FIELDS OF ALGEBRAIC NUMBERS WITH BOUNDED LOCAL DEGREES AND THEIR PROPERTIES

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ABSTRACT. We provide a characterization of infinite algebraic Galois extensions of the rationals with uniformly bounded local degrees, giving a detailed proof of all the results in Checcoli and Zannier's paper (2011) and obtaining relevant generalizations for them. In particular we show that for an infinite Galois extension of the rationals the following three properties are equivalent: having uniformly bounded local degrees at every prime; having uniformly bounded local degrees at all but finitely many primes; having Galois group of finite exponent. The proof of this result enlightens interesting connections with Zel'manov's work on the Restricted Burnside Problem. We give a formula to explicitly compute bounds for the local degrees of an infinite extension in some special cases. We relate the uniform boundedness of the local degrees to other properties: being a subfield of $\mathbb{Q}^{(d)}$, which is defined as the compositum of all number fields of degree at most d over $\mathbb{Q}$; being generated by elements of uniformly bounded degree. We prove that the above properties are equivalent for abelian extensions, but not in general; we provide counterexamples based on group-theoretical constructions with extraspecial groups and their modules, for which we give explicit realizations.

INTRODUCTION

This work is a study of the relations between some properties which can occur for an infinite algebraic extension K of the rationals.

The first property we shall consider is the uniform boundedness of the local degrees of K , namely the existence of a constant b , depending only on the field K , such that for every prime number p and every place v_p of K which extends the p -adic one, the completion of K with respect to v_p is a finite extension of $\mathbb{Q}_p$ of degree bounded by b .

The second property is the inclusion of the extension K in a field $\mathbb{Q}^{(d)}$, which is defined as the compositum in the algebraic closure of the rationals $\mathbb{Q}^{\text{alg}}$ of all number fields of degree at most d over $\mathbb{Q}$.

The inquiry into the relation between these two properties was motivated by Bombieri and Zannier's paper [1], which studies the Northcott property (of the finiteness of elements of bounded absolute Weil height) for certain infinite extensions of $\mathbb{Q}$. The paper considers in particular the field $\mathbb{Q}^{(d)}$, and the Northcott property is proved for the compositum of all abelian extensions of $\mathbb{Q}$ of uniformly bounded degree, while for $\mathbb{Q}^{(d)}$ the question remains open. In this proof a crucial role is played by the uniform boundedness of the local degrees of the field $\mathbb{Q}^{(d)}$.

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The considerations made in [1] led to the question, not free of independent interest, of whether every infinite algebraic extension of $\mathbb{Q}$ with uniformly bounded local degrees is contained in $\mathbb{Q}^{(d)}$ for some positive integer d . This question was negatively answered in [2], Theorem 1.1, where the authors provide a counterexample built up from a certain family of pq -groups provided by A. Lucchini. In this work we ask, more generally, if for an algebraic extension of $\mathbb{Q}$, there are properties which are equivalent to having uniformly bounded local degrees.

We also ask whether the uniform boundedness of the local degrees at every prime is equivalent to the uniform boundedness of the local degrees at *almost every prime*, that is, at all but a finite number of primes.

The study of the structure of $\mathbb{Q}^{(d)}$ revealed some unexpected sides. These arise when we consider a third property for an infinite algebraic extension K of $\mathbb{Q}$, that is, whether every number field contained in K can be generated by elements of degree bounded by a constant depending only on K . A field with this property is certainly contained in $\mathbb{Q}^{(d)}$ for some positive integer d and one could ask whether this condition is equivalent to being a subfield of $\mathbb{Q}^{(d)}$.

In this work we give a complete answer to all these questions, proving the following result.

Theorem 1. *Let $K/\mathbb{Q}$ be an infinite Galois extension. Then the following conditions are equivalent:*

- (1) *K has uniformly bounded local degrees at every prime;*
- (2) *K has uniformly bounded local degrees at almost every prime;*
- (3) *$\text{Gal}(K/\mathbb{Q})$ has finite exponent.*

Moreover, if $K/\mathbb{Q}$ is abelian, then the three properties:

- (a) *K has uniformly bounded local degrees;*
- (b) *K is contained in $\mathbb{Q}^{(d)}$ for some positive integer d ;*
- (c) *every number field contained in K can be generated by elements of uniformly bounded degree*

are equivalent. However, in general, we have that (c) implies (b), which implies (a) and none of the inverse implications holds.

The work is structured in the following way.

Section 1 focuses on the study of the structure of the field $\mathbb{Q}^{(d)}$ and of Galois extensions with uniformly bounded local degrees. We prove that these are exactly those whose Galois group has finite exponent. Moreover, we prove that the existence of a uniform bound for the local degrees at almost every prime implies the uniform boundedness of the local degrees at every prime. The first part of the proof was already done in Checcoli and Zannier's paper [2], Remark 2, and it consists of an application of Chebotarev's density theorem; the second part is based on a theorem of Shafarevich on the number of generators of the Galois group of a p -extension of p -adic fields and on Zelmanov's result on the Restricted Burnside Problem. As already done in [2], Proposition 2.1, the case of abelian extensions is also discussed, proving the first part of Theorem 1. Explicit bounds for the local degrees are provided in some special cases.

In Section 2 we recall the main properties of extraspecial groups. These groups shall enable us to construct examples of fields entailing the validity of the statements of Theorem 1. We also describe some irreducible modules of extraspecial groups over finite fields which shall be used in our constructions.

In Section 3 we prove the main theorem of this work on the nonequivalence of properties (a), (b) and (c), which is a generalization of Theorem 1.1 in [2]. The proof is based on some group-theoretical constructions due to A. Lucchini with extraspecial groups and their modules, followed by an application of Shafarevich's Theorem about the realization of solvable groups of odd order as Galois groups.

The use of Shafarevich's result can actually be avoided by realizing explicitly groups constructed in Section 3, and this will be done in Section 4. These constructions are based on some elementary methods in inverse Galois theory and on a result by Serre.

1. EXTENSIONS WITH UNIFORMLY BOUNDED LOCAL DEGREES

In this section we study some properties of the compositum of fields with bounded degrees, in order to give some motivations to our questions, and we provide a characterization of Galois extensions with uniformly bounded local degrees depending only on the exponent of their Galois group. Finally, we show the equivalence of conditions (a), (b) and (c) of Theorem 1 in the case of abelian extensions.

1.1. Galois extensions with finite exponent. We fix some notation. Let d be a positive integer and let F be a number field. We denote by $F^{(d)}$ the field obtained by taking the compositum of all algebraic extensions of F of degree at most d (over F).

If K is an infinite algebraic extension of $\mathbb{Q}$ and p is a prime number, we define a valuation v_p of K which extends the p -adic valuation over $\mathbb{Q}$ in the following way. We denote by $\mathbb{Q}_p^{\text{alg}}$ an algebraic closure of $\mathbb{Q}_p$, by $\mathbb{C}_p$ its p -adic completion and by w_p the unique valuation of $\mathbb{C}_p$ extending the p -adic valuation of $\mathbb{Q}$. Then we fix an embedding $\sigma : K \hookrightarrow \mathbb{C}_p$ and we define the valuation v_p of K as $v_p(x) := w_p(\sigma(x))$ for every $x \in K$.

The completion of K with respect to v_p is denoted by K_{v_p} and it is an extension of $\mathbb{Q}_p$. We are interested in the case when its degree over $\mathbb{Q}_p$ is bounded, for every prime p and every valuation v_p above p , by a constant independent from p . We have the following definition.

Definition 1.1. An algebraic extension K of $\mathbb{Q}$ has *uniformly bounded local degrees* if there exists a positive integer B such that $[K_{v_p} : \mathbb{Q}_p] < B$ for every prime number p and every valuation v_p of K extending the p -adic valuation of $\mathbb{Q}$.

One of the questions posed in this work is whether every algebraic extension of $\mathbb{Q}$ with uniformly bounded local degrees is contained in $\mathbb{Q}^{(d)}$ for some positive integer d . In order to give a motivation to this we recall that the field $\mathbb{Q}^{(d)}$ has uniformly bounded local degrees. We have the following proposition.

Proposition 1.1. *Let F be a number field of degree n over $\mathbb{Q}$ and let v be any valuation of F . Let w be an extension of v to $F^{(d)}$ and denote by $F_w^{(d)}$ and F_v the completions of $F^{(d)}$ and F with respect to w and v respectively. Then the local degree $[F_w^{(d)} : F_v]$ is bounded solely in terms of n and d .*

Proof. See for instance Proposition 1 of [1]. □

The next theorem is a generalization of Remark 2 of [2] and it provides a characterization of Galois extensions with uniformly bounded local degrees only in terms of the exponent of their Galois group. Moreover, it asserts that the existence of

a uniform bound for the local degrees at almost every prime implies the uniform boundedness of the local degrees at every prime.

Theorem 1.2. *Let L be a number field and K/L an infinite algebraic Galois extension. Then the following conditions are equivalent:*

- (1) K has uniformly bounded local degrees at every prime of L ;
- (2) K has uniformly bounded local degrees at almost every prime of L ;
- (3) $\text{Gal}(K/L)$ has finite exponent.

Proof. Obviously (1) implies (2). Now we suppose that (2) holds; then K has local degrees uniformly bounded by a constant B at every prime except some primes belonging to a finite set S .

We fix a finite Galois extension E of L contained in K and we take $\sigma \in \text{Gal}(E/L)$. By Chebotarev's Density Theorem (see [7], Ch.7, §3, Theorem 7.30) there exists a prime $\wp$ of L with $\wp \notin S$, a prime β of E unramified above $\wp$ and a conjugate τ of σ that generates the decomposition group $D(\beta|\wp)$ which is cyclic and isomorphic to $\text{Gal}(E_\beta/L_\wp)$, where E_β and $L_\wp$ denote the completions of E and L with respect to β and $\wp$ respectively. By assumption, $|\text{Gal}(E_\beta/L_\wp)| \leq B$; thus $\sigma^{B!} = \tau^{B!} = \text{id}$ and $\exp(\text{Gal}(E/L)) \leq B!$. Since $\text{Gal}(K/L)$ is the inverse limit of the family $\{\text{Gal}(E/L)\}_E$, where E varies among the finite Galois extension of L contained in K , we have $\exp(G) \leq B!$ and (3) holds.

Now we suppose that (3) holds for K and we set $\exp(\text{Gal}(K/L)) = b$. We want to prove that the local degrees of K are uniformly bounded at every prime.

We can write K as the compositum of a family of number fields $\{K_m\}_m$, where K_m/L is any finite Galois extension of L contained in K and we set $G_m = \text{Gal}(K_m/L)$.

We let $n = [L : \mathbb{Q}]$ and we fix a prime number p . For every m we denote by L_v and $K_{m,v}$ the completions of L and K_m , respectively, with respect to any valuation v extending the p -adic one. We recall that $\text{Gal}(K_{m,v}/L_v)$, being isomorphic to a subgroup of G_m , has exponent at most b . We have to discuss three cases for the extension $K_{m,v}/L_v$.

If it is unramified, then it is cyclic of order bounded by b ; if it is tamely ramified, then it is a metacyclic extension. Thus its order is bounded by b^2 .

If it is wildly ramified, then the tamely ramified part always has degree at most b^2 . The first ramification group of $K_{m,v}/L_v$ is a p -group of exponent at most b . Moreover, by a theorem of Shafarevich (see [9], Theorem 1) on the number of generators of the Galois group of a p -extension of p -adic fields, this group has at most $nb^2 + 2$ generators. So the problem reduces to the following: *is it true that, for all positive integers b and m , if a finite group H has m generators and exponent b , then the order of H is bounded by a constant which depends only on m and b ?*

This question is known as the Restricted Burnside Problem and was positively answered in 1989 by Efim Zelmanov; for a detailed description of Zelmanov's proof the interested reader should refer to [11]. In view of Zelmanov's result, the wildly ramified part of the extension has a degree which is bounded by a constant depending only on n and b .

We can now conclude our proof. Summing up all the previous results, we get that for every m , the local degree $[K_{m,v} : \mathbb{Q}_p]$ is bounded by a constant depending only on b and $n = [L : \mathbb{Q}]$, which are fixed. Thus the number of all possible completions of the K_m at primes above p is finite and independent from p . The compositum of this finite number of fields has degree over $\mathbb{Q}_p$ which is bounded by a constant

independent of p and it contains the completion of K with respect to any valuation extending the p -adic one. $\square$

The proof of the previous result does not provide an explicit bound for the local degrees of an extension $K/\mathbb{Q}$ of finite exponent. In order to effectively compute such a bound, we need additional information on the order of the wild part of the completion of every finite Galois subextension of K . We have the following result.

Theorem 1.3. *Let $K/\mathbb{Q}$ be an infinite Galois extension of exponent b and let $\{K_m\}_m$ be the family of all finite Galois subextensions of K . Suppose that for every m , every prime number p dividing b and every valuation v of K_m above p we have $v_p(|\text{Gal}(K_{m,v}/\mathbb{Q}_p)|) \leq n$, where v_p denotes the usual p -adic valuation of $\mathbb{Q}$.*

Then K has local degrees bounded by

$$\prod_{i=1}^n A(i),$$

where $A(1) = b^3$ and

$$A(i+1) = b^{(\prod_{j=1}^i A(j)) + 2}.$$

Proof. We fix a prime number p and we consider a valuation w of K extending the p -adic valuation of $\mathbb{Q}$. The field K equals the compositum of all its finite Galois subextensions $\{K_m\}_m$ and, for every m , $v = w|_{K_m}$ is a valuation of K_m above p . We denote by K_w and by $K_{m,v}$ the completions of K and K_m with respect to w and v , respectively.

We fix m and we set $G = \text{Gal}(K_{m,v}/\mathbb{Q}_p)$. We consider the filtration of G via its ramification subgroups

$$(1.1) \quad G \triangleright G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_r = 1,$$

where:

- G/G_0 is cyclic being the Galois group of the unramified part of the extension $K_{m,v}/\mathbb{Q}_p$;
- G_0/G_1 is the Galois group of the tame and totally ramified part of the extension and it is also cyclic;
- G_i/G_{i+1} is p -elementary abelian for every $i \geq 1$.

If p does not divide the exponent b , then the local extension can only be tamely ramified. The maximal unramified subextension of $K_{m,v}$ is the subfield of $K_{m,v}$ fixed by G_0 and we denote it by $K_{m,v}^{G_0}$. Then $K_{m,v}^{G_0}/\mathbb{Q}_p$ is contained in the compositum $L^{\text{ur},b}$ of all unramified extensions of $\mathbb{Q}_p$ of degree dividing b , which has degree $[L^{\text{ur},b} : \mathbb{Q}_p] = b$, since we have only one unramified extension of $\mathbb{Q}_p$ of every fixed degree. Therefore the maximal unramified subextension K_w^{ur} of K_w is also contained in $L^{\text{ur},b}$.

We recall that from local class field theory (see [4], Theorem in Ch. IV, §3.3) there is a one-to-one correspondence between finite abelian extensions of a p -adic field K and open subgroups of finite index of K^* . In particular this correspondence is such that if L/K is the maximal abelian extension of K of exponent b , then $\text{Gal}(L/K) \simeq K^*/(K^*)^b$.

This applies to our situation, since the tame and totally ramified part of the extension $K_{m,v}^{G_1}/K_{m,v}^{G_0}$ is contained in the maximal abelian extension $L^{\text{tame},b}$ of

$L^{\text{ur},b}$ of exponent dividing b and prime to p . Then we have

$$\text{Gal}(L^{\text{tame},b}/L^{\text{ur},b}) \simeq L^{\text{ur},b*}/(L^{\text{ur},b*})^a$$

for some integer a prime to p and smaller than b . Therefore

$$|\text{Gal}(L^{\text{tame},b}/L^{\text{ur},b})| \leq |(\mathbb{Z}/a\mathbb{Z})|^2 \leq b^2.$$

Since this holds for every m , the maximal tame subextension K_w^{tame} of K_w is a subfield of $L^{\text{tame},b}$.

If p divides the exponent b , then one can have wild ramification. However, using the previous arguments, the degree over $\mathbb{Q}_p$ of the maximal tame subextension K_w^{tame} of K_w is bounded by b^3 .

As for the wildly ramified part of the extension, since by hypothesis $v_p(|G|) \leq n$, the group G_1 has order at most p^n and the filtration

$$G_1 \triangleright G_2 \triangleright \dots \triangleright G_r = 1$$

has at most n jumps; this means that $G_i \neq G_{i+1}$ for at most n indices i .

Therefore we have the following tower of fields:

$$\mathbb{Q}_p = K_{m,v}^G \subseteq K_{m,v}^{G_0} \subseteq K_{m,v}^{G_1} \subseteq \dots \subseteq K_{m,v}^{G_r} = K_{m,v}$$

in which the extension $K_{m,v}^{G_{i+1}}/K_{m,v}^{G_i}$ is elementary p -abelian for at most n indexes $i \geq 1$ and trivial otherwise.

We set $L_1 = L^{\text{tame},b}$ and we denote by L_{i+1} the compositum of all p -elementary abelian extensions of L_i . We notice that $K_{m,v}^{G_j}$ is contained in some L_n for every index j . Again from local class field theory, we have that

$$(1.2) \quad \text{Gal}(L_{i+1}/L_i) = (L_i^*)/(L_i^*)^p \simeq (\mathbb{Z}/p\mathbb{Z})^{[L_i:\mathbb{Q}_p] + \epsilon(L_i)},$$

where $\epsilon(L_i) = 1$ if L_i does not contain p -th roots of unity and $\epsilon(L_i) = 2$ otherwise. Since $p < b$ and $[L_1 : \mathbb{Q}_p] \leq b^3 = A(1)$, we have $[L_2 : L_1] \leq b^{b^3+2} = A(2)$ and it is easy to see that $[L_i : L_{i-1}] \leq A(i)$. Since K_w is contained in L_n , the result follows. $\square$

1.2. Abelian extensions. We consider infinite abelian extensions of $\mathbb{Q}$, namely infinite algebraic Galois extensions of $\mathbb{Q}$ with abelian (profinite) Galois group. The following theorem, which was already in [2], Proposition 2.1, gives an answer to our initial question for these extensions.

Theorem 1.4. *For an infinite abelian extension K of $\mathbb{Q}$, the following conditions are equivalent:*

- (a) *K has uniformly bounded local degrees;*
- (b) *there exists a positive integer d such that K is contained in $\mathbb{Q}^{(d)}$;*
- (c) *every finite abelian subextension of K can be generated by elements of uniformly bounded degree.*

Proof. It is trivial to prove that (c) implies (b). The fact that (b) implies (a) follows from Proposition 1.1.

We now assume that K satisfies condition (a). We set $G = \text{Gal}(K/\mathbb{Q})$; then, by Theorem 1.2, $\exp(G) \leq b$ for some positive integer b and

$$G = \varprojlim_m G_m,$$

where $G_m = \text{Gal}(K_m/\mathbb{Q})$ and K_m is any finite abelian extension of $\mathbb{Q}$ contained in K .

For every m , G_m is a finite abelian group, and we can write it as a product of finite cyclic groups

$$G_m = \prod_{i=1}^n U_i.$$

We let H_i be the subgroup of G_m defined as

$$H_i := \prod_{j \neq i} U_j.$$

We have $[G_m : H_i] = |U_i| = \exp(U_i) \leq b$ for all i 's and $\bigcap_{i=1}^n H_i = 1$.

Therefore K_m is the compositum of the fields $\{K_m^{H_i}\}_i$ which satisfy the inequality $[K_m^{H_i} : \mathbb{Q}] = [G_m : H_i] \leq b$. Thus K_m is generated by elements of uniformly bounded degree and K satisfies condition (c), which completes the proof. $\square$

Remark 1.5. We notice that an abelian extension of exponent b has local degrees bounded by b^3 . In fact if $K/\mathbb{Q}$ is an abelian extension of exponent b , the completion of K with respect to a prime p is contained in the maximal abelian extension M of $\mathbb{Q}_p$ of exponent b ; from local class field theory we have that

$$|\text{Gal}(M/\mathbb{Q}_p)| = |\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^b| \leq b^3.$$

2. EXTRASPECIAL GROUPS

In this section we recall definitions and properties of extraspecial groups and their irreducible modules which shall be used later in our constructions. A detailed discussion on extraspecial groups and related topics can be found in [3].

2.1. Definition and properties.

Definition 2.1. Let p be a prime. A p -group G is said to be *extraspecial* if the center $Z(G)$ and the commutator subgroup G' are equal and have order p .

We recall the main properties of extraspecial groups which shall be used in our constructions.

Proposition 2.1. For every odd prime p and every positive integer m there exists an extraspecial group G_m with the following properties:

- (1) G_m has order p^{2m+1} and exponent p ;
- (2) the quotient $G_m/Z(G_m)$ is elementary abelian of order p^{2m} ;
- (3) G_m is isomorphic to $(E_1 \times \dots \times E_m)/N_m$, where $E_1, \dots, E_m$ are extraspecial groups of order p^3 and exponent p , the center of E_i is generated by an element z_i and

$$N_m = \{(z_1^{a_1}, \dots, z_m^{a_m}) \mid \sum_{i=1}^m a_i \equiv 0 \pmod{p}\};$$

- (4) every abelian subgroup of G_m has order smaller than p^{m+1} .

Moreover, every extraspecial group of order p^{2m+1} and exponent p satisfies all the above properties and it is unique up to isomorphisms.

Proof. See [3], Chap. A, §19 and §20. $\square$

Remark 2.2. It is easy to prove that every nonabelian group of order p^3 is extraspecial and that, if p is an odd prime, there exists a unique extraspecial group of order p^3 and exponent p , up to isomorphisms, namely the Heisenberg group with the presentation

$$H = \langle x, y | x^p = y^p = 1, [x, y] \in Z(H) \rangle.$$

2.2. Some irreducible modules of an extraspecial group. We now want to give a description of certain irreducible modules of an extraspecial p -group over a finite field which shall be used to construct examples in the next section.

We fix two odd primes p and q , with p dividing $q - 1$, such that the finite field $\mathbb{F}_q$ contains a primitive p -th root of unity ζ . We consider the extraspecial group H of order p^3 , exponent p and generators x and y and we set $W = (\mathbb{F}_q)^p$. Then W has the structure of an H -module via the following action: if $\{e_1, \dots, e_p\}$ is a basis for W , then x permutes these elements as a cycle of length p and $y \cdot e_i = \zeta^i e_i$.

Now for every positive integer m we denote by G_m the extraspecial group of order p^{2m+1} and exponent p satisfying Proposition 2.1 and by W_m the tensor product of m copies of W . Then G_m is isomorphic to a quotient of the direct product of m copies of H ; thus it acts on W_m , which can be regarded as a G_m -module of dimension p^m over $\mathbb{F}_q$.

We recall the following definition:

Definition 2.3. If G is a group and M is a G -module over a field k , then M is said to be *irreducible* (or *simple*) if it does not have proper nonzero G submodules.

A field k is said to be a *splitting field* for a group G if every irreducible G -module remains irreducible over every extension of k .

We have the following proposition.

Proposition 2.2. W_m is a faithful and irreducible G_m -module.

Proof. We first prove that the module W_m is faithful for G_m . In fact suppose that there exists $g \in G_m$, $g \neq 1$, such that $g \cdot w = 0$ for every $w \in W_m$. We have that, by construction,

$$G_m \simeq (H \times \dots \times H)/N,$$

where N is the normal subgroup of $H \times \dots \times H$ described in Proposition 2.1(3); then $g = [(g_1, \dots, g_m)]_N$ with $g_i \in H$ is a certain class of a quotient of the direct product of m copies of H . We consider a basis $\{e_1, \dots, e_p\}$ for W and we choose $w = e_1 \otimes 1 \otimes \dots \otimes 1$. Then $g \cdot w = \zeta e_i \otimes 1 \otimes \dots \otimes 1$ where ζ is some p -th root of unity. Then $g \cdot w \neq 0$, which is a contradiction.

Now we want to prove that W_m is an irreducible G_m -module. To do this, we first show that W is an irreducible H -module. Again we fix a basis $\mathcal{B} = \{e_1, \dots, e_p\}$ for W and we suppose that M is a nonzero H -submodule of W . We may assume that M is cyclic; this means that there exists an element $w = a_1 e_1 + \dots + a_p e_p \in W$ generating M . We define the elements $w_1, \dots, w_{p-1}$ in the following way: $w_1 := w$ and $w_i := x \cdot (w_{i-1} - y \cdot w_{i-1})$. We notice that $w_i \in M$ for every i and that the

first i coordinates of w_i in the base $\mathcal{B}$ are zero. It follows that w_{p-1} is a multiple of e_p ; therefore M contains e_p and, since $x^i \cdot e_p = e_i$ belongs to M for every i , then $M = W$.

Now, from Corollary 5.21 of [3], we have that $\mathbb{F}_q$ is a splitting field for H (since H has exponent p and $\mathbb{F}_q$ contains a p -th root of unity); in this case, using Theorem 5.22 of [3], we conclude that the tensor product of m copies of W is an irreducible representation of $H^m = H \times \dots \times H$ with kernel the subgroup N introduced above. Therefore it is also an irreducible G_m -module. $\square$

We end this section with some properties of an extraspecial group. These results seem to be *ad hoc* as they will be used only at the end of the next section, when their importance shall become evident.

The first result is about the cardinality of subgroups of extraspecial groups with a bounded number of generators. This will be used to provide an explicit bound for the local degrees of a certain infinite algebraic extension of the rationals, which we shall construct in the next section.

Proposition 2.3. *Let G be an extraspecial group of order p^{2m+1} and let H be a subgroup of G with at most n generators. Then $|H| \leq p^{n+1}$.*

Proof. We suppose first that H contains the center $Z(G)$. Then we have

$$H/Z(G) \subseteq G/Z(G) \simeq (\mathbb{Z}/p\mathbb{Z})^{2m}$$

and $H/Z(G)$ has at most n generators as a subspace of $G/Z(G)$. Hence $|H/Z(G)| \leq p^n$, since it is elementary abelian as a group, and $|H| \leq p^{n+1}$.

Now we suppose that $Z(G) \not\subseteq H$ and we consider $K = HZ(G)$, which is a subgroup of G (since $Z(G)$ is normal in G) that contains the center and has at most $n+1$ generators (n generators from H and one generator from the center which is cyclic by hypothesis). With the previous argument we obtain that $|K/Z(G)| \leq p^{n+1}$ and $|K| \leq p^{n+2}$. Finally, we get $|H| \leq p^{n+1}$. $\square$

The second property is about the intersection of all subgroups of an extraspecial group of bounded index. It will be used to provide subfields of $\mathbb{Q}^{(p^3)}$ which cannot be generated by elements of uniformly bounded degree.

Proposition 2.4. *Let p be an odd prime and G be an extraspecial p -group of order p^{2m+1} . Then the intersection of all subgroups of G of index at most p^m contains $Z(G)$.*

Proof. Let H be a subgroup of G such that $[G : H] \leq p^m$; we have that $|H| \geq p^{m+1}$. Either H contains the center $Z(G)$ or $|HZ(G)| > p^{m+1}$ and from Proposition 2.1, $HZ(G)$ is not abelian; therefore H is also not abelian and it contains two elements which do not commute. This implies that $G' = Z(G) \subseteq H$, since G' is cyclic of order p . $\square$

3. THE MAIN THEOREM

In Section 1 we proved that for an infinite abelian extension of $\mathbb{Q}$ there is a list of properties which are equivalent to the property of having uniformly bounded local degrees. We now want to prove that this is not true in general. The nonequivalence is entailed by the existence of fields constructed using extraspecial groups and their modules. We now state the main result, which will be proved throughout this section and which provides a significant generalization of Theorem 1.1 of [2].

Theorem 2. *Let K be an infinite algebraic Galois extension of $\mathbb{Q}$. Consider the following properties for K :*

- (a) K has uniformly bounded local degrees at almost every prime;
- (b) K is contained in $\mathbb{Q}^{(d)}$ for some positive integer d ;
- (c) every number field contained in K can be generated by elements of uniformly bounded degree over $\mathbb{Q}$;
- (d) there exists a positive integer b and a family $\{K_m\}_m$ of finite Galois extensions of $\mathbb{Q}$ such that K is the compositum of the family and for every m the Galois group of K_m has only minimal normal subgroups of order at most b .

Then:

- (1) (c) $\Rightarrow$ (b) $\Rightarrow$ (a);
- (2) (b) $\Rightarrow$ (d).

However no other implication holds, that is:

- (i) (d) $\not\Rightarrow$ (b);
- (ii) (b) $\not\Rightarrow$ (c);
- (iii) (a) $\not\Rightarrow$ (b).

Remark 3.1. It is immediate to see that if K satisfies (c), then it also satisfies (b) and, from Theorem 1.1, it follows easily that (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

As the proof of the result is quite long, we divide it into four different parts.

3.1. Proof of (2). We want to prove that (b) implies (d). In order to do this we need the following result, which is Proposition 3.2 of [2]. We restate and reprove it here for clarity.

Lemma 3.2. *Let G be a finite group with a minimal normal subgroup W such that $|W| = m$. Suppose that G is a quotient of a group H , where H is a subgroup of a direct product $H_1 \times \dots \times H_s$. Then $|H_i| \geq m$ for some index $i \in \{1, \dots, s\}$.*

Proof. We suppose that $G = H/N$ with $H \leq H_1 \times \dots \times H_s$ and that $|H_i| < m$ for every i . Let us denote by $\pi : H \rightarrow H/N$ the projection map. We set $\overline{H_i} := H \cap 1 \times \dots \times 1 \times H_i \times \dots \times H_s$ and we denote by $G_i = \pi(\overline{H_i})$ the image of $\overline{H_i}$ in G . We notice that the G_i 's are all normal subgroups of G .

We want to show by induction that $W \subseteq G_i$ for every i .

For $i = 1$ this holds by assumption, since $G_1 = G$.

Suppose it is true for G_{i-1} . Now $W \subseteq G_{i-1}$ and $[G_{i-1} : G_i] < m$; thus $U := G_i \cap W$ is normal and nontrivial and so it contains W . Therefore $W \subseteq G_i$. In particular $W \subseteq G_s$, which is a contradiction, since $|G_s| \leq |H_s| < m$. $\square$

Now we are able to prove our result. We suppose that $K \subseteq \mathbb{Q}^{(d)}$ for some integer d . We write K as the compositum of all its finite Galois subextensions $\{K_m\}_m$ and we set $G_m = \text{Gal}(K_m/\mathbb{Q})$. We suppose, moreover, that K does not satisfy property (d). This means that there exists an infinite subsequence N of the positive integers and a strictly increasing sequence of positive integers $\{c_m\}_{m \in N}$ such that G_m has a minimal normal subgroup of order c_m for every $m \in N$.

Thus, in view of Lemma 3.2, whenever G_m is isomorphic to a quotient H/N and H is a subgroup of a direct product $H_1 \times \dots \times H_s$ and $m \in N$, then $|H_i| \geq c_m$ for at least one index i .

We fix $m \in N$ such that $d! < c_m$. We notice that, since K is contained in $\mathbb{Q}^{(d)}$, so is K_m ; then there exist number fields $L_1, \dots, L_s$ such that $K_m \subseteq L_1 \dots L_s$ and $[L_i : \mathbb{Q}] \leq d$.

We denote by L_i^{Gal} the Galois closure of L_i in $\mathbb{Q}^{\text{alg}}$ and by $H_i = \text{Gal}(L_i^{\text{Gal}}/\mathbb{Q})$ its Galois group. We notice that $|H_i| \leq d! < c_m$ by hypothesis.

We have $K_m \subseteq L_1^{\text{Gal}} \dots L_s^{\text{Gal}}$ and $\text{Gal}(L_1^{\text{Gal}} \dots L_s^{\text{Gal}})$ is isomorphic to a subgroup of $H_1 \times \dots \times H_s$ via the restriction map.

Since K_m is a Galois extension,

$$G_m = \text{Gal}(K_m/\mathbb{Q}) \simeq \text{Gal}(L_1^{\text{Gal}} \dots L_s^{\text{Gal}})/N$$

for some normal subgroup N of $\text{Gal}(L_1^{\text{Gal}} \dots L_s^{\text{Gal}})$.

Now $\text{Gal}(L_1^{\text{Gal}} \dots L_s^{\text{Gal}})$ is a subgroup of $(H_1 \times \dots \times H_s)$ with $|H_i| \leq c_m$ for every i , which is a contradiction. Thus K satisfies (d).

3.2. Proof of (i). We have to prove that condition (d) does not imply condition (b). In order to do this we fix a prime p and we consider a family $\{K_m\}_m$ of finite Galois extensions of $\mathbb{Q}$ such that $G_m = \text{Gal}(K_m/\mathbb{Q}) = \mathbb{Z}/p^m\mathbb{Z}$ is the cyclic group of order p^m . We denote by K the compositum of all the K_m .

We notice that $\exp(\text{Gal}(K/\mathbb{Q}))$ is not finite, since the group $\text{Gal}(K/\mathbb{Q})$ contains elements of order p^m for every m . In view of Theorem 1.2, K cannot have uniformly bounded local degrees and therefore it is not contained in $\mathbb{Q}^{(d)}$.

However, for every m , every minimal normal subgroup of G_m has order p and property (d) holds.

3.3. Proof of (ii). We want to show that condition (b) does not imply condition (c). We fix an odd prime number p and we now consider an infinite family $\mathcal{F} = \{F_i\}_i$ of linearly disjoint Galois extensions of $\mathbb{Q}$ (for the definition of a linearly disjoint family of extensions, see [5], Ch. 2, §2.5), where, for every i , $\text{Gal}(F_i/\mathbb{Q}) = H$ is the extraspecial group of order p^3 and exponent p .

We know that such a family exists: in fact a famous theorem of Shafarevich (see [10]) asserts that every solvable group of odd order can be realized over $\mathbb{Q}$. Now, for every integer n , the group $H^n = H \times \dots \times H$ is solvable and hence realizable. We denote by F its realization, that is, F is a Galois extension of $\mathbb{Q}$ with $\text{Gal}(F/\mathbb{Q}) = H^n$. We notice that H^n has n normal subgroups $H_1, \dots, H_n$ isomorphic to H^{n-1} whose intersection is trivial and $H^n/H_i \simeq H$ for all i 's. Thus $F^{H_1}, \dots, F^{H_n}$ are n linearly disjoint realizations of H over $\mathbb{Q}$ and we can make this construction for every integer n .

Now we consider the family of finite Galois extensions $\{K_m\}_m$ constructed in the following way: for every m , denote by $L_m = F_1 \dots F_m$ the compositum of m fields of the family $\mathcal{F}$. By construction

$$\text{Gal}(L_m/\mathbb{Q}) = H \times \dots \times H = H^m.$$

Next note that the subgroup N_m of H^m described in Proposition 2.1 is normal and that H^m/N_m is extraspecial of order p^{2m+1} . We denote by $K_m := L_m^{N_m}$ the subfield of L_m fixed by this subgroup. We have that $K_m/\mathbb{Q}$ is a finite Galois extension with Galois group an extraspecial group of order p^{2m+1} and exponent p .

We denote by K the compositum of all the fields $\{K_m\}_m$. Since by construction every field K_m is contained in $\mathbb{Q}^{(p^3)}$, therefore $K \subseteq \mathbb{Q}^{(p^3)}$ and (b) holds for K .

However, K does not satisfy (c). In fact suppose that every K_m can be generated by elements of degree smaller than d , that is, K_m equals the compositum of all its subextensions of degree at most d over $\mathbb{Q}$.

We fix m such that $d < p^m$ and we denote by $\{H_{m,i}\}_i$ the family of all subgroups of $G_m = \text{Gal}(K_m/\mathbb{Q})$ of index smaller than $c_m := p^m$ in G_m . Then, by the Galois correspondence, $\{K_m^{H_{m,i}}\}_i$ is the family of all subextensions of K_m of degree smaller than c_m over $\mathbb{Q}$. We notice that the compositum $R_m := \prod_i K_m^{H_{m,i}}$ is strictly contained in K_m since, by Proposition 2.4, we have

$$Z(G_m) \subseteq \bigcap_i H_{m,i} = \text{Gal}(K_m/R_m) \neq 1.$$

This contradicts the hypothesis on K_m .

Remark 3.3. To avoid the use of Shafarevich's result, we give, in Section 4, an explicit realization of the family $\mathcal{F}$ and of the fields K_m .

3.4. Proof of (iii). We now want to prove that (a) does not imply (b); that is, we have to show the existence of an extension $K/\mathbb{Q}$ with uniformly bounded local degrees such that K is not contained in $\mathbb{Q}^{(d)}$ for any integer d . This was already proved in [2], Theorem 1.1, and we recall here this proof.

In view of Theorem 1.2 and property (2), in order to construct such a field, it suffices to find a family of finite Galois extensions $\{K_m\}_m$ such that, setting $G_m := \text{Gal}(K_m/\mathbb{Q})$, the family of groups $\{G_m\}_m$ satisfies the following conditions:

- (1) $\exp(G_m)$ is bounded by a constant which does not depend on m ;
- (2) there exists a strictly increasing sequence of positive integers $\{c_m\}_m$ such that G_m has a minimal normal subgroup of order c_m .

Then the compositum K of the fields K_m has the desired property.

In fact condition (1) implies that $\exp(\text{Gal}(K/\mathbb{Q}))$ is finite, which is equivalent to the fact that K has uniformly bounded local degrees.

As for condition (2), it implies that (b) does not hold for K . The proof of this fact is very similar to that of (d) $\not\Rightarrow$ (b), even if condition (2) is not equivalent to condition (d). Suppose that (2) holds and that $K \subseteq \mathbb{Q}^{(d)}$ for some d . Then choose m such that $d! < c_m$. Then $K_m \subseteq \mathbb{Q}^{(d)}$. From previous considerations this implies that G_m is isomorphic to the quotient of a direct product of groups of order bounded by $d!$. But since (2) holds we get a contradiction applying Lemma 3.2.

The construction of K goes as follows.

We take p and q to be two odd primes such that p divides $q - 1$. We denote by E_m the extraspecial group of order p^{2m+1} and exponent p satisfying Proposition 2.1 and by $\mathbb{F}_q[E_m]$ its regular representation, that is,

$$\mathbb{F}_q[E_m] = \left\{ \sum_{g \in E_m} a_g g \mid a_g \in \mathbb{F}_q \right\}.$$

Now we consider the family of finite groups $\{G_m\}_{m \geq 1}$, where $G_m = \mathbb{F}_q[E_m] \rtimes E_m$ and the semidirect product is taken via the natural action by translation of E_m on $\mathbb{F}_q[E_m]$. We can now prove the following lemma.

Lemma 3.4. *The family $\{G_m\}_{m \geq 1}$ satisfies condition (1) with bound pq and condition (2) with $c_m = q^{p^m}$.*

Proof. Since $\exp(G_m) = pq$ for every m , condition (1) holds.

As for condition (2), we consider the irreducible E_m -module W_m constructed in Section 2.2. Since q and p are different primes, by Maschke's theorem (see [3], Ch. A, §11, Thm. 11.5), the group algebra $\mathbb{F}_q[E_m]$ is a semisimple algebra. In particular the irreducible E_m -module W_m appears as a direct summand of $\mathbb{F}_q[E_m]$. By abuse of notation, we denote by W_m a subgroup of $\mathbb{F}_q[E_m]$ isomorphic to W_m and by H_m the subgroup of G_m given by $H_m := W_m \rtimes 1$.

Then the subgroup H_m is a normal subgroup of G_m and it is also minimal, being isomorphic to W_m , which is an irreducible module. Moreover, it has order $|H_m| = q^{p^m}$, and therefore condition (2) is true with $c_m = |H_m| = q^{p^m}$. $\square$

Now, having constructed a family of groups which satisfy the desired properties, we want to find a family of number fields realizing these groups.

It is easy to show that such a family exists: in fact, the groups G_m are pq -groups and, by Burnside's Theorem (see [3], Ch. I, §2, p. 210), they are solvable. Therefore, in view of Shafarevich's Theorem (see [10]), they can be realized over $\mathbb{Q}$ (recall that p and q are odd primes); that is, there exists a family of number fields $\{K_m\}_{m \geq 1}$ such that, for every m , $\text{Gal}(K_m/\mathbb{Q}) \simeq G_m$ and we denote by K the compositum of this family. Then K satisfies (a), but not (b) of Theorem 2.

Remark 3.5. Shafarevich's result is not necessary to prove the existence of such a family, since this will be constructed explicitly in Section 4.2.

Even if Theorem 1.2 is sufficient to prove that the constructed field K has uniformly bounded local degrees, one may be interested in knowing an effective bound for the local degrees of K .

We fix a prime number ℓ and we consider the completion of every K_m at a prime v above ℓ , say $K_{m,v}$. The Galois group of the tame part of the extension $K_{m,v}/\mathbb{Q}_\ell$ is metabelian of order at most $(pq)^3$. The first ramification group G_1 of the local extension has the following structure:

- for $\ell = q$ it is a subgroup of $\mathbb{F}_q[E_m]$ and therefore elementary abelian of order at most $q^{(pq)^3+2}$;
- for $\ell = p$ it is a subgroup of E_m with at most $(pq)^3 + 2$ generators. In view of Proposition 2.3, $|G_1| \leq p^{(pq)^3+3}$.
- G_1 is trivial otherwise.

Now we can apply the formula given in Theorem 1.3 with $n = (pq)^3 + 3$ and we obtain an explicit bound for the local degrees of K .

However this bound can be significantly improved via the following considerations. We again fix a prime ℓ of $\mathbb{Q}$ and we denote by v a valuation of K_m above ℓ . We have to discuss three cases.

- (1) If $\ell \neq p, q$, then, for every m , the extension $K_{m,v}/\mathbb{Q}_\ell$ is tame and it is contained in the compositum of all metabelian extensions of $\mathbb{Q}_\ell$ of exponent dividing pq , which has degree bounded by q^6 .

Then the local degrees of K at prime $\ell \neq p, q$ are bounded by q^6 .

- (2) If $\ell = q$, the unramified part of the extension is contained in the compositum L^{ur} of all unramified extensions of $\mathbb{Q}_q$ of degree dividing pq and this extension has degree bounded by pq .

The tame and totally ramified part of the extension is either trivial or cyclic of degree p . Therefore it is contained in the compositum L^{tame} of all cyclic extensions of L^{ur} of degree p , which has degree bounded again by p^2 .

The wild ramified part, having Galois group which is a subgroup of $\mathbb{F}_q[E_m]$ and therefore elementary abelian, is contained in the compositum of all q -elementary abelian extensions of L^{tame} , which has degree over L^{tame} at most q^{p^3q+2} .

Therefore the local degree of K at q is at most $q^{p^3q+3}p^3 < q^{q^4+6}$.

- (3) If $\ell = p$ we denote by H_m the Galois group of the local extension $K_{m,v}/\mathbb{Q}_p$. Since $\text{Gal}(K_m/\mathbb{Q}) = \mathbb{F}_q[E_m] \rtimes E_m$ is a pq -group with a unique q -Sylow subgroup $\mathbb{F}_q[E_m]$, which is normal, then H_m also possesses a unique q -Sylow subgroup Q_m .

By Schur-Zassenhaus' theorem (see [3], Ch. A, §11, Theorem 11.3), since p and q are different primes, $H_m = Q_m \rtimes P_m$, where P_m is the p -Sylow subgroup, $Q_m \leq \mathbb{F}_q[E_m]$, $P_m \leq E_m$ and the action of P_m on Q_m is induced by the action of E_m on $\mathbb{F}_q[E_m]$ by restriction.

We denote by $K_{m,v}^{Q_m}$ the subfield of $K_{m,v}$ fixed by Q_m . Then the extension $K_{m,v}^{Q_m}/\mathbb{Q}_p$, having metabelian Galois group P_m , is contained in the compositum L of all metabelian p -extensions of $\mathbb{Q}_p$, which has degree $[L : \mathbb{Q}_p] \leq p^{p^2+4} \leq q^{q^2+4}$.

The extension $K_{m,v}/K_{m,v}^{Q_m}$ is an abelian q -extension of a p -adic field with $(p, q) = 1$; thus it is contained in the compositum L' of all tamely ramified extensions of L of degree q and we have $[L' : L] \leq q^2$.

Then the local degree of K at p is bounded by q^{q^2+6} .

Summing up these results, K has local degrees bounded by q^{q^4+6} .

4. EXPLICIT CONSTRUCTIONS

In Section 3 we saw that the existence of certain Galois extensions with special properties was among the basic ingredients of the proof of our main theorem; this was done by using Shafarevich's Theorem on the realizability of solvable groups of odd order over $\mathbb{Q}$.

This section provides explicit realizations of those groups. These constructions are interesting for two reasons: first, they permit us to avoid the use of Shafarevich's result. Secondly, coming back to one of the original motivations of this work, it is proved in [12] that the Northcott property for an infinite algebraic extension K of $\mathbb{Q}$ is strictly related to the behavior of the discriminants of certain finite subextensions of K . Therefore, knowing how to concretely construct fields described in Section 3 could be a step towards understanding whether these extensions might have the Northcott property or not.

Section 4.1 is devoted to describing a method, based on some results in [6], to realize extraspecial p -groups over a field of characteristic zero containing a primitive p -th root of unity. This entails the existence of the family of number fields used in the proof of (ii) of Theorem 2, as pointed out in Remark 3.3. In Section 4.2 we use a method described by Serre in [8] to solve embedding problems with abelian kernel. This result, together with the construction of Section 4.1, provides a realization of the family of Galois extensions needed to prove part (iii) of the main theorem, as announced in Remark 3.5.

4.1. Realizing extraspecial p -groups of exponent p . In this section we generalize a method described in [6] which can be used to realize extraspecial p -groups of exponent p over fields of characteristic zero containing a primitive p -th root of unity. We have the following theorem.

Theorem 4.1. *Let k be a field of characteristic zero containing a primitive p -th root of unity ζ_p . Let $a_1, b_1, \dots, a_m, b_m \in k$ be elements chosen in the following way.*

- (i) *For every i the element a_i is not a p -th power in k and the ideal (a_i) is prime to the ideals (a_j) and (b_j) for every $j < i$.*
- (ii) *Denote by σ_i the generator of $\text{Gal}(k(\sqrt[p]{a_i})/k)$ such that $\sigma_i(\sqrt[p]{a_i}) = \zeta_p \sqrt[p]{a_i}$. Let $\mathfrak{q}_i$ be a prime of k which splits completely in the extension $k(\sqrt[p]{a_i})/k$ and such that $\mathfrak{q}_i \neq \mathfrak{q}_j$ for every $j < i$.*

Denote by $\beta_1^i, \dots, \beta_p^i$ the primes of $k(\sqrt[p]{a_i})$ above $\mathfrak{q}_i$ and by S_i the set of primes of $k(\sqrt[p]{a_i})$ lying above the primes of k dividing (a_j) or (b_j) for $j < i$. Let $x_i \in k(\sqrt[p]{a_i})$ be a solution of the system

$$(4.1) \quad \begin{cases} v_{\beta_1^i}(X) = 1 \\ v_{\beta_j^i}(X) = 0 \quad \forall j \neq 1 \\ v_{\beta}(X) = 0 \quad \forall \beta \in S_i \end{cases}$$

and set $b_i := N_{\sigma_i}(x_i)$, where N_{σ_i} is the norm function of the extension $k(\sqrt[p]{a_i})/k$.

Finally, consider the elements $\omega_1, \dots, \omega_m$ defined as

$$\omega_i := \frac{b_i^{p-1}}{\prod_{j=i}^{p-1} \sigma_i^j(x_i^j)}.$$

Then:

- (a) *$\{k(\sqrt[p]{a_i}, \sqrt[p]{b_i}, \sqrt[p]{\omega_i})\}_i$ is a family of linearly disjoint Galois extensions of k and, for every index i , the group $H_i := \text{Gal}(k(\sqrt[p]{a_i}, \sqrt[p]{b_i}, \sqrt[p]{\omega_i})/k)$ is isomorphic to the extraspecial group of order p^3 and exponent p ;*
- (b) *the extension $k(\sqrt[p]{a_1}, \sqrt[p]{b_1}, \dots, \sqrt[p]{a_m}, \sqrt[p]{b_m}, \sqrt[p]{\omega_1} \dots \omega_m})/k$ is a Galois extension with Galois group the extraspecial group of order p^{2m+1} and exponent p .*

Proof. The first part of property (a) follows from the results of [6]; in fact, setting

$$M_i := k(\sqrt[p]{a_i}, \sqrt[p]{b_i}, \sqrt[p]{\omega_i}),$$

it is proved in [6] that the extensions M_i/k are Galois extensions with Galois group H_i which is an extraspecial group of order p^3 and exponent p .

We want to prove that the extensions M_i are linearly disjoint and that property (b) holds. We prove the result by induction on m . The theorem is trivially true for $m = 1$.

We suppose the theorem to be true for $i < m$ and we want to prove that the extension M_m is linearly disjoint from M_i for every $i < m$. Since the only primes of k which ramify in the extensions $k(\sqrt[p]{a_i})$ and $k(\sqrt[p]{b_i})$ are those above p and those dividing the ideals (a_i) and (b_i) respectively, condition (i) ensures that

$$k(\sqrt[p]{a_m}) \cap k(\sqrt[p]{a_i}) = k(\sqrt[p]{a_m}) \cap k(\sqrt[p]{b_i}) = k$$

for every $i < m$. Since x_m satisfies the system (4.1), we also have

$$k(\sqrt[p]{a_m}) \cap k(\sqrt[p]{b_m}) = k.$$

Moreover, we notice that if β is a prime of k dividing (a_i) or (b_i) for $i < m$, then β does not divide (b_m) ; otherwise (x_m) would be divisible by some elements of S_m . Using the previous arguments we get that $k(\sqrt[p]{a_m}, \sqrt[p]{b_m}) \cap k(\sqrt[p]{a_i}, \sqrt[p]{b_i}) = k$ for every $i < m$.

We notice that an extraspecial group of order p^3 has a unique normal subgroup of order p , which is given by the center; then the extension $k(\sqrt[p]{a_m}, \sqrt[p]{b_m})$ is the unique p -elementary subextension of M_m of order p^2 . Therefore the extensions $M_1, \dots, M_m$ are linearly disjoint.

We now want to prove condition (b). We denote by k_m the compositum of the fields $M_1, \dots, M_m$. Then $\text{Gal}(k_m/k) = H_1 \times \dots \times H_m$ and the field

$$M = k(\sqrt[p]{a_1}, \sqrt[p]{b_1}, \dots, \sqrt[p]{a_m}, \sqrt[p]{b_m}, \sqrt[p]{\omega_1 \dots \omega_m})$$

is a subfield of k_m .

We consider the subgroup N_m of $H_1 \times \dots \times H_m$ described in Proposition 2.1 defined by

$$N_m = \{(z_1^{a_1}, \dots, z_m^{a_m}) | a_1 + \dots + a_m \equiv 0 \pmod{p}\},$$

where $Z(H_i) = \langle z_i \rangle$. It is easy to see that z_i acts in the following way:

- $z_i(\sqrt[p]{\omega_i}) = \zeta_p \sqrt[p]{\omega_i}$;
- $z_i(\sqrt[p]{a_i}) = \sqrt[p]{a_i}$;
- $z_i(\sqrt[p]{b_i}) = \sqrt[p]{b_i}$.

The quotient $(H_1 \times \dots \times H_m)/N_m$ is an extraspecial group of order p^{2m+1} and exponent p and, by Galois correspondence, the extension $k_m^{N_m}/k$ is an extraspecial extension of order p^{2m+1} and exponent p . We claim that $M = k_m^{N_m}$.

The field

$$k(\sqrt[p]{a_1}, \sqrt[p]{b_1}, \dots, \sqrt[p]{a_m}, \sqrt[p]{b_m})$$

has degree p^{2m} over k and is contained in $k_m^{N_m}$, since it is fixed by every element of N_m . Thus $k_m^{N_m}$ must be a p -extension of this field.

If we take $n = (z_1^{a_1}, \dots, z_m^{a_m}) \in N_m$, we have

$$n(\sqrt[p]{\omega_1 \dots \omega_m}) = \zeta_p^{a_1 + \dots + a_m} \sqrt[p]{\omega_1 \dots \omega_m} = \sqrt[p]{\omega_1 \dots \omega_m}$$

since $\sum_{i=1}^m a_i \equiv 0 \pmod{p}$ for $n \in N_m$.

Therefore $M \subseteq k_m^{N_m}$ and, since they have the same degree, they are equal. $\square$

4.2. Extension by the group algebra. This section is meant to provide an explicit realization of the field used in the proof of part (iii) of Theorem 2, as announced in Remark 3.5.

To do this we will resort to considerations from Section 2.2 on irreducible modules of extraspecial groups over certain finite fields, the explicit realization of extraspecial groups given in the previous section and a method by Serre.

We start by fixing two odd different primes p and q . We consider the family of groups $\{E_m\}_m$, where E_m is the extraspecial p -group of order p^{2m+1} and exponent p and we denote by $\mathbb{F}_q[E_m]$ the group algebra of E_m over $\mathbb{F}_q$.

We shall now use the method described in [8], p. 18, to prove that if F is a number field containing the q -th root of unity, L_m/F is a Galois extension with Galois group E_m , and G_m is the semidirect product $G_m = \mathbb{F}_q[E_m] \rtimes E_m$, then the embedding problem for L_m/F and for

$$1 \rightarrow \mathbb{F}_q[E_m] \rightarrow G_m \rightarrow E_m \rightarrow 1$$

has a solution. This means that there exists a Galois extension K_m of L_m such that:

- (1) $\text{Gal}(K_m/L_m) = \mathbb{F}_q[E_m]$;
- (2) K_m/F is Galois with Galois group isomorphic to G_m .

In order to construct K_m , we take a place v of F which splits completely in L_m and we consider a place w of L_m above v . Now we choose an element $x \in L_m$ such that x has w -adic valuation 1, but w_i -adic valuation 0 for every $w_i \neq w$ above v .

We denote by $\Delta := \{\sqrt[q]{\sigma(x)} \mid \sigma \in E_m\}$ and we set $K_m = L_m(\Delta)$. Then the extension K_m/L_m is a Galois extension with Galois group isomorphic to $\mathbb{F}_q[E_m]$. In fact we consider the map

$$\lambda : \mathbb{F}_q[E_m] \rightarrow \text{Gal}(K_m/L_m),$$

$$\sum_{\sigma \in E_m} a_\sigma \sigma \mapsto \tau,$$

where τ is defined as $\tau(\sqrt[q]{\sigma(x)}) = \zeta_q^{a_\sigma} \sqrt[q]{\sigma(x)}$; then λ is clearly the sought for isomorphism.

The extension K_m/F is also a Galois extension: in fact for every $\sigma \in \text{Gal}(L_m/F) = E_m$ and every embedding $\tilde{\sigma} : K_m \rightarrow F^{\text{alg}}$ with $\tilde{\sigma}|_{L_m} = \sigma$ we have that, by construction, $\tilde{\sigma}$ is an automorphism of K_m . Finally, the Galois group $\text{Gal}(K_m/L_m)$ is isomorphic to the semidirect product $\mathbb{F}_q[E_m] \rtimes E_m$, with E_m acting on $\mathbb{F}_q[E_m]$ by translation.

Now we set $F = \mathbb{Q}(\zeta_p, \zeta_q)$ and we take $\{L_m\}_m$ to be the family of Galois extensions of F constructed in Section 4.1 with $\text{Gal}(L_m/F) = E_m$ the extraspecial group of order p^{2m+1} and exponent p . For every m we use the above method to solve the embedding problem for L_m/F and for

$$1 \rightarrow \mathbb{F}_q[E_m] \rightarrow G_m \rightarrow E_m \rightarrow 1$$

and we construct a family of number fields $\{K_m\}_m$ with Galois groups

$$\text{Gal}(K_m/\mathbb{Q}(\zeta_p, \zeta_q)) = \mathbb{F}_q[E_m] \rtimes E_m.$$

In view of Theorem 2 (iii), the compositum K of the family K_m is an infinite algebraic extension of $\mathbb{Q}$ with uniformly bounded local degrees which is not contained in $\mathbb{Q}^{(d)}$ for any positive integer d .

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