FINITE SUMS OF PROJECTIONS IN VON NEUMANN ALGEBRAS

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ABSTRACT. We first prove that in a σ -finite von Neumann factor M , a positive element a with properly infinite range projection R_a is a linear combination of projections with positive coefficients if and only if the essential norm $\|a\|_e$ with respect to the closed two-sided ideal $J(M)$ generated by the finite projections of M does not vanish. We then show that if $\|a\|_e > 1$, then a is a finite sum of projections. Both these results are extended to general properly infinite von Neumann algebras in terms of central essential spectra. Secondly, we provide a necessary condition for a positive operator a to be a finite sum of projections in terms of the principal ideals generated by the excess part $a_+ := (a - I)\chi_a(1, \infty)$ and the defect part $a_- := (I - a)\chi_a(0, 1)$ of a ; this result appears to be new for $B(H)$ also. Thirdly, we prove that in a type II_1 factor a sufficient condition for a positive diagonalizable operator to be a finite sum of projections is that $\tau(a_+) > \tau(a_-)$.

1. INTRODUCTION

The goal of this article is to study the following two related problems in the context of general von Neumann algebras.

- (A) Which positive operators are finite linear combinations of projections with positive coefficients (called *positive combinations of projections*)?
- (B) Which positive operators are *finite sums of projections*?

First we give some historical background about problem (A). Fillmore in [10] proved that all operators in $B(H)$ are linear combinations of projections. This result was extended to properly infinite von Neumann algebras by Percy and Topping [33], to type II_1 factors by Fack and de la Harpe [9] and by Percy and Topping [34], and to von Neumann algebras with no type I direct summands with infinite dimensional center by Goldstein and Paszkiewicz [13]. It was also extended to several types of C^* -algebras by various authors including Marcoux [29], [30] and Marcoux and Murphy [31].

The first result specifically on problem (A) was obtained by Fillmore who proved in [11] that positive *invertible* operators in $B(H)$ (for a separable Hilbert space H) are positive combinations of projections. This result later was extended to all von Neumann algebras with no finite type I direct summands by Bikhchentaev [2, Lemma 5]. In this article we present an alternative proof for the same result and provide an estimate of the number of projections needed (Proposition 2.3).

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It was remarked by Fillmore in [10] that infinite rank compact operators on a separable Hilbert space cannot be positive combinations of projections; Fong and Murphy in [12, Theorem 11] showed that these operators are the only exceptions in $B(H)$.

We prove that for simple σ -finite von Neumann factors, that is, finite factors or type III factors, all positive operators are positive combinations of projections, while for type II_∞ factors the only exceptions are, as in $B(H)$, those operators with infinite range projection that belong to the ideal of relative compact operators (Theorem 2.12, Corollary 3.5). For a general von Neumann algebra M , the ideal of relative compact operators $J(M)$ is the closed two-sided ideal generated by the finite projections of M and was first studied by Breuer [3], [4] and Sonis [35].

For global von Neumann algebras, we obtain a characterization of operators that are positive combinations of projections and have properly infinite range projection in terms of a spectral property of the operator (Theorem 2.12). A reformulation in terms of ideals (Theorem 3.4) uses the notion of essential central spectrum relative to a central ideal. This notion was introduced by Halpern in a cycle of papers including [14] and [15], (see also Stratila and Zsido [36]). The relevant facts are summarized in Section 3.

The key ingredient for our characterization both for the factor and the nonfactor case is Lemma 2.8, new also for $B(H)$, which proves that the direct sum of an arbitrary positive operator and of a locally invertible operator on a “large” subspace is a positive combination of projections. A slightly weaker formulation of this lemma holds also for a large class of C^* -algebras and is used in another paper [23].

Next we give some background on problem (B). Fillmore characterized the positive finite rank operators that are sums of finitely many projections:

Theorem 1.1 ([11, Theorem 1]). *Let $a \in B(H)^+$ have finite rank. Then a is a sum of projections if and only if $\text{Tr}(a) \geq \text{rank}(a)$ and $\text{Tr}(a) \in \mathbb{N}$.*

As reported in a survey article [39, Theorem 4.12] by Wu quoting unpublished joint work of Choi and Wu in 1988, Choi and Wu proved that positive operators with essential norm greater than 1 are *finite sums of projections*. This result is presented together with other related results in their recent preprint [5, Theorem 2.2]. The special case of αI with $\alpha > 1$ follows also from a delicate analysis which characterizes the numbers α for which αI is a sum of at least n projections (see [27] and several other papers).

In recent years related topics became quite active again. In [8], Dykema, Freeman, Kornelson, Larson, Ordower, and Weber, motivated by problems in frame theory, proved by different techniques that all positive operators with essential norm greater than 1 are *strong sums of projections*. Their work inspired other papers, including [1], [26], and work by three of the authors of the present article ([21], [22], [23], [25], and [24]).

In [21], Kaftal, Ng, and Zhang obtained a characterization of strong sums of projections in properly infinite σ -finite von Neumann factors (thus including $B(H)$) and for *diagonalizable* operators, also in type II factors. These characterizations were given in terms of the *excess* and *defect* operators a_+ and a_- , which will also play an important role in the present paper (see Definition 5.3 and Theorem 5.4 for a precise statement).

In [22], the same authors presented a characterization of *strict* sums of projections in the multiplier algebras of σ -unital purely infinite simple C^* -algebras and

in [23] they provided sufficient conditions for operators to be finite sums of projections in the multiplier algebras of σ -unital purely infinite simple C^* -algebras. [24] proved that all operators with norm larger than 1 in σ -unital purely infinite simple C^* -algebras with torsion K_0 are finite sums of projections.

The full problem (B), i.e., the complete characterization of the positive operators that are *finite* sums of projections is still open even in the $B(H)$ case. Here we present some sufficient and some necessary conditions for von Neumann algebras.

For σ -finite factors a sufficient condition is the natural $B(H)$ -analog of having the essential norm larger than 1. The essential norm here is the one relative to the compact ideal $J(M)$, namely the norm in the quotient algebra $M/J(M)$ (see Corollary 4.4). In the type III case, $J(M) = \{0\}$ and hence the condition is that the operator have operator norm larger than 1.

For global algebras we replace the essential norm with the “essential central norm”, with respect to a certain central ideal (Theorem 4.3). The reader interested mainly in factors or, in particular, in $B(H)$, can simply assume that all the “central” objects mentioned are scalars, or more precisely, scalar multiples of the identity.

Our approach is based on Lemma 4.2, where we show that the direct sum of an arbitrary positive operator and a scalar multiple greater than 1 of a “large” projection is always a finite sum of projections. As a consequence, we obtain in Theorem 4.3 that if the essential central norm of an operator $a \in M^+$ is greater than and bounded away from the identity, then a is a finite sum of projections. Restricted to the special case of $B(H)$, our approach gives an alternative proof of the result of Choi and Wu [5], and hence, of Dykema and all [8].

Then in Theorem 5.5 we find a necessary condition for a positive operator a to be a finite sum of projections, formulated in terms of the two-sided (nonclosed) principal ideals of M generated by the excess and defect operators of a . This condition is new for $B(H)$ and generalizes some results in [5] (see Remark 5.10 for more details).

From the initial work of Fillmore, Pearcy, and Topping, the finite case has always proven to be more delicate than the properly infinite case. In [21, Theorem 1] we considered positive *diagonalizable* operators in a II_1 factor M with trace τ and proved that the inequality $\tau(a_+) \geq \tau(a_-)$ is necessary and sufficient for a to be a strong sum of projections. Via a different and considerably more complicated analysis, we show in the present article (Theorem 6.4) that the strict inequality $\tau(a_+) > \tau(a_-)$ is sufficient also for a to be a finite sum of projections. However, the problem remains open when a is not diagonalizable or when $\tau(a_+) = \tau(a_-)$.

Notation. For every $x \in \mathbb{R}$, we will denote by $[x]$ the smallest integer $n \geq x$ and by $\lfloor x \rfloor$ the largest integer $n \leq x$, i.e., the *integer part* of x .

For every selfadjoint element a , χ_a will denote the (projection-valued) spectral measure of a and R_a the range projection of a .

We shall use standard notation for von Neumann algebras: M will denote a von Neumann algebra represented on a Hilbert space H . In general, we will not assume that M is a factor nor that it is σ -finite (i.e., countably decomposable). For semifinite algebras, τ will denote a faithful, normal, semifinite trace and $Z = M \cap M'$ will denote the center of M . If $p \in M$, $c(p)$ will denote the central support of p and M_p the restriction of pMp to the subspace pH . We will identify M_p with pMp and $J(M)$ will denote the closed two-sided ideal generated by all the finite projections of M . If $M = B(H)$, then $J(M)$ is the ideal $K(H)$ of the compact operators on H .

If M is of type III, then $J(M) = \{0\}$. The essential norm with respect to the ideal $J(M)$, i.e., the quotient norm in $M/J(M)$, is denoted by $\|\cdot\|_e$.

2. POSITIVE COMBINATIONS OF PROJECTIONS

Goldstein and Paszkiewicz proved in [13] that a von Neumann algebra has the property that its elements are linear combinations of projections in the algebra if and only if the algebra has no finite type I direct summands with infinite dimensional center, i.e., if it is a direct sum of a properly infinite algebra, a type II_1 algebra, and at most a finite number of matrix algebras.

Using his characterization of sums of two projections [11, Corollary of Theorem 2], Fillmore proved in [11, Corollary of Theorem 3] that every positive invertible operator acting on a separable Hilbert space can be decomposed into a positive combination of projections.

It is easy to see that this property does not extend to general von Neumann algebras.

Lemma 2.1. *Let $\mathcal{A}$ be an infinite dimensional abelian von Neumann algebra. Then there are positive invertible operators in $\mathcal{A}$ that are not linear combinations of projections.*

Proof. Since $\mathcal{A}$ is infinite dimensional, it contains an infinite sequence $\{e_k\}$ of mutually orthogonal projections with $\sum_{k=1}^{\infty} e_k = I$. Let $\mu_k \in [1, 2]$ be a sequence with infinitely many distinct entries. Let $a := \sum_{k=1}^{\infty} \mu_k e_k$. Then a is a positive invertible operator in $\mathcal{A}$. We claim that a is not a linear combination of projections in $\mathcal{A}$. Assume by contradiction that $a = \sum_{j=1}^n \lambda_j p_j$ for some $\lambda_j \in \mathbb{C}$ and projections $p_j \in \mathcal{A}$. Since the projections commute, we can subdivide them into mutually orthogonal projections and thus obtain that $a = \sum_{j=1}^{n'} \lambda'_j p'_j$ with p'_j mutually orthogonal projections in $\mathcal{A}$. But then, on the one hand, we obtain that the spectrum $\text{Sp}(a) = \{\lambda'_j\}_1^{n'}$ is finite and on the other hand that $\text{Sp}(a) = \overline{\{\mu_k\}_1^{\infty}}$ is infinite, a contradiction. $\square$

Abelian von Neumann algebras are a special case of finite type I algebras. In [2, Lemma 5 (C)], Bikchentaev proved using [13, Theorem 3] that in any von Neumann algebra with no finite type I direct summands, positive invertible operators are always positive combinations of projections.

In the following proposition we present a different proof of this result, while providing the explicit estimate of the number of projections that is needed in Theorem 6.4 below. Our proof is an adaptation of Fong's $B(H)$ argument in [12, Lemma 8]. It is based on the notion of the following algebra constants N_o and V_o , which he introduced for $B(H)$, but which also exist for many other operator algebras:

Definition 2.2. An operator algebra M has constants N_o and V_o if every selfadjoint operator $a \in M$ can be decomposed into a real linear combination $a = \sum_{j=1}^{N_o} \alpha_j p_j$ of N_o projections $p_j \in M$ such that

$$\sum_{j=1}^{N_o} |\alpha_j| \leq V_o \|a\|.$$

The existence and the estimates of the constants N_o and V_o for von Neumann algebras with no finite type I direct summands with infinite dimensional center are

given in Theorems 1–3 in [13], explicitly for N_o , and implicitly (i.e., as a simple consequence of the statements of the theorems) for V_o :

- If M is properly infinite, then $N_o = 6$ and $V_o = 8$;
 (2.1) if M is of type II_1 , then $N_o = 12$ and $V_o = 14$;
 if M is the direct sum of m matrix algebras, then $N_o = m + 4$ and
 $V_o = m + 4$.

Proposition 2.3. *Let M be a von Neumann algebra that has no finite type I direct summands with infinite dimensional center and hence has constants N_o and V_o . Assume that $a \in M$ is positive and is invertible. Then a is a positive combination of $N_o + \lceil V_o(\|a\|\|a^{-1}\| - 1) \rceil + 1$ projections in M .*

Proof. Set $\nu := \frac{1}{\|a^{-1}\|}$. Then $a \geq \nu I$ and $\chi_a[\nu, \|a\|] = 1$. Partition the interval $[\nu, \|a\|]$ into $n = \lceil V_o(\frac{\|a\|}{\nu} - 1) \rceil$ equal subintervals with endpoints

$$v = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n = \|a\|$$

and set

$$e_k := \begin{cases} \chi_a[\nu, \lambda_1], & k = 1, \\ \chi_a(\lambda_{k-1}, \lambda_k], & 2 \leq k \leq n. \end{cases} \quad \text{and} \quad b := a - \sum_{j=1}^n \lambda_{j-1} e_j.$$

Then $\sum_{j=1}^n e_j = I$, $b \geq 0$, and $\|b\| \leq \frac{\|a\| - \nu}{n} \leq \frac{\nu}{V_o}$. By the hypothesis that M has constants N_o and V_o , we can decompose $b = \sum_{i=1}^{N_o} \alpha_i q_i$ into a real linear combination of N_o projections $q_i \in M$ satisfying the condition that $\sum_{i=1}^{N_o} |\alpha_i| \leq V_o \|b\| \leq \nu$. But then

$$\begin{aligned} a &= \sum_{j=1}^n (\lambda_{j-1} - \nu) e_j + \nu I + \sum_{i=1}^{N_o} \alpha_i q_i \\ &= \sum_{j=1}^n (\lambda_{j-1} - \nu) e_j + \sum_{\alpha_i \geq 0} \alpha_i q_i + \sum_{\alpha_i < 0} (-\alpha_i)(I - q_i) + \left(\nu - \sum_{\alpha_i < 0} (-\alpha_i)\right) I. \end{aligned}$$

As desired, the number of projections in the above linear combination does not exceed $N_o + n + 1$ and all the coefficients are nonnegative. $\square$

A positive element a such that $a \geq \nu R_a$ for some $\nu > 0$ is invertible in the reduced algebra M_{R_a} . We call such an element *locally invertible*.

If p is a projection in M , we identify M_p with the hereditary subalgebra pMp of M . Clearly, if $a \in M_p$ is a linear combination of projections in M_p , then a is also a linear combination of projections in M . The converse, however, does not hold: a can be a linear combination of projections in M , but as the restriction to pH of a projection q fails to be a projection unless q commutes with p , it does not follow that a is a linear combination of projections in M_p . If however $a = \sum_{j=1}^n \lambda_j p_j$ is a positive combination of projections in M , then $p_j \leq \frac{1}{\lambda_j} a \leq \frac{\|a\|}{\lambda_j} p$ and hence $p_j \in M_p$ for all j . Thus a is also a positive combination of projections in M_p . To summarize:

Lemma 2.4. *Let $a \in M^+$. Then a is a positive combination of projections in M if and only if a is a positive combination of projections in M_{R_a} .*

Thus an immediate consequence of Proposition 2.3 and its proof is the following.

Corollary 2.5. *Let $a \in M^+$ and assume that it is locally invertible, i.e., $a \geq \nu R_a$ for some $\nu \geq 0$, and that M_{R_a} has constants N_o and V_o . Then a is a positive combination of $N_o + \lceil V_o(\frac{\|a\|}{\nu} - 1) \rceil + 1$ projections in M .*

Recall that M_{R_a} has constants N_o and V_o if and only if M_{R_a} has no finite type I direct summands with infinite dimensional center. To guarantee that this holds for every $a \in M$, we must assume that M itself has no (finite or infinite) type I direct summands with infinite dimensional center.

Corollary 2.6. *A von Neumann algebra M has the property that every positive locally invertible operator in M can be decomposed into a positive combination of projections in M if and only if it has no type I direct summands with infinite dimensional center.*

Proof. Assume that M has no type I direct summands with infinite dimensional center and let $a \in M^+$ be locally invertible. Then a is invertible in the reduced algebra M_{R_a} . Now M_{R_a} also has no type I direct summands with infinite dimensional center and hence, a fortiori, has no finite type I direct summands with infinite dimensional center. Thus the conclusion follows from Proposition 2.3 and Lemma 2.4.

For the opposite implication, assume by contradiction that M has a type I direct summand M_g with infinite dimensional center $(M \cap M')_g$, for some central projection g . By definition there is an abelian projection $p \in M$ with central support $c(p) = g$, and hence $M_p = (M \cap M')_p$ is an infinite dimensional abelian algebra. By Lemma 2.1, M_p contains positive invertible operators that are not a positive linear combination of projections in M_p . Seen as elements of M , they are positive locally invertible and, by Lemma 2.4, they are not positive combinations of projections in M . $\square$

Question 2.7. In the case of finite type I algebras with infinite dimensional center we are not aware of conditions that determine which elements are linear combinations or are positive combinations of projections.

Invertibility is of course not a necessary condition for a positive operator in $B(H)$ to be a positive combination of projections; indeed by the spectral theorem, all positive finite rank operators are positive combinations of projections. Fillmore noted that compact operators with infinite rank cannot be a positive combination of projections [10, Remark 5]. Fong and Murphy showed in [12, Lemma 9] that these are the only exceptions. To extend this result to von Neumann algebras, we will need to follow a different approach. The key step is provided by the following lemma, which shows that a direct sum of a positive operator with a positive invertible operator of “sufficiently large” range is a positive combination of projections.

Lemma 2.8. *Let e and f be orthogonal projections in M with $e \prec f$ and such that M_f has no finite type I direct summands with infinite dimensional center. Let $b = be, d = df$ be positive elements in M such that $d \geq \nu f$ for some $\nu > 0$. Then $a := b + d$ is a positive combination of projections in M .*

Proof. If $b = 0$, then $a = d$ is locally invertible (i.e., invertible in the algebra M_f), thus a is positive combination of projections in M by Proposition 2.3 and Lemma 2.4. Assume from now on that $b \neq 0$.

We first prove the statement under the additional assumption that $\nu > \|b\|$.

Choose a partial isometry $v \in M$ for which $v^*v = e$ and $f' := vv^* \leq f$. Define the projections

$$q_- := \begin{pmatrix} \frac{b}{\|b\|} & -\left(\frac{b}{\|b\|} - \frac{b^2}{\|b\|^2}\right)^{1/2} v^* \\ -v\left(\frac{b}{\|b\|} - \frac{b^2}{\|b\|^2}\right)^{1/2} & f' - \frac{1}{\|b\|} v b v^* \end{pmatrix}$$

and

$$q_+ := \begin{pmatrix} \frac{b}{\|b\|} & \left(\frac{b}{\|b\|} - \frac{b^2}{\|b\|^2}\right)^{1/2} v^* \\ v\left(\frac{b}{\|b\|} - \frac{b^2}{\|b\|^2}\right)^{1/2} & f' - \frac{1}{\|b\|} v b v^* \end{pmatrix}.$$

Then $q_- + q_+ = \frac{2}{\|b\|}b + 2f' - \frac{2}{\|b\|}v b v^*$, and hence

$$a = \frac{\|b\|}{2}q_- + \frac{\|b\|}{2}q_+ + d - \|b\|f' + v b v^*.$$

Since

$$d - \|b\|f' + v b v^* \geq d - \|b\|f' \geq (\nu - \|b\|)f,$$

it follows by Proposition 2.3 and Lemma 2.4 that the locally invertible element $d - \|b\|f' + v b v^*$ is a positive combinations of projections in M , and hence, so is a .

Now we remove the assumption that $\nu > \|b\|$. Let $0 < \nu' < \nu$. Since $d = d - \nu'f + \nu'f$ and since $d - \nu'f \geq (\nu - \nu')f$ is locally invertible and hence a positive combination of projections, we can assume without loss of generality that $d = \nu f$. By passing to direct summands, we can consider separately the case when f is properly infinite, and when it is finite. If f is properly infinite, we can decompose it into the sum $\sum_{j=1}^k f_j$ of k mutually orthogonal projections $f_j \sim f$ with $k > \frac{\|b\|}{\nu}$. Then

$$a = \sum_{j=1}^k \left(\frac{1}{k}b + \nu f_j \right)$$

is the sum of k positive operators satisfying the hypotheses of this lemma and the additional condition that $\|\frac{1}{k}b\| < \nu$. Thus from the first part of the proof they are positive combination of projections and hence so is a .

If f is finite, then decompose e into the sum of projections $e' := \chi_b(0, \frac{\nu}{2})$ and $e'' := \chi_b[\frac{\nu}{2}, \infty)$ which both commute with b . By the first part of the proof, $be' + d$ is a positive combination of projections in M . For the “remainder” be'' , notice that $e'' \leq e \prec f$, hence $e'' \sim f''$ for some projection $f'' \leq f$. Since $M_{f''}$ cannot have any type I summand (which would be necessarily finite) with infinite dimensional center, neither can $M_{e''}$. Since $be'' \geq \frac{\nu}{2}e''$ is locally invertible in $M_{e''}$, it follows again by Proposition 2.3 and Lemma 2.4 that be'' is a positive combination of projections in M , and hence so is $a = be'' + be' + d$. $\square$

Remark 2.9. The first part of the proof with the additional assumption that $\nu > \|b\|$ holds without changes also for any C^* -algebra for which all positive locally invertible elements are positive combinations of projections. These algebras include, among others, all properly infinite simple σ -unital C^* -algebras and their multiplier algebras. We will focus on such C^* -algebras in [23].

Notice that the operator a in Lemma 2.8 satisfies the condition

$$(2.2) \quad \exists \delta > 0 \text{ such that } \chi_a(0, \delta) \prec \chi_a[\delta, \infty).$$

It is easy to see, and it can also be obtained as a simple consequence of Lemma 2.10 below, that if M_{R_a} is properly infinite, then condition (2.2) is equivalent to

$$(2.3) \quad \exists \delta > 0 \text{ such that } \chi_a[\delta, \infty) \sim R_a.$$

In Theorem 2.12, we will see that condition (2.2) is sufficient for a to be a positive combination of projections and is also necessary when R_a is properly infinite.

First, we need a result for which we could not find an explicit reference. It is the generalization of the well-known fact for $B(H)$ that if the supremum of a finite number of projections is infinite, then at least one of the projections must be infinite.

Lemma 2.10. *Assume that M is properly infinite and that $\bigvee_{j=1}^n p_j \sim I$ for some projections $p_j \in M$. Then there is a family of mutually orthogonal central projections g_j , some of which may be zero, with $\sum_{j=1}^n g_j = I$ and for which $p_j g_j \sim g_j$ for every j .*

Proof. The proof is by induction on n . Let us first prove the claim for $n = 2$. Assume first that $p_1 \vee p_2 = I$. Since I is properly infinite, we can find a projection $q \sim q^\perp \sim I$. By the comparison property of projections, there is a central projection g such that

$$(2.4) \quad (q \wedge p_1)g \prec (q^\perp \wedge p_1^\perp)g,$$

$$(2.5) \quad (q^\perp \wedge p_1^\perp)g^\perp \prec (q \wedge p_1)g^\perp.$$

Then

$$\begin{aligned} g &\sim qg && \text{(since } q \sim I\text{)} \\ &= (q \wedge p_1)g + (q - q \wedge p_1)g \\ &\prec (q^\perp \wedge p_1^\perp)g + (q - q \wedge p_1)g && \text{(by (2.4), since } q^\perp \wedge p_1^\perp \perp q - q \wedge p_1\text{)} \\ &\sim (q^\perp \wedge p_1^\perp)g + (q \vee p_1 - p_1)g && \text{(by Kaplanski's Parallelogram Law,} \\ &&& \text{since } q^\perp \wedge p_1^\perp \perp q \vee p_1 - p_1\text{)} \\ &= p_1^\perp g \\ &= (p_1 \vee p_2 - p_1)g \\ &\sim (p_2 - p_1 \wedge p_2)g && \text{(by Kaplanski's Parallelogram Law)} \\ &\leq p_2 g \\ &\leq g. \end{aligned}$$

Thus $p_2 g \sim g$. Similarly, $p_1 g^\perp \sim g^\perp$.

Next, consider the case when $p_1 \vee p_2 \sim I$, i.e., when there is an isometry w such that $w^*(p_1 \vee p_2)w = I$. Then $w^*p_1w \vee w^*p_2w = I$; hence by the first part of the proof, there is a central projection g such that $w^*p_1wg \sim g$ and $w^*p_2wg^\perp \sim g^\perp$. But then $p_1g \sim g$ and $p_2g^\perp \sim g^\perp$, which concludes the case $n = 2$.

Now assume that the property holds for $n - 1$ and that $\bigvee_{j=1}^n p_j \sim I$. Then $p_1 \vee \left(\bigvee_{j=2}^n p_j\right) \sim I$ and hence by the result for $n = 2$, there is a central projection g_1 for which $p_1 g_1 \sim g_1$ and

$$(2.6) \quad \left(\bigvee_{j=2}^n p_j\right)g_1^\perp \sim g_1^\perp.$$

If $g_1^\perp = 0$, i.e., $g_1 = I$, we choose $g_j = 0$ for $j \geq 2$ and we are done. If $g_1^\perp \neq 0$, then by (2.6), $\bigvee_{j=2}^n (p_j g_1^\perp) \sim g_1^\perp$; hence by the induction hypothesis applied to the projection $\{p_j g_1^\perp\}_2^n$ in the properly infinite algebra $M_{g_1^\perp}$, we obtain a decomposition of $g_1^\perp = \sum_{j=2}^n g_j$ into mutually orthogonal central projections in $M_{g_1^\perp}$ for which

$$p_j g_j = p_j g_1^\perp g_j \sim g_j.$$

Of course, the central projections $\{g_j\}_2^n$ thus found in the center of $M_{g_1^\perp}$ can be identified with central projections in M (e.g., see [6, Ch. I, Sect. 2, Corollaire, Proposition 2]), which concludes the proof. $\square$

The following lemma is based on and partially overlaps with the proof of [18, Lemma 2.6].

Lemma 2.11. *Let $a \in M^+$ and assume that $a = \sum_j \lambda_j p_j$ for a (finite or infinite) collection of scalars $\lambda_j > 0$ and projections $p_j \in M$, where the sum converges in the strong topology in case the collection is infinite. Assume furthermore that $\delta := \inf \lambda_j > 0$. Then for every j , $\chi_a(0, \delta) \prec R_a - p_j$ and $p_j \prec \chi_a[\delta, \infty)$.*

Proof. If $a \geq \lambda p$ for some $\lambda > 0$ and some projection $p \in M$, then

$$\begin{aligned} \|a^{1/2}\xi\| &\geq \lambda^{1/2}\|\xi\| \quad \forall \xi \in pH, \\ \|a^{1/2}\xi\| &< \lambda^{1/2}\|\xi\| \quad \forall 0 \neq \xi \in \chi_{a^{1/2}}(0, \lambda^{1/2})H. \end{aligned}$$

Since $\chi_a(0, \lambda) = \chi_{a^{1/2}}(0, \lambda^{1/2})$, we see that $\chi_a(0, \lambda)H \cap pH = \{0\}$; i.e., we have $\chi_a(0, \lambda) \wedge p = 0$. But then, by Kaplanski's Parallelogram Law,

$$\begin{aligned} \chi_a(0, \lambda) &= \chi_a(0, \lambda) - \chi_a(0, \lambda) \wedge p \\ &\sim (R_a - p) - (R_a - p) \wedge (R_a - \chi_a(0, \lambda)) \\ &\leq R_a - p \end{aligned}$$

and thus

$$\chi_a(0, \lambda) \prec R_a - p.$$

Also

$$\begin{aligned} p &= p - p \wedge \chi_a(0, \lambda) \\ &\sim (R_a - \chi_a(0, \lambda)) - (R_a - \chi_a(0, \lambda)) \wedge (R_a - p) \\ &\leq R_a - \chi_a(0, \lambda) \\ &= \chi_a[\lambda, \infty), \end{aligned}$$

and thus

$$p \prec \chi_a[\lambda, \infty).$$

Since $a \geq \lambda_j p_j$ for every j , we thus have

$$\chi_a(0, \delta) \leq \chi_a(0, \lambda_j) \prec R_a - p_j \quad \text{and} \quad p_j \prec \chi_a[\lambda_j, \infty) \leq \chi_a[\delta, \infty).$$

$\square$

Theorem 2.12. *Let M be a von Neumann algebra and let $a \in M^+$.*

- (i) *Assume that M_{R_a} has no finite type I direct summands with infinite dimensional center and that there is a $\delta > 0$ such that $\chi_a(0, \delta) \prec \chi_a[\delta, \infty)$. Then a is a positive combination of projections in M .*
- (ii) *If R_a is properly infinite, then a is a positive combination of projections in M if and only if there is a $\delta > 0$ such that $\chi_a(0, \delta) \prec \chi_a[\delta, \infty)$.*

- (iii) If M_{R_a} is a finite sum of finite factors or of σ -finite type III factors, then a is always a positive combination of projections in M .

Proof. (i) Assume that $\chi_a(0, \delta) \prec \chi_a[\delta, \infty)$ for some δ and set

$$\begin{aligned} e &:= \chi_a(0, \delta), & f &:= \chi_a[\delta, \infty), \\ b &:= a\chi_a(0, \delta) = ae, & d &:= a\chi_a[\delta, \infty) = af. \end{aligned}$$

Then $a = b + d$ and $R_a = e + f$. Since $c(e) \leq c(f)$, it follows that $c(f) = c(R_a)$. Assume by contradiction that M_f had a finite direct summand M_{fg} of type I with infinite dimensional center for some central projection $g \leq c(f)$. Then it would follow that M_g is a type I algebra with infinite dimensional center and that fg is a finite projection. But then eg too is finite and hence so is R_ag . But then M_{R_ag} too would be a finite type I algebra with infinite dimensional center against the assumption. Thus the hypotheses of Lemma 2.8 applied to $a = b + d$ are satisfied, and hence a is a positive combination of projections.

- (ii) We prove that the condition is sufficient. Since R_a is properly infinite, then M_{R_a} has no finite direct summands of any type, so a is a positive combination of projections in M by (i).

Conversely, assume that a is a positive combination of projections, i.e., that $a = \sum_{j=1}^n \lambda_j p_j$ for some scalars $\lambda_j > 0$ and projections p_j . To simplify notation, assume that $R_a = I$, i.e., $\bigvee_1^n p_j = I$. Let $\delta = \min_j \lambda_j$. Then $p_j \prec \chi_a[\delta, \infty)$ for all $1 \leq j \leq n$ by Lemma 2.11 and by Lemma 2.10, there is a family of mutually orthogonal central projections g_j with $\sum_{j=1}^n g_j = I$ for which $p_j g_j \sim g_j$. Then $g_j \prec \chi_a[\delta, \infty) g_j$ for all j , and hence, $\chi_a[\delta, \infty) \sim I$. Thus (2.2) holds.

- (iii) Assume, without loss of generality, that $R_a = I$ and that M is a factor.

If M is of type I_n , i.e., a matrix algebra, then the conclusion follows from the spectral theorem.

If M is of type II_1 and τ is the canonical trace, then $\tau(\chi_a(0, \delta)) \rightarrow 0$ for $\delta \rightarrow 0$. Choose a $\delta > 0$ so that $\tau(\chi_a(0, \delta)) \leq \tau(\chi_a[\delta, \infty))$. Then $\chi_a(0, \delta) \prec \chi_a[\delta, \infty)$ and hence a is a positive combination of projections by (i).

If M is a σ -finite type III factor, then all nonzero projections are equivalent, and hence, the condition (2.2) holds trivially. Again, the conclusion follows from (i). $\square$

The following example shows that condition (2.2) may not be necessary when M is neither properly infinite nor a finite sum of finite factors.

Example 2.13. Assume that $M = \sum_{j=1}^{\infty} M_j$ is an infinite direct sum of finite factors M_j such that each M_j contains three mutually orthogonal and equivalent nonzero projections e_j, f_j, g_j . Choose a sequence $(0, 1) \ni \lambda_j \rightarrow 1$ and define

$$a_j := (1 - \lambda_j)e_j + (1 - \lambda_j)f_j + (1 + 2\lambda_j)g_j$$

and $a := \bigoplus_{j=1}^{\infty} a_j$. By Theorem 1.1, each a_j is the sum of three projections, and hence, so is a . In particular, a is a positive combination of projections. However,

for every $0 < \delta < 1$, we have

$$\begin{aligned}\chi_a(0, \delta) &= \bigoplus_{j=1}^{\infty} \chi_{a_j}(0, \delta) = \bigoplus_{1-\lambda_j < \delta} (e_j + f_j), \\ \chi_a[\delta, \infty) &= \bigoplus_{j=1}^{\infty} \chi_{a_j}[\delta, \infty) = \bigoplus_{j=1}^{\infty} g_j + \bigoplus_{1-\lambda_j \geq \delta} (e_j + f_j).\end{aligned}$$

Since $e_j + f_j \not\leq g_j$, it follows that $\chi_a(0, \delta) \not\leq \chi_a[\delta, \infty)$.

3. CENTRAL ESSENTIAL SPECTRA

Fong and Murphy in [12, Theorem 11] proved that if $a \in B(H)^+$ (with H an infinite dimensional and separable Hilbert space) and R_a is infinite, then a is a positive combination of projections if and only if $a \notin K(H)$, i.e., if and only if $\|a\|_e > 0$, where $\|a\|_e$ denotes the essential norm. A natural extension of this result is to the case of σ -finite properly infinite von Neumann factors, by replacing $K(H)$ by the ideal of relative compact operators $J(M)$ generated by the finite projections of M . This can be deduced easily from Theorem 2.12 and is a special case of Theorem 3.4 below.

For von Neumann algebras that are not σ -finite, however, we need more general *central ideals* and if the algebras are not factors, we need the corresponding center-valued version of the essential spectrum, the *essential central spectrum*. The need for the latter is illustrated by the following example.

Example 3.1. Let $M = \bigoplus_{k=1}^{\infty} B(H)$, where H is an infinite dimensional separable Hilbert space and let $a = \bigoplus_{k=1}^{\infty} a_k \in M^+$. Assume that R_a is properly infinite. Then a is a positive combination of projections if and only if for some $\nu > 0$,

$$C_u(a) := \bigoplus_{k=1}^{\infty} (\|a_k\|_e I_k) \geq \nu c(R_a).$$

Notice that $J(M) = \bigoplus_{k=1}^{\infty} K(H)$; hence $\|a\|_e = \sup_k \|a_k\|_e$. Thus in particular the condition that $\|a\|_e > 0$ is not sufficient for a to be a positive combination of projections.

Proof. For the necessity of the condition, let $a = \sum_{j=1}^n \lambda_j p_j$ for some $\lambda_j > 0$ and projections $p_j = \bigoplus_{k=1}^{\infty} p_{jk}$. Then $a_k = \sum_{j=1}^n \lambda_j p_{jk}$ for every k . Let

$$\pi : B(H) \rightarrow B(H)/K(H)$$

denote the canonical quotient map. Then since $\pi(a_k) = \sum_{j=1}^n \lambda_j \pi(p_{jk})$, it follows that $\|a_k\|_e := \|\pi(a_k)\| \geq \lambda_j \|\pi(p_{jk})\|$ for every k and j . Now $\|\pi(p_{jk})\| \in \{0, 1\}$ and by the assumption that R_a is properly infinite, all the summands a_k are either 0 or have infinite rank, i.e., $(R_a)_k = R_{a_k} = \bigvee_{j=1}^n p_{jk}$ is either 0 or an infinite projection. Thus if $a_k \neq 0$, then there is at least one index j for which p_{jk} is infinite, in which case $\pi(p_{jk}) \neq 0$ and hence $\|\pi(p_{jk})\| = 1$. But then, $\|a_k\|_e \geq \min \lambda_j$. Let $\nu := \min \lambda_j$. Then $\nu > 0$ and $\bigoplus_{k=1}^{\infty} (\|a_k\|_e I_k) \geq \nu c(R_a)$.

For the sufficiency of the condition, notice that for any $0 < \delta < \nu$ and for every k for which $a_k \neq 0$ and hence $\|a_k\|_e \geq \nu$ it follows that $\chi_{a_k}[\delta, \infty) \sim I_k$ and hence

$$\chi_a[\delta, \infty) = \bigoplus_k \chi_{a_k}[\delta, \infty) \sim \bigoplus_{a_k \neq 0} I_k \sim R_a.$$

Thus condition (2.2) is satisfied and hence by Theorem 2.12, a is a positive combination of projections. $\square$

The operator $C_u(a) = \bigoplus_{k=1}^{\infty} (\|a_k\|_e I_k)$ is the largest element in what is called the *essential central spectrum* of a , which in the case of a direct sum of factors is easy to describe as:

$$Z\text{-Sp}(a) := \left\{ \bigoplus_{k=1}^{\infty} (\mu_k I_k) \mid \mu_k \in \sigma_e(a_k) \right\}.$$

For arbitrary von Neumann algebras, essential central spectra were introduced in the study of central ideals by Halpern [14]. Example 3.1 suggests a reformulation of condition (2.2) in Theorem 2.12 (ii) in terms of central ideals and essential central spectra.

For the reader's convenience we summarize the notions from [14] concerning central ideals and essential central spectra that will be needed here. The reader interested only in σ -finite factors can skip the remainder of this section.

An ideal $\mathcal{J}$ in a von Neumann algebra M is called *central* if whenever $\{g_\gamma\}_{\gamma \in \Gamma}$ is a family of mutually orthogonal central projections g_γ and $\{a_\gamma\}_{\gamma \in \Gamma}$ is a norm-bounded family of elements of $\mathcal{J}$, then $a := \sum_{\gamma \in \Gamma} a_\gamma g_\gamma$ belongs to $\mathcal{J}$.

Denote by $Z := M \cap M'$ the center of the von Neumann algebra M . For every properly infinite projection $p \in M$ and every central projection $f \geq c(p)$ (recall that $c(p)$ denotes the central support of p), let $\mathcal{J}_f(p)$ be the norm closed two-sided ideal of M whose set of projections is

$$\{q \in M \mid q \leq f \text{ and } pq \prec qg \text{ for some projection } g \in Z \Rightarrow pg = 0\}.$$

Thus by definition

$$\mathcal{J}_f(p) = \mathcal{J}_{c(p)}(p) \oplus M_{f-c(p)}$$

and a projection q belongs to $\mathcal{J}_{c(p)}(p)$ if and only if $q \prec p$ and $qq \not\prec pq$ for all central projections $0 \neq g \leq c(p)$. Then $\mathcal{J}_f(p)$ is central and every central ideal $\mathcal{J}$ has this form. If we further assume that $\overline{\mathcal{J}_f(p)}^w = Mf$, then f is unique and p is unique up to Murray-von Neumann equivalence ([14, Theorem 2.4, Proposition 2.5]).

To simplify notation, identify Z with $C(X)$, with X a compact Hausdorff extremely disconnected space. X is called the spectrum of the abelian algebra Z and is identified by the Gelfand theorem with the collection of the closed two-sided maximal ideals of Z . For every $\zeta \in X$, let $[\zeta]$ denote the closed two-sided ideal of M generated by ζ . For every closed two-sided ideal $\mathcal{J}$ of M , $\mathcal{J} + [\zeta]$ is also a closed two-sided ideal of M and $\pi_{\mathcal{J} + [\zeta]}$ denotes the canonical quotient map $M \rightarrow M/(\mathcal{J} + [\zeta])$.

Given a central ideal $\mathcal{J}$ and an element $a \in M$, the essential central spectrum of a is defined as the collection

$$Z\text{-Sp}(a) := \{z \in Z \mid z(\zeta) \in \text{Sp}(\pi_{\mathcal{J} + [\zeta]}(a)) \text{ for all } \zeta \in X\}.$$

Then $Z\text{-Sp}(a)$ is always a nonempty subset of the center (more precisely, of the center of $M_{c(R_a)}$) and is closed in the strong operator topology [14, Theorem 3.5, Proposition 3.9]. If $a = a^*$, then $Z\text{-Sp}(a)$ is selfadjoint and has [14, Proposition 3.12] a maximal element $C_u(a) \in Z\text{-Sp}(a)$ given for all $\zeta \in X$ by

$$C_u(a)(\zeta) := \max \text{Sp}(\pi_{\mathcal{J} + [\zeta]}(a)).$$

When a is positive, $C_u(a)$ takes the role of an “essential central norm”. Since we consider only one central ideal at a time, to simplify notation we will not mark explicitly the dependence on $\mathcal{J}$ of $Z\text{-Sp}(a)$ or of the maximal element $C_u(a)$.

Recall the well-known property in $B(H)$ (H infinite dimensional separable): if $a \in B(H)^+$, then $\|a\|_e \geq \nu$ if and only if for every $\epsilon > 0$ the spectral projection $\chi_a[\nu - \epsilon, \infty)$ is infinite (equivalently, $\chi_a[\nu - \epsilon, \infty) \sim I$). It is implicit in [14] that this characterization extends to the essential central spectrum in the context of properly infinite von Neumann algebras, but for the reader’s convenience we make the connection explicit, listing only the facts that we need for this article.

From now on, fix an $a \in M^+$ with properly infinite range projection R_a and set $\mathcal{J} := \mathcal{J}_{c(R_a)}(R_a)$. Notice that the operator $\bigoplus_{k=1}^{\infty} (\|a_k\|_e I_k)$ in Example 3.1 is indeed the element $C_u(a)$ relative to this ideal $\mathcal{J}$.

Lemma 3.2. *If $z \in Z\text{-Sp}(a)$, $\epsilon > 0$ and $p := \chi_{a-z}[-\epsilon, \epsilon]$, then $R_a \prec p$.*

Proof. To simplify notation, assume that $c(R_a) = I$. Reasoning as in [14, Proposition 2.9], let f be the maximal central projection for which $R_a f \prec pf$. If $R_a g \prec pf^\perp g$ for some central projection g , then also $R_a f^\perp g \prec pf^\perp g$. By the maximality of f , it follows that $f^\perp g = 0$, and hence that $R_a g = 0$. By definition, $pf^\perp \in \mathcal{J}$. By [14, Proposition 3.13], $f^\perp = 0$ and thus $R_a \prec p$. $\square$

Proposition 3.3. *Let $\nu > 0$. Then $C_u(a) \geq \nu c(R_a)$ if and only if $\chi_a[\nu - \epsilon, \infty) \sim R_a$ for all $0 < \epsilon < \nu$.*

Proof. Without loss of generality, assume that $c(R_a) = I$. For a fixed $0 < \epsilon < \nu$, to simplify notation let $z := C_u(a)$ and $p := \chi_{a-z}[-\epsilon, \epsilon]$. First assume that $z = C_u(a) \geq \nu c(R_a)$. Since p commutes with $a - z$, it also commutes with a . Then

$$(\nu - \epsilon)p \leq zp - \epsilon p \leq ap \leq zp + \epsilon p \leq (\|z\| + \epsilon)p.$$

Hence

$$p \leq \chi_a[\nu - \epsilon, \|z\| + \epsilon] \leq \chi_a[\nu - \epsilon, \infty) \leq \chi_a(0, \infty) = R_a.$$

But $R_a \prec p$ by Lemma 3.2; hence $R_a \sim \chi_a[\nu - \epsilon, \infty)$.

Assume now that for every $0 < \epsilon < \nu$ we have $r := \chi_a[\nu - \epsilon, \infty) \sim R_a$. Then $a \geq ar \geq (\nu - \epsilon)r$. Thus for every $\zeta \in X$ we have

$$\pi_{\mathcal{J}+[\zeta]}(a) \geq (\nu - \epsilon)\pi_{\mathcal{J}+[\zeta]}(r),$$

and hence

$$C_u(a)(\zeta) = \max\{\text{Sp}(\pi_{\mathcal{J}+[\zeta]}(a))\} = \|\pi_{\mathcal{J}+[\zeta]}(a)\| \geq (\nu - \epsilon)\|\pi_{\mathcal{J}+[\zeta]}(r)\|.$$

Since $\pi_{\mathcal{J}+[\zeta]}(r)$ is a projection, $\|\pi_{\mathcal{J}+[\zeta]}(r)\|$ is either 0 or 1. Reasoning by contradiction, assume that for some $\zeta_o \in X$, we have $\|\pi_{\mathcal{J}+[\zeta_o]}(r)\| = 0$. The function

$$X \ni \zeta \rightarrow \|\pi_{\mathcal{J}+[\zeta]}(r)\| \in \mathbb{R}$$

is continuous by [14, Theorem 3.2], and hence $\|\pi_{\mathcal{J}+[\zeta]}(r)\|$ vanishes on a closed set containing ζ_o . This set is clopen because the spectrum of any abelian von Neumann algebra is extremely disconnected; thus the characteristic function of this set is also continuous. Hence that characteristic function is identified with a projection g in the center of M . Then by [14, Lemma 3.1], $rg \in \mathcal{J}$. Since $r \sim R_a$, reasoning as in [14, Proposition 2.9], it follows that $g = 0$, a contradiction. Thus $\|\pi_{\mathcal{J}+[\zeta]}(r)\| = 1$ for all ζ , i.e., $C_u(a)(\zeta) \geq \nu - \epsilon$ for all ζ . Thus $C_u(a) \geq (\nu - \epsilon)I$ and hence $C_u(a) \geq \nu I$. $\square$

Notice that the existence of $\nu > 0$ such that $\chi_a[\nu - \epsilon, \infty) \sim R_a$ for all $0 < \epsilon < \nu$ is obviously equivalent to the existence of $\delta > 0$ such that $\chi_a[\delta, \infty) \sim R_a$ (condition (2.3)), which in turn, as we have already remarked, is equivalent to condition (2.2). In other words, we can reformulate condition (ii) in Theorem 2.12 in terms of essential central spectra as follows.

Theorem 3.4. *Let $a \in M^+$ have properly infinite range projection R_a and let $\mathcal{J} = \mathcal{J}_{c(R_a)}(R_a)$. Then the following are equivalent:*

- (i) *a is a positive combination of projections in M .*
- (ii) *There is a $\delta > 0$ for which $\chi_a(0, \delta) \prec \chi_a[\delta, \infty)$.*
- (iii) *There is a $\delta > 0$ for which $\chi_a[\delta, \infty) \sim R_a$.*
- (iv) *$C_u(a)$ is locally invertible; i.e., there is a $\nu > 0$ for which $C_u(a) \geq \nu c(R_a)$.*

In the case when M is a properly infinite σ -finite factor, and R_a is infinite, then $C_u(a) = \|a\|_e I$, where $\|\cdot\|_e$ is the essential norm relative to the relative compact ideal $J(M)$. Thus condition (iv) is equivalent to asking that $a \notin J(M)$. This is the generalization of the Fong and Murphy characterization in $B(H)$ in [12, Theorem 11].

Corollary 3.5. *Let M be a properly infinite σ -finite factor and let $a \in M^+$ have a properly infinite range projection R_a . Then a is a positive combination of projections if and only if $a \notin J(M)$.*

4. FINITE SUMS OF PROJECTIONS IN PROPERLY INFINITE ALGEBRAS

In this section we prove, for general properly infinite von Neumann algebras, an analog of the $B(H)$ result in [5] that positive operators with essential spectrum larger than 1 are finite sums of projections. Our approach, which is different from the one by Choi and Wu, is to decompose the operator into an infinite sum of blocks, where each block is a sum of not more than a fixed number of projections and where the blocks separated by more than one index are orthogonal. The following simple folklore lemma shows how to then reassemble this infinite sum into a finite sum of projections.

Lemma 4.1. *Let $\{b_j\}$ be a sequence of positive elements of M such that*

- (i) *$R_{b_i} R_{b_j} = 0$ for all $|i - j| > 1$,*
- (ii) *b_j is the sum of N_j projections in M , and*
- (iii) *$N := \sup N_j < \infty$.*

Then $b := \sum_{j=1}^{\infty} b_j$ is the sum of $2N$ projections in M .

Proof. Let

$$b_o := \sum_{j=1}^{\infty} b_{2j-1} \quad \text{and} \quad b_e := \sum_{j=1}^{\infty} b_{2j}.$$

By hypothesis, the range projections of the elements b_{2j-1} are mutually orthogonal, and so are those of the elements b_{2j} . Since $\sup_j \|b_j\| \leq N$, both series converge in the strong operator topology and hence $b = b_o + b_e$. Set $b_j = \sum_{k=1}^N q_{j,k}$, where the $q_{j,k}$ are projections in M (some of these projections can vanish). For each $1 \leq k \leq N$, let

$$q_{o,k} := \sum_{j=1}^{\infty} q_{2j-1,k} \quad \text{and} \quad q_{e,k} := \sum_{j=1}^{\infty} q_{2j,k}.$$

Both series being the sum of mutually orthogonal projections converge to a projection. Thus $b_o = \sum_{k=1}^N q_{o,k}$ and $b_e = \sum_{k=1}^N q_{e,k}$ are each the sum of N projections in M and hence $b = b_o + b_e$ is the sum of $2N$ projections in M . $\square$

The following is one of the key lemmas in this paper. The core of the argument is based on Fillmore's characterization of finite sums of finite projections (see Theorem 1.1).

Lemma 4.2. *Let M be a von Neumann algebra and e and f be orthogonal projections in M such that $e \prec f$ and f is properly infinite. Let $a = be + \alpha f$, where $\alpha > 1$ and $b = be = eb \geq 0$. Then a is the sum of finitely many projections in M .*

Proof. We first reduce the problem to the case when b is a scalar multiple of e and $f = \sum_{k=1}^{\infty} e_k$ is an infinite sum of projections equivalent to e .

Decompose $f = f' + f''$ into a direct sum of two equivalent projections,

$$f' \sim f'' \sim f.$$

Since f and hence f' are properly infinite, $M_{f'}$ has no direct finite summands. Thus Lemma 2.8 applies and $b + \alpha f'$ is a positive combination of projections, say,

$$b + \alpha f' = \sum_{j=1}^n \beta_j p_j$$

for some $n \in \mathbb{N}$, $\beta_j > 0$, and projections $p_j \in M$. Further decompose the properly infinite projection $f'' = \sum_{j=1}^n f_j$ into a sum of n mutually orthogonal equivalent projections $f_j \sim f'' \sim f$. Then

$$a = \sum_{j=1}^n (\beta_j p_j + \alpha f_j),$$

where $p_j \leq e + f' \perp f''$ and hence $p_j \perp f_j$ for all j . Furthermore, $p_j \prec f \sim f_j$ and f_j is properly infinite. Thus after this first part of the reduction, we can assume, without loss of generality, that $a = \beta e + \alpha f$ for some $\beta > 0$.

Next, since the projections $fc(e)$ and $f(c(f) - c(e))$ either vanish or are properly infinite, we can further decompose them into two infinite sums of mutually orthogonal projections $e'_k \sim e$ and $e''_k \sim f(c(f) - c(e))$, that is,

$$fc(e) = \sum_{k=1}^{\infty} e'_k \quad \text{and} \quad f(c(f) - c(e)) = \sum_{k=1}^{\infty} e''_k.$$

Then

$$a = \beta e + \alpha f(c(e)) + \alpha f(c(f) - c(e)) = \beta e + \alpha \sum_{k=1}^{\infty} e'_k + \alpha \sum_{k=1}^{\infty} e''_k.$$

Both summands have the form $\beta e_0 + \alpha \sum_{k=1}^{\infty} e_k$, where the projections $\{e_k\}_0^{\infty}$ are mutually orthogonal and equivalent, $\beta \geq 0$, and $\alpha > 1$. Thus this completes the reduction. Explicitly, we assume without loss of generality that

$$a = \beta e_0 + \alpha \sum_{k=1}^{\infty} e_k, \quad e_k \sim e_0, \quad \beta \geq 0, \quad \alpha > 1.$$

The equivalence of the projections $\{e_k\}_0^{\infty}$ defines an embedding of $B(H)$ in M (H separable infinite dimensional) under which the projections $\{e_k\}_0^{\infty}$ correspond

to rank-one projections in $B(H)$. Thus to simplify notation, assume henceforth that $M = B(H)$ and that all the projections e_k have rank one.

Since

$$a = (\beta - \lfloor \beta \rfloor)e_o + (\alpha - \lceil \alpha - 2 \rceil) \sum_{k=1}^{\infty} e_k + \lfloor \beta \rfloor e_o + \lceil \alpha - 2 \rceil \sum_{k=1}^{\infty} e_k$$

and $\lfloor \beta \rfloor e_o + \lceil \alpha - 2 \rceil \sum_{k=1}^{\infty} e_k$ are already finite sums of projections, we can assume, without loss of generality, that $1 < \alpha \leq 2$ and that $0 \leq \beta < 1$. Set $\beta_0 := \beta$, $n_0 = 0$ and $N := \lfloor \frac{\alpha^2}{\alpha-1} \rfloor$.

Let

$$\begin{aligned} n_1 &:= \left\lceil \frac{1 - \beta_0}{\alpha - 1} \right\rceil, \\ \beta_1 &:= \beta_0 + n_1 \alpha - \lfloor \beta_0 + n_1 \alpha \rfloor, \\ a_1 &:= \beta_0 e_{n_0} + \alpha \sum_{j=n_0+1}^{n_1-1} e_j + (\alpha - \beta_1) e_{n_1}. \end{aligned}$$

Here and in the sequel, we adopt the convention of dropping any sum where the upper index of summation is strictly less than the lower index of summation. So, if $n_1 = 1$ (which occurs when $1 - \beta \leq \alpha - 1$), then the above formula just reads $a_1 = \beta_0 e_0 + (\alpha - \beta_1) e_1$.

By definition, $0 \leq \beta_1 < 1 < \alpha$ and thus a_1 is a positive finite rank operator with

$$\begin{aligned} \text{Tr}(a_1) &= \beta_0 + (n_1 - 1)\alpha + \alpha - \beta_1 = \lfloor \beta_0 + n_1 \alpha \rfloor \in \mathbb{N}, \\ R_{a_1} &= \begin{cases} e_{n_0} + \sum_{j=1}^{n_1} e_j & \text{if } \beta_0 \neq 0 \\ \sum_{j=1}^{n_1} e_j & \text{if } \beta_0 = 0 \end{cases} \end{aligned}$$

and hence

$$\text{rank}(a_1) = \begin{cases} n_1 + 1 & \text{if } \beta_0 \neq 0 \\ n_1 & \text{if } \beta_0 = 0 \end{cases} \leq n_1 + 1.$$

By definition of $\lceil \cdot \rceil$,

$$\frac{1 - \beta_0}{\alpha - 1} \leq n_1 < \frac{1 - \beta_0}{\alpha - 1} + 1 = \frac{\alpha - \beta_0}{\alpha - 1};$$

hence

$$n_1 + 1 \leq \beta_0 + n_1 \alpha < \beta_0 + \frac{(\alpha - \beta_0)\alpha}{\alpha - 1} = \frac{\alpha^2 - \beta_0}{\alpha - 1} \leq \frac{\alpha^2}{\alpha - 1}.$$

From the first inequality, it follows that

$$\text{rank}(a_1) \leq n_1 + 1 \leq \lfloor \beta_0 + n_1 \alpha \rfloor = \text{Tr}(a_1).$$

From the remaining chain of inequalities it follows that

$$\text{Tr}(a_1) = \lfloor \beta_0 + n_1 \alpha \rfloor \leq \lfloor \frac{\alpha^2}{\alpha - 1} \rfloor = N.$$

Since $\text{rank}(a_1) \leq \text{Tr}(a_1) \in \mathbb{N}$, Theorem 1.1 applies and we conclude that a_1 is the sum of $\text{Tr}(a_1) \leq N$ rank-one projections.

Now

$$\begin{aligned} a - a_1 &= \beta_0 e_{n_0} + \alpha \sum_{j=n_0+1}^{\infty} e_j - \left(\beta_0 e_{n_0} + \alpha \sum_{j=n_0+1}^{n_1-1} e_j + (\alpha - \beta_1) e_{n_1} \right) \\ &= \beta_1 e_{n_1} + \alpha \sum_{j=n_1+1}^{\infty} e_j \end{aligned}$$

has the same form as the beginning operator a and with the same constant α . Thus repeating the construction, let

$$\begin{aligned} n_2 &:= n_1 + \left\lceil \frac{1 - \beta_1}{\alpha - 1} \right\rceil, \\ \beta_2 &:= \beta_1 + (n_2 - n_1)\alpha - \lfloor \beta_1 + (n_2 - n_1)\alpha \rfloor, \\ a_2 &:= \beta_1 e_{n_1} + \alpha \sum_{j=n_1+1}^{n_2-1} e_j + (\alpha - \beta_2) e_{n_2}. \end{aligned}$$

Then by the same argument as above, a_2 is a positive operator with

$$\begin{aligned} \text{rank}(a_2) &= \begin{cases} n_2 - n_1 + 1 & \text{if } \beta_1 \neq 0 \\ n_2 - n_1 & \text{if } \beta_1 = 0 \end{cases} \leq n_2 - n_1 + 1, \\ \text{Tr}(a_2) &= \lfloor \beta_1 + (n_2 - n_1)\alpha \rfloor \in \mathbb{N} \end{aligned}$$

and

$$\text{rank}(a_2) \leq \text{Tr}(a_2) \leq N.$$

So, again, by Theorem 1.1, a_2 is a sum of not more than N projections. Notice that

$$R_{a_1} R_{a_2} = \begin{cases} 0 & \beta_1 = 0 \\ e_{n_1} & \beta_1 \neq 0; \end{cases}$$

thus in general $R_{a_1} R_{a_2}$ does not vanish. We can iterate this process to construct for each k the positive operator

$$a_k = \beta_{k-1} e_{n_{k-1}} + \alpha \sum_{j=n_{k-1}+1}^{n_k-1} e_j + (\alpha - \beta_k) e_{n_k},$$

which is the sum of at most N projections. By construction, $a = \sum_{j=1}^{\infty} a_j$. An immediate but key observation is that already $R_{a_1} R_{a_3} = 0$ and in general

$$R_{a_j} R_{a_{j'}} = 0 \quad \text{for } |j - j'| > 1.$$

Thus by Lemma 4.1, a is a sum of finitely many projections. □

Now we proceed to one of our main results.

Theorem 4.3. *Let M be a von Neumann algebra. Assume that $a \in M^+$ has a properly infinite range projection R_a and $C_u(a) \geq \nu c(R_a)$ for some $\nu > 1$. Then a is a finite sum of projections in M .*

Proof. Let $1 < \alpha < \nu$. By Proposition 3.3, $\chi_a[\alpha, \infty) \sim R_a$. By [15, Lemma 3.2], $\chi_a[\alpha, \infty) = p + q$ for some projections p and q that commute with a and for which $p \sim q \sim R_a$. Then in particular,

$$ap = a\chi_a[\alpha, \infty)p \geq \alpha\chi_a[\alpha, \infty)p = \alpha p$$

and similarly, $aq \geq \alpha q$. Let

$$\begin{aligned} a_1 &:= a(R_a - p - q) + (a - \alpha I)q + \alpha p, \\ a_2 &:= (a - \alpha I)p + \alpha q \end{aligned}$$

so that $a = a_1 + a_2$. Since p and q and hence $R_a - p - q$ commute with a , it follows that a_1 and a_2 are both the direct sum of positive operators. Since $p \sim q \sim R_a$, it follows that $R_a - p \prec p$ and $p \prec q$; thus both a_1 and a_2 satisfy Lemma 4.2 and hence are both finite sums of projections. Thus a too is a finite sum of projections. $\square$

As already mentioned, in the σ -finite factor case, the essential central norm reduces to the (scalar) essential norm relative to the ideal $J(M)$.

Corollary 4.4. *Assume that M is an infinite σ -finite factor and $a \in M^+$. A sufficient condition for a to be a finite sum of projections in M is that*

- (i) $\|a\|_e > 1$ when M is of type I_∞ (usual essential norm of $B(H)$);
- (ii) $\|a\|_e > 1$ when M is of type II_∞ (essential norm relative to the ideal $J(M)$);
- (iii) $\|a\| > 1$ when M is of type III (operator norm).

The following examples show that we cannot relax the condition that $C_u(a)$ is bounded away from I .

First, notice that if $a \in B(H)^+$ and R_a is infinite, then the condition $\|a\|_e \geq 1$ is necessary for a to be a finite sum of projections, since then at least one of the projections must be infinite, and hence, has essential norm 1. However, the following simple example shows that the condition $\|a\|_e \geq 1$ is not sufficient.

Example 4.5. Let $a = I + k$ with $k \in K(H)^+$, $0 < \text{Tr}(k) < \infty$ but $\text{Tr}(k) \notin \mathbb{N}$. Then $\|a\|_e = 1$. By [21, Theorem 17], a is not an infinite sum of projections, hence cannot be a finite sum of projections. In fact even if $\text{Tr}(k) \in \mathbb{N}$, if k has infinite rank, then a is not a finite sum of projections by Corollary 5.8 below.

Next notice that even the condition $C_u(a)(\zeta) > 1$ for all $\zeta \in X$ is not sufficient.

Example 4.6. Let H be an infinite dimensional separable Hilbert space, let $M := \bigoplus_{n=1}^\infty B(H)$ and let $a := \bigoplus_{n=1}^\infty (1 + \frac{1}{n})I$. Then $C_u(a) = a \geq I$, but it is not bounded away from I . By [27, Theorem 3], for $n \geq 5$, $(1 + \frac{1}{n})I$ is the sum of not less than $n + 1$ projections, and thus, a cannot be the sum of finitely many projections in M .

5. NECESSARY CONDITIONS

First, we extend to general von Neumann algebras a result that we have obtained for σ -finite factors and strong sums of projections in [21, Proposition 3.1]. For the $B(H)$ case that result is equivalent to [5, Theorem 1.2]. Denote by g the maximal finite central projection of M and by Φ the canonical center-valued trace on M_g .

Proposition 5.1. *Let $a \in M^+$ and $n \in \mathbb{N}$. Then the following conditions are equivalent:*

- (i) *There is a partial isometry $v \in M$ with $v^*v = R_a$ and a decomposition of the identity $I = \sum_{j=1}^n q_j$ into n mutually orthogonal nonzero projections $q_j \in M$ for which $q_j v a v^* q_j = q_j$ for $1 \leq j \leq n$.*
- (ii) *$a = \sum_{j=1}^n p_j$ is the sum of n nonzero projections $p_j \in M$, and there is a decomposition of the identity $I = \sum_{j=1}^n q_j$ into n mutually orthogonal nonzero projections $q_j \in M$ for which $p_j \sim q_j$ for $1 \leq j \leq n$.*

- (iii) $a = \sum_{j=1}^n p_j$ is the sum of n nonzero projections $p_j \in M$, $\Phi(ag) = g$, and $R_a g^\perp \sim g^\perp$.

Proof. (i) $\Leftrightarrow$ (ii). This is, basically, the same proof as in [21, Proposition 3.1], but restricted to finite sums. For the reader's convenience we briefly reproduce it here. Assume that (i) holds. For every $1 \leq j \leq n$, let $w_j := q_j v a^{1/2}$. Then $w_j w_j^* = q_j v a^{1/2} a^{1/2} v^* q_j = q_j$ and hence $p_j := w_j^* w_j$ is a projection equivalent to q_j . Then

$$\begin{aligned} \sum_{j=1}^n p_j &= \sum_{j=1}^n a^{1/2} v^* q_j v a^{1/2} = a^{1/2} v^* \left(\sum_{j=1}^n q_j \right) v a^{1/2} \\ &= a^{1/2} v^* v a^{1/2} = a^{1/2} R_a a^{1/2} = a. \end{aligned}$$

Conversely, assume (ii) holds. Let $w_j \in M$ be partial isometries for which $w_j w_j^* = q_j$ and $w_j^* w_j = p_j$ and let $b := \sum_{j=1}^n w_j$. Then

$$b^* b = \sum_{i,j=1}^n w_i^* w_j = \sum_{j=1}^n p_j = a.$$

Let $b = v a^{1/2}$ be the polar decomposition of b . Then $v^* v = R_a$ and

$$q_j v a^{1/2} = q_j b = \sum_{i=1}^n q_j w_i = w_j;$$

hence $q_j v a v^* q_j = q_j$.

(ii) $\Leftrightarrow$ (iii). It is enough to prove the claim separately for the cases when M is properly infinite and when it is finite.

First assume that M is properly infinite (i.e., $g = 0$) and assume that (ii) holds. By Lemma 2.10, there is a decomposition of the identity $I = \sum_{j=1}^n g_j$ into central projections g_j such that $q_j g_j \sim g_j$. Then for every j ,

$$R_a g_j = \left(\bigvee_{i=1}^n p_i \right) g_j \geq p_j g_j \sim q_j g_j \sim g_j.$$

Thus $R_a g_j \sim g_j$ for all j and hence $R_a \sim I$. Next assume that (iii) holds, i.e., that $R_a \sim I$. Again by Lemma 2.10, there is a decomposition of the identity $I = \sum_{j=1}^n g_j$ into central projections g_j such that $p_j g_j \sim g_j$. By the assumption that M is properly infinite, we can decompose each $g_j = \sum_{i=1}^n e_{ij}$ into a sum of mutually orthogonal and equivalent projections $e_{ij} \sim g_j$. Then for each $i \neq j$, $p_i g_j \prec e_{ij}$; thus we can choose projections p_{ij} so that $p_i g_j \sim p_{ij} \leq e_{ij}$. Set $p_{jj} := e_{jj} + \sum_{i \neq j} (e_{ij} - p_{ij})$. Then

$$p_j g_j \sim g_j \sim e_{jj} \leq p_{jj} \leq g_j;$$

hence also $p_j g_j \sim p_{jj}$. The projections p_{ij} are mutually orthogonal, $g_j = \sum_{i=1}^n p_{ij}$ for all j , and hence $\sum_{i,j=1}^n p_{ij} = I$. Let $q_i := \sum_{j=1}^n p_{ij}$. Thus $\sum_{i=1}^n q_i = I$. Furthermore, $q_i g_j = p_{ij} \sim p_i e_j$ for all j and hence $q_i \sim p_i$ for all i , which proves (ii).

Now assume that M is finite (i.e., $g = I$) and that (ii) holds. Then

$$\Phi(a) = \Phi\left(\sum_{j=1}^n p_j\right) = \sum_{j=1}^n \Phi(p_j) = \sum_{j=1}^n \Phi(q_j) = \Phi(I) = I,$$

which proves (iii). Finally, assume that (iii) holds, i.e., $I = \Phi(a) = \sum_{j=1}^n \Phi(p_j)$. Set $q_1 := p_1$. Then $\Phi(p_2) \leq I - \Phi(p_1) = \Phi(I - p_1)$; hence $p_2 \prec I - q_1$ and thus there is a projection q_2 with $p_2 \sim q_2 \leq I - p_1$. Proceeding inductively, we find mutually orthogonal projections $q_j \sim p_j$. Then

$$\Phi\left(I - \sum_{j=1}^n q_j\right) = I - \sum_{j=1}^n \Phi(q_j) = I - \sum_{j=1}^n \Phi(p_j) = 0,$$

whence $\sum_{j=1}^n q_j = I$. □

We also need the following result about two-sided not necessarily closed ideals.

Lemma 5.2. *Let M be a von Neumann algebra, let $I = \sum_{j=1}^n q_j$ for some finite collection of mutually orthogonal projections $q_j \in M$, and let $\Psi(x)$ be the map on M defined by $\Psi(x) := \sum_{j=1}^n q_j x q_j \in M$. Then*

- (i) Ψ is a faithful, normal, and trace-preserving conditional expectation onto $\bigoplus_{j=1}^n q_j M q_j$. In particular, it is linear and positive.
- (ii) If J is a two-sided ideal of M (not necessarily closed) and $a \in M^+$, then $a \in J$ if and only if $\Psi(a) \in J$.

Proof. (i) Well known.

- (ii) If $a \in J$, then obviously, $\Psi(a) = \sum_{j=1}^n q_j a q_j \in J$. Assume that $\Psi(a) \in J$. Since J is hereditary, it follows that $q_j a q_j \in J$ for every j .

Let

$$J^{1/2} := \text{span}\{x \in M^+ \mid x^2 \in J\}.$$

It is well known that $J^{1/2}$ is also an ideal. (This fact can be easily verified using the characterization of the positive part of an ideal as a hereditary and additive collection of positive operators; see for instance [37, 3.21].) For every $1 \leq i, j \leq n$,

$$|q_i a^{1/2} q_j|^2 = (q_i a^{1/2} q_j)^* (q_i a^{1/2} q_j) = q_j a^{1/2} q_i a^{1/2} q_j \leq q_j a q_j \in J,$$

whence $|q_i a^{1/2} q_j| \in J^{1/2}$, and hence, $q_i a q_j \in J^{1/2}$. As a consequence,

$$a^{1/2} = \sum_{i,j=1}^n q_i a^{1/2} q_j \in J^{1/2}$$

and hence $a \in J$. □

Notice that if $n = \infty$, then $\Psi(a) \in J$ no longer implies that $a \in J$, as we see by considering noncompact positive matrices with a compact main diagonal.

Definition 5.3. Given $a \in M^+$, let

$$\begin{aligned} a_- &:= (I - a)\chi_a(0, 1) && \text{the defect operator associated with } a, \\ a_+ &:= (a - I)\chi_a(1, \infty) && \text{the excess operator associated with } a. \end{aligned}$$

Notice that

$$(5.1) \quad a = a_+ - a_- + R_a.$$

When M is a σ -finite factor, a characterization of a strong sum of projections in M , i.e., of the sums of possibly infinitely many projections in M , with the convergence in the strong operator topology, was obtained in terms of the defect and excess operators in a previous work by three of the authors:

Theorem 5.4 ([21, Theorem 1]). *Let M be a σ -finite factor and $a \in M^+$.*

- (i) *Let M be of type I. Then a is a strong sum of projections if and only if either $\text{Tr}(a_+) = \infty$ or $\text{Tr}(a_-) \leq \text{Tr}(a_+) < \infty$ and $\text{Tr}(a_+) - \text{Tr}(a_-) \in \mathbb{N} \cup \{0\}$.*
- (ii) *Let M be of type II and a be diagonalizable (i.e., $a = \sum_{j=1}^{\infty} \gamma_n e_n$ for $\gamma_n \geq 0$ and $e_n \in M$ mutually orthogonal projections). Then a is a strong sum of projections if and only if $\tau(a_+) \geq \tau(a_-)$, where τ is a faithful semifinite normal trace on M . The condition is necessary even when a is not diagonalizable.*
- (iii) *Let M be of type III. Then a is a strong sum of projections if and only if either $\|a\| > 1$ or a is a projection.*

We will present a necessary condition for a to be a finite sum of projections in the case when M is properly infinite. This condition is new even in the case of $B(H)$.

Recall that $J(M)$ is the closed two-sided ideal generated by the finite M . Denote by $F(M)$ the *finite rank* ideal, namely the generally nonclosed ideal of the elements of M that have finite range projection. Then $F(M)$ and $J(M)$ share the same collection of projections, namely the collection of all the finite projections of M .

Theorem 5.5. *Let M be a von Neumann algebra, $a \in M^+$, and R_a be properly infinite. Assume that a is a finite sum of projections in M . Then*

- (i) $\tau(a_+) \geq \tau(a_-)$ for any (positive) trace τ on M^+ if M is semifinite;
- (ii) $c(R_{a_+}) \geq c(R_{a_-})$;
- (iii) if $a_+ \in J(M)$, then $a_- \in J(M)$ and $(c(R_{a_+}) - c(R_{a_-}))a_+ \in F(M)$;
- (iv) if M is a direct sum of finitely many factors, $a_+ \in J(M)$, and $a_- \neq 0$, then a_+ and a_- generate the same two-sided (nonclosed) principal ideal of M .

Proof. (i) By passing if necessary to M_{R_a} , assume without loss of generality that $R_a = I$ and that M is properly infinite. Then condition (iii) of Proposition 5.1 is satisfied and hence by condition (i) *ibid* there is a decomposition of the identity $I = \sum_{j=1}^n q_j$ into mutually orthogonal projections $q_j \in M$ and an isometry v such that $q_j v a v^* q_j = q_j$ for all j . Equivalently, in the notation of Lemma 5.2,

$$\Psi(v a v^*) = I.$$

Then by (5.1),

$$\Psi(I) = I = \Psi(v a v^*) = \Psi(v(a_+ - a_- + I)v^*) = \Psi(v a_+ v^*) - \Psi(v a_- v^*) + \Psi(v v^*)$$

and hence

$$(5.2) \quad \Psi(v a_+ v^*) = \Psi(v a_- v^*) + \Psi(I - v v^*).$$

Notice that the three operators $\Psi(v a_+ v^*)$, $\Psi(v a_- v^*)$, and $\Psi(I - v v^*)$ are positive.

Thus if τ is a trace on M^+ , we have

$$\tau(a_+) = \tau(va_+v^*) = \tau(\Psi(va_+v^*)) \geq \tau(\Psi(va_-v^*)) = \tau(va_-v^*) = \tau(a_-).$$

(ii) Since $\Psi(va_+v^*) \geq \Psi(va_-v^*)$, and $c(R_{a_+})^\perp$ is central, we have

$$\begin{aligned} 0 &= \Psi(vc(R_{a_+})^\perp a_+v^*) = c(R_{a_+})^\perp \Psi(va_+v^*) \\ &\geq c(R_{a_+})^\perp \Psi(va_-v^*) = \Psi(vc(R_{a_+})^\perp a_-v^*) \geq 0. \end{aligned}$$

Thus $\Psi(vc(R_{a_+})^\perp a_-v^*) = 0$. Since $vc(R_{a_+})^\perp a_-v^* \geq 0$ and Ψ is faithful, it follows that $vc(R_{a_+})^\perp a_-v^* = 0$. Since v is an isometry, $c(R_{a_+})^\perp a_- = 0$ and hence $c(R_{a_+}) \geq c(R_{a_-})$.

(iii) If $a_+ \in J(M)$, it follows that $\Psi(va_+v^*) \in J(M)$. Since we have that $\Psi(va_-v^*) \leq \Psi(va_+v^*)$ and $\Psi(I - vv^*) \leq \Psi(va_+v^*)$, it also follows that $\Psi(va_-v^*) \in J(M)$ and $\Psi(I - vv^*) \in J(M)$. Then by Lemma 5.2, $va_-v^* \in J(M)$ and $I - vv^* \in J(M)$. Since v is an isometry, $a_- \in J(M)$. Furthermore, all projections in $J(M)$ are finite; hence

$$(5.3) \quad I - vv^* \in F(M).$$

By (ii), $c(R_{a_+}) - c(R_{a_-})$ is a central projection and

$$(c(R_{a_+}) - c(R_{a_-}))\Psi(va_-v^*) = \Psi(v(c(R_{a_+}) - c(R_{a_-}))a_-v^*) = 0.$$

Thus by (5.2) and (5.3) we have

$$\begin{aligned} \Psi(v(c(R_{a_+}) - c(R_{a_-}))a_+v^*) &= (c(R_{a_+}) - c(R_{a_-}))\Psi(va_+v^*) \\ &= (c(R_{a_+}) - c(R_{a_-}))\Psi(I - vv^*) \in F(M), \end{aligned}$$

whence we conclude by the same reasoning that $(c(R_{a_+}) - c(R_{a_-}))a_+ \in F(M)$.

(iv) The same argument as in (iii) shows that if a_+ belongs to some two-sided ideal J , then also $a_- \in J$. Conversely, assume that $a_- \in J$ for some nonzero ideal J . If M is a factor, we have that $F(M) \subset J$ (see [17, Theorem 6.8.3]) and the same holds if M is a direct sum of finitely many factors. Since by (5.3), $I - vv^* \in J$ and hence $\Psi(I - vv^*) \in J$, we have by (5.2) that $\Psi(va_+v^*) \in J$ and hence by Lemma 5.2 that $va_+v^* \in J$ and thus $a_+ \in J$. $\square$

If M has an infinite dimensional center, then (iv) may be false. This is due to the fact that there are ideals, even principal ideals generated by projections with central support the identity, that do not contain $F(M)$. The situation is illustrated by the following example.

Example 5.6. Let $M := \bigoplus_{j=1}^\infty B(H)$, where H is an infinite dimensional separable Hilbert space, and let $I = \sum_{j=1}^\infty e_j$ be a decomposition of the identity into mutually orthogonal rank-one projections e_j . For every $n \geq 3$ set

$$a_n := \frac{1}{2}e_1 + \frac{3}{2}e_2 + 2\sum_{j=3}^n e_j + \sum_{j=n+1}^\infty e_j$$

and define $a_1 = a_2 = \frac{1}{2}e_1 + \frac{3}{2}e_2$. By Fillmore's theorem 1.1, $\frac{1}{2}e_1 + \frac{3}{2}e_2$ is the sum of two projections, say $p_1 + p_2$, and hence a_n is also the sum of two projections

$p_1 + \sum_{j=3}^n e_j$ and $p_2 + \sum_{j=3}^\infty e_j$. Thus $a := \bigoplus_1^\infty a_n$ is also the sum of two projections. Now,

$$a_+ = \bigoplus_1^\infty \left(\frac{1}{2} e_2 + \sum_{j=3}^n e_j \right),$$

$$a_- = \bigoplus_1^\infty \frac{1}{2} e_1 \quad \text{and} \quad c(a_-) = I.$$

Thus

$$a_+ \in F(M) = \{b = \bigoplus_1^\infty b_n \in M \mid b_n \in F(B(H)) \ \forall n\}.$$

On the other hand, it is easy to see that the principal ideal generated by a_+ is $F(M)$ and the principal ideal generated by a_- is

$$F_o(M) := \{b = \bigoplus_1^\infty b_n \in F(M) \mid \sup_n \text{rank}(b_n) < \infty\}$$

and $F(M) \not\subset F_o(M)$. Thus $a_+ \in J(M)$, $a_- \neq 0$, but a_+ and a_- do not generate the same principal ideal.

In the special case of $B(H)$ we have

Corollary 5.7. *Let H be an infinite dimensional separable Hilbert space, and assume that $a \in B(H)^+$ is a finite sum of projections and that $a_+ \in K(H)$. Then*

- (i) $a_- \in K(H)$.
- (ii) If $a_- = 0$, then a_+ has finite rank.
- (iii) If $a_- \neq 0$, then a_+ and a_- generate the same principal ideal in $B(H)$.

Corollary 5.8. *Let H be an infinite dimensional separable Hilbert space and let k_1 and k_2 be nonzero positive compact operators.*

- (i) $a := I + k_1$ is a finite sum of projections if and only if k_1 has finite rank and integer trace.
- (ii) If $b := I + (k_1 \oplus -k_2)$ is a finite sum of projections, then k_1 and k_2 generate the same principal ideal in $B(H)$ and either $\text{Tr}(k_1) = \infty$ or $\text{Tr}(k_1) - \text{Tr}(k_2) \in \mathbb{N} \cup \{0\}$.

Proof. (i) By definition, $a_- = 0$ and $a_+ = k_1$. Thus if a is a finite sum of projections, by Corollary 5.7 (ii), k_1 must have finite rank. By [21, Theorem 1] (see Theorem 5.4) $\text{Tr}(k_1) = \text{Tr}(a_+) - \text{Tr}(a_-) \in \mathbb{N} \cup \{0\}$. Conversely, if k_1 has finite rank and integer trace, then $R_{k_1} + k_1$ is a finite sum of projections by Theorem 1.1 and hence so is $a = R_{k_1}^\perp + R_{k_1} + k_1$.

- (ii) By definition, $b_+ = k_1$ and $b_- = k_2$; hence the conclusion follows from Corollary 5.7 (iii) and [21, Theorem 1] (see Theorem 5.4). $\square$

Example 5.9. Corollary 5.8 permits us to construct simple examples of operators that are infinite sums of projections but not finite sums of projections.

In (i), choose k_1 with infinite rank and either infinite trace or integer trace.

In (ii), choose k_1 and k_2 that generate different principal ideals but that have the same trace (finite or infinite) and hence by Theorem 5.4 are strong sums of projections. To construct such operators, we can choose $k_1 := \text{diag}(\xi_n)$ and $k_2 := \text{diag}(\eta_n)$ with both sequences ξ_n, η_n decreasing monotonically to 0, $\sum_1^\infty \xi_n = \sum_1^\infty \eta_n \leq \infty$

and both sequences satisfying the $\Delta_{1/2}$ -condition, i.e., $\sup \frac{\xi_n}{\xi_{2n}} < \infty$ and $\sup \frac{\eta_n}{\eta_{2n}} < \infty$. Then k_1 and k_2 generate the same ideal if and only if $\alpha\xi_n \leq \eta_n \leq \beta\xi_n$ for all n and for some $0 < \alpha \leq \beta$. For instance, we can choose $\xi_n = \frac{1}{n}$ and $\eta_n = \frac{1}{n^{1/2}}$ or $\xi_n = \frac{1}{2^n}$ and $\eta_n = \frac{6}{\pi^2 n^2}$. For background information on operator ideals in $B(H)$ we refer the reader to [7] (see also [20]).

Remark 5.10. There is some overlap between Corollaries 5.7, 5.8, and Example 5.9 and certain results in [5].

- (i) If $a \in B(H)^+$ is a finite sum of projections and $\|a\|_e = 1$, then $a_+ \in K(H)$ and hence $a_- \in K(H)$ by Corollary 5.7 (i). Thus $a - R_a = a_+ - a_- \in K(H)$. This is [5, Lemma 3.1].
- (ii) If a is a finite sum of projections and $a_+ - a_- \in K(H)$ has infinite rank, then by Corollary 5.7 (ii) and (iii) both a_+ and a_- have infinite rank. This is the content of [5, Theorem 3.4].
- (iii) Corollary 5.8 (ii) provides a generalization of [5, Theorem 3.6].
- (iv) Furthermore, Example 5.9 is similar to, but more general than [5, Example 3.9].

The problem of characterizing the positive operators of $B(H)$ that are finite sums of projections remains open.

Question 5.11. Is there a natural necessary and sufficient condition for the operator $b = I + (k_1 \oplus -k_2)$ of Corollary 5.8 to be a finite sum of projections?

Notice that the technique used in [21] to decompose such an operator b into an infinite sum of rank one projections does not seem to provide a natural way to assemble those projections into a finite sum of projections.

6. FINITE SUMS OF PROJECTIONS IN A TYPE II_1 FACTOR

In [21] (see Theorem 5.4 here) we have proven that if M is a type II_1 factor with canonical trace τ and $a \in M$ is a strong sum of projections, then

$$(6.1) \quad \tau((a - I)\chi_a(1, \infty)) \geq \tau((I - a)\chi_a(0, 1)).$$

This condition is also sufficient in the cases when a is diagonalizable, that is, $a = \sum_n \gamma_n e_n$ for some family of mutually orthogonal projections $e_n \in M$ and scalars $\gamma_n > 0$. In the latter case, the key step in the proof [21, Lemma 5.1] was the special case when

$$(6.2) \quad a = (1 - \lambda)f + (1 + \mu)e$$

for a pair of mutually orthogonal projections e and f in M and scalars $0 \leq \lambda \leq 1$ and $\mu > 0$. For this case, condition (6.1) can be rewritten as

$$(6.3) \quad \mu\tau(e) \geq \lambda\tau(f)$$

and the proof in [21, Lemma 5.1] was reduced to the case when equality holds in (6.3).

Using a different approach we will show that if M is a type II_1 factor and a is diagonalizable and strict inequality holds in (6.1), then a is a finite sum of projections. We will start again with the special case (6.2).

As in Lemma 4.2, the strategy of the proof is to decompose a into a sum of two positive operators, b_o and the “remainder” a_1 with the same form as a . b_o is a positive combination of mutually orthogonal equivalent projections constructed so

that b_o satisfies Fillmore's necessary and sufficient condition (see Theorem 1.1) for being a finite sum of projections. The construction is then iterated and the crux is to establish a bound on the number of projections needed in each block, so to satisfy the conditions of Lemma 4.1.

Lemma 6.1. *Let M be a type II_1 factor, e and f be mutually orthogonal projections, $0 \leq \lambda \leq 1$, and $\mu > 0$, and $a := (1 - \lambda)f + (1 + \mu)e$. If $\mu\tau(e) > \lambda\tau(f)$, then a is a finite sum of projections in M . Furthermore, there is an upper bound on the number of projections needed that depends only on μ if $f = 0$ and on μ and $\frac{\mu\tau(e)}{\tau(f)} - \lambda$ if $f \neq 0$.*

Proof. The cases when $\lambda = 1$ (resp., $\lambda = 0$) coincide with (resp., can be immediately reduced to) the case when $f = 0$, that is, $a = (1 + \mu)e$. Hence assume that $0 < \lambda < 1$. First assume that μ is not an integer. We will handle the case when μ is an integer at the end of the proof as a simple consequence of the construction in the noninteger case. We construct recursively infinite sequences of projections e_j and f_j in M and scalars λ_j , starting with $e_{-1} := e + f$, $e_o := e$, $f_o := f$, $\lambda_o := \lambda$, and setting for all $j \geq 0$,

$$(6.4) \quad a_j := (1 - \lambda_j)f_j + (1 + \mu_j)e_j,$$

$$(6.5) \quad b_j := a_j - a_{j+1},$$

$$(6.6) \quad \mu_{j+1} := \begin{cases} \mu_j - \lfloor \mu_j \rfloor & f_j = 0 \\ \mu_j & f_j \neq 0 \end{cases} \quad \text{starting with } \mu_o = \mu,$$

$$(6.7) \quad \delta_j := \frac{\mu_j\tau(e_j)}{\tau(f_j)} - \lambda_j \quad f_j \neq 0,$$

$$(6.8) \quad \gamma_j := \min\left\{\frac{\delta_j}{2}, \frac{1}{2}\right\} \quad f_j \neq 0,$$

so that the following conditions are satisfied for all $j \geq 0$:

$$(6.9) \quad e_j \neq 0,$$

$$(6.10) \quad e_j f_j = 0,$$

$$(6.11) \quad f_j + e_j \leq e_{j-1},$$

$$(6.12) \quad 0 < \lambda_j < 1,$$

$$(6.13) \quad \frac{1}{2} \leq \delta_j \leq 2 + \mu \quad \text{for all } j \geq 1 \text{ for which } f_j \neq 0,$$

$$(6.14) \quad \mu_j\tau(e_j) > \lambda_j\tau(f_j),$$

$$(6.15) \quad \tau(e_{j+1}) \leq \left(1 - \frac{\mu_j - \lfloor \mu_j \rfloor}{2}\right)\tau(e_j),$$

$$(6.16) \quad R_{b_j}e_{j+1} = 0 \quad \text{if } \mu_j = \mu_{j+1},$$

$$(6.17) \quad 0 \neq b_j \text{ is a sum of } N_j \text{ projections in } M,$$

$$(6.18) \quad N_j \leq \begin{cases} 1 + \mu & j = 0, f = 0 \\ \frac{4(2+\mu)^2}{\mu} \max\{\delta_o, \frac{1}{\delta_o}\} & j = 0, f \neq 0 \\ \frac{4(2+\mu)^3}{\mu} & j \geq 1. \end{cases}$$

Once this construction is achieved, we see from condition (6.6) that $\mu_j \neq \mu_{j+1}$ for at most one index j_o . Indeed, if $\mu > 1$ and if $f_j = 0$ for some $j \geq 0$, set j_o to be the

first such index. Then

$$(6.19) \quad \mu_j = \begin{cases} \mu & j \leq j_o \\ \mu - \lfloor \mu \rfloor & j > j_o. \end{cases}$$

If $\mu < 1$ or if $f_j \neq 0$ for all j , then $\mu_j = \mu$ for all j and in this case, set $j_o = \infty$.

But then

$$\mu_j - \lfloor \mu_j \rfloor = \mu - \lfloor \mu \rfloor \neq 0 \quad \text{for all } j;$$

hence condition (6.15) implies that $\tau(e_j) \rightarrow 0$ and thus $e_j \xrightarrow{s} 0$. From (6.11), it also follows that $f_j \xrightarrow{s} 0$ and since the sequences $\{1 - \lambda_j\}$ and $\{1 + \mu_j\}$ are bounded by (6.12) and (6.6), it follows that $a_j \xrightarrow{s} 0$. By (6.5), $a = \sum_{i=o}^j b_i + a_{j+1}$ for all j and hence

$$a = \sum_{j=0}^{\infty} b_j.$$

Now notice that for all $j \geq 0$,

$$\begin{aligned} R_{b_j} &\leq R_{a_j} && \text{(by (6.5)),} \\ &= f_j + e_j && \text{(by (6.4)),} \\ &\leq e_{j-1} && \text{(by (6.11)).} \end{aligned}$$

Whenever $\mu_j = \mu_{j+1}$, i.e., $j \neq j_o$, then by (6.16) we have

$$R_{b_j} \perp e_{j+1} \geq R_{b_{j+2}}.$$

By (6.18) and (6.19), $\sup_j N_j = N < \infty$. At the end of the proof, we will find an explicit upper bound for N also including the simple case when $\mu \in \mathbb{N}$.

Thus the sequence $\{b_j\}_{j \neq j_o}$ satisfies the conditions of Lemma 4.1; hence $\sum_{j \neq j_o} b_j$ is a sum of at most $2N$ projections and thus a is a sum of at most $3N$ projections.

For the initial step of the construction, namely the construction of e_1, f_1, λ_1 , we need to consider separately the simpler case when $f = 0$ and the key case when $f \neq 0$.

Case ($\mathbf{f} = \mathbf{0}$). This is the case when $a = (1 + \mu)e$. Then (6.6) prescribes that $\mu_1 := \mu_o - \lfloor \mu_o \rfloor$ and we set $\lambda_1 := 1 - \mu_1$, thus satisfying (6.12). Since M is of type II_1 , we can choose a projection $f_1 \leq e_o$ in M with $\tau(f_1) = \frac{\mu_1 \tau(e_o)}{2}$ and set $e_1 := e_o - f_1$, thus satisfying (6.10) and (6.11). Furthermore $\tau(e_1) = (1 - \frac{\mu_o - \lfloor \mu_o \rfloor}{2})\tau(e_o)$, thus also satisfying (6.9) and (6.15). Then

$$\mu_1 \tau(e_1) - \lambda_1 \tau(f_1) = \frac{\mu_1}{2} \tau(e_o) = \tau(f_1) > 0 \quad \text{(thus satisfying (6.14))}$$

$$a_1 = (1 - \lambda_1)f_1 + (1 + \mu_1)e_1 \quad \text{(as prescribed by (6.4))}$$

$$= (1 + \mu_1)e_o - f_1$$

$$b_o := a_o - a_1 \quad \text{(as prescribed by (6.5))}$$

$$= \lfloor \mu_o \rfloor e_o + f_1 \quad \text{(thus satisfying (6.17) and (6.18))}$$

$$\text{with } N_o = \lfloor \mu_o \rfloor + 1$$

$$R_{b_o} = \begin{cases} e_o & \mu_o > 1 \\ f_1 & \mu_o < 1. \end{cases}$$

If $\mu_1 = \mu_o$, i.e., $\mu_o < 1$, then $R_{b_o} = f_1 \perp e_1$, which satisfies (6.16). Thus conditions (6.9)–(6.15) are satisfied.

Furthermore, $f_1 \neq 0$ and $\delta_1 = \frac{\mu_1 \tau(e_1) - \lambda_1 \tau(f_1)}{\tau(f_1)} = 1$, thus satisfying (6.13).

Case ($\mathbf{f} \neq \mathbf{0}$). Now we proceed with the first step of the construction in the key case when $f_o = f \neq 0$. Define nonnegative integers k, n , and m and the positive real number α as follows:

$$(6.20) \quad k := \left\lfloor \frac{2 + \mu_o}{\gamma_o} \right\rfloor, \quad \text{i.e.,} \quad \frac{2 + \mu_o - \gamma_o}{\gamma_o} < k \leq \frac{2 + \mu_o}{\gamma_o}.$$

Then $k \geq 4$ since $\mu_o > 0$ and $\gamma_o \leq \frac{1}{2}$ and hence $\frac{2 + \mu_o}{\gamma_o} > 4$.

$$(6.21) \quad n := \left\lceil \frac{k(\lambda_o + \delta_o - \gamma_o)}{\mu_o} \right\rceil - 1, \quad \text{i.e.,} \quad \frac{\lambda_o + \delta_o - \gamma_o}{\mu_o} \leq \frac{n+1}{k} < \frac{\lambda_o + \delta_o - \gamma_o}{\mu_o} + \frac{1}{k}.$$

Then $n \geq 0$ because $\frac{k(\lambda_o + \delta_o - \gamma_o)}{\mu_o} > 0$ since $\lambda_o > 0$, $\delta_o - \gamma_o > 0$, and $k > 0$.

$$(6.22) \quad m := \lfloor (n+1)\mu_o - k\lambda_o \rfloor + 1, \quad \text{i.e.,} \quad (n+1)\mu_o - k\lambda_o < m \leq (n+1)\mu_o - k\lambda_o + 1.$$

Then $m \geq 1$ because by (6.21) we have $\frac{n+1}{k} \geq \frac{\lambda_o + \delta_o - \gamma_o}{\mu_o} > \frac{\lambda_o}{\mu_o}$ and hence $(n+1)\mu_o - k\lambda_o > 0$.

$$(6.23) \quad \alpha := k\lambda_o - n\mu_o + m.$$

Then by (6.22),

$$(6.24) \quad \mu_o < \alpha \leq 1 + \mu_o.$$

As a consequence,

$$(6.25) \quad (n+1) \frac{\tau(f_o)}{k} < \tau(e_o).$$

Indeed

$$\begin{aligned} (n+1) \frac{\tau(f_o)}{k} &< \left(\frac{\lambda_o + \delta_o - \gamma_o}{\mu_o} + \frac{\gamma_o}{2 + \mu_o - \gamma_o} \right) \tau(f_o) && \text{(by (6.21) and (6.20))} \\ &= \left(\frac{\lambda_o + \delta_o}{\mu_o} + \gamma_o \left(\frac{1}{2 + \mu_o - \gamma_o} - \frac{1}{\mu_o} \right) \right) \tau(f_o) \\ &< \frac{\lambda_o + \delta_o}{\mu_o} \tau(f_o) && \text{(since } 0 < \gamma_o \leq \frac{1}{2} < 2) \\ &= \tau(e_o) && \text{(by (6.7)).} \end{aligned}$$

Use the fact that M is of type II_1 to partition $f_o = \sum_{j=1}^k p_j$ into k mutually orthogonal projections p_j , each with trace $\tau(p_j) = \frac{\tau(f_o)}{k}$ and to find $n+1$ mutually orthogonal equivalent projections $q_i \leq e_o$, each with the same trace $\tau(q_i) = \frac{\tau(f_o)}{k}$.

Now set

$$e_1 := e_o - \sum_{i=1}^{n+1} q_i \quad (\text{thus by (6.25) } \tau(e_1) > 0,$$

satisfying (6.9))

$$f_1 := \begin{cases} q_{n+1} & \text{if } \alpha \neq 1 + \mu_o \\ 0 & \text{if } \alpha = 1 + \mu_o \end{cases} \quad (\text{thus satisfying (6.10) and (6.11)})$$

$$\lambda_1 := \begin{cases} \alpha - \mu_o & \text{if } \alpha \neq 1 + \mu_o \\ \text{arbitrary in } (0, 1) & \text{if } \alpha = 1 + \mu_o \end{cases} \quad (\text{thus satisfying (6.12)})$$

$$\mu_1 = \mu_o \quad (\text{as prescribed by (6.6)})$$

$$a_1 = (1 - \lambda_1)f_1 + (1 + \mu_1)e_1 \quad (\text{as prescribed by (6.4)})$$

$$= (1 + \mu_o)e_o - (1 + \mu_o) \sum_{i=1}^n q_i - \alpha q_{n+1}$$

$$b_o = a_o - a_1 \quad (\text{as prescribed by (6.5)})$$

$$= \begin{cases} (1 - \lambda_o)f_o + (1 + \mu_o) \sum_{i=1}^n q_i + \alpha q_{n+1} & \text{if } n > 0 \\ \sum_{j=1}^k (1 - \lambda_o)p_j + \alpha q_{n+1} & \text{if } n = 0. \end{cases}$$

Then

$$\begin{aligned} R_{b_o} &= \sum_{j=1}^k p_j + \sum_{i=1}^{n+1} q_i \\ &= f_o + e_o - e_1 \end{aligned} \quad (\text{thus satisfying (6.16)})$$

and

$$\begin{aligned} \frac{\tau(e_1)}{\tau(e_o)} &= 1 - \frac{(n+1)\tau(f_o)}{k\tau(e_o)} && (\text{by the definition of } e_1) \\ &= 1 - \frac{(n+1)\mu_o}{k(\lambda_o + \delta_o)} && (\text{by (6.7)}) \\ &\leq 1 - \frac{\lambda_o + \delta_o - \gamma_o}{\lambda_o + \delta_o} && (\text{by (6.21)}) \\ &< \frac{\gamma_o}{\delta_o} && (\text{since } \lambda_o > 0) \\ &\leq \frac{1}{2} && (\text{by (6.8)}) \\ &< 1 - \frac{\mu_o - \lfloor \mu_o \rfloor}{2} && (\text{thus satisfying (6.15)}). \end{aligned}$$

Since all the $k + n + 1$ mutually orthogonal projections in the decomposition of b_o have the same trace, they are equivalent and we can view b_o as belonging to a copy of $M_{k+n+1}(\mathbb{C})$ embedded (not unittally) in M . Under this identification, b_o has $\text{rank}(b_o) = k + n + 1$ and by (6.23),

$$\text{Tr}(b_o) = k(1 - \lambda_o) + n(1 + \mu_o) + \alpha = k + n + m \geq k + n + 1 = \text{rank}(b_o).$$

Since k , n , and m are integers, so is $\text{Tr}(b_o)$. Then by Theorem 1.1, b_o is the sum of $N_o := k + n + m$ (equivalent) projections in M and condition (6.17) is satisfied. We claim that

$$(6.26) \quad N_o \leq \frac{4(2 + \mu_o)^2}{\mu_o} \max\{\delta_o, \frac{1}{\delta_o}\},$$

thus satisfying (6.18). Indeed

$$\begin{aligned} N_o &= k + n + m \\ &\leq k + n + (n + 1)\mu_o - k\lambda_o + 1 && \text{(by (6.22))} \\ &= (n + 1)(1 + \mu_o) + (1 - \lambda_o)k \\ &< \left(\frac{k(\lambda_o + \delta_o - \gamma_o)}{\mu_o} + 1 \right) (1 + \mu_o) + (1 - \lambda_o)k && \text{(by (6.21))} \\ &= k \left(\frac{(\delta_o - \gamma_o)(1 + \mu_o)}{\mu_o} + \frac{\lambda_o}{\mu_o} + 1 \right) + 1 + \mu_o \\ &< k \left(\frac{(\delta_o - \gamma_o)(1 + \mu_o)}{\mu_o} + \frac{1}{\mu_o} + 1 \right) + 1 + \mu_o && \text{since } \lambda_o < 1 \\ &= (1 + \mu_o) \left(\frac{k}{\mu_o} (1 + \delta_o - \gamma_o) + 1 \right) \\ &\leq (1 + \mu_o) \left(\frac{(2 + \mu_o)(1 + \delta_o - \gamma_o)}{\mu_o \gamma_o} + 1 \right) && \text{(by (6.20))} \\ &= \begin{cases} (1 + \mu_o) \left(\frac{(2 + \mu_o)(1 + 2\delta_o)}{\mu_o} + 1 \right) & \delta_o \geq 1 \\ (1 + \mu_o) \left(\frac{(2 + \mu_o)(1 + \frac{2}{\delta_o})}{\mu_o} + 1 \right) & \delta_o \leq 1 \end{cases} && \text{(by (6.8))} \\ &= (1 + \mu_o) \left(\frac{(2 + \mu_o)(1 + 2 \max\{\delta_o, \frac{1}{\delta_o}\})}{\mu_o} + 1 \right) \\ &= \frac{2(1 + \mu_o)}{\mu_o} \left((2 + \mu_o) \max\{\delta_o, \frac{1}{\delta_o}\} + 1 + \mu_o \right) \\ &\leq \frac{4(2 + \mu_o)^2}{\mu_o} \max\{\delta_o, \frac{1}{\delta_o}\}. \end{aligned}$$

It may be interesting to notice but not essential for the remainder of the proof that a similar argument yields the inequality

$$N_o > \left(\frac{\delta_o}{2\mu_o} + 1 \right) \left(\frac{4}{\delta_o} - 1 \right) - 1$$

that shows that N_o must indeed be “large” both when δ_o is “large” and when δ_o is “small”. Thus to control the number of projections N_j in the iterated construction, we need upper and lower bounds on δ_j as given by (6.13).

To obtain such bounds for δ_1 (in the case when $f_1 \neq 0$, i.e., when $\alpha < 1 + \mu_o$) and at the same time verify that then the strict inequality (6.14) holds, we start

from the following identity:

$$\begin{aligned}
 \delta_1 &= \frac{\mu_1 \tau(e_1)}{\tau(f_1)} - \lambda_1 && \text{(by (6.7))} \\
 (6.27) \quad &= \frac{\mu_o \left(\tau(e_o) - \frac{n+1}{k} \tau(f_o) \right)}{\frac{\tau(f_o)}{k}} - \alpha + \mu_o && \text{(by the definition of } e_1, f_1, \lambda_1 \text{ and } \mu_1) \\
 &= \frac{\mu_o k \tau(e_o)}{\tau(f_o)} - (n+1)\mu_o - (k\lambda_o - n\mu_o + m) + \mu_o && \text{(by (6.23))} \\
 &= k\delta_o - m && \text{(by (6.7)).}
 \end{aligned}$$

Thus

$$\begin{aligned}
 \delta_1 &< k\delta_o - (n+1)\mu_o + k\lambda_o && \text{(by (6.22))} \\
 &\leq k\gamma_o && \text{(by (6.21))} \\
 &\leq 2 + \mu_o && \text{(by (6.20))}
 \end{aligned}$$

and

$$\begin{aligned}
 \delta_1 &\geq k\delta_o - (n+1)\mu_o + k\lambda_o - 1 && \text{(by (6.22))} \\
 &= k(\lambda_o + \delta_o) - (n+1)\mu_o - 1 \\
 &> k(\lambda_o + \delta_o) - k(\lambda_o + \delta_o - \gamma_o) - \mu_o - 1 && \text{(by (6.21))} \\
 &= k\gamma_o - \mu_o - 1 \\
 &\geq 1 - \gamma_o && \text{(by (6.20))} \\
 &\geq \frac{1}{2} && \text{(by (6.8)).}
 \end{aligned}$$

Thus (6.13) is satisfied. In particular, $\delta_1 > 0$ and hence condition (6.14) is satisfied when $f_1 \neq 0$. Notice that (6.14) is trivially satisfied when $f_1 = 0$ since $e_1 \neq 0$ by (6.9).

Notice now that the remainder $a_1 = (1 - \lambda_1)f_1 + (1 + \mu_1)e_1$ has precisely the same form as the original element a_o , but while δ_o (in the case when $f_o \neq 0$) was arbitrary, δ_1 (in the case when $f_1 \neq 0$) is bounded above and below by (6.13) and hence $\max\{\delta_1, \frac{1}{\delta_1}\} \leq 2 + \mu_1$. Thus we can apply to a_1 the construction of Case ($f = 0$) if $f_1 = 0$ or Case ($f \neq 0$) if $f_1 \neq 0$ and obtain a better estimate for the upper bound of N_2 . Indeed, from (6.26) applied to this second step,

$$\begin{aligned}
 N_2 &\leq \begin{cases} 1 + \mu_1 & f_1 = 0 \\ \frac{4(2+\mu_1)^2}{\mu_1} \max\{\delta_1, \frac{1}{\delta_1}\} & f_1 \neq 0 \end{cases} \\
 &\leq \frac{4(2 + \mu_1)^3}{\mu_1} \leq \max \left\{ \frac{4(2 + \mu)^3}{\mu}, \frac{4(2 + \mu - \lfloor \mu \rfloor)^3}{\mu - \lfloor \mu \rfloor} \right\},
 \end{aligned}$$

thus satisfying (6.18) for $j \geq 1$. The recurrence then proceeds exactly as in the case $j = 1$, thus concluding the proof for the case when $\mu \notin \mathbb{N}$.

Finally, assume that $\mu \in \mathbb{N}$. If $f \neq 0$ we can apply the construction of Case ($f \neq 0$) without any changes. Then if $f_1 \neq 0$ we can apply the construction again. There are two possibilities.

Either f_j never vanishes, i.e., the process continues indefinitely applying only the construction in Case ($f \neq 0$), in which case $\mu = \mu_j$ for all j and the same conclusion and the same bounds for N_j as in the first part of the proof hold, or there is some j for which $f_j = 0$. Let j_o be the first such index. In that case,

$$a_{j_o} = (1 + \mu_{j_o})e_{j_o} = (1 + \mu)e_{j_o}$$

is already the sum of $1 + \mu$ projections and the “remainder” a_{j_o+1} vanishes. Thus the process terminates at step j_o . But for all $0 \leq j \leq j_o$, $\mu_j = \mu$, the bounds for N_j (see (6.18)) and the relations between the range projections of the elements b_j and the projections e_j that we have obtained in the first part of the proof hold, and hence, so does the conclusion.

Now to summarize, we have in all cases (i.e., whether μ is an integer or not),

$$N_o \leq \begin{cases} 1 + \mu & f = 0 \\ \frac{4(2+\mu)^2}{\mu} \max\{\delta_0, \frac{1}{\delta_0}\} & f \neq 0, \end{cases}$$

$$\sup_{j \geq 1} N_j \leq \begin{cases} \frac{4(2+\mu)^3}{\mu} & j_o = \infty \\ \max\left\{\frac{4(2+\mu)^3}{\mu}, \frac{4(2+\mu - \lfloor \mu \rfloor)^3}{\mu - \lfloor \mu \rfloor}\right\} & j_o < \infty. \end{cases}$$

But then, the bound $N = \max\{N_o, \sup_{j \geq 1} N_j\}$ on the number of needed projections for each block, and hence the bound $3N$ on the number of projections needed to decompose a , is seen to indeed depend only on μ if $f = 0$, and on μ and $\delta_o = \frac{\mu\tau(e)}{\tau(f)} - \lambda$ when $f \neq 0$. $\square$

Remark 6.2. (i) If $a = (1 + \frac{m}{n})e$ with $m, n \in \mathbb{N}$, then instead of proceeding with the Case ($f = 0$) construction, etc., we can decompose directly e into n mutually orthogonal equivalent projections. Then by identifying a with an element of $M_n(\mathbb{C})$, we see that $\text{rank}(a) = n$ and $\text{Tr}(a) = n + m$; hence a is the sum of $n + m$ (equivalent) projections in M . In other words, if μ is rational we can terminate the construction in Lemma 6.1 at any step where $f_j = 0$.

(ii) If $a = (1 - \lambda)f + (1 + \mu)e$ and $\frac{\mu}{\lambda} = \frac{\tau(f)}{\tau(e)} = \frac{m}{n}$ for some $m, n \in \mathbb{N}$, then by decomposing e into m mutually orthogonal equivalent projections and f into n mutually orthogonal equivalent projections, the same reasoning as in (i) shows that a is a sum of $n + m$ equivalent projections. This observation shows that the strict inequality $\tau(a_+) > \tau(a_-)$ is not necessary for a to be a finite sum of projections.

Before we consider the general case of a positive diagonalizable operator with $\tau(a_+) > \tau(a_-)$, we need the following elementary lemma.

Lemma 6.3. Assume that $\sum_{i=1}^m \xi_i = \sum_{j=1}^n \eta_j$ for some $m, n \in \mathbb{N}$ and $\xi_i, \eta_j > 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$. Then for all $1 \leq i \leq m$, $1 \leq j \leq n$, there are decompositions $\xi_i = \sum_{k=1}^{m_i} \xi_{i,k}$, $\eta_j = \sum_{h=1}^{n_j} \eta_{j,h}$, where $\xi_{i,k} > 0$, $\eta_{j,h} > 0$ and such that $\{\xi_{i,k}\}_{1 \leq i \leq m, 1 \leq k \leq m(i)} = \{\eta_{j,h}\}_{1 \leq j \leq n, 1 \leq h \leq n(j)}$.

Proof. (1) The proof is by induction on the number $m + n$ of summands. Assume without loss of generality that $m \leq n$. To simplify notation, assume also that the finite sequences ξ_i and η_j are already in monotone nonincreasing order. Then $m\xi_1 \geq n\eta_n$. Thus if $\eta_n \geq \xi_1$, then $m = n$, $\xi_i = \eta_j$ for all i and j and there is nothing to prove. Thus assume that $\eta_n < \xi_1$ and let

$\xi'_1 := \xi_1 - \eta_n$. Then $\xi'_1 + \sum_{i=2}^m \xi_i = \sum_{j=1}^{n-1} \eta_j$ satisfies the same hypotheses but with a number $m + n - 1$ of summands. Routine arguments conclude the proof. $\square$

Theorem 6.4. *Let M be a type II_1 factor and let $a \in M$ be a positive diagonalizable operator. Then a sufficient condition for a to be a finite sum of projections in M is that $\tau((a - I)\chi_a(1, \infty)) > \tau((I - a)\chi_a(0, 1))$.*

Proof. Assume without loss of generality that 1 is not an eigenvector of a and hence that

$$a = \sum_{j=1}^{N_1} (1 - \lambda_j) f_j + \sum_{i=1}^{N_2} (1 + \mu_i) e_i,$$

where $N_1 \in \mathbb{N} \cup \{\infty\} \cup \{0\}$ (again, adopting the convention that a sum such as $\sum_{j=1}^0$ is dropped), $N_2 \in \mathbb{N} \cup \{\infty\}$, $\mu_i > 0$, $0 < \lambda_i < 1$ for all i and j , and $\{e_i, f_j\}$ are mutually orthogonal nonzero projections in M whose sum is the range projection R_a of a . Then

$$a_+ = (I - a)\chi_a(0, 1) = \sum_{j=1}^{N_1} \lambda_j f_j \quad \text{and} \quad a_- = (a - I)\chi_a(1, \infty) = \sum_{i=1}^{N_2} \mu_i e_i.$$

Set

$$\gamma := \tau(a_+) - \tau(a_-) = \sum_{i=1}^{N_2} \mu_i \tau(e_i) - \sum_{j=1}^{N_1} \lambda_j \tau(f_j).$$

Thus $\gamma > 0$ by hypothesis. Assume that $N_2 = N_1 = \infty$ (the other cases are simpler and are left to the reader). Set $h = \lceil 14\|a\| \rceil - 1$. By hypothesis, $\|a\| > 1$ and hence $h \geq 14$. Choose $N \in \mathbb{N}$ so that

$$\sum_{i=N+1}^{\infty} \tau(e_i) \leq \min \left\{ \frac{\gamma}{3(\|a\| - 1)}, \frac{\gamma}{3h} \right\}.$$

Let $e_\infty := \sum_{i=N+1}^{\infty} e_i$ and $a_o := \sum_{i=N+1}^{\infty} (1 + \mu_i) e_i$. Then $a_o \geq e_\infty = R_{a_o}$ and we can apply Corollary 2.5 with $\nu = 1$, and by (2.1) with $N_o = 12$, $V_o = 14$. Thus a_o can be expressed as a positive linear combination of not more than

$$N_o + \lceil V_o \left(\frac{\|a_o\|}{\nu} - 1 \right) \rceil + 1 \leq 12 + \lceil 14\|a\| - 14 \rceil + 1 = h$$

projections. Set $a_o = \sum_{k=1}^n \alpha_k p_k$ for some collection of $n \leq h$ nonzero projections $p_k \in M$ and positive scalars α_k . Necessarily, $\lfloor \alpha_k \rfloor \leq \alpha_k \leq \|a_o\| \leq \|a\|$. By decomposing

$$a_o = \sum_{k=1}^n \lfloor \alpha_k \rfloor p_k + \sum_{k=1}^n (\alpha_k - \lfloor \alpha_k \rfloor) p_k$$

and noticing that the first summand is the sum of at most $n\|a\|$ projections, to simplify notation, assume that $0 < \alpha_k < 1$ for each $1 \leq k \leq n$. Set $\beta_k = 1 - \alpha_k$. Thus

$$(6.28) \quad a = \sum_{i=1}^N (1 + \mu_i) e_i + \sum_{j=1}^{\infty} (1 - \lambda_j) f_j + \sum_{k=1}^n (1 - \beta_k) p_k.$$

Notice that the projections p_k are not mutually orthogonal, but since $p_k \leq e_\infty$ for all k , each p_k is orthogonal to all the projections e_i for $1 \leq i \leq N$ (and also to all the projections f_j , although we do not need the latter fact). Furthermore,

$$\sum_{k=1}^n \beta_k \tau(p_k) < \sum_{k=1}^n \tau(p_k) \leq n \tau(e_\infty) \leq \frac{\gamma}{3}.$$

We also have

$$\sum_{i=N+1}^{\infty} \mu_i \tau(e_i) \leq \sum_{i=N+1}^{\infty} (\|a\| - 1) \tau(e_i) = (\|a\| - 1) \tau(e_\infty) \leq \frac{\gamma}{3}.$$

But then,

$$\sum_{i=1}^N \mu_i \tau(e_i) - \sum_{j=1}^{\infty} \lambda_j \tau(f_j) - \sum_{k=1}^n \beta_k \tau(p_k) = \gamma - \sum_{k=1}^n \beta_k \tau(p_k) - \sum_{i=N+1}^{\infty} \mu_i \tau(e_i) \geq \frac{\gamma}{3} > 0.$$

Set

$$\begin{aligned} \rho &:= \sum_{i=1}^N \mu_i \tau(e_i) - \sum_{j=1}^{\infty} \lambda_j \tau(f_j) - \sum_{k=1}^n \beta_k \tau(p_k), \\ \sigma &:= \sum_{j=1}^{\infty} \tau(f_j) + \sum_{k=1}^n \tau(p_k). \end{aligned}$$

Then $\rho, \sigma > 0$ and

$$\begin{aligned} \sum_{i=1}^N \mu_i \tau(e_i) &= \sum_{j=1}^{\infty} \lambda_j \tau(f_j) + \sum_{k=1}^n \beta_k \tau(p_k) + \rho \\ &= \sum_{j=1}^{\infty} \lambda_j \tau(f_j) + \sum_{k=1}^n \beta_k \tau(p_k) + \frac{\rho}{\sigma} \left(\sum_{j=1}^{\infty} \tau(f_j) + \sum_{k=1}^n \tau(p_k) \right) \\ (6.29) \quad &= \sum_{j=1}^{\infty} \left(\lambda_j + \frac{\rho}{\sigma} \right) \tau(f_j) + \sum_{k=1}^n \left(\beta_k + \frac{\rho}{\sigma} \right) \tau(p_k). \end{aligned}$$

Choose $m \in \mathbb{N}$ for which $\sum_{j=m+1}^{\infty} (\lambda_j + \frac{\rho}{\sigma}) \tau(f_j) \leq \mu_1 \tau(e_1)$. Since M is of type II_1 , we can find mutually orthogonal projection $e_{1,j} \leq e_1$ for all $j \geq m+1$ such that

$$(6.30) \quad \tau(e_{1,j}) = \frac{1}{\mu_1} \left(\lambda_j + \frac{\rho}{\sigma} \right) \tau(f_j).$$

Set for $j \geq m+1$,

$$a(j) := (1 + \mu_1) e_{1,j} + (1 - \lambda_j) f_j \quad \text{and} \quad a' := \sum_{j=m+1}^{\infty} a(j).$$

Since by (6.30),

$$\frac{\mu_1 \tau(e_{1,j})}{\tau(f_j)} - \lambda_j = \frac{\rho}{\sigma} > 0,$$

it follows that $a(j)$ satisfies the conditions of Lemma 6.1 and hence is the sum of finitely many projections. Moreover, for each j , there is an upper bound $N(j)$ on the number of projections needed that depends only on μ_1 and $\frac{\mu_1 \tau(e_{1,j})}{\tau(f_j)} - \lambda_j$, both

of which are constant for all j . Thus $N(j)$ is also constant, say $N(j) \equiv N'$. As a consequence, a' is also a sum of finitely many projections.

Now set

$$(6.31) \quad \begin{aligned} a'' &:= a - a', \\ e_1'' &:= e_1 - \sum_{j=m+1}^{\infty} e_{1,j}. \end{aligned}$$

Then

$$(6.32) \quad a'' = (1 + \mu_1)e_1'' + \sum_{i=2}^N (1 + \mu_i)e_i + \sum_{j=1}^m (1 - \lambda_j)f_j + \sum_{k=1}^n (1 - \beta_k)p_k.$$

Now we have the identity

$$\begin{aligned} \mu_1\tau(e_1'') + \sum_{i=2}^N \mu_i\tau(e_i) &= \mu_1\tau(e_1) - \mu_1 \sum_{j=m+1}^{\infty} \tau(e_{1,j}) + \sum_{i=2}^N \mu_i\tau(e_i) \quad (\text{by (6.31)}) \\ &= \sum_{i=1}^N \mu_i\tau(e_i) - \sum_{j=m+1}^{\infty} \left(\lambda_j + \frac{\rho}{\sigma}\right)\tau(f_j) \quad (\text{by (6.30)}) \\ &= \sum_{j=1}^m \left(\lambda_j + \frac{\rho}{\sigma}\right)\tau(f_j) + \sum_{k=1}^n \left(\beta_k + \frac{\rho}{\sigma}\right)\tau(p_k) \quad (\text{by (6.29)}). \end{aligned}$$

Now apply Lemma 6.3 to this identity, that is, decompose each of the summands so to “match” the subsummands and notice that every decomposition of the (scalar) summands in this identity leads to a corresponding decomposition in the projections. For instance, if for some i we have the decomposition $\mu_i\tau(e_i) = \sum_{k=1}^{m(i)} \xi_{i,k}$, then, using the fact that M is a type II_1 factor we can correspondingly decompose e_i into the sum $e_i = \sum_{k=1}^{m(i)} e_{i,k}$ of mutually orthogonal projections with traces $\tau(e_{i,k}) = \frac{\xi_{i,k}}{\mu_i}$. Similarly, decompose correspondingly all of the other projections f_j and p_k . To simplify notation, list as $\{f'_i\}_1^{m'}$ (resp., $\{p'_i\}_{m'+1}^{n'}$) all the projections obtained from the decomposition of the projections $\{f_j\}_1^m$ (resp., $\{p_k\}_1^n$) and therefore list as $\{e'_i\}_1^{n'}$ the projections obtained from the decomposition of the projections e_1'' and $\{e_j\}_1^N$ and label accordingly the corresponding coefficients. Explicitly,

$$(6.33) \quad \mu'_i\tau(e'_i) = \begin{cases} \left(\lambda'_i + \frac{\rho}{\sigma}\right)\tau(f'_i) & 1 \leq i \leq m', \\ \left(\beta'_i + \frac{\rho}{\sigma}\right)\tau(p'_i) & m' + 1 \leq i \leq n'. \end{cases}$$

Using this decomposition, we rewrite (6.32) as

$$\begin{aligned} a'' &= \sum_{i=1}^{n'} (1 + \mu'_i)e'_i + \sum_{i=1}^{m'} (1 - \lambda'_i)f'_i + \sum_{i=m'+1}^{n'} (1 - \beta'_i)p'_i \\ &= \sum_{i=1}^{m'} \left((1 + \mu'_i)e'_i + (1 - \lambda'_i)f'_i \right) + \sum_{i=m'+1}^{n'} \left((1 + \mu'_i)e'_i + (1 - \beta'_i)p'_i \right). \end{aligned}$$

Thus a'' is now decomposed into the sum of the n' positive operators

$$\begin{cases} (1 + \mu'_i)e'_i + (1 - \lambda'_i)f'_i & 1 \leq i \leq m', \\ (1 + \mu'_i)e'_i + (1 - \beta'_i)p'_{i'} & m' + 1 \leq i \leq n'. \end{cases}$$

Recall that while the projections p_k may not be mutually orthogonal, they are orthogonal to each of the projections e_i , and hence each of the projections $p'_{i'}$ into which they have been decomposed are orthogonal to each of the projections e'_j . In particular e'_i is orthogonal to $p'_{i'}$ for every $m' + 1 \leq i \leq n'$ and, of course, e'_i is orthogonal to f'_i for every $1 \leq i \leq m'$. By (6.33) we have

$$\mu'_i \tau(e'_i) > \begin{cases} \lambda'_i \tau(f'_i) & 1 \leq i \leq m' \\ \beta'_i \tau(p'_{i'}) & m' + 1 \leq i \leq n'. \end{cases}$$

Thus by Lemma 6.1, each of these positive operators is a finite sum of projections, and hence so is a'' . This completes the proof. $\square$

An obvious consequence of this theorem is

Corollary 6.5. *If M is a type II_∞ factor, $a \in M^+$ is diagonalizable, R_a is finite, and $\tau(a_+) > \tau(a_-)$, then a is a finite sum of projections.*

Remark 6.6. The condition that $\tau(a_+) > \tau(a_-)$ alone (i.e., without also requiring that R_a be finite) is not sufficient for a to be a finite sum of projections (whether a is diagonalizable or not). For instance, if $a_+ \in J(M)$, $\tau(a_+) = \infty$ (necessarily R_a is infinite), but $\tau(a_-) < \infty$, then a_- belongs to the trace-class ideal while a_+ does not, so by Theorem 5.5, a is not a finite sum of projections.

We have used the condition that a is diagonalizable in order to reduce the problem to the simpler case of Lemma 6.1. However, this condition is clearly not necessary for a to be a finite sum of projections. For instance, if $0 \leq a \leq 2I$ and there is a unitary u for which $uau^* = 2I - a$, then a , whether diagonalizable or not, is the sum of two projections by [11, Corollary of Theorem 2] (see [21, Proposition 2.10] for the von Neumann algebra case).

Question 6.7. Can the condition that a is diagonalizable be removed?

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