. Theorem 5.12 is the main theorem of this paper that explains the product rule of two Schur P -functions. The proof of Theorem 5.12 is given in Section 6. In Section 7, we argue on skew Schur P -functions, especially on the conjecture by Serrano. In the Appendix, we provide the descriptions of the lowest

weight SSDT and the highest weight SSDT and also prove that they are really the right ones.

Remark 0.1. Recently, D. Grantchrov, J. Jung, S.-J. Kang, M. Kashiwara and M. Kim have shown that the set of SSDT's forms a crystal of the quantum queer superalgebra $U_q(\mathfrak{q}(n))$ [5]. Their SSDT's are different from the SSDT's we deal with in the present paper. We, however, believe that there must be a plactic monoid isomorphism between two sets; that is, they are essentially the same. They also proved a result similar to the main theorem (Theorem 5.12) of the present paper on the decomposition of the product of two Schur P -functions using crystal basis theory.

1. PRELIMINARIES

We introduce necessary notation, terms and two combinatorial models for Schur P -functions: semistandard marked shifted tableaux and semistandard decomposition tableaux. Moreover, the relation between two models, due to Serrano [16], is summarized since it will be required to develop our argument.

We let $\mathcal{DP}_n$ denote the set of partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with distinct parts where $\ell \leq n$. We also let $\mathcal{DP} = \bigcup_n \mathcal{DP}_n$, the set of all partitions with distinct parts.

For $\lambda \in \mathcal{DP}$, the *length* $\ell(\lambda)$ of λ is defined as the number of positive parts of λ , and the *size* $|\lambda|$ of λ is the sum of all parts, $|\lambda| = \sum_i \lambda_i$. The *shifted diagram* $\mathcal{S}(\lambda)$ of λ is an array of boxes with λ_i boxes in the i th row shifted $(i-1)$ units to the right with respect to the top row: $\mathcal{S}(\lambda) = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq \ell(\lambda), i \leq j \leq \lambda_i + i - 1\}$. For $\lambda, \mu \in \mathcal{DP}$, we write $\mu \leq \lambda$ if $\mathcal{S}(\mu) \subseteq \mathcal{S}(\lambda)$. For $\mu \leq \lambda \in \mathcal{DP}$, the *skew shifted diagram* of λ/μ is defined as $\mathcal{S}(\lambda/\mu) = \mathcal{S}(\lambda) - \mathcal{S}(\mu)$. The *main diagonal* of a (skew) shifted diagram is formed by the boxes indexed (i, i) .

Definition 1.1. A *semistandard marked shifted tableau* T of shape $\lambda \in \mathcal{DP}$ on n letters is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet $\{1' < 1 < 2' < 2 < \cdots < n' < n\}$ such that:

- (1) the entries are weakly increasing along each column and each row of T ,
- (2) each row contains at most one i' for every $i = 1, \dots, n$,
- (3) each column contains at most one i for every $i = 1, \dots, n$, and
- (4) there is no i' on the main diagonal.

A *skew semistandard marked shifted tableau* T of shape λ/μ , for $\mu \leq \lambda \in \mathcal{DP}$, on n letters is a filling of $\mathcal{S}(\lambda/\mu)$ with letters from the alphabet $\{1' < 1 < 2' < 2 < \cdots < n' < n\}$ such that the same conditions (1), (2), (3), and (4) for semistandard marked shifted tableau are satisfied. For convenience, we use *marked shifted tableau* instead of 'semistandard' marked shifted tableau, and *skew marked shifted tableau* instead of skew 'semistandard' marked shifted tableau.

We let $\mathcal{Y}_n(\lambda)$ be the set of marked shifted tableaux of shape λ on n letters and $\mathcal{Y}_n(\lambda/\mu)$ be the set of skew marked shifted tableau of shape λ/μ on n letters.

For each $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}^n$, we let $x^\omega = x_1^{\omega_1} x_2^{\omega_2} \cdots x_n^{\omega_n}$. For a filling T of a (skew) shifted diagram with letters from the alphabet $\{1' < 1 < \cdots < n' < n\}$, the *weight* of T is defined as $\omega(T) = (\omega_1(T), \dots, \omega_n(T)) \in \mathbb{Z}^n$, where $\omega_i(T)$ is the number of i 's and (i') 's in T . We give a definition of Schur P -functions in terms of marked shifted tableaux.

Definition 1.2. For a strict partition $\lambda \in \mathcal{DP}$, we define the *Schur P-function* $P_\lambda(x_1, x_2, \dots, x_n)$ as the generating function of marked shifted tableaux of shape λ :

$$P_\lambda(x_1, x_2, \dots, x_n) = \sum_{T \in \mathcal{Y}_n(\lambda)} x^{\omega(T)}.$$

A marked shifted tableau T in $\mathcal{Y}_n(\lambda)$ is called *standard* if each of $1, 2, \dots, |\lambda|$ appear exactly once in T . The *mixed insertion* that is defined by Serrano in [16] extending the mixed insertion of Haiman for permutations [6] gives a correspondence between the words and pairs of a marked shifted tableau and a standard shifted tableau. We introduce the process of mixed insertion in detail.

Definition 1.3. For a marked shifted tableau T on n letters and an alphabet $x \in \{1, 2, \dots, n\}$, the *insertion of x into T* obtaining a new marked shifted tableau $(T \leftarrow x)$ is defined as follows:

Let $S := T$, $z := x$, and $(p, q) = (0, 0)$.

do while $z \neq 0$

if z **is unprimed**

let y be the leftmost element on the $(p+1)$ st row of S that is strictly greater than z

- i. if there is no such y , then place z at the end of the $(p+1)$ st row of S to make (new) S and let $z := 0$;
- ii. if y is on the main diagonal of S (in which case y must be unprimed), then let p be the column number of the position of y , (new) S be the tableau in which y is replaced by z in S , and $z := y'$ the primed y ;
- iii. otherwise:
 - if y is primed, then let p be the column number of the position of y , let (new) S be the tableau in which y is replaced by z in S , and let $z := y$;
 - if y is unprimed, then let p be the row number of the position of y , let (new) S be the tableau in which y is replaced by z in S , and let $z := y$.

if z **is primed**

let y be the topmost element on the $(p+1)$ st column of S that is strictly greater than z

- i. if there is no such y , then place z at the end of the $(p+1)$ st column of S to make (new) S and let $z := 0$;
- ii. otherwise:
 - if y is primed, then let p be the column number of the position of y , (new) S be the tableau in which y is replaced by z in S , and let $z := y$;
 - if y is unprimed, then let p be the row number of the position of y , (new) S be the tableau in which y is replaced by z in S , and let $z := y$.

end do

$(T \leftarrow x) := S$.

Definition 1.4 (Mixed insertion). For a word $w = w_1 w_2 \cdots w_m$ on the alphabet $\{1, 2, \dots, n\}$, the *mixed insertion tableau* $P_{mix}(w)$ and the *mixed recording tableau* $Q_{mix}(w)$ of w are recursively defined:

- (1) $P_0 = \emptyset$, $Q_0 = \emptyset$ are empty tableaux.

- (2) For $i = 1, 2, \dots, m$, $P_i = (P_{i-1} \leftarrow w_i)$ and Q_i is obtained by adding a box with content i on the location of a new box added to P_{i-1} to get P_i .
- (3) $P_{mix}(w) = P_m$ and $Q_{mix}(w) = Q_m$.

Proposition 1.5 ([16]). *The correspondence $w \leftrightarrow (P_{mix}(w), Q_{mix}(w))$ gives a bijection between the set of (finite) words on the alphabet $\{1, 2, \dots, n\}$ and the set of pairs of a marked shifted tableau and a standard shifted tableau of the same shape.*

Example 1.6. Let $w = 33234213$. Then $P_1 = \boxed{3}$, $P_2 = \boxed{3 \mid 3}$ and $P_3 = (P_2 \leftarrow 2) = \begin{array}{|c|c|} \hline 2 & 3' \\ \hline & 3 \\ \hline \end{array}$, since 2 bumps 3 on the main diagonal and 3 is bumped out as primed 3 and is inserted in the next column to bump 3 to the next row. Thus we have

$$P_4 = \begin{array}{|c|c|c|} \hline 2 & 3' & 3 \\ \hline & 3 & \\ \hline \end{array} \quad P_5 = \begin{array}{|c|c|c|c|} \hline 2 & 3' & 3 & 4 \\ \hline & 3 & & \\ \hline \end{array} \quad P_6 = \begin{array}{|c|c|c|c|} \hline 2 & 2 & 3' & 4 \\ \hline & 3 & 3 & \\ \hline \end{array}$$

$$P_7 = \begin{array}{|c|c|c|c|} \hline 1 & 2' & 3' & 4 \\ \hline & 2 & 3' & \\ \hline & & 3 & \\ \hline \end{array} \quad P_8 = \begin{array}{|c|c|c|c|} \hline 1 & 2' & 3' & 3 \\ \hline & 2 & 3' & 4 \\ \hline & & 3 & \\ \hline \end{array}.$$

$$\text{Therefore we have } P_{mix}(w) = \begin{array}{|c|c|c|c|} \hline 1 & 2' & 3' & 3 \\ \hline & 2 & 3' & 4 \\ \hline & & 3 & \\ \hline \end{array} \quad \text{and} \quad Q_{mix}(w) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline & 3 & 6 & 8 \\ \hline & & 7 & \\ \hline \end{array}.$$

The set of explicit basic relations for the equivalence relation on the set of words whose classes are the ones that have the same mixed insertion tableau P has been found, and a nice set of representatives called *semistandard decomposition tableaux* of the classes are defined in [16]:

A word $w = w_1 w_2 \cdots w_m$ on the set of alphabets $\{1, 2, \dots, n\}$ is a *hook word* if there is $1 \leq m' \leq m$ such that

$$w_1 > w_2 > \cdots > w_{m'} \leq w_{m'+1} \leq \cdots \leq w_m.$$

We use $(w \downarrow)$ to represent the subword $w_1 w_2 \cdots w_{m'}$ of w and $(w \uparrow)$ to represent $w_{m'+1} \cdots w_m$.

Definition 1.7. A *semistandard decomposition tableau (SSDT)* R of shape $\lambda \in \mathcal{DP}$ on n letters is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet $\{1 < 2 < \cdots < n\}$ such that:

- (1) the word R_i obtained by reading the i th row of R from left to right is a hook word of length λ_i , and
- (2) R_i is a hook word of maximum length in the concatenation $R_\ell R_{\ell-1} \cdots R_i$ for all $i = 1, 2, \dots, \ell - 1$, where $\ell = \ell(\lambda)$.

Let $\mathcal{D}_n(\lambda)$ be the set of semistandard decomposition tableaux of shape $\lambda \in \mathcal{DP}$ on n letters. The *reading word* $\text{read}(R)$ of $R \in \mathcal{D}_n(\lambda)$ is the concatenation $R_\ell R_{\ell-1} \cdots R_1$ of the hook subwords.

The *weight* of a filling R of $\mathcal{S}(\lambda)$ with letters from $\{1, 2, \dots, n\}$ is $\omega(R) = (\omega_1(R), \dots, \omega_n(R))$, where $\omega_i(R)$ is the number of i 's in R .

Proposition 1.8 (Serrano, [16]). *For an SSDT R in $\mathcal{D}_n(\lambda)$, $P_{mix}(\text{read}(R))$ is a marked shifted tableau of shape λ . Moreover, the function $Y : \mathcal{D}_n(\lambda) \rightarrow \mathcal{Y}_n(\lambda)$*

defined by $Y(R) = P_{\text{mix}}(\text{read}(R))$ is a weight preserving bijection between $\mathcal{Y}_n(\lambda)$ and $\mathcal{D}_n(\lambda)$. Therefore

$$(1.9) \quad P_\lambda(x_1, x_2, \dots, x_n) = \sum_{T \in \mathcal{Y}_n(\lambda)} x^{\omega(T)} = \sum_{R \in \mathcal{D}_n(\lambda)} x^{\omega(R)}.$$

For $T \in \mathcal{Y}_n(\lambda)$, let $D(T) = Y^{-1}(T)$ be the corresponding SSDT in $\mathcal{Y}_n(\lambda)$.

The following is useful to check if a given shifted tableau is an SSDT.

Proposition 1.10 ([16], Proposition 2.10). *A filling R of $\mathcal{S}(\lambda)$ with letters from $\{1 < 2 < \dots < n\}$ is an SSDT if and only if:*

- (1) *the word R_i obtained by reading the i th row of R from left to right is a hook word of length λ_i , and*
- (2') *R_i is a hook word of maximum length in $R_{i+1}R_i$ for all $i = 1, \dots, \ell(\lambda) - 1$.*

Example 1.11. Let $R =$

4	2	1	3
	3	2	3
		3	

. Then by Proposition 1.10, R is an SSDT

of shape $(4, 3, 1)$.

The reading word of R is $\text{read}(R) = 3\,3\,2\,3\,4\,2\,1\,3$, and $P_{\text{mix}}(\text{read}(R)) =$

1	2'	3'	3
	2	3'	4
		3	

by Example 1.6. Hence $Y(R) =$

1	2'	3'	3
	2	3'	4
		3	

.

Proposition 1.12. *For $\lambda \in \mathcal{DP}_n$, we let $\lambda_i = 0$ if $\ell(\lambda) < i$. Then the following holds:*

- (1) *There is a unique marked shifted tableau of weight $(\lambda_1, \lambda_2, \dots, \lambda_n)$ in $\mathcal{Y}_n(\lambda)$ and a unique SSDT of weight $(\lambda_1, \lambda_2, \dots, \lambda_n)$ in $\mathcal{D}_n(\lambda)$.*
- (2) *There is a unique marked shifted tableau of weight $(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$ in $\mathcal{Y}_n(\lambda)$ and a unique SSDT of weight $(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$ in $\mathcal{D}_n(\lambda)$.*

Proof. Due to the definition of a marked shifted tableau, there is a unique way to fill in $\mathcal{S}(\lambda)$ with λ_i i 's for each i : Fill in the i th row with λ_i i 's. Therefore, the first part is proved. For a proof of the second part, we use the well-known fact that $P_\lambda(x_1, \dots, x_n)$ is a symmetric polynomial whose combinatorial proof is given in Section 2. Note that $x_1^{\lambda_n} x_2^{\lambda_{n-1}} \dots x_n^{\lambda_1}$ is obtained by applying the longest permutation π_0 , which maps i to $n+1-i$, to $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$. Hence, by the symmetry of P_λ , the second part is proved. $\square$

Definition 1.13. For a strict partition $\rho \in \mathcal{DP}_n$:

- (1) The unique SSDT in $\mathcal{D}_n(\rho)$ of weight $(\rho_1, \rho_2, \dots, \rho_n)$ is called the *highest weight SSDT* and denoted $V_\rho(n)$.
- (2) The unique SSDT in $\mathcal{D}_n(\rho)$ of weight $(\rho_n, \rho_{n-1}, \dots, \rho_1)$ is called the *lowest weight SSDT* and denoted $U_\rho(n)$.

Remark 1.14. The notion of the highest weight SSDT does not depend on the number n of variables, whereas the lowest weight SSDT does. Hence we use V_ρ instead of $V_\rho(n)$ for the highest weight SSDT, while we use U_ρ for the lowest weight SSDT instead of $U_\rho(n)$ if there is no confusion on the number of variables (or on the number of letters available).

2. BENDER-KNUTH TYPE INVOLUTION ON $\mathcal{D}_n(\lambda)$

Bender-Knuth involution, defined on the set of semistandard tableaux of ordinary shape, gives a combinatorial proof of the symmetry of Schur functions [1]. It is well known that $P_\lambda(x_1, \dots, x_n)$ is a symmetric function. One can argue this by the fact that $P_\lambda(x_1, \dots, x_n)$ is a specialized Hall-Littlewood function ([13], III) or by using the Bender-Knuth type involutions on $\mathcal{Y}_n(\lambda)$ defined by Stembridge ([18], Sec.6).

Since SSDT's are a new combinatorial model for Schur P -functions, it is natural to ask if it is possible to define natural (Bender-Knuth type) involutions on $\mathcal{D}_n(\lambda)$, the set of SSDT's. We show that the symmetric group action defined by Lascoux and Schützenberger [11] on the set of words on the alphabet $\{1, 2, \dots, n\}$ defines an action on the subset $\{\text{read}(R) \mid R \in \mathcal{D}_n(\lambda)\}$ of the set of all words: Lascoux-Schützenberger involutions are Bender-Knuth type involutions on $\mathcal{D}_n(\lambda)$, and this is another combinatorial proof of the symmetry of P_λ .

We first introduce Lascoux-Schützenberger involutions σ_k , $k = 1, \dots, n-1$, on the set of words on the alphabet $\{1, 2, \dots, n\}$. (See Definition 2.2.)

For a word $w = w_1 w_2 \cdots w_m$ on the alphabet $\{1, 2, \dots, n\}$, we say that $w_i = k+1$ and $w_j = k$, $i < j$, are *consecutively k -paired* in w if there is no k or $k+1$ between w_i and w_j . Let $w^{(0)} = w$ and $w^{(s)}$, $s \geq 1$, be the word obtained by consecutively deleting k -paired $(k+1)$'s and k 's in $w^{(s-1)}$. We call w_i and w_j , $i < j$, *k -paired* in w if they are consecutively k -paired in some $w^{(s)}$. We also say that $w_i \in \{k, k+1\}$ is *k -paired* in w if there is w_j such that w_i and w_j (or w_j and w_i) are k -paired in w .

The following lemma is immediate from the definition of k -pairedness.

Lemma 2.1. *For a word $w = w_1 w_2 \cdots w_m$ on the alphabet $\{1, 2, \dots, n\}$, the subword consisting of $w_i \in \{k, k+1\}$ which are not k -paired form an increasing subword.*

Definition 2.2 (Lascoux and Schützenberger, [11]). For $1 \leq k \leq n-1$ define σ_k on the set of words on $\{1, \dots, n\}$ as follows.

For a word $w = w_1 w_2 \cdots w_m$, $w_i \in \{1, 2, \dots, n\}$, let the number of k 's ($(k+1)$'s, respectively) which are not k -paired in w be a (and b , respectively). Then non- k -paired k 's and $(k+1)$'s in w form a nondecreasing subword of

$$w : \underbrace{k \cdots k}_a \underbrace{k+1 \cdots k+1}_b.$$

Replace the subword of w consisting of non- k -paired k 's and $(k+1)$'s by b k 's and a $(k+1)$'s, $\underbrace{k \cdots k}_b \underbrace{k+1 \cdots k+1}_a$, to obtain the word $\sigma_k(w)$.

For $R \in \mathcal{D}_n(\lambda)$, let $\sigma_k(R)$ be the tableau of shape λ such that $\text{read}(\sigma_k(R)) = \sigma_k(\text{read}(R))$.

Example 2.3. If $\lambda = (7, 5, 4)$ and

$$R = \begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 5 & 4 & 2 & 1 & 1 & 3 \\ \hline & 6 & 5 & 2 & 1 & 4 & \\ \hline & & 5 & 1 & 2 & 3 & \\ \hline \end{array} \in \mathcal{D}_6(\lambda),$$

then $\text{read}(R) = w = 5 \underline{1} \textcircled{2} 3 \ 6 \ 5 \ \textcircled{2} \textcircled{1} 4 \ 6 \ 5 \ 4 \ \textcircled{2} \textcircled{1} \textcircled{1} 3$, where the circled numbers are 1-paired in w and the underlined 1 is not 1-paired in w . We replace the under-

lined 1 in w by 2 to obtain the reading word $\sigma_1(w)$ of $\sigma_1(R)$, a tableau of shape λ :

$$\sigma_1(w) = 5 \underline{2} \textcircled{2} 3 \ 6 \ 5 \ \textcircled{2} \textcircled{1} 4 \ 6 \ 5 \ 4 \ \textcircled{2} \textcircled{1} \textcircled{1} 3,$$

$$\sigma_1(R) = \begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 5 & 4 & 2 & 1 & 1 & 3 \\ \hline & 6 & 5 & 2 & 1 & 4 & \\ \hline & & 5 & 2 & 2 & 3 & \\ \hline \end{array}.$$

Lemma 2.4. *For $R \in \mathcal{D}_n(\lambda)$ with $\text{read}(R) = w = w_1 w_2 \cdots w_m$ and for $1 \leq k \leq n-1$, let $R^* = \sigma_k(R)$ and $\text{read}(R^*) = w^* = w_1^* w_2^* \cdots w_m^*$. Then the following holds:*

- (1) w_i is k -paired in w if and only if w_i^* is k -paired in w^* . Moreover, if w_i is k -paired in w , then $w_i^* = w_i$.
- (2) If $w_{i_1} = k+1, w_{i_2} = k$ are k -paired in w , then every $w_i \in \{k, k+1\}$, $i_1 < i < i_2$, is k -paired in w .
- (3) If $w_{i_1}, w_{i_2} \in \{k, k+1\}$, $i_1 < i_2$, are in a row R_j and both of them are not k -paired in w , then every $w_i \in \{k, k+1\}$ such that $i_1 < i < i_2$ is not k -paired in w .

Proof. (1) and (2) are clear from the definition, and we give a proof of (3).

Suppose that $w_{i_1} = k+1$. If $w_{i_1} \in (w \downarrow)$ and there is k after w_{i_1} , then the first k after w_{i_1} is paired with w_{i_1} ; hence there is no k after w_{i_1} in R_j and $w_i = w_{i_2} = k+1$. Therefore w_i cannot be k -paired in w by (2). If $w_{i_1} \in (w \uparrow)$, then there can be no k after w_{i_1} in R_j by the definition of $(w \uparrow)$; hence $w_i = w_{i_2} = k+1$ and w_i cannot be k -paired in w by (2).

Suppose that $w_{i_1} = k$. If $w_i = k$, then there cannot be $k+1$ between w_{i_1} and w_i , and if w_i is k -paired in w , then w_{i_1} has to be k -paired in w by (2). Therefore, w_i is not k -paired in w . If $w_i = k+1$, then w_{i_2} must be $k+1$, and there cannot be k between w_i and w_{i_2} . If w_i is k -paired in w , then w_{i_2} must be k -paired in w by (2) again. Hence w_i is not k -paired in w . $\square$

In the following we prove that $\sigma_k(R)$ is again an SSDT for an SSDT R , showing that the Lascoux-Schützenberger action is also an action on $\mathcal{D}_n(\lambda)$ and Schur P -functions are symmetric.

Theorem 2.5. *For each $k = 1, \dots, n-1$, σ_k defined in Definition 2.2 is an involution on $\mathcal{D}_n(\lambda)$. Moreover, the following holds: For an SSDT $R \in \mathcal{D}_n(\lambda)$,*

$$\omega_i(\sigma_k(R)) = \begin{cases} \omega_i(R) & \text{if } i \neq k, k+1, \\ \omega_{k+1}(R) & \text{if } i = k, \\ \omega_k(R) & \text{if } i = k+1. \end{cases}$$

Proof. First note that σ_k is an involution on the set of words: $\sigma_k^2 = 1$. Hence it is an involution on $\mathcal{D}_n(\lambda)$ once we show that $\mathcal{D}_n(\lambda)$ is invariant under the action of σ_k . Let $R^* = \sigma_k(R)$.

It is clear from the definition of σ_k that the weight of R^* is obtained by exchanging the k th and the $(k+1)$ st entries of $\omega(R)$.

We use Lemma 2.4(3) to prove that the j th row R_j^* of R^* form a hook word. If there is no non- k -paired k or $k+1$ in R_j , then $R_j^* = R_j$ and R_j^* is a hook word. We therefore assume that there are some non- k -paired k or $k+1$ in $R_j = u_1 u_2 \cdots u_r$, and let

$$U = \{i \mid u_i \text{ is a non-}k\text{-paired } k \text{ or } k+1 \text{ in } w\} = \{i_1 < i_2 < \cdots < i_{r_0}\}.$$

Then $u_{i_1} \cdots u_{i_{r_0}}$ is a nondecreasing subword of R_j consisting of k 's and $(k+1)$'s, and if u_i is a k -paired k or $k+1$ in w , then by Lemma 2.4(3), either $i < i_1$ or $i_{r_0} < i$. In R_j , only $u_{i_1} \cdots u_{i_{r_0}}$ will be changed by applying σ_k to R , and it is easy to see that R_j^* is again a hook word. Moreover, the lengths of $(R_j \downarrow)$ and $(R_j^* \downarrow)$ are the same.

By Proposition 1.10, it only remains to prove that R_j^* is a hook word of maximum length in $R_{j+1}^* R_j^*$ for each $j = 1, 2, \dots, \ell(\lambda) - 1$. Let $R_{j+1} R_j = w = w_1 w_2 \cdots w_m$ and $R_{j+1}^* R_j^* = w^* = w_1^* w_2^* \cdots w_m^*$. For a subset $A \subseteq \{1, 2, \dots, m\}$, we let w_A (or w_A^*) be a subword of w (or w^* , respectively) consisting of w_i 's (w_i^* 's, respectively) for $i \in A$. We let $S = \{i \mid w_i \in \{k, k+1\}\} = \{i \mid w_i^* \in \{k, k+1\}\}$ be the (ordered) set of indices, where either k or $k+1$ appears in w or equivalently in w^* . We define some subsets of S according to the property of the corresponding w_i 's or equivalently w_i^* 's:

$$U_{j+1} = \{i \in S \mid w_i \in R_{j+1}, w_i \text{ is not } k\text{-paired in } \text{read}(R)\},$$

$$U_j = \{i \in S \mid w_i \in R_j, w_i \text{ is not } k\text{-paired in } \text{read}(R)\},$$

$$P = P_{j+1} \cup P_j \cup \{i, i+1 \mid w_i = k+1 \in R_j, w_{i+1} = k \in R_j\}, \text{ where}$$

$$P_{j+1} = \{i \in S \mid w_i = k+1 \in R_{j+1} \text{ and is } k\text{-paired with a } k \in R_j \text{ in } \text{read}(R)\},$$

$$P_j = \{i \in S \mid w_i = k \in R_j \text{ and is } k\text{-paired with a } k+1 \in R_{j+1} \text{ in } \text{read}(R)\}.$$

Then by Lemma 2.1, $w_{U_{j+1} \cup U_j}$ is a nondecreasing word of k 's and $k+1$'s. Note that $|P_{j+1}| = |P_j|$, and if there is $w_i = k+1 \in R_{j+1}$ that is k -paired with a k in R_a , $a < j$, then $U_j = \emptyset$ by Lemma 2.4(2). Also note that if $A \cap (U_{j+1} \cup U_j) = \emptyset$ for a subset $A \subseteq \{1, 2, \dots, m\}$, then $w_A = w_A^*$ because σ_k changes only w_i 's for $i \in (U_{j+1} \cup U_j)$ by Lemma 2.4(1).

We show that if $u^* = w_{i_1}^* \cdots w_{i_{m'}}^*$ is a hook subword of w^* , then there is a corresponding hook subword u of w of the same length, which completes the proof since R is an SSDT. Let $I = \{i_1, \dots, i_{m'}\}$. If $I \cap (U_{j+1} \cup U_j) = \emptyset$, then $u = w_{i_1} \cdots w_{i_{m'}} = u^*$ is a hook subword of w . Therefore we assume that $U = J \cap (U_{j+1} \cup U_j)$ is not empty. We assume that $I \cap U_{j+1} \neq \emptyset$, and the proof for the case $I \cap U_j \neq \emptyset$ can be done in a similar way. Note that we can take a partition of I , $I = P_\alpha \cup (I \cap U_{j+1}) \cup (I \cap P) \cup (I \cap U_j) \cup P_\beta$, where elements in P_α are less than the elements in U_{j+1} and so on, and elements of P_α and P_β are all k -paired in $\text{read}(R)$. Note also that $w_i^* = k+1$ for all $i \in P_\beta$ since $I \cap U_{j+1} \neq \emptyset$, by Lemma 2.4(2). If $w_j^* = k$ for $j \in P_\beta$, then $w_j = w_j^*$ is k -paired with a $k+1$ in R_a for $a > i+1$.

Let the first (smallest) and the last (largest) elements in $\{w_i^* \mid i \in I \cap U_{j+1}\}$ be s^* and t^* , respectively. Note, by Lemma 2.1 and the definition of σ_k , that $w_i^* \leq w_{i'}^*$ if $i \leq i' \in I \cap U_{j+1}$. There are three possibilities: $(s^*, t^*) = (k, k+1)$, $(s^*, t^*) = (k, k)$, and $(s^*, t^*) = (k+1, k+1)$. Let s and t be the corresponding elements in w of s^* and t^* , respectively.

When $(s^*, t^*) = (k, k+1)$, the possible (s, t) 's are $(k, k+1)$, (k, k) , $(k+1, k+1)$, and it is easy to see that $w_{i_1} \cdots w_{i_{m'}}$ is itself a hook subword of w in these cases.

When $(s^*, t^*) = (k, k)$, if $(s, t) = (k, k)$, then $w_{i_1} \cdots w_{i_{m'}}$ is a hook subword of w . If either $(s, t) = (k, k+1)$ or $(s, t) = (k+1, k+1)$, then $w_{i_1} \cdots w_{i_{m'}}$ may not be a hook subword of w if $I \cap P \neq \emptyset$ and the smallest w_i^* in $\{w_i^* \mid i \in I \cap P\}$ is k . (If $I \cap P = \emptyset$, then $w_{I \cap (U_{j+1} \cup U_j)}$ is a nondecreasing word on k and $k+1$ and w_{P_β} is a word of only $(k+1)$'s, and there is nothing to care about.) We therefore assume that $I \cap P \neq \emptyset$ and the smallest w_i^* in $\{w_i^* \mid i \in I \cap P\}$ is k . Then $w_{I \cap P}^*$ is a word

of only k 's, by the definition of P , and there must be at least $|I \cap P|$ elements in $I^c \cap P$, where $w_i^* = w_i = k + 1$ for $i \in I^c \cap P$. Let u be the subword of w obtained by replacing $|I \cap P|$ k 's in the position at $I \cap P$ by $k + 1$'s.

When $(s^*, t^*) = (k + 1, k + 1)$, we can use the fact that P_α contains at most one $k + 1$ to show that $w_{i_1} \cdots w_{i_m}$ is a hook subword of w . $\square$

Remark 2.6. As stated in Remark 0.1, Grantchrov, Jung, Kang, Kashiwara and Kim have shown that the set $\mathcal{D}_n(\lambda)$ of SSDT's forms a crystal of the quantum queer superalgebra $U_q(\mathfrak{q}(n))$. Due to the theory developed by them [4], it is also known that each connected component of the $U_q(\mathfrak{q}(n))$ -crystal $\mathcal{D}_n(\lambda)$ is a $U_q(\mathfrak{gl}(n))$ -crystal. The Weyl group (S_n for $\mathfrak{gl}(n)$) action defined by Kashiwara [8] on a $U_q(\mathfrak{gl}(n))$ -crystal is known to be the same action of σ_k on the words. Hence σ_k action on $\mathcal{D}_n(\lambda)$ is the same as the Weyl group action of Kashiwara.

Example 2.7. If

$$R = \begin{array}{cccccccccccccccc} 9 & 8 & 7 & 6 & 5 & 4 & \textcircled{2} & 1 & 1 & 1 & 1 & 1 & 1 & \underline{3} & \underline{3} & 4 & 4 \\ & 8 & 7 & 6 & 5 & 4 & 1 & 1 & \textcircled{2} & \textcircled{2} & \textcircled{2} & \underline{3} & \textcircled{3} & 4 & 5 & & \\ & & 8 & 7 & 6 & 5 & \textcircled{2} & \underline{2} & \underline{3} & \textcircled{3} & \textcircled{3} & \textcircled{3} & 4 & & & & \\ & & & 5 & \underline{2} & \underline{2} & \textcircled{3} & 4 & & & & & & & & & \end{array}$$

is an SSDT in $D_9(\lambda)$, $\lambda = (17, 14, 11, 5)$, then

$$\sigma_2(R) = R^* = \begin{array}{cccccccccccccccc} 9 & 8 & 7 & 6 & 5 & 4 & \textcircled{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \underline{3} & \underline{3} & 4 & 4 \\ & 8 & 7 & 6 & 5 & 4 & 1 & 1 & \textcircled{2} & \textcircled{2} & \textcircled{2} & \underline{3} & \textcircled{3} & 4 & 5 & & & \\ & & 8 & 7 & 6 & 5 & \textcircled{2} & \underline{2} & \underline{2} & \textcircled{3} & \textcircled{3} & \textcircled{3} & 4 & & & & \\ & & & 5 & \underline{2} & \underline{2} & \textcircled{3} & 4 & & & & & & & & & \end{array}.$$

In R and R^* , 2-paired numbers are circled and 2's and 3's that are not 2-paired are underlined. In $\text{read}(R)$ the subword of unpaired 2's and 3's is 2223333, so we replace the corresponding part of R^* by 2222333.

Let us consider the second and the third rows of R^* and R . Adopting the notation from the proof of Theorem 2.5,

$$R_3^* R_2^* = w^* = 8 \ 7 \ 6 \ 5 \ \textcircled{2} \underline{2} \ \underline{2} \ \textcircled{3} \textcircled{3} \textcircled{3} \ 4 \ 8 \ 7 \ 6 \ 5 \ 4 \ 1 \ 1 \ \textcircled{2} \textcircled{2} \textcircled{2} \underline{3} \ \textcircled{3} \ 4 \ 5,$$

$$R_3 R_2 = w = 8 \ 7 \ 6 \ 5 \ \textcircled{2} \underline{2} \ \underline{3} \ \textcircled{3} \textcircled{3} \textcircled{3} \ 4 \ 8 \ 7 \ 6 \ 5 \ 4 \ 1 \ 1 \ \textcircled{2} \textcircled{2} \textcircled{2} \underline{3} \ \textcircled{3} \ 4 \ 5.$$

Observe that $P_\alpha = \{5\}$, $U_3 = \{6, 7\}$, $P = \{8, 9, 10, 19, 20, 21\}$, $U_2 = \{22\}$, and $P_\beta = \{23\}$.

Let

$$u^* = w_1^* w_2^* w_3^* w_4^* w_5^* w_6^* w_7^* w_{20}^* w_{21}^* w_{22}^* w_{23}^* w_{24}^* w_{25}^* = 8 \ 7 \ 6 \ 5 \ 2 \ 2 \ 2 \ 2 \ 3 \ 3 \ 4 \ 5$$

be a hook subword of w^* . Then the corresponding subword

$$w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_{20} w_{21} w_{22} w_{23} w_{24} w_{25} = 8 \ 7 \ 6 \ 5 \ 2 \ 2 \ 3 \ 2 \ 2 \ 3 \ 3 \ 4 \ 5$$

of w is not a hook subword because $w_{U_3}^* = 22$ is replaced by $w_{U_3} = 23$ while $I \cap P = \{20, 21\} \neq \emptyset$. In this case, following the proof of Theorem 2.5, we can let

$$u = w_1 w_2 w_3 w_4 w_5 w_6 w_7 \underline{w_8 w_9} w_{22} w_{23} w_{24} w_{25} = 8 \ 7 \ 6 \ 5 \ 2 \ 2 \ 3 \ 3 \ 3 \ 3 \ 3 \ 4 \ 5$$

be the corresponding hook subword of w of the same length with u^* .

In Theorem 2.9, we prove that if we apply σ_k only to a part of $\text{read}(R)$ for an SSDT R , then we obtain another SSDT under a certain special condition. We first prove a lemma that is necessary in the proof of Theorem 2.9.

Lemma 2.8. *Let $u = u_1 u_2 \cdots u_r$ be a hook word on the alphabet $\{1, 2, \dots, n\}$. For given $1 \leq k \leq n-1$ and $a > 0$, there is a word $\tilde{u} = \tilde{u}_1 \tilde{u}_2 \cdots \tilde{u}_{r+a}$ on the alphabet $\{0, 1, \dots, n, n+1\}$ such that*

- (1) $u\tilde{u} = \text{read}(R)$, where R is an SSDT of shape $(r+a, r)$ and
- (2) there are a $(k+1)$'s and no k 's in $\tilde{u}$.

Proof. We let $(u \downarrow) = u_1 \cdots u_{r'}$ and $(u \uparrow) = u_{r'+1} \cdots u_r$. Let $(u \downarrow)^f$ be the subword of $(u \downarrow)$ consisting of letters at least $k+1$ and $(u \downarrow)^b$ be the rest of $(u \downarrow)$. We also let $(u \uparrow)^f$ be the subword of $(u \uparrow)$ consisting of letters at most $k+1$ and $(u \uparrow)^b$ be the rest of $(u \uparrow)$. The lengths of $(u \downarrow)^f$ and $(u \uparrow)^f$ are denoted by c and d , respectively. We define $\tilde{u}$ as follows

$$\tilde{u}_i = \begin{cases} n+1 & \text{if } i = 1, \\ u_i + 1 & \text{if } 2 \leq i \leq c, \\ u_i - 1 & \text{if } c < i \leq r', \\ 0 & \text{if } r' < i \leq r' + d, \\ k+1 & \text{if } r' + d < i \leq r' + d + a, \\ u_{i-a} & \text{if } r' + d + a < i \leq r + a. \end{cases}$$

It is easy to check that $\tilde{u}$ satisfies the given conditions in the statement of the lemma. $\square$

Theorem 2.9. *For an SSDT $R \in \mathcal{D}_n(\lambda)$ with $\text{read}(R) = w = w_1 w_2 \cdots w_m$, let $w_i = k$ and $\sigma_k^i(R)$ be the tableau obtained by applying σ_k to $w_{i+1} \cdots w_m$, leaving $w_1 \cdots w_i$ unchanged. That is, $\sigma_k^i(R)$ is of shape λ and $\text{read}(\sigma_k^i(R)) = w_1 \cdots w_i \hat{w}_{i+1} \cdots \hat{w}_m$, where $\sigma_k(w_{i+1} \cdots w_m) = \hat{w}_{i+1} \cdots \hat{w}_m$. If w_i is not k -paired in w , then $\sigma_k^i(R)$ is an SSDT in $\mathcal{D}_n(\lambda)$.*

Proof. Let w_i appear in the j th row R_j of R and let $\hat{R} = \sigma_k^i(R)$. We let $R_j^f = (\hat{R}_j)^f$ be the first part of R_j or $\hat{R}_j$ before w_i appears including w_i , and $R_j^b ((\hat{R}_j)^b$, respectively) be the last part of R_j ($\hat{R}_j$, respectively) after w_i appears in R_j ($\hat{R}_j$, respectively). We also let $R^u (\hat{R}^u$, respectively) be the subtableau of R ($\hat{R}$, respectively) consisting of the first $j-1$ rows, and $R^d (\hat{R}^d$, respectively) be the subtableau consisting of the last $\ell(\lambda) - j$ rows from the $(j+1)$ st row.

First note that R^u together with R_j^b on the bottom is an SSDT of shape $(\lambda_1, \dots, \lambda_{j-1}, d)$, where d is the length of R_j^b . Let that SSDT $R^u \cup R_j^b$; then $\sigma_k(R^u \cup R_j^b) = \hat{R}^u \cup (\hat{R}_j)^b$ is an SSDT by Theorem 2.5. Therefore $\hat{R}^u$ is an SSDT of shape $(\lambda_1, \dots, \lambda_{j-1})$. Moreover $\hat{R}^d = R^d$ is an SSDT of shape $(\lambda_{j+1}, \dots, \lambda_{\ell(\lambda)})$. We, therefore, only need to show that

- i) $\hat{R}_{j-1}$ and $\hat{R}_j$ form an SSDT and ii) $\hat{R}_j$ and $\hat{R}_{j+1}$ form an SSDT.

Note the following:

- (a) Every $(k+1)$ in R_j^f or R_l , $l > j$, is k -paired in w since w_i is not k -paired in w .
- (b) Every k in R_j^b is not k -paired in w by Lemma 2.4(2) and hence is not k -paired in $w_{i+1} \cdots w_m$.
- (c) A $(k+1)$ in R_j^b is k -paired in $w_{i+1} \cdots w_m$ if and only if it is k -paired in w .

Let a be the number of k 's in R_j^f that is not k -paired in w . Then $a > 1$ since $w_i = k \in R_j^f$ is not k -paired in w . Let b be the number of k 's in R_{j+1} that is not k -paired in w .

Let us show i) first. Let R_0 be a hook word on the alphabet $\{0, 1, \dots, n, n+1\}$ of length $\lambda_1 + a$ which contains exactly a $(k+1)$'s and no k 's and $R_1 R_0$ is a reading word of an SSDT of shape $(\lambda_1 + a, \lambda_1)$. Such an R_0 exists by Lemma 2.8 and $(k+1)$'s in R_0 are not k -paired in R_0 . If we make $\widetilde{R}^u$ by adding R_0 to the top and R_j to the bottom of R^u , then $\widetilde{R}^u$ is an SSDT. If we only consider the k 's and $(k+1)$'s, then we can make the following observation: a non- k -paired k 's are added in front and a non- k -paired $(k+1)$'s are added at the end of the word $w_{i+1}w_{i+2}\cdots w_m$ to obtain $\text{read}(\widetilde{R}^u)$. With the observations (a), (b) and (c), we can see that the last two rows of $\sigma_k(\widetilde{R}^u)$ coincide with $\widehat{R}_{j-1}$ and $\widehat{R}_j$. This also shows, by Theorem 2.5 if we consider σ_k as a permutation in the symmetric group on $\{0, 1, \dots, n+1\}$, that $\widehat{R}_{j-1}$ and $\widehat{R}_j$ form an SSDT.

To show ii), consider a hook word R'_0 on the alphabet $\{0, 1, \dots, n, n+1\}$ of length $\lambda_1 + a + b$ which contains exactly $(a+b)$ $(k+1)$'s and no k 's and $R_1 R'_0$ is a reading word of an SSDT of shape $(\lambda_1 + a + b, \lambda_1)$. Then $(k+1)$'s in R'_0 are not k -paired in R'_0 . Let $\overline{R}^u$ be the SSDT by adding R'_0 to the top and R_j and R_{j+1} to the bottom of R^u . Then $\overline{R}^u$ is an SSDT and the last two rows of $\sigma_k(\overline{R}^u)$ coincide with $\widehat{R}_j$ and $\widehat{R}_{j+1}$. This shows, by Theorem 2.5, that $\widehat{R}_j$ and $\widehat{R}_{j+1}$ form an SSDT. $\square$

Remark 2.10. If $w_i = k$ is k -paired in w , then $\sigma_k^i(R)$ is not necessarily an SSDT in $\mathcal{D}_n(\lambda)$ for $R \in \mathcal{D}_n(\lambda)$: $R = \begin{array}{|c|c|c|c|c|c|} \hline 3 & \textcircled{2} & 1 & 2 & 2 & 4 \\ \hline 3 & 2 & 3 & 3 & & \\ \hline \end{array}$ is an SSDT in $D_4((6, 4))$, but

$\sigma_2^6(R) = \begin{array}{|c|c|c|c|c|c|} \hline 3 & \textcircled{2} & 1 & \underline{3} & \underline{3} & 4 \\ \hline 3 & 2 & 3 & 3 & & \\ \hline \end{array}$ is not an SSDT.

3. SCHUR P -FUNCTIONS

There are many equivalent ways to represent Schur P -functions, and we introduce some of them in Proposition 3.1. We suppose that there are n variables $x_1, x_2, \dots, x_n$, and for a sequence of nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_k)$, $k \leq n$, we let x^α be the product $\prod_{i=1}^k x_i^{\alpha_i}$. We let S_n be the symmetric group on the set $\{1, 2, \dots, n\}$ and let $\pi \in S_n$ act on n variables by permuting the indices: $\pi(x_i) = x_{\pi(i)}$. The *Pfaffian* of a $2m \times 2m$ skew symmetric matrix $A = (a_{ij})$ is

$$\text{Pf}(A) = \sum_{\pi \in \bar{S}_{2m}} \varepsilon(\pi) \prod_{i=1}^m a_{\pi(2i-1) \pi(2i)},$$

where $\bar{S}_{2m} \subset S_{2m}$ is the set of permutations π such that $\pi(2i-1) < \pi(2i)$ and $\pi(1) < \pi(3) < \cdots < \pi(2m-3) < \pi(2m-1)$, and $\varepsilon(\pi)$ is the sign of the permutation π .

For a sequence of positive integers $\alpha = (\alpha_1, \dots, \alpha_k)$, $k \leq n$, let $B_\alpha(x_1, \dots, x_n)$ denote the $n \times k$ matrix $(x_i^{\alpha_{k+1-j}})$ and

$$A_\alpha(x_1, \dots, x_n) = \begin{pmatrix} A_0(x_1, \dots, x_n) & B_\alpha(x_1, \dots, x_n) \\ -B_\alpha(x_1, \dots, x_n)^t & 0 \end{pmatrix},$$

where $A_0(x_1, \dots, x_n)$ is the skew-symmetric matrix $\left(\frac{x_i - x_j}{x_i + x_j}\right)_{1 \leq i, j \leq n}$. We also define Pf_0 and Pf_α as

$$\text{Pf}_0(x_1, \dots, x_n) = \begin{cases} \text{Pf}(A_0(x_1, \dots, x_n)) & \text{if } n \text{ is even,} \\ \text{Pf}(A_0(x_1, \dots, x_n, 0)) & \text{if } n \text{ is odd} \end{cases}$$

and

$$\text{Pf}_\alpha(x_1, \dots, x_n) = \begin{cases} \text{Pf}(A_\alpha(x_1, \dots, x_n)) & \text{if } n + k \text{ is even,} \\ \text{Pf}(A_\alpha(x_1, \dots, x_n, 0)) & \text{if } n + k \text{ is odd.} \end{cases}$$

The following expressions for Schur P -functions are in [13, III 8] and [7]. We let $S_{n-\ell}$ be the subgroup of S_n consisting of permutations on the set $\{\ell+1, \ell+2, \dots, n\}$.

Proposition 3.1. *For $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_\ell > 0)$ and $n \geq \ell$, the Schur P -function P_λ can be written as follows:*

$$(3.2) \quad P_\lambda(x_1, x_2, \dots, x_n) = \frac{\text{Pf}_\lambda(x_1, \dots, x_n)}{\text{Pf}_0(x_1, \dots, x_n)}$$

$$(3.3) \quad = \frac{1}{(n-\ell)!} \sum_{\pi \in S_n} \pi \left(x^\lambda \prod_{i \leq \ell, i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \right)$$

$$(3.4) \quad = \sum_{\pi \in S_n / S_{n-\ell}} \pi \left(x^\lambda \prod_{i \leq \ell, i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \right).$$

The following is well known ([13]):

$$\text{Lemma 3.5.} \quad \text{Pf}_0(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} \frac{x_i - x_j}{x_i + x_j}.$$

Definition 3.6. By abusing notation we also let

$$(3.7) \quad P_\alpha(x_1, x_2, \dots, x_n) = \frac{1}{(n-k)!} \sum_{\pi \in S_n} \pi \left(x^\alpha \prod_{i \leq k, i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \right)$$

for any sequence of positive integers $\alpha = (\alpha_1, \dots, \alpha_k)$, $k \leq n$.

4. PRODUCT OF SCHUR P -FUNCTIONS

Since Schur P -functions form a basis of the subring they generate in the ring of all symmetric polynomials, the product of two Schur P -functions P_λ and P_μ can be expanded as a linear sum of Schur P -functions:

$$(4.1) \quad P_\lambda P_\mu = \sum_{\nu \in \mathcal{DP}} f_{\lambda\mu}^\nu P_\nu,$$

where $f_{\lambda\mu}^\nu$ are the *shifted Littlewood-Richardson coefficients* (*shifted LR-coefficients*) that are known to be nonnegative integers. A combinatorial rule for $f_{\lambda\mu}^\nu$'s in terms of marked shifted tableaux is given by Stembridge [17]. *Littlewood-Richardson coefficients* (*LR-coefficients*) for the product of Schur functions have been extensively studied, and there are several nice rules to describe them [12, 19, 2]. The plactic monoid that Serrano defined in [16] has motivated us to describe the shifted LR-numbers in terms of *words* as Stembridge did for (ordinary) LR-coefficients in terms of *words* in [19].

In this section we argue how we can decompose the product of two Schur P -functions by employing the materials from the previous sections.

From now on, we always list n variables in the following order: $x_n, x_{n-1}, \dots, x_1$. We therefore have the following.

For $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_\ell > 0)$ and $\ell \leq n$,

$$\begin{aligned} \text{Pf}_\lambda(x_n, x_{n-1}, \dots, x_1) &= (-1)^{\frac{n(n-1)}{2}} \text{Pf}_\lambda(x_1, x_2, \dots, x_n) \\ \text{Pf}_0(x_n, x_{n-1}, \dots, x_1) &= \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{x_j + x_i} = (-1)^{\frac{n(n-1)}{2}} \text{Pf}_0(x_1, x_2, \dots, x_n) \\ P_\lambda(x_n, x_{n-1}, \dots, x_1) &= \frac{\text{Pf}_\lambda(x_n, \dots, x_1)}{\text{Pf}_0(x_n, \dots, x_1)} = P_\lambda(x_1, x_2, \dots, x_n) \\ &= \frac{1}{(n-\ell)!} \sum_{\pi \in S_n} \pi \left(x_n^{\lambda_1} x_{n-1}^{\lambda_2} \cdots x_{n-\ell+1}^{\lambda_\ell} \prod_{n-\ell+1 \leq j, i < j} \frac{x_j + x_i}{x_j - x_i} \right) \\ &= \sum_{\pi \in S_n / S_{n-\ell}} \pi \left(x_n^{\lambda_1} x_{n-1}^{\lambda_2} \cdots x_{n-\ell+1}^{\lambda_\ell} \prod_{n-\ell+1 \leq j, i < j} \frac{x_j + x_i}{x_j - x_i} \right), \end{aligned}$$

where $S_{n-\ell}$ is the symmetric group on $\{1, 2, \dots, n-\ell\}$ this time.

Remember that we also defined P_α for a sequence of positive integers $\alpha = (\alpha_1, \dots, \alpha_k)$, $k \leq n$:

$$(4.2) \quad \begin{aligned} &P_\alpha(x_n, x_{n-1}, \dots, x_1) \\ &= \frac{1}{(n-k)!} \sum_{\pi \in S_n} \pi \left(x_n^{\alpha_1} x_{n-1}^{\alpha_2} \cdots x_{n-k+1}^{\alpha_k} \prod_{n-k+1 \leq j, i < j} \frac{x_j + x_i}{x_j - x_i} \right). \end{aligned}$$

For convenience, we use an abbreviation:

$$D_n = \text{Pf}_0(x_n, x_{n-1}, \dots, x_1).$$

Lemma 4.3. For a sequence of positive integers $\alpha = (\alpha_1, \dots, \alpha_k)$, $k \leq n$,

$$(4.4) \quad \begin{aligned} &D_n \cdot P_\alpha(x_n, x_{n-1}, \dots, x_1) \\ &= \frac{1}{(n-k)!} \sum_{\pi \in S_n} \varepsilon(\pi) \pi \left(x_n^{\alpha_1} x_{n-1}^{\alpha_2} \cdots x_{n-k+1}^{\alpha_k} \prod_{1 \leq i < j \leq n-k} \frac{x_j - x_i}{x_j + x_i} \right). \end{aligned}$$

Proof. By Lemma 3.5,

$$\text{Pf}_0(x_n, x_{n-1}, \dots, x_1) = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{x_j + x_i} = \varepsilon(\pi) \prod_{1 \leq i < j \leq n} \frac{x_{\pi(j)} - x_{\pi(i)}}{x_{\pi(j)} + x_{\pi(i)}}$$

holds for all $\pi \in S_n$. Therefore, we can obtain the desired result if we use (4.2) to calculate $D_n \cdot P_\alpha(x_n, x_{n-1}, \dots, x_1)$. $\square$

The following is immediate from the previous lemma.

Corollary 4.5. If α' is a sequence of positive integers obtained by exchanging two consecutive numbers α_i and α_{i+1} from $\alpha = (\alpha_1, \dots, \alpha_k)$, $k \leq n$, then

$$D_n \cdot P_{\alpha'}(x_n, \dots, x_1) = -D_n \cdot P_\alpha(x_n, \dots, x_1).$$

Hence $D_n \cdot P_\alpha(x_n, x_{n-1}, \dots, x_1) = 0$ if α has two equal parts.

From now on we fix two strict partitions λ and μ with lengths ℓ and l , respectively, and the number of variables n such that $\ell, l \leq n$.

Lemma 4.6. For $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$, $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l > 0)$ and $\ell, l \leq n$,

$$D_n \cdot P_\lambda(x_n, \dots, x_1) P_\mu(x_n, \dots, x_1) \\ = \sum_{T \in \mathcal{Y}_n(\mu)} \left(\frac{1}{(n-\ell)!} \sum_{\pi \in S_n} \varepsilon(\pi) \pi \left(x_n^{\lambda_1} \cdots x_{n-\ell+1}^{\lambda_\ell} x^{\omega(T)} \prod_{1 \leq i < j \leq n-\ell} \frac{x_j - x_i}{x_j + x_i} \right) \right).$$

Proof.

$$D_n \cdot P_\lambda(x_n, \dots, x_1) P_\mu(x_n, \dots, x_1) \\ = \left(\frac{1}{(n-\ell)!} \sum_{\pi \in S_n} \varepsilon(\pi) \pi \left(x_n^{\lambda_1} \cdots x_{n-\ell+1}^{\lambda_\ell} \prod_{1 \leq i < j \leq n-\ell} \frac{x_j - x_i}{x_j + x_i} \right) \right) \\ \times \left(\sum_{T \in \mathcal{Y}_n(\mu)} x^{\omega(T)} \right) \\ = \sum_{T \in \mathcal{Y}_n(\mu)} \left(\frac{1}{(n-\ell)!} \sum_{\pi \in S_n} \varepsilon(\pi) \pi \left(x_n^{\lambda_1} \cdots x_{n-\ell+1}^{\lambda_\ell} x^{\omega(T)} \prod_{1 \leq i < j \leq n-\ell} \frac{x_j - x_i}{x_j + x_i} \right) \right).$$

The first identity holds because of Lemma 4.3 and equation (1.9), while the second identity holds since $P_\mu(x_n, \dots, x_1)$ is invariant under the action of S_n . $\square$

Definition 4.7 (Notation). Let $i \leq j$ be integers in $\{1, 2, \dots, n\}$. For a marked shifted tableau $T \in \mathcal{Y}_n(\mu)$, $T|_i^j$ is the (skew) subtableau of T with entries in $\{i' < i < \cdots < j' < j\}$ and $sh(T|_i^j)$ is the shape of $T|_i^j$. For an SSDT $R \in \mathcal{D}_n(\mu)$, $R|_i^j$ is the *subword* of $read(R)$ with letters from the alphabet $\{i, i+1, \dots, j\}$, and we use $sh(R|_i^j)$ for the shape of the mixed insertion tableau $P_{mix}(R|_i^j)$.

For $\rho \leq \mu \in \mathcal{DP}$, we let $\mathcal{Y}_i^j(\mu/\rho) \subseteq \mathcal{Y}_n(\mu/\rho)$ be the set of skew marked shifted tableaux of shape μ/ρ with entries only in $\{i' < i < \cdots < j' < j\}$.

Remark 4.8. With the notation introduced in Definition 4.7, we make the following observations:

- (1) For any $\rho \leq \mu \in \mathcal{DP}$, $\mathcal{Y}_n(\rho) = \mathcal{Y}_1^n(\rho)$ and $\mathcal{Y}_n(\mu/\rho) = \mathcal{Y}_1^n(\mu/\rho)$.
- (2) For $T \in \mathcal{Y}_n(\mu)$ and $j \leq n$, $T|_1^j \in \mathcal{Y}_n(\rho)$ for some $\rho \leq \mu \in \mathcal{DP}$.

The following is a simple yet very important observation for our work.

Lemma 4.9. For $T \in \mathcal{Y}_n(\mu)$ let $R = Y(T)$ be the corresponding SSDT in $\mathcal{D}_n(\mu)$ so that $T = P_{mix}(read(R))$. Then $T|_1^{n-\ell} = P_{mix}(R|_1^{n-\ell})$.

Proof. For a marked shifted tableau U of shape $\rho \leq \mu$, if $x > n - \ell$, then $(U \leftarrow x)|_1^{n-\ell} = U|_1^{n-\ell}$, and if $x \leq n - \ell$, then $(U \leftarrow x)|_1^{n-\ell} = (U|_1^{n-\ell} \leftarrow x)$. Since $T = P_{mix}(R|_1^n)$, by the definition of mixed insertion (Definition 1.4), we can see that $T|_1^{n-\ell} = P_{mix}(R|_1^{n-\ell})$ must hold. $\square$

Remark 4.10. Lemma 4.9 is not true if 1 is replaced by $i \neq 1$.

To continue manipulation of $D_n \cdot P_\lambda(x_n, \dots, x_1)P_\mu(x_n, \dots, x_1)$ after Lemma 4.6, we examine the marked shifted tableaux of shape μ in more detail:

Observe that $\mathcal{Y}_n(\mu)$ is in bijection with the set of triples (ρ, T_1, T_2) such that $\rho \leq \mu$, $T_1 \in \mathcal{Y}_1^{n-\ell}(\rho)$ and $T_2 \in \mathcal{Y}_{n-\ell+1}^n(\mu/\rho)$. This is because every $T \in \mathcal{Y}_n(\mu)$ can be divided into two parts (T_1, T_2) according to the quantity of the entries, and any triple (ρ, T_1, T_2) makes a tableau in $\mathcal{Y}_n(\mu)$.

In this correspondence, if $R = D(T) \in \mathcal{D}_n(\mu)$ is the corresponding SSDT of T , then

- (1) $\rho = sh(T_1) = sh(P_{mix}(R|_1^{n-\ell}))$, by Lemma 4.9,
- (2) $\omega(T_1) = \omega(R|_1^{n-\ell})$ and $\omega(T_2) = \omega(R|_{n-\ell+1}^n)$.

We therefore can write

$$(4.11) \quad \sum_{T \in \mathcal{Y}_n(\mu)} x^{\omega(T)} = \sum_{\rho \leq \mu} \sum_{(T_1, T_2)} x^{\omega(T_1) + \omega(T_2)} = \sum_{\rho \leq \mu} \sum_{sh(R|_1^{n-\ell}) = \rho} x^{\omega(R|_1^{n-\ell})} x^{\omega(R|_{n-\ell+1}^n)},$$

where (T_1, T_2) varies in $\mathcal{Y}_1^{n-\ell}(\rho) \times \mathcal{Y}_{n-\ell+1}^n(\mu/\rho)$ and R in $\mathcal{D}_n(\mu)$.

For a fixed strict partition $\rho \leq \mu$, T_1 and T_2 are independent of each other and the set of corresponding T_2 's is exactly $\mathcal{Y}_{n-\ell+1}^n(\mu/\rho)$. Hence we have proved the following.

Lemma 4.12. *For a fixed $\rho \leq \mu$ and a given $T_1 \in \mathcal{Y}_1^{n-\ell}(\rho)$,*

$$|\{R \in \mathcal{D}_n(\mu) : P_{mix}(R|_1^{n-\ell}) = T_1\}| = |\mathcal{Y}_{n-\ell+1}^n(\mu/\rho)|.$$

Moreover,

$$(4.13) \quad \sum_{T_2 \in \mathcal{Y}_{n-\ell+1}^n(\mu/\rho)} x^{\omega(T_2)} = \sum_{\substack{R \in \mathcal{D}_n(\mu) \\ P_{mix}(R|_1^{n-\ell}) = T_1}} x^{\omega(R|_{n-\ell+1}^n)}.$$

Definition 4.14. An SSDT $R \in \mathcal{D}_n(\mu)$ is ℓ -essential if $\omega(R|_1^{n-\ell}) = (\rho_{n-\ell}, \dots, \rho_1)$, where $\rho = sh(R|_1^{n-\ell})$.

Example 4.15. When $n = 3$, $R = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline & 2 & \\ \hline \end{array}$ is 1-essential:

$$(\rho_1, \rho_2) = sh(R|_1^2) = (2, 1) \text{ since } P_{mix}(R|_1^2) = P_{mix}(2 \ 2 \ 1) = \begin{array}{|c|c|} \hline 1 & 2' \\ \hline & 2 \\ \hline \end{array}, \text{ and } \omega(R|_1^2) = (1, 2) = (\rho_2, \rho_1).$$

The following is clear since there is a unique SSDT, the one with lowest weight, with weight $(\rho_{n-\ell}, \dots, \rho_1)$ in $\mathcal{D}_1^{n-\ell}(\rho)$.

Lemma 4.16. *$R \in \mathcal{D}_n(\mu)$ is ℓ -essential if and only if $P_{mix}(R|_1^{n-\ell}) = U_\rho(n - \ell)$ when $\rho = sh(R|_1^{n-\ell})$.*

Corollary 4.17. *If $n = \ell + 1$, then every SSDT is ℓ -essential.*

Proof. Since $n = \ell + 1$, $n - \ell = 1$ and $R|_1^{n-\ell}$ is a word of all 1's. Therefore $P_{mix}(R|_1^1)$ is the marked shifted tableau with 1's on the first row only. $\square$

We now can write P_μ using ℓ -essential SSDT's:

Theorem 4.18.

$$(4.19) \quad \sum_{T \in \mathcal{Y}_n(\mu)} x^{\omega(T)} = \sum_{\substack{R \in \mathcal{D}_n(\mu) \\ R \text{ is } \ell\text{-essential}}} \left(P_{sh(R|_1^{n-\ell})}(x_{n-\ell}, \dots, x_1) x^{\omega(R|_{n-\ell+1}^n)} \right).$$

Proof. Define an equivalence relation $\sim$ on $\mathcal{D}_n(\mu)$ as follows:

$$R \sim R' \text{ if and only if } P_{mix}(R|_1^{n-\ell}) = P_{mix}(R'|_1^{n-\ell}),$$

and let $[R]$ be the equivalence class that contains R .

Then for a fixed $\rho \leq \mu$,

$$(4.20) \quad \sum_{(T_1, T_2)} x^{\omega(T_1) + \omega(T_2)} = \sum_{\substack{[R] \in (\mathcal{D}_n(\mu)/\sim) \\ sh(R|_1^{n-\ell}) = \rho}} x^{\omega(R|_1^{n-\ell})} \sum_{R' \in [R]} x^{\omega(R'|_{n-\ell+1}^n)},$$

where (T_1, T_2) varies in $\mathcal{Y}_1^{n-\ell}(\rho) \times \mathcal{Y}_{n-\ell+1}^n(\mu/\rho)$.

For any class $[R] \in \mathcal{D}_n(\mu)/\sim$, because of equation (4.13), we have

$$\sum_{R' \in [R]} x^{\omega(R'|_{n-\ell+1}^n)} = \sum_{T_2 \in \mathcal{Y}_{n-\ell+1}^n(\mu/\rho)} x^{\omega(T_2)}.$$

Therefore, we have

$$\begin{aligned} \sum_{T \in \mathcal{Y}_n(\mu)} x^{\omega(T)} &= \sum_{\rho \leq \mu} \sum_{(T_1, T_2)} x^{\omega(T_1) + \omega(T_2)} \\ &= \sum_{\rho \leq \mu} \left(\sum_{\substack{[R] \in (\mathcal{D}_n(\mu)/\sim) \\ sh(R|_1^{n-\ell}) = \rho}} x^{\omega(R|_1^{n-\ell})} \right) \left(\sum_{T_2 \in \mathcal{Y}_{n-\ell+1}^n(\mu/\rho)} x^{\omega(T_2)} \right) \\ &= \sum_{\rho \leq \mu} P_\rho(x_{n-\ell}, \dots, x_1) \sum_{T_2 \in \mathcal{Y}_{n-\ell+1}^n(\mu/\rho)} x^{\omega(T_2)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sum_{\substack{R \in \mathcal{D}_n(\mu) \\ R \text{ is } \ell\text{-essential}}} \left(P_{sh(R|_1^{n-\ell})}(x_{n-\ell}, \dots, x_1) x^{\omega(R|_{n-\ell+1}^n)} \right) \\ &= \sum_{\rho \leq \mu} \sum_{\substack{R \text{ is } \ell\text{-essential} \\ sh(R|_1^{n-\ell}) = \rho}} P_\rho(x_{n-\ell}, \dots, x_1) x^{\omega(R|_{n-\ell+1}^n)} \\ &= \sum_{\rho \leq \mu} P_\rho(x_{n-\ell}, \dots, x_1) \sum_{P_{mix}(R|_1^{n-\ell}) = U_\rho} x^{\omega(R|_{n-\ell+1}^n)}. \end{aligned}$$

Again, by equation (4.13), $\sum_{P_{mix}(R|_1^{n-\ell}) = U_\rho} x^{\omega(R|_{n-\ell+1}^n)} = \sum_{T_2 \in \mathcal{Y}_{n-\ell+1}^n(\mu/\rho)} x^{\omega(T_2)}$, and we have shown that two sides of the given equation are equal. $\square$

Definition 4.21. We let $r(\lambda) = (0, \dots, 0, \lambda_\ell, \dots, \lambda_2, \lambda_1) \in \mathbb{Z}^n$ be the *reverse* of λ . Then $x^{r(\lambda)} = x_n^{\lambda_1} \dots x_{n-\ell+1}^{\lambda_\ell}$.

We consider Lemma 4.6 and equation (4.19) together:

$$\begin{aligned}
 & D_n \cdot P_\lambda(x_n, \dots, x_1) P_\mu(x_n, \dots, x_1) \times (n - \ell)! \\
 &= \sum_{T \in \mathcal{Y}_n(\mu)} \sum_{\pi \in S_n} \varepsilon(\pi) \pi \left(x^{r(\lambda)} x^{\omega(T)} \prod_{1 \leq i < j \leq n-\ell} \frac{x_j - x_i}{x_j + x_i} \right) \\
 &= \sum_{\substack{R \in \mathcal{D}_n(\mu) \\ R \text{ is } \ell\text{-essential}}} \sum_{\pi \in S_n} \varepsilon(\pi) \pi \left(x^{r(\lambda) + \omega(R|_{n-\ell+1}^n)} P_{sh(R|_1^{n-\ell})}(x_{n-\ell}, \dots, x_1) \right. \\
 &\quad \left. \times \prod_{1 \leq i < j \leq n-\ell} \frac{x_j - x_i}{x_j + x_i} \right)
 \end{aligned}$$

Lemma 4.22. For a sequence of positive integers $\alpha = (\alpha_1, \dots, \alpha_k)$, $k \leq n - \ell$,

$$\begin{aligned}
 & P_\alpha(x_{n-\ell}, \dots, x_1) \prod_{1 \leq i < j \leq n-\ell} \frac{x_j - x_i}{x_j + x_i} \\
 &= \frac{1}{(n - \ell - k)!} \sum_{\pi \in S_{n-\ell}} \varepsilon(\pi) \pi \left(x_{n-\ell}^{\alpha_1} x_{n-\ell+1}^{\alpha_2} \cdots x_{n-\ell-k+1}^{\alpha_k} \prod_{1 \leq i < j \leq n-\ell-k} \frac{x_j - x_i}{x_j + x_i} \right).
 \end{aligned}$$

Proof. For any $\pi \in S_{n-\ell}$, $\prod_{1 \leq i < j \leq n-\ell} \frac{x_j - x_i}{x_j + x_i} = \varepsilon(\pi) \pi \left(\prod_{1 \leq i < j \leq n-\ell} \frac{x_j - x_i}{x_j + x_i} \right)$. Therefore, the proof is completed if we use equation (4.2) to write P_α . $\square$

By Lemma 4.22 we can carry out further calculation as follows:

$$\begin{aligned}
 & D_n \cdot P_\lambda(x_n, \dots, x_1) P_\mu(x_n, \dots, x_1) \times (n - \ell)! \\
 &= \sum_{\substack{R \in \mathcal{D}_n(\mu) \\ R \text{ is } \ell\text{-essential}}} \sum_{\pi \in S_n} \varepsilon(\pi) \pi \left(x^{r(\lambda) + \omega(R|_{n-\ell+1}^n)} P_{sh(R|_1^{n-\ell})}(x_{n-\ell}, \dots, x_1) \right. \\
 &\quad \left. \times \prod_{1 \leq i < j \leq n-\ell} \frac{x_j - x_i}{x_j + x_i} \right) \\
 &= \sum_R \sum_{\pi \in S_n} \frac{\varepsilon(\pi)}{(n - \ell - k_R)!} \pi \left(x^{r(\lambda) + \omega(R|_{n-\ell+1}^n)} \sum_{\tau \in S_{n-\ell}} \varepsilon(\tau) \tau \left(x_{n-\ell}^{\alpha_1^R} \cdots x_{n-\ell-k_R+1}^{\alpha_{k_R}^R} \right. \right. \\
 &\quad \left. \left. \times \prod_{i < j \leq n-\ell-k_R} \frac{x_j - x_i}{x_j + x_i} \right) \right) \\
 &= \sum_R \sum_{\pi \in S_n} \sum_{\tau \in S_{n-\ell}} \frac{\varepsilon(\pi\tau)}{(n - \ell - k_R)!} \pi\tau \left(x^{r(\lambda) + \omega(R|_{n-\ell+1}^n)} x_{n-\ell}^{\alpha_1^R} \cdots x_{n-\ell-k_R+1}^{\alpha_{k_R}^R} \right. \\
 &\quad \left. \times \prod_{1 \leq i < j \leq n-\ell-k_R} \frac{x_j - x_i}{x_j + x_i} \right)
 \end{aligned}$$

$$= \sum_R (n-\ell)! \sum_{\pi \in S_n} \frac{\varepsilon(\pi)}{(n-\ell-k_R)!} \pi \left(x^{r(\lambda)+\omega(R|_{n-\ell+1}^n)} x_{n-\ell}^{\alpha_1^R} \cdots x_{n-\ell-k_R+1}^{\alpha_{k_R}^R} \right. \\ \left. \times \prod_{1 \leq i < j \leq n-\ell-k_R} \frac{x_j - x_i}{x_j + x_i} \right),$$

where $\alpha^R = (\alpha_1^R, \dots, \alpha_{k_R}^R) = sh(R|_1^{n-\ell})$ for each ℓ -essential R . Note that $x^{r(\lambda)+\omega(R|_{n-\ell+1}^n)}$ is a monomial of $x_n, x_{n-1}, \dots, x_{n-\ell+1}$ for each ℓ -essential R and $\tau(x^{r(\lambda)+\omega(R|_{n-\ell+1}^n)}) = x^{r(\lambda)+\omega(R|_{n-\ell+1}^n)}$ for $\tau \in S_{n-\ell}$. One has to be aware that α_i^R is the number of $(n-\ell-i+1)$'s in $R|_{n-\ell+1}^n$ for all $i = 1, 2, \dots, k_R$, by the definition of ℓ -essentiality.

Finally, we obtain the following equation:

Theorem 4.23. For $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$, $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l > 0)$ and $\ell, l \leq n$,

(4.24)

$$D_n \cdot P_\lambda(x_n, \dots, x_1) P_\mu(x_n, \dots, x_1) = \sum_{\substack{R \in \mathcal{D}_n(\mu) \\ R \text{ is } \ell\text{-essential}}} D_n \cdot P_{\lambda+r(\omega(R))}(x_n, \dots, x_1),$$

where $r(\omega(R)) = (0, \dots, 0, \omega_{\ell+k_R}(R), \dots, \omega_1(R)) \in \mathbb{Z}^n$ for

$$\omega(R) = (\omega_1(R), \dots, \omega_{\ell+k_R}(R)).$$

Example 4.25. If $\lambda = (2)$, $\mu = (3, 1)$, $n = 3$, then $\ell = 1$ and there are 24 SSDT's in $\mathcal{D}_3(\mu)$ (equivalently, there are 24 marked shifted tableaux in $\mathcal{Y}_3(\mu)$). There are, however, only 7 ℓ -essential SSDT's:

SSDT R	<table><tr><td>2</td><td>1</td><td>2</td></tr><tr><td></td><td>2</td><td></td></tr></table>	2	1	2		2		<table><tr><td>3</td><td>2</td><td>2</td></tr><tr><td></td><td>2</td><td></td></tr></table>	3	2	2		2		<table><tr><td>3</td><td>2</td><td>1</td></tr><tr><td></td><td>2</td><td></td></tr></table>	3	2	1		2		<table><tr><td>2</td><td>1</td><td>3</td></tr><tr><td></td><td>2</td><td></td></tr></table>	2	1	3		2		<table><tr><td>3</td><td>2</td><td>3</td></tr><tr><td></td><td>2</td><td></td></tr></table>	3	2	3		2		<table><tr><td>3</td><td>2</td><td>2</td></tr><tr><td></td><td>3</td><td></td></tr></table>	3	2	2		3		<table><tr><td>3</td><td>2</td><td>3</td></tr><tr><td></td><td>3</td><td></td></tr></table>	3	2	3		3	
	2	1	2																																														
	2																																																
3	2	2																																															
	2																																																
3	2	1																																															
	2																																																
2	1	3																																															
	2																																																
3	2	3																																															
	2																																																
3	2	2																																															
	3																																																
3	2	3																																															
	3																																																
$r(\omega(R))$	(0, 3, 1)	(1, 3, 0)	(1, 2, 1)	(1, 2, 1)	(2, 2, 0)	(2, 2, 0)	(3, 1, 0)																																										
$\lambda +$	(2, 3, 1)	(3, 3, 0)	(3, 2, 1)	(3, 2, 1)	(4, 2, 0)	(4, 2, 0)	(5, 1, 0)																																										
$r(\omega(R))$																																																	

For example, $R = \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline & 2 & \\ \hline \end{array}$ is not ℓ -essential since $P_{mix}(R|_1^2) = P_{mix}(212) =$

$\begin{array}{|c|c|c|} \hline 1 & 2' & 2 \\ \hline \end{array} \in \mathcal{Y}_2((3, 0))$ is not the lowest weight tableau in $\mathcal{Y}_2((3, 0))$. $R = \begin{array}{|c|c|c|} \hline 3 & 1 & 1 \\ \hline & 1 & \\ \hline \end{array}$

is not ℓ -essential since $P_{mix}(R|_1^2) = P_{mix}(111) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array} \in \mathcal{Y}_2((3, 0))$ is not the lowest weight tableau $\begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline \end{array}$ in $\mathcal{Y}_2((3, 0))$.

By Theorem 4.23, we have

$$\begin{aligned} & D_3 \cdot P_{(2,0,0)}(x_3, x_2, x_1) P_{(3,1,0)}(x_3, x_2, x_1) \\ &= D_3 \cdot P_{(2,3,1)}(x_3, x_2, x_1) + D_3 P_{(3,3,0)}(x_3, x_2, x_1) + D_3 \cdot P_{(3,2,1)}(x_3, x_2, x_1) \\ &\quad + D_3 \cdot P_{(3,2,1)}(x_3, x_2, x_1) + D_3 P_{(4,2,0)}(x_3, x_2, x_1) \\ &\quad + D_3 \cdot P_{(4,2,0)}(x_3, x_2, x_1) + D_3 \cdot P_{(5,1,0)}(x_3, x_2, x_1). \end{aligned}$$

By Corollary 4.5, we have two relations,

$$D_3 \cdot P_{(2,3,1)}(x_3, x_2, x_1) = -D_3 \cdot P_{(3,2,1)}(x_3, x_2, x_1)$$

and

$$D_3 \cdot P_{(3,3,0)}(x_3, x_2, x_1) = 0.$$

We therefore obtain

$$\begin{aligned} D_3 \cdot P_{(2,0,0)}(x_3, x_2, x_1) P_{(3,1,0)}(x_3, x_2, x_1) \\ = D_3 \cdot P_{(3,2,1)}(x_3, x_2, x_1) + 2D_3 \cdot P_{(4,2,0)}(x_3, x_2, x_1) + D_3 \cdot P_{(5,1,0)}(x_3, x_2, x_1). \end{aligned}$$

In Section 5, we provide a combinatorial rule to explain which $(D_n \cdot P_{\lambda+r(\omega(R))})$'s survive in the expansion of $D_n \cdot P_\lambda P_\mu$.

5. COMBINATORICS ON ESSENTIAL TABLEAUX

In Theorem 4.23, we expanded the product $(D_n \cdot P_\lambda)P_\mu$ as a sum of $(D_n \cdot P_\alpha)$'s for sequences of positive integers α . In this section we find a way to write $(D_n \cdot P_\lambda)P_\mu$ as a sum of $(D_n \cdot P_\nu)$'s for $\nu \in \mathcal{DP}$. We find a corresponding pair of each $D_n \cdot P_\alpha$ for $\alpha \notin \mathcal{DP}$ so that they cancel each other. Remember that we have n variables and $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$, $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l > 0)$, $\ell, l \leq n$, are two strict partitions in $\mathcal{DP}$.

For a word $u = u_1 u_2 \dots u_m$ on the alphabet $\{1, 2, \dots, n\}$, the *weight* $\omega(u) = (\omega_1(u), \dots, \omega_n(u))$ of u is an element in $\mathbb{Z}^n$ whose i th element $\omega_i(u)$ is the number of i 's in u . We also use $r(\omega(u))$ to denote the *reverse weight* of u : $r(\omega(u)) = (\omega_n(u), \dots, \omega_1(u))$. Moreover, for $\alpha = (\alpha_1, \dots, \alpha_k)$ with $k \leq n$, we let $r(\alpha) = (\alpha_n, \alpha_{n-1}, \dots, \alpha_2, \alpha_1) \in \mathbb{Z}^n$, where α_i is set to be 0 for $i > k$. For a word $w = w_1 w_2 \dots w_m$ we define $\sigma_k^i(w)$ to be the word obtained by applying σ_k only to the subword $w_{i+1} \dots w_n$, as we defined σ_k^i on SSDT's in Theorem 2.9.

Definition 5.1. We say that a word u is λ -bad if there is i such that $\lambda + r(\omega(u_1 \dots u_i)) \notin \mathcal{DP}$.

Definition 5.2. An SSDT $R \in \mathcal{D}_n(\mu)$ is λ -bad if $\text{read}(R)$ is λ -bad. An SSDT $R \in \mathcal{D}_n(\mu)$ is λ -good if it is not λ -bad.

Example 5.3. Among 7 ℓ -essential SSDT's given in Example 4.25, where $\lambda = (2, 0, 0)$,

$$R_1 = \begin{array}{|c|c|c|} \hline 2 & 1 & 2 \\ \hline & 2 & \\ \hline \end{array}, R_2 = \begin{array}{|c|c|c|} \hline 3 & 2 & 2 \\ \hline & 2 & \\ \hline \end{array} \quad \text{and} \quad R_3 = \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline & 2 & \\ \hline \end{array} \quad \text{are } \lambda\text{-bad:}$$

$$\text{read}(R_1) = 2212, r(\omega(22)) = (0, 2, 0) \text{ and } (2, 0, 0) + (0, 2, 0) \notin \mathcal{DP},$$

$$\text{read}(R_2) = 2322, r(\omega(2322)) = (1, 3, 0) \text{ and } (2, 0, 0) + (1, 3, 0) \notin \mathcal{DP},$$

$$\text{read}(R_3) = 2213, r(\omega(22)) = (0, 2, 0) \text{ and } (2, 0, 0) + (0, 2, 0) \notin \mathcal{DP}.$$

We will show that only “ λ -good” ℓ -essential SSDT's survive in the expansion of $(D_n \cdot P_\lambda)P_\mu$, and Example 4.25 serves as a small evidence of this.

Let $\mathcal{B}(\mu)$ be the set of λ -bad ℓ -essential tableaux in $\mathcal{D}_n(\mu)$. For the proof of our assertion, we define an (sign reversing) involution on $\mathcal{B}(\mu)$ and we need some careful observations on ℓ -essential SSDT's first. Remember that $Y : \mathcal{D}_n(\rho) \rightarrow \mathcal{Y}_n(\rho)$ is a weight preserving bijection between the set of SSDT's and the set of marked shifted tableaux, defined as $Y(R) = P_{\text{mix}}(\text{read}(R))$.

Lemma 5.4. *For a strict partition $\rho = (\rho_1, \dots, \rho_L) \in \mathcal{DP}_n$, let $Y(U_\rho(n))$ be the lowest weight marked shifted tableau in $\mathcal{Y}_n(\rho)$. Then, for each $i = n, n-1, \dots, n-L+1$, the subtableau of $Y(U_\rho)$ consisting of the entries on $\{1' < 1 < \dots < i' < i\}$ is of shape $(\rho_{n+1-i}, \dots, \rho_L)$. Hence the subtableau of $Y(U_\rho(n))$ consisting of (i') 's and i 's is of shape $(\rho_{n+1-i}, \dots, \rho_L)/(\rho_{n+2-i}, \dots, \rho_L)$.*

Proof. For each $i = n, n-1, \dots, n-L+1$, it is easy to see that $\mathcal{S}((\rho_{n+1-i}, \dots, \rho_L)/(\rho_{n+2-i}, \dots, \rho_L))$ is a connected border strip with ρ_{n+1-i} boxes, that is, it is (border) connected and does not contain a 2×2 subdiagram. Moreover, they form a (set) partition of $\mathcal{S}(\rho)$. Therefore, there is a unique way to fill in the boxes in $\mathcal{S}((\rho_{n+1-i}, \dots, \rho_L)/(\rho_{n+2-i}, \dots, \rho_L))$ with (i') 's and i 's. It is obvious that the collection of all fillings of each border strip makes a tableau in $\mathcal{Y}_n(\rho)$ of weight $r(\rho)$, which must be $Y(U_\rho(n))$. $\square$

Corollary 5.5. *For a strict partition $\rho = (\rho_1, \dots, \rho_L) \in \mathcal{DP}_n$, the number of (i') 's and i 's in the first row of $Y(U_\rho(n))$ is $(\rho_{n+1-i} - \rho_{n+2-i})$ for each $i = n, n-1, \dots, n-L+1$. Therefore the subtableau obtained by deleting the first row from $Y(U_\rho(n))$ is the lowest weight marked shifted tableau $Y(U_{\rho'}(n))$ where $\rho' = (\rho_2, \dots, \rho_L)$.*

Proof. It is clear that we still have a marked shifted tableau U' when we delete the first row of $Y(U_\rho(n))$ and it is of shape ρ' . We count the number of (i') 's and i 's in U' for $i = n, n-1, \dots, n-L+2$. For each $i = n, n-1, \dots, n-L+2$, the number of i 's in the first row of $Y(U_\rho(n))$ is $(\rho_{n+1-i} - \rho_{n+2-i})$ by Lemma 5.4. Hence the number of (i') 's and i 's in U' is $\rho_{n+1-i} - (\rho_{n+1-i} - \rho_{n+2-i}) = \rho_{n+2-i}$. This shows that $U' = Y(U_{\rho'}(n))$. $\square$

Definition 5.6. A word $w = w_1 w_2 \dots w_m$ is a *strict lattice word* on $\{n_1, n_1 + 1, \dots, n_2\}$, $n_1 \leq n_2$, if for any $1 \leq i \leq m$, if $w_i \neq n_2$, the number of $(w_i + 1)$'s is strictly greater than the number of w_i 's in the subword $w_1 \dots w_i$.

Lemma 5.7. *If $w = w_1 w_2 \dots w_m$ is a strict lattice word on $\{1, 2, \dots, n_2\}$, then $P_{mix}(w) = Y(U_\rho(n_2))$ when $\rho = sh(P_{mix}(w))$.*

Proof. We use an induction on the length m of the given strict lattice word. If $m = 1$ or $m = 2$, then w must be a word on n_2 only, and the assertion is true. Suppose that the assertion is true for $m-1 \geq 2$ and consider a strict lattice word $w = w_1 w_2 \dots w_m$. Then since $w' = w_1 w_2 \dots w_{m-1}$ is a strict lattice word on $\{1, 2, \dots, n_2\}$, by the induction hypothesis, $P_{mix}(w') = U_{\rho'}(n_2)$ when $\rho' = sh(P_{mix}(w'))$. We now consider $P_{mix}(w) = (P_{mix}(w') \leftarrow w_m)$ of shape ρ which is obtained by adding one box to a row of ρ' . Let $w_m = a$. Then the difference between the number of $(a+1)$'s and the number of a 's in w' must be at least 2 unless $a \neq n_2$. If $a = n_2$, then $\rho = (\rho'_1 + 1, \rho'_2, \dots, \rho'_{n_2})$ and the reverse weight of w is exactly ρ . Hence we assume that $a < n_2$. Since the number of $((a+1)')$'s and $(a+1)$'s on the first row of $Y(U_{\rho'})$ is the difference between the number of $(a+1)$'s and the number of a 's in w' , the number of $(a+1)'$'s and $(a+1)$'s on the first row of $Y(U_{\rho'})$ must be at least two and at least one of them must be unprimed. Hence, by inserting $w_m = a$ into $P_{mix}(w')$, a will bump the first $(a+1)'$ to the next column, bumping the second $(a+1)'$ to the next column, and so on until $(a+1)'$ bump $a+1$ out to the second row. By Corollary 5.5 the subtableau obtained by deleting the first row of $P_{mix}(w')$ is the lowest weight tableau of shape $(\rho'_2, \dots, \rho'_{n_2})$ on $2, 3, \dots, n_2$. Again, if $a+1 = n_2$, that is, $a = n_2 - 1$, then $a+1$ will be placed

at the end of the first row of the subtableau, and if $a + 1 \neq n_2$, then there must be at least one $a + 2$ on the first row of the subtableau and $a + 2$ will be bumped out to the next row. By continuing this process, we can see that if $a + r = n_2$, then $a + r$ will be placed at the end of the $(r + 1)$ st row of $P_{mix}(w')$, which shows $\rho_{r+1} = \rho'_{r+1} + 1$. Note that $r + 1 = n_2 + 1 - a$. Hence $sh(P_{mix}(w)) = \rho$, which is the reverse of the weight of the word w . $\square$

Lemma 5.8. *For a strict partition $\rho \in \mathcal{DP}_{n-\ell}$, there is a one-to-one correspondence between the set of standard tableaux of shape ρ and the set of strict lattice words on $\{1, \dots, n - \ell\}$ of weight $(\rho_{n-\ell}, \dots, \rho_2, \rho_1)$.*

Proof. For a given strict lattice word on $\{1, \dots, n - \ell\}$ of weight $(\rho_{n-\ell}, \dots, \rho_2, \rho_1)$, define the corresponding standard tableau Q as follows: $w_i = a$ is the j th a in w if and only if the j th entry of the $(n - \ell - a + 1)$ st row of Q is i . Then it is clear that this defines a one-to-one correspondence. $\square$

The following shows another equivalent condition for an SSDT to be ℓ -essential.

Proposition 5.9. *$R \in \mathcal{D}_n(\mu)$ is ℓ -essential if and only if $R|_1^{n-\ell}$ is a strict lattice word on the set $\{1, 2, \dots, n - \ell\}$.*

Proof. First note that by Lemma 4.16 $R \in \mathcal{D}(\mu)$ is ℓ -essential if and only if $P_{mix}(R|_1^{n-\ell}) = U_\rho$ when $\rho = sh(R|_1^{n-\ell})$. Hence Lemma 5.7 shows that if $R|_1^{n-\ell}$ is a strict lattice word on $\{1, \dots, n - \ell\}$, then R is ℓ -essential. Suppose that R is ℓ -essential, or equivalently $P_{mix}(R|_1^{n-\ell}) = U_\rho(n - \ell)$ when $\rho = sh(R|_1^{n-\ell})$. By Proposition 1.5 and Lemma 5.8 we can conclude that only strict lattice words can satisfy the property $P_{mix}(w) = U_\rho(n - \ell)$ when $\rho = sh(P_{mix}(w))$. Therefore $R|_1^{n-\ell}$ must be a strict lattice word. $\square$

The following is immediate from the previous lemma. Remember that λ has ℓ positive parts.

Corollary 5.10. *For $R \in \mathcal{B}(\mu)$, let $u = u_1 u_2 \cdots u_m = read(R)$ and i_0 be the first i such that $\lambda + r(\omega(u_1 \cdots u_i)) \notin \mathcal{DP}$. Then $u_{i_0} \geq n - \ell$.*

We define a map on the set of λ -bad ℓ -essential μ -SSDT's, which will be proved to be a sign reversing involution.

Definition 5.11. For $R \in \mathcal{B}(\mu)$, let $u = u_1 u_2 \cdots u_m = read(R)$ and i_0 be the first i such that $\lambda + r(\omega(u_1 \cdots u_i)) \notin \mathcal{DP}$ and $u_{i_0} = k$.

- (1) If $u_{i_0} \geq n - \ell + 1$, then let $R^\diamond = \sigma_k^{i_0}(R)$.
- (2) If $u_{i_0} = n - \ell$, then let $d_R \geq 0$ be the smallest such that $\sigma_k^{i_0} \sigma_{k-1} \cdots \sigma_{k-d_R} \cdots \sigma_{k-1} \sigma_k^{i_0}(R)$ is ℓ -essential, where $d_R = 0$ means $\sigma_k^{i_0}(R)$ is ℓ -essential, and let

$$R^\diamond = \sigma_k^{i_0} \sigma_{k-1} \cdots \sigma_{k-d_R} \cdots \sigma_{k-1} \sigma_k^{i_0}(R).$$

We finally state the main theorem.

Theorem 5.12. *The map defined on $\mathcal{B}(\mu)$ sending R to $R^\diamond$ is an involution, and*

$$D_n \cdot P_{\lambda+r(\omega(R^\diamond))} = -D_n \cdot P_{\lambda+r(\omega(R))}.$$

Hence we have, for $\lambda, \mu \in \mathcal{DP}_n$ with $\ell(\lambda) = \ell$,

$$P_\lambda(x_n, \dots, x_1) P_\mu(x_n, \dots, x_1) = \sum_R P_{\lambda+r(\omega(R))}(x_n, \dots, x_1),$$

where the sum runs over the λ -good ℓ -essential SSDT's in $\mathcal{D}_n(\mu)$.

Lemma 5.13. *If an SSDT $R \in \mathcal{D}_n(\mu)$ is λ -good, then R is ℓ -essential, where $\ell = \ell(\lambda)$.*

Proof. Suppose that R is not ℓ -essential. Let $\text{read}(R) = u_1 u_2 \cdots u_m$. Then $R|_1^{n-\ell}$ is not a strict word on $\{1, 2, \dots, n - \ell\}$ and there must be some j_0 and k such that $\omega_{k+1}(u_1 u_2 \cdots u_{j_0}) \leq \omega_k(u_1 u_2 \cdots u_{j_0}) \neq 0$. This implies, since $1 \leq k < n - \ell$, that the $(n + 1 - k)$ th entry is at least the $(n - k)$ th entry in $\lambda + r(\omega(u_1 u_2 \cdots u_{j_0}))$. This shows that R is λ -bad. $\square$

Corollary 5.14.

$$P_\lambda(x_n, \dots, x_1) P_\mu(x_n, \dots, x_1) = \sum_R P_{\lambda+r(\omega(R))}(x_n, \dots, x_1),$$

where the sum runs over the λ -good SSDT's in $\mathcal{D}_n(\mu)$. Hence $f_{\lambda\mu}^\nu$ is the number of λ -good SSDT's in $\mathcal{D}(\mu)$ with weight $r(\nu - \lambda)$.

Remark 5.15. We describe the expansion of $P_\lambda P_\mu$ only in terms of λ -good SSDT's in Corollary 5.14. However, the notion of ℓ -essentiality is indispensable to develop the theory: ℓ -essential SSDT's are the representatives of $(D_n \cdot P_{\lambda+r(\omega(R))})$'s that appear in the expansion of $D_n \cdot P_\lambda P_\mu$; see Theorem 4.23. Theorem 4.23 is the starting point of our combinatorial work, which provides the objectives with which we can work.

6. PROOF OF THE MAIN THEOREM

We prove Theorem 5.12 in this section by proving a technical lemma on strict lattice words, or equivalently ℓ -essential SSDT's.

Let $R \in \mathcal{B}(\mu)$ be a λ -bad ℓ -essential SSDT $\in \mathcal{D}(\mu)$ and $\text{read}(R) = u = u_1 u_2 \cdots u_m$ be the reading word of R with letters in $\{1, 2, \dots, n\}$. The index i_0 is the smallest i such that $\lambda + r(\omega(u_1 \cdots u_i)) \notin \mathcal{DP}$.

The first observation we make is the following.

Lemma 6.1. *Let $u_{i_0} = k$. Then:*

- (1) $k \geq n - \ell$,
- (2) the k th entry and the $(k + 1)$ st entry of $\lambda + r(\omega(u_1 \cdots u_{i_0}))$ are the same,
- (3) u_{i_0} is not k -paired in u , and hence $\sigma_k^{i_0}(R)$ is an SSDT by Theorem 2.9.

Proof. The first part is a restatement of Corollary 5.10. The definition of i_0 shows the validity of the second part since R is ℓ -essential and λ is a strict partition.

For the proof of the third part, suppose that $u_{i_0} = k$ is k -paired in u . Observe that every k and $k + 1$ between u_i and u_{i_0} is k -paired by Lemma 2.4(2), and hence the number of $(k + 1)$'s and the number of k 's in $u_i \cdots u_{i_0}$ are the same. Thus, by the second part of the current lemma, the k th entry and the $(k + 1)$ st entry of $\lambda + r(\omega(u_1 \cdots u_{i-1}))$ are the same. This is a contradiction since there must be $j_0 < i < i_0$ such that $u_{j_0} = k$ and $\lambda + r(\omega(u_1 \cdots u_{j_0})) \notin \mathcal{DP}$. $\square$

Due to Theorem 2.5 and Theorem 2.9, R° 's defined in Definition 5.11 are always SSDT's once it is shown that they are well defined. Because of the definitions of i_0 , $\sigma_k^{i_0}$, $\sigma_{k'}^{i_0}$'s and Corollary 4.5 it is also clear that $D_n \cdot P_{\lambda+r(\omega(R^\circ))} = -D_n \cdot P_{\lambda+r(\omega(R))}$ once it is shown that the map $R \mapsto R^\circ$ is well defined.

Hence we only show the well-definedness of R° and check that R° 's are λ -bad and ℓ -essential.

Remember that we let $u_{i_0} = k$. There are two cases: $k \geq n - \ell + 1$ or $k = n - \ell$.

If $k \geq n - \ell + 1$, then $\sigma_k^{i_0}$ does not change any $u_i \leq n - \ell$. Hence $R^\circ = \sigma_k^{i_0}(R)$ is ℓ -essential by Lemma 4.16. Moreover the first i_0 entries in $u = \text{read}(R)$ are not changed, and R° is also λ -bad with the same i_0 . Therefore it is clear that the map $R \mapsto R^\circ$ is a well-defined involution on the subset of R 's with $u_{i_0} \geq n - \ell + 1$.

We now assume that $k = n - \ell$ and let

$$\lambda + r(\omega(R)) = (\alpha_1, \dots, \alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_n)$$

so that $\alpha_\ell = \lambda_\ell + (\text{number of } (n - \ell + 1)\text{'s in } R)$ and $\alpha_{\ell+1}$ is the number of $(n - \ell)$'s in R .

Definition 6.2. For a word $w = w_1 \cdots w_m$ on the alphabet $\{1, 2, \dots, n\}$ and $a, b \in \{1, \dots, n\}$, we say that a *dominates* b in w if the number of a 's is strictly larger than the number of b 's in every subword $w_1 \cdots w_i$ of w , in which at least one a or b appears.

The following is immediate from the definitions worthy to state.

Lemma 6.3. Let $w = w_1 \cdots w_m$ be a word on $\{1, 2, \dots, n\}$. Then the following holds:

- (1) w is a strict lattice word on $\{n_1, n_1 + 1, \dots, n_2\}$, $1 \leq n_1 \leq n_2 \leq n$, if and only if $(a + 1)$ dominates a in w for each $a = n_2 - 1, n_2 - 2, \dots, n_1 + 1$.
- (2) If w is a strict lattice word on $\{n_1, n_1 + 1, \dots, n_2\}$, $1 \leq n_1 \leq n_2 \leq n$, then $\sigma_{n'}(w) = w$ for each $n_1 \leq n' < n_2$.

Now we prove the key lemma for the proof of the main theorem:

Lemma 6.4. Let $w = w_1 w_2 \cdots w_m$ be a word on $\{1, 2, \dots, n\}$ and suppose that the subword $w|_1^{n_2}$ of w is a strict lattice word on $\{1, 2, \dots, n_2\}$ for $n_2 < n$. Let $\omega(w) = (\omega_1, \dots, \omega_{n_2}, \omega_{n_2+1}, \dots, \omega_n)$ and assume that there is i such that $(n_2 + 1)$ dominates n_2 in $w_1 \cdots w_{i-1}$ and $\omega_{n_2}(w_1 w_2 \cdots w_i) = \omega_{n_2+1}(w_1 w_2 \cdots w_i) \neq 0$.

Then there is $d \geq 0$ such that the subword $\tilde{w}|_1^{n_2}$ of

$$\tilde{w} = (\sigma_{n_2} \sigma_{n_2-1} \cdots \sigma_{n_2-d} \cdots \sigma_{n_2-1} \sigma_{n_2})(w)$$

is a strict lattice word on $\{1, 2, \dots, n_2\}$. Moreover, the following holds:

- (1) d is the smallest such that $(n_2 - d)$ dominates $(n_2 - d - 1)$ in $\sigma_{n_2-d} \cdots \sigma_{n_2}(w)$.
- (2) If $\omega_{n_2-d} \leq \omega_{n_2+1}$, then, when $\delta = \omega_{n_2+1} - \omega_{n_2-d}$, δ r 's in w are replaced by $(r - 1)$'s, for $r = n_2 + 1, n_2, \dots, n_2 - d + 1$ to obtain $\tilde{w}$. The i th replaced r precedes the i th replaced $(r - 1)$ in w for $r = n_2 + 1, n_2, \dots, n_2 - d + 2$.
- (3) If $\omega_{n_2-d} > \omega_{n_2+1}$, then, when $\delta = \omega_{n_2-d} - \omega_{n_2+1}$, δ r 's in w are replaced by $(r + 1)$'s, for $r = n_2, n_2 - 1, \dots, n_2 - d$ to obtain $\tilde{w}$. The i th replaced r precedes the i th replaced $(r - 1)$ in w for $r = n_2, \dots, n_2 - d + 1$.
- (4) $\omega(\tilde{w})$ is obtained by exchanging the $(n_2 - d)$ th entry and the $(n_2 + 1)$ st entry from $\omega(w)$.

Proof. We use an induction on n_2 . If $n_2 = 1$, then $\tilde{w} = \sigma_1(w)$ satisfies the given conditions with $d = 0$. Suppose that the assertion is true for an $n_2 \geq 1$ and a word $w = w_1 w_2 \cdots w_m$ is given, in which the subword $w|_1^{n_2+1}$ is a strict lattice word on $\{1, 2, \dots, n_2 + 1\}$, $n_2 + 1 < n$, and i_0 is the smallest index such that $(n_2 + 2)$ dominates $(n_2 + 1)$ in $w_1 w_2 \cdots w_{i_0-1}$ and $\omega_{n_2+1}(w_1 w_2 \cdots w_{i_0}) = \omega_{n_2+2}(w_1 w_2 \cdots w_{i_0}) \neq 0$. Let $w' = \sigma_{n_2+1}(w) = w'_1 \cdots w'_m$. If w' is a strict lattice word on $\{1, 2, \dots, n_2 + 1\}$, then the assertion holds with $d = 0$. Suppose that w' is *not* a strict lattice word

on $\{1, 2, \dots, n_2 + 1\}$. Then $(n_2 + 1)$ fails to dominate n_2 in w' and there must be an index i_1 , that is, the smallest among them, such that $\omega_{n_2}(w'_1 \cdots w'_{i_1}) = \omega_{n_2+1}(w'_1 \cdots w'_{i_1}) \neq 0$. Moreover, this implies that $\omega_{n_2+1} > \omega_{n_2+2}$; otherwise extra $(n_2 + 1)$'s appear in w' and $(n_2 + 1)$ certainly dominates n_2 in w' . Since i_0 is the smallest such that $\omega_{n_2+1}(w_1 w_2 \cdots w_{i_0}) = \omega_{n_2+2}(w_1 w_2 \cdots w_{i_0}) \neq 0$ and $(n_2 + 2)$ dominates $(n_2 + 1)$ in $w_1 \cdots w_{i_0-1}$, every $(n_2 + 1)$ and $(n_2 + 2)$ is $(n_2 + 1)$ -paired in $w_1 \cdots w_{i_0}$. Therefore, σ_{n_2+1} does not change $w_1 \cdots w_{i_0}$; we have $w'_1 \cdots w'_{i_0} = w_1 \cdots w_{i_0}$ and $(n_2 + 1)$ dominates n_2 in $w'_1 \cdots w'_{i_0}$. We can conclude that $i_1 > i_0$.

By inductive hypothesis, since $w'|_1^{n_2}$ is a strict lattice word on $\{1, \dots, n_2\}$ and $(n_2 + 1)$ fails to dominate n_2 in w' , there is $d \geq 0$ such that

$$w'' = (\sigma_{n_2} \sigma_{n_2-1} \cdots \sigma_{n_2-d} \cdots \sigma_{n_2-1} \sigma_{n_2})(w')$$

is a strict lattice word on $\{1, \dots, n_2\}$, where d is the smallest such that $(n_2 - d)$ dominates $(n_2 - d - 1)$ in $(\sigma_{n_2-d} \cdots \sigma_{n_2-1} \sigma_{n_2})(w')$.

We claim that $\sigma_{n_2+1}(w'')|_1^{n_2+1}$ is a strict lattice word on $\{1, \dots, n_2 + 1\}$.

To prove the claim we only need to show that $(n_2 + 1)$ dominates n_2 in $\sigma_{n_2+1}(w'')$. Let $\tilde{w} = \sigma_{n_2+1}(w'')$. Note again that every $(n_2 + 1)$ and $(n_2 + 2)$ in $w_1 \cdots w_{i_0}$ is $(n_2 + 1)$ -paired and $w'_1 \cdots w'_{i_0} = w_1 \cdots w_{i_0}$, while $(\omega_{n_2+1} - \omega_{n_2+2})$ $(n_2 + 1)$ -unpaired $(n_2 + 1)$'s from $w_{i_0+1} \cdots w_{i_1} \cdots w_m$ are replaced by $(n_2 + 2)$'s to obtain w' . This is why $(n_2 + 1)$ no longer dominates n_2 in w' .

Let $\sigma = \sigma_{n_2} \sigma_{n_2-1} \cdots \sigma_{n_2-d} \cdots \sigma_{n_2-1} \sigma_{n_2}$ so that $w'' = \sigma(w')$. Then, by the definition of i_1 and the second part of Lemma 6.3,

$$(6.5) \quad w'_1 \cdots w'_{i_0} w''_{i_0+1} \cdots w'_{i_1} = w'_1 \cdots w'_{i_0} w'_{i_0+1} \cdots w'_{i_1} = w_1 \cdots w_{i_0} w'_{i_0+1} \cdots w'_{i_1},$$

and the weight of w' is $(\omega_1, \dots, \omega_{n_2}, \omega_{n_2+2}, \omega_{n_2+1}, \dots, \omega_n)$.

Suppose that $\omega_{n_2-d} \leq \omega_{n_2+2}$. Then by condition (2) of the current lemma, $\delta (= \omega_{n_2+2} - \omega_{n_2-d})$ $(n_2 + 1)$'s in w' are replaced by n_2 's, δ n_2 's in w' are replaced by $(n_2 - 1)$'s, and so on, by applying σ to w' to obtain w'' .

An important observation is that every $(n_2 + 1)$ in $w'_{i_1+1} \cdots w'_m$ is $(n_2 + 1)$ -paired in w' . Consider the subword $w_{j_1} \cdots w_{j_s} w_{j_{s+1}} \cdots w_{j_{s+t}} w_{j_{s+t+1}} \cdots w_{j_{s+t+1}}$, where $w_{j_1} = \cdots = w_{j_{s+t}} = (n_2 + 1)$ and $w_{j_{s+t+1}} = \cdots = w_{j_{s+t+s}} = (n_2 + 2)$, of w consisted with $(n_2 + 1)$ -unpaired $(n_2 + 1)$'s and $(n_2 + 2)$'s in w . Note that $t = \omega_{n_2+1} - \omega_{n_2+2}$, the number of $(n_2 + 1)$'s that are replaced by $(n_2 + 2)$'s by applying σ_{n_2+1} . By the definition of i_1 , some $(n_2 + 1)$'s in $w_1 \cdots w_{i_1}$ must be replaced by $(n_2 + 2)$'s by applying σ_{n_2+1} , and this means that $i_0 < j_{s+1} < i_1$ and every $(n_2 + 1)$ -unpaired $(n_2 + 1)$ in $w_{i_1+1} \cdots w_m$ becomes $(n_2 + 1)$ -unpaired $(n_2 + 2)$ in w' .

Hence δ $(n_2 + 1)$'s in w' , replaced by n_2 's by applying σ , are $(n_2 + 1)$ -paired ones in w' because of (6.5). This implies that applying σ to w' will make δ $(n_2 + 2)$'s (originally $(n_2 + 1)$ -paired in w') $(n_2 + 1)$ -unpaired in w'' . Let us look at the subword

$$\begin{aligned} & w_{j_1} \cdots w_{j_s} w_{j_{s+1}} \cdots w_{j_{s+t}} w_{j_{s+t+1}} \cdots w_{j_{s+t+1}} \\ &= \underbrace{(n_2 + 1) \cdots (n_2 + 1)}_s \underbrace{(n_2 + 1) \cdots (n_2 + 1)}_{\omega_{n_2+1} - \omega_{n_2+2}} \underbrace{(n_2 + 2) \cdots (n_2 + 2)}_s \end{aligned}$$

of $(n_2 + 1)$ -unpaired $(n_2 + 1)$'s and $(n_2 + 2)$'s in w again. The corresponding subword of w' (and also of w'') is

$$\begin{aligned} & w'_{j_1} \cdots w'_{j_s} w'_{j_{s+1}} \cdots w'_{j_{s+t}} w'_{j_{s+t+1}} \cdots w'_{j_{s+t+1}} \\ &= w''_{j_1} \cdots w''_{j_s} w''_{j_{s+1}} \cdots w''_{j_{s+t}} w''_{j_{s+t+1}} \cdots w''_{j_{s+t+1}} \end{aligned}$$

$$= \underbrace{(n_2 + 1) \cdots (n_2 + 1)}_s \underbrace{(n_2 + 2) \cdots (n_2 + 2)}_{\omega_{n_2+1} - \omega_{n_2-2}} \underbrace{(n_2 + 2) \cdots (n_2 + 2)}_s.$$

Since δ $(n_2 + 1)$'s in $w'_{i_1+1} \cdots w'_m$, which are $(n_2 + 1)$ -paired in w' , are replaced by n_2 in w'' , there will be δ new $(n_2 + 1)$ -unpaired $(n_2 + 2)$'s in w'' . That is, the subword of $(n_2 + 1)$ -unpaired $(n_2 + 1)$'s and $(n_2 + 2)$'s in w'' is

$$\underbrace{(n_2 + 1) \cdots (n_2 + 1)}_s \underbrace{(n_2 + 2) \cdots (n_2 + 2)}_{\omega_{n_2+1} - \omega_{n_2-2} + \delta} \underbrace{(n_2 + 2) \cdots (n_2 + 2)}_s.$$

Hence the subword of $(n_2 + 1)$ -unpaired $(n_2 + 1)$'s and $(n_2 + 2)$'s in $\sigma_{n_2+1}(w'')$ is

$$\underbrace{(n_2 + 1) \cdots (n_2 + 1)}_s \underbrace{(n_2 + 1) \cdots (n_2 + 1)}_{\omega_{n_2+1} - \omega_{n_2-2} + \delta} \underbrace{(n_2 + 2) \cdots (n_2 + 2)}_s.$$

An important point is that $(n_2 + 1)$'s in w , that were replaced by $(n_2 + 2)$'s in w' and w'' , will become $(n_2 + 1)$'s again; there are $(\omega_{n_2+1} - \omega_{n_2+2})$ of them. Moreover, extra δ $(n_2 + 2)$'s in w , which were $(n_2 + 1)$ -paired in w and w' , become $(n_2 + 1)$'s in $\tilde{w}$. If we let $w_{p_1} w_{p_2} \cdots w_{p_\delta}$ be the subword of w of those δ special entries, then $w_{p_1} = \cdots = w_{p_\delta} = n_2 + 2$, and each of them are paired with some $(n_2 + 1)$'s. If we let $w_{q_1} w_{q_2} \cdots w_{q_\delta}$ be the subword of w of those $(n_2 + 1)$'s, then $p_r < q_r$ for all $r = 1, 2, \dots, \delta$; w_{p_r} does not have to be $(n_2 + 1)$ -paired with w_{q_r} though. From the argument that we develop, we can see that $\tilde{w}_{p_r} = n_2 + 1$ and $\tilde{w}_{q_r} = n_2$ and $\tilde{w}_{q_r}$'s are the only new n_2 's in $\tilde{w}$; they were $(n_2 + 1)$'s in w . To show that $(n_2 + 1)$ dominates n_2 in $\tilde{w}$, we look at n_2 's in $\tilde{w}$. There are two kinds of n_2 's:

- the one that has not been changed by taking the map σ , that is, $n_2 = \tilde{w}_j = w''_j = w'_j = w_j$, and
- the one that has been changed by σ , that is, $n_2 = \tilde{w}_j = w''_j = \sigma(w'_j)$, where $w'_j = w_j = n_2 + 1$.

Note that n_2 's of the second type are exactly $\tilde{w}_{q_r}$'s for $r = 1, \dots, \delta$. For $n_2 = \tilde{w}_j$ in $\tilde{w}$ of the first type, by the assumption on w , the number of $(n_2 + 1)$'s in $w_1 \cdots w_j$ is strictly greater than the number of n_2 's in $w_1 \cdots w_j$. Thus the number of $(n_2 + 1)$'s of the first type in $\tilde{w}_1 \cdots \tilde{w}_j$, which was originally $(n_2 + 1)$ in w , is strictly greater than the number of n_2 's of the first type in $\tilde{w}_1 \cdots \tilde{w}_j$. For $n_2 = \tilde{w}_{q_r}$ in $\tilde{w}$ of the second type, $\tilde{w}_{p_r} = n_2 + 1$ precedes $\tilde{w}_{q_r}$. One has to note that $i_1 < p_1$ and there are $(n_2 + 1)$'s in $w_1 \cdots w_{i_1}$. This shows that $(n_2 + 1)$ dominates n_2 in $\tilde{w}$ when $\omega_{n_2+2} > \omega_{n_2-d}$.

Properties (1) and (2) of the lemma follow from the argument on w_{p_r} 's and w_{q_r} 's and the inductive hypothesis that assumes the properties of σ .

The proof that $(n_2 + 1)$ dominates n_2 in $\tilde{w}$ when $\omega_{n_2+2} < \omega_{n_2-d}$ and a proof of the assertion (3) can be done in a similar way.

Property (4) of the lemma is clear from Theorem 2.5. $\square$

Example 6.6. Let

$$w = 4 \ 3 \ 3 \ 2 \ 3 \ 4 \ 4 \ 2 \ 1 \ 3 \ 3 \ 4$$

be a word with $\omega(w) = (1, 2, 5, 4)$. It is easy to check that w is a strict lattice word on $\{1, 2, 3\}$, while $\omega_4(w_1 w_2) = \omega_3(w_1 w_2)$ in the subword $w_1 w_2 = 4 \ 3$ so that $i_0 = 2$. Apply σ_3 to w :

$$w' = \sigma_3(w) = 4 \ 3 \ 3 \ 2 \ 4 \ 4 \ 4 \ 2 \ 1 \ 3 \ 3 \ 4.$$

One can check that $w' = \sigma_3(w)$ is not a strict lattice word on $\{1, 2, 3\}$ and $i_1 = 8$ is the first index such that 3 fails to dominate 2 in $w'_1 \cdots w'_i$; $\omega_3(w'_1 \cdots w'_8) = \omega_2(w'_1 \cdots w'_8)$. By applying σ_2 to w' , we obtain

$$w'' = \sigma_2(w') = 4 \ 3 \ 3 \ 2 \ 4 \ 4 \ 4 \ 2 \ 1 \ 2 \ 2 \ 4,$$

in which 2 dominates 1. Hence we apply σ_3 to $\sigma_2(w')$:

$$\tilde{w} = \sigma_3 \sigma_2 \sigma_3(w) = 4 \ 3 \ 3 \ 2 \ 3 \ 3 \ 3 \ 2 \ 1 \ 2 \ 2 \ 4,$$

that is, a strict lattice word on $\{1, 2, 3\}$ with weight $\omega(\tilde{w}) = (1, 4, 5, 2)$. In the equation below, the underlined letters are the ones replaced by different letters, and we can check that condition (2) in Lemma 6.4 is satisfied:

$$w = 4 \ 3 \ 3 \ 2 \ 3 \ 4 \ 4 \ 2 \ 1 \ 3 \ 3 \ 4, \quad \tilde{w} = 4 \ 3 \ 3 \ 2 \ 3 \ 3 \ 3 \ 2 \ 1 \ 2 \ 2 \ 4.$$

We are ready to complete the proof of Theorem 5.12:

Proof of Theorem 5.12. For a given $R \in \mathcal{B}(\mu)$ with $\text{read}(R) = u = u_1 u_2 \cdots u_m$ and $u_{i_0} = n - \ell$, let

$$\begin{aligned} \lambda + r(\omega(R)) &= (\alpha_1, \dots, \alpha_\ell, \alpha_{\ell+1}, \dots, \alpha_n) \text{ so that} \\ \omega(R) &= (\alpha_n, \alpha_{n-1}, \dots, \alpha_{\ell+1}, \alpha_\ell - \lambda_\ell, \alpha_{\ell-1} - \lambda_{\ell-1}, \dots, \alpha_1 - \lambda_1). \end{aligned}$$

We also let $e(u)$ be the word obtained by adding λ_ℓ $(n - \ell + 1)$'s in front of the word u :

$$e(u) = \underbrace{(n - \ell + 1) \cdots (n - \ell + 1)}_{\lambda_\ell} u_1 u_2 \cdots u_m.$$

Then $e(u)|_1^{n-\ell}$ is a strict lattice word on $\{1, 2, \dots, n - \ell\}$ since R is an ℓ -essential SSDT. Moreover,

$$\omega(e(u)) = (\alpha_n, \alpha_{n-1}, \dots, \alpha_{\ell+1}, \alpha_\ell, \alpha_{\ell-1} - \lambda_{\ell-1}, \dots, \alpha_1 - \lambda_1),$$

and i_0 is the first index such that $(n - \ell + 1)$ fails to dominate $(n - \ell)$ in $e(u)$: $(n - \ell + 1)$ dominates $(n - \ell)$ in $\underbrace{(n - \ell + 1) \cdots (n - \ell + 1)}_{\lambda_\ell} u_1 \cdots u_{i_0-1}$ while

$$\begin{aligned} \omega_{n-\ell}(\underbrace{(n - \ell + 1) \cdots (n - \ell + 1)}_{\lambda_\ell} u_1 \cdots u_{i_0}) \\ = \omega_{n-\ell+1}(\underbrace{(n - \ell + 1) \cdots (n - \ell + 1)}_{\lambda_\ell} u_1 \cdots u_{i_0}). \end{aligned}$$

Therefore by Lemma 6.4 there is $d \geq 0$ such that

$$\widetilde{e(u)} = (\sigma_{n-\ell} \cdots \sigma_{n-\ell-d} \cdots \sigma_{n-\ell})(e(u))$$

is a strict lattice word on $\{1, 2, \dots, n - \ell\}$, and $\omega(\widetilde{e(u)})$ is obtained from $\omega(e(u))$ by exchanging the $(n - \ell - d)$ th and the $(n - \ell + 1)$ st entry. Furthermore, we can observe the following:

- (i) $\sigma_{n-\ell} \cdots \sigma_{n-\ell-d} \cdots \sigma_{n-\ell}$ does not alter the first part, $(n - \ell + 1) \cdots (n - \ell + 1) u_1 \cdots u_{i_0}$ of $e(u)$. Hence $(\sigma_{n-\ell}^{i_0} \cdots \sigma_{n-\ell-d} \cdots \sigma_{n-\ell}^{i_0})(u)$ is the word obtained by first removing λ_ℓ $(n - \ell + 1)$'s from $\widetilde{e(u)}$ and is a strict lattice word on $\{1, 2, \dots, n - \ell\}$.

- (ii) For $(\sigma_{n-\ell}^{i_0} \cdots \sigma_{n-\ell-d} \cdots \sigma_{n-\ell}^{i_0})(u)$ to be a strict lattice word on $\{1, 2, \dots, n-\ell\}$, $(n-\ell-d)$ has to dominate $(n-\ell-d-1)$ in $(\sigma_{n-\ell-d} \cdots \sigma_{n-\ell}^{i_0})(u)$. Moreover, we showed that if $(n-\ell-d)$ dominates $(n-\ell-d-1)$ in $(\sigma_{n-\ell-d} \cdots \sigma_{n-\ell}^{i_0})(u)$, then $(\sigma_{n-\ell}^{i_0} \cdots \sigma_{n-\ell-d} \cdots \sigma_{n-\ell}^{i_0})(u)$ is a strict lattice word on $\{1, 2, \dots, n-\ell\}$ in Lemma 6.4. This proves that the condition on d_R given in Definition 5.11, that d_R is the smallest such that $(\sigma_{n-\ell}^{i_0} \cdots \sigma_{n-\ell-d} \cdots \sigma_{n-\ell}^{i_0})(u)$ is ℓ -essential, is satisfied once we find the smallest d such that $(n-\ell-d)$ dominates $(n-\ell-d-1)$ in $(\sigma_{n-\ell-d} \cdots \sigma_{n-\ell}^{i_0})(R)$ as in Lemma 6.4.

By the observations (i) and (ii) we made, we can see that $\text{read}(R^\diamond)$ is the word obtained by first removing $\lambda_\ell (n-\ell+1)$'s from $\widetilde{e(u)}$. It is also easy to see that $\omega(R^\diamond)$ is obtained by exchanging the $(n-\ell-d)$ th and the $(n-\ell+1)$ st entry of $\omega(u)$ and $\lambda + r(\omega(R^\diamond))$ is obtained by exchanging the $(\ell+d+1)$ st entry and the ℓ th entry of $\lambda + r(\omega(R))$. Therefore, by Corollary 4.5, we can conclude that $D_n \cdot P_{\lambda+r(\omega(R^\diamond))} = -D_n \cdot P_{\lambda+r(\omega(R))}$. This completes the proof of the main theorem. $\square$

Remark 6.7. If $\alpha = \lambda + r(\omega(R))$ and $\alpha_\ell = \alpha_{\ell+d+1}$ for d in Definition 5.11, then $R^\diamond = R$ by Lemma 6.4(2).

Example 6.8. In Example 5.3, we showed that there are 3 λ -bad ℓ -essential SSDT's, where $\lambda = (2, 0, 0)$ and $\mu = (3, 1, 0)$;

$$R_1 = \begin{array}{|c|c|c|} \hline 2 & 1 & 2 \\ \hline & 2 & \\ \hline \end{array}, \quad R_2 = \begin{array}{|c|c|c|} \hline 3 & 2 & 2 \\ \hline & 2 & \\ \hline \end{array} \quad \text{and} \quad R_3 = \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline & 2 & \\ \hline \end{array}.$$

It is easy to check that in R_1 , $i_0 = 2$, $k = u_{i_0} = 2 = n - \ell$ and $\lambda + r(\omega(R_1)) = (2, 3, 1)$. Since $\sigma_2^2(R_1) = \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline & 2 & \\ \hline \end{array}$ is R_3 and ℓ -essential, $R_1^\diamond = R_3$. This also shows that $R_3^\diamond = R_1$ and

$$D_n \cdot P_{\lambda+r(\omega(R_1))} = -D_n \cdot P_{\lambda+r(\omega(R_3))}.$$

For R_2 , $i_0 = 4$, $k = u_{i_0} = 2 = n - \ell$ and $\lambda + r(\omega(R_2)) = (3, 3, 0)$. Since $\sigma_2^4(R_2) = R_2$, we have $R_2^\diamond = R_2$ and

$$D_n \cdot P_{\lambda+r(\omega(R_2))} = 0.$$

Example 6.9. We give an example of when $u_{i_0} > n - \ell$ so that we apply the first case of Definition 5.11. When $n = 4$, $\lambda = (3, 2, 0, 0)$ and $\mu = (3, 1, 0, 0)$, let $R =$

$$\begin{array}{|c|c|c|} \hline 3 & 2 & 3 \\ \hline & 2 & \\ \hline \end{array} \in \mathcal{D}_4(\mu).$$

Then R is ℓ -essential since $n - \ell = 2$ and $R|_1^2 = 2 \ 2$ is a strict lattice word on $\{1, 2\}$. Also, R is λ -bad since $\lambda + r(\omega(2 \ 3)) = (3, 3, 1, 0) \notin \mathcal{DP}$ and $i_0 = 2$, $u_{i_0} = 3 > n - \ell$. By Definition 5.11, $R^\diamond = \sigma_3^2(R) = \begin{array}{|c|c|c|} \hline 3 & 2 & 4 \\ \hline & 2 & \\ \hline \end{array}$. Moreover, we can check that $\lambda + r(\omega(R^\diamond)) = (4, 3, 2, 0)$ while $\lambda + r(\omega(R)) = (3, 4, 2, 0)$.

7. SKEW SCHUR P -FUNCTIONS

We consider skew Schur P -functions in this section, especially the conjectures on skew Schur P -functions that Serrano stated in [16].

For $\mu \leq \lambda \in \mathcal{DP}$, the skew Schur P -function $P_{\lambda/\mu}(x_1, x_2, \dots, x_n)$ on n variables is defined as a generating function of skew marked shifted tableaux in $\mathcal{Y}_n(\lambda/\mu)$ (see

Definition 1.1):

$$(7.1) \quad P_{\lambda/\mu}(x_1, x_2, \dots, x_n) = \sum_{T \in \mathcal{Y}_n(\lambda/\mu)} x^{\omega(T)}.$$

With the definition of *skew Schur P -functions*, Lemma 4.12 can be translated into the following.

Lemma 7.2. *For $\mu \leq \lambda$ and a given $T_1 \in \mathcal{Y}_1^{n-\ell}(\mu)$,*

$$P_{\lambda/\mu}(x_{n-\ell+1}, \dots, x_n) = \sum_{T_2 \in \mathcal{Y}_{n-\ell+1}^n(\lambda/\mu)} x^{\omega(T_2)} = \sum_{\substack{R \in \mathcal{D}_1^n(\lambda) \\ P_{mix}(R|_1^{n-\ell}) = T_1}} x^{\omega(R|_1^{n-\ell+1})}.$$

We first introduce known facts on the expansion of skew Schur P -functions as linear sums of Schur P -functions, mainly in terms of *rectification*.

To introduce the notion of rectification, we extend $\mathcal{Y}_n(\lambda/\mu)$ to $\tilde{\mathcal{Y}}_n(\lambda/\mu)$, the set of fillings of $\mathcal{S}(\lambda/\mu)$ with $\{1' < 1 < 2' < 2 < \dots < n' < n\}$ satisfying conditions (1), (2) and (3) in Definition 1.1. Shifted tableaux in $\tilde{\mathcal{Y}}_n(\lambda/\mu)$ may have primed numbers on the main diagonal and $\mathcal{Y}_n(\lambda/\mu) \subset \tilde{\mathcal{Y}}_n(\lambda/\mu)$.

Worley [3] and Sagan [14] defined the shifted analogue of sliding operations on $\tilde{\mathcal{Y}}_n(\lambda/\mu)$ as follows. For $\lambda, \mu \in \mathcal{DP}$ such that $\mu \leq \lambda$, an *inner corner* of $\mathcal{S}(\lambda/\mu)$ is a box $(i, j) \in \mathcal{S}(\mu)$ such that $\mathcal{S}(\mu) - \{(i, j)\}$ is a diagram of a partition in $\mathcal{DP}$. For $T \in \tilde{\mathcal{Y}}_n(\lambda/\mu)$ and an inner corner (i, j) of $\mathcal{S}(\lambda/\mu)$, the *shifted jeu de taquin slide* into the corner (i, j) performs the following to obtain another shifted skew tableau T' : Slide the number either on the south or the east of the corner so that the resulting filling (with a hole) is a shifted skew tableau if we ignore the hole. Repeat sliding the number either on the south or on the east of the hole (either $(i+1, j)$ or $(i, j+1)$) produced by sliding a number in the previous step. There are two special slides to be taken care of when the hole is on the main diagonal:

$$\begin{array}{|c|c|} \hline \cdot & i' \\ \hline i & \cdot \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline i & i \\ \hline \cdot & \cdot \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline \cdot & i' \\ \hline i' & i \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline i' & i \\ \hline \cdot & \cdot \\ \hline \end{array}.$$

For a skew shifted tableau $T \in \mathcal{Y}_n(\lambda/\mu)$, *rectification* of T is the shifted tableau of normal shape $\rho \in \mathcal{DP}$, obtained by successively applying shifted jeu de taquin slides to T . It is known that the rectification is well defined (independent of the order of choice of inner corners) [3, 14].

The following is well known; see [3], Sec. 8 of [17].

Proposition 7.3. *For $\mu \leq \lambda \in \mathcal{DP}_n$,*

- (1) $f_{\mu\rho}^\lambda$ is the number of skew shifted tableaux in $\tilde{\mathcal{Y}}_n(\lambda/\mu)$ which rectifies to a given shifted tableau in $\tilde{\mathcal{Y}}_n(\rho)$.
- (2) $P_{\lambda/\mu}(x_1, \dots, x_n) = \sum_{\rho \leq \lambda} f_{\mu\rho}^\lambda 2^{\ell(\mu) + \ell(\rho) - \ell(\lambda)} P_\rho(x_1, \dots, x_n)$, where the weight of $T \in \mathcal{Y}(\lambda/\mu)$ is contributed to P_ρ for the shape ρ of the rectification of T .

Remark 7.4. $\tilde{\mathcal{Y}}_n(\lambda)$ is a combinatorial model for the Schur Q -functions ([13]):

$$Q_\lambda(x_1, \dots, x_n) = \sum_{T \in \tilde{\mathcal{Y}}_n(\lambda)} x^{\omega(T)}.$$

In [16], Serrano introduces the shifted *plactic* monoid in which plactic Schur P -functions $\mathcal{P}_\lambda$ are defined so that the correspondence $\mathcal{P}_\lambda \leftrightarrow P_\lambda$ defines an isomorphism between two algebras generated by plactic Schur P -functions and by Schur P -functions, respectively.

Definition 7.5. The relation on the set of (finite) words on $\{1, 2, \dots, n\}$ in which two words are related if and only if their P_{mix} are the same is an equivalence relation. The *plactic Schur P -function* is

$$\mathcal{P}_\lambda = \sum_{T \in \mathcal{Y}_n(\lambda)} [T],$$

where $[T]$ is the class of words w such that $P_{mix}(w) = T$.

Plactic skew Schur P -functions are introduced below, and conjectures are made on them by Serrano.

Definition 7.6 (Serrano, [16]). For a skew marked shifted tableau $T \in \mathcal{Y}_n(\lambda/\mu)$, $\text{mread}(T)$ is defined as $\text{read}(D(\hat{T}))|_1^n$, where $\hat{T}$ is a shifted tableau of shape λ on the set of extended alphabets $\{-n < -n' < \dots < -1 < -1' < 1' < 1 < \dots < n' < n\}$ with $\hat{T}(i, j) = T(i, j)$ for $(i, j) \in \lambda/\mu$, and $D : \mathcal{Y}_n(\lambda) \rightarrow \mathcal{D}_n(\lambda)$ is naturally extended to the set of marked shifted tableaux with contents in $\{-n < -n' < \dots < -1 < -1' < 1' < 1 < \dots < n' < n\}$.

Definition 7.7 (Serrano, [16]). For $\mu \leq \lambda$, the *shifted plactic skew Schur P -function* is defined as

$$\mathcal{P}_{\lambda/\mu} = \sum_{T \in \mathcal{Y}_n(\lambda/\mu)} [P_{mix}(\text{mread}(T))].$$

Conjecture 7.8 (Serrano, [16]). (1) $\mathcal{P}_{\lambda/\mu}$ belongs to the ring generated by the shifted plactic Schur P -functions.

(2) Fix a marked shifted tableau U of shape ν . The coefficient of P_ν in $\mathcal{P}_{\lambda/\mu}$ is equal to the number of skew shifted Young tableaux T with $P_{mix}(\text{mread}(T)) = U$.

The second part of the conjecture is stated as a corollary of the first part in [16]. We give two counterexamples to show that both statements in Conjecture 7.8 are not true.

Example 7.9 (Counterexample of Conjecture 7.8(1)). Let $\lambda = (3, 2)$, $\mu = (2, 0)$

and $\nu = (3, 0)$. Let $T_1 = \begin{array}{|c|c|c|} \hline \cdot & \cdot & 1' \\ \hline & 1 & 1 \\ \hline \end{array} \in \mathcal{Y}(\lambda/\mu)$; then $P_{mix}(\text{mread}(T_1)) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array} \in \mathcal{Y}(\nu)$. Therefore, if $\mathcal{P}_{\lambda/\mu}$ belongs to the ring generated by the shifted plactic Schur P -functions, then $\begin{array}{|c|c|c|} \hline 1 & 1 & 2' \\ \hline \end{array} \in \mathcal{Y}(\nu)$ must be $P_{mix}(\text{mread}(T_2))$ for some $T_2 \in \mathcal{Y}(\lambda/\mu)$. There is, however, no such T_2 ; there are only four skew shifted tableaux of weight $(2, 1)$ in $\mathcal{Y}(\lambda/\mu)$, and none of them gives $\begin{array}{|c|c|c|} \hline 1 & 1 & 2' \\ \hline \end{array}$.

Example 7.10 (Counterexample of Conjecture 7.8(2)). Let $\lambda = (6, 5, 2, 1)$, $\mu =$

$(3, 2, 1, 0)$, $\nu = (5, 3, 0, 0)$, and $U = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline & 2 & 2 & 2 & \\ \hline \end{array} \in \mathcal{Y}(\nu)$.

There are 4 skew shifted tableaux T in $\mathcal{Y}(\lambda/\mu)$ such that $P_{mix}(\text{mread}(T)) = U$:

$\cdot$	$\cdot$	$\cdot$	$1'$	1	1
$\cdot$	$\cdot$	$\cdot$	$1'$	2	2
		$\cdot$	1		
			2		

$\cdot$	$\cdot$	$\cdot$	$1'$	1	1
	$\cdot$	$\cdot$	$1'$	$2'$	2
		$\cdot$	1		
			2		

$\cdot$	$\cdot$	$\cdot$	$1'$	1	$2'$
	$\cdot$	$\cdot$	$1'$	$2'$	2
		$\cdot$	$1'$		
			1		

$\cdot$	$\cdot$	$\cdot$	$1'$	1	1
	$\cdot$	$\cdot$	$1'$	$2'$	2
		$\cdot$	$1'$		
			2		

However, the coefficient of P_ν in $P_{\lambda/\mu}$, which is equal to $2^{\ell(\mu)+\ell(\nu)-\ell(\lambda)} \times f_{\mu\nu}^\lambda = 2 \times f_{\mu\nu}^\lambda$, is 2. One can check that $f_{\mu\nu}^\lambda = 1$ using the main theorem (Theorem 5.12) or the well-known combinatorial rule by Stembridge [17].

Remark 7.11. A reason why the conjecture by Serrano does not work is because P_{mix} is not compatible with the operation of taking subwords of certain alphabets other than the smallest:

For a word w on the alphabet $\{1, 2, \dots, n\}$,

- (1) $(P_{mix}(w))|_1^b = P_{mix}(w|_1^b)$ for any b (see Lemma 4.9), but
- (2) the rectification of $(P_{mix}(w))|_a^b$ is not equal to $P_{mix}(w|_a^b)$ for $a \neq 1$.

For example, consider the word $w = 13312$. Then $P_{mix}(w) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline & 3 & 3 \\ \hline \end{array}$,

and the rectification of $(P_{mix}(w))|_2^3 = \begin{array}{|c|c|c|} \hline \cdot & \cdot & 2 \\ \hline & 3 & 3 \\ \hline \end{array}$ is $\begin{array}{|c|c|c|} \hline 2 & 3 & 3 \\ \hline \end{array} \in \mathcal{Y}(3,0)$, while

$P_{mix}(w|_2^3) = P_{mix}(332) = \begin{array}{|c|c|} \hline 2 & 3' \\ \hline & 3 \\ \hline \end{array} \in \mathcal{Y}(2,1)$.

Remark 7.12 (Open problems). (1) For $\mu \leq \lambda$, give a new definition of the *shifted plactic skew Schur P -function* $\mathcal{P}_{\lambda/\mu}$ (see Definition 7.6) so that it belongs to the ring generated by plactic Schur P -functions and describe the multiplicities of P_ρ 's in the expansion of $\mathcal{P}_{\lambda/\mu}$ in a nice way.

- (2) Find a bijection between the set of λ -good SSDT's in $\mathcal{D}(\mu)$ with weight $r(\nu - \lambda)$ and the set of LRS tableaux of shape ν/λ of content μ , both of which count $f_{\lambda\mu}^\nu$.

APPENDIX: LOWEST AND HIGHEST WEIGHT SSDT'S

We provide descriptions of the lowest weight SSDT and the highest weight SSDT in $\mathcal{D}_n(\rho)$. We assume that $\ell(\rho) = n$ for simpler descriptions, but it is a trivial work to generalize the given descriptions to the case $\ell(\rho) \leq n$.

Definition 8.1 (The lowest weight SSDT). For a given strict partition ρ of length ℓ , let $\rho_0 = (\ell, \ell - 1, \dots, 1)$ be a subpartition of ρ .

Step 1. Fill in $\mathcal{S}(\rho_0)$ so that the i th row is $\ell \ \ell - 1 \ \dots \ i$.

Step 2. Fill in $\mathcal{S}(\rho/\rho_0)$ so that

- i. $(\rho_{\ell+1-i} - i)$ i 's appear in the first $(\rho_{\ell+1-i} - i)$ columns of $\mathcal{S}(\rho/\rho_0)$ and
- ii. each column of $\mathcal{S}(\rho/\rho_0)$ is strictly increasing.

Let U_ρ be the resulting tableau.

Example 8.2. For $\rho = (9, 8, 6, 4, 3)$,

$$U_\rho = \begin{array}{cccccccc} 5 & 4 & 3 & 2 & 1 & 1 & 1 & 3 & 4 \\ & 5 & 4 & 3 & 2 & 2 & 2 & 4 & 5 \\ & & 5 & 4 & 3 & 3 & 3 & 5 & \\ & & & 5 & 4 & 4 & 4 & & \\ & & & & 5 & 5 & 5 & & \end{array}$$

Definition 8.3 (The highest weight SSDT). For a given strict partition ρ of length ℓ , let $\rho_1 = (\ell - 1, \ell - 2, \dots, 1)$ be a subpartition of ρ . Then $\mathcal{S}(\rho/\rho_1)$ is of ordinary (nonshifted) shape of length ℓ . For each $i = 1, 2, \dots, \ell$, let $H_i = \{(i, j) \mid j \geq i\} \cup \{(j, i) \mid j > i\}$ be the hook of the diagonals (i, i) of $\mathcal{S}(\rho/\rho_1)$ and $h_i = |H_i|$.

Step 1. Fill in $\mathcal{S}(\rho/\rho_1)$ so that the i th hook H_i is filled with i 's.

Step 2. Fill in $\mathcal{S}(\rho_1)$ with $(\rho_i - h_i)$ i 's for $i = 1, 2, \dots, \ell$ so that each row is strictly decreasing and each column is weakly increasing.

Let V_ρ be the resulting tableau.

Example 8.4. When $\rho = (16, 13, 11, 10, 5, 3, 2)$,

$$V_\rho = \begin{array}{cccccccccccccccc} 7 & 6 & 5 & 4 & 3 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 7 & 6 & 5 & 4 & 3 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & & & \\ & & 6 & 5 & 4 & 3 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & & & & \\ & & & 5 & 4 & 3 & 1 & 2 & 3 & 4 & 4 & 4 & 4 & & & & \\ & & & & 5 & 4 & 1 & 2 & 3 & & & & & & & & \\ & & & & & 4 & 1 & 2 & & & & & & & & & \\ & & & & & & 1 & 2 & & & & & & & & & \end{array}$$

Lemma 8.5. The algorithms in Definitions 8.1 and 8.3 are well defined. Moreover, U_ρ and V_ρ are fillings of $\mathcal{S}(\rho)$ with letters in $\{1, 2, \dots, \ell\}$ whose weights are $r(\rho) = (\rho_\ell, \rho_{\ell-1}, \dots, \rho_1)$ and $\rho = (\rho_1, \dots, \rho_\ell)$, respectively.

Proof. We only need to show that Step 2 in Definition 8.3 can be done in a unique way; other parts of the assertion are immediate from the definition.

Let d be the number of diagonal boxes in $\mathcal{S}(\rho - \rho_1)$ so that $h_1 > h_2 > \dots > h_d > 0$ and let $h_j = 0$ for $d < j \leq \ell$. For each $i \leq d$, let $(H_i)_a = \{(i, j) \mid j \geq i\} \subseteq H_i$ and $(H_i)_l = \{(j, i) \mid j > i\} \subseteq H_i$ be the *arm* and the *leg* of the hook H_i , respectively. Also let $(h_i)_a = |(H_i)_a|$ and $(h_i)_l = |(H_i)_l|$. Then $h_i = (h_i)_a + (h_i)_l$.

Let $O_i = \rho_i - h_i$ for all $i = 1, 2, \dots, \ell$. It is easy to see that $O_1 = \rho_1 - h_1 = 0$ and $\rho_i = (h_i)_a + \ell - 1$ for $i = 1, \dots, d$. Hence, $O_i = \ell - 1 - (h_i)_l$ for $i = 1, \dots, d$. Let $*$ be the first index such that $(h_*)_l = 0$. Then $*$ = d or $*$ = $d + 1$, and we obtain

$$0 = O_1 < O_2 < \dots < O_{*-1} < O_* = \ell - 1$$

since $(h_1)_l > (h_2)_l > \dots > (h_{*-1})_l > (h_*)_l = 0$. Note that if $*$ = $d + 1$, then $O_* = \rho_{d+1} - h_{d+1} = \rho_{d+1} = \ell - 1$.

We have $O_i = \rho_i$ for $i > *$, hence $O_{*+1} > \dots > O_\ell > 0$. Moreover, by the definition of $*$, there is no diagonal entry in the $(* + 1)$ st row of $\mathcal{S}(\rho/\rho_1)$ and $O_{*+1} = \rho_{*+1} \leq (\ell - 1 - *) + (* - 1) = \ell - 2$. Therefore, we have the following relation:

$$0 = O_1 < O_2 < \dots < O_{*-1} < O_* = \ell - 1 > O_{*+1} > \dots > O_\ell > 0.$$

We claim that $O_{*+i} \neq O_j$ for any $1 \leq i \leq \ell - *$ and $1 \leq j \leq * - 1$; that is, $\rho_{*+i} + (h_j)_l \neq \ell - 1$ for any i, j . There are two cases:

- i. the $(* + i)$ th row of ρ and $(H_j)_l$ has a nonempty intersection,
- ii. the $(* + i)$ th row of ρ and $(H_j)_l$ does not intersect.

Assume the first case. Then $\rho_{*+i} + (h_j)_l \geq (\ell - 1 - * - i) + j + (* + i - j) = \ell$, where $(\ell - 1 - * - i)$ is the number of boxes in the $(* + i)$ th row of $\mathcal{S}(\rho_1)$, j is the minimum number of boxes in the $(* + i)$ th row of $\mathcal{S}(\rho/\rho_1)$ and $(* + i - j)$ is the minimum number of boxes where $(h_j)_l$ counts in the j th column of $\mathcal{S}(\rho/\rho_1)$ under the given condition. We therefore showed that $\rho_{*+i} + (h_j)_l$ cannot be $\ell - 1$. When the second case occurs, with a similar reasoning to the first case, we can show that $\rho_{*+i} + (h_j)_l \leq (\ell - 1 - * - i) + j - 1 + (* + i - j - 1) = \ell - 2$ and $\rho_{*+i} + (h_j)_l$ can not be $\ell - 1$.

As a consequence of the claim we have $\{O_2, O_3, \dots, O_\ell\} = \{1, 2, \dots, \ell - 1\}$, and we finally prove that there is a unique way to fill in $\mathcal{S}(\rho_1)$ with O_i i 's for $i = 1, \dots, \ell$ so that each row is strictly decreasing and each column is weakly increasing. If we rotate $\mathcal{S}(\rho_1)$ by 90° counterclockwise, then we have an ordinary shape of the partition $(\ell - 1, \ell - 2, \dots, 1)$, and the filling of $\mathcal{S}(\rho_1)$ with strictly decreasing rows and weakly increasing columns corresponds to the filling of the ordinary shape with strictly increasing columns and weakly increasing rows. Due to the theory of classical Schur functions, which is symmetric, such a filling with $\sigma(i)$ i 's for $i = 1, \dots, \ell - 1$ and a permutation σ uniquely exists. This completes the proof. $\square$

It is not hard to check that U_ρ and V_ρ are SSDT's in $\mathcal{D}(\rho)$, and so we omit the proof. We finally can state the following.

Theorem 8.6. *For a strict partition ρ , U_ρ is the lowest weight SSDT and V_ρ is the highest weight SSDT in $\mathcal{D}(\rho)$.*

ACKNOWLEDGEMENT

The author would like to thank Dongho Moon and Eun-Kyoung Jung. D. Moon suggested the problem to find a way of describing the multiplicity $f_{\lambda\mu}^\nu$ in terms of words. The discussions with D. Moon and E. Jung were very helpful in developing the research at the first stage.

The author is grateful to Myungho Kim, Ji Hye Jung and Seok-Jin Kang for valuable discussions on the subject during the preparation of this paper. They (together with Dimitar Grantcharov and Masaki Kashiwara) were developing crystal basis theory for the quantum queer superalgebra $U_q(\mathfrak{q}(n))$ while the author was struggling to find a more natural definition of ℓ -essentialness. The idea of using the lowest weight tableau instead of the highest weight tableau was *essential* in carrying on the proofs in a nicer way. The author also owes special thanks to D. Grantcharov and M. Kashiwara.

The author thanks the Korean Institute for Advanced Study for providing support and an excellent research environment, where many important parts of the present work were carried out.

The author also thanks the anonymous referee for a careful reading and helpful comments.

REFERENCES

1. Edward A. Bender and Donald E. Knuth, *Enumeration of plane partitions*, J. Combinatorial Theory Ser. A **13** (1972), 40–54. MR0299574 (45:8622)
2. Anders Skovsted Buch, Andrew Kresch, and Harry Tamvakis, *Littlewood-Richardson rules for Grassmannians*, Adv. Math. **185** (2004), no. 1, 80–90. MR2058780 (2005e:05154)
3. D.R. Worley, *A theory of shifted young tableaux*, Ph.D. thesis, MIT, 1984.
4. Dimitar Grantcharov, Ji Hye Jung, Seok-Jin Kang, Masaki Kashiwara, and Myungho Kim, *Quantum queer superalgebra and crystal bases*, Proc. Japan Acad. Ser. A Math. Sci. **86** (2010), no. 10, 177–182. MR2752232 (2011m:17037)
5. D. Grantcharov, J. Jung, S.-J. Kang, M. Kashiwara, and M. Kim, *Crystal bases for the quantum queer superalgebra and semistandard decomposition tableaux*, arXiv:1103.1456.
6. Mark D. Haiman, *On mixed insertion, symmetry, and shifted Young tableaux*, J. Combin. Theory Ser. A **50** (1989), no. 2, 196–225. MR989194 (90j:05014)
7. V. N. Ivanov, *Interpolation analogues of Schur Q -functions*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **307** (2004), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 10, 99–119, 281–282. MR2050689 (2004m:05268)
8. Masaki Kashiwara, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73** (1994), no. 2, 383–413. MR1262212 (95c:17024)
9. Allen Knutson and Terence Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090. MR1671451 (2000c:20066)
10. Allen Knutson, Terence Tao, and Christopher Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48 (electronic). MR2015329 (2005f:14105)
11. Alain Lascoux and Marcel-P. Schützenberger, *Le monoïde plaxique*, Noncommutative structures in algebra and geometric combinatorics (Naples, 1978), Quad. “Ricerca Sci.”, vol. 109, CNR, Rome, 1981, pp. 129–156. MR646486 (83g:20016)
12. D. E. Littlewood and A. R. Richardson, *Group characters and algebra*, Phi. Trans. A (1934), 99–141.
13. I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky. MR1354144 (96h:05207)
14. Bruce E. Sagan, *Shifted tableaux, Schur Q -functions, and a conjecture of R. Stanley*, J. Combin. Theory Ser. A **45** (1987), no. 1, 62–103. MR883894 (88f:05011)
15. I. Schur, *Über die darstellung der symmetrischen und der alternierenden gruppe durch gebrochene lineare substitutionen*, J. Reine Angew. Math. **139** (1911), 155–250.
16. Luis Serrano, *The shifted plactic monoid*, Math. Z. **266** (2010), no. 2, 363–392. MR2678632 (2011i:05268)
17. John R. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), no. 1, 87–134. MR991411 (90k:20026)
18. ———, *On symmetric functions and the spin characters of S_n* , Topics in algebra, Part 2 (Warsaw, 1988), Banach Center Publ., vol. 26, PWN, Warsaw, 1990, pp. 433–453. MR1171291 (93e:20018)
19. ———, *A concise proof of the Littlewood-Richardson rule*, Electron. J. Combin. **9** (2002), no. 1, Note 5, 4 pp. (electronic). MR1912814 (2003e:05141)

DEPARTMENT OF MATHEMATICS, AJOU UNIVERSITY, SUWON 443-749, KOREA
E-mail address: chosj@ajou.ac.kr