DIAMETER OF GRAPHS OF REDUCED WORDS AND GALLERIES

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ABSTRACT. For finite reflection groups of types A and B , we determine the diameter of the graph whose vertices are reduced words for the longest element and whose edges are braid relations. This is deduced from a more general theorem that applies to supersolvable hyperplane arrangements.

1. INTRODUCTION

The symmetric group $W = \mathfrak{S}_n$ on n letters has a well-known Coxeter presentation, with generating set $S = \{s_1, \dots, s_{n-1}\}$ consisting of the adjacent transpositions $s_i = (i, i+1)$, satisfying the *braid relations*

$$\begin{aligned} \text{(i)} \quad & s_i s_j = s_j s_i && \text{for } |i - j| \geq 2, \\ \text{(ii)} \quad & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} && \text{for } 1 \leq i \leq n-2, \end{aligned}$$

together with the condition that each s_i is an involution. Given any w in W , a *reduced decomposition* for w is a sequence $(s_{i_1}, \dots, s_{i_\ell})$ of the generators S for which $w = s_{i_1} \cdots s_{i_\ell}$ that attains the minimum possible length $\ell =: \ell(w)$.

There is a well-studied graph $G(w)$ whose vertex set is the set $\mathcal{R}(w)$ of all reduced decompositions of w , and whose edges correspond to the applicable braid relations (i) and (ii) above. A theorem of Tits [30] (see also [10, Theorem 3.3.(ii)]) says that for any finite Coxeter group (W, S) and any w in W , this graph $G(w)$ is *connected*. A particularly interesting special case occurs when w is the unique longest element w_0 of W . For $W = \mathfrak{S}_4$, the graph $G(w_0)$ is illustrated in Figure 1.1, where each reduced word is abbreviated by its subscript sequence (e.g., 121321 for $(s_1, s_2, s_1, s_3, s_2, s_1)$), and with braid relations of type (i) darkened.

The graph $G(w_0)$ and some of its generalizations were shown to have further *graph-theoretic* connectivity in work of Athanasiadis, Edelman and Reiner [3], and Athanasiadis and Santos [2]. This was motivated by earlier *topological* connectivity results surrounding a closely related poset, appearing first in a conjecture of Baues [6] on loop spaces, which was proven in work of Billera, Kapranov, and Sturmfels [7] and Björner [9]. We also mention here a few ancillary results about the graph $G(w_0)$. Tits [31] gave explicit generators for its fundamental group. Stanley [28] was the first to show that its vertex set $\mathcal{R}(w_0)$ is equinumerous with the standard Young tableaux of shape $(n-1, n-2, \dots, 2, 1)$. In [23], the average degree of a vertex of $G(w_0)$ with respect to only the edges of type (ii) was shown to

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FIGURE 1.1. The graph $G(w_0)$ for $W = \mathfrak{S}_4$.

be 1. Manin and Schechtman [19], Ziegler [32], Felsner [17], and Shapiro, Shapiro and Vainshtein [26] have studied, in the guise of the *higher Bruhat order* $B(n, 2)$, the quotient graph of $G(w_0)$ in which one contracts down all its edges of type (i).

However, the *diameter* of $G(w_0)$, seems to have been considered only very recently. Autord and Dehornoy [4, Proposition 1] show that for $W = \mathfrak{S}_n$, the diameter of $G(w_0)$ grows asymptotically in n as a constant times n^4 . Our main result, Theorem 1.1, shows the diameter is exactly $\frac{1}{24}(n-2)(n-1)n(3n-5)$, which is the number of codimension-two subspaces arising as intersections of two hyperplanes $x_i = x_j$ in the *reflection arrangement* associated to $W = \mathfrak{S}_n$.

1.1. Diameter of $G(w_0)$ and supersolvable hyperplane arrangements. The graph $G(w_0)$ for any finite Coxeter group (W, S) has a natural generalization to the context of *real hyperplane arrangements* that first arose in work of Deligne [15], and later Salvetti [25], on the topology of the complexified complements of these arrangements. This was generalized further to the context of *oriented matroids* by Cordovil and Moreira [14]; to decrease technicalities and enhance readability, we will mainly adhere to the language of hyperplane arrangements in this paper. We review the arrangement viewpoint here in order to state Theorem 1.1; see [12, §4.4, pp. 184–186] and Remark 2.5 below for further discussion.

Let $\mathcal{A}$ be an arrangement of finitely many linear hyperplanes in $\mathbb{R}^d$ that is *central* and *essential*, meaning that $\bigcap_{H \in \mathcal{A}} H = \{0\}$. Let $L = \bigsqcup_{i=0}^d L_i$ be its graded poset of intersection subspaces, ordered via reverse inclusion.

Define a graph structure G_1 on the set $\mathcal{C}$ of chambers of $\mathcal{A}$, in which two chambers c, c' are connected by an edge when they are *separated* by exactly one hyperplane H in L_1 . It is well known, and will be recalled in Section 3, why this graph G_1 always has diameter exactly $|L_1| = |\mathcal{A}|$, that is, the number of hyperplanes.

Now choose a particular base chamber c_0 , and let $\mathcal{R}$ denote the set of all minimal galleries r (that is, geodesics in G_1) from c_0 to $-c_0$. There is a graph structure G_2 on this set $\mathcal{R}$, in which two galleries r, r' are connected by an edge when they are separated (in a sense made precise in Section 2) by exactly one codimension-two intersection subspace X in L_2 . This graph G_2 is known to be connected (see Remark 2.5 below), and it will be shown in Section 3 that its diameter is always *at least* $|L_2|$, raising the following question.

Main question. *For real hyperplane arrangements $\mathcal{A}$ and a choice of base chamber c_0 , does the graph G_2 of minimal galleries from c_0 to $-c_0$ have diameter $|L_2|$?*

Remark 2.6 discusses why the answer to this question is *affirmative* for hyperplane arrangements in dimension at most 3, and even for oriented matroids of rank at most 3, by a result of Cordovil, but *negative* for oriented matroids in rank 4, by an example of Richter-Gebert. After acceptance of this paper for publication, computations by Rob Edman provided a negative answer for the 4-dimensional *cyclic arrangement* of 8 hyperplanes, that is, the hyperplanes in $\mathbb{R}^4$ normal to the vectors $v_i := (1, t_i, t_i^2, t_i^3)$ for real numbers $t_1 < t_2 < \cdots < t_8$. Specifically, in this example, one finds several choices of base chamber c_0 (including the chamber c_0 where vectors have positive dot product with all 8 of the v_i) for which the graph G_2 of minimal galleries from c_0 to $-c_0$ has diameter $30 > 28 = \binom{8}{2} = |L_2|$.

Note that when $\mathcal{A}$ is the arrangement of reflecting hyperplanes for a finite real reflection group W , the choice of base chamber c_0 is immaterial, as W acts simply transitively on the chambers $\mathcal{C}$. Also, in this case the graph G_2 is easily seen to be exactly the graph of reduced words for the longest element w_0 in W described above.

The following main result answers the main question affirmatively for reflection arrangements of types A, B , as well as the more general *supersolvable arrangements*. See Section 4 for undefined terms in its statement.

Theorem 1.1. *When $\mathcal{A}$ is a supersolvable hyperplane arrangement, and the base chamber c_0 is chosen incident to a modular flag, the graph G_2 has diameter $|L_2|$.*

In particular, for the reflection arrangements of type A_{n-1} and B_n and the dihedral groups $I_2(m)$, the graphs of reduced words for w_0 have diameters given by the values of $|L_2|$ shown in Table 1.1.

Unfortunately, types A_{n-1}, B_n , and $I_2(m)$ are the *only* irreducible real reflection groups¹ whose reflection arrangements are supersolvable; see Barcelo and Ihrig [5, Theorem 5.1].

We remark here on some of the data related to Table 1.1. For type D_4 , computer calculations² by Rob Edman show that one has $|\mathcal{R}(w_0)| = 2316$, in agreement with [28, §7]. In addition, these calculations exhibit many reduced words for w_0 giving rise to nodes of the graph G_2 which are L_2 -*accessible* in the sense of Definition 3.9

¹Note that taking *products* of hyperplane arrangements [20, Def. 2.13] which are supersolvable preserves supersolvability, and every finite real reflection group has its reflection arrangement equal to a product of reflection arrangements for *irreducible* real reflection groups.

²These computations give $|\mathcal{R}(w_0)|$ in type D_5 as 12985968, and in type D_6 as 3705762080.

TABLE 1.1

W	$ L_2 $	Does G_2 have diameter $ L_2 $?
A_{n-1}	$\frac{1}{24}n(n-1)(n-2)(3n-5)$	Yes, by Theorem 1.1.
B_n	$\frac{1}{6}n(n-1)(3n^2-5n+1)$	Yes, by Theorem 1.1.
D_n	$\frac{1}{6}n(n-1)(3n^2-11n+13)$	Yes, for $n \leq 4$; unknown generally.
E_8	4900	Unknown.
E_7	1281	Unknown.
E_6	390	Unknown.
F_4	122	Unknown.
H_4	722	Unknown.
H_3	31	Yes, by Theorem 2.7.
$I_2(m)$	1	Yes, trivially.

and Proposition 3.12 below, showing that the diameter is $|L_2|$. Interestingly, none of the L_2 -accessible nodes come from reduced words that are *lexicographically first* among all reduced words, no matter how one linearly orders the Coxeter generators. For F_4 , these calculations show that $|\mathcal{R}(w_0)| = 2144892$, making the full diameter calculation harder, but again, the computer has checked that none of the lexicographically first reduced words gives an L_2 -accessible node.

The remainder of the paper is structured as follows. Section 2 establishes formal definitions for the graph G_2 to be studied, remarking on its connectivity, its diameter in low dimension, as well as its relation to monotonone path zonotopes. Section 3 introduces the notion of a *set-valued metric* on a graph, which is then applied in Section 4 to prove Theorem 1.1. Section 5 explains how some of these results adapt for the graphs $G(w)$ when w is not the longest element w_0 .

2. ARRANGEMENTS AND THE GRAPHS G_1, G_2

We review here some of the theory of hyperplane arrangements; see Orlik and Terao [20] and Stanley [29] for good references.

As in the Introduction, $\mathcal{A} = \{H_1, \dots, H_N\}$ will be an arrangement of hyperplanes in $\mathbb{R}^d$, which is *central* and *essential*, that is, $\bigcap_{i=1}^N H_i = \{0\}$. The *intersection poset* L for $\mathcal{A}$ is the collection of intersection subspaces $X = \bigcap_{i \in I} H_i$ of subsets of the hyperplanes, ordered by *reverse* inclusion. This makes L a *geometric lattice* (see [20, §2.1]), and therefore *graded* or *ranked*. Let $L = \bigsqcup_{i=0}^d L_i$ be the decomposition into its ranks, so that

- the bottom rank L_0 contains only the empty intersection $\mathbb{R}^d$ itself,
- the set of atoms L_1 is the set of hyperplanes $\{H_1, \dots, H_n\} = \mathcal{A}$, and
- the top rank L_d contains only the zero subspace $\{0\}$.

The complement $\mathbb{R}^d \setminus \mathcal{A}$ decomposes into a collection $\mathcal{C}$ of connected components called *chambers*. Given two chambers c, c' , define their separation set

$$L_1(c, c') := \{H \in L_1 = \mathcal{A} : H \text{ separates } c \text{ from } c'\}.$$

FIGURE 2.1. The two possibilities for a minimal gallery r/X from c_0/X to $-c_0/X$ through $\mathcal{A}_X$ when X has codimension-two, each shown as a dashed path.

Definition 2.1 (The graph G_1). Given an arrangement $\mathcal{A}$, define a graph G_1 whose vertex set is the set of chambers $\mathcal{C}$, and having an edge between two chambers $\{c, c'\}$ exactly when $|L_1(c, c')| = 1$.

A (*minimal*) *gallery* from chamber c to chamber c' is a *geodesic* (shortest path)

$$c := c_0, c_1, \dots, c_{d-1}, c_d := c'$$

in this graph G_1 . Fixing one particular choice of a *base chamber* c_0 , let $\mathcal{R}$ denote the set of all minimal galleries r from c_0 to $-c_0$.

We wish to discuss how the codimension-two intersection subspaces in L_2 can *separate* minimal galleries. A minimal gallery r in $\mathcal{R}$ must cross *every* hyperplane $H_1, \dots, H_N$ of $\mathcal{A}$ exactly once, and is completely determined by the linear order in which they are crossed. Given any intersection subspace X , one defines the *localized arrangement* of hyperplanes in the quotient space $\mathbb{R}^d/X$

$$\mathcal{A}_X := \{H/X : H \in \mathcal{A} \text{ and } H \supseteq X\}.$$

Note that the intersection lattice for $\mathcal{A}_X$ may be identified with the lower interval $[\mathbb{R}^d, X]$ within L . For each chamber c of $\mathcal{A}$, there is a unique chamber c/X of $\mathcal{A}_X$ that contains all additive cosets of the subspace X represented by points of c . A minimal gallery r in $\mathcal{R}$ from c_0 to $-c_0$ induces a minimal gallery r/X from c_0/X to $-c_0/X$ in $\mathcal{A}_X$. In particular, when X has codimension two, so that $\mathcal{A}_X$ is an arrangement of lines through the origin in the 2-dimensional plane $\mathbb{R}^d/X$, every minimal gallery r has exactly two possibilities for the induced minimal gallery r/X from c_0/X to $-c_0/X$ in $\mathcal{A}_X$; see Figure 2.1.

Given two minimal galleries r, r' in $\mathcal{R}$ and a codimension-two subspace X in L_2 , say that X *separates* r from r' if $r/X \neq r'/X$. Define their separation set

$$L_2(r, r') := \{X \in L_2 : X \text{ separates } r \text{ from } r'\}.$$

Definition 2.2 (The graph G_2). Given the arrangement $\mathcal{A}$ and the chosen base chamber c_0 in $\mathcal{C}$, define a graph G_2 whose vertex set is the set $\mathcal{R}$ of minimal galleries c_0 to $-c_0$, and having an edge between two galleries $\{r, r'\}$ exactly when $|L_2(r, r')| = 1$.

FIGURE 2.2. Labelling the edges $\{r, r'\}$ in $G(w_0)$ for w_0 by the unique element X in $L_2(r, r')$

The following proposition points out how separation sets $L_i(-, -)$ for $i = 1, 2$ encode the chambers $\mathcal{C}$ and galleries $\mathcal{R}$.

Proposition 2.3. *Given any fixed base chamber c_0 , the set $L_1(c_0, c)$ determines the chamber c uniquely. Given any fixed base gallery $r_0 \in \mathcal{R}$ from c_0 to $-c_0$, the set $L_2(r_0, r)$ determines the gallery r uniquely.*

Proof. The first assertion is clear (and implicit in the discussion of [16, §I]) since $L_1(c_0, c)$ determines on which side of each hyperplane H of $\mathcal{A}$ the chamber c lies.

For the second assertion, as noted earlier, since $L_1(c_0, -c_0) = L_1 = \mathcal{A}$, the gallery r from c_0 to $-c_0$ must cross *every* hyperplane H of $\mathcal{A}$, and r is determined by the linear order in which it crosses these hyperplanes. This linear order is determined by knowing for each pair H, H' which of the two is crossed first. The latter is determined from the order in which r crosses the hyperplanes of the localized arrangement $\mathcal{A}_X$ for the codimension-two subspace $X := H \cap H'$, and this is encoded by the separation set $L_2(r_0, r)$. $\square$

Example 2.4. The reflection arrangement of type A_{n-1} , corresponding to the symmetric group $W = \mathfrak{S}_n$, has ambient space isomorphic to $\mathbb{R}^d$ for $d = n - 1$; one identifies $\mathbb{R}^{n-1}$ with the quotient of $\mathbb{R}^n$ (having coordinates $x_1, \dots, x_n$) by the subspace $x_1 = x_2 = \dots = x_n$. Its hyperplanes are $H_{ij} := \{x_i = x_j\}$ for $1 \leq i < j \leq n$, and its codimension-two intersection subspaces in L_2 are either of type (i) $X_{ij,kl} := \{x_i = x_j, x_k = x_l\}$ or of type (ii) $X_{ijk} := \{x_i = x_j = x_k\}$, corresponding to the braid relations of types (i), (ii) from the Introduction. The

graph $G(w_0)$ from (1.1) is redrawn in Figure 2.2, with each edge $\{r, r'\}$ labeled by the unique codimension-two subspace X separating r from r' .

We close this section with three remarks on the graph G_2 . All of these can be safely skipped by the reader solely interested in the proof of Theorem 1.1.

Remark 2.5 (On the connectivity of G_2). It is not obvious that the graph G_2 is connected for every real hyperplane arrangement $\mathcal{A}$ and every choice of base chamber c_0 . However, as mentioned in the Introduction, this connectivity of G_2 was proven at the following successively stronger levels of generality:

- for real reflection arrangements by Tits [30],
- for real simplicial arrangements by Deligne [15],
- for all real arrangements by Salvetti [25], and
- for oriented matroids by Cordovil and Moreira [14].

Remark 2.6 (On the diameter of G_2 in low dimension). When the arrangement $\mathcal{A}$ lives in $\mathbb{R}^d$ for $d \leq 2$, regardless of the choice of base chamber c_0 , the graph G_2 is trivial, consisting of a single vertex for $d = 1$, and consisting of two vertices connected by a single edge for $d = 2$.

When $d = 3$, regardless of the choice of base chamber c_0 , the diameter of G_2 is exactly $|L_2|$ by the following result of Cordovil [14, Theorem 2.5] (cited there as being implicit in [13, Theorem 2.1]), and proven even more generally for rank 3 oriented matroids:

Theorem 2.7 (Cordovil). *In a 3-dimensional real central hyperplane arrangement $\mathcal{A}$ (or even a rank 3 oriented matroid), any two minimal galleries r, r' from a chamber c_0 to its opposite $-c_0$ can be connected by a sequence of at most $|L_2(\mathcal{A})|$ elementary deformations.*

When $d = 4$, this assertion *fails* at the level of generality of oriented matroids; the authors thank Jim Lawrence for pointing out how this follows from an important counterexample of J. Richter-Gebert, which we recapitulate here; see [12] for most of the oriented matroid terminology left undefined.

A crucial notion is that of a *strong map* $N \rightarrow M$, where N, M are oriented matroids on the same ground set E ; this is defined [12, §7.7] by requiring that the *covectors* of M , as a subset of $\{0, +, -\}^E$, form a subset of the covectors of N . This combinatorially abstracts the arrangement picture as follows. When N comes from a collection of vectors $\{v_e\}_{e \in E}$ in $\mathbb{R}^d$, thought of as the normal vectors to the hyperplanes of $\mathcal{A}$, then its covectors are the sign vectors $(\operatorname{sgn} f(v_i))_{i \in E}$ attained when varying over all linear functionals f in $(\mathbb{R}^d)^*$; equivalently, they index the (relatively open) cones of all dimensions in the decomposition of $\mathbb{R}^d$ by the hyperplanes of $\mathcal{A}$. Then a strong map $N \rightarrow M$ abstracts the situation where M comes from the image vectors $\{\varphi(v_e)\}_{e \in E}$ under some linear map $\mathbb{R}^d \xrightarrow{\varphi} \mathbb{R}^{d'}$.

Richter-Gebert constructs in [24, §3] a certain rank 4 non-realizable, non-Euclidean oriented matroid that he calls $R(12)$, having 12 pseudohyperplanes in general position. He shows [24, Corollary 3.5] that there is a strong map from $R(12) \rightarrow M$, where M is a uniform rank 2 oriented matroid, such that the topes (maximal covectors) of M thought of as a subset of the topes of $R(12)$ cannot be contained in the topes of any pseudohyperplane that extends $R(12)$ by a single element (disproving a conjecture of M. Las Vergnas; see [24, Corollary 3.5]).

Now pick c_0 to be any tope of M , which is necessarily also a tope of $R(12)$, and pick $r, -r$ to be the two unique minimal galleries from c_0 to $-c_0$ passing through topes of M , that is, $r, -r$ pass through the 12 hyperplanes of M or $R(12)$ in exactly reversed orders. Hence if there existed a sequence of $\binom{12}{2} = |L_2(R(12))|$ elementary deformations connecting r to $-r$, this would lead to a *simple allowable sequence* of permutations of length 12 in the sense of Goodman and Pollack; see [12, Chapter 6]. Such an allowable sequence would then give rise to the topes of a uniform rank 3 oriented matroid, containing the topes of M , and coming from a pseudohyperplane that extends $R(12)$ by a single element, contradicting Richter-Gebert's result.

Remark 2.8 (On the relation to monotone path zonotopes). We explain here how Billera and Sturmfels' theory of *fiber polytopes* [8] offers an enlightening perspective on the graph G_2 , implying good behavior for certain of its subgraphs. The reader is referred to Ziegler [33, Lectures 7 and 9] for definitions and terminology omitted in this discussion.

Consider the (central, essential) arrangement $\mathcal{A}$ in $\mathbb{R}^d$ as the *normal fan* for the *zonotope* $Z(\mathcal{A})$ which is generated by functionals $\alpha_H \in (\mathbb{R}^d)^*$ that cut out the hyperplanes H , that is, it is the *Minkowski sum* $Z(\mathcal{A}) = \sum_{H \in \mathcal{A}} [-\alpha_H, \alpha_H]$ of the line segments $[-\alpha_H, \alpha_H]$. Then the graph G_1 defined in Definition 2.1 is exactly the 1-skeleton of this zonotope $Z(\mathcal{A})$.

Now assume that the functionals α_H have been chosen to be positive on points in the chosen base chamber c_0 of $\mathcal{A}$. Then any point f in (the interior of) $-c_0$ gives a linear functional that achieves its minimum, maximum values on $Z(\mathcal{A})$ at the vertices whose normal cones are the chambers $c_0, -c_0$. Let $I = f(Z(\mathcal{A}))$ be the interval inside $\mathbb{R}$ which is the image of $Z(\mathcal{A})$ under this functional f . Then the *fiber polytope/monotone path polytope*

$$Z_2 := \Sigma(Z(\mathcal{A}) \xrightarrow{f} I) = \Sigma_f(Z(\mathcal{A}))$$

discussed by Billera and Sturmfels in [8, Theorem 5.3] is a $(d-1)$ -dimensional polytope with several interesting properties.

The 1-skeleton of Z_2 turns out to be a certain subgraph of the graph G_2 from Definition 2.2. Specifically, minimal galleries c_0 to $-c_0$ correspond to *f-monotone paths* γ in the 1-skeleton of $Z(\mathcal{A})$. The vertices of Z_2 correspond to the subset of *f-monotone paths* γ that are *coherent*, in the sense that there exists some linear functional g whose maximum over each fiber $f^{-1}(x) \cap Z(\mathcal{A})$ for $x \in I$ is achieved uniquely at the point $f^{-1}(x) \cap \gamma$; see [8, Theorem 2.1].

Furthermore, [8, Theorem 2.4, Theorem 4.1] imply that Z_2 is a *zonotope*, generated by the vectors $\{v_{H,H'}\}_{H \neq H' \in \mathcal{A}}$ where $v_{H,H'} := f(\alpha_H)\alpha_{H'} - f(\alpha_{H'})\alpha_H$. One can check that, for *generic choices* of f within the interior of $-c_0$, any two such generating vectors v_{H_1,H_2} and v_{H_3,H_4} for Z_2 are scalar multiples of each other exactly when the codimension-two intersection subspaces $H_1 \cap H_2$ and $H_3 \cap H_4$ are the same subspace X in L_2 .

Hence Z_2 will then be a zonotope having exactly $|L_2|$ distinct parallelism classes among its generating vectors, and the 1-skeleton of Z_2 will be a geometrically distinguished subgraph of G_2 having the expected diameter $|L_2|$.

3. SET-VALUED METRICS ON GRAPHS

We introduce some easy observations that apply to the question of diameter for the graphs G_1, G_2 defined in the previous section.

Definition 3.1. Let $G = (V, E)$ be a simple graph on vertex set V , meaning that E is a set of unordered pairs $\{x, y\}$ with $x \neq y \in V$.

The graph-theoretic *distance* $d_G(x, y)$ is the minimum length d of a path

$$(1) \quad x = v_0, v_1, \dots, v_{d-1}, v_d = y$$

with $\{v_i, v_{i+1}\} \in E$ for each i . Call such a shortest path a *geodesic*.

The *diameter* of G is the maximum value of $d_G(x, y)$ over all $x, y \in V$.

Note that $d_G(-, -)$ satisfies the usual properties of a *metric* on V , that is,

- $d_G(x, x) = 0$,
- $d_G(x, y) = d_G(y, x)$, and
- $d_G(x, z) \leq d_G(x, y) + d_G(y, z)$.

We also make the trivial observation that if $\alpha : V \rightarrow V$ is a *graph automorphism*, meaning a bijection such that for every edge $\{x, y\}$ in E , the image $\{\alpha(x), \alpha(y)\}$ is also in E , then α takes geodesics to geodesics and preserves distances:

$$d_G(\alpha(x), \alpha(y)) = d_G(x, y).$$

Definition 3.2. For a connected simple graph $G = (V, E)$ and a set Ω , say that a function

$$\begin{aligned} \Omega(-, -) : V \times V &\longrightarrow 2^\Omega \\ (x, y) &\longmapsto \Omega(x, y) \end{aligned}$$

is a *set-valued metric* on G if

- (a) $\Omega(x, y) = \Omega(y, x)$,
- (b) whenever $\{x, y\}$ is an edge in E , one has $|\Omega(x, y)| = 1$, and
- (c) $\Omega(x, z) = \Omega(x, y) \triangle \Omega(y, z)$,

where here

$$A \triangle B := (A \setminus B) \sqcup (B \setminus A)$$

denotes the symmetric difference of sets. In particular, the first and third conditions imply that $\Omega(x, x) = \emptyset$ for any x in V .

Here is an equivalent rephrasing of a set-valued metric $\Omega(-, -)$ on G : it is a labelling $\Omega(x, y)$ of each edge $e = \{x, y\}$ in E with an element of Ω in such a way that when one traverses any closed path of edges in the graph, each label appears an even number of times. For *any* pair of vertices x, y in V , not necessarily connected by an edge, one defines $\Omega(x, y)$ to be the set of labels that appear an odd number of times on any path from x to y .

Example 3.3. Given a real hyperplane arrangement $\mathcal{A}$, and the graphs G_i for $i = 1, 2$ defined in Definitions 2.1 and 2.2, one can easily check that the function $L_i(-, -)$ for $i = 1, 2$ gives a set-valued metric.

We begin with two observations about set-valued metrics.

Proposition 3.4. A connected simple graph $G = (V, E)$ supports at least one set-valued metric if and only if G is bipartite.

Proof. Given a set-valued metric $\Omega(-, -)$ on G , choosing any vertex x_0 in V , one has that G is bipartite with vertex bipartition $V = V_0 \sqcup V_1$ in which

$$V_i := \{y \in V : |\Omega(x_0, y)| \equiv i \pmod{2}\}.$$

Conversely, for G bipartite with vertex bipartition $V = V_0 \sqcup V_1$, one can define a trivial set-valued metric $\Omega(-, -) : V \times V \longrightarrow 2^{\{e\}}$, where $\{e\}$ is a singleton, via

$$\Omega(x, y) := \begin{cases} \emptyset & \text{if } x, y \in V_0 \text{ or } x, y \in V_1, \\ \{e\} & \text{otherwise.} \end{cases} \quad \square$$

Proposition 3.5. *A simple graph $G = (V, E)$ with a set-valued metric $\Omega(-, -)$, has $d_G(x, y) \geq |\Omega(x, y)|$ for all $x, y \in V$.*

Proof. A path of length d in G from x to y as in (1) leads to a path

$$\emptyset = \Omega(x, v_0), \Omega(x, v_1), \dots, \Omega(x, v_{d-1}), \Omega(x, v_d) = \Omega(x, y),$$

where each pair of sets $\Omega(x, v_i), \Omega(x, v_{i+1})$ differs in one element, namely the unique element of $\Omega(v_i, v_{i+1})$. Thus $d \geq |\Omega(x, y)|$. $\square$

Definition 3.6. For a set-valued metric $\Omega(-, -)$ on a simple graph $G = (V, E)$, say an involution $x \mapsto -x$ on the vertex set V is $(\mathbb{Z}_2-)$ *equivariant* if

$$(2) \quad \Omega(x, -y) = \Omega \setminus \Omega(x, y)$$

for all x, y in V . This is equivalent, by property (b) in Definition 3.2 of set-valued metrics, to requiring only the special case of (2) where $y = x$, that is, equivariance requires only $\Omega(x, -x) = \Omega$ for all x in V .

Example 3.7. Continuing Example 3.3, the graphs G_i for $i = 1, 2$ endowed with the set-valued metrics $L_i(-, -)$ also have equivariant involutions derived from the linear map $x \mapsto -x$ on $\mathbb{R}^d$. For G_1 , the involution sends the chamber $c \in \mathcal{C}$ to the antipodal chamber $-c$. For G_2 , the involution sends the minimal gallery r

$$c_0, c_1, \dots, c_{d-1}, c_d := -c_0$$

to the minimal gallery $-r$ which visits the antipodes of the same chambers in the reverse order:

$$c_0 = -c_d, -c_{d-1}, \dots, -c_1, -c_0.$$

Equivalently, r and $-r$ cross the hyperplanes of $\mathcal{A}$ in exactly the *opposite* linear orders. For example, in the reflection arrangement of type A_{n-1} , one can check that this involution on galleries sends a reduced word $r = (s_{i_1}, s_{i_2}, \dots, s_{i_{\ell-1}}, s_{i_\ell})$ for w_0 to the reduced word $-r = (s_{n-i_\ell}, s_{n-i_{\ell-1}}, \dots, s_{n-i_2}, s_{n-i_1})$.

Proposition 3.8. *A simple graph $G = (V, E)$ with a set-valued metric $\Omega(-, -)$ and an equivariant involution always has diameter at least $|\Omega|$.*

Proof. By Proposition 3.5

$$d_G(x, -x) \geq |\Omega(x, -x)| = |\Omega \setminus \Omega(x, x)| = |\Omega|. \quad \square$$

Definition 3.9. For a set-valued metric $\Omega(-, -)$ on a simple graph $G = (V, E)$, say that a vertex x_0 in V is Ω -*accessible* if $d_G(x_0, y) = |\Omega(x_0, y)|$ for every $y \in V$.

Example 3.10. Continuing Examples 3.3 and 3.7, it was observed by Edelman (see [16, Proposition 1.1]) that for every real hyperplane arrangement $\mathcal{A}$, every vertex in the graph G_1 is L_1 -accessible: given any two chambers c, c' , a straight-line path between generic points in c, c' gives a path of length $|L_1(c, c')|$ between their corresponding vertices of G_1 .

Example 3.11. Of the 16 reduced words for w_0 in $W = \mathfrak{S}_4$ shown in Figure 1.1, there are exactly four which *do not* give L_2 -accessible vertices for the graph G_2 shown, namely the four words

$$(3) \quad \{213213, \quad 231231, \quad 132132, \quad 312312\}.$$

We check that none of the four words r_0 in this set (3) is L_2 -accessible. Scrutiny of Figures 1.1 and 2.2 shows that there are exactly two words at the maximum distance $7 (= |L_2|)$ from such an r_0 , namely its antipodal word $-r_0$, and a second word $r \neq -r_0$ having $d_{G_2}(r_0, r) = 7 > 5 = |L_2(r_0, r)|$. In particular, any such pair $\{r_0, r\}$ provides an example that answers negatively a question of Autord and Dehornoy [4, Question 1.9].

On the other hand, one can check using Figures 1.1 and 2.2, via brute force (mitigated by some symmetry), that all 12 of the other words r_0 are L_2 -accessible.

Our goal in Section 4 will be to show that for supersolvable arrangements $\mathcal{A}$, when one chooses the base chamber $c_0 \in \mathcal{C}$ incident to a chain of modular flats, there is a choice of base gallery $r_0 \in \mathcal{R}$ which is L_2 -accessible. Therefore, in this case, the diameter for G_2 will be determined by the next proposition.

Proposition 3.12. *Assume one has a simple graph $G = (V, E)$ with a set-valued metric $\Omega(-, -)$, and an involution $v \mapsto -v$ on V which is both equivariant and a graph automorphism of G .*

If V contains an Ω -accessible vertex x_0 , then the diameter of G is exactly $|\Omega|$.

Proof. By Proposition 3.8, it suffices to show that $d_G(x, y) \leq |\Omega|$ for all x, y . This follows from these equalities and inequalities, justified below:

$$\begin{aligned} 2d_G(x, y) &= d_G(x, y) + d_G(x, y) \\ &\stackrel{(1)}{\leq} d_G(x, x_0) + d_G(x_0, y) + d_G(x, -x_0) + d_G(-x_0, y) \\ &\stackrel{(2)}{=} d_G(x, x_0) + d_G(x_0, y) + d_G(-x, x_0) + d_G(x_0, -y) \\ &= (d_G(x, x_0) + d_G(-x, x_0)) + (d_G(x_0, y) + d_G(x_0, -y)) \\ &\stackrel{(3)}{=} (|\Omega(x, x_0)| + |\Omega(-x, x_0)|) + (|\Omega(x_0, y)| + |\Omega(x_0, -y)|) \\ &\stackrel{(4)}{=} |\Omega| + |\Omega| = 2|\Omega|. \end{aligned}$$

Inequality (1) twice uses the triangle inequality for the metric $d_G(-, -)$. Equality (2) twice uses the assumption that $v \mapsto -v$ is a graph automorphism. Equality (3) four times uses the Ω -accessibility of x_0 . Equality (4) twice uses the assumption of equivariance. $\square$

Applying Propositions 3.8 and 3.12 to the graphs G_1, G_2 immediately gives the following.

Corollary 3.13. *For any real hyperplane arrangement $\mathcal{A}$, the graph G_1 has diameter exactly $|L_1|$. For any choice of a base chamber c_0 of $\mathcal{A}$, the graph G_2 has diameter at least $|L_2|$.*

4. SUPERSOLVABLE ARRANGEMENTS

We wish to first review the definition and some properties of supersolvable arrangements [27], [20, §2.3], [11, §4], and then apply this to prove Theorem 1.1.

Definition 4.1. Given a finite geometric lattice L with rank function ρ , say that x in L is a *modular element* of L if $\rho(x \vee y) + \rho(x \wedge y) = \rho(x) + \rho(y)$ for all y in L . The lattice L is called *supersolvable* if it contains an M -chain, that is, a maximal chain of modular elements.

A hyperplane arrangement $\mathcal{A}$ is called *supersolvable* when its intersection lattice L is supersolvable. An inductive rephrasing, due to Björner, Edelman and Ziegler [11, Thm. 4.3], will be more useful for our purposes. Given a real (central, essential) hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^d$, an element ℓ of L_{d-1} is called a *coatom*. Thus coatoms ℓ are *lines* obtained by intersecting the hyperplanes.

Proposition 4.2 ([11, Thm. 4.3]). *Let $\mathcal{A}$ be a hyperplane arrangement.*

- (i) *A coatom ℓ of L is a modular coatom if and only if for every pair H, H' of distinct hyperplanes of $\mathcal{A}$ not containing ℓ there exists a hyperplane H'' of $\mathcal{A}$ containing both ℓ and $H \cap H'$ (that is to say, the hyperplane $H'' = \ell + H \cap H'$ is in $\mathcal{A}$).*
- (ii) *A hyperplane arrangement $\mathcal{A}$ is supersolvable if and only if it satisfies the following inductive definition: either $\mathcal{A}$ has rank $d = 1$ or its intersection lattice L contains a modular coatom ℓ for which the localized arrangement $\mathcal{A}_\ell$ of rank $d - 1$ is supersolvable.*

Example 4.3. When $d = 2$ the arrangement $\mathcal{A}$ is always supersolvable, as any of its hyperplanes (=lines) is a modular coatom.

Example 4.4. Recall from Example 2.4 that the reflection arrangement $\mathcal{A}$ of type A_{n-1} , corresponding to $W = \mathfrak{S}_n$, lives in $\mathbb{R}^n / \{x_1 = x_2 = \cdots = x_n\}$, and has hyperplanes $H_{ij} := \{x_i = x_j\}$. The line ℓ defined by $\{x_1 = x_2 = \cdots = x_{n-1}\}$ is a modular coatom for A_{n-1} : any two typical hyperplanes H_{in}, H_{jn} for $i, j < n$ that do not contain ℓ will have $\ell + (H_{in} \cap H_{jn}) = H_{ij}$, which is another hyperplane in the arrangement A_{n-1} . The localization $\mathcal{A}_\ell$ is isomorphic to the reflection arrangement of type A_{n-2} , and hence one can iterate this construction to show that the arrangement of type A_{n-1} is supersolvable.

Example 4.5. The reflection arrangement $\mathcal{A}$ of type B_n lives in $\mathbb{R}^n$, and consists of all hyperplanes of the form

$$\begin{aligned} H_{ij}^+ &:= \{x_i = +x_j\}, \\ H_{ij}^- &:= \{x_i = -x_j\}, \\ H_{ii}^- &:= \{x_i = 0\}. \end{aligned}$$

Let us check that the line ℓ defined by $\{x_1 = x_2 = \cdots = x_{n-1} = 0\}$ is a modular coatom for B_n . Pairs of hyperplanes not containing ℓ either come from choosing

- two indices $i, j \leq n - 1$ and two signs $\alpha, \beta = \pm 1$, giving hyperplanes $H_{in}^\alpha, H_{jn}^\beta$ which satisfy $\ell + (H_{in}^\alpha \cap H_{jn}^\beta) = H_{ij}^{\alpha\beta}$, a hyperplane of B_n ; or
- an index $i \leq n - 1$ and a sign α , giving hyperplanes H_{nn}^-, H_{in}^α , which satisfy $\ell + (H_{nn}^- \cap H_{in}^\alpha) = H_{ii}^-$, a hyperplane of B_n .

The localization $\mathcal{A}_\ell$ is isomorphic to the reflection arrangement of type B_{n-1} , and hence one can iterate this construction to show that the arrangement of type B_n is supersolvable.

Given an intersection subspace X , define the map π_X sending chambers of $\mathcal{A}$ to their corresponding chamber in the localization $\mathcal{A}_X$:

$$\begin{aligned}\pi_X : \mathcal{C} &=: \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{A}_X) \\ c &\longmapsto c/X.\end{aligned}$$

Say that a chamber c is *incident* to a subspace X if the closure of c intersects X in a subcone of the same dimension as X .

Proposition 4.6. *Assume that ℓ is a modular coatom for $\mathcal{A}$, and $c \in \mathcal{C}$ is a chamber incident to ℓ .*

(i) *(Compare with the discussion before Theorem 4.4 of [11]) There is a linear order on the fiber*

$$\pi_\ell^{-1}(\pi_\ell(c)) = \{c_1(=c), c_2, c_3, \dots, c_t\}$$

such that the sets $L_1(c, c_i)$ for $i = 1, 2, \dots, t$ are nested:

$$\emptyset = L_1(c, c_1) \subset L_1(c, c_2) \subset \dots \subset L_1(c, c_t) = \mathcal{A} \setminus \mathcal{A}_\ell.$$

This induces a linear order $H_1, H_2, \dots$ on $\mathcal{A} \setminus \mathcal{A}_\ell$ such that H_i is the unique hyperplane in $L_1(c, c_i) \setminus L_1(c, c_{i-1})$.

(ii) *Using the linear order $H_1, H_2, \dots$ on $\mathcal{A} \setminus \mathcal{A}_\ell$ from part (i), if $i < j < k$ and if the chamber c incident to ℓ is also incident to $\ell + H_i \cap H_k$, then*

$$(4) \quad H_i \cap H_j = H_i \cap H_k = H_j \cap H_k.$$

Proof of assertion (i). Assume for the sake of contradiction that there exist two chambers c_i, c_j with $\pi_\ell(c_i) = \pi_\ell(c_j) = \pi_\ell(c)$ and two hyperplanes H_i, H_j not containing ℓ for which

$$(5) \quad \begin{aligned}H_i &\in L_1(c, c_i) \setminus L_1(c, c_j), \\ H_j &\in L_1(c, c_j) \setminus L_1(c, c_i).\end{aligned}$$

By the modularity of the coatom ℓ , the hyperplane $H := \ell + H_i \cap H_j$ is in $\mathcal{A}$. Consider the codimension-two subspace

$$X := H_i \cap H_j = H \cap H_i = H \cap H_j$$

and the local picture for the lines and chambers

$$H/X, H_i/X, H_j/X, c/X, c_i/X, c_j/X$$

within the localized rank two arrangement $\mathcal{A}_X$. Then (5) together with the assumption that H contains the line ℓ incident to c forces this local picture to be as in Figure 4.1(i). In particular, it forces H to separate c_i, c_j , and since $\ell \subset H$, this contradicts $\pi_\ell(c_i) = \pi_\ell(c_j)$. $\square$

Proof of assertion (ii). Assume for the sake of contradiction that $i < j < k$ and c is incident to $\ell + H_i \cap H_k$, but (4) fails, so that the intersection $X := H_i \cap H_j \cap H_k$ is of codimension-three, not two. Note that ℓ is contained in none of H_i, H_j, H_k . Therefore since ℓ is a modular coatom, each of the following three hyperplanes containing both ℓ and X must also be a hyperplane in $\mathcal{A}$:

$$\begin{aligned}H_{ij} &:= \ell + H_i \cap H_j, \\ H_{ik} &:= \ell + H_i \cap H_k, \\ H_{jk} &:= \ell + H_j \cap H_k.\end{aligned}$$

FIGURE 4.1. (i) Local picture illustrating why the fibers $\pi_\ell^{-1}(\pi_\ell(c))$ are linearly ordered by inclusion of $L(c, -)$.

(ii) Local picture illustrating why incidence of c to $\ell + H_i \cap H_k$ and $i < j < k$ forces the equality (4).

Now consider the local picture for

$$H_i/X, H_j/X, H_k/X, H_{ij}/X, H_{ik}/X, H_{jk}/X, \ell/X$$

within $\mathcal{A}_X$, which after an invertible linear transformation of $\mathbb{R}^d/X$, can be made to look as in Figure 4.1(ii).

Recalling that c is incident to the line ℓ , the condition $i < j < k$, forces c/X to be in the chamber shown, so that as one starts in c and moves away from ℓ staying within the same chamber of $\mathcal{A}_\ell$, one crosses the hyperplanes H_i, H_j, H_k in this order; if c/X lies in any of the other five chambers incident to ℓ/X in this figure, one will cross H_i, H_j, H_k in a different order.

However, Figure 4.1(ii) also shows that this location for c/X contradicts the incidence of c to $H_{ik} = \ell + H_i \cap H_k$. $\square$

Definition 4.7. A (maximal) flag $F = \{X_i\}_{i=0}^d$ is a chain of intersection subspaces in L

$$\{0\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{d-1} \subset X_d = \mathbb{R}^d$$

in which X_i is of dimension i . Say that a chamber c is *incident* to the flag F if c is incident to each of the X_i . If c_0 is a chamber incident to a flag F , say that a minimal gallery r from c_0 to $-c_0$ is *incident* to the flag F if it first crosses the unique hyperplane in $\mathcal{A}_{X_{d-1}}$ (namely X_{d-1} itself), then crosses the hyperplanes in $\mathcal{A}_{X_{d-2}} \setminus \mathcal{A}_{X_{d-1}}$, etc., always crossing the hyperplanes in $\mathcal{A}_{X_j}$ before those in $\mathcal{A}_{X_i} \setminus \mathcal{A}_{X_j}$ whenever $i < j$.

According to the definition of supersolvability (Definition 4.1) and its rephrasing in Proposition 4.2, $\mathcal{A}$ is supersolvable if and only if it has an M -chain or *modular flag*, in which X_1 is a modular coatom for $\mathcal{A}$, while X_2/X_1 is a modular coatom in the localized arrangement $\mathcal{A}_{X_1}$, and generally X_i/X_{i-1} is a modular coatom in the localized arrangement $\mathcal{A}_{X_{i-1}}$.

Proposition 4.8. *Let ℓ be a coatom of the intersection lattice L for an arrangement $\mathcal{A}$. The map*

$$\begin{aligned} \mathcal{C}(\mathcal{A}) &\xrightarrow{\pi_\ell} \mathcal{C}(\mathcal{A}_\ell) \\ c &\longmapsto c/\ell \end{aligned}$$

becomes two-to-one when restricted to the subset of chambers of $\mathcal{A}$ incident to ℓ .

Consequently,³ when $\mathcal{A}$ is a real central essential arrangement in $\mathbb{R}^d$, there are exactly 2^d chambers incident to each maximal flag F .

Proof. The two chambers of a fiber $\pi_\ell^{-1}(\pi_\ell(c))$ that contain ℓ are the two chambers c^+, c^- in the fiber whose closures contain ℓ^+, ℓ^- , the two rays (half-lines) comprising the line ℓ . $\square$

Theorem 4.9. *Let $\mathcal{A}$ be a real (central, essential) hyperplane arrangement in $\mathbb{R}^d$ which is supersolvable, let $F := \{X_i\}_{i=0}^d$ be a modular flag for $\mathcal{A}$, and let c_0 be any chamber incident to F .*

- (i) *There is a unique minimal gallery r_0 from c_0 to $-c_0$ incident to F .*
- (ii) *This minimal gallery r_0 is an L_2 -accessible vertex for the graph G_2 on the galleries $\mathcal{R}$ from c_0 to $-c_0$.*

Proof of assertion (i). Proceed by induction on d , with the base case $d = 1$ being trivial. In the inductive step, let ℓ^+ be the half-line of ℓ contained in the closure of c_0 . Then the unique minimal gallery r_0 from c_0 to $-c_0$ incident to F is constructed as follows, in order to have it cross all the hyperplanes in $\mathcal{A}_\ell$ first:

- (a) Apply induction to the $(d-1)$ -dimensional supersolvable arrangement $\mathcal{A}_\ell$, to find the unique gallery from c_0/ℓ to $-c_0/\ell$ incident to F/ℓ .
- (b) Begin the gallery r_0 by lifting each chamber c/ℓ of $\mathcal{A}_\ell$ in this gallery from (a) to the unique chamber c in $\mathcal{A}$ whose closure contains ℓ^+ .
- (c) After going through the chambers in this lifted gallery from (b), ending in a gallery called c incident to ℓ , one now has no choice about how to complete the rest of r_0 : one must cross the hyperplanes $\mathcal{A} \setminus \mathcal{A}_\ell$ in the linear order given by Proposition 4.6(i). $\square$

For example, for the reflection arrangement of type A_3 , the base word/gallery $r_0 = 121321$, indexing the vertex at the bottom left in Figure 1.1, is the unique gallery r_0 incident to the modular flag described in Example 4.4; this gallery is discussed further below in Example 4.10. Figure 4.2 shows the reflection arrangement of type B_3 , with the unique gallery r_0 incident to a certain modular flag labeled.

³Compare this with the discussion before Theorem 4.4 of [11].

FIGURE 4.2. The reflection arrangement of type B_3 , drawn as great circles on a unit sphere. The chamber c_0 and its antipode $-c_0$ are labeled, as well as the three galleries r_0, r, r' from c_0 to $-c_0$ which appear in the proof of Theorem 4.9.

Proof of assertion (ii). Proceed by induction on d , with the base case $d = 2$ being trivial. In the inductive step, it suffices to show that for any gallery $r \neq r_0$ from $-c_0$ to c_0 , there exists another gallery r' having $L_2(r_0, r') \subset L_2(r_0, r)$ and $|L_2(r, r')| = 1$. There are two cases.

Case 1. The gallery r crosses all the hyperplanes in $\mathcal{A}_\ell$ before crossing any hyperplanes of $\mathcal{A} \setminus \mathcal{A}_\ell$.

Then just as with r_0 , the gallery r must cross the hyperplanes in $\mathcal{A} \setminus \mathcal{A}_\ell$ in the linear order given by Proposition 4.6. Hence the galleries r and r_0 can differ only in their initial segments $\hat{r}, \hat{r}_0$ where they cross all the hyperplanes of $\mathcal{A}_\ell$. Applying induction on dimension to the quotient galleries $\hat{r}/\ell, \hat{r}_0/\ell$, there exists a gallery $\hat{r}'/\ell$ from c_0/ℓ to $-c_0/\ell$ in $\mathcal{A}_\ell$ having $L_2(\hat{r}_0/\ell, \hat{r}'/\ell) \subset L_2(\hat{r}_0/\ell, \hat{r}/\ell)$ and $|L_2(\hat{r}/\ell, \hat{r}'/\ell)| = 1$. The desired gallery r' from c_0 to $-c_0$ is then obtained by first lifting $\hat{r}'/\ell$ as in Proposition 4.8, and then completing it by crossing the hyperplanes in $\mathcal{A} \setminus \mathcal{A}_\ell$ in the linear order given by Proposition 4.6.

Case 2. The gallery r crosses some hyperplane of $\mathcal{A} \setminus \mathcal{A}_\ell$ before it has finished crossing all the hyperplanes in $\mathcal{A}_\ell$.

Then there must exist at least one ordered pair of hyperplanes (H, H') crossed consecutively by r that has $H \in \mathcal{A} \setminus \mathcal{A}_\ell$ and $H' \in \mathcal{A}_\ell$. Find the earliest occurrence⁴ of such a pair (H, H') .

Thus r begins by crossing some (possibly empty) sequence of hyperplanes in $\mathcal{A}_\ell$, reaching some chamber c incident to ℓ , and then immediately thereafter crosses a

⁴Compare the rest of this proof with the proof of [4, Lemma 1.2], which it generalizes.

sequence of hyperplanes $H_1, H_2, \dots, H_{t-1}, H_t =: H$ that are all in $\mathcal{A} \setminus \mathcal{A}_\ell$, before crossing H' . See Figure 4.2 for an illustration of a typical gallery r , with the chamber c and the hyperplanes $H_1, H_2, \dots, H_{t-1}, H_t (= H), H'$ labeled as in this proof; in this example, $t = 4$.

Let $X := H \cap H'$, a codimension-two subspace in L_2 . Note that as $H' = \ell + X$, this H' is the unique hyperplane in the rank 2 subarrangement $\mathcal{A}_X$ that also lies in $\mathcal{A}_\ell$. We wish to determine *exactly* when the other hyperplanes $\mathcal{A}_X \setminus \{H'\}$ are crossed by the gallery r .

Note that since the quotient gallery r/ℓ would cross H'/ℓ to leave c/ℓ , this chamber c/ℓ must be incident to H'/ℓ . As c is incident to ℓ , this means that c is incident to H' .

Now since r visits the chamber c which is incident to H' , but then crosses another hyperplane H that contains X before crossing H' , it must be that r crosses *every other* hyperplane of the rank 2 arrangement $\mathcal{A}_X$ before crossing H' .

On the other hand, since r only crossed hyperplanes in $\mathcal{A}_\ell$ before reaching c , it must be that

$$\mathcal{A}_X \setminus \{H'\} \subset \{H_1, H_2, \dots, H_{t-1}, H_t (= H)\}.$$

Also note that the hypotheses of Proposition 4.6(ii) are satisfied by c, ℓ and by any two hyperplanes H_i, H_k lying in $\mathcal{A}_X \setminus \{H'\}$, since c is incident to

$$H' = \ell + X = \ell + H_i \cap H_k.$$

This means that for any j with $i < j < k$ one has H_j also in $\mathcal{A}_X \setminus \{H'\}$. Thus the hyperplanes of $\mathcal{A}_X \setminus \{H'\}$ occur *consecutively* within the list $(H_1, \dots, H_{t-1}, H_t)$. That is, we have shown that there is some index $s \leq t$ for which

$$\{H_s, H_{s+1}, \dots, H_{t-1}, H_t (= H), H'\} = \mathcal{A}_X.$$

Now let r' be the gallery obtained from r by reversing this consecutive sequence of crossings $(H_s, H_{s+1}, \dots, H_{t-1}, H_t (= H), H')$ of the hyperplanes in $\mathcal{A}_X$. By construction, $L_2(r, r') = \{X\}$.

We must also check that $L_2(r_0, r') \subset L_2(r_0, r)$, i.e. that X is not in $L_2(r_0, r')$. This follows because r_0 is incident to F , so it must cross the hyperplane $H' \in \mathcal{A}_\ell$ before it can cross the hyperplane $H \in \mathcal{A} \setminus \mathcal{A}_\ell$. Thus in regard to its order of crossing the hyperplanes of $\mathcal{A}_X$, the gallery r_0 agrees with r' , not with r . $\square$

Theorem 1.1 is now immediate from Theorem 4.9(ii) and Proposition 3.12.

Example 4.10. Continuing Example 4.4, for the arrangement $\mathcal{A}$ of type A_{n-1} , one can choose as modular flag $F := \{X_i\}_{i=0}^{n-1}$, where

$$X_i := \{x_1 = x_2 = \dots = x_{n-i}\}.$$

The chambers $c_w \in \mathcal{C}$ may be indexed by permutations w in $\mathfrak{S}_n$, with defining inequalities $x_{w(1)} < \dots < x_{w(n)}$. The chamber c_0 corresponding to the identity permutation is incident to the above modular flag F . The unique gallery r_0 from

c_0 to $-c_0$ incident to F crosses the hyperplanes in this order:

$$\begin{aligned} &H_{12}, \\ &H_{13}, H_{23}, \\ &H_{14}, H_{24}, H_{34}, \\ &H_{15}, H_{25}, H_{35}, H_{45}, \dots \end{aligned}$$

Using the Coxeter generators $S = \{s_1, \dots, s_{n-1}\}$ for $W = \mathfrak{S}_n$, in which s_i is the adjacent transposition $(i, i+1)$, this gallery r_0 corresponds to the following reduced decomposition for w_0 :

$$\begin{aligned} &(s_1, \\ &s_2, s_1, \\ &s_3, s_2, s_1, \\ &s_4, s_3, s_2, s_1, \dots). \end{aligned}$$

Example 4.11. Continuing Example 4.5, for the arrangement $\mathcal{A}$ of type B_n , one can choose as modular flag $F := \{X_i\}_{i=0}^n$, where

$$X_i := \{x_1 = x_2 = \dots = x_{n-i} = 0\}.$$

The chambers $c_w \in \mathcal{C}$ may be indexed by signed permutations w with defining inequalities $0 < \epsilon_1 x_{w(1)} < \dots < \epsilon_n x_{w(n)}$ if w sends the standard basis vector e_j to $\epsilon_j e_{w(j)}$ with $\epsilon_j \in \{\pm 1\}$. The chamber c_0 corresponding to the identity permutation is incident to the above modular flag F . The unique gallery r_0 from c_0 to $-c_0$ incident to F crosses the hyperplanes in this order:

$$\begin{aligned} &H_{11}^-, \\ &H_{12}^-, H_{22}^-, H_{12}^+, \\ &H_{23}^-, H_{13}^-, H_{33}^-, H_{13}^+, H_{23}^+, \\ &H_{34}^-, H_{24}^-, H_{14}^-, H_{44}^-, H_{14}^+, H_{24}^+, H_{34}^+, \dots \end{aligned}$$

Choose Coxeter generators $S = \{s_0, s_1, \dots, s_{n-1}\}$ for the hyperoctahedral group $W = B_n$ of signed permutations acting on the coordinates of $\mathbb{R}^n$, such that s_i is the adjacent transposition $(i, i+1)$, as before, and s_0 is the sign change in the first coordinate x_1 . Then this gallery r_0 corresponds to the following reduced decomposition for w_0 :

$$\begin{aligned} &(s_0, \\ &s_1, s_0, s_1, \\ &s_2, s_1, s_0, s_1, s_2, \\ &s_3, s_2, s_1, s_0, s_1, s_2, s_3, \dots). \end{aligned}$$

Remark 4.12. Consider the following possible hypotheses on a real (central, essential) hyperplane arrangement $\mathcal{A}$ and one of its chambers c_0 :

- (i) The chamber c_0 is a simplicial cone in the sense that its walls have linearly independent normal vectors.
- (ii) The weak order on the chambers $\mathcal{C}$, considered by Edelman [16], in which $c \leq c'$ when $L_1(c_0, c) \subseteq L_1(c_0, c')$, is a lattice.

- (iii) $\mathcal{A}$ is *simplicial*, meaning that every chamber is a simplicial cone.
- (iv) $\mathcal{A}$ is a real reflection arrangement.
- (v) The arrangement $\mathcal{A}$ is supersolvable, and the chamber c_0 is incident to one of its modular flags.

Incorporating well-known results for reflection arrangements with various results from Björner, Edelman and Ziegler [11], one has the following implications:

$$\begin{array}{ccc}
 (iv) & \rightarrow & (iii) \\
 & \searrow & \\
 & & (ii) \rightarrow (i) \\
 & \nearrow & \\
 (v) & &
 \end{array}$$

Bearing in mind that Theorem 1.1 assumes hypothesis (v), and the d -vertex-connectivity of the graph G_2 proven in [3, Theorem 1.1] assumes hypothesis (iii), it is reasonable to ask whether any of the extra hypotheses (i), (ii), (iii), (iv) imply that the lower bound of $|L_2|$ for the diameter of the graph G_2 is tight.

5. ON THE GRAPHS $G(w)$

In this section we show how the previous methods generalize to the graph $G(w)$ of reduced words for w an element of a finite reflection group W , as discussed in the Introduction. Although these methods lead to some bounds on the distance functions and diameters of $G(w)$ in the groups of type A, B , we do not determine these diameters exactly.

As in Section 2, let $\mathcal{A}$ be a (central, essential) hyperplane arrangement in $\mathbb{R}^d$, with set of chambers $\mathcal{C}$. Given two chambers $c, c' \in \mathcal{C}$, recall that $L_1(c, c')$ denotes the set of hyperplanes $H \in L_1 = \mathcal{A}$ that separate c from c' or, equivalently, for which the chambers $c/H, c'/H$ in the localized rank 1 arrangement $\mathcal{A}_H$ are antipodal. Further define

$$\begin{aligned}
 L_2(c, c') &:= \{X \in L_2 : c/X, c'/X \text{ are antipodal chambers in } \mathcal{A}_X\} \\
 &= \{X \in L_2 : \mathcal{A}_X \subseteq L_1(c, c')\}.
 \end{aligned}$$

Denote by $\mathcal{R}(c, c')$ the set of all minimal galleries r from c to c' .

Note that any minimal gallery r in $\mathcal{R}(c, c')$ must cross each of the hyperplanes in $L_1(c, c')$ exactly once, and is completely determined by the linear order in which these hyperplanes are crossed. For each codimension-two subspace X in L_2 , there are two possibilities:

- If $X \notin L_2(c, c')$, then the hyperplanes of $\mathcal{A}_X$ must be crossed by r in a unique linear order, namely the order in which one crosses the hyperplanes of $\mathcal{A}_X \cap L_1(c, c')$ when walking from c/X to c'/X in rank 2.
- If $X \in L_2(c, c')$, then the hyperplanes of $\mathcal{A}_X$ can be crossed by r in one of two possible linear orders, as in Figure 2.1.

Consequently, given two minimal galleries $r, r' \in \mathcal{R}(c, c')$, one can again speak of the separation set $L_2(r, r')$ as the subset of codimension-two subspaces X in $L_2(c, c')$ on which r, r' disagree.⁵

⁵We apologize for the slightly confusing overuse of “ L_2 ” in our notation: we now have not only the set L_2 , which is the collection of all codimension-two intersection subspaces of $\mathcal{A}$, but

Definition 5.1 (The graph $G_2(c, c')$). Given the arrangement $\mathcal{A}$ and two chambers $c, c' \in \mathcal{C}$, define a graph $G_2(c, c')$ whose vertex set is the set $\mathcal{R}(c, c')$ of minimal galleries c to c' , and having an edge between two galleries $\{r, r'\}$ exactly when $|L_2(r, r')| = 1$.

The previous discussion shows that the map $L_2 : \mathcal{R}(c, c') \times \mathcal{R}(c, c') \longrightarrow 2^{L_2(c, c')}$ provides a set-valued metric on $G_2(c, c')$, taking values in $L_2(c, c')$. That discussion also shows that the generalization of the second assertion in Proposition 2.3 still holds: having fixed a base gallery r_0 in $\mathcal{R}(c, c')$, then any other gallery r in $\mathcal{R}(c, c')$ is uniquely determined by its separation set $L_2(r_0, r)$. One also has this immediate consequence of Propositions 3.4 and 3.5.

Corollary 5.2. *The graph $G := G_2(c, c')$ is always bipartite, and its distance function satisfies $d_G(r, r') \geq |L_2(r, r')|$.*

Remark 5.3. Whenever $c_0, c, c' \in \mathcal{C}$, satisfy $L_1(c_0, c) \subset L_1(c_0, c')$, one can define an injection $\mathcal{R}(c_0, c) \hookrightarrow \mathcal{R}(c_0, c')$: fix any minimal gallery r from c to c' , and then concatenation with r as a suffix gives such an injection. It is easily seen that this leads to an embedding of the graph $G_2(c_0, c) \hookrightarrow G_2(c_0, c')$ as a vertex-induced subgraph.

In particular, although the graphs $G_2(c_0, c)$ with their set-valued metric do not in general have a $\mathbb{Z}_2$ -equivariant involution, they are always vertex-induced subgraphs of the graph $G_2(c_0, -c_0)$, which does have such an involution.

Remark 5.4. When $\mathcal{A}$ is the arrangement of reflecting hyperplanes for a finite real reflection group W , the (simply) transitive W -action on the chambers means that any graph $G_2(c, c')$ is isomorphic to the graph $G(w) := G_2(c_0, w(c_0))$ for some fixed choice of a base chamber c_0 and some group element w in W . It is not hard to check that this graph $G(w)$ is the graph of reduced words for w discussed in the Introduction.

Furthermore, in this situation, the sets

$$\begin{aligned} L_1(w) &:= L_1(c_0, w(c_0)), \\ L_2(w) &:= L_2(c_0, w(c_0)) \end{aligned}$$

have the following reflection group interpretations:

- $L_1(w)$ is the usual (left-)inversion set of w , that is, the collection of positive roots α_H for W which are sent to negative roots by w^{-1} .
- $L_2(w)$ is the collection of rank two subroot systems Φ_X having the property that w^{-1} sends every positive root in Φ_X to a negative root.

One then has the following extension of Theorem 4.9.

Theorem 5.5. *Let $\mathcal{A}$ be a real (central, essential) hyperplane arrangement in $\mathbb{R}^d$ which is supersolvable, let $F := \{X_i\}_{i=0}^d$ be a modular flag for $\mathcal{A}$, let c_0 be any chamber incident to F , and let c be any other chamber.*

also two functions called $L_2(-, -)$, namely

$$\begin{aligned} \mathcal{C} \times \mathcal{C} &\longrightarrow 2^{L_2} \\ (c, c') &\longmapsto L_2(c, c'), \\ \mathcal{R}(c, c') \times \mathcal{R}(c, c') &\longrightarrow 2^{L_2(c, c')} \\ (r, r') &\longmapsto L_2(r, r'). \end{aligned}$$

We hope that context resolves any confusion that arises within this section.

- (i) There is a unique minimal gallery r_0 from c_0 to c incident to F .
- (ii) This minimal gallery r_0 is an L_2 -accessible vertex for the graph $G_2(c_0, c)$ on the galleries $\mathcal{R}$ from c_0 to c .

Proof. The proof of assertion (i) is by induction on d exactly as in the proof of Theorem 4.9. This unique gallery r_0 is obtained by applying induction to the $(d-1)$ -dimensional supersolvable arrangement $\mathcal{A}_\ell$, lifting the unique gallery from c_0/ℓ to c/ℓ incident to F/ℓ in order to first cross all hyperplanes in $L_1(c_0, c) \cap \mathcal{A}_\ell$. One must complete it by then crossing the hyperplanes in $L_1(c_0, c) \setminus \mathcal{A}_\ell$ in the linear order which is the restriction of the one from Proposition 4.6(i).

The proof of assertion (ii) is also by induction on d exactly as in the proof of Theorem 4.9. One wishes to show that for any gallery $r \neq r_0$ from c_0 to c , there will be another gallery r' having $L_2(r_0, r') \subset L_2(r_0, r)$ and $|L_2(r, r')| = 1$. Again there are two cases, depending upon whether (Case 1) or not (Case 2) the gallery r crosses all the hyperplanes of $L_1(c_0, c) \cap \mathcal{A}_\ell$ before crossing any hyperplanes of $L_1(c_0, c) \setminus \mathcal{A}_\ell$. There is no essential change in the proof of Case 1.

In Case 2, one must note that the exhibited gallery r' having $L_2(r, r') = \{X\}$ satisfies in addition that X lies in $L_2(c_0, c)$. This is immediate from the fact that r crossed *every* hyperplane of $\mathcal{A}_X$ on its way from c_0 to c , so that $\mathcal{A}_X \subset L_1(c_0, c)$, that is, X lies in $L_2(c_0, c)$. $\square$

Example 5.6. Continuing Example 4.10, for the reflection arrangement $\mathcal{A}$ of type A_5 , choose as modular flag $F := \{X_i\}_{i=0}^5$, where $X_i := \{x_1 = x_2 = \cdots = x_{6-i}\}$. The chamber $c_0 := \{x_1 < \cdots < x_6\}$ corresponding to the identity permutation is incident to the above modular flag F . Let $w = 316425 \in \mathfrak{S}_6$. The unique gallery r_0 from c_0 to $c := w(c_0)$ incident to F crosses the hyperplanes in this order:

$$\begin{aligned} &H_{23}, H_{13}, \\ &H_{24}, \\ &H_{56}, H_{26}, H_{46}. \end{aligned}$$

The corresponding sequence of chambers in the gallery is indexed by the permutations

$$123456 \xrightarrow{H_{23}} 132456 \xrightarrow{H_{13}} 312456 \xrightarrow{H_{24}} 314256 \xrightarrow{H_{56}} 314265 \xrightarrow{H_{26}} 314625 \xrightarrow{H_{46}} 316425,$$

which corresponds to the following reduced decomposition for w

$$\begin{aligned} &(s_2, s_1, \\ &s_3, \\ &s_5, s_4, s_3). \end{aligned}$$

Corollary 5.7. Let $\mathcal{A}$ be a real (central, essential) hyperplane arrangement in $\mathbb{R}^d$ which is supersolvable, let $F := \{X_i\}_{i=0}^d$ be a modular flag for $\mathcal{A}$, and let c_0 be any chamber incident to F . Let c be any other chamber, and let r_0 be the unique minimal gallery from c_0 to c incident to F .

Then in the graph $G := G_2(c_0, c)$ of minimal galleries from c_0 to c , any two galleries r, r' satisfy

$$\begin{aligned} |L_2(r, r')| &\leq d_G(r, r') \leq |L_2(r_0, r)| + |L_2(r_0, r')| \\ &\leq 2|L_2(c_0, c)|. \end{aligned}$$

Thus the diameter of G is at most $2|L_2(c_0, c)|$.

In particular, when $\mathcal{A}$ is a reflection arrangement of type A, B or dihedral type $I_2(m)$, so that $G = G(w)$ for some $w \in W$, any two reduced words r, r' satisfy

$$\begin{aligned} |L_2(r, r')| &\leq d_G(r, r') \leq |L_2(r_0, r)| + |L_2(r_0, r')| \\ &\leq 2|L_2(w)|. \end{aligned}$$

Thus the diameter of G is at most $2|L_2(w)|$.

Example 5.8. Example 3.11 and the four words listed in (3) show that both the upper and lower bounds on $d_G(r, r')$ given in Corollary 5.7 need not be tight, even for $G = G(w_0)$ in type A_{n-1} .

In light of Theorem 1.1 one might wonder whether the upper bound of $2|L_2(w)|$ for the diameter of $G(w)$ in types A, B can be improved to an upper bound of $|L_2(w)|$. However, we mention here some examples in types A_3, B_3 showing that even an upper bound of $|L_2(w)|$ for the diameter of $G(w)$ is not always tight.

In type A_3 , where W is the symmetric group $\mathfrak{S}_4$, the permutation $w = 3412$ has two reduced words

$$\mathcal{R}(w) = \left\{ (s_2, s_3, s_1, s_2), \right. \\ \left. (s_2, s_1, s_3, s_2) \right\}$$

so that the graph $G(w)$ is a single edge, having diameter 1. However, in this case, $L_2(w) = \{X_{13,24}, X_{14,23}\}$, so that $|L_2(w)| = 2$.

One encounters a similar phenomenon in type B_3 , for the signed permutation w that maps the standard basis vectors e_1, e_2, e_3 , respectively, to $-e_3, -e_2, -e_1$, respectively. This w has only two reduced words

$$\mathcal{R}(w) = \{ (s_0, s_1, s_0, s_2, s_1, s_0), \\ (s_0, s_1, s_2, s_0, s_1, s_0) \}$$

with respect to the Coxeter generators $S = \{s_0, s_1, s_2\}$ from Example 4.11, so that the graph $G(w)$ is a single edge, having diameter 1. But $L_2(w)$ consists of the three subspaces of codimension-two of the form $\{x_i = 0, x_j + x_k = 0\}$ with $\{i, j, k\} = \{1, 2, 3\}$, so that $|L_2(w)| = 3$.

These examples suggest the following conjecture.⁶

Conjecture 5.9. For an element w in the symmetric group $\mathfrak{S}_n$,

$$\frac{1}{2}|L_2(w)| \leq \text{diameter}(G(w)) \leq |L_2(w)|.$$

For an element w in the hyperoctahedral group B_n ,

$$\frac{1}{3}|L_2(w)| \leq \text{diameter}(G(w)) \leq |L_2(w)|.$$

Remark 5.10. The authors would like to thank an anonymous referee for pointing out the following connection between Theorems 4.9, 5.5, and the work of Armstrong [1] on *sorting orders* in Coxeter groups.

Let W be a real reflection group W , with reflection arrangement $\mathcal{A}$, and assume $\mathcal{A}$ is supersolvable with a choice of modular flag F . Theorem 4.9 shows there is a unique minimal gallery r_0 from c_0 to $-c_0$ incident to F , which corresponds to a

⁶This conjecture has been checked only for the symmetric groups S_n with $n \leq 5$, and for the hyperoctahedral groups B_n with $n \leq 3$.

particular reduced word $\mathbf{w}_0$ for the longest element w_0 in W . Theorem 5.5 then shows that for any other chamber c , there is again a unique minimal gallery r from c_0 to c incident to F . If w is the unique element of W for which $c = w(c_0)$, then this minimal gallery r corresponds to a particular reduced word $\mathbf{w}$ for w .

Meanwhile in Armstrong's work, *any* choice of a reduced word $\mathbf{w}_0$ for w_0 induces, for each w in W , a particular reduced word $\mathbf{w}$ for w , which he calls the $\mathbf{w}_0$ -sorted word for w . Specifically, $\mathbf{w}$ is the lexicographically leftmost subword of $\mathbf{w}_0$ that gives a reduced word for w .

One can show that these two constructions are the same: the word $\mathbf{w}$ corresponding to the gallery r incident to F is the same as the $\mathbf{w}_0$ -sorted word for w . This can be shown using an induction on the rank d , similar to the one employed in the proof of Theorems 4.9 and 5.5, together with the compatibility of parabolic coset factorization with weak Bruhat orders, and [1, Thm. 4.2].

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