ON THE L -FUNCTION OF MULTIPLICATIVE CHARACTER SUMS

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ABSTRACT. Let $\mathbb{F}_q$ be a finite field and let $\chi_1, \dots, \chi_r$ be multiplicative characters on $\mathbb{F}_q$. Suppose there are homogeneous polynomials $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$ in $\mathbb{F}_q[x_1, \dots, x_n]$ and suppose that $f_1, \dots, f_r$ define smooth hypersurfaces in $\mathbb{P}^{n-1}$ that have normal crossings. When the character sum $S = \sum_{x \in \mathbb{P}^{n-1}(\mathbb{F}_q)} \chi_1(f_1(x)) \dots \chi_r(f_r(x))$ is well defined, we compute the p -adic Dwork cohomology of the L -function associated to S . In particular, we give a lower bound for the p -adic Newton polygon of the L -function and give a formula for the Hilbert series of the non-vanishing cohomology in terms of n, r, q and the d_j 's.

1. INTRODUCTION

Suppose $f_1, \dots, f_r$ are homogeneous polynomials in n variables with coefficients in $\mathbb{F}_q$. For each $j = 1, \dots, r$, let $\deg f_j = d_j$ and let $\chi_j : \mathbb{F}_q^\times \rightarrow \mathbb{C}_p^\times$ be a multiplicative character on $\mathbb{F}_q$ such that $\prod_{j=1}^r \chi_j^{d_j} = \chi_0$, the trivial character. Define the character sum

$$(1-1) \quad S(f, \chi) = S(f_1, \dots, f_r, \chi_1, \dots, \chi_r) = \sum_{x \in \mathbb{P}^{n-1}(\mathbb{F}_q)} \chi_1(f_1(x)) \dots \chi_r(f_r(x)).$$

For each integer $m \geq 2$, we can define multiplicative characters $\chi_1^{(m)}, \dots, \chi_r^{(m)}$ on $\mathbb{F}_{q^m}$, by composition of the characters χ_j with the norm map from $\mathbb{F}_{q^m}$ to $\mathbb{F}_q$, to get the associated sum

$$(1-2) \quad S_m(f, \chi) = \sum_{x \in \mathbb{P}^{n-1}(\mathbb{F}_{q^m})} \chi_1^{(m)}(f_1(x)) \dots \chi_r^{(m)}(f_r(x)).$$

These character sums generate the L -function

$$(1-3) \quad L(f, \chi, \mathbb{P}^{n-1}; t) = \exp \left(\sum_{m=1}^{\infty} S_m(f, \chi) t^m / m \right).$$

By a theorem of Katz [11], if $(p, d_j) = 1$ for each j and $\prod_1^r f_j = 0$ defines a normal crossing divisor (each f_j defines a smooth hypersurface in $\mathbb{P}^{n-1}$ and the normal vectors are linearly independent at any point where subsets of $\{f_1, \dots, f_r\}$ intersect), then $L(f, \chi, \mathbb{P}^{n-1}; t)^{(-1)^n}$ is a polynomial.

Received by the editors January 23, 2009 and, in revised form, August 28, 2011.
 2010 *Mathematics Subject Classification.* Primary 11L40, 14F30.

Key words and phrases. Character sums, L -function, p -adic cohomology, Newton polygon.

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We also define the associated character sums in affine space

$$(1-4) \quad S_m^*(f, \chi) = \sum_{x \in \mathbb{A}^n(\mathbb{F}_{q^m})} \chi_1^{(m)}(f_1(x)) \cdots \chi_r^{(m)}(f_r(x))$$

for all $m \geq 1$, with the relation $S_m^*(f, \chi) = (q^m - 1) \cdot S_m(f, \chi)$. The L -functions are then related by the equation

$$L(f, \chi, \mathbb{A}^n; t) = \frac{L(f, \chi, \mathbb{P}^{n-1}; qt)}{L(f, \chi, \mathbb{P}^{n-1}; t)}.$$

Take an additive character ψ and let χ be a non-trivial multiplicative character. The Gauss sum associated to ψ and χ is defined by $g(\chi, \psi) = \sum_{a \in \mathbb{F}_q} \chi(a) \psi(a)$. If $\chi_j \neq \chi_0$ for all $j = 1, \dots, r$, then

$$(1-5) \quad \sum_{x \in \mathbb{F}_q^n} \chi_1(f_1(x)) \cdots \chi_r(f_r(x)) \\ = \left(\prod_{j=1}^r g(\chi_j^{-1}, \psi) \right)^{-1} \cdot \sum_{(x, y) \in \mathbb{F}_q^{n+r}} \chi_1^{-1}(y_1) \cdots \chi_r^{-1}(y_r) \cdot \psi(y_1 f_1(x) + \cdots + y_r f_r(x)),$$

where $y_1, \dots, y_r \in \mathbb{F}_q^\times$. Therefore, up to the product of Gauss sums, it is equivalent to study sums of the form

$$\sum_{(x, y) \in \mathbb{A}^{n+r}(\mathbb{F}_{q^m})} \chi_1^{(m)}(y_1) \cdots \chi_r^{(m)}(y_r) \cdot \psi^{(m)}(y_1 f_1(x) + \cdots + y_r f_r(x)),$$

where $\psi^{(m)} = \psi \circ \text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$. Accordingly, we define $T_m^*(f, \chi)$ as the sum

$$(1-6) \quad \sum_{(x, y) \in \mathbb{A}^{n+r}(\mathbb{F}_q^m)} \left[(\chi_1^{-1})^{(m)}(y_1) \cdots (\chi_r^{-1})^{(m)}(y_r) \right] \cdot \psi^{(m)} \left(\sum_{j=1}^r y_j f_j(x) \right)$$

and its associated L -function $L(f, \chi, \psi, T^*; t) = \exp(\sum_{m=1}^{\infty} T_m^*(f, \chi) t^m / m)$.

In Chapter 2 we use Dwork's theory to relate this L -function to the characteristic polynomial of a Frobenius operator acting on a certain complex of differential forms. The Frobenius operator induces an action on the cohomology of the complex and in Chapter 3 we compute this cohomology (see Theorem 3.1).

In Chapter 4 we compute the Hilbert series associated to the non-vanishing cohomology space (equation (4-15)), and also generate a formula for the degree of the above L -function (see (4-19)). The dimensions that occur as coefficients of the Hilbert series are used in Chapter 5 to estimate a lower bound for the p -adic Newton polygon of the polynomial $L(f, \chi, \mathbb{P}^{n-1}; t)$ (see Theorem 5.1).

Examples are contained in Chapter 6 for the values $n = 1, 2$ and 3 .

A brief history of the study of the sums of type (1-2) is given in [2]. Character sums of type (1-6) are studied in [4]. In certain cases these sums can be used to count points on an associated variety and are therefore used to study the zeta function. For example, using the normal crossing divisor in $\mathbb{P}^{n-1}$ defined by $f_1(x, z) = x$, $f_2(x, z) = x - z$, $f_3(x, z) = x - \lambda z$ and $f_4(x, z) = z$ we can analyze the elliptic curve defined by the equation $y^2 = (x)(x-1)(x-\lambda)$ for some $\lambda \neq 0, 1$ in $\mathbb{F}_q$, as the number of $\mathbb{F}_{q^m}$ -rational points on this curve is $N_m = q^m + \sum_{x \in \mathbb{F}_{q^m}} \chi^{(m)}(x(x-1)(x-\lambda))$,

where $\chi^{(m)}$ is the quadratic character on $\mathbb{F}_{q^m}$. We note that in this particular case the cohomology can be computed explicitly and agrees with the formulas of Chapters 4 and 5.

2. THE DWORK COMPLEX

Let p be a prime number and let $q = p^a$ for any positive integer a . Denote the field of characteristic p and cardinality q by $\mathbb{F}_q$. The multiplicative norm map from $\mathbb{F}_q$ to $\mathbb{F}_p$ will be denoted by $N_{\mathbb{F}_q/\mathbb{F}_p}$, and the additive trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$ is denoted by $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$. We let $\mathbb{Q}_p$ denote the field of p -adic numbers, with ring of integers $\mathbb{Z}_p$. The p -adic absolute value on $\mathbb{Q}_p$ will be denoted by $|\cdot|_p$ and the p -ordinal will be denoted by ord_p .

Define $K = \mathbb{Q}_p(\zeta_{q-1})$, the unramified extension of $\mathbb{Q}_p$ of degree a with residue field $\mathbb{F}_q$. Let ζ_p denote a p th root of unity and define $\Gamma_1 = \mathbb{Q}_p(\zeta_p)$, a totally ramified extension of $\mathbb{Q}_p$ of degree $p-1$. If we take π satisfying $\pi^{p-1} = -p$, then π generates the maximal ideal in $\mathcal{O}_{\Gamma_1}$ with residue field $\mathcal{O}_{\Gamma_1}/(\pi)\mathcal{O}_{\Gamma_1} = \mathbb{F}_p$. We let $\Gamma_0 = K(\pi)$. For extensions using fractional powers of π , we let b be a positive rational number and write $\frac{(p-1)b}{q-1} = \frac{r_1}{r_2}$. Denote $\tilde{\pi} = \pi^{1/r_2}$ and define the extensions $\tilde{\Gamma}_1 = \Gamma_1(\tilde{\pi})$ and $\tilde{\Gamma}_0 = \Gamma_0(\tilde{\pi})$.

We denote the Teichmüller character on $\mathbb{F}_q$ by $\hat{\chi}$. The Teichmüller representative of the element of $\mathbb{F}_q^\times$ with least positive residue i will be denoted $\text{Teich}(i)$ or $\hat{i}$. On the other hand, the reduction of an element $x \in \mathbb{Q}_p(\zeta_{p-1})^\times$ to the corresponding element of $\mathbb{F}_q^\times$ will be denoted $\bar{x}$. For a multiplicative character χ on $\mathbb{F}_q$ we can get a character on $\mathbb{F}_{q^m}$ by composing χ with the norm map from $\mathbb{F}_{q^m}$ to $\mathbb{F}_q$. In this case we will denote such a character by $\chi^{(m)}$. Similarly, for each $m > 1$ we can get an additive character $\psi^{(m)}$ on $\mathbb{F}_{q^m}$ by composing the additive trace map $\text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ with an additive character ψ on $\mathbb{F}_q$.

Consider the following infinite dimensional spaces of power series in several variables. We abbreviate a monomial in $n + r$ variables as

$$A(u, v)x^u y^v = A(u_1, \dots, u_n, v_1, \dots, v_r)x_1^{u_1} \dots x_n^{u_n} y_1^{v_1} \dots y_r^{v_r},$$

where $A(u, v) \in \tilde{\Gamma}_0$. Let $|v| = \sum_{j=1}^r v_j$. For a positive rational number b and a positive integer d define

$$C(b, d) = \left\{ \sum_{u, v} A(u, v) \pi^{(p-1)b|v|/d} x^u y^{v/d} : |A(u, v)|_p \rightarrow 0 \text{ as } u, v \rightarrow \infty \right\}.$$

This space is complete with respect to the norm $\sup_{u, v} |A(u, v)|_p$ and we take as an orthonormal basis $\{\pi^{(p-1)b|v|/d} x^u y^{v/d}\}$.

We will actually work in some subspaces of $C(b, d)$ given by the following direct product decomposition. Let $\delta = (\delta_1, \dots, \delta_r)$, with $0 \leq \delta_j < d$. For each such δ we define

$$(2-1) \quad C(b, d, \delta) = \left\{ \sum_{u, v} A(u, v) \pi^{(p-1)b|v|/d} x^u y^{v/d} : v_j \equiv \delta_j \pmod{d} \right\}.$$

It follows that $C(b, d) = \bigoplus_{\delta} C(b, d, \delta)$, a product of d^r subspaces. The r -tuple δ will be associated to the original multiplicative characters in the following way. If $\hat{\chi}(z) = \hat{z}$, then we define δ_j using the relation $\chi_j = \hat{\chi}^{\delta_j}$, for each $j = 1, \dots, r$.

Using the splitting functions of Dwork, we use certain power series to construct a lifting of characters from characteristic p to characteristic zero. The Artin-Hasse exponential function is given by the series $E(t) = \exp\left(\sum_{j=0}^{\infty} \frac{t^{p^j}}{p^j}\right)$ and converges for all $|t|_p < 1$. It has $p-1$ zeros with p -ordinal $\frac{1}{p-1}$. We choose one and call it γ . Now define the series $\Theta(t) = E(\gamma t)$. This is a splitting function as described by Dwork [8]. Writing as a power series $\Theta(t) = \sum_{i=0}^{\infty} \lambda_i t^i$, it can be shown that $\lambda_0 = 1$ and $\gamma^i | \lambda_i$ for each i , so that $\text{ord}_p(\lambda_i) \geq \frac{i}{(p-1)}$. Since $\Theta(1)^p = 1$ and since $\Theta(1) \equiv 1 + \gamma \pmod{\gamma^2}$ in $\mathcal{O}_{\Gamma_1}$ it follows that $\Theta(1)$ is a primitive p th root of unity.

Writing $f_j = \sum_{u \in \mathcal{U}_j} a_{u,j} x^u$ we define the series

$$(2-2) \quad G(x, y) = \prod_{\substack{u \in \mathcal{U}_j \\ 1 \leq j \leq r}} \Theta(\hat{a}_{u,j} x^u y_j) \quad \text{and} \quad H(x, y) = \prod_{k=0}^{a-1} G(x^{p^k}, y^{p^k}).$$

It can be shown that $G \in C(b, 1)$ for all $b < 1/(p-1)$ and $H \in C(b, 1)$ for all $b < p/(q(p-1))$. Also multiplication by H is stable on $C(b/q, d, \delta)$ for each δ and it is bounded on bounded subsets of $C(b, d)$, hence continuous.

Now consider the inclusion map $C(pb, d) \rightarrow C(b, d)$. The associated diagonal matrix has entries $(\pi^{pb-b}, \pi^{2pb-2b}, \pi^{3pb-3b}, \dots)$ and is therefore completely continuous in the sense of [17].

If δ is the least non-negative residue of $p\delta'$ modulo d , then for any element $\xi = \sum_{u,v} A(u, v) x^u y^{v/d}$ in $C(b, d, \delta)$ we define the map $\Psi : C(b, d, \delta) \rightarrow C(pb, d, \delta')$ by the formula

$$(2-3) \quad \Psi(\xi) = \sum_{u,v} A(pu, pv) x^u y^{v/d}.$$

We now define the Frobenius morphism $\alpha = \Psi^a \circ H$ as the composition

$$(2-4) \quad C(b, d, \delta) \xrightarrow{i} C(b/q, d, \delta) \xrightarrow{H} C(b/q, d, \delta) \xrightarrow{\Psi^a} C(b, d, \delta),$$

where the map on the left is inclusion and the map in the center is multiplication by H . From the definition of Θ and since $G(x, y) \circ \Psi = \Psi \circ G(x^p, y^p)$, if $d|(q-1)$ it follows that $(\Psi \circ G)^a = \Psi^a \circ H$. Since Ψ is also continuous on $C(b, d)$, the composition $\alpha = \Psi^a \circ H$ is a completely continuous endomorphism of $C(b, d, \delta)$ for all $b < p/(p-1)$.

We define the related operator β , an a th root of α , for which p -adic information on the coefficients is much easier to obtain. Let $\tau \in \text{Gal}(\Gamma_0/\Gamma_1)$ such that $\tau(\pi) = \pi$ and let $\beta = \Psi \circ \tau^{-1} \circ G$, which satisfies $\beta^a = \alpha$.

The Dwork trace formula gives an equality between the trace of α and twisted exponential sums.

Theorem 2.1 (Dwork Trace Formula). *For any $m \geq 1$,*

$$(2-5) \quad \begin{aligned} & (q^m - 1)^{n+r} \text{Tr}(\alpha^m | C(b, d, \delta)) \\ &= \sum_{(x,y) \in \mathbb{F}_{q^m}^{n+r}} (\hat{\chi}^{(m)}(y_1))^{-\frac{(q-1)\delta_1}{d}} \dots (\hat{\chi}^{(m)}(y_r))^{-\frac{(q-1)\delta_r}{d}} \cdot H(\hat{x}, \hat{y}). \end{aligned}$$

Proof. See [1]. □

Moreover, if $d = q-1$, then (2-5) is equal to the sum $T_m^*(f, \chi)$ in (1-6).

From [17], the classification of the operator α as completely continuous allows the application of

$$\det(I - tM) = \exp \left(- \sum_{j=1}^{\infty} \text{Tr}(M^j) t^j / j \right).$$

This determinant is a p -adic entire function.

We can analyze α further by extending its action onto a de Rham-type complex of differential forms with the space $C(b, d, \delta)$ as coefficients. Let

(2-6)

$$\Omega_{C(b, d, \delta)}^k \langle Y \rangle = \bigoplus_{\ell+m=k} \bigoplus_{\substack{1 \leq i_1 < \dots < i_\ell \leq n \\ 1 \leq j_1 < \dots < j_m \leq r}} C(b, d, \delta) dx_{i_1} \wedge \dots \wedge dx_{i_\ell} \wedge \frac{dy_{j_1}}{y_{j_1}} \wedge \dots \wedge \frac{dy_{j_m}}{y_{j_m}}.$$

For some $g \in C(b, d, \delta)$ define $d_{\langle Y \rangle} : \Omega_{C(b, d, \delta)}^k \langle Y \rangle \longrightarrow \Omega_{C(b, d, \delta)}^{k+1} \langle Y \rangle$ by (dropping $\wedge$ from the notation)

$$(2-7) \quad d_{\langle Y \rangle} (g dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}) \\ = \left(\sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i + \sum_{j=1}^r y_j \frac{\partial g}{\partial y_j} \frac{dy_j}{y_j} \right) dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}.$$

Using $d_{\langle Y \rangle}$ we now define a homomorphism $\Delta : \Omega_{C(b, d, \delta)}^k \langle Y \rangle \longrightarrow \Omega_{C(b, d, \delta)}^{k+1} \langle Y \rangle$ as follows. For an integer $s > 0$ let $\gamma_s = \sum_{i=0}^s \gamma^{p^i} / p^i$ and for $\hat{F}$, the Teichmüller lifting of $F = \sum_{j=1}^r y_j f_j(x)$, we set

$$(2-8) \quad \tilde{F}(x, y) = \sum_{s=0}^{\infty} \gamma_s \hat{F}^{\tau^s} (x^{p^s}, y^{p^s}) \\ = \sum_{s=0}^{\infty} \gamma_s \sum_{j=1}^r y_j^{p^s} \hat{f}_j^{\tau^s} (x^{p^s}),$$

which is an element of $C(b, 1)$. It follows that $d_{\langle Y \rangle}(\tilde{F}) \in \Omega_{C(b, 1)}^1 \langle Y \rangle$. Now let $\Delta : \Omega_{C(b, d, \delta)}^k \langle Y \rangle \longrightarrow \Omega_{C(b, d, \delta)}^{k+1} \langle Y \rangle$ be given by

$$(2-9) \quad \Delta(\omega) = d_{\langle Y \rangle}(\omega) + d_{\langle Y \rangle}(\tilde{F}) \wedge \omega,$$

which yields the complex

$$(2-10) \quad (\Omega_{C(b, d, \delta)}^\bullet \langle Y \rangle, \Delta).$$

Let $g \in C(b, d, \delta)$. Extending the Frobenius morphism α we can define a map of complexes $\alpha_\bullet : \Omega_{C(b, d, \delta)}^\bullet \langle Y \rangle \longrightarrow \Omega_{C(b, d, \delta)}^\bullet \langle Y \rangle$ by linearity on $C(b, d, \delta)$ and the formula

$$(2-11) \quad \alpha_k (g dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}) \\ = \frac{q^{n+r-k}}{x_{i_1} \dots x_{i_\ell}} \alpha(x_{i_1} \dots x_{i_\ell} g) dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}.$$

A straightforward computation shows that $\Delta \circ \alpha = \alpha \circ \Delta$ so that $\alpha_\bullet$ is a chain map on the complex (2-10). Also note that β induces a similar map of complexes given by

$$(2-12) \quad \begin{aligned} \beta_k(gdx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}) \\ = \frac{p^{n+r-k}}{x_{i_1} \dots x_{i_\ell}} \beta(x_{i_1} \dots x_{i_\ell} g) dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}. \end{aligned}$$

Now since α is a chain map, it will induce a map on the cohomology of the complex (2-10). Moreover from [17], since each morphism α_k is completely continuous we have the relation

$$(2-13) \quad \sum_{k=0}^{n+r} (-1)^k \text{Tr}(\alpha_k | \Omega_{C(b,d,\delta)}^k \langle Y \rangle) = \sum_{k=0}^{n+r} (-1)^k \text{Tr}(\alpha_k | H^k(\Omega_{C(b,d,\delta)}^\bullet \langle Y \rangle)).$$

Combining the above with Theorem 2.1 we can apply the following version of the trace formula due to Robba [15]:

$$(2-14) \quad L(f, \chi, \psi, T^*; t) = \prod_{k=0}^{n+r} \det \left(I - t\alpha_k | H^k(\Omega_{C(b,d,\delta)}^\bullet \langle Y \rangle, \Delta) \right)^{(-1)^{k+1}}.$$

3. COMPUTATION OF COHOMOLOGY

Before analyzing the action of α and the determinants in (2-14), we compute the cohomology of the Dwork complex. In this chapter we prove the following.

Theorem 3.1. *Suppose $f_1, \dots, f_r$ define a divisor with normal crossings with degrees satisfying $(p, d_j) = 1$. Then $H^k(\Omega_{C(b,d,\delta)}^\bullet \langle Y \rangle, \Delta) = 0$ for $0 \leq k \leq n+r-2$ and $H^{n+r-1}(\Omega_{C(b,d,\delta)}^\bullet \langle Y \rangle, \Delta) \cong H^{n+r}(\Omega_{C(b,d,\delta)}^\bullet \langle Y \rangle, \Delta)$.*

3.1. Reduction to characteristic p . We first simplify the computation, with a reduction, using the fact that the operator α is built from homogeneous elements. Define a grading on $\tilde{\Gamma}_0[x, y]$ by

$$(3-1) \quad \deg_1 x_i = 1, \text{ for } i = 1, \dots, n \text{ and } \deg_1 y_j = -d_j, \text{ for } j = 1, \dots, r.$$

Let $\hat{F}$ be the Teichmüller lifting of $F(x, y) = \sum_{j=1}^r y_j f_j(x)$, which will be a polynomial in $\mathcal{O}_{\Gamma_0}[x, y]$. We can see that all of the monomials in $\hat{F}$ satisfy $\deg_1 = 0$. Therefore G and H defined in (2-2), satisfy $\deg_1 = 0$. Denote the subspace of $C(b, d, \delta)$ consisting of elements having $\deg_1 = 0$ by $C(b, d, \delta)^{(0)}$. It can be shown that for $d = q-1$ and for each acceptable choice of b and δ , that $\text{Tr}(\alpha | C(b, q-1, \delta)) = \text{Tr}(\alpha | C(b, q-1, \delta)^{(0)})$.

By defining

$$(3-2) \quad \deg_1 dx_i = 1 \quad \text{and} \quad \deg_1 dy_j = -d_j,$$

we can extend this grading to $\Omega_{C(b,q-1,\delta)}^\bullet \langle Y \rangle$, where we denote the subspace of forms satisfying $\deg_1 = 0$ by $\Omega_{C(b,q-1,\delta)}^\bullet \langle Y \rangle^{(0)}$. We have $\alpha_k(\Omega_{C(b,q-1,\delta)}^k \langle Y \rangle^{(0)}) \subseteq \Omega_{C(b,q-1,\delta)}^k \langle Y \rangle^{(0)}$ and moreover $\Delta(\Omega_{C(b,q-1,\delta)}^k \langle Y \rangle^{(0)}) \subseteq \Omega_{C(b,q-1,\delta)}^{k+1} \langle Y \rangle^{(0)}$, so that $\Omega_{C(b,q-1,\delta)}^\bullet \langle Y \rangle^{(0)}$ is a subcomplex of $\Omega_{C(b,q-1,\delta)}^\bullet \langle Y \rangle$ satisfying the condition $\det(I - t\alpha_k | \Omega_{C(b,q-1,\delta)}^k \langle Y \rangle / \Omega_{C(b,q-1,\delta)}^k \langle Y \rangle^{(0)}) = 1$. Therefore we can reduce the computation of cohomology to that of the subcomplex $(\Omega_{C(b,q-1,\delta)}^\bullet \langle Y \rangle^{(0)}, \Delta)$.

The space $C(b, q-1, \delta)$ has a natural decreasing filtration by $s \in \mathbb{Z}$ given by coefficients in the principal ideals $\tilde{\pi}^s \mathcal{O}_{\tilde{\Gamma}_0}$. Let

(3-3)

$$\mathcal{F}^s C(b, q-1, \delta) = \left\{ \sum_{u,v} A(u, v) \pi^{(p-1)b|v|/(q-1)} x^u y^{v/(q-1)} \mid A(u, v) \in \tilde{\pi}^s \mathcal{O}_{\tilde{\Gamma}_0} \text{ for all } (u, v) \right\}.$$

We can extend this filtration to the Dwork complex by defining

$$(3-4) \quad \mathcal{F}^s \Omega_{C(b, q-1, \delta)}^k \langle Y \rangle = \bigoplus_{\ell+m=k} \bigoplus_{\substack{1 \leq i_1 < \dots < i_\ell \leq n \\ 1 \leq j_1 < \dots < j_m \leq r}} \mathcal{F}^s C(b, q-1, \delta) dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}.$$

Let $\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^\bullet \langle Y \rangle^{(0)}$ denote the algebra of differential forms (logarithmic in the y 's) with coefficients in the polynomial ring $\mathbb{F}_q[x, y]$ and with all terms having $\deg_1 = 0$. If $\bar{A}(u, v)$ is the reduction of $A(u, v)$ modulo the ideal $\tilde{\pi} \mathcal{O}_{\tilde{\Gamma}_0}$, we can define a homomorphism of modules from $\mathcal{F}^0 \Omega_{C(b, q-1, \delta)}^k \langle Y \rangle^{(0)}$ to $\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^k \langle Y \rangle^{(0)}$ by

$$(3-5) \quad \begin{aligned} & \sum_{u,v} A(u, v) \pi^{(p-1)b|v|/(q-1)} x^u y^{v/(q-1)} dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}} \\ & \mapsto \sum_{u,v} \bar{A}(u, v) x^u y^{v/(q-1)} dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}. \end{aligned}$$

The kernel of this homomorphism is $\mathcal{F}^1 \Omega_{C(b, q-1, \delta)}^k \langle Y \rangle^{(0)}$ and therefore we have an isomorphism

$$(3-6) \quad \mathcal{F}^0 \Omega_{C(b, q-1, \delta)}^k \langle Y \rangle^{(0)} / \mathcal{F}^1 \Omega_{C(b, q-1, \delta)}^k \langle Y \rangle^{(0)} \cong \Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^k \langle Y \rangle^{(0)}.$$

Under this reduction, $\pi^{(p-1)b\gamma^{-1}} \Delta \equiv d_{\langle Y \rangle} F \wedge \pmod{\mathcal{F}^1 \Omega_{C(b, q-1, \delta)}^k \langle Y \rangle^{(0)}}$, so that we have an isomorphism of complexes:

$$(3-7) \quad \begin{aligned} & \left(\mathcal{F}^0 \Omega_{C(b, q-1, \delta)}^k \langle Y \rangle^{(0)} / \mathcal{F}^1 \Omega_{C(b, q-1, \delta)}^k \langle Y \rangle^{(0)}, \pi^{(p-1)b\gamma^{-1}} \Delta \right) \\ & \cong \left(\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^k \langle Y \rangle^{(0)}, d_{\langle Y \rangle} (F) \wedge \right). \end{aligned}$$

It will suffice to compute the cohomology of this associated graded complex (see Section 3.5).

Now, since each v_j satisfies $v_j \equiv \delta_j \pmod{q-1}$, we can work with integer powers of the y 's if we remove a factor of $y_1^{\delta_1/(q-1)} \dots y_r^{\delta_r/(q-1)}$ and work over the space $\mathbb{F}_q[x, y]^{(D)}$, where $D = \frac{1}{q-1} \sum_{j=1}^r d_j \delta_j \in \mathbb{N}$. For the rest of this chapter we will remove this factor and work on the corresponding spaces $\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^k \langle Y \rangle^{(D)}$ without affecting the action of α on cohomology.

We write (3-7), adjusted for D , as a double complex

$$\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^k \langle Y \rangle^{(D)} = \bigoplus_{\ell+m=k} \left(\Omega^{(\ell, m)} \right)^{(D)},$$

where

$$(3-8) \quad (\Omega^{(\ell,m)})^{(D)} = \bigoplus_{\substack{1 \leq i_1 < \dots < i_\ell \leq n \\ 1 \leq j_1 < \dots < j_m \leq r}} \mathbb{F}_q[x, y]^{(D-\ell)} dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}.$$

In this double complex there are $n+1$ columns indexed by ℓ and $r+1$ rows indexed by m and the original complex is recovered by summing down the diagonals. The operator $d_{\langle Y \rangle}(F)$ can be divided into components $\partial_v = \sum_{j=1}^r f_j dy_j = \sum_{j=1}^r y_j f_j dy_j / y_j$ and $\partial_h = \sum_{j=1}^r y_j df_j$, which satisfy the identity $\partial_v \partial_h + \partial_h \partial_v = 0$.

We first consider the cohomology of each of the columns.

Proposition 3.1. *For each $\ell = 0, \dots, n$, we have $H^k((\Omega^{(\ell, \bullet)})^{(D)}, \partial_v) = 0$ for $k < r$.*

Proof. To verify the elements $\{y_j f_j\}_{j=1}^r$ form a regular sequence on $\mathbb{F}_q[x, y]^{(D-\ell)}$, we check that the following multiplication is injective for each $k = 1, \dots, r$:

$$(3-9) \quad \mathbb{F}_q[x, y]^{(D-\ell)} / \sum_{j=1}^{k-1} y_j f_j \mathbb{F}_q[x, y]^{(D-\ell)} \xrightarrow{y_k f_k} \mathbb{F}_q[x, y]^{(D-\ell)} / \sum_{j=1}^{k-1} y_j f_j \mathbb{F}_q[x, y]^{(D-\ell)}.$$

Separating out the powers of the y_j 's we can write

$$\mathbb{F}_q[x, y]^{(D-\ell)} / \sum_{j=1}^{k-1} y_j f_j \mathbb{F}_q[x, y]^{(D-\ell)} = \bigoplus_{v_1, \dots, v_r \geq 0} y_1^{v_1} \dots y_r^{v_r} [\mathbb{F}_q[x] / I_v]^{(D-\ell + \sum v_j d_j)},$$

where $I_v = I(v_1, \dots, v_r)$ is the ideal of $\mathbb{F}_q[x]$ generated by the f_i for which $1 \leq i \leq k-1$ and v_i is strictly greater than zero.

Suppose first that the set $\{i : 1 \leq i \leq k-1 \text{ and } v_i > 0\}$ has cardinality $< n$. It follows that $\{f_i : 1 \leq i \leq k-1 \text{ and } v_i > 0\} \cup \{f_k\}$ form a regular sequence on $\mathbb{F}_q[x]$ by the normal crossing divisor hypothesis on $f_1, \dots, f_r$. Therefore the multiplication in (3-9) is injective.

On the other hand suppose the set $\{i : 1 \leq i \leq k-1 \text{ and } v_i > 0\}$ has cardinality $\geq n$. Without loss of generality, we can assume that $\{1, \dots, n\}$ is in this set. The normal crossing divisor hypothesis on $f_1, \dots, f_r$ implies that $f_1, \dots, f_n$ form a regular sequence in $\mathbb{F}_q[x_1, \dots, x_n]$. Since the Hilbert series for the quotient space $\mathbb{F}_q[x] / (f_1, \dots, f_n)$ is given by $\prod_{i=1}^n (1 + t + t^2 + \dots + t^{d_i-1})$, which is a polynomial of degree $d_1 + \dots + d_n - n$, it follows that in dimensions $k > \sum_{j=1}^n d_j - n$, the quotient $[\mathbb{F}_q[x] / (f_1, \dots, f_n)]^{(k)}$ is zero. Therefore the multiplication is trivially injective. $\square$

3.2. Vanishing theorems and division lemmas for differential forms. In this section we compile various results from commutative algebra involving the vanishing of differential forms which are used in the computation of the cohomology along the top horizontal row of the double complex $(\Omega^{(\bullet, \bullet)})^{(D)}$.

For fixed elements $\omega_1, \dots, \omega_s \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^1$ write

$$\omega_1 \wedge \dots \wedge \omega_s = \sum_{1 \leq i_1 < \dots < i_s \leq n} g_{i_1, \dots, i_s} dx_{i_1} \wedge \dots \wedge dx_{i_s},$$

and let I denote the ideal of $\mathbb{F}_q[x]$ generated by the $g_{i_1, \dots, i_s}$.

We first recall the Saito-de Rham division lemma.

Proposition 3.2. *Suppose $\omega \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k$ such that $\omega_1 \wedge \dots \wedge \omega_s \wedge \omega = 0$.*

(a) There exists an integer $M \geq 0$ such that if $g \in I^M$, then

$$g\omega = \sum_{j=1}^s \omega_j \wedge \varphi_j$$

for some $\varphi_1, \dots, \varphi_s \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{k-1}$.

(b) If $k < \text{depth}(I)$, then there exist $\varphi_1, \dots, \varphi_s \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{k-1}$ such that

$$\omega = \sum_{j=1}^s \omega_j \wedge \varphi_j.$$

Proof. See the main theorem of [16]. $\square$

The next result is a generalization of Proposition 3.2 due to Adolphson and Sperber.

Proposition 3.3. Let $\omega \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k$ satisfy $\omega_j \wedge \omega = 0$ for each $j = 1, \dots, s$.

(a) There exists an integer $M \geq 0$ with the property that if $g \in I^M$, then

$$g\omega = \omega_1 \wedge \dots \wedge \omega_s \wedge \varphi$$

for some $\varphi \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{k-s}$.

(b) If $\text{depth}(I) > 0$ and $k < \text{depth}(I) + s - 1$, then there exists $\varphi \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{k-s}$ such that

$$\omega = \omega_1 \wedge \dots \wedge \omega_s \wedge \varphi.$$

Proof. See [5], Proposition 2.2. $\square$

Now consider the spaces

$$\begin{aligned} \tilde{\Omega}^k &= \mathbb{F}_q[y_1, \dots, y_s] \otimes \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k \\ &= \bigoplus_{v_1, \dots, v_s \geq 0} y_1^{v_1} \dots y_s^{v_s} \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k \end{aligned}$$

and the subspaces

$$(\tilde{\Omega}^k)^{(J)} = \bigoplus_{v_1 + \dots + v_s = J} y_1^{v_1} \dots y_s^{v_s} \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k.$$

In [5], Proposition 3.2 and Proposition 3.3 are used to deduce the following:

Proposition 3.4. For $\omega_1, \dots, \omega_s$ and I as before and $\partial(\omega) = (y_1\omega_1 + \dots + y_s\omega_s) \wedge \omega$:

(a) For each $J \geq 1$ and $k \geq 0$, there exists $M \geq 0$ such that

$$I^M H^k((\tilde{\Omega}^\bullet)^{(J)}, \partial) = 0.$$

(b) If $1 \leq J < \text{depth}(I)$ and $k < \text{depth}(I) + s - 1$, then

$$H^k((\tilde{\Omega}^\bullet)^{(J)}, \partial) = 0.$$

Proof. See [5], Proposition 2.10. $\square$

The last few results involve the vanishing of logarithmic forms.

Proposition 3.5. *Suppose that $h_1, \dots, h_s$ are homogeneous polynomials in $\mathbb{F}_q[x] = \mathbb{F}_q[x_1, \dots, x_n]$ that define a divisor with normal crossings. Let t be an integer such that $t + k \leq n - 1$ and let $\eta \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k$ such that*

$$dh_{j_1} \wedge \cdots \wedge dh_{j_t} \wedge \eta \equiv 0 \pmod{h_{j_1}, \dots, h_{j_t}}.$$

Then

$$\eta \equiv 0 \pmod{dh_{j_1}, \dots, dh_{j_t}, h_{j_1}, \dots, h_{j_t}}.$$

Proof. See [12], Lemma 2.2.2. □

Proposition 3.5 is the main tool used to prove the following vanishing theorem.

Proposition 3.6. *Suppose $h_1, \dots, h_s$ are non-constant, homogeneous polynomials defining a normal crossing divisor. Let $0 \leq k \leq n - 2$ and suppose $\eta \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k$ such that $dh_j \wedge \eta \equiv 0 \pmod{h_j}$ for each $j = 1, \dots, s$. Then η can be written as*

$$\eta = h_1 \cdots h_s \left\{ \eta_0 + \sum_{j=1}^s \frac{dh_j}{h_j} \wedge \eta_j + \cdots \right. \\ \left. \cdots + \sum \frac{dh_{j_1}}{h_{j_1}} \wedge \cdots \wedge \frac{dh_{j_k}}{h_{j_k}} \wedge \eta_{j_1, \dots, j_k} \right\},$$

where $\eta_0 \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k$ and $\eta_j \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{k-1}$ and $\eta_{j_1, \dots, j_t} \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{k-t}$ for $1 \leq j_1 < \cdots < j_t \leq s$ and $t = 2, \dots, k$.

Proof. See [12], Lemma 2.2.3. □

The main result of this section is a generalization of Proposition 3.6, which will be used in the computation of the cohomology of the double complex.

Proposition 3.7. *Let $h_1, \dots, h_s$ be as above. Fix an integer S so that $0 \leq S \leq s$ and suppose we have $\eta \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k$, for $0 \leq k \leq n - 2 - S$, such that*

$$dh_j \wedge dh_1 \wedge \cdots \wedge dh_S \wedge \eta \equiv 0 \pmod{h_j, h_1, \dots, h_S}$$

for $S + 1 \leq j \leq s$. Then η is of the form

(3-10)

$$\eta = \sum_{i=1}^S h_i \gamma_i + \sum_{i=1}^S dh_i \wedge \xi_i \\ + h_{S+1} \cdots h_s \left\{ \eta_0 + \sum_{j=S+1}^s \frac{dh_j}{h_j} \wedge \eta_j + \sum_{S+1 \leq j_1 < j_2 \leq s} \frac{dh_{j_1}}{h_{j_1}} \wedge \frac{dh_{j_2}}{h_{j_2}} \wedge \eta_{j_1, j_2} + \cdots \right\},$$

where $\gamma_i \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^k$, $\xi_i \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{k-1}$, and $\eta_{j_1, \dots, j_t} \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{k-t}$ for $0 \leq t \leq k$.

Proof. The proof proceeds by an induction on S . As a basis step the case $S = 0$ is Proposition 3.6. Suppose now that the proposition is true for some $S > 0$ and also suppose that

$$dh_j \wedge dh_1 \wedge \cdots \wedge dh_S \wedge dh_{S+1} \wedge \eta \equiv 0 \pmod{(h_j, h_1, \dots, h_{S+1})}$$

for $S + 2 \leq j \leq s$. To show that the proposition is true for $S + 1$ we use induction on $s - (S + 1)$, the number of factors in $h_{S+2} \cdots h_s$. For the basis step let $s = S + 2$,

so that $dh_1 \wedge \cdots \wedge dh_{S+1} \wedge dh_{S+2} \wedge \eta \equiv 0 \pmod{(h_1, \dots, h_{S+2})}$. By Proposition 3.5 we can write

$$\eta = \sum_{i=1}^{S+1} h_i \gamma'_i + \sum_{i=1}^{S+1} dh_i \wedge \xi'_i + h_{S+2} \gamma'_{S+2} + dh_{S+2} \wedge \xi'_{S+2}$$

for some γ'_i, ξ'_i . Therefore

$$\eta = \sum_{i=1}^{S+1} h_i \gamma'_i + \sum_{i=1}^{S+1} dh_i \wedge \xi'_i + h_{S+2} \left\{ \gamma'_{S+2} + \frac{dh_{S+2}}{h_{S+2}} \wedge \xi'_{S+2} \right\},$$

which proves the proposition for $s - (S + 1) = 1$ and verifies the basis step of the induction (on $s - (S + 1)$).

For the induction step, we assume that the proposition is true for $S + 1$ and $s - 1$. In this case we may write

$$(3-11) \quad \eta = \sum_{i=1}^{S+1} h_i \gamma_i + \sum_{i=1}^{S+1} dh_i \wedge \xi_i + h_{S+2} \cdots h_{s-1} \left\{ \eta_0 + \sum_{S+2 \leq i \leq s-1} \frac{dh_i}{h_i} \wedge \eta_i + \cdots \right\}.$$

We also assume that

$$dh_j \wedge dh_1 \wedge \cdots \wedge dh_{S+1} \wedge \eta \equiv 0 \pmod{(h_j, h_1, \dots, h_{S+1})}$$

for $j = S + 2, \dots, s$. We want to show that the proposition is now true for s . In order to do this we let N be the largest integer such that $\eta_{i_1 \dots i_t} = 0$ for any $t \geq N + 1$ in (3-11) and proceed by induction on N .

Suppose $N = 0$. Then we can write (3-11) as

$$(3-12) \quad \eta = \sum_{i=1}^{S+1} h_i \gamma_i + \sum_{i=1}^{S+1} dh_i \wedge \xi_i + (h_{S+2} \cdots h_{s-1}) \eta_0.$$

By assumption $dh_s \wedge dh_1 \wedge \cdots \wedge dh_{S+1} \wedge \eta \equiv 0 \pmod{(h_s, h_1, \dots, h_{S+1})}$, which implies $(h_{S+2} \cdots h_{s-1}) dh_s \wedge dh_1 \wedge \cdots \wedge dh_{S+1} \wedge \eta_0 \equiv 0 \pmod{(h_s, h_1, \dots, h_{S+1})}$. Since we are trying to verify the proposition for $S + 1$, we are using the hypothesis $k \leq n - 2 - (S + 1) = n - (S + 3)$. Therefore $S + 3 \leq n - k$.

The normal crossing divisor hypothesis assures that any sequence of h_i 's of length less than n is a regular sequence, so in particular any sequence of length $S + 3$ will be a regular sequence. It follows that the sequence $(h_{S+2}, h_s, h_1, \dots, h_{S+1})$ is regular and therefore in the above relation we can make the substitution $h_{S+2} Q = h_s Q_s + h_1 Q_1 + \cdots + h_{S+1} Q_{S+1}$, for some Q_i 's, where Q is in the ideal $(h_s, h_1, \dots, h_{S+1})$. In this case we can then remove the factor of h_{S+2} . We can repeat the argument with h_{S+3} in place of h_{S+2} , then h_{S+4} and so on, up to h_{s-1} . Therefore we can remove the factor $h_{S+2} \cdots h_{s-1}$, and the above relation reduces to $dh_s \wedge dh_1 \wedge \cdots \wedge dh_{S+1} \wedge \eta_0 \equiv 0 \pmod{(h_s, h_1, \dots, h_{S+1})}$. Now by applying Proposition 3.5, it will follow that η_0 is congruent to 0 in the ideal $(h_s, h_1, \dots, h_{S+1}, dh_s, dh_1, \dots, dh_{S+1})$, so that $\eta_0 = \sum_{i=1}^{S+1} h_i \gamma'_i + \sum_{i=1}^{S+1} dh_i \wedge \xi'_i + h_s \gamma'_s + dh_s \wedge \xi'_s$ for some $\gamma'_1, \dots, \gamma'_s$ and $\xi'_1, \dots, \xi'_s$.

Substituting into (3-12) we get

$$\begin{aligned} \eta = & \sum_{j=1}^{S+1} h_j \gamma_j + \sum_{j=1}^{S+1} dh_j \wedge \xi_j \\ & + h_{S+2} \dots h_{s-1} \left\{ \sum_{j=1}^{S+1} h_j \gamma'_j + \sum_{j=1}^{S+1} dh_j \wedge \xi'_j + h_s \gamma'_s + dh_s \wedge \xi'_s \right\}. \end{aligned}$$

Therefore η is of the form

$$\eta = \sum_{j=1}^{S+1} h_j \gamma''_j + \sum_{j=1}^{S+1} dh_j \wedge \xi''_j + h_{S+2} \dots h_{s-1} h_s \left\{ \gamma'_s + \frac{dh_s}{h_s} \wedge \xi'_s \right\},$$

where $\gamma''_j = \gamma_j + (h_{S+2} \dots h_{s-1}) \gamma'_j$ and $\xi''_j = \xi_j + (h_{S+2} \dots h_{s-1}) \xi'_j$, which proves the proposition for $N = 0$.

For the induction step we assume that the proposition is true for $N - 1$ (i.e. true for $S + 1$, $s - 1$ and $N - 1$). Suppose η is in the form

$$\begin{aligned} (3-13) \quad \eta = & \sum_{j=1}^{S+1} h_j \gamma_j + \sum_{j=1}^{S+1} dh_j \wedge \xi_j + h_{S+2} \dots h_{s-1} \left\{ \eta_0 + \sum_{j=S+2}^{s-1} \frac{dh_j}{h_j} \wedge \eta_j + \dots \right. \\ & \left. + \sum_{S+2 \leq j_1 < \dots < j_N \leq s-1} \frac{dh_{j_1}}{h_{j_1}} \wedge \dots \wedge \frac{dh_{j_N}}{h_{j_N}} \wedge \eta_{j_1 \dots j_N} \right\} \end{aligned}$$

and also suppose for $S + 2 \leq j \leq s$ that

$$dh_j \wedge dh_1 \wedge \dots \wedge dh_{S+1} \wedge \eta \equiv 0 \pmod{(h_j, h_1, \dots, h_{S+1})}.$$

For simplicity we consider the index $(j_1, \dots, j_N) = (S + 2, \dots, S + N + 1)$ in the last term of (3-13). In this case,

$$\begin{aligned} \eta - (h_{S+N+2} \dots h_{s-1}) dh_{S+2} \wedge \dots \wedge dh_{S+N+1} \wedge \eta_{S+2, \dots, S+N+1} \\ \equiv \sum_{i=1}^{S+1} h_i \gamma_i + \sum_{i=1}^{S+1} dh_i \wedge \xi_i \pmod{(h_{S+2}, \dots, h_{S+N+1})}. \end{aligned}$$

Multiplying both sides by $dh_s \wedge dh_1 \wedge \dots \wedge dh_{S+1}$ it follows that

$$\begin{aligned} (h_{S+N+2} \dots h_{s-1}) dh_s \wedge dh_1 \wedge \dots \wedge dh_{S+N+1} \wedge \eta_{S+2, \dots, S+N+1} \\ \equiv 0 \pmod{(h_s, h_1, \dots, h_{S+N+1})}. \end{aligned}$$

We can remove the factor $h_{S+N+2} \dots h_{s-1}$ using the same method as above.

Now, it follows by Proposition 3.5 that there exist $\tilde{\gamma}_i$ and $\tilde{\xi}_i$ for $i = 1, \dots, S + N + 1$ and $i = s$, such that

$$\eta_{S+2, \dots, S+N+1} = \sum_{i=1}^{S+N+1} h_i \tilde{\gamma}_i + \sum_{i=1}^{S+N+1} dh_i \wedge \tilde{\xi}_i + h_s \tilde{\gamma}_s + dh_s \wedge \tilde{\xi}_s.$$

Similarly for any index $(j_1, \dots, j_N)$ of length N we can write

$$(3-14) \quad \begin{aligned} \eta_{j_1, \dots, j_N} &= \sum_{k=1}^{S+1} h_k \widetilde{\gamma}_{j_1, \dots, j_N; k} + \sum_{k=1}^{S+1} dh_k \wedge \widetilde{\xi}_{j_1, \dots, j_N; k} \\ &\quad + \sum_{k=1}^N h_{j_k} \widehat{\gamma}_{j_1, \dots, j_N; k} + \sum_{k=1}^N dh_{j_k} \wedge \widehat{\xi}_{j_1, \dots, j_N; k} \\ &\quad + h_s \widetilde{\gamma}_{j_1, \dots, j_N; s} + dh_s \wedge \widetilde{\xi}_{j_1, \dots, j_N; s}. \end{aligned}$$

Substitution into (3-13) yields

$$(3-15) \quad \begin{aligned} \eta &= \sum_{j=1}^{S+1} h_j \gamma_j + \sum_{j=1}^{S+1} dh_j \wedge \xi_j + h_{S+2} \dots h_{s-1} \left\{ \eta_0 + \sum_{j=S+2}^{s-1} \frac{dh_j}{h_j} \wedge \eta_j + \dots \right. \\ &\quad + \sum_{S+2 \leq j_1 < \dots < j_{N-1} \leq s-1} \frac{dh_{j_1}}{h_{j_1}} \dots \frac{dh_{j_{N-1}}}{h_{j_{N-1}}} \wedge \eta_{j_1, \dots, j_{N-1}} \\ &\quad + \sum_{S+2 \leq j_1 < \dots < j_N \leq s-1} \frac{dh_{j_1}}{h_{j_1}} \dots \frac{dh_{j_N}}{h_{j_N}} \wedge \left\{ \sum_{k=1}^N h_{j_k} \widetilde{\gamma}_{j_1, \dots, j_N; k} \right. \\ &\quad \left. \left. + \sum_{k=1}^N dh_{j_k} \wedge \widetilde{\xi}_{j_1, \dots, j_N; k} + h_s \widetilde{\gamma}_{j_1, \dots, j_N; s} + dh_s \wedge \widetilde{\xi}_{j_1, \dots, j_N; s} \right\} \right\}. \end{aligned}$$

Letting

$$(3-16) \quad \begin{aligned} \rho &= \eta - h_{S+2} \dots h_{s-1} \left\{ \sum_{S+2 \leq j_1 < \dots < j_N \leq s-1} \frac{dh_{j_1}}{h_{j_1}} \dots \frac{dh_{j_N}}{h_{j_N}} (h_s \widetilde{\gamma}_{j_1, \dots, j_N; s} \right. \\ &\quad \left. + dh_s \wedge \widetilde{\xi}_{j_1, \dots, j_N; s}) \right\}, \end{aligned}$$

we see that ρ satisfies the induction hypothesis and can therefore be written as

$$(3-17) \quad \begin{aligned} \rho &= \sum_{j=1}^{S+1} h_j \widehat{\gamma}_j + \sum_{j=1}^{S+1} dh_j \wedge \widehat{\xi}_j \\ &\quad + h_{S+2} \dots h_s \left\{ \rho_0 + \dots + \sum_{S+2 \leq j_1 < \dots < j_{N-1} \leq s-1} \frac{dh_{j_1}}{h_{j_1}} \dots \frac{dh_{j_{N-1}}}{h_{j_{N-1}}} \wedge \rho_{j_1, \dots, j_{N-1}} \right\}, \end{aligned}$$

for some $\widehat{\gamma}_j$, $\widehat{\xi}_j$ and $\rho_0, \dots, \rho_{j_1, \dots, j_{N-1}}$. Substituting this back into (3-16) it follows that η is of the form

$$(3-18) \quad \begin{aligned} \eta &= \sum_{j=1}^{S+1} h_j \overline{\gamma}_j + \sum_{j=1}^{S+1} dh_j \wedge \overline{\xi}_j \\ &\quad + h_{S+2} \dots h_s \left\{ \overline{\eta}_0 + \dots + \sum_{S+2 \leq j_1 < \dots < j_{N+1} \leq s} \frac{dh_{j_1}}{h_{j_1}} \dots \frac{dh_{j_{N+1}}}{h_{j_{N+1}}} \wedge \overline{\eta}_{j_1, \dots, j_{N+1}} \right\}, \end{aligned}$$

for some $\bar{\gamma}_j, \bar{\xi}_j, \bar{\eta}_j, \dots, \bar{\eta}_{j_1, \dots, j_{N+1}}$ which closes the triple induction and proves the proposition. $\square$

3.3. Computation of horizontal cohomology. From the results of Section 3.1, we have that the cohomology of the double complex (3-8) is equal to the cohomology of the complex across the top row (after the vertical cohomology has been computed),

$$(3-19) \quad 0 \longrightarrow H^r((\Omega^{(0, \bullet)})^{(D)}, \partial_v) \xrightarrow{\partial_h} \dots \xrightarrow{\partial_h} H^r((\Omega^{(n, \bullet)})^{(D)}, \partial_v) \longrightarrow 0,$$

where $H^k(\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^\bullet \langle Y \rangle^{(0)}, d_{\langle Y \rangle} F \wedge) \cong H^{k-r}(H^r((\Omega^{(\bullet, \bullet)})^{(D)}, \partial_v), \partial_h)$.

Along with (3-1) and (3-2), let $\deg_2$ be the second half of a bigrading $(\deg_1, \deg_2)$ on $\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^k \langle Y \rangle$ by setting

$$(3-20) \quad \deg_2 x_i = \deg_2 dx_i = 0, \quad \text{for } i = 1, \dots, n \quad \text{and}$$

$$(3-21) \quad \deg_2 y_j = \deg_2 dy_j = 1, \quad \text{for } j = 1, \dots, r.$$

From (3-8) and the definition of ∂_v , we have in the top position of each column (separating out powers of the y_j 's)

$$(3-22) \quad \begin{aligned} & H^r((\Omega^{(\ell, \bullet)})^{(D)}, \partial_v) \\ = & \bigoplus_{\{1 \leq i_1 < \dots < i_\ell \leq n\}} \bigoplus_{\{v_j \geq 0\}} y_1^{v_1} \dots y_r^{v_r} [\mathbb{F}_q[x]/(I)]^{(D-\ell+\sum v_j d_j)} dx_{i_1} \dots dx_{i_\ell} \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r}, \end{aligned}$$

where I is the ideal of $\mathbb{F}_q[x]$ generated by the f_j for which $v_j > 0$.

Since ∂_h shifts the bidegree by $(0, 1)$, we compute the cohomology of sequences of the form

$$(3-23) \quad 0 \longrightarrow H^r(\Omega^{(0, \bullet)})^{(D, J)} \xrightarrow{\partial_h} \dots \xrightarrow{\partial_h} H^r(\Omega^{(n, \bullet)})^{(D, J+n)} \longrightarrow 0,$$

where

$$(3-24) \quad H^r((\Omega^{(\ell, \bullet)})^{(D)}) = \bigoplus_{J=0}^{\infty} H^r((\Omega^{(\ell, \bullet)})^{(D, J)}).$$

We can now state the main result of this section.

Theorem 3.2. *If $f_1, \dots, f_r \in \mathbb{F}_q[x_1, \dots, x_n]$ satisfy the definition of a divisor with normal crossings in $\mathbb{P}^{n-1}(\mathbb{F}_q)$, then*

$$H^\ell(H^r(\Omega^{(\bullet, \bullet)})^{(D)}, \partial_h) = 0 \quad \text{for } \ell < n - 1.$$

Proof. We break up the proof into cases based on the value of ℓ and the value of J .

Case 1. $J = 0$ and $0 \leq \ell \leq n - 2$. Consider the sequence

$$H^r((\Omega^{(\ell-1, \bullet)})^{(D, -1)}) \xrightarrow{\partial_h} H^r((\Omega^{(\ell, \bullet)})^{(D, 0)}) \xrightarrow{\partial_h} H^r((\Omega^{(\ell+1, \bullet)})^{(D, 1)}).$$

It is clear that the negative value in $\deg_2$ implies that the left most space is zero, so we compute the kernel of the map

$$\begin{aligned} \partial_h : \bigoplus_{\{1 \leq i_1 < \dots < i_\ell \leq n\}} \mathbb{F}_q[x]^{(D-\ell)} dx_{i_1} \dots dx_{i_\ell} \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r} \\ \longrightarrow \bigoplus_{j=1}^r \bigoplus y_j [\mathbb{F}_q[x]/(f_j)]^{(D-\ell+d_j-1)} dx_i dx_{i_1} \dots dx_{i_\ell} \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r}, \end{aligned}$$

where the outer sum of the bottom row is over all subsets of $\{1, \dots, n\}$ of the type $i \cup \{i_1, \dots, i_\ell\}$.

Let $\omega \in \bigoplus \mathbb{F}_q[x]^{(D-\ell)} dx_{i_1} \dots dx_{i_\ell} \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r}$, where the sum is over all indices $1 \leq i_1 < \dots < i_\ell \leq n$. Now ω is in the kernel of ∂_h if and only if $df_j \wedge \omega \equiv 0 \pmod{f_j}$ for every j . Since we are assuming that $\ell \leq n-2$, we can apply Proposition 3.6 and we can write ω as (dropping the dy/y 's)

$$\omega = (f_1 \dots f_r) \left\{ \omega_0 + \sum_1^r \frac{df_j}{f_j} \wedge \omega_j + \sum \frac{df_{j_1}}{f_{j_1}} \wedge \frac{df_{j_2}}{f_{j_2}} \wedge \omega_{j_1, j_2} + \dots \right\},$$

where ω_0 is an ℓ -form, each of the ω_j 's is an $(\ell-1)$ -form and each ω_{j_1, j_2} is an $(\ell-2)$ -form, and so on. Moreover, these are the only such ω .

The first summand $(f_1 \dots f_r)\omega_0$ has a value of $\deg_1$ that is too large, as $\deg_1 = \sum d_j > D$, so that $\omega_0 = 0$. In the subsequent terms $(f_1 \dots f_r) \sum_{j=1}^r \frac{df_j}{f_j} \wedge \omega_j$, we have for any j ,

$$\deg_1 \left((f_1 \dots \hat{f}_j \dots f_r) df_j \wedge \omega_j \right) \geq \sum_{j=1}^r d_j > D,$$

since $\deg_1 f_j = \deg_1 df_j$.

Similarly, in each of the remaining terms we have

$$\deg_1 \left((f_1 \dots f_r) \frac{df_{j_1}}{f_{j_1}} \dots \frac{df_{j_t}}{f_{j_t}} \right) = d_1 + \dots + d_r.$$

Now since $D < d_1 + \dots + d_r$ we have that

$$\deg_1 \left((f_1 \dots f_r) \frac{df_{j_1}}{f_{j_1}} \dots \frac{df_{j_t}}{f_{j_t}} \right) > D,$$

which is a contradiction of the choice of coefficients in $\mathbb{F}_q[x]^{(D-\ell)}$. By this contradiction, if any such form is in the kernel of ∂_h it is necessarily zero. Therefore $H^k(H^r((\Omega^{\bullet, \bullet})^{(D, 0)}, \partial_h)) = 0$ for $k < n-1$.

Case 2. $1 \leq J$ and $0 \leq \ell \leq n-2$. We are considering forms with one or more factors of the y_j raised to varying powers v_j . As before $J = \sum_{j=1}^r v_j$. We let κ be an integer, $1 \leq \kappa \leq r$, that represents the number of factors of y_j in a given form with non-zero exponents. For each of these products as $y_{j_1}^{v_{j_1}} \dots y_{j_\kappa}^{v_{j_\kappa}}$ we associate the form $\omega(0, \dots, v_{j_1}, \dots, 0, v_{j_\kappa}, \dots, 0)$. We will condense the r -tuple $(0, \dots, v_{j_1}, \dots, 0, v_{j_\kappa}, \dots, 0)$ to the κ -tuple of non-zero terms $(v_{j_1}, \dots, v_{j_\kappa})$ when the context allows and write $\omega(v_{j_1}, \dots, v_{j_\kappa})$.

For fixed J , we index the terms of $\omega \in H^r((\Omega^{(\ell, \bullet)})^{(D, J)})$ by ascending order of κ and write

$$(3-25) \quad \omega = \sum_{j=1}^r y_j^J \omega(0, \dots, J, \dots, 0) + \sum_{\{1 \leq j_1 < j_2 \leq r\}} \sum_{\{v_{j_1} + v_{j_2} = J\}} y_{j_1}^{v_{j_1}} y_{j_2}^{v_{j_2}} \omega(v_{j_1}, v_{j_2}) \\ + \dots + \sum_{1 \leq j_1 < \dots < j_J \leq r} y_{j_1} \dots y_{j_J} \omega(0, \dots, 1, \dots, 0, \dots, 1).$$

The forms in (3-25) will satisfy (again dropping the factor of $\frac{dy_1}{y_1} \dots \frac{dy_r}{y_r}$)

$$\omega(v_{j_1}, \dots, v_{j_\kappa}) \in \bigoplus_{1 \leq i_1 < \dots < i_\ell \leq n} [\mathbb{F}_q[x]/(f_{j_1}, \dots, f_{j_\kappa})]^{(D + v_{j_1} d_{j_1} + \dots + v_{j_\kappa} d_{j_\kappa} - \ell)} dx_{i_1} \dots dx_{i_\ell}.$$

Suppose now that for a differential form ω as in (3-25) we have $\omega \in \ker(\partial_h)$. Consider first the summand of ω where $\kappa = 1$. If we fix one of the indices j such that $\omega(J) = \omega(0, \dots, J, \dots, 0) \neq 0$ and apply ∂_h we get $(y_1 df_1 + \dots + y_r df_r) \wedge y_j^J \omega(J) = y_1 y_j^J df_1 \wedge \omega(J) + \dots + y_j^{J+1} df_j \wedge \omega(J) + \dots + y_r y_j^J df_r \wedge \omega(J)$. As in the first case, for each $j = 1, \dots, r$, the resulting $(\ell + 1)$ -form associated to the factor y_j^{J+1} satisfies $df_j \wedge \omega(J) \equiv 0 \pmod{f_j}$. Hence, by Proposition 3.6, there exists $\eta(J) \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{(\ell-1)}$ such that $\omega(J) \equiv df_j \wedge \eta(J) \pmod{f_j}$. Making this substitution for each $j = 1, \dots, r$, we conclude that summands of the type (the first sum on the right-hand side of (3-25)) are in the image of ∂_h and that ω can be reduced to the form $\omega =$

$$\sum_{\{1 \leq j_1 < j_2 \leq r\}} \sum_{\{v_{j_1} + v_{j_2} = J\}} y_{j_1}^{v_{j_1}} y_{j_2}^{v_{j_2}} \omega(v_{j_1}, v_{j_2}) + \dots + \sum_{1 \leq j_1 < \dots < j_J \leq r} y_{t_1} \dots y_{t_J} \omega(1, \dots, 1).$$

Now for $2 \leq \kappa < J$, any monomial in $\sum y_j df_j \wedge \omega$ that contains only $y_{j_1}, \dots, y_{j_\kappa}$ comes from sums of forms of the type $y_{j_t} df_{j_t} \wedge y_{j_1}^{v_{j_1}} \dots y_{j_\kappa}^{v_{j_\kappa}} \omega(v_{j_1}, \dots, v_{j_\kappa})$, involving some or all of the $y_{j_1}, \dots, y_{j_\kappa}$. Since we are assuming $\omega \in \ker \partial_h$, we will have

$$\sum_{t=1}^{\kappa} y_{j_t} df_{j_t} \wedge \sum_{|v|=J, v_{j_t} > 0} y_{j_1}^{v_{j_1}} \dots y_{j_\kappa}^{v_{j_\kappa}} \omega(v_{j_1}, \dots, v_{j_\kappa}) \equiv 0 \pmod{(f_{j_1}, \dots, f_{j_\kappa})}.$$

If $J \leq n - 2$ we can apply Proposition 3.4 to find forms $\eta(v'_{j_1}, \dots, v'_{j_\kappa})$ such that

$$\sum_{|v|=J, v_{j_t} > 0} y_{j_1}^{v_{j_1}-1} \dots y_{j_\kappa}^{v_{j_\kappa}-1} \omega(v_{j_1}, \dots, v_{j_\kappa}) \\ \equiv \sum_{t=1}^{\kappa} y_{j_t} df_{t_j} \wedge \sum_{|v'|=J-\kappa-1} y_{j_1}^{v'_{j_1}} \dots y_{j_\kappa}^{v'_{j_\kappa}} \eta(v'_{j_1}, \dots, v'_{j_\kappa}) \pmod{(f_{t_1}, \dots, f_{t_\kappa})},$$

where the sum is over the set of all monomials containing only $y_{j_1}, \dots, y_{j_\kappa}$ having $\sum d_{j_t} v'_{j_t} = J - 1 - \kappa$.

If $J \geq n - 1$, the quotients $[\mathbb{F}_q[x]/(f_{j_1}, \dots, f_{j_\kappa})]^{(D + v_{j_1} d_{j_1} + \dots + v_{j_\kappa} d_{j_\kappa} - \ell)}$ will all be zero. The degree of the Hilbert polynomial of the quotient is $\sum_{j=1}^{\kappa} d_j - \kappa$ which is less than $D + v_{j_1} d_{j_1} + \dots + v_{j_\kappa} d_{j_\kappa} - \ell$ if $J \geq n - 1$, $0 \leq \ell \leq n - 2$ and $D > 0$.

If we now let

$$\eta = y_{j_1} \dots y_{j_\kappa} \sum_{|v'|=J-\kappa-1, v'_{j_t} \geq 0} y_{j_1}^{v'_{j_1}} \dots y_{j_\kappa}^{v'_{j_\kappa}} \eta(v'_{j_1}, \dots, v'_{j_\kappa}),$$

then $\omega - \sum_{j=1}^r y_j df_j \wedge \eta$ has no terms involving exactly the variables $y_{j_1}, \dots, y_{j_\kappa}$. This argument can be repeated for any terms in (3-25) involving fewer than κ different y_j 's, so by induction on κ , we can reduce to the case where

$$\omega = \sum_{1 \leq j_1 < \dots < j_J \leq r} y_{j_1} \dots y_{j_J} \omega(j_1, \dots, j_J),$$

where any remaining terms contain J distinct y_j 's all of exponent 1.

Here we will no longer index by the exponent v_j , since they are either 0 or 1, but rather by the subscript j for which there is a factor of y_j .

We will reduce ω further by using induction on the number of terms of the form $y_{j_1} \dots y_{j_J} \omega(j_1, \dots, j_J)$. Suppose that ω has one term, say $\omega = y_1 \dots y_J \omega(1, \dots, J)$. Then

$$(3-26) \quad \begin{aligned} \sum_{j=1}^r y_j df_j \wedge \omega &= \sum_{j=1}^J y_1 \dots y_j^2 \dots y_J df_j \wedge \omega(1, \dots, J) \\ &\quad + \sum_{j=J+1}^r y_j y_1 \dots y_J df_j \wedge \omega(1, \dots, J). \end{aligned}$$

If $\omega \in \ker(\partial_h)$ we can deduce from the top sum on the right in (3-26) that $df_j \wedge \omega(1, \dots, J) \equiv 0 \pmod{(f_1, \dots, f_J)}$ and therefore by Proposition 3.3 there exists $\eta \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{\ell-J}$ such that

$$(3-27) \quad \omega(1, \dots, J) \equiv df_1 \wedge \dots \wedge df_J \wedge \eta \pmod{(f_1, \dots, f_J)}.$$

Now substituting this into the lower sum on the right in (3-26) we have

$$(y_j y_1 \dots y_J) df_j \wedge df_1 \wedge \dots \wedge df_J \wedge \eta \equiv 0 \pmod{(f_j, f_1, \dots, f_J)},$$

which implies

$$df_j \wedge df_1 \wedge \dots \wedge df_J \wedge \eta \equiv 0 \pmod{(f_j, f_1, \dots, f_J)}$$

for $J+1 \leq j \leq r$.

Using Proposition 3.7 we can write

$$(3-28) \quad \begin{aligned} \eta &= \sum_{j=1}^J f_j \gamma_j + \sum_{j=1}^J df_j \wedge \xi_j \\ &\quad + f_{J+1} \dots f_r \left\{ \eta_0 + \sum_{j=J+1}^r \frac{df_j}{f_j} \wedge \eta_j + \sum_{J+1 \leq j_1 < j_2 \leq r} \frac{df_{j_1}}{f_{j_1}} \wedge \frac{df_{j_2}}{f_{j_2}} \wedge \eta_{j_1, j_2} + \dots \right\}, \end{aligned}$$

where $\gamma_j \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{(\ell-J)}$, $\xi_j \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{(\ell-J-1)}$, and $\eta_{j_1, \dots, j_s} \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{(\ell-J-s)}$.

If we let $\eta' = \eta - [\sum_{j=1}^J f_j \gamma_j + \sum_{j=1}^J df_j \wedge \xi_j]$, then (3-27) holds with η replaced by η' . Also $df_j \wedge \eta' \equiv 0 \pmod{(f_j)}$, for each $j = J+1, \dots, r$. Therefore $\partial_h(y_2 \dots y_J df_2 \wedge \dots \wedge df_J \wedge \eta') = \omega$ so that ω is in the image of ∂_h .

For the induction step, we first put an ordering on the differential forms $y_{j_1}, \dots, y_{j_J} \omega(j_1, \dots, j_J)$. In this ordering we let $y_1^{v_1} \dots y_r^{v_r} > y_1^{w_1} \dots y_r^{w_r}$ if $v_k > w_k$ for some k and $v_j = w_j$ for all $j < k$. Choose $t_1, \dots, t_J$ so that the monomial $y_{t_1} \dots y_{t_J}$ is maximal in the ordering among all of the remaining non-zero terms of ω .

Supposing that $\partial_h(\omega) = 0$, it follows that $df_j \wedge \omega(t_1, \dots, t_J) \in (f_{t_1}, \dots, f_{t_J})$, for each $j \in \{t_1, \dots, t_J\}$. On the other hand, for each $j \notin \{t_1, \dots, t_J\}$, where $j < t_1$, we know that $df_j \wedge \omega(t_1, \dots, t_J)$ is the unique coefficient of $y_j y_{t_1} \dots y_{t_J}$ in $\partial_h(\omega)$. If a term such as this should come from $df_{t_j} \wedge \omega(j, t_1, \dots, \hat{t}_j \dots t_J)$ with $j < t_1$, then $y_j y_{t_1} \dots \hat{y}_{t_j} \dots y_{t_J} > y_{t_1} \dots y_{t_J}$ in the ordering, which contradicts the maximality. By Proposition 3.3 we can write

$$\omega(t_1, \dots, t_J) = df_{t_1} \wedge \dots \wedge df_{t_J} \wedge \eta(t_1, \dots, t_J)$$

for some $\eta(t_1, \dots, t_J)$. Now by Proposition 3.7, $\eta(t_1, \dots, t_J)$ is of the form

$$\begin{aligned} \eta(t_1, \dots, t_J) = & \sum_{i=1}^J f_{t_i} \gamma_{t_i} + \sum_{i=1}^J df_{t_i} \wedge \xi_{t_i} \\ & + (f_1 \dots f_{t_1-1}) \left\{ \eta_0 + \sum_{i=1}^{t_1-1} \frac{df_i}{f_i} \wedge \eta_i + \dots \right\}. \end{aligned}$$

In the above, if we let $\tau(t_1, \dots, t_J) = (f_1 \dots f_{t_1-1}) \left\{ \eta_0 + \sum_{i=1}^{t_1-1} \frac{df_i}{f_i} \wedge \eta_i + \dots \right\}$, then

$$\begin{aligned} \partial_h(y_{t_2} \dots y_{t_J} (df_{t_2} \wedge \dots \wedge df_{t_J} \wedge \tau(t_1, \dots, t_J))) \\ = \sum_{k < t_1} y_k y_{t_2} \dots y_{t_J} df_k \wedge df_{t_2} \wedge \dots \wedge df_{t_J} \wedge \tau(t_1, \dots, t_J) \\ + y_{t_1} y_{t_2} \dots y_{t_J} df_{t_1} \wedge \dots \wedge df_{t_J} \wedge \tau(t_1, \dots, t_J) \\ + \sum_{k > t_1} y_k y_{t_2} \dots y_{t_J} df_k \wedge df_{t_2} \wedge \dots \wedge df_{t_J} \wedge \tau(t_1, \dots, t_J) \end{aligned}$$

so that $\omega - \partial_h(\sigma(t_1, \dots, t_J))$ will contain only monomials that are smaller in the ordering than the maximal $y_{t_1} \dots y_{t_J}$. The terms in the first row on the right-hand side of the equation are congruent to 0 modulo $(f_k, f_{t_2}, \dots, f_{t_J})$. The second row is $y_{t_1} \dots y_{t_J} \omega(t_1, \dots, t_J)$ and in the third row all of the terms are less than $y_{t_1} \dots y_{t_J} \omega(t_1, \dots, t_J)$ in the ordering. Therefore the maximal term is in the image of ∂_h and won't contribute to the cohomology. This completes the induction step and finishes Case 2. Therefore $H^k((H^r(\Omega^{\bullet, \bullet}))^{(D, J)}) = 0$ for all J and $k < n - 1$. $\square$

3.4. Dimensions $n+r-1$ and $n+r$. In this section we establish an isomorphism between the two remaining non-zero cohomology spaces.

Proposition 3.8. *If $f_1, \dots, f_r$ form a normal crossing divisor, then*

$$H^{n+r-1}(\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^{\bullet} \langle Y \rangle^{(D)}, d_{\langle Y \rangle} F \wedge) \cong H^{n+r}(\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^{\bullet} \langle Y \rangle^{(D)}, d_{\langle Y \rangle} F \wedge),$$

and in particular, for each J ,

$$H^{n+r-1}(\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^{\bullet} \langle Y \rangle^{(D, J)}) \cong H^{n+r}(\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^{\bullet} \langle Y \rangle^{(D, J)}).$$

Moreover $H^{n+r}(\Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^{\bullet} \langle Y \rangle^{(D, J)}) = 0$ for all $J \geq n$.

Proof. Abbreviate the notation by letting $\Omega^k \langle Y \rangle^{(D)} = \Omega_{\mathbb{F}_q[x, y]/\mathbb{F}_q}^k \langle Y \rangle^{(D)}$. Following the method of Section 4 of [5] define the map $\Upsilon : \Omega^k \langle Y \rangle^{(D)} \longrightarrow \Omega^{k-1} \langle Y \rangle^{(D)}$ by

linearity on the polynomials in $\mathbb{F}_q[x, y]^{(D-\ell)}$ and by the formula

$$\begin{aligned}
 (3-30) \quad & \Upsilon(dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \wedge \frac{dy_{j_1}}{y_{j_1}} \wedge \cdots \wedge \frac{dy_{j_m}}{y_{j_m}}) \\
 &= \sum_{t=1}^{\ell} (-1)^{t-1} x_{i_t} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_t}} \wedge \cdots \wedge dx_{i_\ell} \wedge \frac{dy_{j_1}}{y_{j_1}} \wedge \cdots \wedge \frac{dy_{j_m}}{y_{j_m}} \\
 &+ \sum_{s=1}^m (-1)^{\ell+s-1} (-d_{j_s}) dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \wedge \frac{dy_{j_1}}{y_{j_1}} \wedge \cdots \wedge \widehat{\frac{dy_{j_s}}{y_{j_s}}} \wedge \cdots \wedge \frac{dy_{j_m}}{y_{j_m}},
 \end{aligned}$$

where $\widehat{dx_i}$ and $\widehat{\frac{dy_{j_s}}{y_{j_s}}}$ are terms to be omitted.

Since $\Upsilon^2 = 0$ we can define the complex

$$0 \longrightarrow \Omega^{n+r}\langle Y \rangle^{(D)} \xrightarrow{\Upsilon} \cdots \xrightarrow{\Upsilon} \Omega^0\langle Y \rangle^{(D)} \longrightarrow 0,$$

which is isomorphic to the Koszul complex on $\mathbb{F}_q[x, y]$ defined by the elements $x_1, \dots, x_n, -d_1, \dots, -d_r$.

It is a straightforward computation to show for any differential forms ω_1, ω_2 , with $\omega_1 \in \Omega^k\langle Y \rangle^{(D)}$, that $\Upsilon(\omega_1 \wedge \omega_2) = \Upsilon(\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge \Upsilon(\omega_2)$. For a polynomial f in $\mathbb{F}_q[x_1, \dots, x_n]$ we have $\Upsilon(d_{\langle Y \rangle} f) = \deg(f) \cdot f$. Therefore if $F(x, y) = \sum_{j=1}^r y_j f_j(x)$ we have $\Upsilon(d_{\langle Y \rangle} F) = 0$ and $\Upsilon(d_{\langle Y \rangle} F \wedge \omega) + d_{\langle Y \rangle} F \wedge \Upsilon(\omega) = 0$.

For $k = 0, \dots, n+r-1$ define the spaces $\tilde{\Omega}^k\langle Y \rangle^{(D)} = \Upsilon(\Omega^{k+1}\langle Y \rangle^{(D)})$. Since $x_1, \dots, x_n$ form a regular sequence in $\mathbb{F}_q[x]$ of depth n and since $(d_j, p) = 1$ for each j , the complex $\Omega^\bullet\langle Y \rangle^{(D)}$ will be exact and therefore we have for $k > 0$ that $\tilde{\Omega}^k\langle Y \rangle^{(D)} = \ker(\Upsilon : \Omega^k\langle Y \rangle^{(D)} \rightarrow \Omega^{k-1}\langle Y \rangle^{(D-1)})$.

For any $\tilde{\omega} \in \tilde{\Omega}^k\langle Y \rangle^{(D)}$, the relations above imply that $dF \wedge \tilde{\omega} \in \tilde{\Omega}^{k+1}\langle Y \rangle^{(D)}$ so that $(\tilde{\Omega}^\bullet\langle Y \rangle^{(D)}, dF \wedge)$ defines a complex.

Similarly we get a third complex $(\hat{\Omega}^\bullet\langle Y \rangle^{(D)}, \phi)$ by setting

$$\hat{\Omega}^k\langle Y \rangle^{(D)} = \tilde{\Omega}^{k-1}\langle Y \rangle^{(D)},$$

for any $k \geq 1$ and for $k = 0$ (using the bigrading) set $(\hat{\Omega}^0\langle Y \rangle)^{(D, J)} = 0$. The boundary map defining this complex ϕ is given by $\phi(\hat{\omega}) = -d_{\langle Y \rangle} F \wedge \hat{\omega}$ for $k \geq 1$ and $\phi = 0$ for $k = 0$.

We get a short exact sequence of complexes

$$0 \longrightarrow \tilde{\Omega}^\bullet\langle Y \rangle^{(D)} \longrightarrow \Omega^\bullet\langle Y \rangle^{(D)} \xrightarrow{\Upsilon'} \hat{\Omega}^\bullet\langle Y \rangle^{(D)} \longrightarrow 0,$$

where the arrow on the left is the inclusion and on the right we define $\Upsilon' = \Upsilon$ for $k > 0$, and for $k = 0$, Υ' is the zero map.

Letting Ξ denote the connecting homomorphism we consider the corresponding long exact sequence in cohomology:

$$\begin{aligned}
 \cdots &\xrightarrow{\Xi} H^k(\tilde{\Omega}^\bullet\langle Y \rangle^{(D)}) \xrightarrow{i} H^k(\Omega^\bullet\langle Y \rangle^{(D)}) \xrightarrow{\Upsilon'} H^k(\hat{\Omega}^\bullet\langle Y \rangle^{(D)}) \\
 &\xrightarrow{\Xi} H^{k+1}(\tilde{\Omega}^\bullet\langle Y \rangle^{(D)}) \xrightarrow{i} \cdots
 \end{aligned}$$

Notice that since Ξ is defined in terms of ϕ that it will shift the bidegree by $(0, 1)$.

We can now substitute $H^k(\hat{\Omega}^\bullet\langle Y \rangle) = H^{k-1}(\tilde{\Omega}^\bullet\langle Y \rangle)$ for each $k \geq 1$. We have already verified that for every J , $H^k(\Omega^\bullet\langle Y \rangle)^{(D, J)} = 0$, if $k \leq n+r-2$ and $k > n+r$.

Therefore for each J the long exact cohomology sequence reduces to

$$\begin{aligned} 0 &\longrightarrow H^{n+r-1}(\tilde{\Omega}^\bullet\langle Y\rangle)^{(D,J)} \longrightarrow H^{n+r-1}(\Omega^\bullet\langle Y\rangle)^{(D,J)} \\ &\longrightarrow H^{n+r-2}(\tilde{\Omega}^\bullet\langle Y\rangle)^{(D,J)} \longrightarrow H^{n+r}(\tilde{\Omega}^\bullet\langle Y\rangle)^{(D,J+1)} \\ &\longrightarrow H^{n+r}(\Omega^\bullet\langle Y\rangle)^{(D,J+1)} \longrightarrow H^{n+r-1}(\tilde{\Omega}^\bullet\langle Y\rangle)^{(D,J+1)} \longrightarrow 0. \end{aligned}$$

Since $\Upsilon(\Omega^{n+r-1}) \subseteq \Omega^{n+r-2}$ and $H^{n+r-2}(\Omega^\bullet) = 0$, we have that the space $H^{n+r-2}(\tilde{\Omega}^\bullet\langle Y\rangle)^{(D,J)} = 0$. Also since $\tilde{\Omega}^{n+r} = \Upsilon(\Omega^{n+r+1}) = 0$, it follows that $H^{n+r}(\tilde{\Omega}^\bullet\langle Y\rangle)^{(D,J+1)} = 0$. This induces isomorphisms $H^{n+r-1}(\tilde{\Omega}^\bullet\langle Y\rangle)^{(D,J)} \cong H^{n+r-1}(\Omega^\bullet\langle Y\rangle)^{(D,J)}$ and $H^{n+r-1}(\tilde{\Omega}^\bullet\langle Y\rangle)^{(D,J)} \cong H^{n+r}(\Omega^\bullet\langle Y\rangle)^{(D,J)}$.

For the proof of the last assertion we take a form in $H^{n+r}(\Omega^\bullet\langle Y\rangle)^{(D,J)}$ which will be a sum of elements from spaces

$$(y_1^{v_1} \dots y_r^{v_r}) (\mathbb{F}_q[x]/(f_{i_1}, \dots, f_{i_k}))^{(D+\sum d_j v_j - n, J)} (dx_i)_{i=1}^n (dy_j/y_j)_{j=1}^r.$$

The degree of the Hilbert polynomial of this space uses the degrees in the quotient for which the corresponding v_j 's are not zero, that is, $\sum_{j=1}^\kappa (d_j - 1) = \sum_{j=1}^\kappa d_j - \kappa$. Now we need to find J so that

$$D + \sum_{j=1}^\kappa d_j v_j - n > \sum_{j=1}^\kappa d_j - \kappa.$$

Any quotient space with such a degree would be zero. Moving the sum on the right and combining with the left leaves $D + \sum_{j=1}^\kappa d_j (v_j - 1) - n$, which is greater than or equal to $D + \sum_{j=1}^\kappa (v_j - 1) - n$ since each degree d_j is positive. So substituting $J = \sum_{j=1}^\kappa v_j$ we have

$$D + J - \kappa - n > -\kappa,$$

which is true whenever $D + J > n$, that is, whenever $J \geq n$. $\square$

3.5. Bases for cohomology spaces. In this section we show that the computations in characteristic p , from the previous sections, completely determine the cohomology of the original complex $\Omega_{C(b,q-1,\delta)}^\bullet\langle Y\rangle^{(0)}$ in characteristic zero.

For $J = 0, \dots, n-1$ let μ_J be the dimension of $H^{n+r}(\Omega_{\mathbb{F}_q[x,y]/\mathbb{F}_q}^\bullet\langle Y\rangle)^{(D,J)}$ as above. Choose integers $u_i(J, k)$ and $v_j(J, k)$ so that for $1 \leq k \leq \mu_J$, representatives of each cohomology class can be given by the monomial forms

$$\bar{\xi}_k^{(J)}(dx) \left(\frac{dy}{y} \right) = \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k)} dx_1 \dots dx_n \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r}.$$

Now define $\xi_k^{(J)} = \pi^{(p-1)bJ} \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k)}$ to be the lifting of the $\bar{\xi}_k^{(J)}$ into characteristic zero. With the factor of $\pi^{(p-1)bJ}$ these elements will be in $\mathcal{F}^0 C(b, q-1, \delta)^{(D)}$. Also these monomials will be the coefficients in the preimage of $\bar{\xi}_k^{(J)}$ under the map (3-5).

Proposition 3.9. *For $k = 1, \dots, \mu_J$ and $J = 0, \dots, n-1$, the classes $[\xi_k^{(J)}]$ generate a Γ_0 -basis for $H^{n+r}(\Omega_{C(b,q-1,\delta)}^\bullet\langle Y\rangle)^{(D)}$.*

Proof. The classes $[\xi_k^{(J)}]$ are linearly independent over Γ_0 . If not, we can suppose there are $a_{k,J} \in \Gamma_0$ and $\eta \in \Omega_{C(b,q-1,\delta)}^{n+r-2}\langle Y\rangle^{(D)}$ such that $\sum_{k,J} a_{k,J} \Upsilon(\xi_k^{(J)}) =$

$\pi^{(p-1)b}\gamma^{-1}\Delta(\eta)$. Multiplying by an appropriate power of π if needed, we can further assume that the $a_{k,J} \in \mathcal{O}_{\Gamma_0}$ with at least one $a_{k,J} \notin \pi\mathcal{O}_{\Gamma_0}$.

If $\eta \notin \mathcal{F}^0\Omega_{C(b,q-1,\delta)}^{n+r-2}\langle Y \rangle^{(D)}$, choose $s > 0$ so that $\eta \in \mathcal{F}^{-s}\Omega_{C(b,q-1,\delta)}^{n+r-2}\langle Y \rangle^{(D)}$. Now by the relation $\sum_{k,J} a_{k,J}\Upsilon(\xi_k^{(J)}) = \pi^{(p-1)b}\gamma^{-1}\Delta(\eta)$ we have that $\pi^{(p-1)b}\gamma^{-1}\Delta(\pi^s\eta) \in \mathcal{F}^1\Omega_{C(b,q-1,\delta)}^{n+r-2}\langle Y \rangle^{(D)}$. So by the reduction map (3-5) $d_{\langle Y \rangle}(F) \wedge \overline{\pi^s\eta} = 0$.

Since $H^{n+r-2}(\Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^\bullet\langle Y \rangle^{(D)}) = 0$, there exists $\zeta \in \Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^{n+r-3}\langle Y \rangle^{(D)}$ such that $\overline{\pi^s\eta} = d_{\langle Y \rangle}(F) \wedge \zeta$. Now choose a lifting $\hat{\zeta} \in \mathcal{F}^0\Omega_{C(b,q-1,\delta)}^{n+r-3}\langle Y \rangle^{(D)}$. Then $\pi^s\eta - \pi^{(p-1)b}\gamma^{-1}\Delta(\hat{\zeta}) \in \mathcal{F}^1\Omega_{C(b,q-1,\delta)}^{n+r-3}\langle Y \rangle^{(D)}$, so that $\eta_1 = \eta - \pi^{(p-1)b}\gamma^{-1}\Delta(\pi^{-s}\hat{\zeta}) \in \mathcal{F}^{-(s-1)}\Omega_{C(b,q-1,\delta)}^{n+r-2}\langle Y \rangle^{(D)}$. Notice that $\pi^{(p-1)b}\gamma^{-1}\Delta(\eta_1) = \pi^{(p-1)b}\gamma^{-1}\Delta(\eta)$ and in particular that $\sum_{k,J} a_{k,J}\Upsilon(\xi_k^{(J)}) = \pi^{(p-1)b}\gamma^{-1}\Delta(\eta_1)$.

We can repeat this process inductively and reduce s to 0, so that we may assume the above relation with $\eta_1 \in \mathcal{F}^0\Omega_{C(b,q-1,\delta)}^{n+r-2}\langle Y \rangle^{(D)}$.

It follows using (3-5) that $\sum_{k,J} \bar{a}_{k,J}\Upsilon(\bar{\xi}_k^{(J)}) = d_{\langle Y \rangle}(F) \wedge \bar{\eta}$, with at least one $\bar{a}_{k,J} \neq 0$. Therefore the set of elements $\Upsilon(\bar{\xi}_k^{(J)})$ are linearly dependent in the space $H^{n+r-1}(\Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^\bullet\langle Y \rangle^{(D)})$. However, the set of $\bar{\xi}_k^{(J)}$ are linearly independent in the space $H^{n+r}(\Omega_{\mathbb{F}_q[x]/\mathbb{F}_q}^\bullet\langle Y \rangle^{(D)})$ and Υ is an isomorphism, which gives a contradiction. This contradiction shows that the set of $\Upsilon([\xi_k^{(J)}])$ are linearly independent in the space $H^{n+r-1}(\Omega_{C(b,q-1,\delta)}^\bullet\langle Y \rangle^{(D)})$; hence the set of $[\xi_k^{(J)}]$ must be linearly independent in $H^{n+r}(\Omega_{C(b,q-1,\delta)}^\bullet\langle Y \rangle^{(D)})$.

To show that the set $\{[\xi_k^{(J)}] : 1 \leq k \leq \mu_J, 0 \leq J \leq n-1\}$ spans the space $H^{n+r}(\Omega_{C(b,q-1,\delta)}^\bullet\langle Y \rangle^{(D)})$, we take $\eta \in \mathcal{F}^0\Omega_{C(b,q-1,\delta)}^{n+r}\langle Y \rangle^{(D)}$ and show that it must be of the form

$$\eta = \sum_{i,j} c_{i,j} \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k)} dx_1 \dots dx_n \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r} + \pi^{(p-1)b}\gamma^{-1}\Delta(\zeta)$$

for some $c_{i,j} \in \tilde{\Gamma}_0$ and $\zeta \in \mathcal{F}^0\Omega_{C(b,q-1,\delta)}^{n+r-1}\langle Y \rangle^{(D)}$. Using the isomorphism in (3-5) we can write

$$(3-30) \quad \eta = \sum_{i,j} c_{i,j,1} \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k)} dx_1 \dots dx_n \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r} + \pi\eta_1 + \pi^{(p-1)b}\gamma^{-1}\Delta(\zeta_1),$$

where $c_{i,j,1} \in \tilde{\Gamma}_0$, and $c_{i,j,1} \prod x_i^{u_i} \prod y_j^{v_j} \in \mathcal{F}^0C(b,d,\delta)^{(0)}$; also $\eta_1 \in \mathcal{F}^0\Omega_{C(b,d,\delta)}^{n+r}\langle Y \rangle^{(D)}$ and $\zeta_1 \in \mathcal{F}^0\Omega_{C(b,d,\delta)}^{n+r-1}\langle Y \rangle^{(D)}$.

Suppose we can also write

$$(3-31) \quad \eta = \sum_{i,j} c_{i,j,s} \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k)} dx_1 \dots dx_n \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r} + \pi^s\eta_s + \pi^{(p-1)b}\gamma^{-1}\Delta(\zeta_s),$$

where each $c_{i,j,s} \in \tilde{\Gamma}_0$, and $c_{i,j,s} \prod x_i^{u_i} \prod y_j^{v_j} \in \mathcal{F}^0 C(b, q-1, \delta)^{(0)}$; also $\eta_s \in \mathcal{F}^0 \Omega_{C(b, q-1, \delta)}^{n+r} \langle Y \rangle^{(D)}$ and $\zeta_s \in \mathcal{F}^0 \Omega_{C(b, q-1, \delta)}^{n+r-1} \langle Y \rangle^{(D)}$ such that

$$(c_{i,j,s} - c_{i,j,s-1}) \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k)} \in \mathcal{F}^{s-1} \Omega_{C(b, q-1, \delta)}^{n+r} \langle Y \rangle^{(D)}$$

and $\zeta_s - \zeta_{s-1} \in \mathcal{F}^{s-1} \Omega_{C(b, q-1, \delta)}^{n+r-1} \langle Y \rangle^{(D)}$.

We can also write for any $s \geq 1$,

$$\eta_s = \sum_{i,j} c'_{i,j} \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k)} dx_1 \dots dx_n \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r} + \pi \eta_{s+1} + \pi^{(p-1)b} \gamma^{-1} \Delta(\zeta'_s),$$

with $c'_{i,j} \in \tilde{\Gamma}_0$, and $c'_{i,j} \prod x_i^{u_i} \prod y_j^{v_j} \in \mathcal{F}^0 C(b, q-1, \delta)$; also we have $\eta_{s+1} \in \mathcal{F}^0 \Omega_{C(b, q-1, \delta)}^{n+r} \langle Y \rangle^{(D)}$ and $\zeta'_s \in \mathcal{F}^0 \Omega_{C(b, q-1, \delta)}^{n+r-1} \langle Y \rangle^{(D)}$.

If we set $c_{i,j,s+1} = c_{i,j,s} + \pi^s c'_{i,j}$ and $\zeta_{s+1} = \zeta_s + \pi^s \zeta'_s$, then substituting into (3-31) we have

$$\begin{aligned} \eta &= \sum_{i,j} c_{i,j,s+1} \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k)} dx_1 \dots dx_n \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r} \\ &\quad + \pi^{(s+1)} \eta_{s+1} + \pi^{(p-1)b} \gamma^{-1} \Delta(\zeta_{s+1}). \end{aligned}$$

Now if we let $c_{i,j} = \lim_{s \rightarrow \infty} c_{i,j,s} \in \Gamma_0$ and $\zeta = \lim_{s \rightarrow \infty} \zeta_s \in \mathcal{F}^0 \Omega_{C(b, q-1, \delta)}^{n+r-1} \langle Y \rangle^{(D)}$, then we have

$$\eta = \sum_{i,j} c_{i,j} \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k)} dx_1 \dots dx_n \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r} + \pi^{(p-1)b} \gamma^{-1} \Delta(\zeta),$$

which completes the proof. $\square$

If we now put the factor of $\prod_{j=1}^r y_j^{\delta_j/(q-1)}$ back in we won't have affected the cohomology, so we have that for $0 \leq J \leq n-1$ and $1 \leq k \leq \mu_J$,

$$\begin{aligned} \widehat{\xi}_k^{(J)}(dx) \left(\frac{dy}{y} \right) &= \pi^{\frac{(p-1)b(\sum_{j=1}^r \delta_j)}{(q-1)}} \prod_{j=1}^r y_j^{\delta_j/(q-1)} \xi_k^{(J)}(dx) \left(\frac{dy}{y} \right) \\ &= \pi^{\frac{(p-1)b(J(q-1) + \sum_{j=1}^r \delta_j)}{(q-1)}} \prod_{i=1}^n x_i^{u_i(J,k)} \prod_{j=1}^r y_j^{v_j(J,k) + \frac{\delta_j}{q-1}} dx_1 \dots dx_n \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r} \end{aligned}$$

form a basis for $H^{n+r}(\Omega_{C(b, q-1, \delta)}^\bullet \langle Y \rangle)^{(0)}$.

4. HILBERT SERIES

Grading by $\deg_2 = J$, we can use combinatorial methods to get a formula for the Hilbert series of the Dwork complex $\Omega^\bullet \langle Y \rangle^{(D, \bullet)}$, which we use to then deduce a formula for the Hilbert series of the non-zero cohomology space of dimension $n+r$. This formula (4-15) depends on q , the degrees d_j and the original choice of multiplicative characters.

We fix a choice of characters, that is, we fix a δ , and we denote the Hilbert series for the $(n+r)$ -dimensional cohomology by $H_\delta(t)$. It will follow from our formula that $H_\delta(t)$ is a polynomial of degree $n-1$, and we write $H_\delta(t) = \sum_{J=0}^{n-1} \mu_J t^J$, where $\mu_J = \dim H^{n+r}(\Omega_{\mathbb{F}_q[x,y]/\mathbb{F}_q}^\bullet \langle Y \rangle)^{(D, J)}$.

Consider $\Omega^\bullet \langle Y \rangle^{(D, \bullet)}$ as the sum of complexes

$$(4-1) \quad 0 \longrightarrow \Omega^0 \langle Y \rangle^{(D, J-n-r)} \longrightarrow \dots \longrightarrow \Omega^{n+r} \langle Y \rangle^{(D, J)} \longrightarrow 0,$$

for $0 \leq J < \infty$. From the definition of $\deg_2$, it is clear that $\Omega^k \langle Y \rangle^{(D, J)} = 0$ for any k whenever $J < 0$. We define the Hilbert series of $\Omega^\bullet \langle Y \rangle^{(D, \bullet)}$ as

$$(4-2) \quad K(t) = \sum_{J=0}^{\infty} \left(\sum_{k=0}^{n+r} (-1)^{k+n+r} d(k, J) \right) t^J,$$

where $d(k, J) = \dim \Omega^k \langle Y \rangle^{(D, J+k-n-r)}$, the dimension of the k th element in the J th row in (4-1).

By definition $\Omega^k \langle Y \rangle^{(D, J)}$ is a sum of spaces of the form

$$\mathbb{F}_q[x, y]^{(D-\ell)} dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}.$$

Therefore, we start by counting elements in a canonical basis for such spaces given by forms of the type $x_1^{u_1} \dots x_n^{u_n} y_1^{v_1} \dots y_r^{v_r} dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}}$. The restriction on $\deg_1$ gives $D = u_1 + \dots + u_n - d_1 v_1 - \dots - d_r v_r + \ell$, and for $\deg_2$ we have $J = v_1 + \dots + v_r$.

Fixing $v_1, \dots, v_r$ and ℓ , we have that the number of sets $\{u_1, \dots, u_n\}$ satisfying the condition on $\deg_1$ is given by the binomial coefficient

$$\binom{d_1 v_1 + \dots + d_r v_r - \ell + D + n - 1}{n - 1}.$$

Defining the polynomial

$$(4-3) \quad P_\ell(v_1, \dots, v_r) = \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (d_1 v_1 + \dots + d_r v_r - \ell + D + k),$$

it is clear that

$$P_\ell(v_1, \dots, v_r) = \binom{d_1 v_1 + \dots + d_r v_r - \ell + D + n - 1}{n - 1}.$$

Now consider the series in r variables, generated by $P_\ell(v_1, \dots, v_r)$, associated to the index $j_1, \dots, j_m$,

$$(4-4) \quad K_\ell(j_1, \dots, j_m; t_1, \dots, t_r) = \sum_{v_1, \dots, v_r=0}^{\infty} P_\ell(v_1, \dots, v_r) t_1^{v_1} \dots t_r^{v_r}.$$

Therefore for fixed ℓ and fixed $j_1, \dots, j_m$, by setting each $t_j = t$ we have

$$(4-5) \quad K_\ell(j_1, \dots, j_m; t, \dots, t) = \sum_{J=0}^{\infty} \left(\dim_{\mathbb{F}_q} (\mathbb{F}_q[x, y] dx_{i_1} \dots dx_{i_\ell} \frac{dy_{j_1}}{y_{j_1}} \dots \frac{dy_{j_m}}{y_{j_m}})^{(D, J)} \right) t^J.$$

Summing down the columns of (4-1) we can now use (4-5) to write

$$(4-6) \quad K(t) = \sum_{\ell=0}^n \sum_{m=0}^r \sum_{\{1 \leq j_1 < \dots < j_m \leq r\}} (-1)^{n+r-\ell-m} \binom{n}{\ell} t^{n+r-\ell-m} K_\ell(j_1, \dots, j_m; t, \dots, t).$$

Note that for any set of indices of length m , we have $K_\ell(j_1, \dots, j_m; t, \dots, t) = K_\ell(j'_1, \dots, j'_m; t, \dots, t)$ for all $m = 0, \dots, r$. Therefore we can simplify (4-6) slightly, by setting $K_\ell(m; t_1, \dots, t_r) = K_\ell(j_1, \dots, j_m; t_1, \dots, t_r)$ for each m , which yields

$$(4-7) \quad K(t) = \sum_{\ell=0}^n \sum_{m=0}^r (-1)^{n+r-\ell-m} \binom{n}{\ell} \binom{r}{m} t^{n+r-\ell-m} K_\ell(m; t, \dots, t).$$

Since P_ℓ is a polynomial, in $v_1, \dots, v_r$, of degree $n-1$ we can write

$$(4-8) \quad P_\ell(v_1, \dots, v_r) = \sum_{e_1 + \dots + e_r \leq n-1} A_\ell(e_1, \dots, e_r) v_1^{e_1} \dots v_r^{e_r},$$

for some rational coefficients $A_\ell(e_1, \dots, e_r)$. If we set $E = e_1 + \dots + e_r$ and let $S_i(z_1, \dots, z_{n-1})$ denote the i th elementary symmetric function in $n-1$ variables, then one can prove (by induction on r) that these coefficients satisfy

$$(4-9) \quad A_\ell(e_1, \dots, e_r) = \frac{(-1)^{n-1-E} E!}{(n-1)! e_1! \dots e_r!} S_{n-1-E}(\ell - D - (n-1), \dots, \ell - D - 1) d_1^{e_1} \dots d_r^{e_r}.$$

Making this substitution we can now write

$$(4-10) \quad K_\ell(m; t_1, \dots, t_r) = \sum_{E \leq n-1} A_\ell(e_1, \dots, e_r) \sum_{v_1, \dots, v_r=0}^{\infty} v_1^{e_1} \dots v_r^{e_r} t_1^{v_1} \dots t_r^{v_r}.$$

If we define the polynomial $Q_e(t)$ by the relation on Taylor series

$$(4-11) \quad \sum_{v=0}^{\infty} v^e t^v = \left(t \frac{d}{dt} \right)^e \frac{1}{1-t} = \frac{Q_e(t)}{(1-t)^{e+1}},$$

then we can rewrite (4-10) as

$$(4-12) \quad K_\ell(m; t, \dots, t) = \sum_{E \leq n-1} A_\ell(e_1, \dots, e_r) \frac{Q_{e_1}(t) \dots Q_{e_r}(t)}{(1-t)^{E+r}}.$$

Using the binomial coefficients from the expansion of $(1-t)^r$ and fixing $e_1, \dots, e_r$ we also have

$$(4-13) \quad \sum_{m=0}^r (-1)^{r-m} \binom{r}{m} t^{r-m} Q_{e_1}(t) \dots Q_{e_r}(t) = (1-t)^r Q_{e_1}(t) \dots Q_{e_r}(t).$$

These substitutions reduce the Hilbert series of $\Omega^\bullet \langle Y \rangle^{(D, \bullet)}$ to

$$(4-14) \quad K(t) = \sum_{\ell=0}^n (-1)^{n-\ell} \binom{n}{\ell} t^{n-\ell} \sum_{E \leq n-1} A_\ell(e_1, \dots, e_r) \frac{Q_{e_1}(t) \dots Q_{e_r}(t)}{(1-t)^E}.$$

Now from vanishing of the cohomology except for dimensions $n+r-1$ and $n+r$, we have

$$\begin{aligned} \sum_{k=0}^{n+r} (-1)^k \dim \Omega^k \langle Y \rangle^{(D, \bullet)} &= \sum_{k=0}^{n+r} (-1)^k \dim H^k(\Omega^\bullet \langle Y \rangle^{(D, \bullet)}) \\ &= (-1)^{n+r-1} \dim H^{n+r-1}(\Omega^\bullet \langle Y \rangle^{(D, \bullet)}) \\ &\quad + (-1)^{n+r} \dim H^{n+r}(\Omega^\bullet \langle Y \rangle^{(D, \bullet)}). \end{aligned}$$

In particular for each J ,

$$\begin{aligned} \sum_{k=0}^{n+r} (-1)^k \dim \Omega^k \langle Y \rangle^{(D, J+k-n-r)} &= (-1)^{n+r} \sum_{k=0}^{n+r} (-1)^k d(k, J) \\ &= (-1)^{n+r-1} \dim H^{n+r-1}(\Omega^\bullet \langle Y \rangle^{(D, J-1)}) \\ &\quad + (-1)^{n+r} \dim H^{n+r}(\Omega^\bullet \langle Y \rangle^{(D, J)}). \end{aligned}$$

It follows that $K(t) = (1-t)H_\delta(t)$. So by (4-14) we have that

$$\begin{aligned} (4-15) \quad H_\delta(t) &= \sum_{\ell=0}^n (-1)^{n-\ell} \binom{n}{\ell} t^{n-\ell} \sum_{E \leq n-1} A_\ell(e_1, \dots, e_r) \frac{Q_{e_1}(t) \cdots Q_{e_r}(t)}{(1-t)^{E+1}} \\ &= \sum_{E \leq n-1} \left(\sum_{\ell=0}^n (-1)^{n-\ell} \binom{n}{\ell} A_\ell(e_1, \dots, e_r) t^{n-\ell} \right) \frac{Q_{e_1}(t) \cdots Q_{e_r}(t)}{(1-t)^{E+1}}. \end{aligned}$$

Using (4-15) we can also develop a related formula for $H_\delta(1)$ that gives the degree of the L -function associated to the sum (1-6) and will be independent of δ . Define the polynomial $\tilde{A}_{e_1, \dots, e_r}(t)$ by

$$\tilde{A}_{e_1, \dots, e_r}(t) = \sum_{\ell=0}^n (-1)^{n-\ell} \binom{n}{\ell} A_\ell(e_1, \dots, e_r) t^{n-\ell},$$

which simplifies $H_\delta(t)$ to

$$H_\delta(t) = \sum_{E \leq n-1} \tilde{A}_{e_1, \dots, e_r}(t) \frac{Q_{e_1}(t) \cdots Q_{e_r}(t)}{(1-t)^{E+1}}.$$

Now from the definitions of $A_\ell(e_1, \dots, e_r)$ and $\tilde{A}_{e_1, \dots, e_r}(t)$ we have that $(1-t)^{E+1}$ divides $\tilde{A}_{e_1, \dots, e_r}(t)$. Indeed, consider $A_\ell(e_1, \dots, e_r)$ as a polynomial in the variable ℓ of degree $n-1-E$. We can therefore write $\tilde{A}_{e_1, \dots, e_r}(t)$ as a linear combination of the polynomials

$$(4-16) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} k^i t^k = \left(t \frac{d}{dt} \right)^i (1-t)^n$$

for $i = 0, \dots, n-1-E$, which are all divisible by $(1-t)^{E+1}$.

By Taylor's formula the value at $t = 1$ of $\tilde{A}_{e_1, \dots, e_r}(t)/(1-t)^{E+1}$ is equal to the value at $t = 1$ of

$$\frac{(-1)^{E+1}}{(E+1)!} \left(\frac{d}{dt} \right)^{E+1} \tilde{A}_{e_1, \dots, e_r}(t),$$

which is equal to the value at $t = 1$ of

$$(4-17) \quad \frac{(-1)^{E+1}}{(E+1)!} \left(t \frac{d}{dt} \right)^{E+1} \tilde{A}_{e_1, \dots, e_r}(t).$$

Similarly, the polynomial $\left(t \frac{d}{dt} \right)^{E+1} \tilde{A}_{e_1, \dots, e_r}(t)$ can be written as a linear combination of the polynomials in (4-16) for $i = E+1, \dots, n$. In this linear combination we compute the coefficient of $\left(t \frac{d}{dt} \right)^n (1-t)^n$ to be

$$\frac{d_1^{e_1} \cdots d_r^{e_r}}{(n-1-E)! e_1! \cdots e_r!}.$$

It is clear that the values at $t = 1$ for the polynomials in (4-16) are all zero for $i < n$, while for $i = n$ the value at $t = 1$ is $(-1)^n n!$. Therefore the value at $t = 1$ of (4-17) is

$$(4-18) \quad (-1)^{n+E+1} \binom{n}{E+1} \frac{d_1^{e_1} \cdots d_r^{e_r}}{e_1! \cdots e_r!}.$$

Now, since $Q_e(1) = e!$ it follows that

$$(4-19) \quad H_\delta(1) = \sum_{E \leq n-1} (-1)^{n+E+1} \binom{n}{E+1} d_1^{e_1} \cdots d_r^{e_r}.$$

5. NEWTON POLYGON OF $L(f, \chi; t)$

Recall that if $v_j \equiv \delta_j \pmod{q-1}$, then $p\delta'_j \equiv v_j \pmod{q-1}$. Recursively, for $I = 0, \dots, a-1$, we define $\delta^{(I)} = (\delta_1^{(I)}, \dots, \delta_r^{(I)}) = (\delta^{(I-1)})'$ so that $p^I \delta^{(I)} \equiv v_j \pmod{q-1}$. For each combination of δ 's, i.e. for different combinations of multiplicative characters, the dimensions μ_J may be different and we define $\mu_J^{(I)}$ accordingly. Construct an array with a rows and $H_\delta(1)$ columns. Row 1 will be associated to $\delta^{(0)}$, row 2 with $\delta^{(1)}$, and so on. Each row will have integer entries ranging from 0 to $n-1$, which are taken $\mu_J^{(I)}$ at a time and ranked in ascending order. For $1 \leq s \leq H_\delta(1)$, let $\bar{J}_s$ be the average of the entries in the s th column.

Theorem 5.1. *Let $f_1, \dots, f_r$ be non-constant homogeneous polynomials with coefficients in $\mathbb{F}_q$ and with $d_j = \deg f_j$. Assume $(p, d_j) = 1$ for each $j = 1, \dots, r$. If $\chi_1, \dots, \chi_r$ are non-trivial multiplicative characters on $\mathbb{F}_q$ such that $\prod_{j=1}^r \chi_j^{d_j} = \chi_0$ and if the set of solutions to the equation $\prod_{j=1}^r f_j(x) = 0$ defines a divisor with normal crossings, then the Newton polygon of the L -function in (1-3) (with respect to ord_q) sits on or above the Newton polygon of the polynomial*

$$(5-1) \quad \prod_{s=1}^{H_\delta(1)} (1 - q^{\bar{J}_s} t).$$

Proof. Applying Theorem 3.1 to (2-14) gives

$$L(f, \chi, \psi, T^*; t) = \prod_{k=n+r-1}^{n+r} \det \left(I - t\alpha_k | H^k(\Omega_{C(b,d,\delta)}^\bullet \langle Y \rangle, \Delta) \right)^{(-1)^{k+1}}.$$

For the rest of this chapter we abbreviate the notation and let $\det(I - t\alpha_{n+r}) = \det(I - t\alpha_{n+r} | H^{n+r}(\Omega_{C(b,q-1,\delta)}^\bullet \langle Y \rangle)^{(0)})$, and we note that the determinants $\det(I - t\alpha_{n+r-1})$ and $\det(I - q\alpha_{n+r})$ are equal following the method of [6], Theorem 6.16.

We will analyze α_{n+r} using the related operator β_{n+r} . The appearance of Ψ , in the definition of β , will have the effect of permuting the spaces $C(b, q-1, \delta)$ indexed by certain δ 's. As a result, we need to consider a separate maps of the type $\beta : C(b, q-1, \delta)^{(0)} \rightarrow C(b, q-1, \delta')^{(0)}$. This induces the map $\beta_{n+r} : \Omega_{C(b,q-1,\delta^{(I)})}^{n+r} \langle Y \rangle^{(0)} \rightarrow \Omega_{C(b,q-1,\delta^{(I+1)})}^{n+r} \langle Y \rangle^{(0)}$. For a fixed, acceptable choice of characters $\chi_1, \dots, \chi_r$ we will denote the associated cohomology $H^{n+r}(\delta)$, where δ represents the r -tuple $\delta_1, \dots, \delta_r$ as before. Now in general, for $I = 1, \dots, a-1$, as β_{n+r} cycles through the spaces $C(b, q-1, \delta^{(I)})$ we will write $\beta_{n+r} : H^{n+r}(\delta^{(I)}) \rightarrow H^{n+r}(\delta^{(I+1)})$, where it is understood that $\delta = \delta^{(0)}$. It follows that β_{n+r} is an endomorphism on the space $\mathbf{H}^{n+r} = \bigoplus_{I=0}^{a-1} H^{n+r}(\delta^{(I)})$, and also that the map α_{n+r}

is both a $\tilde{\Gamma}_0$ -linear and $\tilde{\Gamma}_1$ -linear endomorphism of $\mathbf{H}^{n+r}$ and also of each of the spaces $H^{n+r}(\delta^{(I)})$.

With notation similar to Section 3.5, for $0 \leq I \leq a-1$, $0 \leq J \leq n-1$ and $1 \leq k \leq \mu_J^{(I)}$ we let u_i and v_j be integers depending on k, J and I so that if

$$\xi_{k,I}^{(J)} = \prod_{i=1}^n x_i^{u_i} \prod_{j=1}^r y_j^{v_j + \delta_j^{(I)}/(q-1)},$$

then the forms $\xi_{k,I}^{(J)} dx_1 \dots dx_n \frac{dy_1}{y_1} \dots \frac{dy_r}{y_r}$ will generate a Γ_0 -basis for each of the $H^{n+r}(\delta^{(I)})$. We normalize each of these monomials with a factor of the form π^ϵ so that $\pi^\epsilon \xi_{k,I}^{(J)} \in \mathcal{F}^0 C(b/p, q-1, \delta^{(I)})^{(0)}$. Since $\beta = \Psi \circ \tau^{-1} \circ G$ maps $C(b/p, q-1, \delta^{(I)})$ into $C(b, q-1, \delta^{(I+1)})$ and since β is made up of homogeneous elements, it will map $\mathcal{F}^0 C(b/p, q-1, \delta^{(I)})$ to $\mathcal{F}^0 C(b, q-1, \delta^{(I+1)})$.

We choose a basis $\gamma_1, \dots, \gamma_a$ for $\tilde{\Gamma}_0$ over $\tilde{\Gamma}_1$ satisfying p -adic directness (see [9], Section 3c), that is, for any $c_1, \dots, c_a \in \tilde{\Gamma}_1$, $\text{ord}_p \sum_{s=1}^a c_s \gamma_s = \min\{\text{ord}_p c_s\}$.

It follows that the classes of $\gamma_s \pi^\epsilon \xi_{k,I}^{(J)}$ will form a $\tilde{\Gamma}_1$ -basis for $H^{n+r}(\chi, \delta^{(I)})$. We denote the $a \cdot H_\delta(1) \times a \cdot H_\delta(1)$ matrix for $\beta_{n+r} : H^{n+r}(\chi, \delta^{(I)}) \rightarrow H^{n+r}(\chi, \delta^{(I+1)})$ by $[\mathbf{B}(I; I+1)]$, with entries $\mathbf{B}(s, u, v, I; s', u', v', I+1)$, where (dropping the $dx_1 \wedge \dots \wedge dx_n \wedge \frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_r}{y_r}$),

$$(5-2) \quad \beta_{n+r}(\gamma_s \pi^\epsilon \xi_{k,I}^{(J)}) = \sum_{s', k', J'} \mathbf{B}(s, u, v, I; s', u', v', I+1) \gamma_{s'} \pi^{\epsilon'} \xi_{k', I+1}^{(J')}.$$

These a matrices will form the only non-zero blocks of the matrix of the $\tilde{\Gamma}_1$ -endomorphism $\beta_{n+r} : \mathbf{H}^{n+r} \rightarrow \mathbf{H}^{n+r}$, which takes the form

$$(5-3) \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & \dots & \dots & [\mathbf{B}(a-1; 0)] \\ [\mathbf{B}(0; 1)] & 0 & & & 0 \\ \vdots & [\mathbf{B}(1; 2)] & & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & [\mathbf{B}(a-2; a-1)] & 0 \end{bmatrix}.$$

It is straightforward to show that $\mathbf{B}^a$ is of the form

$$(5-4) \quad \mathbf{B}^a = \begin{bmatrix} B_1 & 0 & \dots & \dots & 0 \\ 0 & B_2 & & & 0 \\ \vdots & & B_3 & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & B_a \end{bmatrix},$$

where the blocks down the diagonal are a product of all a of the blocks $[\mathbf{B}(I; I+1)]$ in some order. The matrix for $\alpha_{n+r} : H^{n+r}(\chi, \delta) \rightarrow H^{n+r}(\chi, \delta)$, call it $\mathbf{A}$, which is equal to the block of $\mathbf{B}^a$ labeled B_1 in (5-4), is written

$$(5-5) \quad \mathbf{A} = [\mathbf{B}(a-1; 0)] \cdot [\mathbf{B}(a-2; a-1)] \cdot \dots \cdot [\mathbf{B}(1; 2)] \cdot [\mathbf{B}(0; 1)].$$

Write $\det(I - t\mathbf{B}) = 1 + \sum_{k=1}^N b_k t^k$, where $N = a^2 \cdot H_\delta(1)$ is the number of rows of $\mathbf{B}$. Recall that $b_k = 0$ unless $a|k$, which says that $\det(I - t\mathbf{B})$ is actually a

polynomial in t^a , so we can rewrite

$$(5-6) \quad \det(I - t\mathbf{B}) = \prod_{i=1}^{N/a} (1 - \lambda_i t^a).$$

Now since the eigenvalues of $\mathbf{B}^a$ are just a th powers of the eigenvalues of $\mathbf{B}$ we have

$$(5-7) \quad \det(I - t\mathbf{B}^a) = \prod_{i=1}^{N/a} (1 - \lambda_i t)^a.$$

Also in $\mathbf{B}^a$ we see that each non-zero block down the diagonal is conjugate to $\mathbf{A}$ so that $[\det(I - t\mathbf{A})]^a = \det\left(I - t \prod_{j=1}^a B_j\right)$, where the product on the right is of the blocks of (5-4) in any acceptable order. Therefore $[\det(I - t\mathbf{A})]^a = \det(I - t\mathbf{B}^a)$, so that using (5-6) and (5-7) we get

$$(5-8) \quad \begin{aligned} \det(I - t\mathbf{A}) &= \det(I - \mathbf{B}t^{1/a}) \\ &= \prod_{i=1}^{N/a} (1 - \lambda_i t) \\ &= 1 + b_a t + \cdots + b_N t^{N/a}. \end{aligned}$$

From Section 5 of [17] we have the relation

$$(5-9) \quad b_m = (-1)^m \sum_{\sigma \in S_m} \sum_{\sigma \in S_m} \text{sign}(\sigma) \prod_{k=1}^m \mathbf{B}(j_k, u_k, v_k I; j_{\sigma(k)}, u_{\sigma(k)}, v_{\sigma(k)}, I+1),$$

where the outer sum is taken over all sets $\{(j_k, u_k, v_k)\}_{k=1}^m$ of m distinct triples in the set $\{1, \dots, a\} \times \{\bigcup_{I=0}^{a-1} H^{n+r}(\delta^{(I)})\}$ (basis elements taken one at a time from each $H^{n+r}(\delta^{(I)})$ in sets of size a), and the inner sum is taken over all permutations σ on m letters, with signature $\text{sign}(\sigma)$. Moreover we may assume that $x^{u_k} y^{v_k} \in H^{n+r}(\chi_1^{(I)}, \dots, \chi_r^{(I)})$ and $x^{u_{\sigma(k)}} y^{v_{\sigma(k)}} \in H^{n+r}(\chi_1^{(I+1)}, \dots, \chi_r^{(I+1)})$ for some I and k .

Now the image of a basis element under β_{n+r} will be in the filtered space $\mathcal{F}^0 \Omega_{C(b, q-1, \delta^{(I+1)})}^{n+r} \langle Y \rangle^{(0)}$, so it is a sum of terms $A(u', v') \pi^{\frac{(p-1)b|v'|}{q-1}} x^{u'} y^{v'}$, where $A(u', v') \in \tilde{\pi}^0 \mathcal{O}_{\tilde{\Gamma}_0}$. Therefore, using (5-2), the factor $\mathbf{B}(s, u, v, I; s', u', v', I+1) \pi^{\epsilon'}$ will be divisible by $\pi^{\frac{(p-1)b[\sum v'_j(q-1) + \delta_j^{(I+1)}]}{q-1}}$, and therefore $\mathbf{B}(s, u, v, I; s', u', v', I+1)$ is divisible by

$$\pi^{\frac{(p-1)b[\sum v'_j(q-1) + \delta_j^{(I+1)}]}{q-1} - \frac{(p-1)b[\sum v'_j(q-1) + \delta_j^{(I+1)}]}{p(q-1)}} = \pi^{\frac{(p-1)b[\sum v'_j(q-1) + \delta_j^{(I+1)}]}{q-1}} \cdot (1 - \frac{1}{p}).$$

Now in each of the non-zero blocks $[\mathbf{B}(I; I+1)]$, if we put the basis elements in ascending order ranked by J , then the first $a\mu_0^{(I)}$ rows will be divisible by $\pi^{\frac{(p-1)b[\sum \delta_j^{(I+1)}]}{q-1}} (1-1/p)$, and the next $a\mu_1^{(I)}$ rows by $\pi^{\frac{(p-1)b[(q-1) + \sum \delta_j^{(I+1)}]}{q-1}} (1-1/p)$, and so on, with the last $a\mu_{n-1}^{(I)}$ rows divisible by $\pi^{\frac{(p-1)b[(n-1)(q-1) + \sum \delta_j^{(I+1)}]}{q-1}} (1-1/p)$.

We construct an array with a rows, corresponding to each $I = 0, \dots, a-1$ and $H_\delta(1)$ columns corresponding to the divisors of the $a\mu_J^{(I)}$ th row of the matrix $\mathbf{B}$. The first $\mu_0^{(0)}$ entries of row 1 are all equal to $\pi^{\frac{(p-1)b(0(q-1) + \sum \delta_j^{(I)})}{q-1}} (1-1/p)$, and the

next $\mu_1^{(0)}$ entries are $\pi^{\frac{(p-1)b(1(q-1)+\sum \delta_j^{(I)})}{q-1}(1-1/p)}$, and so on, filling out the first row. The rest of the rows are constructed similarly. We note at this point that these values are independent of $b < p/(p-1)$ and we can simplify the array by taking the limit as $b \rightarrow p/(p-1)$, which will cancel out the factor of $(1-1/p)$. This simplifies each of the entries to $\pi^{J+\sum \delta_j^{(I)}/(q-1)}$ for appropriate J and I .

Simplify this array by taking just the exponent of π , so that the first $\mu_0^{(0)}$ entries of the first row are $0 + \sum \delta_j^{(0)}/(q-1)$, the next $\mu_1^{(0)}$ entries of the first row are $1 + \sum \delta_j^{(0)}/(q-1)$, and so on. For the entry in the I th row and the k th column, we label these entries $J(I, k) + \sum \delta_j^{(I)}/(q-1)$. In this case, the average down the k th column is

$$\frac{1}{a} \sum_{I=0}^{a-1} \left(J(I, k) + \sum_{j=1}^r \delta_j^{(I)}/(q-1) \right) = \frac{1}{a} \sum_{I=0}^{a-1} J(I, k) + \frac{1}{a} \sum_{I=0}^{a-1} \sum_{j=1}^r \delta_j^{(I)}/(q-1),$$

for each $k = 1, \dots, H_\delta(1)$. From Stickelberger's theorem we see that the last term on the right is the q -ordinal of the product of Gauss sums in the formula for $T^*(f, \chi, \psi)$. We remove this factor from each entry of the array, so that the average down each column is reduced to $\bar{J}_k = \frac{1}{a} \sum_{I=0}^{a-1} J(I, k)$.

Since we now have a lower bound on the q -ordinal of each entry in the matrix, we can get a lower bound for each b_m by taking m columns of the array at a time. It follows from [3] that coordinates for the Newton polygon of $\det(I - t\mathbf{A})$ are given by $(0, 0)$ and

$$(5-10) \quad \left(\sum_{k=0}^M \bar{J}_k, \frac{1}{a} \sum_{k=0}^M k \bar{J}_k \right),$$

$M = 0, 1, \dots, n-1$. Since we have ordered the array as increasing from left to right, these vertices are the same as the vertices obtained by taking as the first coordinate pair $(1, \bar{J}_1)$, the second $(2, \bar{J}_2)$, and so on, which is equivalent to the statement in the theorem. $\square$

6. EXAMPLES

6.1. Stickelberger's theorem. Let $n = 1$ and $r = 1$. The space $\mathbb{P}^0$ consists of a single point, and the polynomial $f(x) = x^d$ satisfies the definition of a normal crossing divisor. The corresponding affine character sum is

$$\sum_{x \in \mathbb{F}_q^\times} \chi(x^d) = q - 1.$$

Since we are assuming that $\chi^d = \chi_0$, we must have $d|(q-1)$.

The related sum $T^*(f, \chi, \psi)$ gives

$$\begin{aligned} \sum_{x, y \in (\mathbb{F}_q)^\times} \chi^{-1}(y) \psi(yx^d) &= g(\chi^{-1}, \psi) \cdot \sum_{x \in \mathbb{F}_q^\times} \chi(x^d) \\ &= (q-1) \cdot g(\chi^{-1}, \psi). \end{aligned}$$

If we write $\chi = \hat{\chi}^\delta$, where $\hat{\chi}$ is the Teichmüller character on $\mathbb{F}_q$, then $1 \leq \delta \leq q-2$ and we can write $\delta = j_0 + j_1p + \dots + j_{a-1}p^{a-1}$ to define the integers $j_0, \dots, j_{a-1}$.

The estimates for T^* give

$$\begin{aligned}\mathrm{ord}_q(g(\chi^{-1}, \psi)) &\geq \frac{1}{a} \left(\frac{\delta^{(0)} + \delta^{(1)} + \cdots + \delta^{(a-1)}}{q-1} \right) \\ &= \frac{j_0 + j_1 + \cdots + j_{a-1}}{a(p-1)}.\end{aligned}$$

Using the same formula with $(q-1) - \delta^{(I)}$ replacing $\delta^{(I)}$ we get the estimate

$$\begin{aligned}\mathrm{ord}_q g(\chi, \psi) &\geq \frac{1}{a} \sum_{I=0}^{a-1} \frac{(q-1) - \delta^{(I)}}{q-1} \\ &= 1 - \frac{j_0 + j_1 + \cdots + j_{a-1}}{a(p-1)}.\end{aligned}$$

If we take the strict inequalities, we have $\mathrm{ord}_q(g(\chi, \psi) \cdot g(\chi^{-1}, \psi)) > 1$, which gives a contradiction to the identity $g(\chi, \psi) \cdot g(\chi^{-1}, \psi) = q \cdot \chi(-1)$.

6.2. The case $n = 2$. We assume that $f_1, \dots, f_r$ are homogeneous polynomials in $\mathbb{F}_q[x, z]$. The f_j will satisfy the conditions of a normal crossing divisor if each f_j can be factored into non-repeated linear factors over $\overline{\mathbb{F}}_q$ and also if the factors are unique to each f_j . The associated character sum is

$$\sum_{(x,z) \in \mathbb{P}^1(\mathbb{F}_q)} \chi_1(f_1(x, z)) \cdots \chi_r(f_r(x, z)),$$

where $\prod_{j=1}^r \chi_j^{d_j} = \chi_0$. In this case the Hilbert series of the top cohomology is given by $H_\delta(t) = (D-1) + (r-D-1)t$.

For example, if $r = 4$, then there are three possible values for D , which are $D = 1, 2$ or 3 . If $D = 1$, then $H_\delta(t) = 2t$ and the lower bound for the Newton polygon of $L(t)$ is given by the vertices $(0,0)$ and $(2,0)$.

If $D = 2$, then $H_\delta(t) = 1 + t$ and the lower bound is given by the vertices $(0,0)$, $(1,0)$ and $(2,1)$.

In the last case where $D = 3$, we have $H_\delta(t) = 2$ and the lower bound is given by the vertices $(0,0)$ and $(2,0)$.

6.3. The case $n = 3$. In this case the normal crossing divisor will be a collection of smooth curves in $\mathbb{P}^2(\mathbb{F}_q)$, having transversal intersections. For $f_1, \dots, f_r \in \mathbb{F}_q[x, y, z]$ the character sum

$$\sum_{(x,y,z) \in \mathbb{P}^2} \chi_1(f_1(x, y, z)) \cdots \chi_r(f_r(x, y, z)),$$

where $\prod \chi_j^{d_j} = \chi_0$, will produce a Hilbert series for $H^{r+3}(\Omega^\bullet(Y))^{(0)}$ of the form $H_\delta(t) = a_0 + a_1 t + a_2 t^2$, where

$$a_0 = \frac{(D-1)(D-2)}{2}, \quad a_1 = 1 - D^2 + \sum_{j=1}^r \left(D d_j + \frac{1}{2} d_j (d_j - 3) \right),$$

and

$$a_2 = \frac{(D+1)(D+2)}{2} + \sum_{j=1}^r \left(\frac{1}{2} d_j (d_j - 3) - D d_j \right) + \sum_{1 \leq j_1 < j_2 \leq r} d_{j_1} d_{j_2}.$$

In order to illustrate the effect of the original choice of multiplicative characters, we will set $q = 81$ and $r = 3$. We first take $\delta_j = \delta'_j$ for each j , and in the second case we suppose that $\delta_j \neq \delta'_j$ but $\delta_j = \delta''_j$.

Taking the quadratic character χ on $\mathbb{F}_{81}$ and setting $\chi = \chi_j$ for each j will satisfy the conditions of the first case. If we assume for this example that $d_j = 2$ for each j the quadratic character also satisfies the necessary condition that $\prod_{j=1}^3 \chi_j^2 = \chi_0$. Therefore setting each $\delta_j = (q-1)/2$ it follows that $D = 3$. This results in the polynomial $(1-t)(1-qt)^7(1-q^2t)$. This polynomial has Newton polygon defined by the vertices $(0,0)$, $(1,0)$, $(8,7)$ and $(9,9)$.

Now consider the multiplicative characters

$$\chi_1 = \widehat{\chi}^{30}, \quad \chi_2 = \widehat{\chi}^{60}, \quad \text{and} \quad \chi_3 = \widehat{\chi}^{30}.$$

Computing the 3-tuples $\delta^{(I)}$ for $I = 0, 1, 2, 3$ gives that $\delta^{(0)} = \delta^{(2)} = (30, 60, 30)$ and $\delta^{(1)} = \delta^{(3)} = (10, 20, 10)$. The first set of values $(30, 60, 30)$ gives $D = 3$ and the set $(10, 20, 10)$ gives $D = 1$. For either set we have $H_\delta(1) = 9$, which is the degree of the associated L -function. Computing the two Hilbert series gives

$$H_{\delta^{(0)}}(t) = 1 + 7t + t^2 \quad \text{and} \quad H_{\delta^{(1)}}(t) = 3t + 6t^2.$$

The result is the polynomial $(1 - q^{\frac{1}{2}}t)(1 - qt)^2(1 - q^{\frac{3}{2}}t)^5(1 - q^2t)$ whose Newton polygon is determined by the vertices $(0,0)$, $(1,1/2)$, $(3,5/2)$, $(8,10)$, and $(9,12)$.

6.4. Rational functions. One can also consider the case of a rational function f/g using the character sum $\sum_{x \in \mathbb{P}^2} \chi_1(f(x))\chi_2(g(x))$, where $\chi_2 = \chi_1^{-1}$. If we require that $\deg f = \deg g = d$, then $D = d$ and the Hilbert series is

$$H_\delta(t) = \frac{(d-1)(d-2)}{2} + (d-1)(2d-1)t + \frac{(d-1)(d-2)}{2}t^2.$$

We compute a few for small values of d . If $d = 2$, then $H_\delta(1) = 3$ and $H_\delta(t) = 3t$. This gives a lower bound for the Newton polygon of the L -function of $(0,0)$ and $(3,3)$, which has only one segment of length 3 and slope 1.

If $d = 3$, then $H(1) = 12$ and $H(t) = 1 + 10t + t^2$, which results in the polynomial $(1-t)(1-qt)^{10}(1-q^2t)$. The lower bound is then given by a polygon that has one segment of length 1 and slope 0, a segment of length 10 and slope 1, and a segment of length 1 and slope 2 defined by the vertices $(0,0)$, $(1,0)$, $(11,10)$, $(12,12)$.

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