CONNES EMBEDDINGS AND VON NEUMANN REGULAR CLOSURES OF AMENABLE GROUP ALGEBRAS

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ABSTRACT. The analytic von Neumann regular closure $R(\Gamma)$ of a complex group algebra $\mathbb{C}\Gamma$ was introduced by Linnell and Schick. This ring is the smallest $*$ -regular subring in the algebra of affiliated operators $U(\Gamma)$ containing $\mathbb{C}\Gamma$. We prove that all the algebraic von Neumann regular closures corresponding to sofic representations of an amenable group are isomorphic to $R(\Gamma)$. This result can be viewed as a structural generalization of Lück's approximation theorem.

The main tool of the proof which might be of independent interest is that an amenable group algebra $K\Gamma$ over any field K can be embedded to the rank completion of an ultramatricial algebra.

1. INTRODUCTION

1.1. Regular rank rings. In this paper all rings are considered unital. Regular rings were introduced by John von Neumann; these are the rings where any principal right ideal is generated by an idempotent (see [9]). A $*$ -regular ring $\mathcal{R}$ is a ring with involution and $a^*a = 0$ implies that $a = 0$. In a $*$ -regular ring any principal right ideal is generated by a unique projection [11]. A $*$ -regular ring $\mathcal{R}$ is *proper* if $\sum_{i=1}^n a_i a_i^* = 0$ implies that all of the a_i 's are equal to 0. Note that $\text{Mat}_{d \times d}(\mathcal{R})$ is regular if and only if $\mathcal{R}$ is regular, nevertheless for a $*$ -regular ring $\mathcal{R}$ $\text{Mat}_{d \times d}(\mathcal{R})$ is $\star$ -regular if and only if $\mathcal{R}$ is proper [1]. Since $\text{Mat}_{k \times k}(\text{Mat}_{d \times d}(\mathcal{R})) = \text{Mat}_{kd \times kd}(\mathcal{R})$, $\mathcal{R}$ is proper if and only if all the matrix rings over $\mathcal{R}$ are proper.

A rank function on a regular ring $\mathcal{R}$ is function $\text{rk} : \mathcal{R} \rightarrow \mathbb{R}$ satisfying the following conditions:

- (1) $0 \leq \text{rk}(a) \leq 1$.
- (2) $\text{rk}(a) = 0$ if and only if $a = 0$.
- (3) $\text{rk}(a + b) \leq \text{rk}(a) + \text{rk}(b)$.
- (4) $\text{rk}(ab) \leq \text{rk}(a), \text{rk}(b)$.
- (5) If e, f are orthogonal idempotents, then $\text{rk}(e + f) = \text{rk}(e) + \text{rk}(f)$.

The most important examples of regular rank rings are matrix rings over division rings D . Recall that for $A \in \text{Mat}_{d \times d}(D)$, $\text{rk}(A) = \frac{\text{rank}(A)}{d}$, where rank is the usual matrix rank. In this case, the values of the rank are always rational. The rank defines a metric on the regular ring by $d(x, y) = \text{rk}(x - y)$. The completion of this metric is a regular rank ring as well. Note that for the completion of ultramatricial algebras (see Section 4) the values of the rank can be any real number in between

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zero and one [9]. Let $\mathcal{N}$ be a finite tracial von Neumann algebra; then, its Ore localization with respect to its non-zero divisors $U(\mathcal{N})$ is a $\star$ -regular ring. The elements of this ring are called affiliated operators (see [19]). The rank of an affiliated operator is the trace of the idempotent that generates the right ideal generated by the operator. Note that if $A \in U(\mathcal{N})$, then

$$(1) \quad \text{rk}(A) = 1 - \lim_{\lambda \rightarrow 0} \text{tr}_{\mathcal{N}}(E_{\lambda}),$$

where $\int_0^\infty \lambda dE_{\lambda}$ is the spectral decomposition of the unbounded operator A^*A . This shows that if $i : \mathcal{N} \rightarrow \mathcal{M}$ is a trace-preserving homomorphism between finite tracial von Neumann algebras, then its Ore localization $\tilde{i} : U(\mathcal{N}) \rightarrow U(\mathcal{M})$ is a rank preserving $\star$ -homomorphism. By [21, Lemma 2.2], $U(\mathcal{N})$ is just the rank completion of $\mathcal{N}$.

Note that $U(\mathcal{N})$ is always proper (see Section 3).

If $\mathcal{R}$ is a regular rank ring, then there is a unique natural extension of the rank to a matrix rank of $\text{Mat}_{k \times k}(\mathcal{R})$ [10]. Note that a matrix rank rk_m has the same property as the rank rk except that $0 \leq \text{rk}_m(M) \leq k$.

1.2. The Connes embedding problem. Let $\mu = \{d_1 < d_2 < \dots\}$ be an infinite sequence of positive integers. Then one can consider the ultraproduct of the matrix algebras $\{\text{Mat}_{d_i \times d_i}(\mathbb{C})\}_{i=1}^\infty$ as tracial algebras in the following way (see [18]).

Let ω be a non-principal ultrafilter on the natural numbers and let $\lim_\omega$ be the corresponding ultralimit. First, consider the algebra of bounded elements

$$\mathcal{B} = \{(a_1, a_2, \dots) \in \prod_{i=1}^\infty \text{Mat}_{d_i \times d_i}(\mathbb{C}) \mid \sup \|a_i\| < \infty\}.$$

Now let $\mathcal{I} \triangleleft \mathcal{B}$ be the ideal of elements $\{a_i\}_{i=1}^\infty$ such that $\lim_\omega \frac{\text{tr}(a_n^* a_n)}{d_n} = 0$. Then $\mathcal{B}/\mathcal{I} = \mathcal{M}_\mu$ is a type II_1 -von Neumann factor with trace defined by

$$\text{Tr}_\omega[\{a_i\}_{i=1}^\infty] = \lim_\omega \frac{\text{tr}(a_n)}{d_n}.$$

The following conjecture is generally referred to as the Connes embedding problem: Is it true that a type- II_1 von Neumann algebra with a separable predual has a trace-preserving embedding to some $\mathcal{M}_\mu$? See the survey of Pestov [18] for further details.

There is a purely algebraic version of the Connes embedding problem first considered in [7]. Namely, we can consider the ultraproduct of the matrix rings $\{\text{Mat}_{d_i \times d_i}(\mathbb{C})\}$ as *rank algebras*.

Definition 1.1. Let $\mathcal{J} \triangleleft \prod_{i=1}^\infty \text{Mat}_{d_i \times d_i}(\mathbb{C})$ be the following ideal,

$$\mathcal{J} = \{ \{a_i\}_{i=1}^\infty \mid \lim_\omega \frac{\text{rank}(a_i)}{d_i} = 0 \}.$$

Then $\prod_{i=1}^\infty \text{Mat}_{d_i \times d_i}(\mathbb{C})/\mathcal{J} = \mathcal{M}_\mu^{\text{alg}}$ is the ultraproduct of the matrix rings $\{\text{Mat}_{d_i \times d_i}(\mathbb{C})\}$.

$\mathcal{M}_\mu^{\text{alg}}$ is a simple complete $\star$ -regular rank ring (see [7, Proposition 3.3]). It is important to note that $\mathcal{M}_\mu^{\text{alg}}$ depends on the choice of the ultrafilter. One can ask of course, whether any countable dimensional regular rank ring embeds to some $\mathcal{M}_\mu^{\text{alg}}$.

1.3. Lück's approximation theorem. Let Γ be a finitely generated residually finite group and let $\Gamma = N_0 \supset N_1 \supset N_2, \dots, \bigcap_{k=1}^{\infty} N_k = \{1\}$ be finite index normal subgroups. Let $\Delta \in \text{Mat}_{d \times d}(\mathbb{Z}\Gamma)$ be a $d \times d$ -matrix over the integer group algebra $\mathbb{Z}\Gamma$. Denote by $\mathcal{N}(\Gamma)$ the von Neumann algebra of Γ . Note that Δ acts on $l^2(\Gamma)^d$ as a bounded operator. Then one can define $\dim_{\Gamma} \text{Ker}(\Delta)$ the von Neumann dimension of the kernel of Δ by

$$\dim_{\Gamma} \text{Ker}(\Delta) := \sum_{i=1}^d \langle P_{\text{Ker}(\Delta)} e_i, e_i \rangle,$$

where $P_{\text{Ker}(\Delta)}$ is the orthogonal projection onto $\text{Ker}(\Delta)$ and e_i is the unit vector supported on the unit element of the i -th component of $l^2(\Gamma)^d$. Let $\pi_k : \mathbb{C}\Gamma \rightarrow \mathbb{C}(\Gamma \backslash N_k)$ be the natural projections. That is $\pi_k(\Delta) \in \text{Mat}_{d \times d}(\mathbb{C}(\Gamma \backslash N_k))$ is a finite dimensional linear transformation. According to Lück's approximation theorem (see [15])

$$(2) \quad \lim_{k \rightarrow \infty} \frac{\dim_{\mathbb{C}} \text{Ker}(\pi_k(\Delta))}{|\Gamma : N_k|} = \dim_{\Gamma} \text{Ker}(\Delta).$$

Note that (2) holds even if the coefficients are algebraic numbers [3]. It is conjectured that (2) holds for any $\Delta \in \text{Mat}_{d \times d}(\mathbb{C}\Gamma)$ as well. The conjecture was confirmed for amenable groups Γ in [8] (see also the remark at the end of Section 8).

1.4. Regular closures. Linnell and Schick [13] proved the following theorem (see Section 3). Let $\mathcal{R}$ be a proper $\star$ -regular ring. Then for any subset $T \subseteq \mathcal{R}$ there exists a smallest $\star$ -regular subring containing T . We call this ring $R(T, \mathcal{R})$ the regular closure of T in $\mathcal{R}$. Let Γ be a countable group; then one can consider the natural embedding of its complex group algebra into its von Neumann algebra $\mathbb{C}(\Gamma) \rightarrow \mathcal{N}(\Gamma)$. Let $\mathcal{U}(\Gamma)$ be the Ore localization of $\mathcal{N}(\Gamma)$. Then $\mathcal{U}(\Gamma)$ is a proper $\star$ -regular ring (see [2]). Therefore one can consider the *analytic regular closure* $R(\mathbb{C}(\Gamma), \mathcal{U}(\Gamma)) = R(\Gamma)$.

Now let $\Gamma = N_0 \supset N_1 \supset \dots, \bigcap_{i=1}^{\infty} N_i = \{1\}$ be finite index normal subgroups of a residually finite group. Let $\pi_i : \mathbb{C}\Gamma \rightarrow \mathbb{C}(\Gamma/N_i)$ be the natural projection as in the previous subsection and let $s_i : \mathbb{C}(\Gamma/N_i) \rightarrow \text{Mat}_{\Gamma/N_i \times \Gamma/N_i}(\mathbb{C})$ be the natural representations by convolutions. Define $r_i = s_i \circ \pi_i : \mathbb{C}\Gamma \rightarrow \text{Mat}_{\Gamma/N_i \times \Gamma/N_i}(\mathbb{C})$. Then we have an injective (see [7, Theorem 1]) $\star$ -homomorphism $r : \mathbb{C}\Gamma \rightarrow \mathcal{M}_{\mu}^{alg}$, where $\mu = \{|\Gamma/N_1|, |\Gamma/N_2|, \dots\}$. Therefore we can consider the *algebraic regular closure* $R(\mathbb{C}\Gamma, \mathcal{M}_{\mu}^{alg})$ for any normal chain of a residually finite group. The main result of this paper is the following theorem.

Theorem 1. *Let Γ be a finitely generated amenable group. Then there is a rank preserving $\star$ -homomorphism*

$$j : R(\Gamma) \rightarrow R(\mathbb{C}\Gamma, \mathcal{M}_{\mu}^{alg})$$

which is the identity map restricted on $\mathbb{C}\Gamma$.

This theorem can be viewed as a structural generalization of Lück's approximation theorem for amenable groups. Indeed, let $\Delta \in \text{Mat}_{d \times d}(\mathbb{C}\Gamma)$. Also, let $\text{rk}_1(\Delta)$ be the matrix rank of Δ in the ring $\text{Mat}_{d \times d}(U(\Gamma))$, respectively; let $\text{rk}_2(\Delta)$ be the matrix rank of Δ in $\text{Mat}_{d \times d}(\mathcal{M}_{\mu}^{alg})$. By Theorem 1,

$$(3) \quad \text{rk}_1(\Delta) = \text{rk}_2(\Delta),$$

since both sides are equal to the matrix rank of Δ in $\text{Mat}_{d \times d}(R(\Gamma))$. Observe that $\text{rk}_1(\Delta) = d - \dim_{\Gamma} \ker(\Delta)$. On the other hand,

$$\text{rk}_2(\Delta) = d - \lim_{\omega} \frac{\dim_{\mathbb{C}} \text{Ker}(\pi_k(\Delta))}{|\Gamma : N_k|}.$$

Since, by (3), $\text{rk}_2(\Delta)$ does not depend on the choice of the ultrafilter, then $\lim_{k \rightarrow \infty} \frac{\dim_{\mathbb{C}} \text{Ker}(\pi_k(\Delta))}{|\Gamma : N_k|}$ must exist and (2) follows.

Actually, we prove a generalization of Theorem 1, where we consider algebraic closures associated to arbitrary sofic representations (see Section 7) of the group Γ . We shall also prove the following theorem.

Theorem 2. *For any finitely generated amenable group and coefficient field K , there exists a rank preserving embedding of the group algebra $K\Gamma$ into the rank completion of an ultramatricial algebra.*

2. THE STRATEGY OF THE PROOF

In our paper we consider four complete proper regular $\star$ -rings: $\overline{\mathcal{A}_{\phi}}$ (this will be defined in Section 4), $U(\Gamma)$, $\mathcal{M}_{\mu}^{alg}$ and $U(\mathcal{M}_{\mu})$. Also, we will define seven rank preserving embeddings:

- (1) $f_1 : \mathbb{C}\Gamma \rightarrow \mathcal{M}_{\mu}^{alg}$.
- (2) $f_2 : \mathbb{C}\Gamma \rightarrow U(\Gamma)$.
- (3) $f_3 : \mathbb{C}\Gamma \rightarrow U(\mathcal{M}_{\mu})$.
- (4) $f_4 : \mathbb{C}\Gamma \rightarrow \overline{\mathcal{A}_{\phi}}$.
- (5) $f_5 : \overline{\mathcal{A}_{\phi}} \rightarrow \mathcal{M}_{\mu}^{alg}$.
- (6) $f_6 : \overline{\mathcal{A}_{\phi}} \rightarrow U(\mathcal{M}_{\mu})$.
- (7) $f_7 : U(\Gamma) \rightarrow U(\mathcal{M}_{\mu})$.

In the final section, we shall prove three identities:

- (1) $f_5 \circ f_4 = f_1$.
- (2) $f_6 \circ f_4 = f_3$.
- (3) $f_7 \circ f_2 = f_3$.

From these identities, the main theorem easily follows. Indeed, $R(\Gamma)$ is the smallest $\star$ -regular ring containing $\mathbb{C}\Gamma$ in $U(\Gamma)$. The ring $R(\Gamma)$ is inside $\overline{\mathcal{A}_{\phi}}$; in fact, it is the minimal $\star$ -regular ring containing $\mathbb{C}\Gamma$ in $\overline{\mathcal{A}_{\phi}}$. On the other hand, $R(\mathbb{C}\Gamma, \mathcal{M}_{\mu}^{alg})$ is also the smallest $\star$ -regular ring containing $\mathbb{C}\Gamma$ in $\overline{\mathcal{A}_{\phi}}$. $\square$

3. VON NEUMANN REGULAR CLOSURES

In this section, we review some results of Linnell and Schick [12], [13] about the von Neumann regular closures in proper $\star$ -regular rings. The starting points of Linnell's paper are the following two observations about finite von Neumann algebras already mentioned in the introduction.

- (1) $U(\mathcal{N})$ is a $\star$ -regular ring, that is, any right ideal is generated by a single projection.
- (2) If $\alpha, \beta \in U(\mathcal{N})$ and $\alpha\alpha^* + \beta\beta^* = 0$, then $\alpha = \beta = 0$.

Although Linnell and Schick consider only group von Neumann algebras, all they used are the two properties above. The following result is a strengthening of the second observation.

Proposition 3.1 ([12, Lemma 2]). *If $\alpha, \beta \in U(\mathcal{N})$, then $(\alpha\alpha^* + \beta\beta^*)U(\mathcal{N}) \supseteq \alpha U(\mathcal{N})$.*

Using a simple induction one also has the following proposition.

Proposition 3.2 ([13, Lemma 2.5]). *If $\alpha_1, \alpha_2, \dots, \alpha_n \in U(\mathcal{N})$, then*

$$\sum_{i=1}^n \alpha_i \alpha_i^* U(\mathcal{N}) \supseteq \alpha_1 U(\mathcal{N}).$$

This leads to the crucial proposition about the existence of the von Neumann regular closures.

Proposition 3.3 ([13, Proposition 3.1]). *Let $\{R_i \mid i \in I\}$ be a collection of $\star$ -regular subrings of $U(\mathcal{N})$. Then $\bigcap_{i \in I} R_i$ is also a $\star$ -regular subring of $U(\mathcal{N})$.*

We also need to show that the proposition above holds for $\mathcal{M}_\mu^{alg}$ as well.

Proposition 3.4. *Let $\{R_i \mid i \in I\}$ be a collection of $\star$ -regular subrings of $\mathcal{M}_\mu^{alg}$. Then $\bigcap_{i \in I} R_i$ is also a $\star$ -regular subring of $\mathcal{M}_\mu^{alg}$.*

Proof. Since $\mathcal{M}_\mu^{alg}$ is a $\star$ -regular ring, we only need to prove that if $\alpha\alpha^* + \beta\beta^* = 0$ in $\mathcal{M}_\mu^{alg}$, then both α and β are equal to 0. Then the proof of Proposition 3.3 works without any change.

Lemma 3.1. *For finite dimensional matrices $A, B \in \text{Mat}_{k \times k}(\mathbb{C})$,*

$$\text{rank}(AA^* + BB^*) \geq \max(\text{rank}(A), \text{rank}(B)).$$

Proof. If $(AA^* + BB^*)(v) = 0$, then $A^*(v) = 0$ and $B^*(v) = 0$. Hence

$$\ker(AA^* + BB^*) \subseteq \ker(A^*) \cap \ker(B^*).$$

Therefore $\text{rank}(AA^* + BB^*) \geq \text{rank}(A^*) = \text{rank}(A)$ and $\text{rank}(AA^* + BB^*) \geq \text{rank}(B^*) = \text{rank}(B)$. $\square$

Now let $A_n, B_n \in \text{Mat}_{d_n \times d_n}(\mathbb{C})$, then

$$\lim_{\omega} \frac{\text{rank}(A_n A_n^* + B_n B_n^*)}{d_n} \geq \lim_{\omega} \frac{\text{rank}(A_n)}{d_n}$$

and

$$\lim_{\omega} \frac{\text{rank}(A_n A_n^* + B_n B_n^*)}{d_n} \geq \lim_{\omega} \frac{\text{rank}(B_n)}{d_n}.$$

Hence the proposition follows. $\square$

4. BRATTELI DIAGRAMS, ULTRAMATRICIAL ALGEBRAS AND TILINGS

Recall that a Bratteli diagram is an oriented countable graph with labeled edges and vertices such that the vertex set is partitioned into finite sets $\{Z_i\}_{i=1}^\infty$ in such a way that:

- If the starting vertex of an edge is in Z_i , then the end vertex is necessarily in Z_{i+1} .
- Each vertex has at least one outgoing edge.
- Each vertex A has a non-negative *size* $S(A)$.
- Each edge (from a vertex A to a vertex B) has a non-negative multiplicity $K(A, B)$ such that for each $B \in Z_{n+1}$, $S(B) = \sum_{A \in Z_n} S(A)K(A, B)$. Note that $K(A, B) = 0$ if there is no edge between A and B .

We call a function $m : \bigcup_{n=1} Z_n \rightarrow [0, 1]$ a *harmonic function* if:

- $\sum_{A \in Z_n} m(A) = 1$ for any $n \geq 1$.
-

$$m(A) = \sum_{B \in Z_{n+1}} \frac{S(A)K(A, B)}{S(B)} m(B)$$

for any $n \geq 1$ and $A \in Z_n$.

4.1. Construction of the ultramatricial algebra. First of all, let $(\bigcup_{n=1}^\infty Z_n, m)$ be a Bratteli diagram with a harmonic function. We shall use the following convention in our paper. If A is a set and K is a field, then we will consider the matrix ring $\text{Mat}_{A \times A}(K)$. Each element of $\text{Mat}_{A \times A}(K)$ is regarded as a kernel map $T : A \times A \rightarrow K$, where:

- $(T + S)(p, q) = T(p, q) + S(p, q)$.
- $TS(p, q) = \sum_{r \in A} T(p, r)S(r, q)$.
- If $K = \mathbb{C}$, then $T^*(p, q) = \overline{T(p, q)}$.
- If $v = \sum_{p \in A} r_p e_p \in K^A$, then $T(v) = \sum_{q \in A} (\sum_{p \in A} T(q, p)r_p)e_q$.

For each $n \geq 0$, we regard each element of Z_n as a finite set of size $|A| = S(A)$. For each $A \in Z_n$ and $B \in Z_{n+1}$, we fix $K(A, B)$ injective maps $f_1^{A, B}, f_2^{A, B}, \dots, f_{K(A, B)}^{A, B}$ from A to B in such a way that

$$\bigcup_{A \in Z_n} \bigcup_{i=1}^{K(A, B)} f_i^{A, B}(A)$$

gives rise to a partition of B . Using this data, we construct an injective homomorphism

$$E_B : \bigoplus_{A \in Z_n} \text{Mat}_{A \times A}(K) \rightarrow \text{Mat}_{B \times B}(K)$$

the following way. If $\bigoplus_{A \in Z_n} T_A \in \bigoplus_{A \in Z_n} \text{Mat}_{A \times A}(K)$, $x \in B$, $x = f_j^{A_i, B}(p)$, where $p \in A_i$, then

$$E_B(\bigoplus_{A \in Z_n} T_A)(e_x) = \sum_{q \in A_i} T(q, p)e_{f_j^{A_i, B}(q)}.$$

Note that $E_B(T_{A_1} \oplus T_{A_2} \oplus \dots \oplus T_{A_{i_n}})$ is a block diagonal matrix, where exactly $K(A_i, B)$ of the blocks are equal to the matrix T_{A_i} .

The embeddings E_B define injective maps

$$\phi_n : \bigoplus_{A \in Z_n} \text{Mat}_{A \times A}(K) \rightarrow \bigoplus_{B \in Z_{n+1}} \text{Mat}_{B \times B}(K)$$

by $\phi_n = \bigoplus_{B \in Z_{n+1}} E_B$. The direct limit $\lim_{\rightarrow} \phi_n$ is the ultramatricial algebra $\mathcal{A}_\phi$. Clearly, $\mathcal{A}_\phi$ is a regular ring ($\star$ -regular if $K = \mathbb{C}$).

The harmonic function defines a rank function rk_ϕ on $\mathcal{A}_\phi$. Namely, if $T_A \in \text{Mat}_{A \times A}$, $A \in Z_n$, then let

$$\text{rk}_\phi(\bigoplus_{A \in Z_n} T_A) = \sum_{A \in Z_n} m(A) \frac{\text{rank}(T_A)}{|A|},$$

where $\text{rank}(T_A)$ is the rank of the matrix T_A . Then it is easy to see that each ϕ_n is a rank preserving $\star$ -isomorphism. Therefore $\mathcal{A}_\phi$ is a rank regular ring.

Now let Γ be a finitely generated group with a symmetric generating system S . The Cayley graph of Γ , $\text{Cay}(\Gamma, S)$ is defined as follows:

- $V(\text{Cay}(\Gamma, S)) = \Gamma$.
- $(a, b) \in E(\text{Cay}(\Gamma, S))$ if $as = b$ for some $s \in S$.

Let $F \subset \Gamma$ be a finite set. Then ∂F is the set of vertices in F that are adjacent to a vertex in the complement of F . The isoperimetric constant of F is defined as

$$i(F) := \frac{|\partial F|}{|F|}.$$

The group Γ is amenable if there exists a Følner-sequence in Γ , that is, a sequence of finite sets $\{F_n\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} i(F_n) = 0$. If $\gamma \in \Gamma$, $F \subset \Gamma$, then γF is called an F -tile.

Definition 4.1 (Bratteli tiling system). A Bratteli tiling system has the following properties:

- The level set Z_n consists of finite subsets of a group Γ , $F_1^n, F_2^n, \dots, F_{i_n}^n$ and the set E_n containing only the unit element. Also, we have $i(F_j^n) \leq \frac{1}{2^n}$ for all j and n .
- For any $n \geq 2$ and $F_j^n \in Z_n$ we have a partition $F_j^n = \bigcup_{i=1}^{a_{n,j}} \gamma_i A_i$, where $A_i \in Z_{n-1}$. That is, we have tiling of F_j^n with the tiles of Z_{n-1} .
- $K(F_l^{n-1}, F_j^n)$ is the number of F_l^{n-1} -tiles in the partition of F_j^n . Also $K(E_{n-1}, F_j^n)$ is the number of E_{n-1} -tiles (single vertices).
- $S(F_j^n) = |F_j^n|$, $S(E_n) = 1$.
- We also suppose that $K(E_{n-1}, F_j^n) \leq \frac{1}{2^n} |F_j^n|$.
- We have a harmonic function $m : \bigcup_{n=1}^\infty Z_n \rightarrow [0, 1]$ such that $m(E_n) \rightarrow 0$ as $n \rightarrow \infty$.

Our main technical tool is the following proposition.

Proposition 4.1. *For any amenable group Γ and generating system S we can construct a Bratteli tiling system with the following property. For any $\epsilon > 0$ and $n > 0$ there exists $\delta = \delta_{\epsilon, n} > 0$ such that if $H \in \Gamma$ is a finite set and $i(H) < \delta$, then one can tile H with translates of the elements Z_n in such a way that if $A \in Z_n$ and T_A^H is the set of points in H covered by a translate of A , then*

$$\left| \frac{|T_A^H|}{|H|} - m(A) \right| < \epsilon.$$

5. PROOF OF PROPOSITION 4.1

First, let us recall the notion of ε -quasitilings. Let $\text{Cay}(\Gamma, S)$ be the Cayley-graph of an amenable group Γ as above. Let $H \subset \Gamma$ be a finite set and $A_1, A_2, \dots, A_n$ be subsets of H . We say that $\{A_i\}_{i=1}^n$ ε -cover H if

$$\frac{|\bigcup_{i=1}^n A_i|}{|H|} > 1 - \varepsilon.$$

Also, we call $\{A_i\}_{i=1}^n$ ε -disjoint if there exist mutually disjoint sets $\{B_i\}_{i=1}^n$, $B_i \subset A_i$, such that

$$\frac{|B_i|}{|A_i|} > 1 - \varepsilon.$$

The system $\{A_i\}_{i=1}^n$ ε -quasitiles H if it both ε -covers H and is ε -disjoint. The following result of Ornstein and Weiss [16] is crucial for our proof.

Proposition 5.1 (Quasitiling theorem). *Let $F_1 \subset F_2 \subset \dots$ be a Følner-sequence. Then for any $\varepsilon > 0$ there exists $\delta > 0$ and a finite subfamily $F_{n_1} \subset F_{n_2} \subset \dots \subset F_{n_k}$ such that if $i(H) < \delta$, then H can be ε -quasitiled by translates of the F_{n_i} 's.*

Lemma 5.1. *If A, B are finite subsets of a d -regular graph such that $i(A) < \varepsilon$, $B \subset A$ and $\frac{|B|}{|A|} > 1 - \varepsilon$, then*

$$i(B) < (d+1) \frac{\varepsilon}{1-\varepsilon}.$$

Proof. The set ∂B is covered by the union of ∂A and the neighbours of the vertices in $A \setminus B$. Thus $|\partial B| \leq |\partial A| + d|A \setminus B|$. Therefore,

$$i(B) = \frac{|\partial B|}{|B|} \leq \frac{|\partial A| + d|A \setminus B|}{|A|(1-\varepsilon)} \leq (d+1) \frac{\varepsilon}{1-\varepsilon}. \quad \square$$

The following lemma is a corollary of Lemma 5.1.

Lemma 5.2. *Let $\delta > 0$ and $F_1 \subset F_2 \subset \dots$ be a Følner-sequence. Then there exists $c_\delta > 0$ and $n_\delta > 0$ such that if $j \geq n_\delta$, $B \subset F_j$ and $\frac{|B|}{|F_j|} > 1 - c_\delta$, then $i(B) < \delta$.*

Lemma 5.3. *For any $\delta > 0$, there exist finite sets $L_1, L_2, \dots, L_{i_\delta} \subset \Gamma$ and a real constant $\kappa_\delta > 0$ such that for any $1 \leq j \leq i_\delta$:*

- $i(L_j) < \delta$.
- If $i(H) < \kappa_\delta$, then H can be tiled by translates of the L_i 's and less than $\delta|H|$ single points.

Proof. First fix a Følner-sequence $F_1 \subset F_2 \subset \dots$. Pick an $\varepsilon > 0$ such that:

- $(1 - \varepsilon)^2 > 1 - \delta$.
- $\varepsilon < c_\delta$ (with c_δ as in the previous lemma).

Consider the subsequence $F_{n_\delta} \subset F_{n_\delta+1} \subset F_{n_\delta+2} \dots$, where n_δ is the constant of the previous lemma. By Proposition 5.1, there exists a finite subsequence $F_{a_1} \subset F_{a_2} \subset \dots \subset F_{a_k}$ and a constant $\kappa_\delta > 0$ such that if $i(H) < \kappa_\delta$, then H can be ε -quasitiled by translates of the F_{a_i} 's. By the definition of ε -quasitiling, for any translate γF_{a_i} , we can choose a subset $R_{\gamma,i} \subset \gamma F_{a_i}$ in such a way that:

- The chosen subsets are disjoint.
- $\frac{|R_{\gamma,i}|}{|\gamma F_{a_i}|} > 1 - \varepsilon$.
- The union of the chosen subsets has size at least $(1 - \varepsilon)^2 |H|$.

Also, by Lemma 5.2 and the choice of ε , $i(R_{\gamma,i}) < \delta$. Therefore, if $F_1, F_2, \dots, F_{i_\delta} \subset \Gamma$ is just the family of all subsets of the F_{a_i} 's having isoperimetric constant less than δ , then the family and the constant κ_δ satisfy the conditions of our lemma. $\square$

Now we construct a Bratteli system by induction. Suppose that for any $1 \leq m \leq n$ we have a family of tiles $\{F_1^m, F_2^m, \dots, F_{i_m}^m\}$ and real constants $\delta_1 > \delta_2 > \dots > \delta_m > \delta_{n+1}$ such that:

- For any $1 \leq i \leq n+1$, $\delta_i < \frac{1}{2^i}$.
- For any $1 \leq m \leq n$ and $1 \leq j \leq i_m$, $i(F_j^m) < \delta_m$.
- For any $1 \leq m \leq n$, if $i(H) < \delta_{m+1}$, then H can be tiled by translates of the F_i^m 's and less than $\delta_m |H|$ single vertices.

Now using Lemma 5.3, we choose a finite family $\{F_1^{n+1}, F_2^{n+1}, \dots, F_{i_{n+1}}^{n+1}\}$ such that:

- $i(F_j^{n+1}) < \delta_{n+1}$, for any $1 \leq j \leq i_{n+1}$.
- If $i(H) < \kappa_{\delta_{n+1}}$ (see Lemma 5.3), then H can be tiled by translates of the F_i^{n+1} 's and less than $\delta_{n+1}|H|$ vertices.

Finally, set

$$\delta_{n+2} := \min\left\{\frac{\kappa_{\delta_{n+1}}}{2}, \frac{1}{2^{n+3}}, \frac{\delta_{n+1}}{2}\right\}.$$

By the induction above, we obtain a Bratteli system. Note that Z_n consists of the family $\{F_1^n, F_2^n, \dots, F_{i_n}^n\}$ and the singleton (the unit element).

Now we construct a harmonic function m . We fix a Følner-sequence $\{G_n\}_{n=1}^\infty$ that we will use as a tool in the rest of our paper. Let $\{\delta_n\}_{n=1}^\infty$ be the constants as above. Define j_n such that

$$\delta_{j_n+2} \leq i(G_n) < \delta_{j_n+1}.$$

Then pick a tiling of G_n by translates of the elements of Z_{j_n} in such a way that the number of single vertices is less than $(\frac{1}{2^{j_n}})|G_n|$. Then pick a tiling of the Z_{j_n} -tiles by translates of Z_{j_n-1} in such a way that the number of single vertices in any tile T is less than $(\frac{1}{2^{j_n-1}})|T|$. Inductively, we obtain a tiling of G_n by Z_i -translates for any $1 \leq i \leq j_n$. Note that the number of single vertices used in the Z_i -tiling of G_n is less than

$$\left(\sum_{k=i}^{j_n} \frac{1}{2^k}\right)|G_n| \leq \frac{1}{2^{i-1}}|G_n|.$$

For $A \in Z_i$ and for $n \geq 1$ such that $j_n \geq i$, let us denote by $c_n(A)$ the number of vertices in G_n covered by A -translates and let

$$q_n(A) = \frac{c_n(A)}{|G_n|}.$$

Clearly, $\sum_{A \in Z_i} q_n(A) = 1$. By a diagonal argument, we can pick a subsequence $\{q_{n_k}\}_{k=1}^\infty$ such that for any tile A , $\lim_{k \rightarrow \infty} q_{n_k}(A)$ exists. Hence, we may suppose that for any tile A the limit $m(A) = \lim_{n \rightarrow \infty} q_n(A)$ exists.

Lemma 5.4. *The function m is harmonic satisfying $\lim_{i \rightarrow \infty} m(E_i) = 0$.*

Proof. The fact that $\lim_{i \rightarrow \infty} m(E_i) = 0$ follows from our previous observation about the number of single vertices used in the tiling of G_n . Observe that if $A \in Z_i$,

$$q_n(A) = \sum_{B \in Z_{i+1}} \frac{S(A)K(A, B)}{S(B)} q_n(B).$$

Indeed, $\frac{q_n(A)|G_n|}{S(A)}$ is the number of A -tiles in G_n . Similarly, $\frac{q_n(B)|G_n|}{S(B)}$ is the number of B -tiles in G_n . Also, by definition, $K(A, B)$ is the number of A tiles in B .

By taking the limit as $n \rightarrow \infty$ we get that

$$m(A) = \sum_{B \in Z_{i+1}} \frac{S(A)K(A, B)}{S(B)} m(B). \quad \square$$

Now let us show that our Bratteli tiling system satisfies the properties of Proposition 4.1. First we prove a simple lemma.

Lemma 5.5. *For any $n > 0$ and $\delta > 0$ there exists $\lambda = \lambda_{n,\delta} > 0$ and $p = p_{n,\delta} > 0$ with the following property. Let $l > p$ and $J \subseteq G_l$, $\frac{|J|}{|G_l|} > 1 - \lambda$. For $A \in Z_n$, $|A| > 1$, let $j_A^{l,n}$ be the number of vertices in G_l that are covered by an A -translate (in the tiling previously defined) which is completely inside J . Then*

$$\left| \frac{j_A^{l,n}}{|J|} - m(A) \right| < \delta.$$

Proof. The number of points covered by such A -translates that contain at least one point from the complement of J is less than $|G_l \setminus J||A|$. Hence

$$q_l(A) - \frac{|G_l \setminus J|}{|G_l|}|A| \leq \frac{j_A^{l,n}}{|G_l|} \leq q_l(A).$$

Since $\sup_{A \in Z_n} |A| < \infty$ and $q_l(A) \rightarrow m(A)$, the inequality of the lemma holds if l is large enough and $1 - \frac{|J|}{|G_l|}$ small enough. $\square$

Now we finish the proof of our proposition. Let $\varepsilon > 0$, $n > 0$ be the constants of our propositions and $\lambda_{n,\frac{\varepsilon}{4}} > 0$, $p_{n,\frac{\varepsilon}{4}} > 0$ be the corresponding constants in Lemma 5.5.

Pick an $\alpha > 0$ such that:

- $\alpha < \lambda_{n,\frac{\varepsilon}{4}}$.
- $1 - (1 - \alpha)^2 < \frac{\varepsilon}{4 \max_{A \in Z_n} |A|}$.

By Proposition 5.1, we have a constant ρ (note that this constant will be the $\delta_{\varepsilon,n}$ of our proposition) and a subfamily $\{G_{a_1}, G_{a_2}, \dots, G_{a_t}\}$ of the *Følner*-sequence $\{G_n\}_{n=1}^\infty$ such that:

- $a_1 > p_{n,\frac{\varepsilon}{4}}$.
- $\{G_{a_1}, G_{a_2}, \dots, G_{a_t}\}$ α -quasitiles any finite subset $H \subset \Gamma$ with $i(H) < \rho$.

So, let $H \subset \Gamma$ be a finite subset such that $i(H) < \rho$. Let $\bigcup_{k=1}^s J_k \subset H$ be the disjoint union of subsets, such that each J_k covers at least $(1 - \alpha)$ -part of the G_{a_i} -tile in our α -quasitiling. Observe that $\sum_{k=1}^s |J_k| > (1 - \alpha)^2 |H|$. By Lemma 5.5, we can tile each J_k by A -translates, $A \in Z_n$ such that for any $1 \leq k \leq s$,

$$|c_A^{J_k} - m(A)||J_k| < \frac{\varepsilon}{4}|J_k|,$$

where $c_A^{J_k}$ is the number of vertices in J_k covered by an A -tile, completely inside J_k . That is,

$$\left| \sum_{k=1}^s c_A^{J_k} - m(A) \sum_{k=1}^s |J_k| \right| \leq \frac{\varepsilon}{4} \sum_{k=1}^s |J_k|.$$

Hence

$$(4) \quad \left| \sum_{k=1}^s c_A^{J_k} - m(A)|H| \right| \leq \frac{\varepsilon}{4}|H| + (1 - (1 - \alpha)^2)|H| \leq \frac{\varepsilon}{2}|H|.$$

Observe that for any $A \in Z_n$,

$$|T_A^H| - \sum_{k=1}^s c_A^{J_k} \leq (|H| - \sum_{k=1}^s |J_k|)|A| \leq (1 - (1 - \alpha)^2)|H||A| \leq \frac{\varepsilon}{2}|H|.$$

Hence, by (4), $\left| \frac{|T_A^H|}{|H|} - m(A) \right| < \varepsilon$.

6. THE CANONICAL RANK ON AMENABLE GROUP ALGEBRAS

In this section we recall some results from [4]. Let Γ be a finitely generated amenable group and $K\Gamma$ be its group algebra over the field K . Let $\{F_n\}_{n=1}^\infty$ be a Følner-sequence. For $a \in K\Gamma$, let $V_n^a \subset K^{F_n} \subset K\Gamma$ be the vector space of elements z supported on F_n such that $za = 0$. Then

$$\lim_{n \rightarrow \infty} \frac{\dim_K V_n^a}{|F_n|} = k_a$$

exists and is independent of the choice of the Følner-sequence. We call $\text{rk}(a) = 1 - k_a$ the natural rank of a . It is proved in [6, Proposition, page 237] that if $K = \mathbb{C}$, then $\text{rk}(a) = 1 - \dim_\Gamma \text{Ker } M_a$, where $\dim_\Gamma$ is the von Neumann dimension and $\text{Ker } M_a$ is the set of elements $w \in l^2(\Gamma)$ for which $wa = 0$. Note that the rank can be computed slightly differently as well. Let S be a symmetric generating system of Γ and $\text{Cay}(\Gamma, S)$ be the Cayley-graph of Γ . We consider the shortest path metric $d_{\text{Cay}(\Gamma, S)}$ on Γ . Choose r such that $\text{supp}(a) \subset B_r(1)$, where $B_r(1)$ is the r -ball around the unit element in the Cayley-graph and

$$\text{supp}(a) = \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$$

if $a = \sum a_\gamma \gamma$. For a finite set $F \subset \Gamma$, let $\partial_r F$ be the set of elements $x \in F$ such that

$$d_{\text{Cay}(\Gamma, S)}(x, F^c) \leq r.$$

Note that $\partial F = \partial_1(F)$. Clearly, if $b \in K\Gamma$ and $\text{supp}(b) \in F \setminus \partial_r F$, then $\text{supp}(ba) \subset F$. Then for any $s > r$, we have the following lemma.

Lemma 6.1.

$$\text{rk}(a) = \lim_{n \rightarrow \infty} \frac{\dim_K W_{F_n} a}{|F_n|},$$

where W_{F_n} is the set of elements in $K\Gamma$ supported on $F_n \setminus \partial_s F_n$.

Proof. By our definition, $\text{rk}(a) = \lim_{n \rightarrow \infty} \frac{\dim_K K^{F_n} a}{|F_n|}$. Observe that

$$\frac{\dim_K K^{F_n} a - \dim_K W_{F_n} a}{|F_n|} \leq \frac{\dim_K K^{F_n} - \dim_K W_{F_n}}{|F_n|} = \frac{|\partial_s F_n|}{|F_n|}.$$

Since $\lim_{n \rightarrow \infty} \frac{|\partial_s F_n|}{|F_n|} = 0$, the lemma follows. $\square$

7. SOFIC REPRESENTATIONS

7.1. Sofic approximation and sofic representations. In this section we recall the notion of sofic representations from [7]. Let Γ be a finitely generated group with a symmetric generating set S . Let $\{P_n\}_{n=1}^\infty$ be a sequence of finite graphs such that the directed edges are labeled by the elements of S in such a way that if (x, y) is labeled by s , then (y, x) is labeled by s^{-1} . We say that $\{P_n\}_{n=1}^\infty$ is a sofic approximation of Γ if for any natural number $r > 0$ there exists $n_r > 0$ such that:

- If $n \geq n_r$, then for the set V_n^r of vertices $x \in P_n$ for which the ball $B_r(x)$ in P_n is isomorphic to the ball $B_r(1) \in \text{Cay}(\Gamma, S)$ as labeled graphs

$$\frac{|V_n^r|}{|P_n|} > 1 - \frac{1}{r}.$$

A group is called *sofic* if it possesses a sofic approximation. At the moment, no non-sofic group is known. If $\text{Cay}(\Gamma, S)$ is the Cayley-graph of an amenable group and $\{F_n\}_{n=1}^\infty$ is a Følner-sequence, then the induced graphs P_n of the sets F_n form a sofic approximation of Γ . If Γ is residually finite (amenable or not) with normal chain $\{N_k\}_{k=1}^\infty$, $\bigcap_{k=1}^\infty N_k = \{1\}$, then the graph sequence $\text{Cay}(\Gamma/N_k, S)$ forms a sofic approximation of Γ . If $\{P_n\}_{n=1}^\infty$ is an arbitrary sofic approximation of a group Γ , then one can construct an embedding of $K\Gamma$ (K is an arbitrary field) to the ultraproduct of matrix algebras the following way.

Let $\{\text{Mat}_{P_n \times P_n}(K)\}_{n=1}^\infty$ be a sequence of matrix algebras. Let $a \in K\Gamma$, $a = \sum a_\gamma \gamma$ be an element of the group algebra and choose r such that if $a_\gamma \neq 0$, then $\gamma \in B_r(1) \subset \text{Cay}(\Gamma, S)$. Let $\{e_x\}_{x \in P_n}$ be the natural basis of K^{P_n} . If $x \in V_n^r$, then let

$$\psi_n(a)(e_x) = \sum_{\gamma \in B_r(1)} a_\gamma e_{x\gamma}.$$

Note that by our condition on the support of a , and because the r -ball around x is isomorphic as labeled graphs to the corresponding ball in the Cayley graph of Γ , $x\gamma = y$ is meaningful. If $x \notin V_n^r$, let $\psi_n(a)(e_x) = 0$. This way one defines an injective homomorphism $\psi_\mu : K\Gamma \rightarrow \mathcal{M}_\mu^{\text{alg}}$, where $\mu = \{|P_1|, |P_2|, \dots\}$ ([7], Theorem 1). If $K = \mathbb{C}$, the homomorphism above is a $*$ -homomorphism. The map ψ_μ is called the sofic representation associated to the sequence $\{P_n\}_{n=1}^\infty$.

7.2. Sofic approximation of amenable groups. Let $\{P_n\}_{n=1}^\infty$ be a sofic approximation of the amenable group Γ (with symmetric generating set S). For $L > 0$, let V_n^L be the set of vertices x in P_n such that

$$B_L(x) \cong B_L(1) \subset \text{Cay}(\Gamma, S)$$

as S -labeled graphs. If $F \subset B_L(1)$, then for $x \in V_n^L$ we call $\pi(F)$ an F -translate, where $\pi : B_L(1) \rightarrow B_L(x)$ is the S -labeled graph isomorphism mapping 1 to x . In [8, Theorem 2] we proved the following generalization of the Ornstein-Weiss quasitiling theorem.

Proposition 7.1 (Generalized quasitiling theorem). *Let $F_1 \subset F_2 \dots$ be a Følner-sequence. Then for any $\varepsilon > 0$ there exists $L > 0$, $\delta > 0$ and a finite subcollection $F = \{F_{n_1}, F_{n_2}, \dots, F_{n_t}\}$ such that if*

$$\frac{|V_n^L|}{|P_n|} > 1 - \delta,$$

then P_n can be ε -quasitiled by F -translates.

8. EMBEDDING $K\Gamma$ TO THE COMPLETION OF AN ULTRAMATRICIAL ALGEBRA

Let $(\{Z_i\}_{i=1}^\infty, m)$ be a Bratteli tiling system as in Proposition 4.1. Let E_B, ϕ_i be as in Section 4, and $\mathcal{A}_\phi$ be the ultramatricial algebra corresponding to the system $(\{Z_i\}_{i=1}^\infty, m)$. Remember that any $A \in Z_i$ is in fact some finite subset of Γ .

Now we define the maps $\pi_i^A : K\Gamma \rightarrow \text{Mat}_{A \times A}(K)$ the following way. Recall from Section 4 that we regard the elements of $\text{Mat}_{A \times A}(K)$ as kernel maps. Let $a = \sum a_\gamma \gamma \in K\Gamma$, and choose r minimal with $\text{supp}(a) \subset B_r(1)$. If $x \in A \setminus \partial_r(A)$, then let

$$\pi_i^A(a)(e_x) = \sum a_\gamma e_{x\gamma}.$$

Note that by the condition on the support $x\gamma$ is well-defined. If $x \in \partial_r(A)$, then let $\pi_i^A(a)(e_x) = 0$. Finally we define the maps $\pi_i := \bigoplus_{A \in Z_i} \pi_i^A : K\Gamma \rightarrow \bigoplus_{A \in Z_i} \text{Mat}_{A \times A}(K)$. Thus if $|A| = 1$, then $\pi_i^A(a) = 0$.

Lemma 8.1. *For any $a \in K\Gamma$, $\{\pi_i(a)\}_{i=1}^\infty$ is a Cauchy-sequence in $\mathcal{A}_\phi$, where $[\pi_i(a)]$ denotes the image of $\pi_i(a)$ under the natural embedding $\bigoplus_{A \in Z_i} \text{Mat}_{A \times A}(K) \rightarrow \mathcal{A}_\phi$.*

Proof. First of all note that

$$\text{rk}_\phi(\phi_i \circ \pi_i(a) - \pi_{i+1}(a)) = \sum_{B \in Z_{i+1}} m_{i+1}(B) \frac{\text{rank}(E_B \circ \pi_i(a) - \pi_{i+1}^B(a))}{|B|}.$$

Observe that

$$\text{rank}(E_B \circ \pi_i(a) - \pi_{i+1}^B(a)) = |B| - \dim_K \ker(E_B \circ \pi_i(a) - \pi_{i+1}^B(a)).$$

On the other hand,

$$\dim_K \ker(E_B \circ \pi_i(a) - \pi_{i+1}^B(a)) \geq T_B,$$

where T_B is the number of vertices x in B for which

$$E_B \circ \pi_i(a)(e_x) = \pi_{i+1}^B(a)(e_x).$$

Clearly,

$$T_B \geq |B| - |\partial_r B| - \sum_{A \in Z_i} K(A, B) |\partial_r(A)|.$$

Recall that if $|B| > 1$, then $|\partial_r B| \leq |B|2^{-(i-1)}$. Indeed, B is tiled with tiles of isoperimetric constants less than $\frac{1}{2^i}$ and at most $\frac{1}{2^{i+1}}|B|$ singletons. Hence

$$|\partial_r B| \leq |B|2^{-(i-1)}(d+1)^{r+1},$$

where d is the degree of the vertices in the Cayley graph. Also,

$$\sum_{A \in Z_i} K(A, B) |\partial_r(A)| \leq K(E_i, B) + \sum_{A \in Z_i, |A| > 1} K(A, B) |A| 2^{-(i-2)}(d+1)^{r+1}.$$

Therefore,

$$T_B \geq |B| - |B|2^{-(i-1)}(d+1)^{r+1} - |B|2^{-(i+1)} - |B|2^{-(i-2)}(d+1)^{r+1}.$$

Hence,

$$\begin{aligned} \text{rk}_\phi(\phi_i \circ \pi_i(a) - \pi_{i+1}(a)) &\leq 2^{-(i+1)} \\ &+ \sum_{B \in Z_{i+1}, |B| > 1} m_{i+1}(B) (2^{-(i-1)}(d+1)^{r+1} + 2^{-(i+1)} + 2^{-(i-2)}(d+1)^{r+1}). \end{aligned}$$

Thus the lemma follows. □

Lemma 8.2. *Let $a \in K\Gamma, b \in K\Gamma$, then:*

- (1) $\lim_{i \rightarrow \infty} \text{rk}_\phi(\pi_i(a)\pi_i(b) - \pi_i(ab)) = 0$.
- (2) $\lim_{i \rightarrow \infty} \text{rk}_\phi(\pi_i(a) + \pi_i(b) - \pi_i(a+b)) = 0$.
- (3) *If $K = \mathbb{C}$, then $\lim_{i \rightarrow \infty} \text{rk}_\phi(\pi_i(a^*) - \pi_i(a)^*) = 0$.*

Proof. We prove only the first part; the other two statements can be seen in exactly the same way. If $x \in A \setminus \partial_{r+s}(A)$, where $\text{supp}(a) \subset B_r(1)$ and $\text{supp}(b) \subset B_s(1)$, then

$$(\pi_i^A(a)\pi^A(b) - \pi_i^A(ab))(e_x) = 0.$$

Therefore

$$\text{rk}_\phi(\pi_i(a)\pi_i(b) - \pi_i(ab)) \leq \sum_{A \in Z_i} m(A) \frac{|\partial_{r+s}(A)|}{|A|} \leq \frac{1}{2^i} + \max_{A \in Z_i, |A| > 1} \frac{|\partial_{r+s}(A)|}{|A|}.$$

Since $\lim_{i \rightarrow \infty} \max_{A \in Z_i, |A| > 1} \frac{|\partial_{r+s}(A)|}{|A|} = 0$, the lemma follows. $\square$

Let $\pi(a) \in \overline{\mathcal{A}}_\phi$ be the limit of elements $\pi_i(a)$ in the rank completion of the ultramatrixial algebra $\mathcal{A}_\phi$. By the previous lemma π is a homomorphism and if $K = \mathbb{C}$, then π is even a $\star$ -homomorphism. Finally, we prove that

$$\text{rk}_\phi(\pi(a)) = \text{rk}_\Gamma(a).$$

By definition,

$$(5) \quad \text{rk}_\phi(\pi_i(a)) = \sum_{A \in Z_i} m(A) \frac{\text{rank}(\pi_i^A(a))}{|A|}.$$

If $|A| = 1$, then $m(A) \leq 2^{-i}$. By Lemma 6.1, $\lim_{i \rightarrow \infty} \frac{\text{rank}(\pi_i^{A_i}(a))}{|A_i|} = \text{rk}_\Gamma(a)$, where A_i is an arbitrary element of Z_i such that $|A_i| > 1$. Hence by (5), $\text{rk}_\phi(\pi(a)) = \text{rk}_\Gamma(a)$. This finishes the proof of Theorem 2. $\square$

Remark. Let Γ be a residually finite amenable group and K be a field such that $K\Gamma$ has no zero-divisor. Let $\Gamma = N_0 \supset N_1 \supset N_2 \dots, \bigcap_{k=1}^\infty N_k = \{1\}$ be finite index normal subgroups and $\pi_k : \mathbb{C}\Gamma \rightarrow \mathbb{C}(\Gamma/N_k)$ be the natural projections as in Subsection 1.3. Then the following version of (2) holds by [14, Theorem 0.2(i)] (the authors use a slightly different language). For any $\Delta \in \text{Mat}_{d \times d}(K\Gamma)$,

$$\lim_{k \rightarrow \infty} \frac{\dim_K \text{Ker}(\pi_k(\Delta))}{|\Gamma : N_k|} = d - \text{rk}_{S^{-1}K\Gamma}(i(\Delta)),$$

where the skewfield $S^{-1}K\Gamma$ is the Ore localization of $K\Gamma$ and i is the natural embedding. One can actually prove, using similar methods, that for arbitrary finitely generated amenable residually finite groups

$$\lim_{k \rightarrow \infty} \frac{\dim_K \text{Ker}(\pi_k(\Delta))}{|\Gamma : N_k|} = d - \text{rk}_{\mathcal{A}_\phi}(\pi(\Delta)),$$

where $\text{rk}_{\mathcal{A}_\phi}$ is the matrix rank in $\text{Mat}_{d \times d}(\mathcal{A}_\phi)$.

9. THE PROOF OF THEOREM 1

In this section we prove the three main identities of Section 2 finishing the proof of Theorem 1.

9.1. The first identity. We will prove that $f_5 \circ f_4 = f_1$, where:

- f_1 is the map ψ_μ defined in Subsection 7.1 (associated to the sofic approximation below).
- $f_4 = \pi$, where $\pi : \mathbb{C}\Gamma \rightarrow \overline{\mathcal{A}_\phi}$ is the map defined in Section 8.
- The map f_5 will be defined below.

Let $\{P_n\}_{n=1}^\infty$ be the sofic approximation of our group Γ . Our first goal is to construct a map

$$\tau : \mathcal{A}_\phi \rightarrow \prod_{n=1}^\infty \text{Mat}_{P_n \times P_n}(\mathbb{C}),$$

where $\mathcal{A}_\phi$ is the ultramatricial algebra of Section 4. We fix a Bratteli tiling system $(\{Z_n\}_{n=1}^\infty, m)$ for the whole section.

Lemma 9.1. *We have a sequence $\{j(n)\}_{n=1}^\infty$, $\lim_{n \rightarrow \infty} j(n) = \infty$ with the following properties:*

- We have an iterated tiling of P_n starting with a $Z_{j(n)}$ -tiling.
- For any $k > 0$ and $A \in Z_k$,

$$\lim_{n \rightarrow \infty} \frac{|T_A^{P_n}|}{|P_n|} = m(A),$$

where $T_A^{P_n}$ is the number of points in P_n covered by A -tiles in the Z_k -tiling of P_n .

Proof. Let $G_1 \subset G_2 \subset \dots$ be the Følner-sequence in the proof of Proposition 4.1. Recall that for each $n \geq 1$ there exist inductive tilings of the G_n 's such that

$$\lim_{n \rightarrow \infty} \frac{|T_A^{G_n}|}{|G_n|} = m(A).$$

Claim 9.1. We have a sequence $\{\varepsilon(n)\}_{n=1}^\infty$ tending to zero and for each $n \geq 1$ a subfamily $G^n = \{G_{t_1^n} \subset G_{t_2^n} \subset \dots \subset G_{t_{m_n}^n}\}$ such that:

- P_k is $\varepsilon(n)$ -quasitiled by G^n -tiles, if $k \geq n$.
- $\lim_{n \rightarrow \infty} t_1^n = \infty$.

Proof. We define inductively the sequences $\{\varepsilon(n)\}_{n=1}^\infty \subset \mathbb{R}^+$, $\{c_n\}_{n=1}^\infty \subset \mathbb{N}$ and $\{t_1^n\}_{n=1}^\infty \subset \mathbb{N}$ such that $\varepsilon(n) \rightarrow 0$, $t_1^n \rightarrow \infty$. First of all, let $\varepsilon(1) := 1$, $t_1^1 := 1$, $c_1 := 2$. Moreover, given the Følner-sequence $G_n \subset G_{n+1} \subset \dots$ and $\varepsilon = 1/n$, let $L_n > 0$, $\delta_n > 0$ be the corresponding constants, and F^n be the subcollection of the Følner-sequence as in Proposition 7.1. Also, let N_n be the natural number in the definition of the sofic approximation such that P_k can be $1/n$ -quasitiled by F^n -translates for any $k \geq N_n$, as in Proposition 7.1. Since everything can be 1 -quasitiled by any tiling system, we can clearly assume that $N_1 = 1$. Moreover, we can choose G^1 arbitrarily.

Now we make the inductive definition. Suppose that for $i \leq n-1$, $\varepsilon(i)$, c_i , G^i are already defined and $P(k)$ can be $\varepsilon(i)$ -quasitiled by G^i -translates if $k \geq 1$. If $n < N_{c_{n-1}}$, we set $c_n := c_{n-1}$, $\varepsilon(n) := \varepsilon(n-1)$ and $G^n := G^{n-1}$. However, if $n \geq N_{c_{n-1}}$, then we set $c_n := c_{n-1} + 1$, $\varepsilon(n) := 1/c_{n-1}$, $G^n := F^{c_{n-1}}$. It follows from the choices that for any $k \geq n$, $P(k)$ can be $\varepsilon(n)$ -quasitiled by G^n -tiles and $t_1^n \geq c_{n-1}$. Hence we get the desired sequences. $\square$

In the next step, we construct the numbers $j(n)$ and the iterated tilings of the sets P_n , starting with a $Z_{j(n)}$ -tiling. Recall that, for each n , we have in Section 5 defined j_n and an iterated Z_{j_n} -tiling of G_n with the properties we are interested in. We wish to make use of these tilings for the G^n -quasitiles of the quasitiling of P_n we have found. Choose $j(n)$ maximal such that $j(n) \leq j_{t_1^n}, \dots, j \leq j_{t_{m_n}^n}$ and $|A| \leq 1/\sqrt{\epsilon(n)}$ for each $A \in Z_{j(n)}$. Since $\epsilon_n \rightarrow 0$ and all the $t_k^n \geq t_1^n \rightarrow \infty$, it immediately follows that $j(n) \rightarrow \infty$.

By the definition of the quasitiling, we have disjoint subsets $\{J_\alpha\}_{\alpha \in I_n}$ such that each J_α is inside a G^n -translate L_α , covering at least the $(1 - \epsilon(n))$ part of L_α . For any $J_\alpha \subset L_\alpha$, we consider an iterated tiling. First, we take the $Z_{j(n)}$ -tiling of L_α , where L_α is a $G_{t_i^n}$ -translate. We can do it since $j(n) \leq j_{t_i^n}$. We keep all the $Z_{j(n)}$ -tiles that are completely covered by J_α and cover all other points in J_α by singletons. Then we start the iterative tiling procedure. That is, for any $1 \leq i \leq j(n)$, the Z_i -tiling of J_α consists of the Z_i -tiles of L_α such that the corresponding $Z_{j(n)}$ -tile is completely in J_α and some singletons.

Lemma 9.2. *For any index sequence $\{\alpha_n \in I_n\}_{n=1}^\infty$, for any $i \geq 1$ and $A \in Z_i$,*

$$\lim_{n \rightarrow \infty} \frac{|T_A^{J_{\alpha_n}}|}{|J_{\alpha_n}|} = m(A),$$

where $T_A^{J_{\alpha_n}}$ is the number of vertices in J_{α_n} covered by an A -tile in the Z_i -tiling of J_{α_n} .

Proof. We use the same idea as in the proof of Lemma 5.5. The number of vertices in L_{α_n} that are in a $Z_{j(n)}$ -tile not completely in J_{α_n} is less than $|L_{\alpha_n} \setminus J_{\alpha_n}|M$, where M is the maximal size of a $Z_{j(n)}$ -tile. Hence the lemma follows from our condition on the maximal size of a $Z_{j(n)}$ -tile. $\square$

One can immediately see that Lemma 9.1 follows from Lemma 9.2. $\square$

Recall from Section 4 the construction of an injective homomorphism

$$E_P : \bigoplus_{A \in Z_k} \text{Mat}_{A \times A}(\mathbb{C}) \rightarrow \text{Mat}_{P \times P}$$

associated to a Z_k -tiling of a graph P . Now if $i(P_n) \geq k$, then let

$$\tau_{k,n} : \bigoplus_{A \in Z_k} \text{Mat}_{A \times A}(\mathbb{C}) \rightarrow \text{Mat}_{P_n \times P_n}$$

is the homomorphism associated to the Z_k -tiling of P_n given in Lemma 9.1. By the iterative tiling construction, if $j(P_n) \geq k + 1$, then

$$(6) \quad \tau_{k,n} = \tau_{k+1,n} \circ \phi_k,$$

where $\phi_k : \bigoplus_{A \in Z_k} \text{Mat}_{A \times A}(\mathbb{C}) \rightarrow \bigoplus_{B \in Z_{k+1}} \text{Mat}_{B \times B}(\mathbb{C})$ is the injective homomorphism defined in Section 4. Let us define $\tau_{k,n}$ to be the zero homomorphism if $k > j(P_n)$. Hence, we defined a homomorphism

$$\tau^k = (\tau_{k,1}, \tau_{k,2}, \dots) : \bigoplus_{A \in Z_k} \text{Mat}_{A \times A}(\mathbb{C}) \rightarrow \prod_{n=1}^\infty \text{Mat}_{P_n \times P_n}.$$

Lemma 9.3. *For any $\bigoplus_{A \in Z_k} a_A \in \bigoplus_{A \in Z_k} \text{Mat}_{A \times A}(\mathbb{C})$,*

$$\lim_{n \rightarrow \infty} \frac{\text{rank}(\tau_{k,n}(\bigoplus_{A \in Z_k} a_A))}{|P_n|} = \text{rk}_\phi(\bigoplus_{A \in Z_k} a_A).$$

Proof.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\text{rank}(\tau_{k,n}(\bigoplus_{A \in Z_k} a_A))}{|G_n|} &= \lim_{n \rightarrow \infty} \sum_{A \in Z_k} \frac{\frac{|T_A^{P_n}|}{|A|} \text{rank}(a_A)}{|P_n|} \\ &= \sum_{A \in Z_k} m(A) \frac{\text{rank}(a_A)}{|A|} = \text{rk}_\phi(\bigoplus_{A \in Z_k} a_A). \end{aligned}$$

□

Let $\mathcal{Q} \subset \prod_{n=1}^{\infty} \text{Mat}_{P_n \times P_n}$ be the ideal of elements that are eventually zero. Clearly, $\mathcal{Q} \subset \mathcal{I}$ and $\mathcal{Q} \subset \mathcal{J}$, where $\mathcal{I}$ and $\mathcal{J}$ are the ideals introduced in Subsection 1.2. By (6), $\text{Im}(\tau^k - \tau^{k+1} \circ \phi_k) \in \mathcal{Q}$. Hence, the τ^k 's extend to a map $\tau : \mathcal{A}_\phi \rightarrow \prod_{n=1}^{\infty} \text{Mat}_{P_n \times P_n} / \mathcal{Q}$. By Lemma 9.3, the map $\tau_{\text{alg}} = \pi \circ \tau : \mathcal{A}_\phi \rightarrow \mathcal{M}_\mu^{\text{alg}}$ is rank-preserving, where

$$\pi : \prod_{n=1}^{\infty} \text{Mat}_{P_n \times P_n} / \mathcal{Q} \rightarrow \mathcal{M}_\mu^{\text{alg}}$$

is the natural quotient map. Therefore τ_{alg} can be extended to the completion $\overline{\mathcal{A}_\phi}$. That will be our map $f_5 : \overline{\mathcal{A}_\phi} \rightarrow \mathcal{M}_\mu^{\text{alg}}$.

Now let us prove $f_1 = f_5 \circ f_4$. Let $a \in \mathbb{C}\Gamma$, $\text{supp}(a) \subset B_r(1) \subset \text{Cay}(\Gamma, S)$. Then $f_1(a)$ can be represented in $\prod_{n=1}^{\infty} \text{Mat}_{P_n \times P_n}(\mathbb{C})$ by the element $\prod_{n=1}^{\infty} \psi_n(a)$, where ψ_n is defined in Subsection 7.1.

Recall from Section 8 that $f_4(a) = \lim_{k \rightarrow \infty} \pi_k(a)$. Since f_5 is rank preserving, $f_5 \circ f_4(a)$ equals to $\lim_{k \rightarrow \infty} f_5 \circ \pi_k(a)$. Observe that $f_5 \circ \pi_k(a)$ is represented in $\prod_{n=1}^{\infty} \text{Mat}_{P_n \times P_n}(\mathbb{C})$ by

$$\tau^k \circ \pi_k(a) = \prod_{n=1}^{\infty} \psi_n^k(a),$$

where

- $a = \sum a_\gamma \gamma$ and $r > 0$ is the smallest number such that $\text{supp}(a) \subset B_r(1)$.
- $\psi_n^k(a) = 0$, if $j(n) < k$.
- $\psi_n^k(a)(e_x) = \sum_{y \in B_r(1)} a_\gamma e_{x\gamma}$, if $j(n) \geq k$ and $x \in S \setminus \partial_r S$ for some Z_k -translate S in the Z_k -tiling of P_n .

Observe that in $\mathcal{M}_\mu^{\text{alg}}$,

$$\text{rank}[\prod_{n=1}^{\infty} (\psi_n(a) - \psi_n^k(a))] \leq 1 - \lim_{\omega} \frac{|z_n^k(a)|}{|P_n|},$$

where $z_n^k(a)$ is the set of elements $x \in G_n$ for which $\psi_n^k(a)(e_x) = \psi_n(a)(e_x)$. Therefore $f_1 = f_5 \circ f_4$ immediately follows from the lemma below.

Lemma 9.4. $\lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{|P_n \setminus z_n^k(a)|}{|P_n|} = 0$.

Proof. If $x \in P_n \setminus z_n^k(a)$, then either $x \in \partial_r S$, for some Z_k -translate S , $|S| > 1$ or x is a singleton in the Z_k -tiling of P_n . Fix a real constant $\varepsilon > 0$. By the definition of the Bratteli tiling systems, if k is large enough, then $m_k(E_k) < \frac{\varepsilon}{2}$. Hence by Lemma 9.1, for n large enough (depending on k), the number of singletons in the Z_k -tiling of P_n is less than $\frac{\varepsilon}{2}|P_n|$. On the other hand, by the definition of the Bratteli tiling systems, for k large enough, $\frac{|\partial_r(A)|}{|A|} < \frac{\varepsilon}{2}$, if $A \in Z_k$ and $|A| > 1$. Hence the lemma follows. $\square$

9.2. The second identity. We need to prove that $f_6 \circ f_4 = f_3$, where f_4 is defined in the previous subsection. The maps f_3 and f_6 will be defined below.

Let rk_1 , resp. rk_2 , denote the ranks on $\mathcal{M}_\mu$, resp. $\mathcal{M}_\mu^{\text{alg}}$. Let

$$t \in \prod_{n=1}^{\infty} \text{Mat}_{P_n \times P_n}(\mathbb{C}) = (t_1, t_2, \dots),$$

where $\sup_{n \geq 1} \|t_n\| < \infty$. Note that t represents an element $[t]_{\mathcal{M}_\mu}$ in $\mathcal{M}_\mu$ and an element $[t]_{\mathcal{M}_\mu^{\text{alg}}}$ in $\mathcal{M}_\mu^{\text{alg}}$. It is important to note that $\text{rk}_1([t]_{\mathcal{M}_\mu})$ is not necessarily equal to $\text{rk}_2([t]_{\mathcal{M}_\mu^{\text{alg}}})$. Indeed, let $t_n = \frac{1}{n}Id$. Then $[t]_{\mathcal{M}_\mu} = 0$, hence $\text{rk}_1([t]_{\mathcal{M}_\mu}) = 0$. Nevertheless, $\text{rk}_2([t]_{\mathcal{M}_\mu^{\text{alg}}}) = 1$. However, we have the following lemma.

Lemma 9.5. *Let $t = (t_1, t_2, \dots) \in \prod_{n=1}^{\infty} \text{Mat}_{P_n \times P_n}(\mathbb{C})$, where for any $n \geq 1$, t_n is self-adjoint and all the t_n 's together have finitely many distinct eigenvalues $\lambda_0 = 0, \lambda_1, \lambda_2, \dots, \lambda_k$. Then*

$$\text{rk}_1([t]_{\mathcal{M}_\mu}) = \text{rk}_2([t]_{\mathcal{M}_\mu^{\text{alg}}}).$$

Proof. Let $t_{n,i}$ be the multiplicity of λ_i in t_n . Then

$$\text{rk}_2([t]_{\mathcal{M}_\mu^{\text{alg}}}) = \lim_{\omega} (1 - \frac{t_{n,0}}{|P_n|}).$$

The spectral decomposition of $[t]_{\mathcal{M}_\mu}$ is $\sum_{i=1}^k \lambda_i P_i$, where

$$\text{tr}_{\mathcal{M}_\mu}(P_i) = \lim_{\omega} (1 - \frac{t_{n,i}}{|P_n|}).$$

By (1),

$$\text{rk}_1([t]_{\mathcal{M}_\mu}) = \lim_{\omega} (1 - \frac{t_{n,0}}{|P_n|}). \quad \square$$

In general, we have the following inequality.

Proposition 9.1. *Let $t = (t_1, t_2, \dots) \in \prod_{n=1}^{\infty} \text{Mat}_{P_n \times P_n}(\mathbb{C})$ such that $\sup_{n \geq 1} \|t_n\| \leq K$. Then*

$$\text{rk}_1([t]_{\mathcal{M}_\mu}) \leq \text{rk}_2([t]_{\mathcal{M}_\mu^{\text{alg}}}).$$

Proof. Denote by R_n the orthogonal projection onto the kernel of $t_n^* t_n$. Consider the projection $R = (R_1, R_2, \dots) \in \prod_{n=1}^{\infty} \text{Mat}_{P_n \times P_n}(\mathbb{C})$. By definition,

$$(7) \quad \text{rk}_2([t]_{\mathcal{M}_\mu^{\text{alg}}}) = \text{rk}_2([t^* t]_{\mathcal{M}_\mu^{\text{alg}}}) = 1 - \text{rk}_2([R]_{\mathcal{M}_\mu^{\text{alg}}}).$$

By Lemma 9.5, $\text{rk}_2([R]_{\mathcal{M}_\mu^{\text{alg}}}) = \text{rk}_1([R]_{\mathcal{M}_\mu})$. Hence $\text{rk}_2([t]_{\mathcal{M}_\mu^{\text{alg}}}) = 1 - \text{rk}_1([R]_{\mathcal{M}_\mu})$.

Lemma 9.6. *If $E_\lambda, \lambda > 0$ is an orthogonal projection in the spectral decomposition of $[t^* t]_{\mathcal{M}_\mu}$, then $[R]_{\mathcal{M}_\mu} \leq E_\lambda$.*

Proof. We can suppose that $\lambda < K$. Observe, that $(t_n^* t_n) R_n = R_n (t_n^* t_n) = 0$. Hence $[t^* t]_{\mathcal{M}_\mu} [R]_{\mathcal{M}_\mu} = [R]_{\mathcal{M}_\mu} [t^* t]_{\mathcal{M}_\mu} = 0$. Therefore by continuous functional calculus [20, Theorem VII. 1], for any continuous function f on $[0, K]$ such that $f(0) = 1$,

$$(8) \quad f([t^* t]_{\mathcal{M}_\mu}) [R]_{\mathcal{M}_\mu} = [R]_{\mathcal{M}_\mu} f([t^* t]_{\mathcal{M}_\mu}) = [R]_{\mathcal{M}_\mu}.$$

Now let $\{f_n\}_{n=1}^\infty$ be a monotonically decreasing sequence of continuous functions such that $f_n(0) = 1$, and for any $x \in [0, K]$, $\lim_{n \rightarrow \infty} f_n(x) = \chi_{[0, \lambda]}$. Then, $\{f_n([t^* t]_{\mathcal{M}_\mu})\}_{n=1}^\infty$ converges to E_λ in the strong operator topology ([20, Theorem VII 2]), hence

$$E_\lambda [R]_{\mathcal{M}_\mu} = [R]_{\mathcal{M}_\mu} E_\lambda = [R]_{\mathcal{M}_\mu}. \quad \square$$

By the previous lemma,

$$\text{rk}_1([R]_{\mathcal{M}_\mu}) = \text{tr}_{\mathcal{M}_\mu}([R]_{\mathcal{M}_\mu}) \leq \text{tr}_{\mathcal{M}_\mu}(E_\lambda).$$

Hence $\text{rk}_1([t]_{\mathcal{M}_\mu}) = \text{rk}_1([t^* t]_{\mathcal{M}_\mu}) \leq 1 - \text{rk}_1([R]_{\mathcal{M}_\mu})$, thus the proposition follows from (7). $\square$

Let $i_\mu : \mathbb{C}\Gamma \rightarrow \mathcal{M}_\mu$ be defined as $\rho \circ \psi$, where $\psi = \prod_{n=1}^\infty \psi_n$ as in Subsection 7.1 and

$$\rho : B\left(\prod_{n=1}^\infty \text{Mat}_{P_n \times P_n}(\mathbb{C})\right) \rightarrow \mathcal{M}_\mu$$

be the quotient map from the bounded part of the direct product. The map i_μ is trace-preserving and extends to an injective trace-preserving map $\overline{i_\mu} : \mathcal{N}(\Gamma) \rightarrow \mathcal{M}_\mu$ (see e.g. [17, Proposition 7.1]. The map f_3 is just i_μ composed by the embedding $i_{\mathcal{M}}$ of $\mathcal{M}_\mu$ into its Ore localization. We prove that f_3 is rank-preserving later.

Now let us define the map f_6 . Let τ be the map defined in Subsection 9.1. Then let $j = \rho \circ \tau : \mathcal{A}_\phi \rightarrow \mathcal{M}_\mu$ and let $s = i_{\mathcal{M}} \circ j$. Then f_6 is defined as the extension of s to $\overline{\mathcal{A}_\phi}$. We need to show of course that j is rank-preserving. By Lemma 9.3, we only need to show that

$$\text{rk}_1[\tau^k(\underline{a})]_{\mathcal{M}_\mu} = \text{rk}_2[\tau^k(\underline{a})]_{\mathcal{M}_\mu^{\text{alg}}},$$

for any $\underline{a} \in \bigoplus_{A \in Z_k} \text{Mat}_{A \times A}(\mathbb{C})$. However, this follows from Lemma 9.5.

Now we prove $f_3 = f_6 \circ f_4$. This also shows that f_3 is rank-preserving. Again, it is enough to show that

$$(9) \quad \lim_{k \rightarrow \infty} \left[\prod_{n=1}^\infty (\psi_n(a) - \psi_n^k(a)) \right]_{\mathcal{M}_\mu} = 0.$$

We already proved that

$$(10) \quad \lim_{k \rightarrow \infty} \left[\prod_{n=1}^\infty (\psi_n(a) - \psi_n^k(a)) \right]_{\mathcal{M}_\mu^{\text{alg}}} = 0.$$

Hence equation (9) follows from Proposition 9.1.

9.3. The third identity. We need to show that $f_7 \circ f_2 = f_3$. The map f_2 is defined as $i_{\mathcal{N}(\Gamma)} \circ i$, where $i : \mathbb{C}\Gamma \rightarrow \mathcal{N}(\Gamma)$ is the natural embedding, and $i_{\mathcal{N}(\Gamma)}$ is the embedding of $\mathcal{N}(\Gamma)$ into its Ore localization. The map $f_7 : U(\mathcal{N}(\Gamma)) \rightarrow U(\mathcal{M}_\mu)$ is the extension of the injective map $\overline{i_\mu}$ to the Ore localizations. Note that f_7 is rank-preserving by [21, Lemma 2.2]. The map f_3 is defined above. By definition,

$$\overline{i_\mu} \circ i = i_\mu.$$

Since f_7 is an extension of $\overline{i_\mu}$,

$$f_7 \circ i_{\mathcal{N}(\Gamma)} = i_{\mathcal{M}} \circ \overline{i_\mu}.$$

Therefore, $f_7 \circ f_2 = f_3$. □

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