C^* -ALGEBRAS GENERALIZING BOTH RELATIVE CUNTZ-PIMSNER AND DOPLICHER-ROBERTS ALGEBRAS

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ABSTRACT. We introduce and analyse the structure of C^* -algebras arising from ideals in right tensor C^* -precategories, which naturally unify the approaches based on Hilbert C^* -modules and C^* -categories with tensor structure. We establish an explicit intrinsic construction of the algebras considered, prove a number of key results such as a structure theorem and a gauge-invariant uniqueness theorem, and describe the gauge-invariant ideal structure. These results give a new insight into the corresponding statements for relative Cuntz-Pimsner algebras and are applied to Doplicher-Roberts algebras associated with C^* -correspondences.

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INTRODUCTION

C^* -categories, the categorical analogues of (unital) C^* -algebras, arise quite naturally in different problems of representation theory, harmonic analysis, or cohomology theory; cf. [15], [32] and the sources cited there. The recent interest in C^* -categories however is essentially due to a series of papers by S. Doplicher and J. E. Roberts where, motivated by questions arising in quantum field theory, they

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developed an abstract duality for compact groups; cf. [13]. In their scenario an object of the dual group is represented by a certain tensor C^* -category $\mathcal{T}$, and a machinery performing this duality rests on a construction of a C^* -algebra $\mathcal{O}_\rho$, influenced by Cuntz algebras [9], associated functorially to each object ρ of $\mathcal{T}$. This association of $\mathcal{O}_\rho$ can be applied, with no substantial modifications, to the case where $\mathcal{T}$ is just a *right tensor* C^* -category, i.e. a C^* -category for which the set of objects is a unital semigroup, with identity ι , and for any object $\tau \in \mathcal{T}$ there is a $*$ -functor $\otimes 1_\tau : \mathcal{T} \rightarrow \mathcal{T}$ (which intuitively should be thought of as a tensoring on the right with identity 1_τ in the space of morphisms $\mathcal{T}(\tau, \tau)$) such that

$$\otimes 1_\tau : \mathcal{T}(\rho, \sigma) \rightarrow \mathcal{T}(\rho\tau, \sigma\tau) \quad \text{and} \quad \otimes 1_\iota = id, \quad ((a \otimes 1_\tau) \otimes 1_\omega) = a \otimes 1_{\tau\omega},$$

where $\omega, \rho, \sigma \in \mathcal{T}$, $a \in \mathcal{T}(\rho, \sigma)$, and we write $a \otimes 1_\tau$ for an element customarily denoted by $\otimes 1_\tau(a)$. In this paper we adopt a “*contravariant*” convention that $\mathcal{T}(\rho, \sigma)$ stands for the space of morphisms from σ to ρ .

The construction of the single algebra $\mathcal{O}_\rho$, with ρ fixed, relies only on the semigroup $\{\rho^n\}_{n \in \mathbb{N}}$ generated by ρ (we include 0 in $\mathbb{N}$). In such a situation our framework becomes even more transparent by assuming that $\mathcal{T}$ is simply a C^* -category with $\mathbb{N}$ as the set of objects, equipped with a $*$ -functor $\otimes 1 : \mathcal{T} \rightarrow \mathcal{T}$ such that $\otimes 1 : \mathcal{T}(n, m) \rightarrow \mathcal{T}(n+1, m+1)$, $n, m \in \mathbb{N}$. Then it makes sense to denote the *Doplicher-Roberts algebra* (associated to the object $1 \in \mathbb{N}$) by $\mathcal{DR}(\mathcal{T})$. Such algebras are sometimes also called DR-algebras [32].

A somewhat different but closely related and important class of algebras form the C^* -algebras associated with C^* -correspondences, the study of which was initiated by Pimsner [29]. More specifically, a C^* -correspondence X over a C^* -algebra A (sometimes called a Hilbert bimodule) is a right Hilbert A -module equipped with a left action ϕ of A by adjointable operators, and a *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is constructed as a quotient of the *Toeplitz algebra* of X generated by the *Fock representation* of X on the Fock module $\mathcal{F}(X) = \bigotimes_{n=0}^\infty X^{\otimes n}$. Algebras arising in this way are known to comprise various C^* -algebras found in the literature: crossed products by automorphisms, partial crossed products, crossed products by endomorphisms, C^* -algebras of graphs (in particular, Cuntz-Krieger algebras), Exel-Laca algebras, C^* -algebras of topological quivers, and many more. It has to be noted that originally Pimsner in his analysis assumed that the *left action* ϕ on X is *injective*. However, this seemingly only technical assumption turned out to be crucial. In particular, the efforts to remove this restriction resulted in a variety of approaches [27], [1], [11], [12], [17]. We single out two of these. First, the C^* -algebra $\mathcal{O}_X$ introduced by Katsura [17] seems to be the most natural candidate for $\mathcal{O}_X$ in the general case. It is the smallest C^* -algebra among C^* -algebras generated by injective representations of X admitting gauge actions; cf. [20, Prop. 7.14]. Secondly, the so-called *relative Cuntz-Pimsner algebras* $\mathcal{O}(J, X)$ of Muhly and Solel [27] possess traits of being the most general, since by particular choices of an ideal J in A , one can cover all the aforementioned constructions. Moreover, $\mathcal{O}(J, X)$ arises quite naturally, when one tries to understand the ideal structure of Cuntz-Pimsner algebras [11], [27], and when dealing with certain concrete problems of description of C^* -algebras generated by irreversible dynamical systems [8], [21] [23].

The relationship between Pimsner’s algebras and Doplicher-Roberts algebras $\mathcal{DR}(X)$ associated with a C^* -correspondence X was investigated in [14], [11], where it was assumed, as in Pimsner’s paper [29], that the *left action* ϕ on X is *injective*. It was noticed that $\mathcal{DR}(X)$ is closely related to but tends to be larger than $\mathcal{O}_X$.

Namely, there are natural embeddings $\mathcal{O}_X \subset \mathcal{DR}(X) \subset \mathcal{O}_X^{**}$ and the equality $\mathcal{O}_X = \mathcal{DR}(X)$ holds, for instance, if X is finite projective; cf. [14, Prop. 3.2], [11, Cor. 6.3]. In particular, if $\mathcal{O}_X = \mathcal{DR}(X)$, the Doplicher-Roberts construction makes the analysis of the Cuntz-Pimsner algebra $\mathcal{O}_X$ very accessible and proves to be very useful; cf. [16]. It should be stressed, however, that when ϕ is not injective the relation between $\mathcal{DR}(X)$ and any of the algebras $\mathcal{O}(J, X)$ is far more elusive and remains practically untouched. In this article we develop a general approach to overcome these problems.

The main observation is that even though the categorical language of Doplicher and Roberts does not exactly fit into the formalism of Pimsner, in many aspects it is more natural and being *properly adapted* it clarifies the relationship between all the above-named constructions as well as shedding much more light on their structures. To support this point of view let us note that the algebras $\mathcal{O}(J, X)$, $\mathcal{DR}(X)$, $\mathcal{DR}(\mathcal{T})$ are spanned respectively by images of the spaces of “compact” operators, adjointable operators, and abstract morphisms (arrows):

$$\mathcal{K}(X^{\otimes m}, X^{\otimes n}), \quad \mathcal{L}(X^{\otimes m}, X^{\otimes n}), \quad \mathcal{T}(n, m), \quad n, m \in \mathbb{N},$$

where the above building blocks are “glued together” to form the corresponding algebras in procedures based essentially on the “tensoring”: either the natural tensoring on X , or an abstract tensoring on $\mathcal{T}$. Namely, if A is unital, the family $\mathcal{T}_X := \{\mathcal{L}(X^{\otimes m}, X^{\otimes n})\}_{n, m \in \mathbb{N}}$, where $X^{\otimes 0} := A$, with the tensoring on the right by the identity operator on X forms a natural right tensor C^* -category. By definition $\mathcal{DR}(X) := \mathcal{DR}(\mathcal{T}_X)$ and the general Doplicher-Roberts algebra $\mathcal{DR}(\mathcal{T})$ is a C^* -algebra endowed with an action of the unit circle for which the k -spectral subspace is an inductive limit of the inductive sequence

$$\mathcal{T}(r+k, r) \xrightarrow{\otimes 1} \mathcal{T}(r+k+1, r+1) \xrightarrow{\otimes 1} \mathcal{T}(r+k+2, r+2) \xrightarrow{\otimes 1} \dots$$

In particular, we have natural homomorphisms $\iota_{n, m} : \mathcal{T}(n, m) \rightarrow \mathcal{DR}(\mathcal{T})$ that form a representation of the C^* -category $\mathcal{T}$, [15], and clearly this representation determines the structure of $\mathcal{DR}(\mathcal{T})$. Similarly, the Fock representation (or any other universal representation) of X gives rise to homomorphisms $\iota_{n, m} : \mathcal{K}(X^{\otimes m}, X^{\otimes n}) \rightarrow \mathcal{O}(J, X)$ which possess the same properties as the aforesaid representation of $\mathcal{T}$. It is evident and almost symptomatic that any analysis of $\mathcal{O}(J, X)$ leads to an analysis of the family $\{\iota_{n, m}\}_{n, m \in \mathbb{N}}$ of such representations. This traces the fact that within our unified approach we may transfer a very rich and well-developed representation theory of relative Cuntz-Pimsner algebras [27], [11], [19], [20], onto the ground of Doplicher and Roberts. In particular, we may “improve” the construction of $\mathcal{DR}(\mathcal{T})$ so that the universal representation of $\mathcal{T}$ in $\mathcal{DR}(\mathcal{T})$ is injective, even when the right tensoring is not. On the other hand, generalizing the inductive limit construction of $\mathcal{DR}(\mathcal{T})$, which makes its structure very accessible, allows us to clear up the description of the ideal structure of $\mathcal{O}(J, X)$ obtained in [11], [19].

There are two plain but important new principles that shine through our development:

- 1) It is more natural to work with C^* -precategories, the categorical analogues of (not necessarily unital) C^* -algebras, rather than with C^* -categories.
- 2) Not only right tensor C^* -precategories but also their ideals naturally give rise to universal C^* -algebras.

To support 1) note that the unit-existence-requirement embedded into the notion of a category causes dispensable technicalities and lingual consequences such as that an ideal in a C^* -category, [15, Def. 1.6], may not be a C^* -category. Moreover, dealing with a C^* -correspondence X over a nonunital C^* -algebra A leads to the following problem: If ϕ is not nondegenerate there is no obvious right tensoring on the C^* -category $\{\mathcal{L}(X^{\otimes m}, X^{\otimes n})\}_{n,m \in \mathbb{N}}$, as there is no such extension of $\phi : A \rightarrow \mathcal{L}(X)$ up to the multiplier algebra $M(A) = \mathcal{L}(A)$ of A . A natural solution to that problem is considering a smaller C^* -precategory $\mathcal{T}_X$, where $\mathcal{T}_X(0,0) := A$ (then $\mathcal{T}_X$ is a C^* -category iff A is unital).

An important remark because of 2) is that the C^* -precategory $\mathcal{K}_X$, where $\mathcal{K}_X(n,m) := \mathcal{K}(X^{\otimes m}, X^{\otimes n})$, $n, m \in \mathbb{N}$, forms an ideal in the right tensor C^* -precategory $\mathcal{T}_X$, but as a rule it is not a right tensor C^* -precategory itself. Indeed, a typical situation is that the tensor product of a “compact” operator with the identity is no longer “compact”, so $\mathcal{K}_X$ is not preserved under the tensoring $\otimes 1$. As we will show this turns out not to be an obstacle. This is due to the new feature of our construction; it applies well not only to right tensor C^* -precategories but also to their ideals.

The article is organized as follows. We begin in Section 1 with a review of basic facts and objects related to C^* -correspondences that will be of interest to us. In particular, we introduce examples of C^* -correspondences associated with partial morphisms and directed graphs which we will use throughout the paper to give a dynamical and combinatorial interpretation of the theory presented. In Section 2, slightly modifying and extending the terminology of [15], we establish the rudiments of the theory of C^* -precategories. A very useful statement here is Theorem 2.6 which characterizes ideals in C^* -precategories via ideals in C^* -algebras. When applied to the C^* -precategory $\mathcal{K}_X$ this gives a one-to-one correspondence between the ideals in A and $\mathcal{K}_X$; see Proposition 2.17.

One of the most important structures of our analysis, the right tensor C^* -precategories and their representation theory, is undertaken in Section 3. We provide a definition of a *right tensor representation* as a representation of a C^* -precategory which is compatible with the tensoring and, what is very important, this notion makes sense not only for right tensor C^* -precategories but also for their ideals. Proposition 3.13 states that such representations may be considered as generalizations of representations of C^* -correspondences. Following this line of thinking we give a new meaning to the notion of coisometricity introduced by Muhly and Solel [27]; see Definitions 3.18, 3.20, and also Corollary 3.23.

In Section 4 we present three different definitions, and establish their equivalence, of the C^* -algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ of an ideal $\mathcal{K}$ in a right tensor C^* -precategory $\mathcal{T}$ relative to an ideal $\mathcal{J}$. We define $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ as: a universal C^* -algebra with respect to right tensor representations of $\mathcal{K}$ coisometric on $\mathcal{J}$, Definition 4.2; an explicitly constructed algebra with explicit formulas for norm and algebraic operations, Subsection 4.2; and a C^* -algebra obtained via inductive limits formed from a specially constructed right tensor C^* -precategory $\mathcal{K}_{\mathcal{J}}$, page 1841. Such a variety of points of view results in numerous immediate interesting remarks. In particular, it allows us to reveal the relationships between the algebras $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$, $DR(\mathcal{T})$ and algebras admitting circle action; see Section 5.

The fundamental tool in our analysis of the ideal structure of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ is the *Structure Theorem* (Theorem 6.9) which generalizes the main goal of [11]. It states

that the ideal $\mathcal{O}(\mathcal{N})$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ generated by an invariant ideal $\mathcal{N}$ in $\mathcal{T}$ may be naturally identified as $\mathcal{O}_{\mathcal{T}}(\mathcal{K} \cap \mathcal{N}, \mathcal{J} \cap \mathcal{N})$, and the quotient $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})/\mathcal{O}(\mathcal{N})$ identifies as $\mathcal{O}_{\mathcal{T}/\mathcal{N}}(\mathcal{K}/\mathcal{N}, \mathcal{J}/\mathcal{N})$. Moreover, we show that the ideal $\mathcal{N}$ may be replaced by its $\mathcal{J}$ -saturation $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ (a concept that generalizes X -saturation [28] and negative invariance [20]), so that our Structure Theorem actually gives an embedding of the lattice of invariant, $\mathcal{J}$ -saturated ideals in $\mathcal{T}$ into the lattice of gauge-invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. Another application of the Structure Theorem establishes procedures of reduction of relations defining $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$; see Definition 6.10 and Theorem 6.11. This broadens and deepens a topic started in [25] and is indispensable in our further considerations. In Subsection 6.1 we discuss how Theorem 6.9 improves [11, Thm 3.1] and why a gauge-invariant ideal $\mathcal{O}(I)$ in a relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$ is “merely” Morita equivalent to the corresponding relative Cuntz-Pimsner algebra $\mathcal{O}(J \cap I, XI)$.

Our analogue of the *gauge-invariant uniqueness theorem* is Theorem 7.3. It extends the corresponding theorems for relative Cuntz-Pimsner algebras [11, Thm. 4.1], [28, Thm. 5.1], [19, Thm. 6.4], [20, Cor. 11.7]. The main novelty is the use of the right tensor C^* -precategory $\mathcal{K}_{\mathcal{J}}$ constructed in Theorem 4.11. In particular, we establish (in Theorem 7.6) a lattice isomorphism that characterizes the *gauge-invariant ideal structure* of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ in terms of invariant $\mathcal{K}_{\mathcal{J}}$ -saturated ideals in $\mathcal{K}_{\mathcal{J}}$. Under certain additional assumptions, this result allows us to obtain an analogous description in terms of invariant $\mathcal{J}$ -saturated ideals in $\mathcal{K}$ (Theorem 7.8). The general relationship between invariant saturated ideals in $\mathcal{K}$ and $\mathcal{K}_{\mathcal{J}}$ (and thereby also gauge-invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$) is complex; on the level of relative Cuntz-Pimsner algebras it is completely revealed in Theorem 7.17 where the role of T -pairs introduced in [20] is also clarified.

Aiming at a generalization of [11, Thm. 6.6], [14, Thm. 4.1] in Section 8 we examine the general conditions assuring that algebras of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ embed into one another. In this direction we establish two useful results, Propositions 8.2, 8.5, which are nontrivial, inequivalent generalizations of [14, Prop. 3.2], [11, Cor. 6.3]. Motivated by these considerations we introduce an analogue of relative Cuntz-Pimsner algebras: *relative Doplicher-Roberts algebras* $\mathcal{DR}(J, X)$, Definition 8.7. In particular, we give necessary and sufficient conditions under which the natural embedding $\mathcal{O}(J_0, X) \subset \mathcal{DR}(J, X)$ holds.

In the final section, we describe representations of $\mathcal{DR}(J, X)$ that extend representations of $\mathcal{O}(J_0, X)$ (Proposition 9.3) and give criteria under which such a representation is faithful; see Theorem 9.4. Additionally we show that every faithful representation of $\mathcal{O}(J_0, X)$ extends to a faithful representation of $\mathcal{DR}(J, X)$ for an appropriate J ; see Theorem 9.6.

Notational conventions 0.1. Following [20] we denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers, by $\mathbb{C}$ the set of complex numbers, and by S^1 the group of complex numbers with absolute value 1. We use a convention that $\gamma(A, B) = \{\gamma(a, b) \in D \mid a \in A, b \in B\}$ for a map $\gamma: A \times B \rightarrow D$ such as inner products, multiplications or representations. We denote by $\text{span}\{\dots\}$ linear spans of $\{\dots\}$, and by $\overline{\text{span}}\{\dots\}$ the closure of $\text{span}\{\dots\}$.

1. PRELIMINARIES ON C^* -CORRESPONDENCES

We adopt the standard notation and definitions of objects related to (right) Hilbert C^* -modules; cf. [26], [30]. In particular, we denote by X and Y Hilbert

modules over a C^* -algebra A ; $\mathcal{L}(X, Y)$ stands for the space of adjointable operators from X into Y ; and $\mathcal{K}(X, Y)$ is the space of “compact” operators in $\mathcal{L}(X, Y)$, that is, $\mathcal{K}(X, Y) = \overline{\text{span}}\{\Theta_{y,x} : x \in X, y \in Y\}$, where $\Theta_{y,x}(z) = y\langle x, z \rangle_A$, $z \in X$. If I is an ideal in A (by which we always mean a closed two-sided ideal), then XI is both a Hilbert A -submodule of X and a Hilbert I -module, as we have

$$(1.1) \quad XI = \{xi : x \in X, i \in I\} = \{x \in X : \langle x, y \rangle_A \in I \text{ for all } y \in X\};$$

cf. [20, Prop. 1.3]. A natural identification of $\mathcal{K}(XI)$ as a subalgebra of $\mathcal{K}(X)$ was used in [20], [11]. Actually we have

Lemma 1.1. *The equalities*

$$\begin{aligned} \overline{\text{span}}\{\Theta_{y,x} : x \in X, y \in YI\} &= \overline{\text{span}}\{\Theta_{y,x} : x \in XI, y \in Y\} \\ &= \overline{\text{span}}\{\Theta_{y,x} : x \in XI, y \in YI\} \end{aligned}$$

establish natural identifications

$$\mathcal{K}(X, YI) = \mathcal{K}(XI, Y) = \mathcal{K}(XI, YI) \subset \mathcal{K}(X, Y).$$

Proof. This is clear by the definitions; cf. [26, p. 9], [20]. $\square$

The above identifications do not carry over to general adjointable maps due to a possible nonexistence of adjoint operators. In order to clarify this situation, let us note that if I is an ideal in A , then (by (1.1)) for $a \in \mathcal{L}(X, Y)$ the two conditions $a(X) \subset YI$ and $a^*(Y) \subset XI$ are equivalent. We denote by $\mathcal{L}_I(X, Y)$ the space of all adjointable maps $a \in \mathcal{L}(X, Y)$ satisfying these equivalent conditions.

Lemma 1.2. *The inclusions of sets of mapping and restriction of mappings yield six natural linear homomorphisms between the space of morphisms, presented on the diagram*

$$\begin{array}{ccccc} & & \mathcal{L}(XI, YI) & & \\ & \nearrow \text{incls.} & \uparrow \text{restr.} & \nwarrow \text{restr.} & \\ \mathcal{L}(XI, Y) & & \mathcal{L}(X, YI) & & \mathcal{L}(X, Y) \\ & \nwarrow \text{restr.} & \uparrow \text{incls.} & \nearrow \text{incls.} & \\ & & \mathcal{L}_I(X, Y) & & \end{array}$$

where

- the three homomorphisms from $\mathcal{L}_I(X, Y)$ to $\mathcal{L}(XI, Y)$, $\mathcal{L}(X, YI)$, $\mathcal{L}(X, Y)$ are injective,
- homomorphisms from $\mathcal{L}(XI, Y)$ and $\mathcal{L}(X, YI)$ to $\mathcal{L}(XI, YI)$ are injective and the one from $\mathcal{L}(X, Y)$ to $\mathcal{L}(XI, YI)$ in general is not.

Moreover,

- the three intersections in $\mathcal{L}(XI, YI)$ of any two images of the three homomorphisms in b) coincide,
- the three maps from $\mathcal{L}_I(X, Y)$ to $\mathcal{L}(XI, YI)$, through $\mathcal{L}(XI, Y)$, $\mathcal{L}(X, YI)$, $\mathcal{L}(X, Y)$, coincide, and this common map is an isomorphism onto the common intersection in i).

Proof. To see item a) note that if $a \in \mathcal{L}_I(Y, X)$, then a treated as a mapping is an element of $\mathcal{L}(X, YI)$ with the adjoint given by restriction of $a^* \in \mathcal{L}(Y, X)$ to YI . Moreover, the map $\mathcal{L}_I(X, Y) \hookrightarrow \mathcal{L}(XI, Y)$ given by restriction of mappings is

an isometric homomorphism since it can be obtained by passing to adjoints in the inclusion $\mathcal{L}_I(Y, X) \subset \mathcal{L}(Y, XI)$. To prove item b) one can argue in a similar way.

For item i) note that $a \in \mathcal{L}(XI, YI)$ is in the intersection of the images of $\mathcal{L}(XI, Y)$ and $\mathcal{L}(X, YI)$ in $\mathcal{L}(XI, YI)$ if and only if there are $\tilde{a} \in \mathcal{L}(X, YI)$ and $\tilde{a}^* \in \mathcal{L}(Y, XI)$ such that $\tilde{a}|_{XI} = a$ and $\tilde{a}^*|_{YI} = a^*$, but then it follows that $\tilde{a} \in \mathcal{L}_I(X, Y)$, where $\tilde{a}^* = \tilde{a}|^*$. Conversely, for any $\tilde{a} \in \mathcal{L}_I(X, Y)$ its restriction $a = \tilde{a}|_{XI}$ yields an element lying in the intersection of the images of $\mathcal{L}(XI, Y)$ and $\mathcal{L}(X, YI)$ in $\mathcal{L}(XI, YI)$. Similarly one sees that the remaining two intersections in item i) coincide and consist of morphisms from $\mathcal{L}(XI, YI)$ possessing adjointable extensions to elements of $\mathcal{L}(X, Y)$ which then necessarily lie in $\mathcal{L}_I(X, Y)$. This proves i) and ii). $\square$

Remark 1.3. All five injective homomorphisms in the above lemma become inclusions when viewing X and Y as Hilbert modules in a C^* -algebra $\mathcal{M}$; cf. [14, Prop. 2.1]. In general, the six considered maps are not surjective, and the three images of $\mathcal{L}(XI, Y)$, $\mathcal{L}(X, YI)$ and $\mathcal{L}(X, Y)$ in $\mathcal{L}(XI, YI)$ are incomparable.

An important fact for our purposes is

Proposition 1.4. *The C^* -algebra $\mathcal{K}(XI)$ is an ideal in $\mathcal{L}_I(X)$ which in turn is an ideal in the C^* -algebra $\mathcal{L}(X)$.*

For an ideal I in a C^* -algebra A we may consider the quotient space X/XI as a Hilbert A/I -module with an A/I -valued inner product and right action of A/I given by $\langle q(x), q(y) \rangle_{A/I} := q(\langle x, y \rangle_A)$, $q(x)q(a) := q(xa)$, where q denotes both the quotient maps $A \rightarrow A/I$ and $X \rightarrow X/XI$; cf. [11, Lem. 2.1]. Moreover, we have a natural map $q : \mathcal{L}(X) \rightarrow \mathcal{L}(X/XI)$, where $q(a)q(x) = q(ax)$ for $a \in \mathcal{L}(X)$ and $x \in X$.

Lemma 1.5. *The kernel of the map $q : \mathcal{L}(X) \rightarrow \mathcal{L}(X/XI)$ is $\mathcal{L}_I(X)$ and the restriction of q to $\mathcal{K}(X)$ is a surjection onto $\mathcal{K}(X/XI)$ whose kernel is $\mathcal{K}(XI)$.*

Proof. See [20, Lem. 1.6] and the remarks preceding this statement. $\square$

Corollary 1.6 (Lem. 2.6 [11]). *We have a natural isomorphism $\mathcal{K}(X)/\mathcal{K}(XI) \cong \mathcal{K}(X/XI)$.*

For Hilbert modules with left action we use the term C^* -correspondence, and we reserve the term Hilbert bimodule for an object with an additional structure (cf. Definition 1.10 and Proposition 1.11 below). This convention seems to become standard.

Definition 1.7. *A C^* -correspondence X over a C^* -algebra A is a (right) Hilbert A -module equipped with a $*$ -homomorphism $\phi : A \rightarrow \mathcal{L}(X)$. We refer to ϕ as the left action of the C^* -correspondence X and write $a \cdot x := \phi(a)x$.*

Let us fix a C^* -correspondence X over a C^* -algebra A and a Hilbert A -module Y . There is a naturally defined tensor product Hilbert A -module $Y \otimes X$; cf. [26], [30] or [19]. An ideal I in A is called X -invariant if $\varphi(I)X \subset XI$, and for such an ideal the quotient A/I -module X/XI with quotient left action $q(a)q(x) = q(\varphi(a)x)$ becomes a C^* -correspondence over A/I ; cf. [11, Lem. 2.3], [20], [16]. In particular, we may consider two Hilbert A/I -modules $Y/YI \otimes X/XI$ and $(Y \otimes X)/(Y \otimes XI)$.

Lemma 1.8. *For an X -invariant ideal I in A we have a natural isomorphism of Hilbert modules*

$$(Y/YI) \otimes (X/XI) \cong (Y \otimes X)/(Y \otimes XI).$$

Proof. Let $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $i, j \in I$. Then

$$(y_1 + y_2j) \otimes (x_1 + x_2i) = y_1 \otimes x_1 + \left(y_1 \otimes x_2i + y_2 \otimes \varphi(j)x_1 + y_2j \otimes x_2i \right),$$

where by X -invariance of I the term in brackets belongs to $Y \otimes XI$. Thereby the mapping

$$(y + YI) \otimes (x + XI) \mapsto (y \otimes x) + (Y \otimes XI)$$

is well defined. Clearly, it is surjective and A/I -linear. The simple calculation

$$\langle q(y_1) \otimes q(x_1), q(y_2) \otimes q(x_2) \rangle_{A/I} = \langle q(x_1), q(\varphi\langle y_1, y_2 \rangle_A) q(x_2) \rangle_{A/I}$$

$$= q(\langle x_1, \varphi(\langle y_1, y_2 \rangle_A) x_2 \rangle_A) = q(\langle y_1 \otimes x_1, y_2 \otimes x_2 \rangle_A) = \langle q(y_1 \otimes x_1), q(y_2 \otimes x_2) \rangle_{A/I}$$

shows that the above mapping preserves A/I -valued inner products and hence is isometric. $\square$

We have a homomorphism $\mathcal{L}(Y) \ni a \rightarrow a \otimes 1 \in \mathcal{L}(Y \otimes X)$, where

$$(1.2) \quad (a \otimes 1)(y \otimes x) := ay \otimes x, \quad x \in X, y \in Y.$$

The properties of this homomorphism are related to objects that will play an important role throughout this paper. We define

$$J(X) := \phi^{-1}(\mathcal{K}(X)),$$

which is an ideal in A . If J is an ideal in A we define

$$J^\perp = \{a \in A : aJ = \{0\}\},$$

which is also an ideal in A called an *annihilator* of J . It is a unique ideal in A such that $J^\perp \cap J = \{0\}$ and for any ideal I in A we have $J \cap I = \{0\} \implies I \subset J^\perp$.

Lemma 1.9 (cf. [11, Lem. 4.2]). *Suppose that X is a C^* -correspondence over A and Y is a right Hilbert A -module.*

- i) *The map $a \mapsto a \otimes 1$ restricted to $\mathcal{L}_{(\ker \phi)^\perp}(Y)$ is isometric.*
- ii) *If $\{\mu_\lambda\}_\lambda$ is an approximate unit for $\mathcal{K}(Y)$, then $\mu_\lambda \otimes 1$ converges strictly to $1 \in \mathcal{L}(Y \otimes X)$. In particular, $(\mathcal{K}(Y) \otimes 1)\mathcal{K}(Y \otimes X) = \mathcal{K}(Y \otimes X)$.*
- iii) *If $a \in \mathcal{L}_{(\ker \phi)^\perp}(Y)$ and $a \otimes 1 \in \mathcal{K}(Y \otimes X)$, then $a \in \mathcal{K}(Y)$.*
- iv) *If $a \otimes 1 \in \mathcal{K}(Y \otimes X)$ and $a \in \mathcal{K}(Y)$, then $a \in \mathcal{K}(YJ(X))$.*

Proof. i) Let $a \in \mathcal{L}_{(\ker \phi)^\perp}(Y)$. It suffices to prove $\|a\| \leq \|a \otimes 1\|$, and for that purpose take an arbitrary $y \in Y$. Since $\langle ay, ay \rangle_A \in (\ker \phi)^\perp$ and ϕ is isometric on $(\ker \phi)^\perp$ we have $\|\phi(\langle ay, ay \rangle_A)\| = \|ay\|^2$. Positivity of $\phi(\langle ay, ay \rangle_A)$ implies that for each $\varepsilon > 0$, there exists $x \in X$ such that $\|x\| = 1$ and

$$\|\langle x, \phi(\langle ay, ay \rangle_A)x \rangle_A\| \geq \|\phi(\langle ay, ay \rangle_A)\| - \varepsilon = \|ay\|^2 - \varepsilon.$$

Thus

$$\|(a \otimes 1)(y \otimes x)\|^2 = \|\langle x, \phi(\langle ay, ay \rangle_A)x \rangle_A\| \geq \|ay\|^2 - \varepsilon$$

and as $\|(y \otimes x)\| \leq \|y\|$ we get $\|a \otimes 1\| \geq \|a\|$.

- ii) See the proof of [11, Lem. 4.2 (2)].

iii) Let $\{\mu_\lambda\}_\lambda$ be an approximate unit for $\mathcal{K}(Y)$. By item ii) we get

$$0 = \lim_\lambda \|a \otimes 1 - (\mu_\lambda \otimes 1)(a \otimes 1)\| = \lim_\lambda \|a - \mu_\lambda a\|.$$

Hence a is “compact”.

iv) See the second part of the proof of [11, Lem. 4.2 (2)]. $\square$

Following [19] we clarify the relationship between C^* -correspondences and Hilbert bimodules.

Definition 1.10. We say that X is a *Hilbert A -bimodule* if it is at the same time a Hilbert left A -module and a Hilbert right A -module with sesqui-linear forms ${}_A\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_A$ related via the so-called imprimitivity condition:

$$(1.3) \quad x \cdot \langle y, z \rangle_A = {}_A\langle x, y \rangle \cdot z, \quad \text{for all } x, y, z \in X.$$

It follows from (1.3) that the left action in a Hilbert bimodule acts by adjointable maps and hence every Hilbert C^* -bimodule is a C^* -correspondence; cf. [17, 3.3]. In the converse direction we have

Proposition 1.11. *Let X be a C^* -correspondence. The following conditions are equivalent:*

- i) X is a Hilbert A -bimodule (that is, there exists a sesqui-linear form ${}_A\langle \cdot, \cdot \rangle$ for which X together with the left action ϕ becomes a left Hilbert A -module satisfying (1.3)),
- ii) there is a function ${}_A\langle \cdot, \cdot \rangle : X \times X \rightarrow (\ker \phi)^\perp$ such that

$$(1.4) \quad \phi({}_A\langle x, y \rangle) = \Theta_{x, y}, \quad x, y \in X,$$

- iii) the mapping $\phi : (\ker \phi)^\perp \cap J(X) \rightarrow \mathcal{K}(X)$ is onto (hence it is automatically an isomorphism).

The objects in i) and ii) are determined uniquely, the function ${}_A\langle \cdot, \cdot \rangle$ from item ii) coincides with the inner product from item i) and

$$(1.5) \quad {}_A\langle x, y \rangle = \phi^{-1}(\Theta_{x, y}), \quad x, y \in X,$$

where ϕ^{-1} is the inverse to the isomorphism $\phi : (\ker \phi)^\perp \cap J(X) \rightarrow \mathcal{K}(X)$. Moreover denoting $\overline{{}_A\langle X, X \rangle} := \overline{\text{span}\{{}_A\langle x, y \rangle : x, y \in X\}}$ we have

$$\overline{{}_A\langle X, X \rangle} = (\ker \phi)^\perp \cap J(X).$$

Proof. i) $\Rightarrow$ ii) One easily sees that conditions (1.4) and (1.3) are equivalent. Since ϕ is injective on $\overline{{}_A\langle X, X \rangle}$ and $\overline{{}_A\langle X, X \rangle}$ is an ideal we have $\overline{{}_A\langle X, X \rangle} \subset (\ker \phi)^\perp$. Consequently ii) holds.

ii) $\Rightarrow$ iii) By (1.4), ϕ maps the set $\overline{{}_A\langle X, X \rangle}$ onto $\mathcal{K}(X)$ and $\overline{{}_A\langle X, X \rangle} \subset (\ker \phi)^\perp \cap J(X)$. As ϕ is injective on $(\ker \phi)^\perp$ we see that $\phi : (\ker \phi)^\perp \cap J(X) \rightarrow \mathcal{K}(X)$ is an isomorphism. In particular, $\overline{{}_A\langle X, X \rangle} = (\ker \phi)^\perp \cap J(X)$ and (1.5) holds.

iii) $\Rightarrow$ i) It is straightforward to check that ${}_A\langle \cdot, \cdot \rangle$ defined by (1.5) is an inner product for the left A -module X . As we already noted, (1.4) and (1.3) are equivalent, and thus i) holds. $\square$

Example 1.12 (C^* -correspondence of a partial morphism). By a *partial morphism* of a C^* -algebra A we mean a nondegenerate $*$ -homomorphism $\varphi : A \rightarrow M(A_0)$ from A to the multiplier algebra $M(A_0)$ of a hereditary subalgebra A_0 of A ; cf. [17]. We recall that $\varphi : A \rightarrow M(A_0)$ is said to be *nondegenerate* if $\varphi(A)A_0 = A_0$. We

construct a C^* -correspondence X_φ from φ in the following way. We let $X_\varphi := A_0 A$ and put

$$a \cdot x := \varphi(a)x, \quad x \cdot a := xa, \quad \text{and} \quad \langle x, y \rangle_A := x^*y,$$

where $a \in A$, $x, y \in X_\varphi$. Then $J(X_\varphi) = \varphi^{-1}(A_0)$. To assert when X_φ is a Hilbert bimodule we slightly extend R. Exel's definition [10, Def. 3.1] and by a *partial automorphism* of A we shall mean a triple (θ, I, A_0) consisting of an ideal I in A , a hereditary subalgebra A_0 of A and an isomorphism $\theta : I \rightarrow A_0$. A partial automorphism (θ, I, A_0) gives rise to a partial morphism $\varphi : A \rightarrow M(A_0)$ via the formula $\varphi(a)b := \theta(a\theta^{-1}(b))$, $a \in A$, $b \in A_0$, and then we have $I = (\ker \varphi)^\perp \cap \varphi^{-1}(A_0)$. Conversely, if $\varphi : A \rightarrow M(A_0)$ is a partial morphism such that φ restricted to $I := (\ker \varphi)^\perp \cap \varphi^{-1}(A_0)$ is an isomorphism onto A_0 , then φ arises from a partial automorphism (θ, I, A_0) , where $\theta := \varphi|_I$.

Therefore, in view of Proposition 1.11, a C^* -correspondence X_φ is a Hilbert bimodule if and only if φ arises from a partial automorphism θ , and then the “left” inner product is given by

$${}_A \langle x, y \rangle := \theta^{-1}(xy^*).$$

We indicate that every endomorphism $\alpha : A \rightarrow A$ of a C^* -algebra A may be treated as a partial morphism, where $A_0 = \alpha(A)A\alpha(A)$. In this sense a partial morphism φ is an endomorphism iff $\varphi^{-1}(A_0) = A$. We shall denote a C^* -correspondence arising from an endomorphism α by X_α . It is shown in [22, Prop. 1.9] that if A is unital, then $X_\alpha = \alpha(1)A$ is a Hilbert bimodule iff there exists a *complete transfer operator* for α , which is a bounded, positive linear map $\mathcal{L} : A \rightarrow A$ such that

$$\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b) \quad \text{and} \quad \alpha(\mathcal{L}(1)) = \alpha(1), \quad a, b \in A;$$

see [24] ($\mathcal{L}$ exists iff the kernel of α is unital and the range is hereditary; see [22]). If this is the case the “left” inner product is given by

$${}_A \langle x, y \rangle := \mathcal{L}(xy^*).$$

Example 1.13 (C^* -correspondence of a graph). Suppose $E = (E^0, E^1, r, s)$ is a *directed graph* with vertex set E^0 , edge set E^1 , and $r, s : E^1 \rightarrow E^0$ describing the range and the source of edges. A C^* -correspondence X_E of the graph E is defined in the following manner; cf. [12], [11], [28]. The space X_E consists of functions $x : E^1 \rightarrow \mathbb{C}$ for which

$$v \in E^0 \longmapsto \sum_{\{e \in E^1 : r(e)=v\}} |x(e)|^2$$

belongs to $A := C_0(E^0)$, and X_E is a C^* -correspondence over A with the operations

$$\begin{aligned} (x \cdot a)(e) &:= x(e)a(r(e)) \quad \text{for } e \in E^1, \\ \langle x, y \rangle_A(v) &:= \sum_{\{e \in E^1 : r(e)=v\}} \overline{x(e)}y(e) \quad \text{for } v \in E^0, \text{ and} \\ (a \cdot x)(e) &:= a(s(e))x(e) \quad \text{for } e \in E^1. \end{aligned}$$

We note that X_E and A are respectively spanned by the point masses $\{\delta_f : f \in E^1\}$ and $\{\delta_v : v \in E^0\}$. In particular,

$$(\ker \phi)^\perp = \overline{\text{span}}\{\delta_v : s^{-1}(v) \neq \emptyset\}, \quad J(X_E) = \overline{\text{span}}\{\delta_v : |s^{-1}(v)| < +\infty\};$$

cf. for instance [28]. Moreover, if $f, g \in E^1$ and $v \in E^0$ emits finitely many edges, then

$$\phi(\delta_v) = \sum_{\{e \in E^1: s(e)=v\}} \Theta_{\delta_e, \delta_e} \quad \text{and} \quad \Theta_{\delta_f, \delta_g} \neq 0 \iff r(f) = r(g).$$

It follows that X_E is a Hilbert bimodule iff every vertex of the graph E emits and receives at most one edge (equivalently maps r, s are injective). If this is the case the left and right inner products are zero on the complement of $r(E^1)$ and $s(E^1)$, respectively, and

$$\langle x, y \rangle_A(r(e)) = \overline{x(e)}y(e), \quad {}_A\langle x, y \rangle(s(e)) = x(e)\overline{y(e)}.$$

The above formulas and objects naturally generalize to *topological graphs* [18], [17], that is, quadruples $E = (E^0, E^1, d, r)$, where E^0, E^1 are locally compact spaces, $d: E^1 \rightarrow E^0$ is a local homeomorphism and $r: E^1 \rightarrow E^0$ is a continuous map. The C^* -correspondence X_E of the topological graph E is defined by the same formulas as above, with r replaced with d and s with r . It is a Hilbert bimodule iff d, r are homeomorphisms onto open sets and then the bimodule X_E may be treated as a bimodule X_φ arising from a partial morphism $\varphi: C_0(E^0) \rightarrow C_b(r(E^1))$ given by the partial homeomorphism $r \circ d^{-1}: d(E^1) \rightarrow r(E^1)$.

2. C^* -PRECATEGORIES AND THEIR IDEALS

For a notion of a precategory we adopt the standard definition of a category with the only exception that we drop the assumption of existence of identity morphisms. In the present paper we shall be interested in precategories with the class of objects being the set of natural numbers $\mathbb{N}$. However, for future reference and to underscore the role of categorical language, in this section (and only in this section) we shall deal with general precategories.

Definition 2.1. A *precategory* $\mathcal{T}$ consists of a class of objects, denoted here by ρ, σ, τ , etc., a class $\{\mathcal{T}(\sigma, \rho)\}_{\sigma, \rho \in \mathcal{T}}$ of disjoint classes of morphisms (arrows) where $\mathcal{T}(\sigma, \rho)$ stands for the space of morphisms from ρ to σ , and a composition of morphisms $\mathcal{T}(\tau, \sigma) \times \mathcal{T}(\sigma, \rho) \ni (a, b) \rightarrow ab \in \mathcal{T}(\tau, \rho)$, which is associative, in the sense that $(ab)c = a(bc)$ whenever the compositions of morphisms a, b, c are allowable.

One can always equip (if necessary) the sets of morphisms $\mathcal{T}(\sigma, \sigma)$ with identity morphisms in such a way that a given precategory $\mathcal{T}$ becomes a category. We generalize the notion of a C^* -category; cf. [15], [13], in an obvious fashion.

Definition 2.2. A precategory $\mathcal{T}$ is a C^* -precategory if each set of morphisms $\mathcal{T}(\sigma, \rho)$ is a complex Banach space, the composition of morphisms gives us a bilinear map

$$\mathcal{T}(\tau, \sigma) \times \mathcal{T}(\sigma, \rho) \ni (a, b) \rightarrow ab \in \mathcal{T}(\tau, \rho)$$

with $\|ab\| \leq \|a\| \cdot \|b\|$, and there is an antilinear involutive contravariant functor $*$: $\mathcal{T} \rightarrow \mathcal{T}$ such that if $a \in \mathcal{T}(\sigma, \rho)$, then $a^* \in \mathcal{T}(\rho, \sigma)$ and the C^* -equality $\|a^*a\| = \|a\|^2$ holds. A C^* -precategory $\mathcal{T}$ where $\mathcal{T}$ is a category is a C^* -category.

Notational conventions 2.3. If $\mathcal{S}$ is a *sub- C^* -precategory* of $\mathcal{T}$, that is, if $\mathcal{S}$ and $\mathcal{T}$ are two C^* -precategories such that each space $\mathcal{S}(\sigma, \rho)$ is a closed subspace of $\mathcal{T}(\sigma, \rho)$, we shall briefly write $\mathcal{S} \subset \mathcal{T}$. If $\mathcal{S}$ and $\mathcal{T}$ are two sub- C^* -precategories of another C^* -precategory, we denote by $\mathcal{T} \cap \mathcal{S}$ the C^* -precategory, where $(\mathcal{S} \cap \mathcal{T})(\sigma, \rho) := \mathcal{S}(\sigma, \rho) \cap \mathcal{T}(\sigma, \rho)$.

Each space of morphisms $\mathcal{T}(\rho, \rho)$ in a C^* -precategory $\mathcal{T}$ is a C^* -algebra, and by the C^* -equality the functor “ $*$ ” is isometric on every space $\mathcal{T}(\sigma, \rho)$. A C^* -precategory $\mathcal{T}$ is a C^* -category if and only if every C^* -algebra $\mathcal{T}(\rho, \rho)$, $\rho \in \mathcal{T}$, is unital. In general, by adjoining units to C^* -algebras $\mathcal{T}(\rho, \rho)$, $\rho \in \mathcal{T}$, one may obtain “unitization” of $\mathcal{T}$, which is a C^* -category that contains $\mathcal{T}$ as an ideal in the sense of the following definition; cf. [15, Def. 1.6].

Definition 2.4. By a (closed two-sided) *ideal* $\mathcal{K}$ in a C^* -precategory $\mathcal{T}$ we shall mean a collection of Banach subspaces $\mathcal{K}(\sigma, \rho)$ in $\mathcal{T}(\sigma, \rho)$, $\rho, \sigma \in \mathcal{T}$, such that

$$\mathcal{T}(\tau, \sigma)\mathcal{K}(\sigma, \rho) \subset \mathcal{K}(\tau, \rho) \quad \text{and} \quad \mathcal{K}(\tau, \sigma)\mathcal{T}(\sigma, \rho) \subset \mathcal{K}(\tau, \rho),$$

for any $\sigma, \rho, \tau \in \mathcal{T}$.

Arguing as for C^* -algebras; cf. [15, Prop. 1.7], one shows that an ideal $\mathcal{K}$ in a C^* -precategory $\mathcal{T}$ is “selfadjoint” (hence it is a C^* -precategory). Obviously, each space $\mathcal{K}(\rho, \rho)$ is a closed two-sided ideal in the C^* -algebra $\mathcal{T}(\rho, \rho)$. A useful fact is that $\mathcal{K}$ is uniquely determined by these “diagonal ideals”. In order to prove this, we apply the following simple lemma.

Lemma 2.5. *Let $\mathcal{T}$ be a C^* -precategory and let $a \in \mathcal{T}(\sigma, \rho)$. Let $\mathcal{K}(\rho, \rho)$, $\mathcal{K}(\sigma, \sigma)$ be arbitrary ideals in $\mathcal{T}(\rho, \rho)$ and $\mathcal{T}(\sigma, \sigma)$ respectively, and let $\{\mu_\lambda\}_\lambda$, $\{\nu_\lambda\}_\lambda$ be approximate units in $\mathcal{K}(\rho, \rho)$ and $\mathcal{K}(\sigma, \sigma)$ respectively. Then*

$$a^*a \in \mathcal{K}(\rho, \rho) \iff \lim_\lambda a\mu_\lambda = a,$$

$$aa^* \in \mathcal{K}(\sigma, \sigma) \iff \lim_\lambda \nu_\lambda a = a.$$

Proof. If $\lim_\lambda a\mu_\lambda = a$, then $\lim_\lambda a^*a\mu_\lambda = a^*a$ is in $\mathcal{K}(\rho, \rho)$. Conversely, if $a^*a \in \mathcal{K}(\rho, \rho)$, then

$$\|a - a\mu_\lambda\|^2 = \|(a - a\mu_\lambda)^*(a - a\mu_\lambda)\| \leq \|a^*a - a^*a\mu_\lambda\| + \|\mu_\lambda\| \cdot \|a^*a - a^*a\mu_\lambda\|,$$

which implies that $a\mu_\lambda$ converges to a . The second equivalence can be proved analogously. $\square$

Theorem 2.6 (Characterization of ideals in C^* -precategories). *If $\mathcal{K}$ is an ideal in a C^* -precategory $\mathcal{T}$, then for all objects σ and ρ ,*

$$(2.1) \quad \mathcal{K}(\sigma, \rho) = \{a \in \mathcal{T}(\sigma, \rho) : a^*a \in \mathcal{K}(\rho, \rho)\} = \{a \in \mathcal{T}(\sigma, \rho) : aa^* \in \mathcal{K}(\sigma, \sigma)\}.$$

Conversely, if $\{\mathcal{K}(\rho, \rho)\}_{\rho \in \mathcal{T}}$ is a class of ideals $\mathcal{K}(\rho, \rho)$ in $\mathcal{T}(\rho, \rho)$, $\rho \in \mathcal{T}$, such that the equality

$$(2.2) \quad \{a \in \mathcal{T}(\sigma, \rho) : a^*a \in \mathcal{K}(\rho, \rho)\} = \{a \in \mathcal{T}(\sigma, \rho) : aa^* \in \mathcal{K}(\sigma, \sigma)\}$$

holds for every $\sigma, \rho \in \mathcal{T}$, then relations (2.1) define an ideal $\mathcal{K}$ in $\mathcal{T}$.

Proof. Let $\mathcal{K}$ be an ideal in $\mathcal{T}$ and let $a \in \mathcal{T}(\sigma, \rho)$. If $a \in \mathcal{K}(\sigma, \rho)$, then $a^*a \in \mathcal{K}(\rho, \rho)$. Conversely, if $a^*a \in \mathcal{K}(\rho, \rho)$, then using Lemma 2.5 one gets $a \in \mathcal{K}(\sigma, \rho)$. Thus $\mathcal{K}(\sigma, \rho) = \{a \in \mathcal{T}(\sigma, \rho) : a^*a \in \mathcal{K}(\rho, \rho)\}$, and the equality $\mathcal{K}(\sigma, \rho) = \{a \in \mathcal{T}(\sigma, \rho) : aa^* \in \mathcal{K}(\sigma, \sigma)\}$ can be proved analogously.

To prove the second part of the assertion we fix a class of ideals $\{\mathcal{K}(\rho, \rho)\}_{\rho \in \mathcal{T}}$ such that (2.2) holds and use (2.1) to define $\mathcal{K} = \{\mathcal{K}(\sigma, \rho)\}_{\sigma, \rho \in \mathcal{T}}$. If $a \in \mathcal{K}(\sigma, \rho)$, then $a^*a \in \mathcal{K}(\rho, \rho)$ and by Lemma 2.5 for arbitrary $b \in \mathcal{T}(\tau, \sigma)$ we have $(ba)^*(ba) = a^*b^*ba \in \mathcal{K}(\rho, \rho)$, that is, $ba \in \mathcal{K}(\tau, \rho)$. This shows that $\mathcal{T}(\tau, \sigma)\mathcal{K}(\sigma, \rho) \subset \mathcal{K}(\tau, \rho)$. From (2.1) it follows that the star functor preserves $\mathcal{K}$ and thus $\mathcal{K}(\tau, \sigma)\mathcal{T}(\sigma, \rho) = (\mathcal{T}(\rho, \sigma)\mathcal{K}(\sigma, \tau))^* \subset (\mathcal{K}(\rho, \tau))^* = \mathcal{K}(\tau, \rho)$. $\square$

As a first application of the above statement we construct an annihilator of an ideal in a C^* -precategory.

Proposition 2.7. *If $\mathcal{I}$ is an ideal in a C^* -precategory $\mathcal{T}$, then there exists a unique ideal $\mathcal{I}^\perp$ in $\mathcal{T}$ such that*

$$\mathcal{I}^\perp(\rho, \rho) = \mathcal{I}(\rho, \rho)^\perp, \quad \sigma \in \mathcal{T}.$$

Accordingly, we have $\mathcal{I}^\perp \cap \mathcal{I} = \{0\}$ and for any ideal $\mathcal{K}$ in $\mathcal{T}$,

$$\mathcal{I} \cap \mathcal{K} = \{0\} \implies \mathcal{K} \subset \mathcal{I}^\perp,$$

where $\{0\}$ denotes the ideal consisting of zero morphisms.

The ideal $\mathcal{I}^\perp$ will be called an *annihilator* of $\mathcal{I}$.

Proof. In view of Theorem 2.6 we need to verify that the ideals $\mathcal{I}(\rho, \rho)^\perp$, $\rho \in \mathcal{T}$, satisfy condition (2.2). Suppose then that $a \in \mathcal{T}(\sigma, \rho)$ is such that $a^*a \in \mathcal{I}(\rho, \rho)^\perp$. By Lemma 2.5, $a^*b^*ba \in \mathcal{I}(\rho, \rho)^\perp$, for all $b \in \mathcal{T}(\sigma, \sigma)$. In particular, for all $b \in \mathcal{I}(\sigma, \sigma)$ we have $a^*b^*ba \in (\mathcal{I} \cap \mathcal{I}^\perp)(\rho, \rho) = \{0\}$ and since

$$a^*b^*ba = 0 \implies ba = 0 \implies baa^* = 0,$$

it follows that $aa^* \in \mathcal{I}(\sigma, \sigma)^\perp$. $\square$

We now turn to a discussion of maps between C^* -precategories.

Definition 2.8. A *homomorphism* Φ from a C^* -precategory $\mathcal{T}$ to a C^* -precategory $\mathcal{S}$ consists of a mapping $\mathcal{T} \ni \sigma \mapsto \Phi(\sigma) \in \mathcal{S}$ and linear operators $\mathcal{T}(\sigma, \rho) \ni a \mapsto \Phi(a) \in \mathcal{S}(\Phi(\sigma), \Phi(\rho))$, $\sigma, \rho \in \mathcal{T}$, such that

$$\Phi(a)\Phi(b) = \Phi(ab), \quad \Phi(a^*) = \Phi(a)^*, \quad b \in \mathcal{T}(\sigma, \rho), \quad a \in \mathcal{T}(\tau, \sigma), \quad \sigma, \rho, \tau \in \mathcal{T}.$$

The notions such as an *isomorphism*, *endomorphism*, etc., for C^* -precategories are defined in an obvious fashion.

A homomorphism $\Phi : \mathcal{T} \rightarrow \mathcal{S}$ between C^* -categories is a functor iff it is “unital”, that is, for every $\rho \in \mathcal{T}$, Φ maps the unit in $\mathcal{T}(\rho, \rho)$ onto the unit in $\mathcal{S}(\Phi(\rho), \Phi(\rho))$. In general, as the operators $\Phi : \mathcal{T}(\rho, \rho) \rightarrow \mathcal{S}(\Phi(\rho), \Phi(\rho))$ are $*$ -homomorphisms of C^* -algebras, using C^* -equality, one easily gets

Proposition 2.9. *For any homomorphism $\Phi : \mathcal{T} \rightarrow \mathcal{S}$ of C^* -precategories the operators*

$$\Phi : \mathcal{T}(\sigma, \rho) \rightarrow \mathcal{S}(\Phi(\sigma), \Phi(\rho)), \quad \sigma, \rho \in \mathcal{T}$$

are contractions, and if they are injective, then they are isometries.

Clearly, if $\Phi : \mathcal{T} \rightarrow \mathcal{S}$ is a homomorphism and $\mathcal{K}$ is an ideal in $\mathcal{S}$ the collection of sets

$$\Phi^{-1}(\mathcal{K})(\sigma, \rho) := \{a \in \mathcal{T}(\sigma, \rho) : \Phi(a) \in \mathcal{K}(\Phi(\sigma), \Phi(\rho))\}$$

forms an ideal in $\mathcal{T}$, which we shall refer to as the *preimage* of $\mathcal{K}$. In particular, the preimage of the zero ideal will be called a *kernel* of Φ and denoted by $\ker \Phi$.

Proposition 2.10. *If $\mathcal{K}$ is an ideal in a C^* -precategory $\mathcal{T}$, then the precategory $\mathcal{T}/\mathcal{K}$ whose morphisms are given by the quotient spaces*

$$(\mathcal{T}/\mathcal{K})(\sigma, \rho) := \mathcal{T}(\sigma, \rho)/\mathcal{K}(\sigma, \rho)$$

is a C^ -precategory and the quotient maps $q_{\mathcal{T}} : \mathcal{T}(\sigma, \rho) \rightarrow (\mathcal{T}/\mathcal{K})(\sigma, \rho)$ give rise to the quotient homomorphism of C^* -precategories $q_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{K}$.*

Proof. Mimic the argument that shows the corresponding fact for C^* -algebras. $\square$

Definition 2.11. By a *representation of a C^* -precategory $\mathcal{T}$* in a C^* -algebra B we mean a homomorphism $\pi : \mathcal{T} \rightarrow B$, where B is considered as a C^* -precategory with a single object. Equivalently π may be treated as a collection $\{\pi_{\sigma,\rho}\}_{\sigma,\rho \in \mathcal{T}}$ of linear operators $\pi_{\sigma,\rho} : \mathcal{T}(\sigma,\rho) \rightarrow B$ such that

$$\pi_{\sigma,\rho}(a)^* = \pi_{\rho,\sigma}(a^*) \quad \text{and} \quad \pi_{\tau,\rho}(ba) = \pi_{\tau,\sigma}(b)\pi_{\sigma,\rho}(a),$$

for $a \in \mathcal{T}(\sigma,\rho)$, $b \in \mathcal{T}(\tau,\sigma)$. By a *representation of $\mathcal{T}$ in a Hilbert space H* we mean a representation of $\mathcal{T}$ in the C^* -algebra $L(H)$ of all bounded operators. We say that the representation is faithful if all the mappings $\{\pi_{\sigma,\rho}\}_{\sigma,\rho \in \mathcal{T}}$ are injective.

Remark 2.12. The above definition differs from [15, Def. 1.8]. However, as in [15, Prop. 1.14] (which can be easily refined to deal with C^* -precategories) one can show that every C^* -precategory $\mathcal{T}$ may be faithfully represented in a Hilbert space.

The following statement generalizes an elementary, but frequently used, fact that a representation of an ideal in a C^* -algebra extends to a representation of this C^* -algebra.

Proposition 2.13. *Suppose that $\mathcal{K}$ is an ideal in $\mathcal{T}$ and $\{\pi_{\sigma,\rho}\}_{\sigma,\rho \in \mathcal{T}}$ is a representation of $\mathcal{K}$ in a Hilbert space H . If we denote by P_ρ , $\rho \in \mathcal{T}$, the orthogonal projection onto the essential subspaces of $\pi_{\rho,\rho}$:*

$$P_\rho H = \pi_{\rho,\rho}(\mathcal{K}(\rho,\rho))H,$$

then essential subspaces of $\pi_{\sigma,\rho}$, $n \in \mathbb{N}$, are contained in $P_\rho H$, and there is a unique extension $\bar{\pi} = \{\bar{\pi}_{\sigma,\rho}\}_{\sigma,\rho \in \mathcal{T}}$ of π to a representation of a C^ -precategory $\mathcal{T}$ such that essential subspaces of $\bar{\pi}_{\sigma,\rho}$ are contained in $P_\rho H$, $\sigma, \rho \in \mathbb{N}$. This extension is determined by the relations*

$$(2.3) \quad \bar{\pi}_{\sigma,\rho}(a)\pi_{\rho,\rho}(b)h = \pi_{\sigma,\rho}(ab)h,$$

$a \in \mathcal{T}(\sigma,\rho)$, $b \in \mathcal{K}(\rho,\rho)$, $h \in H$.

Proof. By Lemma 2.5 we have $\mathcal{K}(\rho,\sigma) = \mathcal{K}(\rho,\rho)\mathcal{K}(\rho,\sigma)$ and since the essential subspace of $\pi_{\sigma,\rho}$ is $\pi_{\sigma,\rho}(\mathcal{K}(\sigma,\rho))^*H = \pi_{\rho,\sigma}(\mathcal{K}(\rho,\sigma))H$ it is contained in $P_\rho H$.

Since for $a \in \mathcal{T}(\sigma,\rho)$, $b \in \mathcal{K}(\rho,\rho)$, $h \in H$, and an approximate unit $\{\mu_\lambda\}$ for $\mathcal{K}(\rho,\rho)$ we have

$$\begin{aligned} \|\pi_{\sigma,\rho}(ab)h\| &= \lim_{\lambda} \|\pi_{\sigma,\rho}(a\mu_\lambda)\pi_{\rho,\rho}(b)h\| \leq \lim_{\lambda} \|\pi_{\sigma,\rho}(a\mu_\lambda)\| \|\pi_{\rho,\rho}(b)h\| \\ &\leq \|a\| \cdot \|\pi_{\rho,\rho}(b)h\|, \end{aligned}$$

formula (2.3) gives rise to the representation $\bar{\pi}_{\sigma,\rho}$ of $\mathcal{T}(\sigma,\rho)$ on $P_\rho H$. Defining $\bar{\pi}_{\sigma,\rho}$ to be zero on $(P_\rho H)^\perp$ one readily sees that $\bar{\pi} = \{\bar{\pi}_{\sigma,\rho}\}_{\sigma,\rho \in \mathcal{T}}$ is a representation of $\mathcal{T}$ in H . Obviously, every representation $\bar{\pi} = \{\bar{\pi}_{\sigma,\rho}\}_{\sigma,\rho \in \mathcal{T}}$ that extends π satisfies (2.3), which together with the requirement that $\bar{\pi}_{\sigma,\rho}|_{(P_\rho H)^\perp} \equiv 0$, for all $\rho, \sigma \in \mathcal{T}$, determines $\bar{\pi}$ uniquely. $\square$

Example 2.14 (C^* -category of Hilbert modules). Let $\{X_\rho\}_{\rho \in \mathcal{T}}$ be the family of right Hilbert modules over a C^* -algebra A , indexed by a collection of objects $\mathcal{T}$. Then $\mathcal{T}$ with morphisms being the adjointable maps between the Hilbert A -modules:

$$\mathcal{T}(\sigma,\rho) := \mathcal{L}(X_\rho, X_\sigma), \quad \rho, \sigma \in \mathcal{T},$$

becomes a C^* -category. The collection of spaces of “compact” operators

$$\mathcal{K}(\sigma, \rho) := \mathcal{K}(X_\rho, X_\sigma), \quad \rho, \sigma \in \mathcal{T},$$

gives rise to the ideal $\mathcal{K}$ of “compact operators” in $\mathcal{T}$. Moreover, using Proposition 1.4, we see that every ideal J in A naturally gives rise to two ideals in $\mathcal{T}$. One, denoted by $\mathcal{J}$, consists of all the adjointable maps with ranges in the spaces $X_\rho J$, $\rho \in \mathcal{T}$:

$$\mathcal{J}(\sigma, \rho) := \mathcal{L}_J(X_\rho, X_\sigma), \quad \rho, \sigma \in \mathcal{T}.$$

The other one is $\mathcal{K} \cap \mathcal{J}$, that is, $(\mathcal{K} \cap \mathcal{J})(\sigma, \rho) = \mathcal{K}(X_\rho, X_\sigma J)$, $\rho, \sigma \in \mathcal{T}$.

2.1. C^* -precategory $\mathcal{T}_X$ of a C^* -correspondence. Here we examine our model example of a C^* -precategory, which shall be equipped with an additional structure of a right tensor C^* -precategory in Example 3.2. Throughout this subsection we fix a C^* -correspondence X over a C^* -algebra A .

Definition 2.15. We define a C^* -precategory $\mathcal{T}_X$ of the C^* -correspondence X to be the C^* -precategory whose objects are natural numbers $\mathbb{N} = \{0, 1, \dots\}$, and whose spaces of morphisms are the following adjointable maps:

$$\mathcal{T}_X(n, m) := \begin{cases} \mathcal{K}(X^{\otimes m}, X^{\otimes n}), & \text{if } n = 0 \text{ or } m = 0, \\ \mathcal{L}(X^{\otimes m}, X^{\otimes n}), & \text{if } n, m \geq 1, \end{cases}$$

where $X^{\otimes n} := X \otimes \cdots \otimes X$ is the n -fold tensor product and $X^{\otimes 0} := A$ is the standard Hilbert A -module. Moreover, we shall use the Riesz-Fréchet theorem [30, Lem. 2.32] to assume the following identifications:

$$(2.4) \quad \mathcal{K}(A, X^{\otimes n}) = X^{\otimes n}, \quad \mathcal{K}(X^{\otimes n}, A) = \tilde{X}^{\otimes n}, \quad n \in \mathbb{N},$$

where $\tilde{X}$ denotes the dual C^* -correspondence to X , i.e. a left Hilbert A -module equipped with a right action such that there is an anti-linear isomorphism $\flat : X \rightarrow \tilde{X}$ that preserves the corresponding structures ($\tilde{X}^{\otimes n} := \widetilde{X^{\otimes n}}$). Thus we have

$$\mathcal{T}_X(0, 0) = A, \quad \mathcal{T}_X(n, 0) = X^{\otimes n}, \quad \mathcal{T}_X(0, m) = \tilde{X}^{\otimes m}.$$

Obviously, we may consider $\mathcal{T}_X$ as a sub- C^* -precategory in the C^* -category $\mathcal{T} = \{\mathcal{L}(X^{\otimes m}, X^{\otimes n})\}_{n, m \in \mathbb{N}}$, and if A is unital, then actually $\mathcal{T}_X = \mathcal{T}$. The reason why we deal with $\mathcal{T}_X$ rather than $\mathcal{T}$ is explained in Remark 3.3. Likewise in Example 2.14 we associate with an ideal J in A two ideals in $\mathcal{T}_X$.

Definition 2.16. We denote by $\mathcal{K}_X := \{\mathcal{K}(X^{\otimes m}, X^{\otimes n})\}_{n, m \in \mathbb{N}}$ the ideal in $\mathcal{T}_X$ consisting of “compact” operators, and for an ideal J in A we put

$$\begin{aligned} \mathcal{K}_X(J) &:= \{\mathcal{K}(X^{\otimes m}, X^{\otimes n} J)\}_{n, m \in \mathbb{N}}, \\ \mathcal{T}_X(J) &:= \{\mathcal{L}_J(X^{\otimes m}, X^{\otimes n})\}_{n, m \in \mathbb{N}} \cap \mathcal{T}_X. \end{aligned}$$

These ideals give useful estimates for arbitrary ideals in $\mathcal{T}_X$.

Proposition 2.17. Let $\mathcal{J}$ be an ideal in $\mathcal{T}_X$ and put $J := \mathcal{J}(0, 0)$. Then

$$\mathcal{K}_X(J) \subset \mathcal{J} \subset \mathcal{T}_X(J).$$

In particular, the relations

$$(2.5) \quad J = \mathcal{J}(0, 0), \quad \mathcal{J} = \mathcal{K}_X(J)$$

establish a one-to-one correspondence between ideals in A and $\mathcal{K}_X$.

Proof. By Theorem 2.6 an element $x \in X^{\otimes n} = \mathcal{T}_X(n, 0)$ belongs to $\mathcal{J}(n, 0)$ iff $x^*x = \langle x, x \rangle_A \in J$. Thus, by the Hewitt-Cohen Factorization Theorem, we have $\mathcal{J}(n, 0) = X^{\otimes n}J$. Since for $x \in \mathcal{J}(n, 0) = X^{\otimes n}J$ and $y \in \mathcal{T}_X(m, 0) = X^{\otimes m}$ we identify $xy^* \in \mathcal{J}(n, m)$ with the “one-dimensional” operator $\Theta_{x,y} \in \mathcal{K}(X^{\otimes m}, X^{\otimes n}J)$, one sees that $\mathcal{K}(X^{\otimes m}, X^{\otimes n}J) \subset \mathcal{J}(n, m)$ and consequently $\mathcal{K}_X(J) \subset \mathcal{J}$. To prove that $\mathcal{J} \subset \mathcal{T}_X(J)$ let $a \in \mathcal{J}(n, m) \subset \mathcal{L}(X^{\otimes m}, X^{\otimes n})$. Since for arbitrary $x \in \mathcal{T}(n, 0) = X^{\otimes n}$ and $y \in \mathcal{T}(m, 0) = X^{\otimes m}$ we have

$$x^*ay = \langle x, ay \rangle_A \in J,$$

it follows that a takes values in $X^{\otimes n}J$. □

Corollary 2.18. *For any ideal I in A relations (2.5) establish a one-to-one correspondence between ideals J in I and ideals $\mathcal{J}$ in $\mathcal{K}_X(I)$.*

Proof. Use the transitivity of the relation of being an ideal. □

Remark 2.19. For an ideal I in A , XI is naturally considered as a C^* -correspondence over I . Applying Proposition 2.17 to XI we get a one-to-one correspondence between ideals J in I and ideals $\mathcal{J}$ in $\mathcal{K}_{XI}$, given by the relations

$$J = \mathcal{J}(0, 0), \quad \mathcal{J} = \mathcal{K}_{XI}(J).$$

Thus by Corollary 2.18 the ideal structures of $\mathcal{K}_{XI}$ and $\mathcal{K}_X(I)$ are isomorphic, even though $\mathcal{K}_{XI} = \{\mathcal{K}(XI^{\otimes m}, XI^{\otimes n})\}_{n,m \in \mathbb{N}} \neq \mathcal{K}_X(I) = \{\mathcal{K}(X^{\otimes m}, X^{\otimes n}I)\}_{n,m \in \mathbb{N}}$ unless $\phi(I)X = XI$. This phenomenon will be pursued in Subsection 6.1.

We illustrate the introduced objects in the context of C^* -correspondences from Examples 1.12, 1.13.

Example 2.20 (C^* -precategory of a partial morphism). Let X_φ be the C^* -correspondence of a partial morphism $\varphi : A \rightarrow M(A_0)$. We shall call $\mathcal{T}_\varphi := \mathcal{T}_{X_\varphi}$ a C^* -precategory of φ . We have the following natural identifications for $\mathcal{K}_\varphi := \mathcal{K}_{X_\varphi} = \{\mathcal{K}(X_\varphi^{\otimes m}, X_\varphi^{\otimes n})\}_{n,m \in \mathbb{N}}$:

$$(2.6) \quad \mathcal{K}_\varphi(n, m) = \varphi \left(\underbrace{\varphi(\dots \varphi(A_0)A_0)\dots}_n \right) A_0 \underbrace{\varphi \left(\varphi(\dots \varphi(A_0)A_0)\dots \right)}_m A_0,$$

$n, m \in \mathbb{N}$, and in view of Proposition 2.17 every ideal $\mathcal{J}$ in $\mathcal{K}_\varphi$ is of the form

$$\mathcal{J}(n, m) = \varphi \left(\underbrace{\varphi(\dots \varphi(A_0)A_0)\dots}_n \right) A_0 \underbrace{J \varphi \left(\varphi(\dots \varphi(A_0)A_0)\dots \right)}_m A_0, \quad n, m \in \mathbb{N},$$

where $J = \mathcal{J}(0, 0)$ is an ideal in A . In particular, if φ is a partial morphism arising from a partial automorphism (θ, I, A_0) , then denoting by D_n the domain of θ^{-n} ; cf. [10], the above algebras are given by

$$\mathcal{K}_\varphi(n, m) = D_{\max\{m, n\}}, \quad \mathcal{J}(n, m) = J \cap D_{\max\{m, n\}}, \quad n, m \in \mathbb{N}.$$

If in turn φ arises from an endomorphism $\alpha : A \rightarrow A$, we then have

$$\mathcal{K}_\varphi(n, m) = \alpha^n(A)A\alpha^m(A), \quad \mathcal{J}(n, m) = \alpha^n(A)J\alpha^m(A), \quad n, m \in \mathbb{N}.$$

In this event we shall denote the C^* -precategory $\mathcal{T}_\varphi$ by $\mathcal{T}_\alpha$.

Example 2.21 (C^* -precategory of a directed graph). We set $\mathcal{T}_E := \mathcal{T}_{X_E}$, where X_E is a C^* -correspondence associated with a directed graph $E = (E^0, E^1, r, s)$, and we call it a C^* -precategory of E . For $n \geq 1$ we denote by E^n the set of all paths of length n , i.e. the set of sequences $(e_1, e_2, \dots, e_n)$, where $e_i \in E^1$ and $r(e_i) = s(e_{i+1})$, and we put $r^{(n)}(e_1, e_2, \dots, e_n) := r(e_n)$, $s^{(n)}(e_1, e_2, \dots, e_n) := s(e_1)$. The quadruple $E^{(n)} := (E^0, E^n, r^{(n)}, s^{(n)})$ is a directed graph and the C^* -correspondence $X_{E^{(n)}}$ may be naturally identified with $X_E^{\otimes n}$. In particular, $X_E^{\otimes n}$ is spanned by the point masses $\{\delta_\mu : \mu \in E^n\}$ and the ideal of “compact” operators $\mathcal{K}_E := \mathcal{K}_{X_E}$ is spanned by the “matrix units”

$$(2.7) \quad \Theta_{\delta_\mu, \delta_\nu} \quad \text{such that} \quad r^{(n)}(\mu) = r^{(m)}(\nu), \quad \mu \in E^n, \nu \in E^m.$$

Since every ideal in $A = C_0(E^0)$ is determined by its hull contained in E^0 , in view of Proposition 2.17, the equalities

$$(2.8) \quad \mathcal{J}(n, m) = \overline{\text{span}}\{\Theta_{\delta_\mu, \delta_\nu} : r^{(n)}(\mu) = r^{(m)}(\nu) \notin V, \mu \in E^n, \nu \in E^m\},$$

for $n, m \in \mathbb{N}$, establish a one-to-one correspondence between subsets V of E^0 and ideals $\mathcal{J}$ in $\mathcal{K}_E$.

3. RIGHT TENSOR C^* -PRECATEGORIES AND THEIR REPRESENTATIONS

The notion of a (strict) tensor C^* -category in the case that the underlying semigroup is $\mathbb{N}$ could be adapted to C^* -precategories as follows.

Definition 3.1. A C^* -precategory $\mathcal{T}$ with the set of objects $\mathbb{N} = \{0, 1, 2, \dots\}$ together with the designated endomorphism $\otimes 1 : \mathcal{T} \rightarrow \mathcal{T}$ sending n to $n + 1$:

$$\otimes 1 : \mathcal{T}(n, m) \rightarrow \mathcal{T}(n + 1, m + 1)$$

shall be called a *right tensor C^* -precategory*. We say that $\otimes 1$ is a *right tensoring* on $\mathcal{T}$ and write $a \otimes 1$ instead of $\otimes 1(a)$ for a morphism a in $\mathcal{T}$. If $\mathcal{T}$ is a C^* -category we shall refer to it simply as a *right tensor C^* -category*.

Iterating a right tensoring $\otimes 1$ on a C^* -precategory $\mathcal{T}$ one gets the semigroup $\{\otimes 1^k\}_{k \in \mathbb{N}}$ of endomorphisms $\otimes 1^k : \mathcal{T} \rightarrow \mathcal{T}$,

$$\otimes 1^k : \mathcal{T}(n, m) \rightarrow \mathcal{T}(n + k, m + k),$$

where by convention we put $\otimes 1^0 := id$. The model example of a right-tensor C^* -precategory is the C^* -precategory $\mathcal{T}_X$ with a right tensoring induced by the homomorphism $\phi : A \rightarrow \mathcal{L}(X)$ (cf. [14], [19, Def. 1.6, 1.7]), which we now describe in detail.

Example 3.2 (Right tensor C^* -precategory $\mathcal{T}_X$ of a C^* -correspondence X). Let $\mathcal{T}_X$ be as in Definition 2.15. If $n > 0, m > 0$, we have a natural tensoring

$$\mathcal{L}(X^{\otimes m}, X^{\otimes n}) \ni a \mapsto a \otimes 1 \in \mathcal{L}(X^{\otimes(m+1)}, X^{\otimes(n+1)}),$$

given by (1.2). For $a \in \mathcal{T}_X(0, 0) = A$ we put $a \otimes 1 := \phi(a) \in \mathcal{L}(X)$. In order to define the right tensoring on the spaces $\mathcal{T}_X(n, 0)$, $\mathcal{T}_X(0, n)$, for $n > 0$, we use the assumed identifications (2.4) and the mappings $L^{(n)} : X^{\otimes n} \mapsto \mathcal{L}(X, X^{\otimes(n+1)})$ and $D^{(n)} : \mathcal{K}(X^{\otimes n}, A) \mapsto \mathcal{L}(X^{\otimes(n+1)}, X)$ determined by the formulas

$$[L^{(n)}(x)](y) := x \otimes y, \quad x \in X^{\otimes n}, y \in X,$$

$$D^{(n)}(a)(y_1 \otimes y_2) := \phi(a(y_1))y_2, \quad y_1 \in X^{\otimes n}, y_2 \in X,$$

where $a \in \mathcal{K}(X^{\otimes n}, A)$. Alternatively, when $\mathcal{K}(X^{\otimes n}, A)$ is identified with $\tilde{X}^{\otimes n}$ we have $[D^{(n)}(b(x))](y_1 \otimes y_2) = \phi(\langle x, y_1 \rangle_A) y_2$. We put

$$a \otimes 1 := \begin{cases} L^{(n)}(a), & a \in \mathcal{T}_X(n, 0), \\ D^{(n)}(a), & a \in \mathcal{T}_X(0, n). \end{cases}$$

In this way $\mathcal{T}_X$ becomes a right tensor C^* -precategory with right tensoring “ $\otimes 1$ ”. We shall call it a *right tensor C^* -precategory of X* .

Remark 3.3. The $*$ -homomorphism $\varphi : A \rightarrow \mathcal{L}(X)$ need not extend to a $*$ -homomorphism $M(A) \rightarrow \mathcal{L}(X)$ unless it is nondegenerate (or A is unital). In other words, in general there is no obvious right tensoring on the C^* -category $\mathcal{T} := \{\mathcal{L}(X^{\otimes m}, X^{\otimes n})\}_{n,m \in \mathbb{N}}$ and thus the sub- C^* -precategory $\mathcal{T}_X \subset \mathcal{T}$ seems to be a more appropriate object to work with.

Applying the above construction to Examples 2.20, 2.21 we get respectively a *right tensor C^* -precategory $\mathcal{T}_\varphi$ of a partial morphism φ* and a *right tensor C^* -precategory $\mathcal{T}_E$ of a directed graph E* .

Example 3.4 (Right tensor C^* -precategory of a partial morphism). The C^* -precategory $\mathcal{T}_\varphi$ from Example 2.20 is a right tensor C^* -precategory, with right tensoring induced by φ . In particular, if φ arises from an endomorphism $\alpha : A \rightarrow A$ of a C^* -algebra A , the ideal $\mathcal{K}_\varphi = \{\alpha^n(A)A\alpha^m(A)\}_{n,m \in \mathbb{N}}$ is a right tensor C^* -precategory itself and the right tensoring on $\mathcal{K}_\varphi$ assumes the form

$$\alpha^n(A)A\alpha^m(A) \ni a \longrightarrow a \otimes 1 = \alpha(a) \in \alpha^{n+1}(A)A\alpha^{m+1}(A).$$

If additionally A is unital the above formula describes a right tensoring on $\mathcal{T}_\alpha = \mathcal{T}_\varphi$ as then we have $\mathcal{T}_\alpha = \mathcal{K}_\varphi = \{\alpha^n(1)A\alpha^m(1)\}_{n,m \in \mathbb{N}}$.

Example 3.5 (Right tensor C^* -precategory of a directed graph). The C^* -precategory $\mathcal{T}_E$ from Example 2.21 is a right tensor C^* -precategory, where the right tensoring is induced by “composition” of graphs (equivalently by multiplication of incidence matrices). In the event the set of edges E^1 is finite, the C^* -precategory $\mathcal{T}_E$ coincides with the ideal of “compact” operators $\mathcal{K}_E$ and the right tensoring is determined by the formula

$$\Theta_{\delta_\mu, \delta_\nu} \otimes 1 := \sum_{\{e \in E^1 : r^{(n)}(\mu) = s(e)\}} \Theta_{\delta_{\mu e}, \delta_{\nu e}}, \quad \mu \in E^n, \nu \in E^m,$$

where μe and νe are the paths obtained by concatenation.

We now turn to the investigation of representations of ideals in right tensor C^* -precategories which respect the right tensoring. As we shall see in Proposition 3.13 these representations generalize representations of C^* -correspondences.

Definition 3.6. Let $\mathcal{K}$ be an ideal in a right tensor C^* -precategory $\mathcal{T}$. We will say that a representation $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ is a *right tensor representation* if it satisfies

$$(3.1) \quad \pi_{n,m}(a)\pi_{m+k,l}(b) = \pi_{n+k,l}((a \otimes 1^k)b)$$

for all $a \in \mathcal{K}(n, m)$ and $b \in \mathcal{K}(m+k, l)$, $k, l, m, n \in \mathbb{N}$.

Remark 3.7. Since $\mathcal{K}$ is an ideal, the right-hand side of (3.1) makes sense. Furthermore, by taking adjoints one gets the symmetrized version of this equation:

$$\pi_{n,m+k}(a)\pi_{m,l}(b) = \pi_{n,l+k}(a(b \otimes 1^k)),$$

where $a \in \mathcal{K}(n, m+k)$ and $b \in \mathcal{K}(m, l)$, $k, l, m, n \in \mathbb{N}$.

We introduce the class of ideals in a right tensor C^* -precategory $\mathcal{T}$ whose right tensor representations extend naturally to $\mathcal{T}$.

Definition 3.8. We shall say that a right tensoring $\otimes 1$ acts *nondegenerately* on an ideal $\mathcal{K}$ in a right tensor C^* -precategory $\mathcal{T}$, or that $\mathcal{K}$ is $\otimes 1$ -*nondegenerate* if

$$(\mathcal{K}(n, m) \otimes 1)\mathcal{K}(n+1, m+1) = \mathcal{K}(n+1, m+1), \quad n, m \in \mathbb{N};$$

briefly $(\mathcal{K} \otimes 1)\mathcal{K} = \{\mathcal{K}(n, m)\}_{n, m > 0}$.

We use the term “nondegeneracy” because of the similarity to a condition defining the nondegeneracy of a homomorphism of C^* -algebras; cf. Example 1.12. By Lemma 1.9 ii) the ideal $\mathcal{K}_X := \{\mathcal{K}(X^{\otimes m}, X^{\otimes n})\}_{n, m \in \mathbb{N}}$ in $\mathcal{T}_X$ is $\otimes 1$ -nondegenerate, and actually any $\otimes 1$ -nondegenerate ideal $\mathcal{K}$ in a right tensor C^* -precategory $\mathcal{T}$ may be naturally embedded as a sub- C^* -precategory into $\mathcal{T}_X$ associated with the C^* -correspondence given by the relations

$$A =: \mathcal{K}(0, 0), \quad X := \mathcal{K}(1, 0), \quad \langle x, y \rangle_A := x^*y, \quad \phi(a) = a \otimes 1,$$

where $x, y \in X$ and $a \in A$, and then $\mathcal{K}_X \subset \mathcal{K} \subset \mathcal{T}_X$ (we leave the details to the reader, as we shall not use this fact in the sequel).

The following statement is a version of Proposition 2.13 adapted to right tensor C^* -precategories.

Proposition 3.9. Suppose that $\mathcal{K}$ is an ideal in $\mathcal{T}$ and $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ is a right tensor representation of $\mathcal{K}$ in a Hilbert space H . Then the projections P_m , $m \in \mathbb{N}$, onto the essential subspaces of $\pi_{m,m}$ are pairwise commuting and if $\mathcal{K}$ is $\otimes 1$ -nondegenerate, then we actually have

$$(3.2) \quad P_1 \geq P_2 \geq P_3 \geq \cdots.$$

Moreover, if (3.2) holds, then the extension $\bar{\pi} = \{\bar{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ of $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ described in Proposition 2.13 is a right tensor representation of $\mathcal{T}$, and

$$(3.3) \quad \bar{\pi}_{n,m}(a)P_{m+k} = \bar{\pi}_{n+k,m+k}(a \otimes 1^k), \quad P_{n+k}\bar{\pi}_{n,m}(a) = \bar{\pi}_{n+k,m+k}(a \otimes 1^k),$$

for $a \in \mathcal{T}(n, m)$, $k \in \mathbb{N}$.

Proof. Let us note that the equality $\pi_{m,m}(a)\pi_{m+k,m+k}(b) = \pi_{m+k,m+k}((a \otimes 1^k)b)$ implies $P_{m+k}\pi_{m,m}(a)P_{m+k} = \pi_{m,m}(a)P_{m+k}$ and this together with a similar equality for a^* yields $P_{m+k}\pi_{m,m}(a) = \pi_{m,m}(a)P_{m+k}$. Therefore $P_{m+k}P_m = P_mP_{m+k}$.

If $\mathcal{K}$ is $\otimes 1$ -nondegenerate, then $(\mathcal{K}(m, m) \otimes 1)\mathcal{K}(m+1, m+1) = \mathcal{K}(m+1, m+1)$ together with (3.1) imply that $P_mP_{m+1} = P_{m+1}$, that is, $P_m \geq P_{m+1}$. In general, if (3.2) holds, then any element h in $P_{m+k}H$ may be written in the form $\pi_{m,m}(b)\pi_{m+k,m+k}(c)h_0$, $b \in \mathcal{K}(m, m)$, $c \in \mathcal{K}(m+k, m+k)$ and then for $a \in \mathcal{T}(n, m)$

we have

$$\begin{aligned}\bar{\pi}_{n,m}(a)h &= \bar{\pi}_{n,m}(a)\pi_{m,m}(b)\pi_{m+k,m+k}(c)h_0 = \pi_{n,m}(ab)\pi_{m+k,m+k}(c)h_0 \\ &= \pi_{n+k,m+k}((ab) \otimes 1^k c)h_0 = \pi_{n+k,m+k}((a \otimes 1^k)(b \otimes 1^k)c)h_0 \\ &= \bar{\pi}_{n+k,m+k}(a \otimes 1^k)\pi_{m+k,m+k}((b \otimes 1^k)c)h_0 \\ &= \bar{\pi}_{n+k,m+k}(a \otimes 1^k)\pi_{m,m}(b)\pi_{m+k,m+k}(c)h_0 = \bar{\pi}_{n+k,m+k}(a \otimes 1^k)h.\end{aligned}$$

Hence $\bar{\pi}_{n,m}(a)P_{m+k} = \bar{\pi}_{n+k,m+k}(a \otimes 1^k)$ and by passing to adjoints one gets $P_{n+k}\bar{\pi}_{n,m}(a) = \bar{\pi}_{n+k,m+k}(a \otimes 1^k)$. In particular, for $a \in \mathcal{T}(n, m)$ and $b \in \mathcal{T}(m+k, l)$ we have

$$\begin{aligned}\bar{\pi}_{n,m}(a)\bar{\pi}_{m+k,l}(b) &= \bar{\pi}_{n,m}(a)P_{m+k}\bar{\pi}_{m+k,l}(b) = \bar{\pi}_{n+k,m+k}(a \otimes 1^k)\bar{\pi}_{m+k,l}(b) \\ &= \bar{\pi}_{n+k,l}((a \otimes 1^k)b),\end{aligned}$$

that is, $\bar{\pi} = \{\bar{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ is a right tensor representation. $\square$

Corollary 3.10. *If $\mathcal{K}$ is an ideal in $\mathcal{T}$ and $\mathcal{K}$ is $\otimes 1$ -nondegenerate, then we have a one-to-one correspondence between right tensor representations of $\mathcal{K}$ in the Hilbert space H and right tensor representations $\{\bar{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{T}$ in H satisfying*

$$(3.4) \quad \bar{\pi}_{m,m}(\mathcal{T}(m, m))H = \bar{\pi}_{m,m}(\mathcal{K}(m, m))H, \quad m \in \mathbb{N}.$$

Proof. This is clear by Proposition 3.9 and Lemma 2.5. $\square$

3.1. Right tensor representations and representations of C^* -correspondences. Let X be a C^* -correspondence over A . We shall investigate relationships between right tensor representations of the right tensor C^* -precategory $\mathcal{T}_X$, its ideal $\mathcal{K}_X = \{\mathcal{K}(X^{\otimes m}, X^{\otimes n})\}_{m,n \in \mathbb{N}}$, and representations of X .

Definition 3.11. A representation (π, t) of the C^* -correspondence X in a C^* -algebra B consists of a linear map $t : X \rightarrow B$ and a $*$ -homomorphism $\pi : A \rightarrow B$ such that

$$t(x \cdot a) = t(x)\pi(a), \quad t(x)^*t(y) = \pi(\langle x, y \rangle_A), \quad t(a \cdot x) = \pi(a)t(x),$$

for $x, y \in X$ and $a \in A$. If π is faithful (then automatically t is isometric; cf. [12], [28]), we say that the representation (π, t) is *faithful*. If $B = L(H)$ for a Hilbert space H , we say that (π, t) is a *representation of X in H* .

Remark 3.12. The above introduced notion is called a *Toeplitz representation* of X in [12], [11], and an *isometric covariant representation* of X in [27].

The first step is to show that representation (π, t) gives rise to a right tensor representation of $\mathcal{K}_X$ and this, in essence, follows from the results of [16], [11], [12], [29] where it was used in an implicit form.

Proposition 3.13. *If (π, t) is a representation of X in a C^* -algebra B , then there is a unique right tensor representation $\{\pi_{n,m}\}_{m,n \in \mathbb{N}}$ of the ideal $\mathcal{K}_X$ in the right tensor C^* -precategory $\mathcal{T}_X$, such that*

$$(3.5) \quad \pi_{0,0} = \pi, \quad \pi_{1,0} = t.$$

We shall denote this representation by $[\pi, t]$. Every right tensor representation of $\mathcal{K}_X$ is of the form $[\pi, t]$ and

$$\ker[\pi, t] = \mathcal{K}_X(\ker \pi);$$

cf. Definition 2.16.

Proof. Suppose that $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ is a right tensor representation of $\mathcal{K}_X$ such that (3.5) holds. Then in view of Definition 3.6, for $x_i \in X$, $i = 1, \dots, n$, we have

$$(3.6) \quad \pi_{n,0}(x_1 \otimes \cdots \otimes x_n) = t(x_1)t(x_2) \cdots t(x_n).$$

Hence $\pi_{n,0}$, $n > 0$, is uniquely determined by t . Furthermore, $\pi_{n,m}$ is uniquely determined by $\pi_{n,0}$ and $\pi_{m,0}$, since for $x \in X^{\otimes n}$ and $y \in X^{\otimes m}$ we have

$$(3.7) \quad \pi_{n,m}(\Theta_{x,y}) = \pi_{n,0}(x)(\pi_{m,0}(y))^*.$$

This proves the uniqueness of the representation $[\pi, t] = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$. For the existence of $[\pi, t]$ note that formula (3.6) gives rise to the linear map $\pi_{n,0} : X^{\otimes n} \rightarrow B$ such that $(\pi_{n,0}, \pi)$ is a representation of $X^{\otimes n}$; cf. [12, Prop. 1.8] or [11, Lem. 3.6]. Arguing as in [16, Lem. 2.2] or [29, Lem. 3.2], one sees that formula (3.7) defines a contraction $\pi_{n,m} : \mathcal{K}(X^{\otimes m}, X^{\otimes n}) \rightarrow B$ such that the family $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ forms a representation of $\mathcal{K}_X$ in B . To check that $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ is a right tensor representation let $x = \bigotimes_{i=1}^n x_i \in X^{\otimes n}$, $y = \bigotimes_{i=1}^m y_i \in X^{\otimes m}$, $x' = \bigotimes_{i=1}^{m+k} x'_i \in X^{\otimes m+k}$ and $y' = \bigotimes_{i=1}^l y'_i \in X^{\otimes l}$ (we adhere to the convention that the indexes indicate the order of the factors). One readily sees that

$$(\Theta_{x,y} \otimes 1^k) \Theta_{x',y'} = \Theta_{z,y'}$$

where $z = x \langle \bigotimes_{i=1}^m y_i, \bigotimes_{i=1}^m x'_i \rangle \otimes x'_{m+1} \otimes \cdots \otimes x'_{m+k} \in X^{\otimes m+k}$. On the other hand,

$$\begin{aligned} \pi_{n,m}(\Theta_{x,y}) \pi_{m+k,l}(\Theta_{x',y'}) &= \prod_{i=1}^n t(x_i) \left(\prod_{i=1}^m t(y_i) \right)^* \prod_{i=1}^{m+k} t(x'_i) \left(\prod_{i=1}^l t(y'_i) \right)^* \\ &= \prod_{i=1}^n t(x_i) \pi \left(\left\langle \bigotimes_{i=1}^m y_i, \bigotimes_{i=1}^m x'_i \right\rangle_A \right) \prod_{i=1}^k t(x'_{m+i}) \left(\prod_{i=1}^l t(y'_i) \right)^* \\ &= \pi_{n+k,m+k}(\Theta_{z,y'}), \end{aligned}$$

which means that condition (3.1) is satisfied by “rank one” operators and thereby by all morphisms of the ideal $\mathcal{K}_X$.

Clearly, if $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ is a right tensor representation of $\mathcal{K}_X$, then the pair (π, t) , where $\pi_{0,0} := \pi$, $t := \pi_{1,0}$, is a representation of X and we have $[\pi, t] = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$. To investigate the form of $\ker \pi_{n,m}$ note that for $x \in X^{\otimes n}$ we have

$$\|\pi_{n,0}(x)\|^2 = \|\pi_{n,0}(x)^* \pi_{n,0}(x)\| = \|\pi(\langle x, x \rangle_A)\| \leq \|\langle x, x \rangle_A\| = \|x\|^2.$$

It follows that $\pi_{n,0}(x) = 0 \iff \langle x, x \rangle_A \in \ker \pi$. Therefore, by the Hewitt-Cohen Factorization Theorem, $\ker \pi_{n,0} = X^{\otimes n} \ker \pi$. In view of formulas (3.6), (3.7), $a \in \mathcal{K}(X^{\otimes m}, X^{\otimes n})$ belongs to $\ker \pi_{n,m}$ iff the range of a is contained in $\ker \pi_{n,0} = X^{\otimes n} \ker \pi$. $\square$

Remark 3.14. The mapping $\pi_{1,1}$ in the above theorem is determined by the formula

$$\pi_{1,1}(\Theta_{x,y}) = t(x)t(y)^*, \quad x, y \in X.$$

In [27], [11], [12], [28], it is denoted by $\pi^{(1)}$ where it is used to introduce the notion of coisometricity; see Definition 3.21 below.

If (π, t) is a representation of X on a Hilbert space H , then within the notation of Proposition 3.13, the essential subspace of $\pi_{m,m}$, $m = 1, 2, \dots$, is

$$(3.8) \quad \pi_{m,m}(\mathcal{K}(X^{\otimes m}))H = \overline{\text{span}}\{t(x_1) \cdots t(x_m)h : x_1, \dots, x_m \in X, h \in H\}.$$

Hence the projections P_m onto these spaces form a decreasing sequence; cf. [12, Prop. 1.6], [11, 4.3]. This could also be derived from Proposition 3.9, since $\mathcal{K}_X$ is a $\otimes 1$ -nondegenerate ideal in $\mathcal{T}_X$. As a consequence of Propositions 3.13, 3.9 we get

Proposition 3.15. *Every representation (π, t) of a C^* -correspondence X in a Hilbert space H gives rise to a unique right tensor representation $\{\bar{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{T}_X$ such that*

$$(3.9) \quad \bar{\pi}_{0,0} = \pi, \quad \bar{\pi}_{1,0} = t,$$

and

$$(3.10) \quad \bar{\pi}_{m,m}(\mathcal{L}(X^{\otimes m}))H = \bar{\pi}_{m,m}(\mathcal{K}(X^{\otimes m}))H, \quad m = 1, 2, \dots$$

We shall denote this representation by $\overline{[\pi, t]}$. An arbitrary right tensor representation of $\mathcal{T}_X$ in a Hilbert space is of the form $\overline{[\pi, t]}$ if and only if it satisfies (3.10). Moreover, the kernel of $\overline{[\pi, t]}$ is determined by the kernel of π :

$$\ker \overline{[\pi, t]} = \mathcal{T}_X(\ker \pi);$$

see Definition 2.16.

Proof. The only thing requiring a comment is the form of $\ker \overline{[\pi, t]}$, but this follows from (2.3) and the equality $\ker[\pi, t] = \mathcal{K}_X(\ker \pi)$. $\square$

Example 3.16. Let $\alpha : A \rightarrow A$ be an endomorphism of a unital C^* -algebra A . A triple (π, U, B) consisting of a unital $*$ -homomorphism $\pi : A \rightarrow B$ and a partial isometry $U \in B$ such that

$$\pi(\alpha(a)) = U\pi(a)U^*, \quad \text{for all } a \in A, \quad \text{and} \quad U^*U \in \pi(A)',$$

is called a *covariant representation of α in B* ; see [31], [25]. It is known [11], [25] that covariant representations of α are in one-to-one correspondence with representations of the C^* -correspondence $X_\alpha = \alpha(1)A$. Thus in view of Proposition 3.13 every right tensor representation of the right tensor C^* -category $\mathcal{T}_\alpha = \{\alpha^n(1)A\alpha^m(1)\}_{n,m \in \mathbb{N}}$; cf. Examples 2.20, 3.4, assumes the form

$$\pi_{n,m}(a) := U^{*n}\pi(a)U^m, \quad a \in \alpha^n(1)A\alpha^m(1), \quad n, m \in \mathbb{N},$$

for a covariant representation (π, U, B) of α where $\pi = \pi_{0,0}$ and $U = \pi_{0,1}(\alpha(1))$. In particular, the C^* -algebra generated by the image of the C^* -category $\mathcal{T}_\alpha$ under $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ is the C^* -algebra

$$C^*(\pi(A), U) = \overline{\text{span}}\{U^{*n}aU^m : a \in A, m, n \in \mathbb{N}\}$$

generated by $\pi(A)$ and U .

In a more general context we may define a *covariant representation of a partial morphism $\varphi : A \rightarrow M(A_0)$* to be a triple (π, U, H) consisting of a Hilbert space H , a nondegenerate representation $\pi : A \rightarrow L(H)$ and a partial isometry $U \in L(H)$ such that

$$U\pi(a)U^* = \bar{\pi}(\varphi(a)), \quad \text{for all } a \in A, \quad \text{and} \quad U^*U \in \pi(A)',$$

where $\bar{\pi} : M(A_0) \rightarrow H$ is a representation given by the conditions

$$\bar{\pi}(b)|_{(\pi(A_0)H)^\perp} \equiv 0 \quad \text{and} \quad \bar{\pi}(b)\pi(a_0)h = \pi(ba_0)h, \quad b \in M(A_0), \quad a_0 \in A_0, \quad h \in H.$$

Then the final subspace of U is $\pi(A_0)H$. One checks that the triple (π, U, H) defined above the pair (π, t) , where

$$t(x) := U^*\pi(x), \quad x \in X_\varphi = A_0A,$$

is a representation of the C^* -correspondence X_φ . Conversely, if (π, t) is a representation of X_φ in a Hilbert space H , then for $a_0 \in A_0$ and $h \in H$ we have

$$\|t(a_0)h\|^2 = \langle t(a_0)h, t(a_0)h \rangle = \langle \pi(a_0^*a_0)h, h \rangle = \langle \pi(a_0)h, \pi(a_0)h \rangle = \|\pi(a_0)h\|^2.$$

Thus the relations

$$Ut(a_0)h = \pi(a_0)h, \quad a_0 \in A_0, \quad h \in H, \quad \text{and} \quad U|_{(t(A_0)H)^\perp} \equiv 0$$

define a partial isometry $U \in L(H)$. For $a_0 \in A_0$, $a \in A$ and $h \in H$ we have

$$\begin{aligned} U\pi(a)U^*\pi(a_0)h &= U\pi(a)t(a_0)h = Ut(\varphi(a)a_0)h = \pi(\varphi(a)a_0)h \\ &= \bar{\pi}(\varphi(a))\pi(a_0)h, \end{aligned}$$

that is, $U\pi(a)U^* = \bar{\pi}(\varphi(a))$. Furthermore

$$\begin{aligned} \pi(a)U^*Ut(a_0)h &= \pi(a)t(a_0)h = t(\varphi(a)a_0)h = U^*Ut(\varphi(a)a_0)h \\ &= U^*U\pi(a)t(a_0)h. \end{aligned}$$

Hence $U^*U \in \pi(A)'$ and the triple (π, U, H) is a covariant representation of φ . As a consequence, by Proposition 3.15, it follows that right tensor representations $\{\bar{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{T}_\varphi$ satisfying (3.10) are in one-to-one correspondence with covariant representations (π, U, H) of φ . In particular, the C^* -algebra generated by the image of the C^* -precategory $\mathcal{T}_\varphi$ under $\{\bar{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ is contained in the W^* -algebra $W^*(\pi(A), U)$ generated by $\pi(A)$ and U .

Example 3.17. Let $\mathcal{T}_E$ be the right tensor C^* -precategory of a directed graph $E = (E^0, E^1, r, s)$. By a *Toeplitz-Cuntz-Krieger E -family* is meant a collection of mutually orthogonal projections $\{p_v : v \in E^0\}$ together with a collection of partial isometries with orthogonal ranges $\{s_e : e \in E^1\}$ that satisfy the relations

$$s_e^*s_e = p_{r(e)}, \quad s_e s_e^* \leq p_{s(e)}, \quad e \in E^1.$$

Such families are in one-to-one correspondence with representations of X_E ; cf. [17], [28], [6]. Hence, in view of Proposition 3.13, every right tensor representation $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of the ideal $\mathcal{K}_E = \{\mathcal{K}(X_E^{\otimes m}, X_E^{\otimes n})\}_{n,m \in \mathbb{N}}$ in $\mathcal{T}_E$, see Example 2.21, is given by the formula

$$\pi_{n,m}(\Theta_{\delta_\mu, \delta_\nu}) = s_{e_1} \dots s_{e_m} s_{f_n}^* \dots s_{f_1}^*,$$

where $\mu = (e_1, \dots, e_m) \in E^m$, $\nu = (f_1, \dots, f_n) \in E^n$, for a Toeplitz-Cuntz-Krieger E -family $\{p_v : v \in E^0\}$, $\{s_e : e \in E^1\}$, where

$$p_v = \pi_{0,0}(\delta_v), \quad v \in E^0 \quad \text{and} \quad s_e = \pi_{1,0}(\delta_e), \quad e \in E^1.$$

In particular, the algebra generated by the image of the ideal $\mathcal{K}_E$ under the representation $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ is the C^* -algebra

$$C^*(\{p_v : v \in E^0\}, \{s_e : e \in E^1\})$$

generated by the corresponding Toeplitz-Cuntz-Krieger E -family.

3.2. Ideals of coisometricity for right tensor representations. We transfer the concept of coisometricity for representations of C^* -correspondences, introduced by P. Muhly and B. Solel in [27], onto the ground of ideals in right tensor C^* -precategories.

Definition 3.18. We put $J(\mathcal{K}) := (\otimes 1)^{-1}(\mathcal{K}) \cap \mathcal{K}$ for any ideal $\mathcal{K}$ in a right tensor C^* -precategory $\mathcal{T}$. In other words, $J(\mathcal{K}) = \{J(\mathcal{K})(n, m)\}_{n, m \in \mathbb{N}}$ is an ideal in $\mathcal{K}$, where

$$(3.11) \quad J(\mathcal{K})(n, m) := \{a \in \mathcal{K}(n, m) : a \otimes 1 \in \mathcal{K}(n+1, m+1)\}.$$

The ideal $J(\mathcal{K})$ plays the role of the ideal $J(X) = \phi^{-1}(\mathcal{K}(X))$ introduced by Pimsner in [29]. Roughly speaking, it consists of the elements for which the notion of coisometricity makes sense.

Proposition 3.19. *Let $\mathcal{K}$ be an ideal in a right tensor C^* -precategory $\mathcal{T}$. If $\pi = \{\pi_{n, m}\}_{n, m \in \mathbb{N}}$ is a right tensor representation of $\mathcal{K}$ in a C^* -algebra B , then the spaces*

$$\mathcal{J}(n, m) := \left\{a \in J(\mathcal{K})(n, m) : \pi_{n, m}(a) = \pi_{n+1, m+1}(a \otimes 1)\right\}$$

form an ideal in $J(\mathcal{K})$. Moreover, we have $\mathcal{J} \cap (\ker \otimes 1) \subset \ker \pi$ and $\ker \pi \cap J(\mathcal{K}) \subset \mathcal{J}$. In particular, faithfulness of π implies that $\mathcal{J} \subset (\ker \otimes 1)^\perp$.

Proof. The first part of the proposition follows directly from the definition of right tensor representation. To show the second part, note that for $a \in \mathcal{J}(n, m) \cap \ker(\otimes 1)(n, m)$ we have $\pi_{n, m}(a) = \pi_{n+1, m+1}(a \otimes 1) = \pi_{n+1, m+1}(0) = 0$, that is, $a \in (\ker \pi)(n, m)$. If, in turn, $a \in \ker \pi(n, m) \cap J(\mathcal{K})(n, m)$, then

$$\pi_{n+1, m+1}(a \otimes 1)(\pi_{n+1, m+1}(a \otimes 1))^* = \pi_{n, m}(a)(\pi_{n+1, m+1}(a \otimes 1))^* = 0,$$

that is, $\pi_{n+1, m+1}(a \otimes 1) = 0$ and therefore $a \in \mathcal{J}(n, m)$. $\square$

Definition 3.20. Let $\mathcal{K}$ be an ideal in a right tensor C^* -precategory $\mathcal{T}$ and let $\pi = \{\pi_{n, m}\}_{n, m \in \mathbb{N}}$ be a right tensor representation of $\mathcal{K}$. We shall say that π is *coisometric* on an ideal $\mathcal{J} \subset J(\mathcal{K})$ if

$$\pi_{n, m}(a) = \pi_{n+1, m+1}(a \otimes 1), \quad \text{for all } a \in \mathcal{J}(n, m), n, m \in \mathbb{N}.$$

The ideal $\mathcal{J}$ defined in Proposition 3.19 is the biggest ideal on which π is coisometric and we shall call it the *ideal of coisometricity* for π .

We devote the rest of this subsection to discuss and reveal the relationship between the above definition and the following one.

Definition 3.21 (cf. [11], [27]). A representation (π, t) of a C^* -correspondence X is called *coisometric* on an ideal J contained in $J(X) = \phi^{-1}(\mathcal{K}(X))$ if

$$\pi(a) = \pi_{1, 1}(a \otimes 1), \quad \text{for all } a \in J,$$

where $\pi_{1, 1}$ is defined in Remark 3.14. The set $\{a \in J(X) : \pi(a) = \pi_{1, 1}(a \otimes 1)\}$ is the biggest ideal on which (π, t) is coisometric and we shall call it an *ideal of coisometricity* for (π, t) .

As an immediate corollary of Propositions 2.17, 3.19 we get

Proposition 3.22. *If (π, t) is a representation of X in a C^* -algebra B and*

$$(3.12) \quad J = \{a \in J(X) : \pi_{1,1}(\phi(a)) = \pi(a)\},$$

then the ideal of coisometricity for the right tensor representation $[\pi, t]$ of the ideal $\mathcal{K}_X$, cf. Proposition 3.13, is $\mathcal{K}_X(J) = \{\mathcal{K}(X^{\otimes m}, X^{\otimes n}J)\}_{n,m \in \mathbb{N}}$.

Corollary 3.23. *Relations (3.5) together with equality $J = \mathcal{J}(0,0)$ establish a one-to-one correspondence between representations of a C^* -correspondence X coisometric on $J \subset J(X)$ and right tensor representations of $\mathcal{K}_X$ coisometric on $\mathcal{J} \subset J(\mathcal{K}_X)$.*

Corollary 3.24 (Prop. 2.21 [27]). *If $J \subset J(X)$ is an ideal of coisometricity for a representation (π, t) of X , then $J \cap \ker \phi \subset \ker \pi$. Hence if (π, t) is faithful, then $J \subset (\ker \phi)^\perp$.*

With the help of the following lemma we get a version of Proposition 3.22 for the extended representation $[\pi, t] = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ introduced in Proposition 3.15. Here P_m stands for the orthogonal projection onto the subspace (3.8).

Lemma 3.25. *Let (π, t) be a representation of X on a Hilbert space H , and let*

$$J = \{a \in A : \pi_{1,1}(\phi(a)) = \pi(a)\}.$$

Then $J_0 := J \cap J(X)$ is an ideal of coisometricity for (π, t) and

- i) *an element $a \in \mathcal{L}(X^{\otimes m}, X^{\otimes n})$ belongs to $\mathcal{L}_J(X^{\otimes m}, X^{\otimes n})$ if and only if $\pi_{n,m}(a)$ is supported on $P_{m+1}H$, equivalently, $\pi_{n,m}(a) = \pi_{n+1,m+1}(a \otimes 1)$.*
- ii) *If $a \otimes 1 \in \mathcal{K}(X^{\otimes m+1}, X^{\otimes n+1})$ and $\pi_{n,m}(a)$ is supported on $P_{m+1}H$, then a belongs to $\mathcal{L}_{J_0}(X^{\otimes m}, X^{\otimes n})$ and in the case $J_0 \subset (\ker \phi)^\perp$ (which is always the case when (π, t) is faithful) $a \in \mathcal{K}(X^{\otimes m}, X^{\otimes n}J_0)$.*

Proof. i) Let $a \in \mathcal{L}_J(X^{\otimes m}, X^{\otimes n})$. Then for any $x \in X^{\otimes m}$ there exist $y \in X^{\otimes n}$ and $b \in J$ such that $ax = yb$, and we have

$$\begin{aligned} \pi_{n,m}(a)\pi_{m,0}(x) &= \pi_{n,0}(ax) = \pi_{n,0}(yb) = \pi_{n,0}(y)\pi(b) = \pi_{n,0}(y)\pi_{1,1}(\phi(b)) \\ &= \pi_{n+1,1}(y \otimes 1 \cdot \phi(b)) = \pi_{n+1,1}(yb \otimes 1) = \pi_{n+1,1}(ax \otimes 1) \\ &= \pi_{n+1,m+1}(a \otimes 1)\pi_{m,0}(x). \end{aligned}$$

Thus $\pi_{n,m}(a) = \pi_{n+1,m+1}(a \otimes 1)$, which by (3.3) is equivalent to $\pi_{n,m}(a)$ being supported on $P_{m+1}H$.

Conversely, assume a is such that $\pi_{n,m}(a)$ is supported on $P_{m+1}H$ (equivalently $\pi_{n,m}(a) = \pi_{n+1,m+1}(a \otimes 1)$). Multiplying $\pi_{n,m}(a)$ by $\pi_{0,n}(b(x))$, $x \in X^{\otimes n}$, from the left and by $\pi_{m,0}(y)$, $y \in X^{\otimes m}$, from the right we get

$$\pi_{0,n}(b(x))\pi_{n,m}(a)\pi_{m,0}(y) = \pi(b(x) \cdot a \cdot y) = \pi(\langle x, ay \rangle_A).$$

Analogously for $\pi_{n+1,m+1}(a \otimes 1)$ we have

$$\begin{aligned} \pi_{0,n}(b(x))\pi_{n+1,m+1}(a \otimes 1)\pi_{m,0}(y) &= \pi_{1,1}(b(x) \otimes 1 \cdot (a \otimes 1) \cdot y \otimes 1) \\ &= \pi_{1,1}(\langle x, ay \rangle_A \otimes 1) = \pi_{1,1}(\phi(\langle x, ay \rangle_A)). \end{aligned}$$

This implies that $\pi(\langle x, ay \rangle_A) = \pi_{1,1}(\phi(\langle x, ay \rangle_A))$ and hence $\langle x, ay \rangle_A \in J$. By arbitrariness of x and y we conclude that $a \in \mathcal{L}_J(X^{\otimes m}, X^{\otimes n})$.

ii) If $a \otimes 1 \in \mathcal{K}(X^{\otimes m+1}, X^{\otimes n+1})$ and $\bar{\pi}_{n,m}(a)$ is supported on $P_{m+1}H$, then the argument from the proof of i) shows that $\pi(\langle x, ay \rangle_A) = \bar{\pi}_{1,1}(\phi(\langle x, ay \rangle_A))$, and similarly as in the proof of [11, Lem. 4.2 ii)], one sees that $\phi(\langle x, ay \rangle_A) \in \mathcal{K}(X)$. Hence we deduce that $\langle x, ay \rangle_A \in J \cap J(X)$ and as a consequence $a \in \mathcal{L}_{J_0}(X^{\otimes m}, X^{\otimes n})$. If additionally $J_0 \subset (\ker \phi)^\perp$, then $a \in \mathcal{K}(X^{\otimes m}, X^{\otimes n}J_0)$ by Lemma 1.9 iii). $\square$

Proposition 3.26. *If (π, t) is a representation of X on a Hilbert space H and*

$$J = \{a \in A : \bar{\pi}_{1,1}(\phi(a)) = \pi(a)\},$$

then the ideal of coisometricity for the right tensor representation $\overline{[\pi, t]}$ of $\mathcal{T}_X$ is $\mathcal{T}_X(J)$. In particular, if (π, t) is faithful, then J is contained in $(\ker \phi)^\perp$.

Proof. Apply Proposition 3.9 iii) and Lemma 3.25 i). $\square$

Example 3.27. Let $\mathcal{T}_\alpha = \{\alpha^n(1)A\alpha^m(1)\}_{n,m \in \mathbb{N}}$ be a right tensor C^* -category associated with an endomorphism $\alpha : A \rightarrow A$ of a unital C^* -algebra A ; cf. Examples 2.20, 3.4. If (π, U, B) is a covariant representation of α and $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ is the associated right tensor representation of $\mathcal{T}_\alpha$, see Example 3.16, then denoting by $\mathcal{J}$ the ideal of coisometricity for $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ we have

$$\mathcal{J}(n, m) = \alpha^n(1)J\alpha^m(1), \quad \text{where} \quad J = \{a \in A : U^*U\pi(a) = \pi(a)\};$$

cf. [25], [11, Ex. 1.6]. Thus in the terminology of [25] the triple (π, U, B) is called a *covariant representation* (π, U, B) *associated with the ideal* J , and we have a one-to-one correspondence between right tensor representations of $\mathcal{T}_\alpha$ with the ideal of coisometricity $\mathcal{J} = \{\alpha^n(1)J\alpha^m(1)\}_{n,m \in \mathbb{N}}$ and covariant representations of α associated with $J = \mathcal{J}(0, 0)$. Furthermore, if there exists a complete transfer operator $\mathcal{L}$ for α ; cf. Example 3.4, and $J = (\ker \alpha)^\perp$ (equivalently $\mathcal{J} = (\ker \otimes 1)^\perp$), then

$$U\pi(a)U^* = \pi(\alpha(a)), \quad U^*\pi(a)U = \pi(\mathcal{L}(a));$$

that is, (π, U, B) is a covariant representation in the sense of [4], [24], [3]; cf. [25]. In Example 3.16 we have defined the notion of a covariant representation (π, U, H) of a general partial morphism $\varphi : A \rightarrow M(A_0)$ in a Hilbert space H . Every such representation gives rise to the right tensor representation $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of the ideal $\mathcal{K}_\varphi = \{\mathcal{K}(X_\varphi^{\otimes m}, X_\varphi^{\otimes n})\}_{n,m \in \mathbb{N}}$ in $\mathcal{T}_\varphi$, which in turn extends to the right tensor representation $\{\bar{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{T}_\varphi$. The ideals of coisometricity $\mathcal{J}$ and $\mathcal{J}_0$ for $\{\bar{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ and $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ are established by the ideals

$$J = \{a \in A : U^*U\pi(a) = \pi(a)\} \quad \text{and} \quad J_0 = J \cap \varphi^{-1}(A_0),$$

respectively. One may see that, if φ arises from a partial automorphism; cf. Example 1.12, then (π, U, H) is a covariant representation generalizing those introduced in [10] iff $J_0 = (\ker \varphi)^\perp \cap \varphi^{-1}(A_0)$ (equivalently $\mathcal{J}_0 = (\ker \otimes 1)^\perp \cap J(\mathcal{K}_\varphi)$).

Example 3.28. As in Example 3.17, let $\mathcal{T}_E$ be a right tensor C^* -precategory of a directed graph $E = (E^0, E^1, r, s)$, $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ a right tensor representation of $\mathcal{K}_E = \{\mathcal{K}(X_E^{\otimes m}, X_E^{\otimes n})\}_{n,m \in \mathbb{N}}$ and $\{p_v : v \in E^0\}$, $\{s_e : e \in E^1\}$ the corresponding Toeplitz-Cuntz-Krieger E -family. Then the ideal of coisometricity $\mathcal{J}$ for $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ is established via (2.8) by a set of vertices $V \subset E^0$, where

$$p_v = \sum_{s(e)=v} s_e s_e^*, \quad \text{if and only if } v \in V.$$

Authors of [28] called an E -family satisfying $p_v = \sum_{s(e)=v} s_e s_e^*$, for $v \in V$, a *Cuntz-Krieger (E, V) -family*. Thus we have a one-to-one correspondence between Cuntz-Krieger (E, V) -families and right tensor representations of $\mathcal{K}_E$ coisometric on the ideal $\mathcal{J}$ corresponding to V . In particular,

$$\mathcal{J} = J(\mathcal{K}_E) \cap (\ker \otimes 1)^\perp \quad \text{iff} \quad V = \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\},$$

and if these equivalent conditions hold, the corresponding (E, V) -family is called a *Cuntz-Krieger E -family* [28], [5], [6].

4. C^* -ALGEBRA $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$

In this section we present a construction of the title object of the paper, the C^* -algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. Before approaching this task, we formulate a universal definition of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ and briefly discuss its relations with relative Cuntz-Pimsner algebras, various crossed products and graph C^* -algebras.

4.1. Definition and examples of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. The main goal of this section is to construct an object, the existence of which is ensured by the following.

Theorem 4.1 (Characterization of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ as a universal algebra). *For every ideal $\mathcal{K}$ in a right tensor C^* -precategory $\mathcal{T}$ and every ideal $\mathcal{J}$ in $J(\mathcal{K})$ there exists a pair $(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}), \iota)$ consisting of a C^* -algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ and a right tensor representation $\iota = \{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ coisometric on $\mathcal{J}$, such that*

- 1) *the C^* -algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ is generated by the image of $\iota = \{\iota_{n,m}\}_{n,m \in \mathbb{N}}$, i.e.*

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) = C^*(\{\iota_{n,m}(\mathcal{K}(n, m))\}_{n,m \in \mathbb{N}}).$$

- 2) *Every right tensor representation $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ coisometric on $\mathcal{J}$ integrates to a representation Ψ_π of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ given by*

$$\Psi_\pi(\iota_{n,m}(a)) = \pi_{n,m}(a), \quad a \in \mathcal{K}(n, m), \quad n, m \in \mathbb{N}.$$

Moreover, the pair $(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}), \iota)$ is uniquely determined in the sense that if (C, κ) is any other pair consisting of a C^* -algebra C and a right tensor representation $\kappa = \{\kappa_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ in C coisometric on $\mathcal{J}$ that satisfies conditions 1) and 2), then the mappings $\iota_{n,m}(a) \mapsto \kappa_{n,m}(a)$, $a \in \mathcal{K}(n, m)$, give rise to the canonical isomorphism $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) \cong C$.

Proof. The first part of the assertion follows from the well-known fact on universal C^* -algebras; see [7]. The second part is straightforward: by the universality of $(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}), \iota)$ and (C, κ) the mappings $\iota_{n,m}(a) \mapsto \kappa_{n,m}(a)$ and $\kappa_{n,m}(a) \mapsto \iota_{n,m}(a)$, $a \in \mathcal{K}(n, m)$, extend to mutually inverse homomorphisms. $\square$

Definition 4.2. For any ideals $\mathcal{K}, \mathcal{J}$ in a right tensor C^* -precategory $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$, the object $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ described in the statement of Theorem 4.1 will be called a *C^* -algebra of the ideal $\mathcal{K}$ in the right tensor C^* -precategory $\mathcal{T}$ relative to the ideal $\mathcal{J}$* .

Remark 4.3. Uniqueness of $(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}), \iota)$ implies that there is a canonical circle action $\gamma : S^1 \rightarrow \text{Aut}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}))$ by automorphisms of the C^* -algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ given by

$$\gamma_z(\iota_{n,m}(a)) = z^{n-m} \iota_{n,m}(a), \quad a \in \mathcal{K}(n, m), \quad z \in S^1.$$

We shall refer to γ as a *gauge action* on $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$.

Remark 4.4. With analogy to C^* -algebras arising from C^* -correspondences, see Example 4.5 below, it is reasonable to call the algebras

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \{0\}), \quad \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}), \quad \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}(\mathcal{K}))$$

a *Toeplitz algebra* of $\mathcal{K}$, (*Katsura's*) C^* -*algebra of the ideal* $\mathcal{K}$, and a *Cuntz-Pimsner algebra* of $\mathcal{K}$, respectively.

In the event that $\mathcal{T}$ is a right tensor C^* -category, the Doplicher-Roberts algebra $\mathcal{DR}(\mathcal{T})$ together with the family of natural maps $\iota_{n+k,n} : \mathcal{T}(n+k, n) \rightarrow \varinjlim \mathcal{T}(r+k, r)$ (cf. the definition of $\mathcal{DR}(\mathcal{T})$ on page 1844) is naturally identified with the algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{T}, \mathcal{T})$. We postpone a relevant discussion until Section 5 and now present a survey of examples of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ associated with C^* -correspondences.

Example 4.5 (Relative Cuntz-Pimsner algebras). Let $\mathcal{T} = \mathcal{T}_X$ be the right tensor C^* -precategory of a C^* -correspondence X and let $\mathcal{J}$ be an ideal in $J(\mathcal{K}_X)$. We know that $\mathcal{J} = \mathcal{K}_X(J) = \{\mathcal{K}(X^{\otimes m}, X^{\otimes n}J)\}_{n,m \in \mathbb{N}}$, where $J = \mathcal{J}(0, 0) \subset J(X)$; cf. Proposition 2.17. In view of Corollary 3.23 and [11, Prop. 1.3] the algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X, \mathcal{J})$ coincides with the *relative Cuntz-Pimsner algebra* of Muhly and Solel [27]:

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X, \mathcal{J}) = \mathcal{O}(J, X).$$

In particular, $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X, \{0\}) = \mathcal{O}(\{0\}, X)$ is called a *Toeplitz algebra* of X . In the case that ϕ is injective, $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X, J(\mathcal{K}_X)) = \mathcal{O}(J(X), X)$ is the algebra originally introduced by Pimsner, and in the general case, the algebra

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}_X, J(\mathcal{K}_X) \cap (\ker \otimes 1)^{\perp}) = \mathcal{O}(J(X) \cap (\ker \phi)^{\perp}, X)$$

is the C^* -*algebra* $\mathcal{O}_X$ of the correspondence X investigated and popularized by T. Katsura [17], [19], [20].

As subclasses of the above algebras we get

Example 4.6 (Relative graph algebras). Let $\mathcal{T} = \mathcal{T}_E$ be a right tensor C^* -precategory of a directed graph E and let $\mathcal{K}$ be the ideal in $\mathcal{T}$ spanned by the matrix units (2.7). Then any ideal $\mathcal{J}$ in $\mathcal{K}$ is determined by a set of vertices $V \subset E^0$; see (2.8). In view of Example 3.28, $V \subset \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}$ iff $\mathcal{J} \subset J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}$ and if this is the case we have

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) = C^*(E, V),$$

where $C^*(E, V)$ is the *relative graph algebra* introduced in [28, Def. 3.5]. In particular, $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \{0\}) = C^*(E, \emptyset)$ is the *Toeplitz algebra of* E as defined in [12], and

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}) = C^*(E, \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}) = C^*(E),$$

where $C^*(E)$ is the *graph algebra*; cf. [28], [5], [6]. We extend [28, Def. 3.5] and define $C^*(E, V) := \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ for any $V \subset \{v \in E^0 : |s^{-1}(v)| < \infty\}$.

Example 4.7 (Crossed products by partial morphisms). Let $\mathcal{T} = \mathcal{T}_{\varphi}$ be the right tensor C^* -precategory associated with a partial morphism φ . For the ideal $\mathcal{K}$ described by (2.6) we have

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}) = A \rtimes_{\varphi} \mathbb{N},$$

where $A \rtimes_{\varphi} \mathbb{N}$ is the *crossed product by partial morphism* defined in [17]. In particular, if φ arises from a partial automorphism, this algebra coincides with Exel's crossed product by partial automorphism [10].

Example 4.8 (Partial isometric crossed products by endomorphisms). Let $\mathcal{T}$ be the right tensor C^* -category $\mathcal{T}_\alpha = \{\alpha^n(1)A\alpha^m(1)\}_{n,m \in \mathbb{N}}$ associated with an endomorphism $\alpha : A \rightarrow A$ of a unital C^* -algebra A ; cf. Example 3.4. Then every ideal $\mathcal{J}$ in $\mathcal{T}$ is of the form $\mathcal{J} = \{\alpha^n(1)J\alpha^m(1)\}_{n,m \in \mathbb{N}}$ for an ideal J in A , and

$$\mathcal{O}_{\mathcal{T}}(\mathcal{T}, \mathcal{J}) = C^*(A, \alpha; J),$$

where $C^*(A, \alpha; J)$ is the *relative crossed product* considered in [25]; cf. Example 3.27. Thereby, see [25], for $J = A$, $J = (\ker \alpha)^\perp$ and $J = \{0\}$ one arrives at a partial-isometric crossed product from [2], [21] and [31], respectively. Furthermore, if there exists a complete transfer operator $\mathcal{L}$ for α , then

$$C^*(A, \alpha; (\ker \alpha)^\perp) = \mathcal{O}_{\mathcal{T}}(\mathcal{T}, (\ker \otimes 1)^\perp)$$

is the crossed product investigated in [24], [3]. A distinctive property of C^* -algebras of this type is that they are generated by a homomorphic image of A and a partial isometry.

The last two examples indicate that it is reasonable to adopt the following definition which embraces all the crossed products mentioned above.

Definition 4.9. Let $\varphi : A \rightarrow M(A_0)$ be a partial morphism and let J be an ideal in $\varphi^{-1}(A_0)$. We define the *relative crossed product* $C^*(\varphi; J)$ of φ relative to J to be the C^* -algebra $\mathcal{O}(J, X_\varphi)$.

4.2. Construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. Let $\mathcal{K}$ and $\mathcal{J} \subset \mathcal{K}$ be ideals in a right tensor C^* -precategory $\mathcal{T}$. A construction of the C^* -algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ will consist of two steps. First, we describe a matrix calculus that yields a purely algebraic structure, a $*$ -algebra $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. Secondly, we use $\mathcal{J}$ to define a seminorm $\|\cdot\|_{\mathcal{J}}$ on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ such that completing the quotient space $\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$ yields $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$.

4.2.1. An algebraic framework for $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. We let $\mathcal{M}_{\mathcal{T}}$ be the set of all infinite matrices $\{a_{n,m}\}_{n,m \in \mathbb{N}}$, where $a_{n,m} \in \mathcal{T}(n, m)$ are such that for some $N \in \mathbb{N}$, $|n - m| \geq N$ implies that $a_{n,m} = 0$ (at most a finite number of diagonals in $\{a_{n,m}\}_{n,m \in \mathbb{N}}$ are nonzero). Then the standard matrix operations (including matrix multiplication) yield a well-defined $*$ -algebra structure on $\mathcal{M}_{\mathcal{T}}$. We denote by $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ the subset of $\mathcal{M}_{\mathcal{T}}$ consisting of matrices $\{a_{n,m}\}_{n,m \in \mathbb{N}}$ such that

$$a_{n,m} \in \mathcal{K}(n, m), \quad n, m \in \mathbb{N},$$

and there is at most a finite number of elements $a_{n,m}$ which are nonzero. We let the addition, multiplication by scalars, and involution on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ be inherited from $\mathcal{M}_{\mathcal{T}}$; that is, for $a = \{a_{n,m}\}_{n,m \in \mathbb{N}}$ and $b = \{b_{n,m}\}_{n,m \in \mathbb{N}}$ we put

$$(4.1) \quad (a + b)_{n,m} := a_{n,m} + b_{n,m},$$

$$(4.2) \quad (\lambda a)_{n,m} := \lambda a_{n,m},$$

$$(4.3) \quad (a^*)_{n,m} := a_{m,n}^*.$$

A multiplication “ $\star$ ” on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ is more involved. We set

$$(4.4) \quad a \star b := a \cdot \sum_{k=0}^{\infty} \Lambda^k(b) + \sum_{k=1}^{\infty} \Lambda^k(a) \cdot b,$$

where “ $\cdot$ ” stands for the matrix multiplication in $\mathcal{M}_{\mathcal{T}}$ and $\Lambda : \mathcal{M}_{\mathcal{T}}(\mathcal{K}) \rightarrow \mathcal{M}_{\mathcal{T}}$ is defined to act as follows:

$$(4.5) \quad \Lambda(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{0,0} \otimes 1 & a_{0,1} \otimes 1 & a_{0,2} \otimes 1 & \cdots \\ 0 & a_{1,0} \otimes 1 & a_{1,1} \otimes 1 & a_{1,2} \otimes 1 & \cdots \\ 0 & a_{2,0} \otimes 1 & a_{2,1} \otimes 1 & a_{2,2} \otimes 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix};$$

that is, $\Lambda(a)_{n,m} = a_{n-1,m-1} \otimes 1$, for $n, m \geq 1$, and $\Lambda(a)_{n,m} = 0$ otherwise. Note that even though the entries of $\Lambda(a)$ need not belong to $\mathcal{K}$, entries of $a \star b$ are in $\mathcal{K}$ (because $\mathcal{K}$ is an ideal in $\mathcal{T}$) and hence $a \star b$ is well defined.

We denote by $\iota_{n,m}^{\mathcal{M}} : \mathcal{T}(n, m) \rightarrow \mathcal{M}_{\mathcal{T}}$, $n, m \in \mathbb{N}$, the natural embeddings; that is, $\iota_{n,m}^{\mathcal{M}}(a)$ is a matrix $\{a_{i,j}\}_{i,j \in \mathbb{N}}$ satisfying $a_{i,j} = \delta_{i,n} \delta_{j,m} a$, where $\delta_{k,l}$ is the Kronecker symbol.

Proposition 4.10. *The set $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ with operations (4.1), (4.2), (4.3), (4.4) becomes an algebra with involution. Moreover,*

- i) *the family $\{\iota_{n,m}^{\mathcal{M}}\}_{n,m \in \mathbb{N}}$ forms a right tensor representation of $\mathcal{K}$ in $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$;*
- ii) *we have a one-to-one correspondence, established by the relations*

$$\Psi(\iota_{n,m}^{\mathcal{M}}(a_{n,m})) = \pi_{n,m}(a_{n,m}),$$

between right tensor representations $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ and representations Ψ of $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. We shall denote by Ψ_{π} the representation $\Psi_{\pi}(\{a_{n,m}\}_{n,m \in \mathbb{N}}) = \sum_{n,m=0}^{\infty} \pi_{n,m}(a_{n,m})$ corresponding to $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$.

Proof. The mapping $\sum_{k=0}^{\infty} \Lambda^k : \mathcal{M}_{\mathcal{T}}(\mathcal{K}) \rightarrow \mathcal{M}_{\mathcal{T}}$ embeds $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ equipped with operations (4.1), (4.2), (4.3), (4.4) into the $\ast$ -algebra $\mathcal{M}_{\mathcal{T}}$ (equipped with matrix multiplication). Indeed, a moment of thought shows that $\sum_{k=0}^{\infty} \Lambda^k : \mathcal{M}_{\mathcal{T}}(\mathcal{K}) \rightarrow \mathcal{M}_{\mathcal{T}}$ is an injective linear map that preserves involution, and

$$\begin{aligned} \sum_{k=0}^{\infty} \Lambda^k(a \star b) &= \sum_{k=0}^{\infty} \Lambda^k \left(a \cdot \sum_{l=0}^{\infty} \Lambda^l(b) + \sum_{l=1}^{\infty} \Lambda^k(a) \cdot b \right) \\ &= \sum_{k=0}^{\infty} \left(\Lambda^k(a) \sum_{l=0}^{\infty} \Lambda^{k+l}(b) + \sum_{l=1}^{\infty} \Lambda^{k+l}(a) \cdot \Lambda^k(b) \right) \\ &= \sum_{k,l=0}^{\infty} \Lambda^k(a) \Lambda^l(b) = \left(\sum_{k=0}^{\infty} \Lambda^k(a) \right) \left(\sum_{l=0}^{\infty} \Lambda^l(b) \right). \end{aligned}$$

Hence $\sum_{k=0}^{\infty} \Lambda^k : \mathcal{M}_{\mathcal{T}}(\mathcal{K}) \rightarrow \mathcal{M}_{\mathcal{T}}$ is an injective $\ast$ -homomorphism, and therefore $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ is a $\ast$ -algebra.

i) Checking that $\{\iota_{n,m}^{\mathcal{M}}\}_{n,m \in \mathbb{N}}$ is a right tensor representation is straightforward. In particular, the relation $\iota_{n,m}^{\mathcal{M}}(a_{n,m}) \star \iota_{m+k,l}^{\mathcal{M}}(b_{m+k,l}) = \iota_{n+k,l}^{\mathcal{M}}((a_{n,m} \otimes 1^k) b_{m+k,l})$ follows directly from (4.4).

ii) If $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ is a right tensor representation of $\mathcal{K}$, then $\Psi(\iota_{n,m}^{\mathcal{M}}(a_{n,m})) = \pi_{n,m}(a_{n,m})$ determines uniquely a linear map Φ . Since

$$\Psi(\iota_{n,m}^{\mathcal{M}}(a_{n,m})^{\ast}) = \Psi(\iota_{m,n}^{\mathcal{M}}(a_{n,m}^{\ast})) = \pi_{m,n}(a_{n,m}^{\ast}) = \pi_{n,m}(a_{n,m})^{\ast}$$

and

$$\begin{aligned}\Psi(\iota_{n,m}^{\mathcal{M}}(a_{n,m}) \star \iota_{m+k,l}^{\mathcal{M}}(b_{m+k,l})) &= \Psi(\iota_{n+k,l}^{\mathcal{M}}((a_{n,m} \otimes 1^k)b_{m+k,l})) \\ &= \pi_{n+k,l}((a_{n,m} \otimes 1^k)b_{m+k,l}) \\ &= \pi_{n,m}(a_{n,m})\pi_{m+k,l}(b_{m+k,l}) \\ &= \Psi(\iota_{n,m}^{\mathcal{M}}(a_{n,m}))\Psi(\iota_{m+k,l}^{\mathcal{M}}(b_{m+k,l})),\end{aligned}$$

it follows that Ψ is a representation of $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. Conversely, if Ψ is a representation of $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$, then $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ as a “composition” of Ψ and the right tensor representation $\{\iota_{n,m}^{\mathcal{M}}\}_{n,m \in \mathbb{N}}$ is a right tensor representation. $\square$

4.2.2. Grading and seminorms $\|\cdot\|_{\mathcal{J}}$ in $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. We define a seminorm $\|\cdot\|_{\mathcal{J}}$ in $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ using a natural grading of $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. Also we take an opportunity to show that this seminorm could be defined within the inductive limit frames similar to the Doplicher-Roberts approach. For that purpose, for $r \in \mathbb{N}$, $k \in \mathbb{Z}$, $r+k \geq 0$, we put

$$\mathcal{M}(r+k, r) := \{\{a_{n,m}\}_{m,n \in \mathbb{N}} \in \mathcal{M}_{\mathcal{T}}(\mathcal{K}) : a_{n,m} \neq 0 \implies n-m=k, m \leq r\}.$$

Hence an element $a \in \mathcal{M}_{\mathcal{T}}(\mathcal{K})$ is in $\mathcal{M}(r+k, r)$ iff it is of the form

$$\left\{ \begin{array}{c} k \\ r+1 \end{array} \right\} \left(\begin{array}{ccc} & & 0 \\ & a_{k,0} & \ddots \\ & \ddots & \ddots \\ 0 & & a_{r+k,r} \end{array} \right), \text{ if } k \geq 0, \text{ or } \left(\begin{array}{ccc} & \overbrace{a_{0,-k}}^{r+k+1} & 0 \\ & \ddots & \ddots \\ 0 & & a_{r+k,r} \end{array} \right), \text{ if } k < 0.$$

For every $k \in \mathbb{Z}$ we get an increasing family $\{\mathcal{M}(r+k, r)\}_{r \in \mathbb{N}, r+k \geq 0}$ of subspaces of $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$, and the spaces

$$\mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K}) := \bigcup_{\substack{r \in \mathbb{N}, \\ r+k \geq 0}} \mathcal{M}(r+k, r), \quad k \in \mathbb{Z},$$

define a $\mathbb{Z}$ -grading on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. In particular

$$\mathcal{M}_{\mathcal{T}}(\mathcal{K}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K}).$$

The first step in defining $\|\cdot\|_{\mathcal{J}}$ is to equip the family $\{\mathcal{M}(n, m)\}_{n,m \in \mathbb{N}}$ with the structure of a right tensor C^* -precategory. We recall that $q_{\mathcal{J}}$ denotes the quotient map from $\mathcal{T}$ onto $\mathcal{T}/\mathcal{J}$; cf. Proposition 2.10.

Theorem 4.11 (Construction of the right tensor C^* -precategory $\mathcal{K}_{\mathcal{J}}$). *For every $r \in \mathbb{N}$, $k \in \mathbb{Z}$, $r+k \geq 0$, the formula*

$$\|a\|_{r+k,r}^{\mathcal{J}} := \max \left\{ \max_{s=0,\dots,r-1} \left\{ \|q_{\mathcal{J}} \left(\sum_{\substack{i=0, \\ i+k \geq 0}}^s a_{i+k,i} \otimes 1^{s-i} \right)\| \right\}, \left\| \sum_{\substack{i=0, \\ i+k \geq 0}}^r (a_{i+k,i} \otimes 1^{r-i}) \right\| \right\}$$

defines a seminorm on $\mathcal{M}(r+k, r)$, such that the family of completions of quotient spaces

$$\mathcal{K}_{\mathcal{J}} := \{\overline{\mathcal{M}(n, m)} / \|\cdot\|_{n,m}^{\mathcal{J}}\}_{n,m \in \mathbb{N}}$$

form a right tensor C^* -precategory with a right tensoring $\otimes_{\mathcal{J}} 1$ induced by the inclusions $\mathcal{M}(n, m) \subset \mathcal{M}(n+1, m+1)$, $m, n \in \mathbb{N}$. Moreover the right tensoring $\otimes_{\mathcal{J}} 1$ is faithful iff $\mathcal{J} \subset (\ker \otimes 1)^\perp$.

Proof. By (4.3) we have $\mathcal{M}(n, m)^* = \mathcal{M}(m, n)$, $n, m \in \mathbb{N}$. To show that

$$\mathcal{M}(r+k, r) \star \mathcal{M}(r, r+l) \subset \mathcal{M}(r+k, r+l), \quad r \in \mathbb{N}, k, l \in \mathbb{Z},$$

note that $\mathcal{M}(r+k, r) = \text{span}\{\iota_{i+k, i}^{\mathcal{M}}(\mathcal{K}(i+k, i)) : i = 0, \dots, r, i+k \geq 0\}$ and for $a \in \mathcal{K}(i+k, i)$ and $b \in \mathcal{K}(j, j+l)$, $i, j = 0, \dots, r$, the element

$$(4.6) \quad \iota_{i+k, i}^{\mathcal{M}}(a) \star \iota_{j, j+l}^{\mathcal{M}}(b) = \begin{cases} \iota_{j+k, j+l}^{\mathcal{M}}((a \otimes 1^{j-i})b), & \text{if } i \leq j, \\ \iota_{i+k, i+l}^{\mathcal{M}}(a(b \otimes 1^{i-j})), & \text{if } j < i, \end{cases}$$

is in $\mathcal{M}(r+k, r+l)$. Thus $\{\mathcal{M}(n, m)\}_{n, m \in \mathbb{N}}$ is a $*$ -precategory.

It is clear that functions $\|\cdot\|_{r+k, r}^{\mathcal{J}}$ are seminorms on $\mathcal{M}(r+k, r)$. To see that they are submultiplicative let $a = \{a_{n, m}\}_{n, m \in \mathbb{N}} \in \mathcal{M}(r+k, r)$, $b = \{b_{n, m}\}_{n, m \in \mathbb{N}} \in \mathcal{M}(r, r+l)$ and $s = 0, 1, \dots, r+l$. We adopt to the convention that $a_{n, m} \otimes 1^p = 0$ whenever m, n , or p is less than 0. Using (4.6) one gets

$$(a \star b)_{p+k-l, p} = \sum_{i=0}^{p-l} (a_{i+k, i} \otimes 1^{p-l-i}) b_{p-l, p} + \sum_{j=0}^{p-l-1} (a_{p+k-l, p-l} (b_{j, j+l}) \otimes 1^{p-l-j}),$$

for $p = 0, \dots, s$, and after rearranging the addends one obtains

$$(4.7) \quad \sum_{p=0}^s (a \star b)_{p+k-l, p} \otimes 1^{s-p} = \left(\sum_{p=0}^s a_{p+k-l, p-l} \otimes 1^{s-p} \right) \left(\sum_{p=0}^s b_{p-l, p} \otimes 1^{s-p} \right).$$

Accordingly, since $q_{\mathcal{J}}$ preserves the operations in $\mathcal{T}$ and by submultiplicativity of the norm in $\mathcal{T}$, we see that $\|q_{\mathcal{J}}(\sum_{p=0}^s (a \star b)_{p+k-l, p} \otimes 1^{s-p})\|$ is not larger than

$$\|q_{\mathcal{J}}\left(\sum_{p=0}^s a_{p+k-l, p-l} \otimes 1^{s-p}\right)\| \cdot \|q_{\mathcal{J}}\left(\sum_{p=0}^s b_{p-l, p} \otimes 1^{s-p}\right)\|$$

and

$$\left\| \sum_{p=0}^{r+l} (a \star b)_{p+k-l, p} \otimes 1^{s-p} \right\| \leq \left\| \sum_{p=-l}^r a_{p+k, p} \otimes 1^{r-p} \right\| \cdot \left\| \sum_{p=0}^{r+l} b_{p-l, p} \otimes 1^{r+l-p} \right\|.$$

Therefore

$$\|a \star b\|_{r+k, r+l}^{\mathcal{J}} \leq \|a\|_{r+k, r}^{\mathcal{J}} \|b\|_{r, r+l}^{\mathcal{J}}.$$

Putting in (4.7), $k = l$, $b = a^*$ and using the C^* -equality in $\mathcal{T}$ one sees that the above inequalities become equalities, and therefore

$$\|a \star a^*\|_{r+k, r+k}^{\mathcal{J}} = (\|a\|_{r+k, r}^{\mathcal{J}})^2.$$

Accordingly, $\mathcal{K}_{\mathcal{J}} = \{\overline{\mathcal{M}(n, m)} / \|\cdot\|_{n, m}^{\mathcal{J}}\}_{n, m \in \mathbb{N}}$ is a right tensor C^* -precategory.

To see that the inclusion $\mathcal{M}(r+k, r) \subset \mathcal{M}(r+k+1, r+1)$ factors through to the mapping

$$\mathcal{M}(r+k, r) / \|\cdot\|_{r+k, r}^{\mathcal{J}} \xrightarrow{\otimes_{\mathcal{J}} 1} \mathcal{M}(r+k+1, r+1) / \|\cdot\|_{r+k+1, r+1}^{\mathcal{J}},$$

let $a \in \mathcal{M}(r+k, r)$ be such that $\|a\|_{r+k, r}^{\mathcal{J}} = 0$. Then $\sum_{i=0}^s a_{i+k, i} \otimes 1^{s-i} \in \mathcal{J}(s+k, s)$, for all $s = 0, 1, \dots, r$, where for $s = r$ we actually have $\sum_{i=0}^r a_{i+k, i} \otimes 1^{r-i} = 0$. Moreover,

$$\sum_{i=0}^{r+1} a_{i+k, i} \otimes 1^{r+1-i} = \sum_{i=0}^r a_{i+k, i} \otimes 1^{r+1-i} = \left(\sum_{i=0}^r a_{i+k, i} \otimes 1^{r-i} \right) \otimes 1 = 0.$$

Thus $\|a\|_{r+1, r+1}^{\mathcal{J}} = 0$ and $\otimes_{\mathcal{J}} 1$ is a well-defined right tensoring on $\mathcal{K}_{\mathcal{J}}$.

Assume now that $\mathcal{J} \subset (\ker \otimes 1)^{\perp}$ and let $a \in \mathcal{M}(r+k, r)$ be such that $\|a\|_{r+k+1, r+1}^{\mathcal{J}} = 0$. Then $\sum_{i=0}^s a_{i+k, i} \otimes 1^{s-i} \in \mathcal{J}(s+k, s)$, for each $s = 0, 1, \dots, r$, and

$$\sum_{i=0}^{r+1} a_{i+k, i} \otimes 1^{r+1-i} = \left(\sum_{i=0}^r a_{i+k, i} \otimes 1^{r-i} \right) \otimes 1 = 0.$$

In particular, $\sum_{i=0}^r a_{i+k, i} \otimes 1^{r-i} \in \ker(\otimes 1)(r+k, r)$ and $\sum_{i=0}^r a_{i+k, i} \otimes 1^{r-i} \in \mathcal{J}(r+k, r)$.

Since $\mathcal{J} \cap \ker(\otimes 1) = \{0\}$, we get $\sum_{\substack{i=0, \\ i+k \geq 0}}^r a_{i+k, i} \otimes 1^{r-i} = 0$ and thereby $\|a\|_{r+k, r}^{\mathcal{J}} = 0$.

This proves the injectivity of $\otimes_{\mathcal{J}} 1$.

Conversely, if $\mathcal{J} \not\subset (\ker \otimes 1)^{\perp}$, then there exists $a \in (\mathcal{J} \cap \ker \otimes 1)(n, m)$ such that $a \neq 0$. Then $\|\iota_{n, m}^{\mathcal{M}}(a)\|_{n, m}^{\mathcal{J}} \neq 0$ and $\|\iota_{n+1, m+1}^{\mathcal{M}}(a)\|_{n+1, m+1}^{\mathcal{J}} = 0$, that is, $\otimes_{\mathcal{J}} 1$ is not injective. $\square$

The foregoing statement allows us to apply the ‘‘Doplicher-Roberts method’’ of constructing C^* -algebras from right tensor C^* -categories presented in [13]. Namely, for $k \in \mathbb{Z}$, we let $\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{J}, \mathcal{K}) := \varinjlim_{\gamma} \mathcal{K}_{\mathcal{J}}(r+k, r)$ be the Banach space inductive limit of the inductive sequence

$$\mathcal{K}_{\mathcal{J}}(r+k, r) \xrightarrow{\otimes_{\mathcal{J}} 1} \mathcal{K}_{\mathcal{J}}(r+k+1, r+1) \xrightarrow{\otimes_{\mathcal{J}} 1} \mathcal{K}_{\mathcal{J}}(r+k+2, r+2) \xrightarrow{\otimes_{\mathcal{J}} 1} \dots$$

defined for $r \in \mathbb{N}$, $k+r \geq 0$. The algebraic direct sum $\bigoplus_{k \in \mathbb{Z}} \mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K}, \mathcal{J})$ has a natural structure of a $\mathbb{Z}$ -graded $*$ -algebra and we may define $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ to be the C^* -algebra obtained by completing $\bigoplus_{k \in \mathbb{Z}} \mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K}, \mathcal{J})$ in the unique C^* -norm for which the automorphic action defined by the grading is isometric; see [13, Thm. 4.2].

We may also consider $\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K}, \mathcal{J})$ as a completion of the quotient space $\mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K}) / \|\cdot\|_{\mathcal{J}}$, where $\|\cdot\|_{\mathcal{J}}$ is a seminorm described below, and thus obtain another construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$.

Theorem 4.12 (Construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$). *Let $\mathcal{J}$ be an ideal in $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$. The formula*

$$\|a\|_{\mathcal{J}} = \sum_{k \in \mathbb{Z}} \lim_{r \rightarrow \infty} \max \left\{ \max_{s=0, \dots, r-1} \left\{ \|q_{\mathcal{J}} \left(\sum_{\substack{i=0, \\ i+k \geq 0}}^s a_{i+k, i} \otimes 1^{s-i} \right)\| \right\}, \right. \\ \left. \left\| \sum_{\substack{i=0, \\ i+k \geq 0}}^r (a_{i+k, i} \otimes 1^{r-i}) \right\| \right\}$$

defines a submultiplicative $$ -seminorm on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ such that for the enveloping C^* -algebra of the quotient $*$ -algebra $\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$ we have*

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) = \overline{\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}}$$

in the sense that the right tensor representation described in Proposition 4.10 i) composed with the quotient map yields a right tensor representation $\iota = \{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ in $\overline{\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}}$ which is coisometric on $\mathcal{J}$ and obeys the conditions of Theorem 4.1.

Proof. One immediately sees that the norm of the image of an element $a \in \mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K})$ in the inductive limit space $\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K}, \mathcal{J})$ coincides with the value $\|a\|_{\mathcal{J}}$, that is, $\|a\|_{\mathcal{J}} = \lim_{r \rightarrow \infty} \|a\|_{r+k, r}^{\mathcal{J}}$. In particular, $\|\cdot\|_{\mathcal{J}}$ is a submultiplicative $*$ -seminorm on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ and the enveloping C^* -norm on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$ satisfies the conditions of [13, Thm. 4.2]. Since the quotient map $\mathcal{M}_{\mathcal{T}}(\mathcal{K}) \mapsto \mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$ is a $*$ -homomorphism and $\{\iota_{n,m}^{\mathcal{M}}\}_{n,m \in \mathbb{N}}$ is a right tensor representation, their composition is a right tensor representation $\iota = \{\iota_{n,m}\}_{n,m \in \mathbb{N}}$. To see that ι is coisometric on $\mathcal{J}$ take $a \in J(\mathcal{K})(n, m)$, $n, m \in \mathbb{N}$, and notice that

$$\begin{aligned} \iota_{n,m}(a) = \iota_{n+1, m+1}(a \otimes 1) &\iff \|\iota_{n,m}^{\mathcal{M}}(a) - \iota_{n+1, m+1}^{\mathcal{M}}(a \otimes 1)\|_{\mathcal{J}} = 0 \\ &\iff a \in \mathcal{J}(n, m). \end{aligned}$$

Plainly, the image of the representation $\{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ generates the C^* -algebra $\overline{\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}}$. Let us prove universality of $\{\iota_{n,m}\}_{n,m \in \mathbb{N}}$. To this end let $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ be a right tensor representation of $\mathcal{K}$ coisometric on $\mathcal{J}$. We show that the representation Ψ_{π} described in Proposition 4.10 factors through to the representation of $\overline{\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}}$ such that

$$\Psi_{\pi}(\iota_{n,m}(a)) = \pi_{n,m}(a), \quad a \in \mathcal{K}(n, m).$$

By the form of $\|\cdot\|_{\mathcal{J}}$ (it is the sum of seminorms on spectral subspaces) it is enough to show that Ψ_{π} factors through to a representation of the subspace $\mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$, $k \in \mathbb{Z}$. For that purpose let $a \in \mathcal{M}_{\mathcal{T}}^{(k)}(\mathcal{K})$ be such that $\|a\|_{\mathcal{J}} = 0$. Then

$$\sum_{\substack{i=0, \\ i+k \geq 0}}^s a_{i+k, i} \otimes 1^{s-i} \in \mathcal{J}(s, s), \quad s \in \mathbb{N}, \quad \lim_{r \rightarrow \infty} \left\| \sum_{\substack{i=0, \\ i+k \geq 0}}^r a_{i+k, i} \otimes 1^{r-i} \right\| = 0.$$

Thus, for sufficiently large N , we have

$$\begin{aligned}\|\Psi_\pi(a)\| &= \left\| \sum_{\substack{i=0, \\ i+k \geq 0}}^N \pi_{i+k,i}(a_{i+k,i}) \right\| = \|\pi_{N+k,N}(\sum_{\substack{i=0, \\ i+k \geq 0}}^N a_{i+k,i} \otimes 1^{N-i})\| \\ &\leq \left\| \sum_{\substack{i=0, \\ i+k \geq 0}}^N a_{i+k,i} \otimes 1^{N-i} \right\| \longrightarrow 0, \quad \text{as } N \longrightarrow \infty.\end{aligned}$$

□

Remark 4.13. In the process of construction of $\mathcal{O}_\mathcal{T}(\mathcal{K}, \mathcal{J})$, until the above statement, we did not require that $\mathcal{J} \subset J(\mathcal{K})$. Actually for any ideal $\mathcal{J}$ in $\mathcal{T}$, the procedure described in Theorem 4.12 defines the C^* -algebra $\mathcal{O}_\mathcal{T}(\mathcal{K}, \mathcal{J}) = \overline{\mathcal{M}_\mathcal{T}(\mathcal{K})/\|\cdot\|_\mathcal{J}}$. However, without the condition $\mathcal{J} \subset J(\mathcal{K})$ the relationship between representations of $\mathcal{K}$ and $\mathcal{O}_\mathcal{T}(\mathcal{K}, \mathcal{J})$ breaks down.

4.3. Immediate corollaries of the construction of $\mathcal{O}_\mathcal{T}(\mathcal{K}, \mathcal{J})$.

Theorem 4.14 (Norm of elements in spectral subspaces). *If we let $(\mathcal{O}_\mathcal{T}(\mathcal{K}, \mathcal{J}), \iota)$ be the universal pair as in Theorem 4.1 then for every $k \in \mathbb{Z}$, the elements of the form*

$$(4.8) \quad a = \sum_{\substack{j=0, \\ j+k \geq 0}}^{r_0} \iota_{j+k,j}(a_{j+k,j}), \quad r_0 \in \mathbb{N},$$

constitute a dense subspace of the k -th spectral subspace of $\mathcal{O}_\mathcal{T}(\mathcal{K}, \mathcal{J})$, and the norm $\|a\|$ of a is given by the following limit:

$$\lim_{r \rightarrow \infty} \max \left\{ \max_{s=0, \dots, r-1} \left\{ \|q_\mathcal{J} \left(\sum_{\substack{i=0, \\ i+k \geq 0}}^s a_{i+k,i} \otimes 1^{s-i} \right)\| \right\}, \left\| \sum_{\substack{i=0, \\ i+k \geq 0}}^r (a_{i+k,i} \otimes 1^{r-i}) \right\| \right\}.$$

If $\mathcal{J} \subset (\ker \otimes 1)^\perp$ the above formula for $\|a\|$ reduces to the following one:

$$\max \left\{ \max_{s=0, \dots, r_0-1} \left\{ \|q_\mathcal{J} \left(\sum_{\substack{i=0, \\ i+k \geq 0}}^s a_{i+k,i} \otimes 1^{s-i} \right)\| \right\}, \left\| \sum_{\substack{i=0, \\ i+k \geq 0}}^{r_0} (a_{i+k,i} \otimes 1^{r_0-i}) \right\| \right\}.$$

Proof. We have already noticed that the norm of an element $a \in \mathcal{M}_\mathcal{T}^{(k)}(\mathcal{K})$ in the inductive limit space $\mathcal{O}_\mathcal{T}^{(k)}(\mathcal{K}, \mathcal{J})$ coincides with the value $\|a\|_\mathcal{J}$; cf. Theorem 4.12. This proves the first part of the assertion since the spaces $\mathcal{O}_\mathcal{T}^{(k)}(\mathcal{K}, \mathcal{J})$, $k \in \mathbb{Z}$, embed isometrically into $\mathcal{O}_\mathcal{T}(\mathcal{K}, \mathcal{J})$. The last part of the statement follows from the second part of Theorem 4.11. □

Corollary 4.15 (Kernel of universal representation). *The universal representation $\iota = \{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_\mathcal{T}(\mathcal{K}, \mathcal{J})$ is faithful if and only if $\mathcal{J} \subset (\ker \otimes 1)^\perp$, and in general we have*

$$a \in \ker \iota_{n,m} \iff \lim_{k \rightarrow \infty} \|a \otimes 1^k\| = 0 \quad \text{and} \quad a \otimes 1^k \in \mathcal{J} \quad \text{for all } k \in \mathbb{N}.$$

Corollary 4.16. *The universal representation of a C^* -correspondence X in the C^* -algebra $\mathcal{O}(J, X)$ is faithful if and only if $J \subset (\ker \phi)^\perp$.*

The ideal described in Corollary 4.15, the kernel of $\iota = \{\iota_{n,m}\}_{n,m \in \mathbb{N}}$, together with condition (4.9) presented below, plays an essential role in the investigation of faithful representations arising from right tensor representations; cf. Theorem 7.3. In the context of C^* -correspondences, condition (4.9) leads to the notion of a T -pair introduced in [20]; see Subsection 7.1.

Proposition 4.17. *The ideal $\mathcal{J}$ is an ideal of coisometricity for the universal representation of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. Moreover, we have*

$$(4.9) \quad \mathcal{J}(n, n) = \iota_{n,n}^{-1} \left(\sum_{j=1}^k \iota_{n+j, n+j}(\mathcal{K}(n+j, n+j)) \right), \quad \text{for all } n \in \mathbb{N} \text{ and } k > 0.$$

Proof. The form of the ideal of coisometricity for $\{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ follows from the argument in the proof of Theorem 4.12. In particular, if $k > 0$, then $\mathcal{J}(n, n) \subset \iota_{n,n}^{-1} \left(\sum_{j=1}^k \iota_{n+j, n+j}(\mathcal{K}(n+j, n+j)) \right)$ and for the converse inclusion note that

$$\iota_{n,n}(a) = \sum_{j=1}^k \iota_{n+j, n+j}(b_j) \implies \|\iota_{n,n}^{\mathcal{M}}(a) - \sum_{j=1}^k \iota_{n+j, n+j}^{\mathcal{M}}(b_j)\|_{\mathcal{J}} = 0 \implies a \in \mathcal{J}(n, n).$$

□

Corollary 4.18. *Every ideal $\mathcal{J}$ in $J(\mathcal{K})$ is an ideal of coisometricity for a certain right tensor representation of $\mathcal{K}$.*

Corollary 4.19. *Let X be a C^* -correspondence. Every ideal J in $J(X)$ is an ideal of coisometricity for a certain representation (π, t) of X .*

Example 4.20. Let $\varphi : A \rightarrow M(A_0)$ be a partial morphism. In view of Example 3.27, for every ideal J in $\varphi^{-1}(A_0)$ there exists a covariant representation (π, U, H) such that

$$J = \{a \in A : U^*U\pi(a) = \pi(a)\};$$

see the analogous results of [25].

Example 4.21. For any graph $E = (E^0, E^1, r, s)$ and every set of vertices $V \subset \{v \in E^0 : |s^{-1}(v)| < \infty\}$ there exists a Cuntz-Krieger (E, V) -family, which is not an (E, V') -family for any V' bigger than V ; cf. Example 3.28.

5. $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$, $\mathcal{DR}(\mathcal{T})$ AND ALGEBRAS WITH CIRCLE ACTIONS

The *Doplicher-Roberts algebra* $\mathcal{DR}(\mathcal{T})$ of a right tensor C^* -category $\mathcal{T}$ is defined to be the completion of the algebraic direct sum $\bigoplus_{k \in \mathbb{Z}} \mathcal{DR}^{(k)}(\mathcal{T})$, where

$$\mathcal{DR}^{(k)}(\mathcal{T}) := \varinjlim \mathcal{T}(r+k, r),$$

in the unique C^* -norm for which the automorphic action defined by the grading is isometric [13, p. 179]. It is evident that this construction could be successfully applied to C^* -precategories (or even to ideals in C^* -precategories). Thus we slightly

extend existing nomenclature and, for any right tensor C^* -precategory $\mathcal{T}$, call the C^* -algebra $\mathcal{DR}(\mathcal{T})$ defined above a *Doplicher-Roberts algebra of the right tensor C^* -precategory $\mathcal{T}$* .

Proposition 5.1 (Relationship between $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ and $\mathcal{DR}(\mathcal{T})$). *Every algebra $\mathcal{DR}(\mathcal{T})$ is an algebra of the type $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. Namely, we have a natural isomorphism*

$$\mathcal{DR}(\mathcal{T}) \cong \mathcal{O}_{\mathcal{T}}(\mathcal{T}, \mathcal{T}).$$

Conversely, for all ideals $\mathcal{K}, \mathcal{J}$ in $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$ the mappings

$$\mathcal{K}(n, m) \ni a \longmapsto \iota_{n, m}^{\mathcal{M}}(a) \in \mathcal{K}_{\mathcal{J}},$$

where $\{\iota_{n, m}^{\mathcal{M}}\}_{n, m \in \mathbb{N}}$ is the representation from Proposition 4.10 i), induce an isomorphism

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) \cong \mathcal{DR}(\mathcal{K}_{\mathcal{J}}).$$

Proof. This is an immediate consequence of Theorem 4.11 and the definition of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ presented below Theorem 4.11. $\square$

Corollary 5.2. *A C^* -algebra is a Doplicher-Roberts algebra associated with a C^* -category (as defined in [13]) if and only if it is an algebra of type $\mathcal{O}_{\mathcal{T}}(\mathcal{T}, \mathcal{J})$ for a right tensor C^* -category $\mathcal{T}$.*

Proof. If $\mathcal{T}$ is a C^* -category, then the right tensor C^* -precategory $\mathcal{T}_{\mathcal{J}}$ defined in Theorem 4.11 (where $\mathcal{K} = \mathcal{T}$) is a C^* -category. $\square$

It follows that the classes of C^* -algebras of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ and type $\mathcal{DR}(\mathcal{T})$ coincide (in fact they are both equal to the class of all C^* -algebras). Obviously what distinguishes these algebras is an additional structure, particularly the associated gauge action. We show that an arbitrary circle action may be viewed as a gauge action induced by our construction.

Let $\gamma : S^1 \rightarrow \text{Aut } B$ be a circle action on a C^* -algebra B and let $\{B_n\}_{n \in \mathbb{Z}}$ be the family of *spectral subspaces*:

$$B_n = \{b \in B : \gamma_z(b) = z^n b \text{ for } z \in S^1\}.$$

We set $\mathcal{T} = \{\mathcal{T}(n, m)\}_{n, m \in \mathbb{N}}$, where $\mathcal{T}(r + k, r) = B_k$ and $\mathcal{T}(r, r + k) = B_{-k}$ for all $r, k \in \mathbb{N}$. Then $\mathcal{T}$ with operations inherited from B is a C^* -precategory, matricially presented in the following way:

$$(5.1) \quad \mathcal{T} = \begin{pmatrix} B_0 & B_{-1} & B_{-2} & \cdots \\ B_1 & B_0 & B_{-1} & \cdots \\ B_2 & B_1 & B_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We equip $\mathcal{T}$ with a right tensoring $\otimes 1$ which simply slides the elements along the diagonals in (5.1), that is, $\mathcal{T}(n, m) \ni a \xrightarrow{\otimes 1} a \in \mathcal{T}(n + 1, m + 1)$. In this way $\mathcal{T}$ becomes a right tensor C^* -precategory and we have a natural gauge-invariant isomorphism

$$B \cong \mathcal{O}_{\mathcal{T}}(\mathcal{T}, \mathcal{T}) = \mathcal{DR}(\mathcal{T}).$$

Therefore we get

Theorem 5.3. *Every C^* -algebra with a circle action is gauge invariantly isomorphic to an algebra of type $\mathcal{DR}(\mathcal{T})$ (and all the more to an algebra of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$).*

The above statement shows that taking into account gauge actions the class of relative Cuntz-Pimsner algebras $\mathcal{O}(J, X)$ is strictly smaller than the class of algebras of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{T})$. Indeed, we have

Theorem 5.4 (Thm. 3.1 [1]). *A C^* -algebra with a circle action γ is gauge invariantly isomorphic to a relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$ if and only if the action γ is semi-saturated.*

The right tensor C^* -precategory (5.1) has an advantage that the universal representation embeds the spaces $\mathcal{T}(k, 0)$, $k \in \mathbb{N}$, into $\mathcal{O}_{\mathcal{T}}(\mathcal{T}, \mathcal{T})$ as spectral subspaces. We now present necessary and sufficient conditions assuring this in the general case. We start with a statement that illustrates the forthcoming definition.

Proposition 5.5. *Let $\mathcal{K}$ be an ideal in a right tensor C^* -precategory $\mathcal{T}$. The following conditions are equivalent.*

- i) *There is a system of mappings $\mathcal{L} : \{\mathcal{K}(n, m)\}_{n,m=1}^{\infty} \rightarrow \mathcal{K}$, where $\mathcal{L} : \mathcal{K}(n+1, m+1) \rightarrow \mathcal{K}(n, m)$ satisfies the properties*

$$\mathcal{L}(a) \otimes 1 = a \quad (\text{right-inversion of } \otimes 1),$$

$$\mathcal{L}(a(b \otimes 1)) = \mathcal{L}(a)b \quad (\text{transfer operator property}),$$

for all $a \in \mathcal{K}(n+1, m+1)$, and $b \in \mathcal{K}(m, l)$ $n, m, l \in \mathbb{N}$ (we do not explicitly impose any other algebraic properties on $\mathcal{L}$).

- ii) *The morphism $\otimes 1 : J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp} \rightarrow \{\mathcal{K}(n, m)\}_{n,m=1}^{\infty}$ is an epimorphism (then it is automatically an isomorphism).*

If these conditions are satisfied, then $\mathcal{L}$ in item i) is determined uniquely: it is the inverse of the isomorphism $\otimes 1 : J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp} \rightarrow \{\mathcal{K}(n, m)\}_{n,m=1}^{\infty}$. In particular, $\mathcal{L} : \{\mathcal{K}(n, m)\}_{n,m=1}^{\infty} \rightarrow \mathcal{K}$ is a homomorphism of C^ -precategories.*

Proof. i) $\implies$ ii). Let $a \in \mathcal{K}(n+1, m+1)$. Since $\mathcal{L}(a) \otimes 1 = a$ we get $\mathcal{L}(a) \in J(\mathcal{K})(n, m)$. For every $b \in (\mathcal{K} \cap \ker \otimes 1)(m, l)$ we have

$$\mathcal{L}(a)b = \mathcal{L}(a(b \otimes 1)) = \mathcal{L}(0) = \mathcal{L}(0(0 \otimes 1)) = \mathcal{L}(0) \cdot 0 = 0.$$

Hence $\mathcal{L}(a) \in (\ker \otimes 1)^{\perp}$ and it follows that the image of $\mathcal{L}$ is contained in $J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}$. This together with the equality $\mathcal{L}(a) \otimes 1 = a$ implies that $\otimes 1 : J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp} \rightarrow \{\mathcal{K}(n, m)\}_{n,m=1}^{\infty}$ is a surjection. Furthermore, since $\otimes 1$ is isometric on $(\ker \otimes 1)^{\perp}$, the image of $\mathcal{L}$ coincides with $J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}$ and $\mathcal{L}$ coincides with the inverse of the isomorphism $\otimes 1 : J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp} \rightarrow \{\mathcal{K}(n, m)\}_{n,m=1}^{\infty}$.

ii) $\implies$ i). Define $\mathcal{L}$ as the inverse of $\otimes 1 : J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp} \rightarrow \{\mathcal{K}(n, m)\}_{n,m=1}^{\infty}$. Relation $\mathcal{L}(a) \otimes 1 = a$ is trivially satisfied. To show the “transfer operator property” let $a \in \mathcal{K}(n+1, m+1)$ and $b \in \mathcal{K}(m, l)$, $n, m, l \in \mathbb{N}$. Then

$$\mathcal{L}(a(b \otimes 1)) \otimes 1 = a(b \otimes 1) = (\mathcal{L}(a) \otimes 1)(b \otimes 1) = (\mathcal{L}(a)b) \otimes 1$$

and since both $\mathcal{L}(a(b \otimes 1))$ and $\mathcal{L}(a)b$ belong to $J(\mathcal{K}) \cap (\ker \otimes 1)^{\perp}(n, l)$, we get $\mathcal{L}(a(b \otimes 1)) = \mathcal{L}(a)b$. $\square$

Definition 5.6. We say that an ideal $\mathcal{K}$ in a right tensor C^* -precategory *admits a transfer homomorphism* if it satisfies the equivalent conditions of Proposition 5.5.

The role of the above introduced notion is explained by the following.

Theorem 5.7. *Let $\mathcal{K}$ and $\mathcal{J}$ be ideals in a right tensor C^* -precategory $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$. The universal representation embeds each space $\mathcal{K}(k, 0)$, $k \in \mathbb{N}$, into $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ as the k -th spectral subspace for the associated gauge action γ if and only if*

$$\mathcal{J} = J(\mathcal{K}) \cap (\ker \otimes 1)^\perp$$

and the ideal $\mathcal{K}$ admits a transfer homomorphism.

Proof. Assume that $\mathcal{J} = J(\mathcal{K}) \cap (\ker \otimes 1)^\perp$ and $\mathcal{K}$ admits a transfer homomorphism $\mathcal{L}$. By Corollary 4.15 the universal representation $\{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ is faithful. To see that the image of the space $\mathcal{K}(k, 0)$ is the k -spectral subspace of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ let $a \in \mathcal{K}(r+k, r)$, $r > 0$. Since $a = \mathcal{L}(a) \otimes 1$, $\mathcal{L}(a) \in \mathcal{J}$ and ι is coisometric on $\mathcal{J}$ we have $\iota_{r+k,r}(a) = \iota_{r-1+k,r-1}(\mathcal{L}(a))$ and following in this way by induction one gets $\iota_{r+k,r}(a) = \iota_{k,0}(\mathcal{L}^r(a))$.

Now suppose that the representation $\{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ embeds $\mathcal{K}(k, 0)$, $k \in \mathbb{N}$, as the k -spectral subspace of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. Then $\mathcal{J} \subset J(\mathcal{K}) \cap (\ker \otimes 1)^\perp$ by Corollary 4.15. To show the converse inclusion let $a \in J(\mathcal{K}) \cap (\ker \otimes 1)^\perp(n, n)$. Since $\iota_{n,n}(a)$ and $\iota_{n+1,n+1}(a \otimes 1)$ lie in the spectral subspace $\iota_{0,0}(\mathcal{K}(0, 0))$ there exists $b \in \mathcal{K}(0, 0)$ such that $\iota_{0,0}(b) = \iota_{n,n}(a) - \iota_{n+1,n+1}(a \otimes 1)$. Using Theorem 4.14 we get

$$b \otimes 1^n - a \in \mathcal{J}(n, n), \quad b \otimes 1^{n+1} = 0.$$

It follows that $b \otimes 1^n$ belongs both to $(\ker \otimes 1)^\perp(n, n)$ and $\ker \otimes 1(n, n)$. Therefore $b \otimes 1^n = 0$ and consequently $a \in \mathcal{J}(n, n)$. This shows that $\mathcal{J} = J(\mathcal{K}) \cap (\ker \otimes 1)^\perp$ and to prove that $\otimes 1 : J(\mathcal{K}) \cap (\ker \otimes 1)^\perp \rightarrow \{\mathcal{K}(n, m)\}_{n,m=1}^\infty$ is an epimorphism let $a \in \mathcal{K}(n+1+k, n+1)$, $n, k \in \mathbb{N}$. Then there exists $b \in \mathcal{K}(k, 0)$ such that $\iota_{k,0}(b) = \iota_{n+1+k,n+1}(a)$ or equivalently

$$b \otimes 1^r \in \mathcal{J}(r+k, r), \quad r = 0, \dots, n, \quad b \otimes 1^{n+1} = a.$$

Hence $a = c \otimes 1$, where $c = b \otimes 1^n \in \mathcal{J}(n+k, n) = J(\mathcal{K}) \cap (\ker \otimes 1)^\perp(n+k, n)$. $\square$

We interpret this result on the level of C^* -algebras associated with C^* -correspondences.

Proposition 5.8. *An ideal $\mathcal{K}_X = \{\mathcal{K}(X^{\otimes m}, X^{\otimes n})\}_{n,m \in \mathbb{N}}$ in a right tensor C^* -precategory $\mathcal{T} = \mathcal{T}_X$ associated with a C^* -correspondence X over a C^* -algebra A admits a transfer homomorphism if and only if X is a Hilbert A -bimodule.*

Proof. Apply Proposition 1.11 iii) and Proposition 5.5 ii). $\square$

Thus as a corollary to Theorem 5.7 we get

Theorem 5.9 (cf. Prop. 5.17 [19]). *The universal representation of a C^* -correspondence X over A embeds A into a relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$ as the fixed point algebra for the gauge action if and only if X is a Hilbert A -bimodule and*

$$J = (\ker \phi)^\perp \cap J(X).$$

If this is the case, then $(\ker \phi)^\perp \cap J(X) = \overline{{}_A\langle X, X \rangle}$ and $\mathcal{O}(J, X)$ is canonically isomorphic to the crossed-product $A \rtimes_X \mathbb{Z}$ by a Hilbert bimodule X in the sense of [1]. In particular, each space $X^{\otimes n}$, $n \in \mathbb{N}$, embeds into $\mathcal{O}(J, X)$ as the spectral subspace.

Proof. If $J = (\ker \phi)^\perp \cap J(X)$ and X is a Hilbert A -bimodule, then by Theorem 5.7, algebra A and spaces $X^{\otimes n}$, $n > 0$, embed into $\mathcal{O}(J, X)$ as spectral subspaces. Conversely, if A embeds into $\mathcal{O}(J, X)$ as the 0-spectral space, then the argument from the proof of Theorem 5.7 shows that $J = (\ker \phi)^\perp \cap J(X)$ and X forms a Hilbert bimodule; cf. Proposition 1.11. $\square$

By Examples 1.12, 3.27, 4.7, 4.8 we get

Corollary 5.10. *Let $\varphi : A \rightarrow M(A_0)$ be a partial morphism. Algebra A embeds into the crossed product $A \rtimes_\varphi \mathbb{N}$ as the 0-spectral subspace if and only if φ arises from a partial automorphism (in our broader sense; see Example 1.12). If this is the case, then $A \rtimes_\varphi \mathbb{N}$ coincides with the partial crossed product as defined in [10] (where instead of the ideal J one puts the hereditary subalgebra A_0).*

Corollary 5.11. *Let $\alpha : A \rightarrow A$ be an endomorphism of a unital C^* -algebra A . Algebra A embeds into the relative crossed product $C^*(A, \alpha; J)$ as the 0-spectral subspace if and only if α admits a complete transfer operator $\mathcal{L}$ and $J = \mathcal{L}(A) = (\ker \alpha)^\perp$. If this is the case, $C^*(A, \alpha; J)$ coincides with the crossed product $A \rtimes_\alpha \mathbb{Z}$ introduced in [3]; cf. [24].*

By Examples 3.28, 4.6 we get

Corollary 5.12. *Let $E = (E^0, E^1, r, s)$ be a directed graph and let $R(E) = \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}$. Then $A = C_0(E^0)$ embeds into the relative graph algebra $C^*(E, V)$ as the 0-spectral subspace if and only if $V = R(E)$ and r, s are injective. In this case $C^*(E, V)$ coincides with both the graph algebra $C^*(E)$ and the partial crossed product defined by the partial homeomorphism $s \circ r^{-1} : r(E^1) \rightarrow s(E^1)$.*

The above statement could be generalized in a natural manner to topological graphs; cf. Example 1.13. It reflects the fact that as a rule description of the core (the fixed point subalgebra) of a graph algebra in terms of the graph is a nontrivial step.

6. STRUCTURE THEOREM FOR $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$

In this section we generalize the main result of [11]. To this end we introduce and discuss the notions of invariance and saturation for ideals in right tensor C^* -precategories.

Definition 6.1. We say that an ideal $\mathcal{N}$ in a right tensor C^* -precategory $\mathcal{T}$ is *invariant* if $\mathcal{N} \otimes 1 \subset \mathcal{N}$. For such an ideal there is a *quotient right tensor C^* -precategory* $\mathcal{T}/\mathcal{N}$ defined by

$$\begin{aligned} (\mathcal{T}/\mathcal{N})(n, m) &:= \mathcal{T}(n, m)/\mathcal{N}(n, m), \\ (a + \mathcal{N}(n, m)) \otimes 1 &:= a \otimes 1 + \mathcal{N}(n, m), \end{aligned}$$

where $a \in \mathcal{T}(n, m)$, $n, m \in \mathbb{N}$; cf. Proposition 2.10. More generally, if $\mathcal{K}$ is an ideal in $\mathcal{T}$ we say that an ideal $\mathcal{N}$ is $\mathcal{K}$ -invariant if

$$(\mathcal{N}(n, m) \otimes 1)\mathcal{K}(m+1, l) \subset \mathcal{N}(n+1, l), \quad n, m, l \in \mathbb{N};$$

shortly $(\mathcal{N} \otimes 1)\mathcal{K} \subset \mathcal{N}$.

Notational conventions 6.2. If $\mathcal{N}$ and $\mathcal{K}$ are ideals in $\mathcal{T}$, then the image of $\mathcal{K}$ in the quotient C^* -precategory $\mathcal{T}/\mathcal{N}$ shall be denoted by $\mathcal{K}/\mathcal{N}$ (obviously it may be identified with the quotient C^* -precategory $\mathcal{K}/(\mathcal{N} \cap \mathcal{K})$). If $\mathcal{S}$ is a sub- C^* -precategory of $\mathcal{T}$ and $\mathcal{K}$ is an ideal in $\mathcal{T}$, we denote by $\mathcal{S} + \mathcal{K}$ a sub- C^* -precategory of $\mathcal{T}$, where $(\mathcal{S} + \mathcal{K})(n, m) := \mathcal{S}(n, m) + \mathcal{K}(n, m)$, $n, m \in \mathbb{N}$. Similarly, if $\{\mathcal{S}_k\}_{k \in \mathbb{N}}$ is a family of ideals in $\mathcal{T}$, we denote by $\sum_{k=0}^{\infty} \mathcal{S}_k$ the ideal in $\mathcal{T}$, where $(\sum_{k=0}^{\infty} \mathcal{S}_k)(n, m) := \overline{\text{span}}\{a \in \mathcal{S}_k(n, m) : k \in \mathbb{N}\}$. If additionally $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots$, then we write $\varinjlim \mathcal{S}_k$ for $\sum_{k=0}^{\infty} \mathcal{S}_k$. If a is a morphism in $\mathcal{T}(n, m)$, then to say that a is in $\mathcal{S}(n, m)$ we briefly write $a \in \mathcal{S}$.

Lemma 6.3. *Let $\mathcal{N}$ be a sub- C^* -precategory in a right tensor C^* -precategory $\mathcal{T}$. Then, for each $k \in \mathbb{N}$, the spaces*

$$(\mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{T})(n, m) := \overline{\text{span}}\{a(b \otimes 1^k)c : a \in \mathcal{T}(n, p+k), \\ b \in \mathcal{N}(p, r), c \in \mathcal{T}(r+k, m)\}$$

form an ideal in $\mathcal{T}$. In particular, $\mathcal{T}\mathcal{N}\mathcal{T}$ is the smallest ideal containing $\mathcal{N}$ and

$$\tilde{\mathcal{N}} = \sum_{k=0}^{\infty} \mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{T}$$

is the smallest invariant ideal in $\mathcal{T}$ containing $\mathcal{N}$.

Proof. One sees that $\mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{T}$ is a well-defined C^* -precategory and since it is invariant under left and right multiplication, it is an ideal. The ideal $\tilde{\mathcal{N}} = \sum_{k=0}^{\infty} \mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{T}$ is a direct limit of the ascending sequence of ideals $\sum_{k=0}^N \mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{T}$, where

$$(\mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{T}) \subset \mathcal{T}(\mathcal{N} \otimes 1^{k+1})\mathcal{T}.$$

Thus $\tilde{\mathcal{N}}$ is invariant. The minimality of $\tilde{\mathcal{N}}$ and $\mathcal{T}\mathcal{N}\mathcal{T}$ is obvious. $\square$

Remark 6.4. By analogy to the nomenclature concerning graphs; cf. Example 6.25 below, one could call invariant ideals in $\mathcal{T}$ *hereditary ideals* and then for any ideal $\mathcal{N}$ one could call $\tilde{\mathcal{N}} = \sum_{k=0}^{\infty} \mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{T}$ a *hereditation* of $\mathcal{N}$.

Proposition 6.5. *Let $\mathcal{K}$ be an ideal in a right tensor C^* -precategory $\mathcal{T}$. For an ideal $\mathcal{N}$ in $\mathcal{K}$, the following conditions are equivalent:*

- i) $\mathcal{N}$ is $\mathcal{K}$ -invariant,
- ii) $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{K}$ for an invariant ideal $\tilde{\mathcal{N}}$ in $\mathcal{T}$,
- iii) $\mathcal{N} = \ker \pi$ for a right tensor representation π of $\mathcal{K}$.

In particular, if π is a right tensor representation of $\mathcal{K}$ and $\tilde{\mathcal{N}}$ is an invariant ideal in $\mathcal{T}$ such that $\ker \pi = \tilde{\mathcal{N}} \cap \mathcal{K}$, then π factors through to the faithful right tensor representation of the ideal $\mathcal{K}/\tilde{\mathcal{N}} \cong \mathcal{K}/\ker \pi$ in the quotient right tensor C^ -precategory $\mathcal{T}/\tilde{\mathcal{N}}$.*

Proof. i) $\Rightarrow$ ii). Let $\tilde{\mathcal{N}} = \sum_{k=0}^{\infty} \mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{T}$ be the smallest invariant ideal in $\mathcal{T}$ containing $\mathcal{N}$. Note that by $\mathcal{K}$ -invariance of $\mathcal{N}$ we have

$$\mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{T}\mathcal{K} = \mathcal{T}(\mathcal{N} \otimes 1^k)\mathcal{K} \subset \mathcal{T}\mathcal{N} = \mathcal{N}.$$

Therefore $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{K}$.

ii) $\Rightarrow$ iii). Consider the image $\mathcal{K}/\tilde{\mathcal{N}}$ of $\mathcal{K}$ under the quotient homomorphism $q_{\tilde{\mathcal{N}}} : \mathcal{T} \rightarrow \mathcal{T}/\tilde{\mathcal{N}}$. Then one may define π to be the composition of $q_{\tilde{\mathcal{N}}}$ with any faithful right tensor representation of $\mathcal{K}/\tilde{\mathcal{N}}$ (such a representation exists by Corollary 4.15).

iii) $\Rightarrow$ i). If $\mathcal{N} = \ker \pi$ for a right tensor representation of $\mathcal{K}$, $a \in \mathcal{N}(n, m)$ and $b \in \mathcal{K}(m+1, l)$, then $\pi_{n+1, l}((a \otimes 1)b) = \pi_{n, m}(a)\pi_{m+1, l}(b) = 0$, that is, $(a \otimes 1)b \in \mathcal{N}(n+1, l)$ and hence $\mathcal{N}$ is $\mathcal{K}$ -invariant. $\square$

Invariant ideals are closely related with kernels of right tensor representations. We reveal a similar relationship between saturated ideals and ideals of coisometricity for right tensor representations.

Definition 6.6. Let $\mathcal{N}$ and $\mathcal{J}$ be ideals in a right tensor C^* -precategory $\mathcal{T}$. We say that $\mathcal{N}$ is $\mathcal{J}$ -saturated if $\mathcal{J} \cap \otimes 1^{-1}(\mathcal{N}) \subset \mathcal{N}$. In general we put

$$\mathcal{S}_{\mathcal{J}}(\mathcal{N}) := \varinjlim \mathcal{S}_k, \quad \text{where } \mathcal{S}_0 := \mathcal{N} \text{ and } \mathcal{S}_k := \mathcal{J} \cap \otimes 1^{-1}(\mathcal{S}_{k-1}) + \mathcal{S}_{k-1}, \quad k > 0.$$

Then $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ is the smallest $\mathcal{J}$ -saturated ideal containing $\mathcal{N}$ and we shall call it $\mathcal{J}$ -saturation of $\mathcal{N}$.

The saturation works well with invariance.

Lemma 6.7. If $\mathcal{N}$ and $\mathcal{J}$ are ideals in a right tensor C^* -precategory $\mathcal{T}$ and $\mathcal{N}$ is invariant, then the $\mathcal{J}$ -saturation $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ of $\mathcal{N}$ is invariant and $a \in \mathcal{T}(n, m)$ is in $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ if and only if

$$(6.1) \quad a \otimes 1^k \in \mathcal{J} + \mathcal{N}, \quad \text{for all } k \in \mathbb{N}, \quad \text{and} \quad \lim_{k \rightarrow \infty} \|q_{\mathcal{N}}(a \otimes 1^k)\| = 0.$$

More generally, if $\mathcal{J} \subset J(\mathcal{K})$ and $\mathcal{N}$ is $\mathcal{K}$ -invariant for an ideal $\mathcal{K}$ in $\mathcal{T}$, then

$$\mathcal{S}_{\mathcal{J}}(\mathcal{N}) = \mathcal{S}_{\mathcal{J}}(\tilde{\mathcal{N}}) \cap \mathcal{K}$$

for any invariant ideal $\tilde{\mathcal{N}}$ such that $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{K}$. In particular, $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ is $\mathcal{K}$ -invariant.

Proof. The invariance of $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ follows from the inclusions $\mathcal{S}_k \otimes 1 \subset \mathcal{S}_{k-1}$, $k > 0$. Passing to $\mathcal{T}/\mathcal{N}$ one sees that every element in $\mathcal{T}(n, m)$ satisfying (6.1) may be approximated (arbitrarily closely) by elements $a \in \mathcal{T}(n, m)$ such that

$$(6.2) \quad a \otimes 1^k \in \mathcal{J} + \mathcal{N}, \quad \text{for } k = 0, \dots, N-1, \quad \text{and} \quad a \otimes 1^N \in \mathcal{N}, \quad N \in \mathbb{N}.$$

We claim that a satisfies (6.2) iff $a \in \mathcal{S}_N$ and therefore a satisfies (6.1) iff $a \in \mathcal{S}_{\mathcal{J}}(\mathcal{N})$. Indeed, let $a \in \mathcal{T}(n, m)$ satisfy (6.2). Then $a \otimes 1^N \in \mathcal{S}_0 = \mathcal{N}$ and

$$a \otimes 1^{N-1} \in \left(\otimes 1^{-1}(\mathcal{N}) \cap (\mathcal{J} + \mathcal{N}) \right) = (\otimes 1^{-1}(\mathcal{N}) \cap \mathcal{J}) + \mathcal{N} = \mathcal{S}_1.$$

Similarly, for $k = 1, \dots, N$, one sees that

$$a \otimes 1^{N-k} \in \mathcal{S}_k \quad \text{and} \quad a \otimes 1^{N-k-1} \in \mathcal{N} + \mathcal{J}$$

imply that $a \otimes 1^{N-k-1} \in \mathcal{S}_{k+1}$. Hence, by induction, $a \in \mathcal{S}_N$. Conversely, let $a \in \mathcal{S}_N(n, m)$. Then, since $\mathcal{S}_{\mathcal{J}}(\mathcal{N}) \subset \mathcal{J} + \mathcal{N}$, we have $a \otimes 1^k \in \mathcal{J} + \mathcal{N}$, $k = 0, \dots, N-1$, and since $\mathcal{S}_0 \otimes 1 \subset \mathcal{S}_0 = \mathcal{N}$ and $\mathcal{S}_k \otimes 1 \subset \mathcal{S}_{k-1}$, $k > 0$, we get $a \otimes 1^N \in \mathcal{S}_0 = \mathcal{N}$.

To prove the second part of the statement we denote by $\{\mathcal{S}_k(\mathcal{N})\}_{k \in \mathbb{N}}$ and $\{\mathcal{S}_k(\tilde{\mathcal{N}})\}_{k \in \mathbb{N}}$ the increasing sequences of ideals whose direct limits yield respectively $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ and $\mathcal{S}_{\mathcal{J}}(\tilde{\mathcal{N}})$. We shall show our claim by proving that

$$\mathcal{S}_k(\mathcal{N}) = \mathcal{S}_k(\tilde{\mathcal{N}}) \cap \mathcal{K}, \quad \text{for all } k \in \mathbb{N}.$$

This relation trivially holds for $k = 0$, and if we suppose it holds for $k = N - 1$, then using the inclusions $\mathcal{J} \subset J(\mathcal{K}) \subset \mathcal{K}$ we get

$$\begin{aligned}\mathcal{S}_N(\mathcal{N}) &= \mathcal{J} \cap \otimes 1^{-1}(\mathcal{S}_{N-1}(\mathcal{N})) + \mathcal{S}_{N-1}(\mathcal{N}) \\ &= \mathcal{J} \cap \otimes 1^{-1}(\mathcal{S}_{N-1}(\tilde{\mathcal{N}}) \cap \mathcal{K}) + \mathcal{S}_{N-1}(\tilde{\mathcal{N}}) \cap \mathcal{K} \\ &= \mathcal{J} \cap \otimes 1^{-1}(\mathcal{S}_{N-1}(\tilde{\mathcal{N}})) + \mathcal{S}_{N-1}(\tilde{\mathcal{N}}) \cap \mathcal{K} \\ &= \mathcal{S}_N(\tilde{\mathcal{N}}) \cap \mathcal{K}.\end{aligned}$$

□

Proposition 6.8. *Let $\mathcal{J}$ and $\mathcal{K}$ be ideals in a right tensor C^* -precategory $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$. For an ideal $\mathcal{N}$ in $\mathcal{K}$ the following conditions are equivalent:*

- i) $\mathcal{N}$ is $\mathcal{K}$ -invariant and $\mathcal{J}$ -saturated,
- ii) $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{K}$ for an invariant and $\mathcal{J}$ -saturated ideal $\tilde{\mathcal{N}}$ in $\mathcal{T}$,
- iii) $\mathcal{N} = \ker \pi$ for a right tensor representation π of $\mathcal{K}$ coisometric on $\mathcal{J}$.

Proof. i) $\Rightarrow$ ii). In view of Proposition 6.5 it suffices to note that if $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{K}$, then $\mathcal{N}$ is $\mathcal{J}$ -saturated iff $\tilde{\mathcal{N}}$ is $\mathcal{J}$ -saturated, which follows from the inclusion $\mathcal{J} \subset J(\mathcal{K})$.

ii) $\Rightarrow$ iii). Consider the same representation as in the proof of Proposition 6.5, and note that $\tilde{\mathcal{N}}$ is $\mathcal{J}$ -saturated iff the ideal $\mathcal{J}/\tilde{\mathcal{N}}$ in $\mathcal{T}/\tilde{\mathcal{N}}$ is contained in the annihilator of the kernel of the quotient right tensoring on $\mathcal{T}/\tilde{\mathcal{N}}$. Hence, by Corollary 4.15, the considered representation may be chosen to be coisometric on $\mathcal{J}$.

iii) $\Rightarrow$ i). The $\mathcal{K}$ -invariance of $\ker \pi$ follows from Proposition 6.5, and $\mathcal{J}$ -saturation is a direct consequence of the definitions of $J(\mathcal{K})$ and $\mathcal{J}$ -saturation; see Definitions 3.18, 6.6. □

Now we are in a position to prove the main result of this section.

Theorem 6.9 (Structure Theorem). *Let $\mathcal{K}$, $\mathcal{J}$ and $\mathcal{N}$ be ideals in a C^* -precategory $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$ and $\mathcal{N} \subset \mathcal{K}$ is $\mathcal{K}$ -invariant. The subspace*

$$\mathcal{O}(\mathcal{N}) = \overline{\text{span}}\{\iota_{n,m}(a) : a \in \mathcal{N}(n, m), n, m \in \mathbb{N}\} \subset \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$$

generated by the image of $\mathcal{N}$ under the universal representation $\iota = \{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ is an ideal in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ and there are natural isomorphisms

$$(6.3) \quad \mathcal{O}(\mathcal{N}) \cong \mathcal{O}_{\mathcal{T}}(\mathcal{N}, \mathcal{J} \cap \mathcal{N}), \quad \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})/\mathcal{O}(\mathcal{N}) \cong \mathcal{O}_{\mathcal{T}/\tilde{\mathcal{N}}}(\mathcal{K}/\tilde{\mathcal{N}}, \mathcal{J}/\tilde{\mathcal{N}}),$$

where $\tilde{\mathcal{N}}$ is an arbitrary invariant ideal in $\mathcal{T}$ such that $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{K}$. Moreover, for the $\mathcal{J}$ -saturation $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ of $\mathcal{N}$ we have

$$\mathcal{S}_{\mathcal{J}}(\mathcal{N}) = \iota^{-1}(\mathcal{O}(\mathcal{N})).$$

Hence $\mathcal{O}(\mathcal{S}_{\mathcal{J}}(\mathcal{N})) = \mathcal{O}(\mathcal{N})$ and on the right-hand sides of (6.3) the ideals $\mathcal{N}$ and $\tilde{\mathcal{N}}$ may be replaced by their $\mathcal{J}$ -saturation $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ and $\mathcal{S}_{\mathcal{J}}(\tilde{\mathcal{N}})$.

Proof. Using operations (4.1) – (4.4) we see that

$$\mathcal{M}_{\mathcal{T}}(\mathcal{N}) = \text{span}\{\iota_{n,m}(a) : a \in \mathcal{N}(n, m), n, m \in \mathbb{N}\}$$

is a two-sided ideal in the algebra $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ defined in Subsection 4.2.1. Therefore $\mathcal{O}(\mathcal{N}) = \overline{\text{span}}\{\iota_{n,m}(a) : a \in \mathcal{N}(n, m), n, m \in \mathbb{N}\}$ is an ideal in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. To prove that $\mathcal{O}(\mathcal{N}) \cong \mathcal{O}_{\mathcal{T}}(\mathcal{N}, \mathcal{J} \cap \mathcal{N})$ we show that the seminorms $\|\cdot\|_{\mathcal{J}}$ and $\|\cdot\|_{\mathcal{J} \cap \mathcal{N}}$ give the same quotients of $\mathcal{M}_{\mathcal{T}}(\mathcal{N})$; cf. Theorem 4.12. Let $a \in \mathcal{M}_{\mathcal{T}}(\mathcal{N})$. Obviously, $\|a\|_{\mathcal{J} \cap \mathcal{N}} = 0$ implies $\|a\|_{\mathcal{J}} = 0$. Conversely, we may assume that

$a = \sum_{s=0}^r \iota_{s,s}(a_{s,s})$, $a_{s,s} \in \mathcal{N}(s,s)$, $s = 0, \dots, r$, $r \in \mathbb{N}$, and then the condition $\|a\|_{\mathcal{J}} = 0$ is equivalent to

$$\sum_{j=0}^s a_{j,j} \otimes 1^{s-j} \in \mathcal{J}(s,s), \quad s = 1, \dots, r-1, \quad \lim_{r \rightarrow \infty} \sum_{j=0}^r a_{j,j} \otimes 1^{r-j} = 0.$$

By $\mathcal{K}$ -invariance of $\mathcal{N}$ we get

$$\sum_{j=0}^s a_{j,j} \otimes 1^{s-j} \in \mathcal{J} \cap \mathcal{N}(s,s), \quad s = 1, \dots, r-1, \quad \lim_{r \rightarrow \infty} \sum_{j=0}^r a_{j,j} \otimes 1^{r-j} = 0,$$

which is equivalent to $\|a\|_{\mathcal{J} \cap \mathcal{N}} = 0$. Hence $\mathcal{O}(\mathcal{N}) \cong \mathcal{O}(\mathcal{N}, \mathcal{J} \cap \mathcal{N})$.

Let $\tilde{\mathcal{N}}$ be an invariant ideal in $\mathcal{T}$ such that $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{K}$. To construct the isomorphism $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})/\mathcal{O}(\mathcal{N}) \cong \mathcal{O}_{\mathcal{T}/\tilde{\mathcal{N}}}(\mathcal{K}/\tilde{\mathcal{N}}, \mathcal{J}/\tilde{\mathcal{N}})$ we consider the right tensor representation $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}/\tilde{\mathcal{N}}}(\mathcal{K}/\tilde{\mathcal{N}}, \mathcal{J}/\tilde{\mathcal{N}})$ given by $\pi_{n,m} = \iota_{n,m} \circ q_{\tilde{\mathcal{N}}}$, $m, n \in \mathbb{N}$, where $\iota = \{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ denotes the universal representation of $\mathcal{K}/\tilde{\mathcal{N}}$ in $\mathcal{O}_{\mathcal{T}/\tilde{\mathcal{N}}}(\mathcal{K}/\tilde{\mathcal{N}}, \mathcal{J}/\tilde{\mathcal{N}})$. Since for $a \in \mathcal{J}(n,m)$ we have

$$\begin{aligned} q_{\tilde{\mathcal{N}}}(a) \in (\mathcal{J}/\tilde{\mathcal{N}})(n,m) &\iff \iota_{n,m}(q_{\tilde{\mathcal{N}}}(a)) = \iota_{n+1,m+1}(q_{\tilde{\mathcal{N}}}(a) \otimes 1) \\ &\iff \pi_{n,m}(a) = \pi_{n+1,m+1}(a \otimes 1), \end{aligned}$$

it follows that π is coisometric on $\mathcal{J}$ and thereby induces a homomorphism Ψ_{π} from $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ onto $\mathcal{O}_{\mathcal{T}/\tilde{\mathcal{N}}}(\mathcal{K}/\tilde{\mathcal{N}}, \mathcal{J}/\tilde{\mathcal{N}})$. Plainly, Ψ_{π} is zero on $\mathcal{O}(\mathcal{N})$ and hence it factors through to an epimorphism $\Psi_{\pi} : \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})/\mathcal{O}(\mathcal{N}) \rightarrow \mathcal{O}_{\mathcal{T}/\tilde{\mathcal{N}}}(\mathcal{K}/\tilde{\mathcal{N}}, \mathcal{J}/\tilde{\mathcal{N}})$. One proves injectivity of Ψ_{π} by constructing its inverse. Indeed, the formula

$$\omega_{n,m}(a + \tilde{\mathcal{N}}(n,m)) = q(\iota_{n,m}(a)), \quad a \in (\mathcal{K}/\tilde{\mathcal{N}})(n,m), \quad n, m \in \mathbb{N},$$

where $q : \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) \rightarrow \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})/\mathcal{O}(\mathcal{N})$ is the quotient map, defines a right tensor representation ω of $\mathcal{K}/\tilde{\mathcal{N}}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})/\mathcal{O}(\mathcal{N})$ which is coisometric on $\mathcal{J}/\tilde{\mathcal{N}}$. Thus ω integrates to a homomorphism from $\mathcal{O}_{\mathcal{T}/\tilde{\mathcal{N}}}(\mathcal{K}/\tilde{\mathcal{N}}, \mathcal{J}/\tilde{\mathcal{N}})$ to $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})/\mathcal{O}(\mathcal{N})$ which is inverse to Ψ_{π} .

Furthermore, we have $\ker \pi_{n,m} = \iota_{n,m}^{-1}(\mathcal{O}(\mathcal{N}))$ (where π is the representation defined above) and thus by Corollary 4.15, $a \in \mathcal{K}(n,m)$ is in $\iota_{n,m}^{-1}(\mathcal{O}(\mathcal{N}))$ if and only if

$$a \otimes 1^k \in \mathcal{J} + \tilde{\mathcal{N}}, \quad \text{for all } k \in \mathbb{N}, \quad \text{and} \quad \lim_{k \rightarrow \infty} \|q_{\tilde{\mathcal{N}}}(a \otimes 1^k)\| = 0.$$

In view of Lemma 6.7, this proves the second part of the theorem. $\square$

The Structure Theorem has a number of important consequences.

First, $\mathcal{O}(\mathcal{N})$ is a *gauge-invariant ideal* in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$, i.e. it is globally invariant under the associated gauge action, and if $\mathcal{N}$ is $\mathcal{J}$ -saturated it is uniquely determined by $\mathcal{O}(\mathcal{N})$. Hence, denoting by $\text{Lat}_{\mathcal{J}}(\mathcal{K})$ the lattice of $\mathcal{K}$ -invariant and $\mathcal{J}$ -saturated ideals in $\mathcal{K}$, and by $\text{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}))$ the lattice of gauge-invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ we get the natural embedding

$$\text{Lat}_{\mathcal{J}}(\mathcal{K}) \hookrightarrow \text{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})).$$

In general this embedding is not an isomorphism. However, we show in Theorem 7.8 that in certain important cases we have $\text{Lat}_{\mathcal{J}}(\mathcal{K}) \cong \text{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}))$.

Secondly, the zero ideal $\mathcal{N} = \{0\}$ is invariant and hence, dividing $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ by the ideal generated by the $\mathcal{J}$ -saturation $\mathcal{S}_{\mathcal{J}}(\{0\})$ of $\{0\}$, we reduce the relations

defining $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ without affecting the algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ itself. We formalize this remark as follows.

Definition 6.10. Let $\mathcal{J}$ be an ideal in a right tensor C^* -precategory $\mathcal{T}$. We denote the $\mathcal{J}$ -saturation $\mathcal{S}_{\mathcal{J}}(\{0\})$ of the zero ideal by $\mathcal{R}_{\mathcal{J}}$ and call it a *reduction ideal* associated with $\mathcal{J}$.

Theorem 6.11 (Reduction of relations). *Let $\mathcal{K}$ be an ideal in a right tensor C^* -category $\mathcal{T}$ and let $\mathcal{J}$ be an ideal in $\mathcal{J}(\mathcal{K})$. Putting*

$$\mathcal{T}_{\mathcal{R}} := \mathcal{T}/\mathcal{R}_{\mathcal{J}}, \quad \mathcal{K}_{\mathcal{R}} := \mathcal{K}/\mathcal{R}_{\mathcal{J}}, \quad \mathcal{J}_{\mathcal{R}} := \mathcal{J}/\mathcal{R}_{\mathcal{J}},$$

we get the “reduced” right tensor C^ -category $\mathcal{T}_{\mathcal{R}}$, with right tensoring $\otimes_{1_{\mathcal{R}}}$. The “reduced” ideals $\mathcal{K}_{\mathcal{R}}, \mathcal{J}_{\mathcal{R}}$ are such that $\mathcal{J}_{\mathcal{R}} \subset \mathcal{J}(\mathcal{K}_{\mathcal{R}}) \cap (\ker \otimes_{1_{\mathcal{R}}})^{\perp}$ and there is a natural isomorphism*

$$\mathcal{O}(\mathcal{K}, \mathcal{J}) \cong \mathcal{O}(\mathcal{K}_{\mathcal{R}}, \mathcal{J}_{\mathcal{R}}).$$

Proof. This is clear by Theorem 6.9. □

Remark 6.12. We may often reduce even more relations in $\mathcal{T}$ in the sense that for any invariant ideal $\mathcal{R}'$ in $\mathcal{T}$ such that $\mathcal{R}' = \mathcal{R}_{\mathcal{J}} \cap \mathcal{K}$ we have

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) \cong \mathcal{O}_{\mathcal{T}_{\mathcal{R}'}}(\mathcal{K}_{\mathcal{R}'}, \mathcal{J}_{\mathcal{R}'}),$$

where $\mathcal{K}_{\mathcal{R}'} := \mathcal{K}/\mathcal{R}'$ and $\mathcal{J}_{\mathcal{R}'} := \mathcal{J}/\mathcal{R}'$ coincide with $\mathcal{K}_{\mathcal{R}}$ and $\mathcal{J}_{\mathcal{R}}$, but in general $\mathcal{T}_{\mathcal{R}'} = \mathcal{T}/\mathcal{R}'$ is “smaller” than $\mathcal{T}_{\mathcal{R}} = \mathcal{T}/\mathcal{R}_{\mathcal{J}}$.

6.1. Structure theorem for relative Cuntz-Pimsner algebras. As an application of Theorem 6.9 we get a version of the Structure Theorem for relative Cuntz-Pimsner algebras which improves [11, Thm. 3.1]. For that purpose we fix a C^* -correspondence X over a C^* -algebra A and establish the relationships between the relevant ideals in $\mathcal{K}_X$ and A .

Proposition 6.13. *Each of the relations*

$$(6.4) \quad I = \mathcal{N}(0, 0), \quad \mathcal{N} = \mathcal{K}_X(I)$$

establishes a one-to-one correspondence between X -invariant ideals I in A and $\mathcal{K}_X$ -invariant ideals $\mathcal{N}$ in $\mathcal{K}_X$. Moreover, we have a natural isomorphism of C^ -precategories*

$$\mathcal{K}_X/\mathcal{N} \cong \mathcal{K}_{X/XI} = \{\mathcal{K}((X/XI)^{\otimes n}, (X/XI)^{\otimes m})\}_{n,m \in \mathbb{N}}.$$

Proof. For an X -invariant ideal I in A , $\tilde{\mathcal{N}} := \mathcal{T}_X(I)$ is invariant and hence $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{K} = \mathcal{K}_X(I)$ is $\mathcal{K}_X$ -invariant. Conversely, if $\mathcal{N}$ is an ideal in $\mathcal{K}_X$, then by Proposition 2.17, $\mathcal{N} = \mathcal{K}_X(I)$, where $I := \mathcal{N}(0, 0)$. If additionally $\mathcal{N}$ is $\mathcal{K}_X$ -invariant, then I is X -invariant because for $a \in I = \mathcal{K}_X(I)(0, 0)$ and $b = \Theta_{x,y} \in \mathcal{K}_X(1, 1)$ we have

$$(a \otimes 1)b = \phi(a)\Theta_{x,y} = \Theta_{\phi(a)x,y} \in \mathcal{K}_X(I)(1, 1),$$

that is, $\phi(a)x \in XI$. The remaining part of the assertion follows from Lemma 1.8 and Corollary 1.6. □

In the case when $J = J(X) \cap (\ker \phi)^{\perp}$, the notion we are about to introduce coincides with the property called X -saturation in [28, Def. 6.1] and negative invariance in [20, Def. 4.8]; cf. also [20, Def. 4.14].

Definition 6.14. Let I and J be ideals in A . We say that I is J -saturated if $J \cap \phi^{-1}(\mathcal{L}(XI)) \subset I$ or, equivalently, if

$$a \in J \quad \text{and} \quad \phi(a)X \subset XI \implies a \in I.$$

In general we put $S_J(I) := \varinjlim S_k$, where $S_0 := I$ and $S_k := J \cap \phi^{-1}(\mathcal{L}(XS_{k-1})) + S_{k-1}$, $k > 0$. Then $S_J(I)$ is the smallest J -saturated ideal containing I , which we shall call J -saturation of I .

Proposition 6.15. Let J and I be ideals in A . The $\mathcal{K}_X(J)$ -saturation of $\mathcal{K}_X(I)$ coincides with $\mathcal{K}_X(S_J(I))$. In particular, I is J -saturated if and only if $\mathcal{K}_X(I)$ is $\mathcal{K}_X(J)$ -saturated.

Proof. It suffices to check that under the notation of Definitions 6.6, 6.14 we have $\mathcal{K}_X(S_n) = \mathcal{S}_n$, $n \in \mathbb{N}$, which is straightforward. $\square$

In view of the above statement, Corollary 3.23 and Proposition 6.8 we get

Proposition 6.16. Let $J \subset J(X)$. If J is an ideal of coisometricity for a representation (π, t) of X , then $\ker \pi$ is J -saturated and X -invariant. Conversely, every J -saturated and X -invariant ideal in A is the kernel $\ker \pi$ for a certain representation (π, t) of X whose ideal of coisometricity is J .

To state the Structure Theorem for C^* -correspondences in its full force we introduce algebras that generalize relative Cuntz-Pimsner algebras.

Definition 6.17. If I and J are ideals in A such that $J \subset J(XI)$ we put

$$\mathcal{O}_X(I, J) := \mathcal{O}_{\mathcal{T}_X}(\mathcal{K}_X(I), \mathcal{K}_X(J)).$$

This algebra is well defined because $\mathcal{K}_X(J) \subset J(\mathcal{K}_X(I))$ iff $J \subset J(XI)$. In particular, we have $\mathcal{O}(J, X) = \mathcal{O}_X(A, J)$.

We already noted that ideal structures of $\mathcal{K}_{XI} = \{\mathcal{K}(XI^{\otimes n}, XI^{\otimes m})\}_{n, m \in \mathbb{N}}$ and $\mathcal{K}_X(I) = \{\mathcal{K}(X^{\otimes n}, X^{\otimes m}I)\}_{n, m \in \mathbb{N}}$ are isomorphic; see Remark 2.19. By the following results, this could be interpreted as these C^* -precategories are “Morita equivalent” with the “equivalence” established by $L = \{\mathcal{K}(XI^{\otimes m}, X^{\otimes n})\}_{n, m \in \mathbb{N}}$.

Lemma 6.18. Let I be an arbitrary ideal in A . Then

$$L = \{\mathcal{K}(XI^{\otimes m}, X^{\otimes n})\}_{n, m \in \mathbb{N}}$$

is a left ideal in a C^* -precategory $\mathcal{T}_X$, and the following relations hold:

$$L^*L = \mathcal{K}_{XI}, \quad LL^* = \mathcal{K}_X(I),$$

where $L^*L(n, m) = \overline{\text{span}}\{x^*y : x \in L(k, n), y \in L(k, m)\}$ and $LL^*(n, m) = \overline{\text{span}}\{xy^* : x \in L(n, k), y \in L(m, k)\}$, $n, m \in \mathbb{N}$.

Proof. Clearly, $\mathcal{K}_{XI} \subset L$ and hence $\mathcal{K}_{XI} = \mathcal{K}_{XI}^* \mathcal{K}_{XI} \subset L^*L$. The opposite inclusion follows from the fact that for $x, u \in X^{\otimes k}$, $y \in XI^{\otimes n}$ and $v \in XI^{\otimes m}$,

$$(\Theta_{u,v})^* \Theta_{x,y} = \Theta_{v, y \langle x, u \rangle} \in \mathcal{K}(XI^{\otimes n}, XI^{\otimes m}).$$

Thus $L^*L = \mathcal{K}_{XI}$. To show that $LL^* = \mathcal{K}_X(I)$ it suffices to apply Proposition 2.17, since both LL^* and $\mathcal{K}_X(I)$ are ideals in $\mathcal{K}_X$ such that $LL^*(0, 0) = \mathcal{K}_X(I)(0, 0) = I$. $\square$

The above “Morita equivalence” of C^* -precategories yields Morita equivalence of C^* -algebras.

Theorem 6.19 (Morita equivalence of $\mathcal{O}_X(I, I \cap J)$ and $\mathcal{O}_{XI}(I, J \cap I)$). *Let I be an X -invariant ideal in A and let J be an ideal in $J(X)$. The algebras*

$$\mathcal{O}_X(I, J \cap I), \quad \mathcal{O}_{XI}(I, J \cap I) = \mathcal{O}(J \cap I, XI)$$

may be naturally considered as subalgebras of $\mathcal{O}_{\mathcal{T}_X}(\mathcal{T}_X, \mathcal{K}_X(J \cap I))$. They coincide whenever $\varphi(I)X = XI$, and in general they are Morita equivalent with an equivalence established via the subspace

$$\mathcal{L} = \overline{\text{span}}\{\iota_{n,m}(a) : a \in L(n, m)\} \subset \mathcal{O}_{\mathcal{T}_X}(\mathcal{T}_X, \mathcal{K}_X(J \cap I)).$$

Proof. The first part of the statement is evident; cf. Definition 6.17. To see the second part consider the $*$ -algebra $\mathcal{M}_{\mathcal{T}}(\mathcal{T})$ defined in Subsection 4.2.1 for $\mathcal{T} = \mathcal{T}_X$. For any sub- C^* -precategory $\mathcal{S}$ of $\mathcal{T}$ we put

$$\mathcal{M}_{\mathcal{T}}(\mathcal{S}) = \text{span}\{\iota_{n,m}^{\mathcal{M}}(a) : a \in \mathcal{S}(n, m)\}.$$

Then $\mathcal{M}_{\mathcal{T}}(\mathcal{K}(I))$ is a two-sided ideal in $\mathcal{M}_{\mathcal{T}}(\mathcal{T})$, and $\mathcal{M}_{\mathcal{T}}(L)$ is a left ideal in $\mathcal{M}_{\mathcal{T}}(\mathcal{T})$ such that

$$\mathcal{M}_{\mathcal{T}}(L) \star \mathcal{M}_{\mathcal{T}}(L)^* = \mathcal{M}_{\mathcal{T}}(\mathcal{K}(I)).$$

Indeed, since $L \subset \mathcal{K}(I)$ we have $\mathcal{M}_{\mathcal{T}}(L) \star \mathcal{M}_{\mathcal{T}}(L)^* \subset \mathcal{M}_{\mathcal{T}}(\mathcal{K}(I))$, and the opposite inclusion follows from Lemma 6.18. Similarly, $\mathcal{M}_{\mathcal{T}}(\mathcal{K}(XI))$ is a $*$ -subalgebra of $\mathcal{M}_{\mathcal{T}}(\mathcal{T})$ such that

$$\mathcal{M}_{\mathcal{T}}(L)^* \star \mathcal{M}_{\mathcal{T}}(L) = \mathcal{M}_{\mathcal{T}}(\mathcal{K}(XI)).$$

Indeed, since $\mathcal{K}(XI) \subset L$ we have $\mathcal{M}_{\mathcal{T}}(\mathcal{K}(I)) \subset \mathcal{M}_{\mathcal{T}}(L)^* \star \mathcal{M}_{\mathcal{T}}(L)$. To get the opposite inclusion notice that for any $y \in XI^{\otimes n}$, $v \in XI^{\otimes m}$, $u, x_1 \in X^{\otimes l}$, $x_2 \in X^{\otimes k}$, by X -invariance of I , the operator

$$((\Theta_{u,v})^* \otimes 1^k) \Theta_{x_1 \otimes x_2, y} = \Theta_{v \otimes \varphi(\langle u, x_1 \rangle) x_2, y}$$

is in $\mathcal{K}(XI^{\otimes n}, XI^{\otimes(m+k)})$.

Now it suffices to take the quotients with respect to the seminorm $\|\cdot\|_{\mathcal{K}_X(J \cap I)}$ defined in Theorem 4.12 and apply the enveloping procedure. $\square$

We are in a position to formulate the main result of this subsection.

Theorem 6.20 (Structure Theorem for C^* -correspondences). *Suppose J is an ideal in $J(X)$ and let $\mathcal{O}(I)$ denote the ideal in $\mathcal{O}(J, X)$ generated by the image of an X -invariant ideal I under the universal representation. There are natural isomorphisms*

$$(6.5) \quad \mathcal{O}(I) \cong \mathcal{O}_X(I, J \cap I), \quad \mathcal{O}(J, X)/\mathcal{O}(I) \cong \mathcal{O}(J/I, X/XI).$$

Algebras $\mathcal{O}(I)$ and $\mathcal{O}(J \cap I, XI)$ are Morita equivalent and if $\phi(I)X = XI$, then simply $\mathcal{O}(I) \cong \mathcal{O}_X(I, J \cap I) = \mathcal{O}(J \cap I, XI)$. Moreover, the J -saturation $S_J(I)$ of I is X -invariant and

$$S_J(I) = \iota_{0,0}^{-1}(\mathcal{O}(I)).$$

Thus $\mathcal{O}(I) = \mathcal{O}(S_J(I))$ and on the right-hand sides of (6.5) the X -invariant ideal I may be replaced by the X -invariant and J -saturated ideal $S_J(I)$.

Proof. One sees that $\mathcal{O}(I) = \overline{\text{span}}\{\iota_{n,m}(a) : a \in \mathcal{K}_X(I)(n, m), n, m \in \mathbb{N}\}$, and hence the assertion follows from Theorems 6.9, 6.19 and Propositions 6.13, 6.15. $\square$

Remark 6.21. For an X -invariant ideal I in A the subspace $\phi(I)X$ of X may be considered as a C^* -correspondence over I . The argument from [19, Prop. 9.3] shows that $\mathcal{O}(J \cap I, \phi(I)X)$ is Morita equivalent to $\mathcal{O}(I)$. Thus we have three C^* -algebras with natural embeddings

$$\mathcal{O}(J \cap I, \phi(I)X) \subset \mathcal{O}(J \cap I, XI) \subset \mathcal{O}_X(I, J \cap I) \cong \mathcal{O}(I),$$

which are all Morita equivalent, and if $\phi(I)X = XI$, then they are actually equal. One of the advantages of our approach is that we have been able to identify $\mathcal{O}(I)$ precisely as $\mathcal{O}_X(I, J \cap I) = \mathcal{O}_{\mathcal{T}_X}(\mathcal{K}_X(I), \mathcal{K}_X(J \cap I))$ (not only up to Morita equivalence, as it is in [11], [19]).

The reduction procedure from Theorem 6.11 is in full consistency with reduction of relations in C^* -correspondences presented in [25].

Definition 6.22. Let J be an ideal in A . The J -saturation $S_J(\{0\})$ of the zero ideal will be denoted by R_J and called a *reduction ideal* associated to J .

Theorem 6.23 (Reduction of C^* -correspondences). *Let J be an ideal in $J(X)$. The reduction ideal R_J is X -invariant, and putting*

$$X_R := X/XR_J, \quad A_R := A/R_J, \quad J_R := J/R_J,$$

we get the “reduced” C^ -correspondence X_R over A_R such that $J_R \subset J(X_R) \cap \ker(\phi_R)^\perp$, where ϕ_R is the left action on X_R , and*

$$\mathcal{O}(J, X) \cong \mathcal{O}(J_R, X_R).$$

Proof. This is clear by Theorem 6.20. □

Example 6.24. Let $X = X_\varphi$ be the C^* -correspondence associated with a partial morphism $\varphi : A \rightarrow M(A_0)$. One readily checks that an ideal I in A is X -invariant if and only if $\varphi(I)A_0 \subset I$. For an X -invariant ideal I we have a restricted partial morphism $\varphi_I : I \rightarrow M(A_0 \cap I)$ and a quotient partial morphism $\varphi^I : A/I \rightarrow M(A_0/I)$, where

$$\varphi_I = \varphi|_I, \quad \varphi^I(a + I)(a_0 + I) := \varphi(a)a_0 + I, \quad a \in A, a_0 \in A_0.$$

Both φ_I and φ^I are well defined as $A_0 \cap I$ is a hereditary subalgebra of I , A_0/I is a hereditary subalgebra of A/I , $\varphi_I(I)A_0 \cap I = A_0 \cap I$ and $\varphi^I(A/I)(A_0/I) = A_0/I$. Furthermore, we may naturally identify (as C^* -correspondences) X_{φ_I} with X/XI and X_{φ^I} with $\phi(I)X$. Using our general definition of relative crossed products (Definition 4.9), by Theorem 6.20 and Remark 6.21 we get

$$C^*(\varphi; J)/\mathcal{O}(I) \cong C^*(\varphi^I; J/I),$$

where $\mathcal{O}(I)$ is Morita equivalent to $C^*(\varphi_I; J \cap I)$, and J is an ideal in $\varphi^{-1}(A_0)$. If we denote by J_∞ the ideal in J consisting of those elements $a \in J$ for which the iterates $\varphi^n(a)$, $n \in \mathbb{N}$, make sense and belong to J , i.e.

$$a \in J_\infty \iff a \in J, \varphi(a) \in J \cap A_0, \varphi^2(a) \in J \cap A_0, \dots, \varphi^n(a) \in J \cap A_0, \dots,$$

then the reduction ideal $R := R_J$ assumes the following form:

$$R = \overline{\{a \in J_\infty : \exists_{n \in \mathbb{N}} \varphi^n(a) = 0\}};$$

cf. [25]. For the quotient partial morphism $\varphi^R : A/R \rightarrow M(A_0/R)$ we have

$$C^*(\varphi, J) \cong C^*(\varphi^R, J/R).$$

Hence φ^R may be viewed as a natural reduction of the partial morphism φ relative to the ideal J .

Example 6.25. Suppose that $X = X_E$ is the C^* -correspondence of a graph E , I is an ideal in $A = C_0(E^0)$ and $F \subset E^0$ is a complement of the hull of I . Then

$$XI = \overline{\text{span}}\{\delta_e : r(e) \in F\}, \quad \phi(I)X = \overline{\text{span}}\{\delta_e : s(e) \in F\}.$$

It follows that I is X -invariant if and only if F is *hereditary*, that is, if $s(e) \in F \implies r(e) \in F$, for all $e \in E^1$; cf. [6], [5], [28]. When F is hereditary, then (slightly abusing notation) we may consider $E \setminus F := (E^0 \setminus F, r^{-1}(E^0 \setminus F), r, s)$ and $F := (F, s^{-1}(F), r, s)$ as subgraphs of E . In this event X/XI is canonically isomorphic to $X_{E \setminus F}$ and $\phi(I)X$ is canonically isomorphic to X_F ; cf. [11, Ex. 2.4]. We recall that $J(X) = \overline{\text{span}}\{\delta_v : |s^{-1}(v)| < +\infty\}$. Hence Theorem 6.20 implies that for any hereditary subset $F \subset E^0$ and any $V \subset \{v \in E^0 : |s^{-1}(v)| < \infty\}$ there is an ideal $\mathcal{O}(I)$ in the relative graph algebra $\mathcal{O}(E, V)$ such that

$$\mathcal{O}(E, V)/\mathcal{O}(I) \cong \mathcal{O}(E \setminus F, V \setminus F)$$

and $\mathcal{O}(I)$ is Morita equivalent to the relative graph algebra $C^*(F, F \cap V)$. We shall say that a subset $F \subset E^0$ is *V -saturated* if every vertex in $V \subset E^0$ which feeds into F and only F is in F :

$$v \in V \text{ and } \{r(e) : s(e) = v\} \subset F \implies v \in F.$$

By a *V -saturation* of a set F we mean the smallest saturated subset $S_V(F)$ of E^0 containing F . In the case that $V = \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}$ these notions coincide with the ones (without prefix V) defined in [5], [6]. If $J = \overline{\text{span}}\{\delta_v : v \in V\}$, then the reduction ideal $R := R_J$ is spanned by the point masses of the V -saturation $S_V(\emptyset)$ of the empty set ($S_V(\emptyset)$ consists of vertices in V that form paths leading to sinks). As a consequence we have

$$\mathcal{O}(E, V) \cong \mathcal{O}(E \setminus S_V(\emptyset), V \setminus S_V(\emptyset)).$$

Hence $E \setminus S_V(\emptyset)$ may be considered as a natural reduction of the graph E relative to the set V .

7. IDEAL STRUCTURE OF $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$

In this section we prove the gauge invariance theorem for $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ and describe the lattice of gauge-invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. These results generalize the corresponding statements for relative Cuntz-Pimsner obtained in [19], [11], [20] (see the relevant discussions in Subsection 7.1 and Section 9).

Let us fix ideals $\mathcal{K}, \mathcal{J}$ in a C^* -precategory $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K})$ and let $\mathcal{K}_{\mathcal{J}}$ be the C^* -precategory defined in Theorem 4.11. Representation $\{\iota_{n,m}^{\mathcal{M}}\}_{n,m \in \mathbb{N}}$ from Proposition 4.10 yields the injective homomorphism of $\mathcal{K}$ into $\mathcal{K}_{\mathcal{J}}$:

$$(7.1) \quad \mathcal{K}(n, m) \ni a \longmapsto \iota_{n,m}^{\mathcal{M}}(a) \in \mathcal{K}_{\mathcal{J}}(n, m).$$

We use it to adopt the identifications

$$\mathcal{K} \subset \mathcal{K}_{\mathcal{J}}, \quad \mathcal{DR}(\mathcal{K}_{\mathcal{J}}) = \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J});$$

cf. Proposition 5.1.

Proposition 7.1. *We have a one-to-one correspondence between the right tensor representations $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ coisometric on $\mathcal{J}$ and right tensor representations $\tilde{\pi} = \{\tilde{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}_{\mathcal{J}}$ coisometric on $\mathcal{K}_{\mathcal{J}}$. This correspondence is given by*

$$(7.2) \quad \tilde{\pi}_{r,r+k} \left(\sum_{\substack{j=0, \\ j+k \geq 0}}^r \iota_{j+k,j}^{\mathcal{M}}(a_{j+k,j}) \right) = \sum_{\substack{j=0, \\ j+k \geq 0}}^r \pi_{j+k,j}(a_{j+k,j}).$$

Proof. In view of the definition of $\mathcal{K}_{\mathcal{J}}$ the assertion may be verified directly. One may also deduce it from Theorem 4.12 and Propositions 4.10, 5.1. $\square$

It is well known that if two C^* -algebras A, B admit circle actions, then a $*$ -homomorphism $h : A \rightarrow B$ that maps faithfully spectral subspaces of A onto the corresponding spectral subspaces of B is an isomorphism if and only if it is gauge-invariant. Thus the following statement can be thought of as a (stronger) version of what is usually meant by a gauge invariance theorem.

Definition 7.2. A representation $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of an ideal $\mathcal{K}$ in a right tensor C^* -precategory is said to admit a gauge action if for every $z \in S^1$ relations

$$\beta_z(\pi_{n,m}(a)) = z^{n-m} \pi_{n,m}(a), \quad a \in \mathcal{K}(n, m), \quad n, m \in \mathbb{N},$$

give rise to a well-defined $*$ -homomorphism $\beta_z : C^*(\pi) \rightarrow C^*(\pi)$, where $C^*(\pi)$ stands for the C^* -algebra generated by the spaces $\pi_{n,m}(\mathcal{K}(n, m))$, $n, m \in \mathbb{N}$.

Theorem 7.3 (Gauge-invariant uniqueness for $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$). *Let $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ be a right tensor representation of $\mathcal{K}$ coisometric on $\mathcal{J} \subset J(\mathcal{K})$ and let $\mathcal{R}_{\mathcal{J}}$ be the reduction ideal associated with $\mathcal{J}$ (Definition 6.10). The following conditions are equivalent:*

- i) π integrates to a representation faithful on the 0-spectral subspace of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$;
- ii) π integrates to a representation faithful on all of the spectral subspaces of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$;
- iii) $\ker \tilde{\pi}$ coincides with $\mathcal{R}_{\mathcal{J}}$ embedded into $\mathcal{K}_{\mathcal{J}}$ via (7.1);
- iv) $\ker \pi = \mathcal{R}_{\mathcal{J}}$ and for $n \in \mathbb{N}$,

$$(7.3) \quad \mathcal{J}(n, n) = \pi_{n,n}^{-1} \left(\sum_{j=1}^k \pi_{n+j,n+j}(\mathcal{K}(n+j, n+j)) \right), \quad \text{for all } k > 0.$$

In particular, π integrates to the faithful representation of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ if and only if π admits a gauge action, and one of the equivalent conditions i)-iv) holds.

Proof. Equivalence i) $\iff$ ii) follows from C^* -equality and algebraic relations between spectral subspaces; cf. [10]. The representations Ψ_{π} and $\Psi_{\tilde{\pi}}$ of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$, arising from π and $\tilde{\pi}$, coincide and hence we put $\Psi = \Psi_{\pi} = \Psi_{\tilde{\pi}}$. From the direct limit construction of $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$ we see that Ψ is faithful on spectral subspaces iff it is faithful on spaces $\phi_{n,m}(\mathcal{K}_{\mathcal{J}}(n, m))$, $n, m \in \mathbb{N}$, where $\phi = \{\phi_{n,m}\}_{n,m \in \mathbb{N}}$ is a universal representation of $\mathcal{K}_{\mathcal{J}}$ in $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$. The latter requirement is equivalent to the equality $\ker \tilde{\pi} = \ker \phi$, and we claim that $\ker \phi$ coincides with $\mathcal{R}_{\mathcal{J}}$. Indeed, $\phi_{r,r} \left(\sum_{j=0}^r \iota_{j,j}^{\mathcal{M}}(a_j) \right) = 0$ if and only if $\lim_{k \rightarrow \infty} \left\| \sum_{j=0}^r \iota_{j,j}^{\mathcal{M}}(a_j) \right\|_{r+k,r+k}^{\mathcal{J}} = 0$, and

this is equivalent to the conditions

$$\sum_{j=0}^s a_j \otimes 1^{s-j} \in \mathcal{J}, \text{ for } s = 0, \dots, r-1, \text{ and } b \otimes 1^k \in \mathcal{J}, k \in \mathbb{N}, \lim_{k \rightarrow \infty} \|b \otimes 1^k\| = 0,$$

where $b = \sum_{j=0}^r a_j \otimes 1^{r-j}$. In other words, $\sum_{j=0}^r \iota_{j,j}^{\mathcal{M}}(a_j)$ as an element of $\mathcal{K}_{\mathcal{J}}(r, r)$ could be identified with $\iota_{r,r}^{\mathcal{M}}(b)$ and $b \in \mathcal{R}_{\mathcal{J}} = \mathcal{S}_{\mathcal{J}}(\{0\})$; cf. (6.1). This proves our claim and hence ii) $\iff$ iii).

Implication ii) $\implies$ iv) follows from Corollary 4.15 and Proposition 4.17. To obtain iv) $\implies$ iii) suppose that $\sum_{j=0}^r \iota_{j,j}^{\mathcal{M}}(a_j) \in \ker \tilde{\pi}_{r,r}$, that is, $\sum_{j=0}^r \pi_{j,j}(a_j) = 0$. Then $\pi_{0,0}(a_0) = -\sum_{j=1}^r \pi_{j,j}(a_j)$ implies (by (7.3)) that $a_0 \in \mathcal{J}(0, 0)$. Hence $\pi_{0,0}(a_0) = \pi_{1,1}(a_0 \otimes 1)$ and consequently $\pi_{1,1}(a_0 \otimes 1 + a_1) = -\sum_{j=2}^r \pi_{j,j}(a_j)$ implies that $a_0 \otimes 1 + a_1 \in \mathcal{J}(0, 0)$. Proceeding in this way one gets

$$\sum_{j=0}^s a_j \otimes 1^{s-j} \in \mathcal{J}, \text{ for } s = 0, \dots, r-1, \text{ and } \pi_{r,r}(\sum_{j=0}^r a_j \otimes 1^{r-j}) = 0,$$

that is, $\sum_{j=0}^r \iota_{j,j}^{\mathcal{M}}(a_j)$ as an element of $\mathcal{K}_{\mathcal{J}}(r, r)$ coincides with $\iota_{r,r}^{\mathcal{M}}(\sum_{j=0}^r a_j \otimes 1^{r-j})$ and $\sum_{j=0}^r a_j \otimes 1^{r-j} \in \ker \pi = \mathcal{R}_{\mathcal{J}}$. Hence $\ker \tilde{\pi} \subset \mathcal{R}_{\mathcal{J}}$. The inclusion $\mathcal{R}_{\mathcal{J}} \subset \ker \tilde{\pi}$ is obvious. $\square$

Corollary 7.4 (Gauge invariance theorem for $\mathcal{DR}(\mathcal{T})$). *A right tensor representation $\pi = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{T}$ coisometric on $\mathcal{T}$ integrates to the faithful representation of $\mathcal{DR}(\mathcal{T})$ if and only if π admits a gauge action and $\ker \pi = \mathcal{R}_{\mathcal{T}}$.*

Remark 7.5. The condition (7.3) implies that $\mathcal{J}$ is an ideal of coisometricity for π . In many natural situations the ideal of coisometricity automatically satisfies (7.3); cf. Corollary 7.9 below. On the level of Cuntz-Pimsner algebras, the condition (7.3) for $k = 1$ implies it for all $k > 0$; cf. Remark 7.15, and in fact it reduces to equality (9.1), the role of which was discovered by Katsura [19] and led him to the notion of a T -pair (see Proposition 7.14 and Definition 7.16 below).

Continuing the discussion undertaken below Theorem 6.9, we give a complete description of the gauge-invariant ideal structure of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$.

Theorem 7.6 (Lattice structure of gauge-invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$). *We have a one-to-one correspondence between the following objects:*

- i) kernels of right tensor representations $\tilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ coisometric on $\mathcal{K}_{\mathcal{J}}$;
- ii) invariant and $\mathcal{K}_{\mathcal{J}}$ -saturated ideals $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$, that is, ideals satisfying

$$\mathcal{N}_{\mathcal{J}} \otimes_{\mathcal{J}} 1 \subset \mathcal{N}_{\mathcal{J}} \quad \text{and} \quad (\otimes_{\mathcal{J}} 1)^{-1}(\mathcal{N}_{\mathcal{J}}) \subset \mathcal{N}_{\mathcal{J}};$$

- iii) gauge-invariant ideals P in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) = \mathcal{DR}(\mathcal{K}_{\mathcal{J}})$.

These correspondences preserve inclusions and are given by

$$\mathcal{N}_{\mathcal{J}} = \ker \tilde{\pi}, \quad \ker \Psi_{\tilde{\pi}} = \mathcal{O}(\mathcal{N}_{\mathcal{J}}) = P, \quad \mathcal{N}_{\mathcal{J}}(n, m) = \iota_{n,m}^{-1}(P), \quad n, m \in \mathbb{N},$$

where $\iota = \{\iota_{n,m}\}_{n,m \in \mathbb{N}}$ denotes the universal right tensor representation of $\mathcal{K}_{\mathcal{J}}$ in $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$ and $\mathcal{O}(\mathcal{N}_{\mathcal{J}})$ is the closed linear span of the image of $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$. In particular,

$$P \cong \mathcal{DR}(\mathcal{N}_{\mathcal{J}}), \quad P = \overline{\text{span}}\{\iota_{n,m}(a) : a \in \mathcal{N}_{\mathcal{J}}(n, m)\} \subset \mathcal{DR}(\mathcal{K}_{\mathcal{J}}),$$

and we have the lattice isomorphism $\text{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}}) \cong \text{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}))$.

Proof. If $\tilde{\pi}$ is a right tensor representation of $\mathcal{K}_{\mathcal{J}}$, then by Proposition 6.8, $\ker \tilde{\pi}$ is an invariant and $\mathcal{K}_{\mathcal{J}}$ -saturated ideal in $\mathcal{K}_{\mathcal{J}}$. Conversely, if $\mathcal{N}_{\mathcal{J}}$ is an invariant and $\mathcal{K}_{\mathcal{J}}$ -saturated ideal in $\mathcal{K}_{\mathcal{J}}$, then by Theorem 6.9 we have a gauge-invariant homomorphism $\Psi : \mathcal{DR}(\mathcal{K}_{\mathcal{J}}) \rightarrow \mathcal{DR}(\mathcal{K}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}})$ whose kernel is $\mathcal{O}(\mathcal{N}_{\mathcal{J}})$. Hence disintegrating Ψ we get a right tensor representation $\tilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ coisometric on $\mathcal{K}_{\mathcal{J}}$ such that $\mathcal{N}_{\mathcal{J}} = \ker \tilde{\pi}$. This proves the correspondence between the objects in i) and ii).

Let P be an arbitrary gauge-invariant ideal in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) = \mathcal{DR}(\mathcal{K}_{\mathcal{J}})$. The spaces $\mathcal{N}_{\mathcal{J}}(n, m) := \iota_{n, m}^{-1}(P)$, $n, m \in \mathbb{N}$, form an ideal $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$. By definition of the algebra $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$ we have $\iota_{n, m}(a) = \iota_{n+1, m+1}(a \otimes 1)$, for all $a \in \mathcal{K}_{\mathcal{J}}(n, m)$. Thus $\mathcal{N}_{\mathcal{J}}$ is both invariant and $\mathcal{K}_{\mathcal{J}}$ -saturated. Since $\mathcal{O}(\mathcal{N}_{\mathcal{J}}) \subset P$ the identity map factors through to a surjection

$$\Psi : \mathcal{DR}(\mathcal{K}_{\mathcal{J}})/\mathcal{O}(\mathcal{N}_{\mathcal{J}}) \longrightarrow \mathcal{DR}(\mathcal{K}_{\mathcal{J}})/P.$$

As the ideals $\mathcal{O}(\mathcal{N}_{\mathcal{J}})$ and P are gauge invariant the gauge action on $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$ factors through to gauge actions on $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})/\mathcal{O}(\mathcal{N}_{\mathcal{J}})$ and $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})/P$. The epimorphism Ψ intertwines these actions. In view of Theorem 6.9 we may identify $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})/\mathcal{O}(\mathcal{N}_{\mathcal{J}})$ with $\mathcal{DR}(\mathcal{K}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}})$ and then $\Psi(\iota_{n, m}(a + \mathcal{N}_{\mathcal{J}}(n, m))) = \iota_{n, m}(a) + P$, for $a \in \mathcal{K}_{\mathcal{J}}(n, m)$. Thus we see that Ψ is injective on every space $\iota_{n, m}(\mathcal{K}_{\mathcal{J}}(n, m) + \mathcal{N}_{\mathcal{J}}(n, m))$. Therefore Ψ is injective on spectral subspaces of $\mathcal{DR}(\mathcal{K}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}}) = \mathcal{DR}(\mathcal{K}_{\mathcal{J}})/\mathcal{O}(\mathcal{N}_{\mathcal{J}})$, and since it is gauge invariant, it is an isomorphism. Hence $\mathcal{O}(\mathcal{N}_{\mathcal{J}}) = P$. This together with Theorem 6.9 proves the correspondence between the objects in ii) and iii). $\square$

Corollary 7.7 (Lattice structure of gauge-invariant ideals in $\mathcal{DR}(\mathcal{T})$). *There is a lattice isomorphism $\text{Lat}_{\mathcal{T}}(\mathcal{T}) \cong \text{Lat}(\mathcal{DR}(\mathcal{T}))$ between the gauge-invariant ideals in $\mathcal{DR}(\mathcal{T})$ and invariant $\mathcal{T}$ -saturated ideals in $\mathcal{T}$. Moreover an ideal in $\text{Lat}(\mathcal{DR}(\mathcal{T}))$ corresponding to an ideal $\mathcal{N}$ in $\text{Lat}_{\mathcal{T}}(\mathcal{T})$ is isomorphic to $\mathcal{DR}(\mathcal{N})$.*

It follows from Propositions 6.5, 7.1 that for every invariant $\mathcal{K}_{\mathcal{J}}$ -saturated ideal $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$ the intersection $\mathcal{N} := \mathcal{K} \cap \mathcal{N}_{\mathcal{J}}$ (we use our identification $\mathcal{K} \subset \mathcal{K}_{\mathcal{J}}$ given by (7.1)) is a $\mathcal{K}$ -invariant and $\mathcal{J}$ -saturated ideal $\mathcal{N}$ in $\mathcal{K}$. Conversely, any $\mathcal{K}$ -invariant and $\mathcal{J}$ -saturated ideal $\mathcal{N}$ in $\mathcal{K}$ gives rise to the invariant and $\mathcal{K}_{\mathcal{J}}$ -saturated ideal $\mathcal{N}_{\mathcal{J}}[\mathcal{N}]$ in $\mathcal{K}_{\mathcal{J}}$, where

$$\mathcal{N}_{\mathcal{J}}[\mathcal{N}](r, r) := \left\{ \sum_{j=0}^r \iota_{j, j}^{\mathcal{M}}(a_j) : a_j \in \mathcal{N}(j, j) \right\}.$$

For an invariant and $\mathcal{K}_{\mathcal{J}}$ -saturated ideal $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$ we have $\mathcal{N}_{\mathcal{J}}[\mathcal{K} \cap \mathcal{N}_{\mathcal{J}}] \subset \mathcal{N}_{\mathcal{J}}$ but in general $\mathcal{N}_{\mathcal{J}}[\mathcal{K} \cap \mathcal{N}_{\mathcal{J}}] \neq \mathcal{N}_{\mathcal{J}}$. This is exactly the reason why the embedding $\text{Lat}_{\mathcal{J}}(\mathcal{K}) \hookrightarrow \text{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}}) \cong \text{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}))$ in general fails to be an isomorphism. We now introduce conditions under which the aforementioned obstacle vanishes.

Theorem 7.8. *If one of the following conditions holds,*

- i) $\mathcal{K} \subset \mathcal{J} + \ker \otimes 1$,
- ii) $\mathcal{K}$ admits a transfer homomorphism and $\mathcal{J} = J(\mathcal{K}) \cap (\ker(\otimes 1))^{\perp}$,

then every ideal in $\text{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}})$ has the form $\mathcal{N}_{\mathcal{J}}[\mathcal{N}]$, where $\mathcal{N} \in \text{Lat}_{\mathcal{J}}(\mathcal{K})$. As a consequence we get the lattice isomorphisms

$$\text{Lat}_{\mathcal{J}}(\mathcal{K}) \cong \text{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}}) \cong \text{Lat}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})).$$

Moreover, every gauge-invariant ideal P in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ is generated by the image of a $\mathcal{K}$ -invariant ideal $\mathcal{N}$ in $\mathcal{K}$ and $P \cong \mathcal{O}_{\mathcal{T}}(\mathcal{N}, \mathcal{J} \cap \mathcal{N})$.

Proof. Let $\mathcal{N}_{\mathcal{J}} \in \text{Lat}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}})$ and let $\mathcal{N} := \mathcal{K} \cap \mathcal{N}_{\mathcal{J}}$. By Theorem 2.6 (and definition of $\mathcal{K}_{\mathcal{J}}$) it suffices to show that an element a in $\mathcal{N}_{\mathcal{J}}(n, n)$ represented by $\sum_{j=0}^n \iota_{j,j}^{\mathcal{M}}(a_j)$, where $a_j \in \mathcal{K}(j, j)$, may also be represented by $\sum_{j=0}^n \iota_{j,j}^{\mathcal{M}}(b_j)$, where $b_j \in \mathcal{N}(j, j)$, $j = 0, \dots, n$. For abbreviation we shall write equality between a and its representatives.

i). Suppose that $\mathcal{K} \subset \mathcal{J} + \ker \otimes 1$. Then $a_0 = b_0 + j_0$, where $b_0 \in (\ker \otimes 1)(0, 0)$ and $j_0 \in \mathcal{J}(0, 0)$. Plainly $a = \iota_{0,0}^{\mathcal{M}}(b_0) + \iota_{1,1}^{\mathcal{M}}(j_0 \otimes 1 + a_1) + \sum_{j=2}^n \iota_{j,j}^{\mathcal{M}}(a_j)$. Since $j_0 \otimes 1 + a_1$ is in $\mathcal{K} \subset \mathcal{J} + \ker \otimes 1$ we have $j_0 \otimes 1 + a_1 = b_1 + j_1$, where $b_1 \in (\ker \otimes 1)(1, 1)$, $j_1 \in \mathcal{J}(1, 1)$ and then $a = \iota_{0,0}^{\mathcal{M}}(b_0) + \iota_{1,1}^{\mathcal{M}}(b_1) + \iota_{1,1}^{\mathcal{M}}(j_1 \otimes 1 + a_2) + \sum_{j=3}^n \iota_{j,j}^{\mathcal{M}}(a_j)$. Proceeding in this way one gets

$$a = \sum_{j=0}^n \iota_{j,j}^{\mathcal{M}}(b_j), \quad \text{where } b_j \in (\ker \otimes 1)(j, j), \quad j = 0, \dots, n-1, \quad b_n \in \mathcal{K}(n, n).$$

We denote by $\{\mu_{\lambda}^{(j)}\}_{\lambda}$ an approximate unit in $\mathcal{K}(j, j)$, $j = 0, \dots, n$. Using the form of multiplication in $\mathcal{K}_{\mathcal{J}}$ we obtain

$$\iota_{n,n}^{\mathcal{M}}(b_n) = \lim_{\lambda} \iota_{n,n}^{\mathcal{M}}(b_n) \star \iota_{n,n}^{\mathcal{M}}(\mu_{\lambda}^{(n)}) = \lim_{\lambda} a \star \iota_{n,n}^{\mathcal{M}}(\mu_{\lambda}^{(n)}) \in \mathcal{N}_{\mathcal{J}}(n, n),$$

that is, $b_n \in \mathcal{N}(n, n)$. Similar computations show that $\iota_{n-1,n-1}^{\mathcal{M}}(b_{n-1})$ equals

$$\lim_{\lambda} \left(a \star \iota_{n-1,n-1}^{\mathcal{M}}(\mu_{\lambda}^{(n-1)}) - \iota_{n,n}^{\mathcal{M}}(b_n(\mu_{\lambda}^{(n-1)} \otimes 1)) \right) \in \mathcal{N}_{\mathcal{J}}(n, n).$$

Hence, since $(\otimes_{\mathcal{J}} 1)^{-1}(\mathcal{N}_{\mathcal{J}}) \subset \mathcal{N}_{\mathcal{J}}$, we get $\iota_{n-1,n-1}^{\mathcal{M}}(b_{n-1}) \in \mathcal{N}_{\mathcal{J}}(n-1, n-1)$, that is, $b_{n-1} \in \mathcal{N}(n-1, n-1)$. Proceeding in this way one obtains $b_j \in \mathcal{N}(j, j)$, $j = 0, \dots, n$.

ii). Suppose that $\mathcal{J} = J(\mathcal{K}) \cap (\ker(\otimes 1))^{\perp}$ and $\mathcal{K}$ admits a transfer homomorphism $\mathcal{L}$. One sees that

$$a = \iota_{0,0}^{\mathcal{M}}(b_0), \quad \text{where } b_0 = \sum_{j=0}^n \mathcal{L}^{n-j}(a_j) \in \mathcal{K}(0, 0).$$

Since $(\otimes_{\mathcal{J}} 1)^{-1}(\mathcal{N}_{\mathcal{J}}) \subset \mathcal{N}_{\mathcal{J}}$ one gets $\iota_{0,0}^{\mathcal{M}}(b_0) \in \mathcal{N}_{\mathcal{J}}(0, 0)$, that is, $b_0 \in \mathcal{N}(0, 0)$ and the proof is complete. $\square$

Corollary 7.9. *If $\mathcal{K} \subset \mathcal{J} + \ker \otimes 1$, then a right tensor representation π of $\mathcal{K}$ coisometric on $\mathcal{J}$ integrates to the faithful representation of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ if and only if π is gauge-invariant, $\mathcal{J}$ is the ideal of coisometricity for π and $\ker \pi = \mathcal{R}_{\mathcal{J}}$.*

Corollary 7.10. *If $\mathcal{K}$ admits a transfer homomorphism and $\mathcal{J} = J(\mathcal{K}) \cap (\ker(\otimes 1))^{\perp}$, then a right tensor representation π of $\mathcal{K}$ coisometric on $\mathcal{J}$ integrates to the faithful representation of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ if and only if π is faithful and gauge-invariant.*

7.1. Ideal structure of relative Cuntz-Pimsner algebras. Now we show how a complete description of the gauge-invariant ideal structure of relative Cuntz-Pimsner algebras $\mathcal{O}(J, X)$ can be deduced from Theorems 7.6, 7.8. Hopefully it will shed more light on the results of [20]. We start with a useful statement which follows from Theorem 7.8 and contains, as particular cases, [20, Cor. 8.7, Thm. 10.6], [28, Thm. 6.4].

Theorem 7.11. *Let X be a C^* -correspondence over A and let J be an ideal in $J(X)$. If one of the following conditions holds,*

- i) $A = J + \ker \phi$,
- ii) X is a Hilbert bimodule and $J = J(X) \cap (\ker \phi)^\perp$,

then we have an isomorphism between the lattice of gauge-invariant ideals in $\mathcal{O}(J, X)$ and the lattice of X -invariant J -saturated ideals in A :

$$\mathcal{O}(J, X) \triangleright P \mapsto \iota_{0,0}^{-1}(P) \triangleleft A.$$

In particular, every gauge-invariant ideal P in $\mathcal{O}(J, X)$ is generated by the image of an X -invariant ideal I in A and then P is isomorphic to $\mathcal{O}_X(I, J \cap I)$ and Morita equivalent to $\mathcal{O}(J \cap I, XI)$. If the condition ii) holds we actually have $P \cong \mathcal{O}(J \cap I, XI)$.

Proof. If $A = J + \ker \phi$, then on the level of the C^* -precategory $\mathcal{T}_X$ we have $\mathcal{K}_X \subset \mathcal{K}_X(J) + \ker \otimes 1$. If X is a Hilbert bimodule and $J = J(X) \cap (\ker \phi)^\perp$, then by Proposition 5.8, $\mathcal{K}_X$ admits a transfer homomorphism and $\mathcal{K}_X(J) = J(\mathcal{K}_X) \cap (\ker \otimes 1)^\perp$. Moreover, for every X -invariant J -saturated ideal I in A we have $\phi(I)X = XI$; see [20, Prop. 10.2]. Thus it suffices to apply Theorems 7.8, 6.20. $\square$

A general result (without additional assumptions) requires a description of invariant and $\mathcal{K}_{\mathcal{J}}$ -saturated ideals in the right tensor C^* -precategory $\mathcal{K}_{\mathcal{J}}$, where

$$\mathcal{K} := \mathcal{K}_X \quad \text{and} \quad \mathcal{J} := \mathcal{K}_X(J).$$

Equivalently, by Theorem 7.6, we need to describe all the kernels of right tensor representations $\tilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ coisometric on $\mathcal{K}_{\mathcal{J}}$. We recall that each such representation $\tilde{\pi}$ is uniquely determined by a representation (π, t) of X coisometric on J ; see Proposition 3.13 and Proposition 7.1.

Proposition 7.12. *Let (π, t) be a faithful representation of X coisometric on J and let I' be the ideal of coisometricity for (π, t) (then automatically $J \subset I' \subset (\ker \phi)^\perp \cap J(X)$). The kernel of the corresponding representation $\tilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ is uniquely determined by I' . Namely, $\ker \tilde{\pi}(n, n)$, $n \in \mathbb{N}$, consists of elements that can be represented by $\sum_{k=0}^n \iota_{k,k}^{\mathcal{M}}(a_k)$, $a_k \in \mathcal{K}(X^{\otimes k})$, such that*

$$(7.4) \quad \sum_{k=0}^j a_k \otimes 1^{j-k} \in \mathcal{K}(X^{\otimes j} I'), \quad j = 0, \dots, n-1, \quad \sum_{k=0}^n a_k \otimes 1^{n-k} = 0.$$

Proof. Let $a = \sum_{k=0}^n \iota_{k,k}^{\mathcal{M}}(a_k) \in \mathcal{K}_{\mathcal{J}}(n, n)$, $a_k \in \mathcal{K}(X^{\otimes k})$, $k = 0, \dots, n$, $n \in \mathbb{N}$, be such that $\tilde{\pi}(a) = 0$. We consider (π, t) , and thereby also $\tilde{\pi}$, as a representation on a Hilbert space H and take advantage of the family $\{P_m\}_{m \in \mathbb{N}}$ of decreasing projections onto the essential subspaces of $\pi_{m,m}$; cf. page 1829. Within the notation of Proposition 3.15, we get

$$0 = \|\tilde{\pi}(a)\| = \max \left\{ \max_{j=0,1,\dots,n-1} \{\|\tilde{\pi}(a)(P_j - P_{j+1})\|\}, \|\tilde{\pi}(a)P_n\| \right\}$$

and thus

$$(7.5) \quad \bar{\pi}_{j,j} \left(\sum_{k=0}^j a_k \otimes 1^{j-k} \right) (P_j - P_{j+1}) = 0, \quad j = 1, \dots, n-1, \quad \bar{\pi}_{n,n} \left(\sum_{k=0}^n a_k \otimes 1^{n-k} \right) = 0.$$

Since π is faithful, $\{\bar{\pi}_{i,j}\}_{i,j \in \mathbb{N}}$ is faithful. Hence (7.5) implies that $\sum_{k=0}^n a_k \otimes 1^{n-k} = 0$ and each element $\bar{\pi}_{j,j} \left(\sum_{k=0}^j a_k \otimes 1^{j-k} \right)$ is supported on $P_{j+1}H$, $j = 0, \dots, n-1$. In particular, $\left(\sum_{k=0}^{n-1} a_k \otimes 1^{n-1-k} \right) \otimes 1 = \sum_{k=0}^n a_k \otimes 1^{n-k} - a_n \in \mathcal{K}(X^{\otimes n})$ and thus by Lemma 3.25 ii), $\sum_{k=0}^{n-1} a_k \otimes 1^{n-1-k} \in \mathcal{K}(X^{\otimes(n-1)}I')$. Similarly, if we assume that $\sum_{k=0}^{n-m} a_k \otimes 1^{n-m-k} \in \mathcal{K}(X^{\otimes(n-m)}I')$, for certain $m = 0, \dots, n-1$, then

$$\left(\sum_{k=0}^{n-m-1} a_k \otimes 1^{n-m-1-k} \right) \otimes 1 = \sum_{k=0}^{n-m} a_k \otimes 1^{n-m-k} - a_{n-m} \in \mathcal{K}(X^{\otimes(n-m-1)})$$

and hence by Lemma 3.25 ii), $\sum_{k=0}^{n-m-1} a_k \otimes 1^{n-m-1-k} \in \mathcal{K}(X^{\otimes(n-m-1)}I')$. Thus by induction, relations (7.4) are satisfied. $\square$

By passing to quotients we may use the above proposition to get a description of the kernel of $\tilde{\pi}$ in a general situation. To this end we use the following lemma.

Lemma 7.13. *Let (π, t) be a representation of X and let*

$$(7.6) \quad I = \ker \pi, \quad I' = \{a \in A : \pi(a) \in \pi_{1,1}(\mathcal{K}(X))\}.$$

Then representation (π, t) factors through to the faithful representation of X/XI for which the ideal of coisometricity is I'/I . In particular, the following relations hold:

$$(7.7) \quad I \text{ is } X\text{-invariant}, \quad I \subset I' \subset q^{-1}(J(X/XI) \cap (\ker \phi^I)^\perp),$$

where $q : A \rightarrow A/I$ is the quotient map and ϕ^I is the left action on X/XI .

Proof. See [20, Lem. 5.10] and Corollary 3.24. $\square$

Proposition 7.14. *If (π, t) is a representation of X coisometric on J , then the kernel of the corresponding right tensor representations $\tilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ is uniquely determined by the ideals*

$$I = \ker \pi \quad \text{and} \quad I' = \{a \in A : \pi(a) \in \pi_{1,1}(\mathcal{K}(X))\}.$$

Namely, $\ker \tilde{\pi}(n, n)$, $n \in \mathbb{N}$, consists of those elements that can be represented in a form $\sum_{k=0}^n \nu_{k,k}^M(a_k)$, $a_k \in \mathcal{K}(X^{\otimes k})$, $k = 0, \dots, n$, where

$$\sum_{k=0}^j a_k \otimes 1^{j-k} \in \mathcal{K}(X^{\otimes j}I'), \quad j = 0, \dots, n-1, \quad \sum_{k=0}^n a_k \otimes 1^{n-k} \in \mathcal{K}(X^{\otimes n}I).$$

Proof. Pass from (π, t) to the (faithful) quotient representation $(\dot{\pi}, \dot{t})$ of the quotient C^* -correspondence X/XI over A/I . By Proposition 6.13 this corresponds to passing from the right tensor representation $\{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}$ to the (faithful) right tensor representation $\{\dot{\pi}_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}/\mathcal{T}_X(I) = \mathcal{K}_{X/XI}$. Applying Proposition 7.12 to representation $(\dot{\pi}, \dot{t})$ we get the assertion by Lemma 7.13. $\square$

Remark 7.15. Using the above statement one may deduce that for a representation $[\pi, t] = \{\pi_{n,m}\}_{n,m \in \mathbb{N}}$ of $\mathcal{K}_X$, all $n \in \mathbb{N}$ and $k > 0$ we have

$$\pi_{n,n}^{-1} \left(\sum_{j=1}^k \pi_{n+j,n+j}(\mathcal{K}_X(n+j, n+j)) \right) = \pi_{n,n}^{-1}(\pi_{n+1,n+1}(\mathcal{K}_X(n+1, n+1))),$$

and this space is equal to $\mathcal{K}(X^{\otimes n} I')$. Hence for right tensor representations of $\mathcal{K}_X$ it suffices to check the condition (7.3) only for $k = 1$.

We adapt the notion of coisometricity to the notion of a T -pair [20, Def. 5.6].

Definition 7.16. A pair (I, I') of ideals in A satisfying (7.7) is called a T -pair of X . We shall say that a T -pair (I, I') is *coisometric* on an ideal J in A , if $J \subset I'$. In particular, an O -pair introduced in [20, Def. 5.21] is simply a T -pair coisometric on $J = (\ker \phi)^\perp \cap J(X)$.

Let us note that if (I, I') is a T -pair coisometric on J , then I is automatically J -saturated. Indeed, in view of Lemma 7.13 we have

$$J \subset q^{-1}(J(X/XI) \cap (\ker \phi^I)^\perp) \implies J \cap \varphi^{-1}(\mathcal{K}(XI)) \subset I.$$

Furthermore, T -pairs form a lattice with the natural order induced by inclusion.

Theorem 7.17 (Lattice structure of gauge-invariant ideals in $\mathcal{O}(J, X)$; cf. [20]).
We have lattice isomorphisms between the following objects:

- i) T -pairs (I, I') of X coisometric on J ,
- ii) invariant and $\mathcal{K}_{\mathcal{J}}$ -saturated ideals $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$,
- iii) gauge-invariant ideals P in $\mathcal{O}(J, X)$.

The correspondences between the objects in ii) and iii), and i) and ii) are respectively given by the equality $\mathcal{O}(\mathcal{N}_{\mathcal{J}}) = P$, and the equivalence: $a \in \mathcal{K}_{\mathcal{J}}(n, n)$ is in $\mathcal{N}_{\mathcal{J}}(n, n)$, $n \in \mathbb{N}$, iff it may be represented by $\sum_{k=0}^n \iota_{k,k}(a_k)$, where

$$\sum_{k=0}^j a_k \otimes 1^{j-k} \in \mathcal{K}(X^{\otimes j} I'), \quad j = 0, \dots, n-1, \quad \sum_{k=0}^n a_k \otimes 1^{n-k} \in \mathcal{K}(X^{\otimes n} I).$$

Moreover, we have

$$P \cong \mathcal{DR}(\mathcal{N}_{\mathcal{J}}), \quad \mathcal{O}(J, X)/P \cong \mathcal{O}(I'/I, X/XI).$$

Proof. In view of Theorem 7.6 and Proposition 7.14 it suffices to show that the pair of ideals (I, I') from item i) define (via the described equivalence) an invariant and $\mathcal{K}_{\mathcal{J}}$ -saturated ideal $\mathcal{N}_{\mathcal{J}}$ in $\mathcal{K}_{\mathcal{J}}$. To this end note that by (7.7) and Corollary 4.16 the universal representation of X/XI in $\mathcal{O}(I'/I, X/XI)$ is faithful. Composing this representation with quotient maps $X \rightarrow X/XI$, $A \rightarrow A/I$ one gets the representation (π, t) of X such that relations (7.6) are satisfied. The pair (π, t) gives rise to the representation $\tilde{\pi}$ of $\mathcal{K}_{\mathcal{J}}$ whose kernel (by Proposition 7.14) is the desired ideal $\mathcal{N}_{\mathcal{J}}$. Using Theorem 7.3 we get $\mathcal{O}(J, X)/P \cong \mathcal{O}(I'/I, X/XI)$. $\square$

Remark 7.18. The ideals we identified in Theorem 6.20 as algebras of the form $\mathcal{O}_X(I, J \cap I)$ are exactly these gauge-invariant ideals in $\mathcal{O}(J, X)$ that correspond to T -pairs (I, I') , where $I' = I + J$. As we noticed in Theorem 7.11, in many situations all gauge-invariant ideals are of this form.

8. EMBEDDING CONDITIONS FOR $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$

We fix a right tensor C^* -precategory $\mathcal{T}$ and exhibit conditions for algebras of type $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ to be embedded into one another via universal representations. These results have a similar motivation as [14, Prop. 3.2], [11, Prop. 6.3] and shall be applied in Section 9 to algebras associated with C^* -correspondences. They also may be viewed as a description of certain gauge-invariant subalgebras of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. We start with

Proposition 8.1 (Necessary conditions). *For $j = 1, 2$, let $\mathcal{K}_j$ and $\mathcal{J}_j$ be ideals in $\mathcal{T}$ such that $\mathcal{J}_j \subset J(\mathcal{K}_j)$ and $\mathcal{K}_1 \subset \mathcal{K}_2$. Denote by $\{\iota_{n,m}^{(j)}\}_{m,n \in \mathbb{N}}$ the universal representations of $\mathcal{K}_j$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_j, \mathcal{J}_j)$, $j = 1, 2$. The natural homomorphism $\Psi : \mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J}_1) \mapsto \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J}_2)$,*

$$(8.1) \quad \mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J}_1) \ni \iota_{n,m}^{(1)}(a) \xrightarrow{\Psi} \iota_{n,m}^{(2)}(a) \in \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J}_2),$$

is well defined if and only if $\mathcal{J}_1 \subset \mathcal{J}_2$. Moreover, if Ψ is well defined and injective, then

$$\mathcal{J}_1 = \mathcal{J}_2 \cap J(\mathcal{K}_1), \quad \mathcal{R}_{\mathcal{J}_2} \cap \mathcal{K}_1 = \mathcal{R}_{\mathcal{J}_1},$$

where $\mathcal{R}_{\mathcal{J}_j}$ is the reduction ideal associated with $\mathcal{J}_j$, $j = 1, 2$; see Definition 6.10.

Proof. If $\mathcal{J}_1 \subset \mathcal{J}_2$, then Ψ is well defined by the construction of the norm in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_j, \mathcal{J}_j)$, $j = 1, 2$; see Theorem 4.12. Conversely, if Ψ is well defined, then for $a \in J(\mathcal{K}_1)(n, m)$ we have

$$\iota_{n,m}^{(1)}(a) = \iota_{n+1,m+1}^{(1)}(a \otimes 1) \implies \iota_{n,m}^{(2)}(a) = \iota_{n+1,m+1}^{(2)}(a \otimes 1),$$

that is, $\mathcal{J}_1 \subset \mathcal{J}_2$. In the event that Ψ is injective the above implication is an equivalence and thus we have $\mathcal{J}_1 = \mathcal{J}_2 \cap J(\mathcal{K}_1)$. Moreover, representations $\{\iota_{n,m}^{(1)}\}_{m,n \in \mathbb{N}}$ and $\{\iota_{n,m}^{(2)}|_{\mathcal{K}_1}\}_{m,n \in \mathbb{N}} = \{\Psi \circ \iota_{n,m}^{(1)}\}_{m,n \in \mathbb{N}}$ have the same kernels and hence we get $\mathcal{R}_{\mathcal{J}_2} \cap \mathcal{K}_1 = \mathcal{R}_{\mathcal{J}_1}$. $\square$

By the above statement we may narrow our attention down to algebras $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1))$ and $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J})$, where $\mathcal{K}_1 \subset \mathcal{K}_2$ and $\mathcal{J} \subset J(\mathcal{K}_2)$.

Proposition 8.2 (Sufficient conditions on ideals $\mathcal{K}_1$ and $\mathcal{J}$). *Let $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{J}$ be ideals in $\mathcal{T}$ such that $\mathcal{K}_1 \subset \mathcal{K}_2$ and $\mathcal{J} \subset J(\mathcal{K}_2)$. The condition*

$$(8.2) \quad \mathcal{K}_1 \cap J(\mathcal{J}) \subset J(\mathcal{K}_1)$$

(which holds e.g. whenever $\mathcal{J} \subset \mathcal{K}_1$, $\mathcal{J} \subset \mathcal{K}_1^\perp$ or $\mathcal{K}_1$ is invariant) implies that there is a natural embedding

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1)) \subset \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J}).$$

Proof. Notice that $\mathcal{K}_1 \cap J(\mathcal{J}) \subset J(\mathcal{K}_1)$ is equivalent to $\mathcal{J} \cap \mathcal{K}_1 \cap (\otimes 1)^{-1}(\mathcal{J} + \mathcal{K}_1) \subset J(\mathcal{K}_1)$. Indeed, on the one hand, we have $\mathcal{K}_1 \cap J(\mathcal{J}) \subset \mathcal{J} \cap \mathcal{K}_1 \cap (\otimes 1)^{-1}(\mathcal{J} + \mathcal{K}_1)$, and on the other hand, if $a \in \mathcal{J} \cap \mathcal{K}_1 \cap (\otimes 1)^{-1}(\mathcal{J} + \mathcal{K}_1)(n, m)$, then either $a \in J(\mathcal{K}_1)$ or $a \in \mathcal{K}_1 \cap J(\mathcal{J})$: for $a \in \mathcal{T}(n, m)$, $m, n \in \mathbb{N}$, we have

$$(8.3) \quad a \in (\mathcal{K}_1 \cap \mathcal{J}) \text{ and } a \otimes 1 \in (\mathcal{K}_1 + \mathcal{J}) \implies a \in J(\mathcal{K}_1).$$

We show that Ψ given by (8.1) is faithful on the core (and hence on all of the spectral subspaces) of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1))$. To this end let $a \in \mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1))$ be of the

form $a = \sum_{j \in \mathbb{N}} \iota_{j,j}(a_j)$, $a_j \in \mathcal{K}_1(j, j)$, $j \in \mathbb{N}$, and suppose that $\|\Psi(a)\| = 0$. Then by Theorem 4.14 we have

$$\sum_{j=0}^s a_j \otimes 1^{s-j} \in \mathcal{J}(s, s), \quad s \in \mathbb{N}, \quad \lim_{r \rightarrow \infty} \sum_{j=0}^r a_j \otimes 1^{r-j} = 0.$$

Since $a_0 \in (\mathcal{K}_1 \cap \mathcal{J})(0, 0)$ and $a_0 \otimes 1 = (a_0 \otimes 1 + a_1) - a_1 \in (\mathcal{K}_1 + \mathcal{J})(1, 1)$, by (8.3) we get $a_0 \in J(\mathcal{K}_1)(1, 1)$ and consequently $a_0 \otimes 1 + a_1 \in (\mathcal{J} \cap \mathcal{K}_1)(1, 1)$. Suppose now that $\sum_{j=0}^s a_j \otimes 1^{s-j} \in (\mathcal{J} \cap \mathcal{K}_1)(s, s)$ for certain $s \in \mathbb{N}$. Since

$$\left(\sum_{j=0}^s a_j \otimes 1^{s-j} \right) \otimes 1 = \sum_{j=0}^{s+1} a_j \otimes 1^{s+1-j} - a_{s+1} \in (\mathcal{J} + \mathcal{K})(s+1, s+1),$$

by (8.3) we then have $\sum_{j=0}^s a_j \otimes 1^{s-j} \in J(\mathcal{K}_1)(s+1, s+1)$ and consequently $\sum_{j=0}^{s+1} a_j \otimes 1^{s+1-j} \in (\mathcal{J} \cap \mathcal{K}_1)(s+1, s+1)$. Thus by induction we get

$$\sum_{j=0}^s a_j \otimes 1^{s-j} \in (\mathcal{J} \cap J(\mathcal{K}_1))(s, s), \quad s \in \mathbb{N}, \quad \lim_{r \rightarrow \infty} \sum_{j=0}^r a_j \otimes 1^{r-j} = 0,$$

which in view of Theorem 4.14 is equivalent to $\|a\| = 0$.

Clearly, $\Psi : \mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1)) \rightarrow \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J})$ preserves the gauge actions, and hence the injectivity of Ψ on spectral subspaces implies the injectivity of Ψ on the whole algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1))$. $\square$

Corollary 8.3. *For any ideals $J \subset J(X)$ and I in A such that*

$$(8.4) \quad I \cap J(XJ) \subset J(XI)$$

(which holds e.g. whenever $J \subset I$, $J \subset I^\perp$ or I is X -invariant) we have the natural embedding

$$\mathcal{O}_X(I, J \cap J(XI)) \subset \mathcal{O}(J, X).$$

Proof. Apply Proposition 8.2 to $\mathcal{K}_2 = \mathcal{K}_X$, $\mathcal{K}_1 = \mathcal{K}_X(I)$ and $\mathcal{J} = \mathcal{K}_X(J)$. $\square$

We notice that if I is X -invariant, then for $J \subset J(X)$ we have $J \cap J(XI) = J \cap I$ and thereby the foregoing statement implies the inclusion $\mathcal{O}_X(I, J \cap I) \subset \mathcal{O}(J, X)$ from Theorem 6.20. As the next example shows, the condition (8.2), or more precisely its special case (8.4), is essential.

Example 8.4. Suppose $A = A_0 \oplus A_0$, where A_0 is a unital C^* -algebra. Consider the ideals $J = A_0 \oplus A_0$, $I = A_0 \oplus \{0\}$ and the C^* -correspondence $X = X_\alpha$ associated with the endomorphism $\alpha : A \rightarrow A$ given by the formula $\alpha(a, b) = (0, a)$. Then we have

$$I \cap J(XJ) = A_0 \oplus \{0\} \not\subset J(XI) = \{0\} \oplus A_0,$$

so the inclusion (8.4) fails. On the other hand the algebra $\mathcal{O}_X(I, J \cap J(XI))$ cannot be embedded into $C^*(J, X)$ as we have $C^*(J, X) = \{0\}$ and $\mathcal{O}_X(I, J \cap J(XI)) = I$ (the latter relation may be checked using, for instance, Theorem 4.14).

Let us now exploit the condition $\mathcal{R}_{\mathcal{J}} \cap \mathcal{K} = \mathcal{R}_{\mathcal{J} \cap J(\mathcal{K}_1)}$ introduced in Proposition 8.1. It is evident that $\mathcal{J} \subset \mathcal{K}_1$ implies $\mathcal{R}_{\mathcal{J}} = \mathcal{R}_{\mathcal{J} \cap J(\mathcal{K}_1)}$ (it follows from Propositions 8.1, 8.2). Moreover, we have

$$\mathcal{J} \subset (\ker \otimes 1)^\perp \implies \mathcal{R}_{\mathcal{J}} = \mathcal{R}_{\mathcal{J} \cap J(\mathcal{K}_1)} = \{0\}.$$

Thus for C^* -algebras generalizing Katsura's algebras of C^* -correspondences (cf. Remark 4.4) the condition $\mathcal{R}_{\mathcal{J}} = \mathcal{R}_{\mathcal{J} \cap J(\mathcal{K}_1)}$ is trivially satisfied.

Proposition 8.5 (Sufficient conditions on ideal $\mathcal{K}_1$). *Let $\mathcal{K}_1$ be an ideal in $\mathcal{T}$ such that*

$$(8.5) \quad \otimes^{-1}(\mathcal{K}_1) \cap (\ker \otimes 1)^\perp \subset \mathcal{K}_1.$$

Then for every ideal $\mathcal{K}_2$ in $\mathcal{T}$ and every ideal $\mathcal{J}$ in $J(\mathcal{K}_2)$ the natural homomorphism (8.1) establishes the embedding

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1)) \subset \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J})$$

if and only if $\mathcal{R}_{\mathcal{J}} \cap \mathcal{K}_1 = \mathcal{R}_{\mathcal{J} \cap J(\mathcal{K}_1)}$.

Proof. The “only if” part follows from Proposition 8.1. Let us assume that $\mathcal{R}_{\mathcal{J}} \cap \mathcal{K}_1 = \mathcal{R}_{\mathcal{J} \cap J(\mathcal{K}_1)}$. By the reduction procedure described in Theorem 6.11 applied to $\mathcal{J}$ we may actually assume that $\mathcal{R}_{\mathcal{J}} = \{0\}$ and $\mathcal{J} \subset (\ker \otimes 1)^\perp$. It suffices to prove that if $a \in \mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J} \cap J(\mathcal{K}_1))$ is such that $a = \sum_{s=0}^r \iota_{s,s}(a_s)$, $a_s \in \mathcal{K}_1(s, s)$, $s = 0, \dots, r$, and $\|\Psi(a)\| = 0$, then $\|a\| = 0$. In view of Theorem 4.14 the requirement $\|\Psi(a)\| = 0$ is equivalent to

$$(8.6) \quad \sum_{j=0}^s a_j \otimes 1^{s-j} \in \mathcal{J}(s, s), \quad s = 0, \dots, r-1, \quad \sum_{j=0}^r a_j \otimes 1^{r-j} = 0.$$

To show that $\|a\| = 0$ we need to check whether

$$(8.7) \quad \sum_{j=0}^s a_j \otimes 1^{s-j} \in (\mathcal{J} \cap J(\mathcal{K}_1))(s, s), \quad s = 0, \dots, r-1, \quad \sum_{j=0}^r a_j \otimes 1^{r-j} = 0.$$

However, since

$$\left(\sum_{j=0}^{r-1} a_j \otimes 1^{r-1-j} \right) \otimes 1 = \sum_{j=0}^r a_j \otimes 1^{r-j} - a_r = -a_r \in \mathcal{K}_1(r, r),$$

it follows that $\sum_{j=0}^{r-1} a_j \otimes 1^{r-1-j} \in \left(\otimes^{-1}(\mathcal{K}_1) \cap (\ker \otimes 1)^\perp \right)(r-1, r-1) \subset J(\mathcal{K}_1)(r-1, r-1)$. Analogously, if $\sum_{j=0}^s a_j \otimes 1^{s-j} \in J(\mathcal{K}_1)(s, s)$, $s = 0, \dots, r-1$, then

$$\left(\sum_{j=0}^{s-1} a_j \otimes 1^{s-1-j} \right) \otimes 1 = \sum_{j=0}^s a_j \otimes 1^{s-j} - a_{s,s} \in \mathcal{K}_1(r, r)$$

and $\sum_{j=0}^{s-1} a_j \otimes 1^{s-1-j} \in \left(\otimes^{-1}(\mathcal{K}_1) \cap (\ker \otimes 1)^\perp \right)(s-1, s-1) \subset J(\mathcal{K}_1)(s-1, s-1)$. Hence by induction (8.7) holds. $\square$

Item iii) of Lemma 1.9 states that the ideal $\mathcal{K}_X$ in $\mathcal{T}_X$ satisfies (8.5), and thus we get

Proposition 8.6. *Let $\mathcal{T} = \mathcal{T}_X$ be a right tensor C^* -precategory of a C^* -correspondence X and let $\mathcal{S}$ and $\mathcal{J}$ be ideals in $\mathcal{T}$ such that $\mathcal{J} \subset J(\mathcal{K}_2)$ and $\mathcal{K}_X \subset \mathcal{K}_2$. If we put $J := \mathcal{J}(0, 0)$, then the natural homomorphism is an embedding*

$$\mathcal{O}(J \cap J(X), X) \subset \mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J})$$

if and only if $R_J = R_{J \cap J(X)}$, where R_J and $R_{J \cap J(X)}$ are reducing ideals associated to J and $J \cap J(X)$ respectively; see Definition 6.22.

In view of Proposition 5.1, the algebra $\mathcal{O}_{\mathcal{T}_X}(\mathcal{T}_X, \mathcal{T}_X)$ coincides with the *Doplicher-Roberts algebra of a C^* -correspondence* investigated in [11], [14]. It is natural to consider the following “relative version” of such algebras.

Definition 8.7. Suppose X is a C^* -correspondence over A and J is an arbitrary ideal in A . We shall call the C^* -algebra

$$\mathcal{DR}(J, X) := \mathcal{O}_{\mathcal{T}_X}(\mathcal{T}_X, \mathcal{T}_X(J))$$

a *relative Doplicher-Roberts algebra of X relative to J* . Within this notation the algebra considered in [11], [14] is $\mathcal{DR}(A, X)$.

Now Proposition 8.6 can be interpreted as the following generalization of [11, Prop. 6.3], [14, Prop. 3.2].

Corollary 8.8. *The natural homomorphism is an embedding*

$$\mathcal{O}(J \cap J(X), X) \subset \mathcal{DR}(J, X)$$

if and only if $R_J = R_{J \cap J(X)}$. In particular, $\mathcal{O}(J \cap J(X), X)$ embeds into $\mathcal{DR}(J, X)$ whenever $J \subset (\ker \phi)^\perp$ or $J \subset J(X)$.

Example 8.9. Let A_0 be a nonunital C^* -algebra and A_0^+ its minimal unitization. Let us consider the C^* -correspondence $X = A_0 \oplus A_0^+$ over $A = A_0^+ \oplus A_0^+$, where $\langle x, y \rangle_A := x^*y$, $x \cdot a = xa$ and $a \cdot x = \alpha(a)x$ and where $\alpha(a_0 \oplus b_0) = 0 \oplus a_0$. Then $J(X) = A_0^+ \oplus A_0^+$ and for the ideal $J = A$ we have

$$R_J = A_0^+ \oplus A_0^+ \neq A_0 \oplus A_0^+ = R_{J \cap J(X)}.$$

On the other hand, $\mathcal{O}(J \cap J(X), X) \cong \mathbb{C}$ and $\mathcal{DR}(J, X) = \{0\}$.

9. APPLICATION TO DOPLICHER-ROBERTS ALGEBRAS ASSOCIATED WITH C^* -CORRESPONDENCES

Our study in this section is motivated by [11], [14] and our aim is to generalize [11, Thm. 6.6], [14, Thm. 4.1]. We recall that there is a one-to-one correspondence, given by relations

$$\pi = \Psi \circ \iota_{0,0}|_A, \quad t = \Psi \circ \iota_{1,0}|_X,$$

between representations Ψ of the algebra $\mathcal{O}(J, X)$ and representations (π, t) of X coisometric on J . The inverse of this correspondence is given by the equality $\Psi = \Psi_{[\pi, t]}$, where $[\pi, t]$ is the right tensor representation of $\mathcal{K}_X$ defined in Proposition 3.13. For brevity we denote the representation of $\mathcal{O}(J, X)$ corresponding to (π, t) by $\pi \times_J t$. We arrive at the following statement; cf. [20, Cor. 11.7], [28, Thm. 5.1], [19, 6.4].

Theorem 9.1 (Gauge invariance theorem for $\mathcal{O}(J, X)$). *Let (π, t) be a representation of a C^* -correspondence X coisometric on an ideal $J \subset J(X)$. Then $\pi \times_J t$ is a faithful representation of $\mathcal{O}(J, X)$ if and only if (π, t) admits a gauge action, $\ker \pi = R_J$ and*

$$(9.1) \quad J = \{a \in A : \pi(a) \in \pi_{1,1}(\mathcal{K}(X))\}.$$

In particular,

- i) *if $J \subset (\ker \varphi)^\perp$, then $\pi \times_J t$ is faithful if and only if (π, t) is faithful, admits a gauge action and J is an ideal of coisometricity for (π, t) ;*
- ii) *if $J + (\ker \varphi)^\perp = A$, then $\pi \times_J t$ is faithful if and only if (π, t) admits a gauge action, $\ker \pi = R_J$ and J is an ideal of coisometricity for (π, t) .*

Proof. For the first part use Theorem 7.3 and Proposition 7.14. To see items i), ii) apply Proposition 7.12 and Theorem 7.11. $\square$

Remark 9.2. One could state the foregoing theorem in a somewhat more general form similar to Theorem 7.3. Namely, if in the above statement one drops the requirement of admitting a gauge action, one gets necessary and sufficient conditions for $\pi \times_J t$ to be faithful on spectral subspaces of $\mathcal{O}(J, X)$.

If (π, t) is a representation on a Hilbert space H , then $[\pi, t] = \{\pi_{n,m}\}_{m,n \in \mathbb{N}}$ extends to a right tensor representation $[\overline{\pi}, \overline{t}] = \{\overline{\pi}_{n,m}\}_{m,n \in \mathbb{N}}$ of $\mathcal{T}_X$; see Proposition 3.9. Therefore, for an appropriately chosen ideal J , $[\overline{\pi}, \overline{t}]$ integrates to a representation of the relative Doplicher-Roberts algebra $\mathcal{DR}(J, X)$ that we shall denote by $\overline{\pi \times_J t}$. By Proposition 3.15 we have

Proposition 9.3. *Let J be an ideal in A . A representation Ψ of $\mathcal{DR}(J, X)$ on a Hilbert space H is of the form $\overline{\pi \times_J \pi}$ for a representation (π, t) of X if and only if*

$$\Psi(\iota_{n,n}(\mathcal{L}(X^{\otimes n})))H = \Psi(\iota_{n,n}(\mathcal{K}(X^{\otimes n})))H, \quad n \in \mathbb{N},$$

where $\{\iota_{n,m}\}_{m,n \in \mathbb{N}}$ is the universal representation of $\mathcal{T}_X$ in $\mathcal{DR}(J, X)$. If this is the case and additionally $R_J = R_{J_0}$, where $J_0 = J \cap J(X)$, then $\pi \times_J \pi$ is an extension of $\pi \times_{J_0} \pi$ (as we then have $\mathcal{O}(J_0, X) \subset \mathcal{DR}(J, X)$; cf. Corollary 8.8).

We note that

$$\begin{aligned} \{a \in A : \overline{\pi}_{1,1}(\phi(a)) = \pi(a)\} &= \{a \in A : \pi(a)P_1 = \pi(a)\} \\ &= \{a \in A : \pi(a) \in \overline{\pi}_{1,1}(\mathcal{L}(X))\} = \{a \in A : \pi(a)H \subset \overline{t(X)H}\}; \end{aligned}$$

cf. Lemma 3.25 i) or [11, Lem. 1.9], so the forthcoming results may be stated without use of the mapping $\overline{\pi}_{1,1}$. We shall however keep using it as it indicates the relationship with coisometricity for C^* -correspondences. The following statement generalizes [11, Thm. 6.6].

Theorem 9.4 (Extending representations from $\mathcal{O}(J_0, X)$ to $\mathcal{DR}(J, X)$). *Let (π, t) be a representation of X on a Hilbert space H and let J be an ideal in A such that*

$$J \subset \{a \in A : \overline{\pi}_{1,1}(\phi(a)) = \pi(a)\}, \quad \text{and} \quad R_J = R_{J_0}, \quad \text{where} \quad J_0 = J \cap J(X).$$

Then $\mathcal{O}(J_0, X) \subset \mathcal{DR}(J, X)$ and the representation $\overline{\pi \times_J t}$ of $\mathcal{DR}(J, X)$ is an extension of the representation $\pi \times_{J_0} t$ of $\mathcal{O}(J_0, X)$ and

- i) $\overline{\pi \times_J t}$ is faithful on spectral subspaces of $\mathcal{DR}(J, X)$ if and only if $\pi \times_{J_0} t$ is faithful on spectral subspaces of $\mathcal{O}(J_0, X)$ and $J = \{a \in A : \overline{\pi}_{1,1}(\phi(a)) = \pi(a)\}$;
- ii) $\pi \times_J t$ is faithful if and only if $\pi \times_{J_0} t$ is faithful and $J = \{a \in A : \overline{\pi}_{1,1}(\phi(a)) = \pi(a)\}$.

Proof. i). Since $\pi \times_{J_0} t$ is considered as a restriction of $\overline{\pi \times_J t}$, faithfulness of $\overline{\pi \times_J t}$ on spectral subspaces implies such a faithfulness of $\pi \times_{J_0} t$, and then we get $J = \{a \in A : \overline{\pi}_{1,1}(\phi(a)) = \pi(a)\}$ by Proposition 4.17. Conversely, suppose that $J = \{a \in A : \overline{\pi}_{1,1}(\phi(a)) = \pi(a)\}$ and $\pi \times_{J_0} t$ is faithful on spectral subspaces of $\mathcal{O}(J_0, X)$. In this event $\ker \pi = R_J = R_{J_0}$ and hence by passing to quotients (dividing all the associated C^* -precategories by $\mathcal{N} = \mathcal{T}_X(\ker \pi)$) we may assume that $\ker \pi = 0$. Then, by Proposition 3.15, representation $\{\overline{\pi}_{i,j}\}_{i,j \in \mathbb{N}}$ is faithful. To show that $\overline{\pi \times_J t}$ is faithful, let $a = \sum_{s=0}^r \iota_{s,s}(a_s)$, where $a_s \in \mathcal{L}(X^{\otimes s})$, $s = 0, \dots, r$,

$r \in \mathbb{N}$, are such that $\overline{\pi \times_J t}(a) = 0$. Similarly as in the proof of Proposition 7.12 we get

$$\overline{\pi}_{s,s} \left(\sum_{j=0}^s a_j \otimes 1^{s-j} \right) (P_s - P_{s+1}) = 0, \quad s = 1, \dots, r-1, \quad \overline{\pi}_{r,r} \left(\sum_{j=0}^r a_j \otimes 1^{r-j} \right) = 0.$$

In particular, $\sum_{j=0}^r a_j \otimes 1^{r-j} = 0$ and each operator $\overline{\pi}_{s,s} \left(\sum_{j=0}^s a_j \otimes 1^{s-j} \right)$ is supported on $P_{s+1}H$, $s = 0, \dots, r-1$. Thus applying inductively Lemma 3.25 i) we get

$$\sum_{j=0}^s a_j \otimes 1^{s-j} \in \mathcal{L}(X^{\otimes s}, X^{\otimes s} J), \quad s = 1, \dots, r-1, \quad \sum_{j=0}^r a_j \otimes 1^{r-j} = 0,$$

which is equivalent to $\|a\| = 0$.

ii). If $\overline{\pi \times_J t}$ is faithful, then $\pi \times_{J_0} t$ is faithful and $J = \{a \in A : \overline{\pi}_{1,1}(\phi(a)) = \pi(a)\}$ by Proposition 4.17. Conversely, if $J = \{a \in A : \overline{\pi}_{1,1}(\phi(a)) = \pi(a)\}$ and $\pi \times_{J_0} t$ is faithful, then $\overline{\pi \times_J t}$ is faithful on spectral subspaces of $\mathcal{DR}(J, X)$ by item i). Thus it suffices to show that

$$(9.2) \quad \|\overline{\pi \times_J t}(E(a))\| \leq \|\overline{\pi \times_J t}(a)\|, \quad a \in \mathcal{DR}(J, X),$$

where E is the conditional expectation onto the 0-spectral subspace of $\mathcal{DR}(J, X)$; see [13, Thm 4.2]. For that purpose we note three facts. First, we have

$$(9.3) \quad \|\overline{\pi \times_J t}(E(a))\| \leq \|\overline{\pi \times_J t}(a)\|, \quad a \in \mathcal{O}(J_0, X) \subset \mathcal{DR}(J, X),$$

(a weaker version of (9.2)) because $\overline{\pi \times_J t}$ is an extension of $\pi \times_{J_0} t$ and $\pi \times_{J_0} t$ is faithful. Secondly, for any $a \in \mathcal{DR}(J, X)$ and any $r \in \mathbb{N}$ we have

$$(9.4) \quad \|\overline{\pi \times_J t}(a)\| = \max \left\{ \max_{s=0,1,\dots,r-1} \{ \|\overline{\pi \times_J t}(a)(P_s - P_{s+1})\| \}, \|\overline{\pi \times_J t}(a)P_r\| \right\},$$

where $\{P_m\}_{m \in \mathbb{N}}$ is the family of decreasing projections described on page 1829. Third, it is enough to prove (9.2) for elements $a \in \mathcal{DR}(J, X)$ of the form

$$(9.5) \quad a = \sum_{k=-\infty}^{\infty} \sum_{\substack{j=0, \\ j+k \geq 0}}^r \iota_{j+k,j}(a_{j+k,j}), \quad a_{j+k,j} \in \mathcal{L}(X^{\otimes j}, X^{\otimes j+k}), \quad r \in \mathbb{N},$$

as they form a dense subspace of $\mathcal{DR}(J, X)$.

Hence we fix an element a of the form (9.5), $\varepsilon > 0$ and $\xi \in (P_s - P_{s+1})H$, where $s = 0, 1, \dots, r-1$. Since $\pi_{s,s}$ restricted to $P_s H$ is nondegenerate, by the Hewitt-Cohen Factorization Theorem there exist $c \in \mathcal{K}(X^{\otimes s})$ and $\eta \in H$ such that $\xi = \pi_{s,s}(c)\eta$ and

$$\|\pi_{s,s}(c)\| \cdot \|\eta\| \leq (1 + \varepsilon) \cdot \|\xi\|.$$

We have

$$\begin{aligned}
 \|\overline{\pi \times_J t}(E(a))\xi\| &= \|\overline{\pi \times_J t}(E(a))(P_s - P_{s+1})\xi\| = \|\overline{\pi}_{s,s}(\sum_{j=0}^s a_{j,j} \otimes 1^{s-j})\xi\| \\
 &= \|\overline{\pi}_{s,s}(\sum_{j=0}^s a_{j,j} \otimes 1^{s-j})\pi_{s,s}(c)\eta\| = \|\pi_{s,s}((\sum_{j=0}^s a_{j,j} \otimes 1^{s-j})c)\eta\| \\
 &\leq \|\overline{\pi \times_J t}(\sum_{j=0}^s a_{j,j} \otimes 1^{s-j}c)\| \cdot \|\eta\| \\
 &= \|\overline{\pi \times_J t}(E(\sum_{k=-\infty}^{+\infty} \iota_{s+k,s}(\sum_{\substack{j=0, \\ j+k \geq 0}}^s a_{j+k,j} \otimes 1^{s-j}c)))\| \cdot \|\eta\|.
 \end{aligned}$$

Applying (9.3) to $\sum_{k=-\infty}^{+\infty} \iota_{s+k,s}(\sum_{\substack{j=0, \\ j+k \geq 0}}^s a_{j+k,j} \otimes 1^{s-j}c) \in \mathcal{O}(J_0, X)$ we get

$$\begin{aligned}
 \|\overline{\pi \times_J t}(E(a))\xi\| &\leq \|\overline{\pi \times_J t}(\sum_{k=-\infty}^{+\infty} \iota_{s+k,s}(\sum_{\substack{j=0, \\ j+k \geq 0}}^s a_{j+k,j} \otimes 1^{s-j}c))\| \cdot \|\eta\| \\
 &\leq \|\sum_{k=-\infty}^{+\infty} \overline{\pi}_{s+k,s}(\sum_{\substack{j=0, \\ j+k \geq 0}}^s a_{j+k,j} \otimes 1^{s-j})\| \cdot \|\pi_{s,s}(c)\| \cdot \|\eta\| \\
 &\leq \|\sum_{k=-\infty}^{+\infty} \sum_{\substack{j=0, \\ j+k \geq 0}}^r \overline{\pi}_{j+k,j}(a_{j+k,j})(P_s - P_{s+1})\| \cdot (1 + \varepsilon) \cdot \|\xi\| \\
 &= \|\overline{\pi \times_J t}(a)(P_s - P_{s+1})\| \cdot (1 + \varepsilon) \cdot \|\xi\|.
 \end{aligned}$$

Therefore by arbitrariness of ε and ξ we have

$$\|\overline{\pi \times_J t}(E(a))(P_s - P_{s+1})\| \leq \|\overline{\pi \times_J t}(a)(P_s - P_{s+1})\| \quad \text{for } s = 0, \dots, r-1.$$

Similarly one shows that $\|\overline{\pi \times_J t}(E(a))P_r\| \leq \|\overline{\pi \times_J t}(a)P_r\|$. Hence by (9.4), inequality (9.2) holds and the proof is complete. $\square$

The requirement $R_J = R_{J_0}$ is automatically satisfied whenever one deals with faithful representations and, in particular, every faithful representation of $\mathcal{O}(J_0, X)$ extends to a faithful representation of $\mathcal{DR}(J, X)$ for an appropriate ideal J .

Theorem 9.5. *Let (π, t) be a representation of X on the Hilbert space H . Put*

$$J = \{a \in A : \overline{\pi}_{1,1}(\phi(a)) = \pi(a)\} \quad \text{and} \quad J_0 = J \cap J(X).$$

If $\pi \times_{J_0} t$ is faithful on $\mathcal{O}(J_0, X)$ (resp. on spectral subspaces of $\mathcal{O}(J_0, X)$), then $R_J = R_{J_0}$ and representation $\overline{\pi \times_J t}$ is faithful on $\mathcal{DR}(J, X)$ (resp. on spectral subspaces of $\mathcal{DR}(J, X)$).

Proof. Representations $\pi \times_{J_0} t$ and $\overline{\pi \times_J t}$ are intertwined by the natural homomorphism Ψ ; see Proposition 8.1: $(\pi \times_{J_0} t)(a) = \overline{\pi \times_J t}(\Psi(a))$, $a \in \mathcal{O}(J_0, X)$. Hence, if $\pi \times_{J_0} t$ is faithful on spectral subspaces of $\mathcal{O}(J_0, X)$, then so is Ψ . As faithfulness of Ψ on spectral subspaces of $\mathcal{O}(J_0, X)$ implies the equality $R_J = R_{J_0}$, the assertion follows from Theorem 9.4. $\square$

Corollary 9.6. *If $\pi \times_{J_0} t$ is a faithful representation of $\mathcal{O}(J_0, X)$ on a Hilbert space H , then*

$J_0 = J \cap J(X) \quad \text{and} \quad R_J = R_{J_0}, \quad \text{where} \quad J = \{a \in A : \bar{\pi}_{1,1}(\phi(a)) = \pi(a)\},$
and $\pi \times_{J_0} t$ extends to a faithful representation $\overline{\pi \times_J t}$ of $\mathcal{DR}(J, X)$.

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