PARTIAL REGULARITY OF $p(x)$ -HARMONIC MAPS

MARIA ALESSANDRA RAGUSA, ATSUSHI TACHIKAWA, AND HIROSHI TAKABAYASHI

*In memory of the Japanese victims of the earthquake and tsunami that occurred
 on 11 March 2011*

ABSTRACT. Let $(g^{\alpha\beta}(x))$ and $(h_{ij}(u))$ be uniformly elliptic symmetric matrices, and assume that $h_{ij}(u)$ and $p(x)$ (≥ 2) are sufficiently smooth. We prove partial regularity of minimizers for the functional

$$\mathcal{F}(u) = \int_{\Omega} (g^{\alpha\beta}(x) h_{ij}(u) D_{\alpha} u^i D_{\beta} u^j)^{p(x)/2} dx,$$

under the nonstandard growth conditions of $p(x)$ -type. If $g^{\alpha\beta}(x)$ are in the class VMO , we have partial Hölder regularity. Moreover, if $g^{\alpha\beta}$ are Hölder continuous, we can show partial $C^{1,\alpha}$ -regularity.

1. INTRODUCTION

Throughout this note we investigate regularity properties of local minimizers of functionals defined in open subsets Ω of $\mathbb{R}^m$.

We say that a function $u : \Omega \rightarrow \mathbb{R}^n$, $u \in W^{1,1}(\Omega)$ is a *local minimizer* of the functional

$$\mathcal{F}(u) = \int_{\Omega} f(x, u, Du) dx$$

if $|Du| \in L^1_{loc}(\Omega)$ satisfies

$$\mathcal{F}(u; \text{supp} \varphi) \leq \mathcal{F}(u + \varphi; \text{supp} \varphi)$$

for every $\varphi \in W^{1,1}_0(\Omega)$ with compact support in Ω .

Recently there has been an increase in interest in the study of regularity of solutions to elliptic equations, systems and of minimizers of integral functionals of the calculus of variations under nonstandard growth conditions; it is relevant, for instance, in the applications to nonlinear elasticity problems. We highlight the results obtained in the context of regularity theory of minimizers, for instance, by Carbone and Sbordone in [3], Fusco in [12] or Dal Maso and Modica in [8], where for integral functionals of the kind

$$\int_{\Omega} f(x, u, Du) dx$$

where for a given function f is assumed the p -growth condition

$$(1.1) \quad |z|^p \leq f(x, u, z) \leq C(1 + |z|^p), \quad p \geq 1.$$

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Marcellini in [24] replaced this hypothesis considering integrals that are of the kind

$$(1.2) \quad \int_{\Omega} f(Dv) \, dx$$

and satisfy the nonstandard growth condition

$$(1.3) \quad \lambda \sum_{j=1}^n |z^j|^{q_j} \leq f(z) \leq \Lambda \sum_{j=1}^n |z^j|^{q_j}, \quad 2 \leq q_j \leq \frac{2n}{n-2}, \quad j = 1, \dots, n.$$

The study continues in [25] where weak solutions of elliptic equations are of the form

$$\sum_{i=1}^n \frac{\partial}{\partial x^i} a^i(x, Du) = b(x), \quad x \in \Omega,$$

under the hypothesis that a^i satisfy the following (p, q) -growth conditions:

$$\sum_{i,j} a_{z^j}^i(x, z) \xi^i \xi^j \geq \lambda(1 + |\xi|^2)^{(p-2)/2} |\xi|^2, \quad \forall z, \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega,$$

$$|a_{z^j}^i(x, z)| \leq \Lambda(1 + |z|^2)^{(q-2)/2}, \quad \forall z \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega, \quad \forall i, j,$$

with $2 \leq p \leq q < np/(n-2)$. Such types of equations correspond to the Euler-Lagrange equations of the functionals

$$\int f(x, Du) \, dx,$$

with (p, q) -growth condition

$$(1.4) \quad \lambda(1 + |z|^2)^{p/2} \leq f(x, z) \leq \Lambda(1 + |z|^2)^{q/2}, \quad 2 \leq p \leq q < \frac{np}{n-2}.$$

Some examples of weak minimizers of integrals where the (p, q) -growth condition is used have been given in [14], [23]. Later a sharp condition is introduced by Boccardo, Marcellini, Sbordone in [2] to study a special anisotropic case. Other results for anisotropic equations and functional and variational problems with nonstandard growth conditions are later studied by Marcellini in [26], by Morrey in [29] and the references therein.

We also quote the study made by Marcellini in [27] of Lipschitz regularity as well as $C^{1,\alpha}$ regularity obtained for systems and the regularity results for functions dependent on the modulus of the gradient.

Let us also notice that in [28] regularity results are established for minimizers of the integral (1.2) that can be applied not only to the case of arbitrary large exponential growth but also to functions whose behaviour at infinity can be represented by any finite composition of exponential functions.

In 1997, Zhikov in [34] obtained conditions to satisfy Meyers type estimates, for variational problems related to the integrand $|\zeta|^{p(x)}$, where $p(x)$ has a logarithmic continuity modulus.

We mention the study of Coscia and Mingione [5] where Hölder continuous functions $p(x)$ are considered to prove Hölder continuity for local minimizers of the functional

$$\int_{\Omega} |Du|^{p(x)} \, dx,$$

where we point out the vectorial setting of the function u .

Among the many related papers in the literature, in a noncomprehensive way, we recall the study made by Acerbi and Mingione in [1], by Fan and Zhao in [10] and later in [11].

Hölder continuity for local minimizers is also intensely studied by Chiadò Piat and Coscia in [4], for variational integrals where the integrand function $f(x, u, Du)$ satisfies growth assumptions with a variable exponent.

Significant contributions were made by other authors such as Coscia and Mucci in [6], where we find integral representation properties of limits of sequences of integral functionals of the type

$$\int_{\Omega} f(x, Du) dx$$

under nonstandard growth conditions of the type:

$$|z|^{p(x)} \leq f(x, z) \leq C(1 + |z|^{p(x)}),$$

which is a borderline hypothesis lying between the conditions (1.1) and the above condition (1.4).

For real-valued minimizers of an integral functional $\int_{\Omega} f(x, u, Du) dx$, Eleuteri in [9] proves a regularity result under a nonstandard growth condition of $p(x)$ type.

More recently, Cupini, Marcellini and Mascolo, see [7], prove boundedness of minimizers of energy functionals, for instance of the anisotropic type

$$\mathcal{C}(u) = \int_{\Omega} \sum_{i=1}^n |u_{x_i}|^{p_i(x)} dx$$

under sharp assumptions on the exponent p_i in terms of the usual Sobolev conjugate exponent $p^* = \frac{np}{n-p}$ of p . The authors also prove local Lipschitz continuity of minimizers under sharp assumptions on the exponent of anisotropic growth.

In the present paper we set Ω to be an open subset of $\mathbb{R}^m$ and $p: \Omega \rightarrow [2, +\infty)$ a locally Hölder continuous function. We consider the $p(x)$ -energy functional

$$(1.5) \quad \mathcal{F}(u) = \int_{\Omega} f(x, u, Du) = (g^{\alpha\beta}(x) h_{ij}(u) D_{\alpha} u^i D_{\beta} u^j)^{\frac{p(x)}{2}} dx.$$

Here and in the sequel, we use the Einstein convention.

Our work is devoted to the study of partial regularity of the local minimizers of the functional $\mathcal{F}$. We point out that in this note we deal with the vector-valued case $n \geq 1$. The technique to obtain higher integrability properties for local minimizers employs a direct approach as well as classical inequalities by Jensen and Poincaré.

About constant p -growth functionals defined for $u: \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ with general $m, n \geq 2$, roughly speaking, known regularity results differ from each other by the structures of functionals. Let us consider the three types of functionals

$$(1.6) \quad F_1(u) = \int_{\Omega} a(|Du|) dx,$$

$$(1.7) \quad F_2(u) = \int_{\Omega} a(g^{\alpha\beta}(x, u) h_{ij}(x, u) D_{\alpha} u^i D_{\beta} u^j) dx,$$

$$(1.8) \quad F_3(u) = \int_{\Omega} A(x, u, Du) dx,$$

where $(g^{\alpha\beta})$, (h_{ij}) and the Hessian matrix of $A(x, u, \xi)$ with respect to ξ are uniformly positive definite. Moreover, we are assuming the following growth conditions

on $a : [0, \infty) \rightarrow [0, \infty)$ and A :

$$\lambda t^p \leq a(t) \leq \Lambda t^p, \text{ for all } t \in [0, \infty),$$

$$\lambda |\xi|^p \leq A(x, u, \xi) \leq \Lambda |\xi|^p, \text{ for all } (x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}.$$

We have the following results for minimizers of the above types of functionals.

- (I) : (Uhlenbeck [33]) Let u be a minimizer of F_1 ; then $u \in C^{1,\alpha}(\Omega)$.
- (II) : (Giaquinta-Modica [19], Fusco-Hutchinson [13]) Let u be a minimizer of F_2 . Then $u \in C^{1,\alpha}(\Omega_0)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-p-\varepsilon}(\Omega \setminus \Omega_0) = 0$ for some $\varepsilon > 0$. Here, $\mathcal{H}^q$ denotes the q -dimensional Hausdorff measure. In [13], they also showed that under slightly stronger conditions the estimate on the dimension of the singular set can be improved.
- (III) : (Giaquinta-Giusti [16]) Let u be a minimizer of F_3 . Then $u \in C^{1,\alpha}(\Omega_0)$, where Ω_0 is an open subset of Ω with $|\Omega \setminus \Omega_0| = 0$. Here, for a measurable set $D \subset \mathbb{R}^m$, $|D|$ denotes the Lebesgue measure of D .

On the other hand, for $p(x)$ -growth cases, the results of Coscia-Mingione [5] and of Acerbi-Mingione [1] correspond to the above results (I) and (III) respectively. The results of this paper correspond to a part of (II).

2. SOME DEFINITIONS AND THE MAIN RESULT

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary $\partial\Omega$. In the following we write

$$B(x, R) := \{y \in \mathbb{R}^m ; |x - y| < R\}.$$

Definition 2.1. Letting $f \in L^1(\Omega)$, we set the integral mean $f_{x,R}$ by

$$f_{x,R} = \frac{1}{|\Omega \cap B(x, R)|} \int_{\Omega \cap B(x, R)} f(y) dy,$$

where $|\Omega \cap B(x, R)|$ is the Lebesgue measure of $\Omega \cap B(x, R)$.

If we are not interested in specifying which is the center, we only set f_R .

Let us now give the definition of a bounded mean oscillation function (BMO) that first appeared in the note by John and Nirenberg [21].

Definition 2.2. Let $f \in L^1_{\text{loc}}(\Omega)$. We say that f belongs to $BMO(\Omega)$ if the seminorm

$$\|f\|_* \equiv \sup_{B(x, R)} \frac{1}{|\Omega \cap B(x, R)|} \int_{\Omega \cap B(x, R)} |f(y) - f_{x,R}| dy < \infty.$$

Let us recall the definition of the space of vanishing mean oscillation functions, given at first by Sarason in [32]. (See also [30, 31].)

Definition 2.3. Let $f \in BMO(\Omega)$ and

$$\eta(f, R) = \sup_{\rho \leq R} \sup_{B(x, \rho)} \frac{1}{|\Omega \cap B(x, \rho)|} \int_{\Omega \cap B(x, \rho)} |f(y) - f_\rho| dy,$$

where $B(x, \rho)$ ranges over the class of the balls of $\mathbb{R}^m$ of radius ρ . We say that $f \in VMO(\Omega)$ if

$$\lim_{R \rightarrow 0} \eta(f, R) = 0.$$

We consider the $p(x)$ -energy functional defined by (1.5) with the following conditions.

(H 1) There exist constants $\lambda_0, \Lambda_0, \lambda_1, \Lambda_1$ ($0 < \lambda_i < \Lambda_i$, $i = 0, 1$) such that

$$\lambda_0 |\zeta|^2 \leq g^{\alpha\beta}(x) \zeta_\alpha \zeta_\beta \leq \Lambda_0 |\zeta|^2, \quad \lambda_1 |\eta|^2 \leq h_{ij}(u) \eta^i \eta^j \leq \Lambda_1 |\eta|^2$$

for all $x \in \Omega$, $u, \zeta \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$.

(H 2) For every $u, v \in \mathbb{R}^n$,

$$|h_{ij}(u) - h_{ij}(v)| \leq \omega_0(|u - v|^2),$$

where ω_0 is some monotone increasing concave function with $\omega_0(0) = 0$.

(H 3) $g^{\alpha\beta}$ are in the class $L^\infty \cap VMO(\Omega)$.

In the sequel, we put

$$\eta(g, R) := \max_{1 \leq \alpha, \beta \leq m} \eta(g^{\alpha\beta}, R).$$

Moreover, we assume the following conditions on the exponent $p(x)$.

(H 4) For some positive constants γ_1, γ_2 ,

$$(2.1) \quad 2 \leq \gamma_1 \leq p(x) \leq \gamma_2, \quad \text{for all } x, y \in \Omega.$$

(H 5) For some constants $L > 0$ and $\sigma \in (0, 1)$,

$$(2.2) \quad |p(x) - p(y)| \leq L|x - y|^\sigma \quad \text{for all } x, y \in \Omega.$$

In general, if $p(x)$ is a uniformly continuous function in the set Ω , as in [6], we consider

$$W^{1,p(x)}(\Omega) := \{u \in W^{1,1}(\Omega) ; \int_\Omega |Du|^{p(x)} dx < \infty\}$$

and similarly, if Ω has Lipschitz boundary or at least has the cone property and $p(x)$ is continuous in Ω ,

$$W_{\text{loc}}^{1,p(x)}(\Omega) := \{u \in L_{\text{loc}}^1(\Omega) ; |Du|^{p(x)} \in L_{\text{loc}}^1(\Omega)\}.$$

We also define

$$W_0^{1,p(x)}(\Omega) := \{u \in W_0^{1,1}(\Omega) ; \int_\Omega |Du|^{p(x)} dx < \infty\},$$

and for a given map φ ,

$$\varphi + W_0^{1,p(x)}(\Omega) := \{u \in W^{1,p(x)}(\Omega) ; u - \varphi \in W_0^{1,p(x)}(\Omega)\}.$$

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Assume that $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ satisfy the conditions (H 1)–(H 3). Let $u \in W^{1,p(x)}(\Omega)$ be a local minimizer of the functional*

$$\mathcal{F}(u) = \int_\Omega (g^{\alpha\beta}(x) h_{ij}(u) D_\alpha u^i D_\beta u^j)^{\frac{p(x)}{2}} dx,$$

where $p(x) : \Omega \rightarrow [2, \infty)$ satisfies (H 4) and (H 5). Then $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0, 1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-\gamma_1}(\Omega \setminus \Omega_0) = 0$.

Moreover, if $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ are Hölder continuous, then $u \in C^{1,\alpha'}(\Omega_0)$ for some $\alpha' \in (0, 1)$.

3. PRELIMINARY RESULTS

In the following, for the sake of simplicity, we freely set by c any constant which may differ from line to line.

We need some results on the higher integrability of minimizers. We will state them in a more general form than actually needed for the proof of the main results.

Let $g(x, u, \xi)$ be a Carathéodory function on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$ which satisfies, for some positive constants $\lambda < \Lambda$,

$$(3.1) \quad \lambda |\xi|^{p(x)} \leq g(x, u, \xi) \leq \Lambda |\xi|^{p(x)}$$

for all $(x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$.

Let $\omega_1 : [0, +\infty) \rightarrow [0, +\infty)$ be a nondecreasing continuous function with $\omega_1(0) = 0$ which represents the modulus of continuity; namely, ω_1 satisfies

$$(3.2) \quad |p(x) - p(y)| \leq \omega_1(|x - y|).$$

Let us consider the following condition on ω_1 :

$$(3.3) \quad \lim_{r \rightarrow 0} \omega_1(r) \log \left(\frac{1}{r} \right) = \mu < +\infty.$$

The above condition implies

$$(3.4) \quad (1/t)^{\omega_1(t)} = \exp(-\log t \omega_1(t)) \rightarrow e^\mu \text{ as } t \rightarrow 0^+.$$

We also mention that if $p(x)$ is Hölder continuous, then condition (3.3) is fulfilled.

Let $\mathcal{G}$ be a functional defined by

$$(3.5) \quad \mathcal{G}(w; \Omega) := \int_{\Omega} g(x, w, Dw) dx$$

for $w : \Omega \rightarrow \mathbb{R}^n$. We call a map $v \in W_{\text{loc}}^{1,p(x)}(\Omega, \mathbb{R}^n)$ a *local minimizer* of the functional $\mathcal{G}(w, \Omega)$ if $|Dv|^{p(x)} \in L_{\text{loc}}^1(\Omega)$ and

$$\mathcal{G}(v; \text{supp} \varphi) \leq \mathcal{G}(v + \varphi; \text{supp} \varphi)$$

for all $\varphi \in W_0^{1,1}(\Omega, \mathbb{R}^n)$ with compact support.

About the higher integrability of local minimizers, here we state a result which is originally due to Zhikov [34] and is extended by Eleuteri [9] for general functionals as above. The next result is given in [9], only for the real-valued functions. The proof is unchanged for the general case $n > 1$.

Proposition 3.1 ([9, Theorem 3.1]). *Let $\mathcal{G}$ be a functional defined as above. Assume that the exponent $p(x) > 1$ has modulus of continuity ω_1 which satisfies (3.3). Let $u \in W_{\text{loc}}^{1,p(x)}(\Omega, \mathbb{R}^n)$ be a local minimizer of $\mathcal{G}$. Then, there exists a constant $\delta_0 > 0$ such that $|Du|^{(1+\delta_0)p(x)} \in L_{\text{loc}}^1(\Omega)$. Moreover, the estimate*

$$(3.6) \quad \int_{B(y,R)} |Du|^{(1+\delta_0)p(x)} dx \leq c \left(\int_{B(y,2R)} (1 + |Du|^2)^{p(x)/2} dx \right)^{(1+\delta_0)}$$

holds for any $B(y, 2R) \Subset \Omega$.

Here, by virtue of Hölder's inequality, (3.6) holds for any $\delta \in (0, \delta_0)$.

We also need higher integrability results on the neighborhood of the boundary. Let us use the following notation: for $T > 0$ we put

$$\begin{aligned} B^+(T) &:= \{x \in \mathbb{R}^m ; |x| < T, x^m > 0\}, \\ \Gamma(T) &:= \{x \in \mathbb{R}^m ; |x| < T, x^m = 0\}, \\ \partial^+ B^+ &:= \partial B^+(T) \setminus \Gamma(T). \end{aligned}$$

Let $g(x, u, \xi)$ be a Carathéodory function on $B^+(T) \times \mathbb{R}^n \times \mathbb{R}^{mn}$ which satisfies (3.1). Assume that $p(x)$ is a uniformly continuous function on $B^+(T)$.

When we consider the functional $\mathcal{G}$ on $B^+(R)$, as in [22] for the case $p(x) = 2$, let us call a map $v \in \bigcap_{T' < T} W^{1,p(x)}(B^+(T'), \mathbb{R}^n)$ a *local minimizer* of the functional

$$\mathcal{G}(w, B^+(T)) := \int_{B^+(T)} g(x, w, Dw) dx$$

in $B^+(T) \cup \Gamma(T)$ if for every $T' < T$ and any $\varphi \in W_0^{1,1}(B^+(T'), \mathbb{R}^n)$ the following inequality holds:

$$\mathcal{G}(v, B^+(T')) \leq \mathcal{G}(v + \varphi, B^+(T')).$$

Then we have the following lemma on the higher integrability of such local minimizers.

Lemma 3.2. *Assume that the exponent $p(x) > 1$ has modulus of continuity ω_1 which satisfies (3.3). Let $p_1 := \inf_{B^+(T)} p(x)$ and $p_2 := \sup_{B^+(T)} p(x)$, and suppose that*

$$(3.7) \quad (p_2)_* = \frac{mp_2}{m + p_2} < p_1 \text{ (or equivalently } p_1^* = \frac{mp_1}{m - p_1} > p_2).$$

For some $\delta_0 > 0$, let u be a given map in the class $W^{1,(1+\delta_0)p(x)}(B^+(T))$. Let v be a local minimizer of $\mathcal{G}$ in the class

$$\{w \in W^{1,p(x)}(B^+(T)) ; w = u \text{ on } \Gamma(T)\}.$$

Then, there exists a positive constant $\delta_1 < \delta_0$ such that for any $\delta \in (0, \delta_1)$, the local minimizer v satisfies $v \in W^{1,(1+\delta)p(x)}(B^+(T'))$ for any $T' < T$. Moreover, if $x_0 \in B^+(T') \cup \Gamma(T')$ and $r < T - T'$, we have

$$\begin{aligned} (3.8) \quad & \left(\int_{\Omega(x_0, r/2)} (1 + |Dv|^2)^{(1+\delta)p(x)/2} dx \right)^{1/(1+\delta)} \\ & \leq c \int_{\Omega(x_0, r)} (1 + |Dv|^2)^{p(x)/2} dx + c \left(\int_{\Omega(x_0, r)} (1 + |Du|^2)^{(1+\delta)p(x)/2} dx \right)^{1/(1+\delta)}, \end{aligned}$$

where we put

$$\Omega(y, \rho) := B(y, \rho) \cap B^+(T).$$

Proof. For convenience, we extend u, v, Du, Dv to be zero outside $B^+(T)$. Of course, because extended u, v may have discontinuity on $\partial B^+(T)$, they are not always in $W_{\text{loc}}^{1,p(x)}(\mathbb{R}^m)$, and therefore Du, Dv do not necessarily coincide with distributional derivatives of them. On the other hand, since $u = v$ on $\Gamma(T)$, $u - v$ is in the class $W^{1,p(x)}(B(T'))$ and $Du - Dv$ can be regarded as the weak derivatives of $u - v$ in $B(T')$ for any $T' < T$.

For some $T' < T$ let $x_1 \in B(T')$ and s, t, r be positive numbers with $t < s < r < \min\{T - T', 1\}$. In the following let us write

$$\Omega_r := \Omega(x_1, r), \quad B_r := B(x_1, r),$$

and put

$$\rho_1 = \inf_{\Omega_r} p(x) \text{ and } \rho_2 = \sup_{\Omega_r} p(x).$$

Choose a function $\eta \in C_0^\infty(B_s)$ so that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on B_t , $|D\eta| \leq 2/(s-t)$. Put $\varphi = \eta(v-u)$. Then we see that $\varphi \in W_0^{1,1}(B^+(T'))$ and that $\mathcal{G}(v) \leq \mathcal{G}(v-\varphi)$. Since

$$D(v-\varphi) = (1-\eta)Dv - (v-u)D\eta + \eta Du,$$

using (3.1), we get

$$\begin{aligned} \lambda \int_{\Omega_t} |Dv|^{p(x)} dx &\leq \Lambda \int_{\Omega_s} |D(v-\varphi)|^{p(x)} dx \\ &\leq \Lambda \left\{ \int_{\Omega_s \setminus \Omega_t} |Dv|^{p(x)} dx + \int_{\Omega_s} |Du|^{p(x)} dx \right. \\ &\quad \left. + \int_{\Omega_s} \left(\frac{2}{s-t} \right)^{p(x)} |v-u|^{p(x)} dx \right\} \\ &\leq \Lambda \left\{ \int_{\Omega_s \setminus \Omega_t} |Dv|^{p(x)} dx + \int_{\Omega_s} |Du|^{p(x)} dx \right. \\ (3.9) \quad &\quad \left. + 2^{\rho_2} \left(\frac{1}{s-t} \right)^{\rho_2} \int_{\Omega_s} |v-u|^{p(x)} dx \right\}. \end{aligned}$$

Now, we use *the hole filling method*. Adding $\Lambda \int_{\Omega_t} |Dv|^{p(x)} dx$ to both sides and dividing them by $\lambda + \Lambda$, we get

$$\begin{aligned} \int_{\Omega_t} |Dv|^{p(x)} dx &\leq \frac{\Lambda}{\lambda + \Lambda} \int_{\Omega_s} |Dv|^{p(x)} dx + c \int_{\Omega_s} |Du|^{p(x)} dx \\ &\quad + \frac{c}{(s-t)^{\rho_2}} \int_{\Omega_s} |v-u|^{p(x)} dx \\ &\leq \frac{\Lambda}{\lambda + \Lambda} \int_{\Omega_r} |Dv|^{p(x)} dx + c \int_{\Omega_r} |Du|^{p(x)} dx \\ (3.10) \quad &\quad + \frac{c}{(s-t)^{\rho_2}} \int_{\Omega_s} |v-u|^{p(x)} dx. \end{aligned}$$

Using, for example, [15, Lemma 1.1], from (3.10), we can see that

$$\int_{\Omega_\rho} |Dv|^{p(x)} dx \leq c \left((r-\rho)^{-\rho_2} \int_{\Omega_r} |v-u|^{p(x)} dx + \int_{\Omega_r} |Du|^{p(x)} dx \right)$$

holds for any $\rho \in (0, r)$. Taking $\rho = r/2$ in the above estimate, we obtain

$$(3.11) \quad \int_{\Omega_{r/2}} |Dv|^{p(x)} dx \leq c \left(r^{-\rho_2} \int_{\Omega_r} |v-u|^{p(x)} dx + \int_{\Omega_r} |Du|^{p(x)} dx \right).$$

Since we are assuming that $r \leq 1$, we have

$$(3.12) \quad r^{-\rho_2} = r^{p(x)-\rho_2} \cdot r^{-p(x)} \leq r^{\rho_1-\rho_2} \cdot r^{-p(x)}.$$

On the other hand, the assumption (3.3) implies that

$$(3.13) \quad r^{\rho_1-\rho_2} \leq \left(\frac{1}{r} \right)^{\omega_1(2r)} = \exp \left(\omega_1(2r) \log \frac{1}{r} \right) \leq e^{2\mu}$$

for sufficiently small $r > 0$. Combining (3.11), (3.12) and (3.13), we obtain

$$(3.14) \quad \int_{\Omega_{r/2}} |Dv|^{p(x)} dx \leq c \left(\int_{\Omega_r} \left(\frac{|v-u|}{r} \right)^{p(x)} dx + \int_{\Omega_r} |Du|^{p(x)} dx \right).$$

Now, let us estimate the first term of the right-hand side of the above inequality. As in [22], we treat the two cases $x_1^m \leq \frac{3}{4}r$ and $x_1^m > \frac{3}{4}r$ separately.

Case 1. Let us assume that $x_1^m \leq \frac{3}{4}r$. Then $v - u = 0$ in $B_r \setminus B^+(T)$ and $|B_r \setminus B^+(T)| > \gamma|B_r|$ for some constant $\gamma > 0$ depending only on the dimension m . So we can use the Sobolev-Poincaré inequality for $u - v$ to obtain

$$(3.15) \quad \begin{aligned} \int_{B_r} \left| \frac{v-u}{r} \right|^{p(x)} dx &\leq \int_{B_r} \left(1 + \left| \frac{v-u}{r} \right| \right)^{\rho_2} dx \\ &\leq cr^m + cr^{-\rho_2} \int_{B_r} |v-u|^{\rho_2} dx \\ &\leq cr^m + cr^{-\rho_2} \left(\int_{B_r} |Dv - Du|^{\rho_3} dx \right)^{\rho_2/\rho_3}, \end{aligned}$$

where $\rho_3 = (\rho_2)_* = m\rho_2/(m + \rho_2)$. Here, let us mention that by virtue of (3.7) and the definition of ρ_1 , ρ_2 , ρ_3 , we have

$$\rho_3 = \frac{m\rho_2}{m + \rho_2} \leq \frac{mp_2}{m + p_2} < p_1 \leq \rho_1.$$

Then, we can choose a constant $\theta_0 > 1$ so that $\rho_1/\theta_0 > \rho_3$. For any $\theta \in (1, \theta_0)$, using the Hölder inequality, we can estimate the second term (3.15) as follows:

$$(3.16) \quad \begin{aligned} &r^{-\rho_2} \left(\int_{B_r} |Dv - Du|^{\rho_3} dx \right)^{\rho_2/\rho_3} \\ &\leq r^{-\rho_2} \left\{ \left(\int_{B_r} |Dv - Du|^{\rho_1/\theta} dx \right)^{\theta\rho_3/\rho_1} (cr^m)^{1-(\theta\rho_3/\rho_1)} \right\}^{\rho_2/\rho_3} \\ &= cr^{-\rho_2} \left(\int_{B_r} |Dv - Du|^{\rho_1/\theta} dx \right)^{\theta\rho_2/\rho_1} r^{m\{(\rho_2/\rho_3) - (\theta\rho_2/\rho_1)\}} \\ &= cr^{m - \frac{\rho_2}{\rho_1}\theta m} \left(\int_{B_r} |Dv - Du|^{\rho_1/\theta} dx \right)^{\theta\rho_2/\rho_1} \\ &= cr^m \left(\int_{B_r} |Dv - Du|^{\rho_1/\theta} dx \right)^{\theta\rho_2/\rho_1} \\ &= cr^m \left(\int_{B_r} |Dv - Du|^{\rho_1/\theta} dx \right)^{(\rho_2 - \rho_1)\theta/\rho_1} \left(\int_{B_r} |Dv - Du|^{\rho_1/\theta} dx \right)^\theta \\ &=: r^m \cdot A \cdot B. \end{aligned}$$

Using (3.13), we can show the boundedness of A as follows:

$$\begin{aligned}
 A &\leq \left(cr^{-m} \int_{B_r} |Dv - Du|^{\rho_1/\theta} dx \right)^{(\rho_2 - \rho_1)\theta/\rho_1} \\
 &\leq cr^{(\rho_1 - \rho_2)\theta m/\rho_1} \left(\int_{B_r} (1 + |Dv|^{p(x)} + |Du|^{p(x)}) dx \right)^{(\rho_2 - \rho_1)\theta/\rho_1} \\
 &\leq cr^{(\rho_1 - \rho_2)\theta m/\rho_1} \left(\int_{B^+(T')} (1 + |Dv|^{p(x)} + |Du|^{p(x)}) dx \right)^{(\rho_2 - \rho_1)\theta/\rho_1} \\
 &\leq ce^{2\mu\theta m/\rho_1} \left(\int_{B^+(T')} (1 + |Du|^{p(x)}) dx \right)^{(\rho_2 - \rho_1)\theta/\rho_1} \\
 (3.17) \quad &\leq c(\rho_1, \rho_2, \rho_3, m, \theta_0, \mathcal{G}(u, B^+(T'))).
 \end{aligned}$$

Here, we use the minimality of v also. On the other hand, we can estimate B as

$$\begin{aligned}
 B &\leq \left(cr^{-m} \int_{B_r} (|Du|^2 + |Dv|^2)^{\rho_1/(2\theta)} dx \right)^\theta \\
 &\leq \left(cr^{-m} \int_{B_r} (1 + |Du|^2 + |Dv|^2)^{p(x)/(2\theta)} dx \right)^\theta \\
 (3.18) \quad &\leq c + c \left(\int_{B_r} |Du|^{p(x)/\theta} dx \right)^\theta + c \left(\int_{B_r} |Dv|^{p(x)/\theta} dx \right)^\theta.
 \end{aligned}$$

Combining (3.15), (3.16), (3.17) and (3.18), we obtain

$$\begin{aligned}
 \int_{B_r} \left| \frac{v - u}{r} \right|^{p(x)} dx &\leq c + c \left(\int_{B_r} |Dv|^{p(x)/\theta} dx \right)^\theta \\
 (3.19) \quad &\quad + c \left(\int_{B_r} |Du|^{p(x)/\theta} dx \right)^\theta.
 \end{aligned}$$

Since we extended u, v, Du, Dv to be zero outside $B^+(T)$, the estimate (3.14) holds for $B_{r/2}$ and B_r instead of $\Omega_{r/2}$ and Ω_r respectively. So, combining (3.19) with (3.14), we see that

$$\begin{aligned}
 &\int_{B_{r/2}} |Dv|^{p(x)} dx \\
 &\leq c + c \left(\int_{B_r} |Du|^{p(x)/\theta} dx \right)^\theta + c \left(\int_{B_r} |Dv|^{p(x)/\theta} dx \right)^\theta + c \int_{B_r} |Du|^{p(x)} dx \\
 &\leq c + c \int_{B_r} |Du|^{p(x)} dx + c \left(\int_{B_r} |Dv|^{p(x)/\theta} dx \right)^\theta.
 \end{aligned}$$

So, we have the following estimate:

$$\begin{aligned}
 \int_{B_{r/2}} (1 + |Dv|)^{p(x)} dx &\leq c \int_{B_r} (1 + |Du|)^{p(x)} dx \\
 (3.20) \quad &\quad + c \left(\int_{B_r} (1 + |Dv|)^{p(x)/\theta} dx \right)^\theta.
 \end{aligned}$$

Case 2. Let $x_1^m > \frac{3}{4}r$. Then $B_{3r/4} \subset B^+(T)$, and therefore we can proceed as in the proof of [9, Theorem 3.1] slightly modifying the radii, to see that for some

constant $\nu > 1$ the following estimate holds:

$$(3.21) \quad \begin{aligned} \int_{B_{r/2}} (1 + |Dv|^2)^{p(x)/2} dx &\leq c \left(\int_{B_{3r/4}} (1 + |Dv|^2)^{p(x)/(2\nu)} dx \right)^\nu \\ &\leq c \left(\int_{B_r} (1 + |Dv|^2)^{p(x)/(2\nu)} dx \right)^\nu. \end{aligned}$$

From (3.20) and (3.21), putting $k = \min\{\theta, \nu\} (> 1)$, we have for any case,

$$(3.22) \quad \begin{aligned} &\int_{B_{r/2}} (1 + |Dv|^2)^{p(x)/2} dx \\ &\leq c \int_{B_r} (1 + |Du|^2)^{p(x)/2} dx + c \left(\int_{B_r} (1 + |Dv|^2)^{p(x)/k} dx \right)^k. \end{aligned}$$

Now, we are in the position to use the so-called *reverse Hölder inequality with increasing supports* due to Giaquinta-Modica [18]. We can find this famous inequality in many articles. Here, let us use [20, p.299, Theorem 3] with

$$g = (1 + |Dv|^2)^{p(x)/(2k)}, \quad f = (1 + |Du|^2)^{p(x)/(2k)}, \quad q = k, \quad \theta = 0.$$

Then, for some $l > k$, we see that $|Dv|^{p(x)/k} \in L^l_{\text{loc}}(B(T))$ and that the following estimate holds for any $y \in B(T)$:

$$(3.23) \quad \begin{aligned} &\left(\int_{B(y, r/2)} (1 + |Dv|^2)^{p(x)l/(2k)} dx \right)^{1/l} \\ &\leq c \left(\int_{B(y, r)} (1 + |Dv|^2)^{p(x)/2} dx \right)^{1/k} + c \left(\int_{B(y, r)} (1 + |Du|^2)^{p(x)l/(2k)} dx \right)^{1/l}. \end{aligned}$$

Putting $1 + \delta = l/k (> 1)$, we obtain

$$\begin{aligned} &\left(\int_{B(y, r/2)} (1 + |Dv|^2)^{(1+\delta)p(x)/2} dx \right)^{1/(1+\delta)} \\ &\leq c \int_{B(y, r)} (1 + |Dv|^2)^{p(x)/2} dx + c \left(\int_{B(y, r)} (1 + |Du|^2)^{(1+\delta)p(x)/2} dx \right)^{1/(1+\delta)}. \end{aligned}$$

When $y = x_0 \in B^+(T') \cup \Gamma(T')$, the above estimate is nothing but (3.8). $\square$

By virtue of the above lemma, we can show the following estimate.

Corollary 3.3. *Let $D \subset \mathbb{R}^m$ be a bounded domain with smooth boundary ∂D . Let $S > 0$ be a positive number which satisfies the following conditions for every $y \in \partial D$:*

- (C 1) $p_1 = \inf_{B(y, 4S)} p(x)$ and $p_2 = \sup_{B(y, 4S)} p(x)$ satisfy (3.7).
- (C 2) There is a diffeomorphism $\Psi : B(y, 4S) \rightarrow B(T)$ which satisfies

$$\Psi(B(y, 4S) \cap D) \subset B^+(T) \text{ and } \Psi(B(y, 4S) \cap \partial D) = \Gamma(T).$$

Given a map $u \in W^{1, p(x)(1+\delta_0)}(D, \mathbb{R}^n)$, let v be a minimiser of $\mathcal{G}(\cdot, D)$ in the class

$$\{w \in W^{1, p(x)}(D, \mathbb{R}^n) ; w - u \in W^{1, 1}_0(D, \mathbb{R}^n)\}.$$

Then there exists a constant $\delta_1 \in (0, \delta_0)$ such that for any $\delta \in (0, \delta_1]$, we have that $v \in W^{p(x)(1+\delta)}(D, \mathbb{R}^n)$ and that

$$(3.24) \quad \int_D (1 + |Dv|^2)^{(1+\delta)p(x)/2} dx \leq c(1 + |D|^\delta S^{(-m\delta)}) \int_D (1 + |Du|^2)^{(1+\delta)p(x)/2} dx,$$

where c depends only on $m, \lambda, \Lambda, p(x), \mathcal{G}(u, D)$.

Proof. Choose $S > 0$ as above and cover D by a finite number of $B(y, S)$ with $y \in \overline{D}$. If $B(y, S) \subseteq D$, we can use (3.6). For $B(y, S)$ with $B(y, S) \cap \partial D \neq \emptyset$, we flatten $B(y, S) \cap \partial D$ by a diffeomorphism and use (3.8). Summing up these inequalities we get

$$\begin{aligned} & \left(\frac{S}{2}\right)^{-m} \int_D (1 + |Dv|^2)^{(1+\delta)p(x)/2} dx \\ & \leq c \left(S^{-m} \int_D 1 dx\right)^{1+\delta} + c \left(S^{-m} \int_D |Dv|^{p(x)} dx\right)^{1+\delta} \\ & \quad + c S^{-m} \int_D (1 + |Du|)^{(1+\delta)p(x)} dx. \end{aligned}$$

Using the minimality of v in order to estimate the second term of the right-hand side, we obtain

$$\begin{aligned} & \int_D (1 + |Dv|)^{(1+\delta)p(x)} dx \\ & \leq c S^{-m\delta} \left(\int_D |Du|^{p(x)} dx\right)^{1+\delta} + c \int_D (1 + |Du|^2)^{(1+\delta)p(x)/2} dx. \end{aligned}$$

Now, using Hölder's inequality for the first term of the right-hand side, we get the assertion. $\square$

Remark 3.4. In the corollary above, when the domain D is a ball $B(x, R)$, we can choose, for example, $S = R/4$. Hence, the quantity $|D|^\delta S^{-m\delta}$ can be a constant depending only on the dimension m .

In [5], Coscia-Mingione have proved $C^{1,\alpha}$ -regularity of local minimizers of the functional $\int |Du|^{p(x)} dx$. It is easy to see that their result is valid for the local minimizers of

$$\int (A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta u^j)^{p(x)/2} dx$$

with constant coefficient $A_{ij}^{\alpha\beta}$ satisfying

$$\lambda |\xi|^2 \leq A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \text{ for all } \xi \in \mathbb{R}^{mn}.$$

From [5, Proposition 6] and a part of the proof of $C^{1,\alpha}$ -regularity in [5], we have the following proposition.

Proposition 3.5 ([5]). *Let $A_{ij}^{\alpha\beta}$ be as above and define a functional $\mathcal{A}(w)$ by*

$$\mathcal{A}(w, \Omega) := \int_\Omega (A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta u^j)^{p(x)/2} dx.$$

Assume that $p(x)$ is Hölder continuous with exponent $\sigma \in (0, 1)$ and that $p_1 := \inf_\Omega p(x)$ and $p_2 := \sup_\Omega p(x)$ satisfy

$$p_2 < (1 + \delta_0/2)p_2 \leq (1 + \delta_0)p_1,$$

where δ_0 is the constant given by Proposition 3.1 for $\mathcal{A}$. Let $v \in W_{\text{loc}}^{1,p(x)}(\Omega, \mathbb{R}^n)$ be a local minimizer of $\mathcal{A}$. Then, for some $R_2 > 0$ and every $\tau \in (0, 1)$ there exists $c > 0$ such that for all $0 < \rho < R < R_2$ with $B(y, 2R) \Subset \Omega$ we have

$$(3.25) \quad \int_{B(y, \rho)} |Dv|^{\rho_2} dx \leq c \left(\frac{\rho}{R} \right)^{m-\tau} \left[\int_{B(y, 2R)} |Dv|^{\rho_2} dx + R^{m-\tau} \right],$$

where $\rho_2 = \sup_{B(y, 2R)} p(x)$. Moreover, for some $\alpha > 0$,

$$(3.26) \quad \begin{aligned} & \int_{B(y, \rho)} |Dv - (Dv)_\rho|^{\rho_2} dx \\ & \leq c \left[\left(\frac{\rho}{R} \right)^{\alpha \rho_2} \rho^m \int_{B(y, R)} |Dv|^{\rho_2} dx + R^\sigma \int_{B(y, 2R)} (1 + |Dv|^2)^{\rho_2/2} dx \right]. \end{aligned}$$

Remark 3.6. In [5], the authors have written the above estimates with exponent $p_2 = \sup_\Omega p(x)$. However, in the proof of [5, Proposition 6], they use the estimate

$$(3.27) \quad |p_2 - p(x)| \leq cR^\sigma, \text{ for all } x \in B(y, 2R).$$

Accordingly p_2 in the proposition should be $\sup_{B(y, 2R)} p(x)$. On the other hand, it is clear that we can take the exponent $\rho_2 = \sup_{B(y, kR)} p(x)$ ($k > 2$) when $B(y, kR) \Subset \Omega$, since (3.27) holds for such a choice of ρ_2 .

4. PROOF OF THEOREM 2.4

Proof of Theorem 2.4. Let δ_0 be a positive constant given by applying Proposition 3.1 to u , in order to have

$$(4.1) \quad |Du|^{(1+\delta_0)p(x)} \in L_{\text{loc}}^1(\Omega).$$

Without loss of generality we assume that $\delta_0 < 1$. For this δ_0 let $\delta_1 \in (0, \delta_0)$ be the constant given in Corollary 3.3, and take a positive constant $\delta < \delta_1$ sufficiently small so that

$$(4.2) \quad \sigma - m\delta > 0,$$

where σ is the positive constant given in the assumption (H 5) as the Hölder exponent of $p(x)$.

Fix a point $x_0 \in \Omega$ arbitrarily, and take $R_1 > 0$ sufficiently small so that

$$p_1 := \inf_{B(x_0, R_1) \cap \Omega} p(x), \quad p_2 := \sup_{B(x_0, R_1) \cap \Omega} p(x)$$

satisfy (3.7) as well as

$$(4.3) \quad \omega_1(2R_1) := p_2 - p_1 < \delta, \quad (1 + \frac{\delta}{2})p_2 \leq (1 + \delta)p_1.$$

Then, from (4.1) we get

$$(4.4) \quad |Du|^{(1+\frac{\delta}{2})p_2} \in L^1(B(x_0, R_1)).$$

Take a point $x_1 \in B(x_0, R_1)$ arbitrarily. In the following, let us write $B_r := B(x_1, r)$ and

$$(4.5) \quad \rho_1(r) := \inf_{B_r} p(x), \quad \rho_2(r) := \sup_{B_r} p(x).$$

Choosing $R > 0$ so that $B_{2R} \subset B(x_0, R_1)$, let us define two types of “frozen functionals”

$$(4.6) \quad \mathcal{F}_1(v) := \int_{B_R} (g_R^{\alpha\beta} h_{ij}(u_R) D_\alpha v^i D_\beta v^j)^{p(x)/2} dx,$$

$$(4.7) \quad \mathcal{F}_2(v) := \int_{B_R} (g_R^{\alpha\beta} h_{ij}(u_R) D_\alpha v^i D_\beta v^j)^{\rho_2(2R)/2} dx,$$

where $g_R^{\alpha\beta} = g_{x_1, R}^{\alpha\beta}$ and $u_R = u_{x_1, R}$; see Definition 2.1. Let v be a minimizer of $\mathcal{F}_1$ in the class $u + W_0^{1, p(x)}(B_R)$. Then, keeping in mind our choice of δ and using Corollary 3.3 with $D = B_R$, we have for any $\varepsilon \in (0, \delta]$,

$$(4.8) \quad \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)p(x)/2} dx \leq C \int_{B_R} (1 + |Du|^2)^{(1+\varepsilon)p(x)/2} dx.$$

On the other hand, for every $\beta \in (0, 1)$, from Proposition 3.5, recalling Remark 3.6, we can see that the following estimate holds for any $0 < s < R/2$:

$$(4.9) \quad \int_{B_s} |Dv|^{\rho_2(2R)} dx \leq c \left(\frac{s}{R}\right)^{m-\beta} \left[\int_{B_R} |Dv|^{\rho_2(2R)} dx + R^{m-\beta} \right].$$

Using (4.8) with $\varepsilon = \omega_1(2R) (< \omega_1(2R_1) \leq \delta)$ and Proposition 3.1, we can estimate the right-hand side of (4.9) as

$$\begin{aligned} & \int_{B_R} |Dv|^{\rho_2(2R)} dx \\ & \leq c \int_{B_R} (1 + |Dv|^2)^{(1+\omega_1(2R))p(x)/2} dx \\ & \leq c \int_{B_R} (1 + |Du|^2)^{(1+\omega_1(2R))p(x)/2} dx \\ & \leq cR^{-m\omega_1(2R)} \left(\int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx \right)^{1+\omega_1(2R)} \\ & \leq cR^{-m\omega_1(2R)} \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx \\ & \leq cR^{-m\omega_1(2R)} \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx, \end{aligned}$$

where we point out that for the fourth inequality, we used the boundedness of $\int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx$. Since we see that $R^{-m\omega_1(2R)}$ is bounded by virtue of (3.4), from (4.9) and the above estimate, we obtain

$$(4.10) \quad \int_{B_s} |Dv|^{\rho_2(2R)} dx \leq c \left(\frac{s}{R}\right)^{m-\beta} \left[\int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx + R^{m-\beta} \right].$$

Now, put

$$\begin{aligned} (4.11) \quad e_0 &:= g_R^{\alpha\beta} h_{ij}(u_R) D_\alpha v^i D_\beta v^j, \\ f_1(\xi) &:= (g_R^{\alpha\beta} h_{ij}(u_R) \xi_\alpha^i \xi_\beta^j)^{p(x)/2}, \\ f_2(\xi) &:= (g_R^{\alpha\beta} h_{ij}(u_R) \xi_\alpha^i \xi_\beta^j)^{\rho(2R)/2}. \end{aligned}$$

Then by [13, (5.8), (5.10)], we see that

$$\mathcal{F}_2(u) - \mathcal{F}_2(v) \geq \int_{B_R} \frac{\partial f_2}{\partial \xi_\alpha^i}(Dv)(D_\alpha u^i - D_\alpha v^i) dx + c \int_{B_R} |Du - Dv|^{\rho_2(2R)} dx.$$

So, we have

$$(4.12) \quad \int_{B_R} |Du - Dv|^{\rho_2(2R)} dx \leq c(\mathcal{F}_2(u) - \mathcal{F}_2(v)) + \int_{B_R} \frac{\partial f_2}{\partial \xi_\alpha^i}(Dv)(D_\alpha u^i - D_\alpha v^i) dx.$$

Since v is a weak solution of the Euler-Lagrange equation of $\mathcal{F}_1$, using e_0 defined by (4.11), we estimate the second term of the right-hand side of the above inequality as follows:

$$(4.13) \quad \begin{aligned} & \left| \int_{B_R} \frac{\partial f_2}{\partial \xi_\alpha^i}(Dv)(D_\alpha u^i - D_\alpha v^i) dx \right| \\ &= \left| \int_{B_R} \left\{ \frac{\partial f_2}{\partial \xi_\alpha^i}(Dv) - \frac{\partial f_1}{\partial \xi_\alpha^i}(Dv) \right\} (D_\alpha u^i - D_\alpha v^i) dx \right| \\ &\leq \left| \int_{B_R} (\rho_2 e_0^{(\rho_2-2)/2} - p(x) e_0^{(p(x)-2)/2}) \right. \\ &\quad \cdot g^{\alpha\beta}(x_0) h_{ij}(u_R) D_\alpha v^i (D_\beta u^j - D_\beta v^j) dx \left. \right| \\ &\leq c \int_{B_R} \left\{ (\rho_2 - p(x)) e_0^{(\rho_2-2)/2} + p(x) \left| e_0^{(\rho_2-2)/2} - e_0^{(p(x)-2)/2} \right| \right\} \\ &\quad \cdot |Dv| |Du - Dv| dx \\ &\leq c \int_{B_R} \left\{ (\rho_2 - p(x)) e_0^{(\rho_2-1)/2} + p(x) \left| e_0^{(\rho_2-1)/2} - e_0^{(p(x)-1)/2} \right| \right\} \\ &\quad \cdot |Du - Dv| dx \\ &=: \int_{B_R} \{ (i) + (ii) \} |Du - Dv| dx. \end{aligned}$$

By (2.2), we have

$$(4.14) \quad (i) \leq cR^\sigma e_0^{(\rho_2-1)/2}.$$

For (ii), we mention that as (7) in [5] for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$(4.15) \quad |t^r - t^s| \leq C(\varepsilon)(s - r)(1 + t^{(1+\varepsilon)s}), \text{ for all } t \geq 0 \text{ and } s \geq r \geq 1.$$

Using the above inequality with $r = (\rho_2(2R) - 1)/2$ and $s = (p(x) - 1)/2$, we see that

$$(4.16) \quad (ii) \leq \gamma_2 C(\varepsilon)(\rho_2 - p(x))(1 + e_0^{(1+\varepsilon)(\rho_2-1)/2}) \leq cR^\sigma(1 + e_0)^{(1+\varepsilon)(\rho_2-1)/2}.$$

Here we used (2.2) also. From (4.14) and (4.16), we get for some constant c depending on ε ,

$$(4.17) \quad (i) + (ii) \leq cR^\sigma(1 + e_0)^{(1+\varepsilon)(\rho_2-1)/2}.$$

Thus, using Young's inequality, we can see that for $0 \leq R \leq 1$,

$$\begin{aligned}
 & \left| \int_{B_R} \frac{\partial f_2}{\partial \xi_\alpha^i}(Dv)(D_\alpha u^i - D_\alpha v^i) dx \right| \\
 & \leq cR^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)(\rho_2-1)/2} |Du - Dv| dx \\
 (4.18) \quad & \leq \frac{1}{2} \int_{B_R} |Du - Dv|^{\rho_2} dx + cR^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)\rho_2/2} dx.
 \end{aligned}$$

From (4.12) and (4.18), we obtain

$$\begin{aligned}
 & \int_{B_R} |Du - Dv|^{\rho_2(2R)} dx \leq c(\mathcal{F}_2(u) - \mathcal{F}_2(v)) \\
 (4.19) \quad & + cR^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)\rho_2(2R)/2} dx.
 \end{aligned}$$

Now, we can estimate $\int_{B_R} |Du - Dv|^{\rho_2(2R)} dx$ as follows:

$$\begin{aligned}
 & \int_{B_R} |Du - Dv|^{\rho_2(2R)} dx \\
 & \leq c(\mathcal{F}_2(u) - \mathcal{F}_2(v)) + cR^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)\rho_2/2} dx \\
 & \leq c(\mathcal{F}_2(u) - \mathcal{F}_1(u) + \mathcal{F}_1(u) - \mathcal{F}(u) + \mathcal{F}(u) - \mathcal{F}(v) \\
 & \quad + \mathcal{F}(v) - \mathcal{F}_1(v) + \mathcal{F}_1(v) - \mathcal{F}_2(v)) + cR^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)\rho_2/2} dx \\
 & \leq c\{(\mathcal{F}_2(u) - \mathcal{F}_1(u)) + (\mathcal{F}_1(u) - \mathcal{F}(u)) + (\mathcal{F}(v) - \mathcal{F}_1(v)) \\
 & \quad + (\mathcal{F}_1(v) - \mathcal{F}_2(v)) + cR^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)\rho_2/2} dx\} \\
 (4.20) \quad & =: c(I + II + III + IV + V).
 \end{aligned}$$

Using (4.15) with $r = \rho_2(2R)$ and $s = p(x)$, we can estimate I and IV as

$$\begin{aligned}
 |I| & \leq cR^\sigma \int_{B_R} (1 + |Du|^2)^{(1+\varepsilon)\rho_2(2R)/2} dx, \\
 |IV| & \leq cR^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\varepsilon)\rho_2(2R)/2} dx.
 \end{aligned}$$

Let us take $\varepsilon < \delta/2$. Then by the assumption (4.3), we have

$$(4.21) \quad (1 + \varepsilon)\rho_2(2R) \leq \left(1 + \frac{\delta}{2}\right)\rho_2(2R) < (1 + \delta)\rho_1(2R) \leq (1 + \delta)p(x).$$

So, we can estimate I as

$$(4.22) \quad |I| \leq cR^\sigma \int_{B_R} (1 + |Du|^2)^{(1+\delta)p(x)/2} dx.$$

Using Corollary 3.3, we have

$$\begin{aligned}
 |IV|, |V| & \leq cR^\sigma \int_{B_R} (1 + |Dv|^2)^{(1+\delta)p(x)/2} dx \\
 (4.23) \quad & \leq cR^\sigma \int_{B_R} (1 + |Du|^2)^{(1+\delta)p(x)/2} dx.
 \end{aligned}$$

Using Proposition 3.1 we can see that

$$\begin{aligned}
 R^\sigma \int_{B_R} (1 + |Du|^2)^{(1+\delta)p(x)/2} dx \\
 \leq cR^{\sigma-m\delta} \left(\int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx \right)^{1+\delta} \\
 \leq c'R^{\sigma-m\delta} \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx,
 \end{aligned}
 \tag{4.24}$$

where we used the boundedness of $\int |Du|^{p(x)} dx$. Combining (4.22) and (4.23) with (4.24), we obtain

$$|I|, |IV|, |V| \leq cR^{\sigma-m\delta} \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx.$$

Let us estimate $|II + III|$. Writing $q = 1 + \delta$ and $q' = q/(q - 1) = (1 + \delta)/\delta$, using Hölder's inequality, and remembering condition (H 2), we have

$$\begin{aligned}
 |II + III| &\leq c \left[\left(\int_{B_R} \omega_0^{q'} (|u - u_R|^2) dx \right)^{1/q'} + \left(\int_{B_R} |g(x) - g_R|^{q'} dx \right)^{1/q'} \right] \\
 &\quad \cdot \left(\int_{B_R} |Du|^{qp(x)} dx \right)^{1/q} \\
 &+ c \left[\left(\int_{B_R} \omega_0^{q'} (|v - u_R|^2) dx \right)^{1/q'} + \left(\int_{B_R} |g(x) - g_R|^{q'} dx \right)^{1/q'} \right] \\
 &\quad \cdot \left(\int_{B_R} |Dv|^{qp(x)} dx \right)^{1/q} \\
 &=: c(II' + III').
 \end{aligned}
 \tag{4.26}$$

Here and in the sequel, we write

$$g(x) = (g^{\alpha\beta}(x)), \text{ and } |g(x) - g_R| = \left\{ \sum_{\alpha\beta} (g^{\alpha\beta}(x) - g_R^{\alpha\beta})^2 \right\}^{1/2}.$$

Since ω_0 and g are bounded, using Proposition 3.1, Jensen's inequality, Hölder's inequality and the Sobolev-Poincaré inequality, we can estimate II' as follows:

$$\begin{aligned}
 II' &\leq c \left[\omega_0^{1/q'} (cR^{2-m} \int_{B_R} |Du|^2 dx) + \left(\int_{B_R} |g(x) - g_R| dx \right)^{1/q'} \right] \\
 &\quad \cdot \int_{B_{2R}} |Du|^{p(x)} dx \\
 &\leq c \left[\omega_0^{1/q'} \left(\{ cR^{\rho_2(2R)-m} \int_{B_R} |Du|^{\rho_2(2R)} dx \}^{2/\rho_2(2R)} \right) \right. \\
 &\quad \left. + \left(\int_{B_R} |g(x) - g_R| dx \right)^{1/q'} \right] \int_{B_{2R}} |Du|^{p(x)} dx.
 \end{aligned}
 \tag{4.27}$$

Using Corollary 3.3 and proceeding as above, we estimate III' as

$$\begin{aligned}
 III' &\leq c \left[\left(\int_{B_R} \omega_0(|u - u_R|^2 + |u - v|^2) dx \right)^{1/q'} + \left(\int_{B_R} |g(x) - g_R| dx \right)^{1/q'} \right] \\
 &\quad \cdot \left(\int_{B_R} (1 + |Du|^2)^{qp(x)/2} dx \right)^{1/q} \\
 &\leq c \left[\omega_0^{1/q'} \left(\int_{B_R} |u - u_R|^2 dx + \int_{B_R} |u - v|^2 dx \right) \right. \\
 &\quad \left. + \left(\int_{B_R} |g(x) - g_R| dx \right)^{1/q'} \right] \cdot \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx \\
 &\leq c \left[\omega_0^{1/q'} (cR^{2-m} \int_{B_R} |Du|^2 dx + cR^{2-m} \int_{B_R} |Dv|^2 dx) \right. \\
 (4.28) \quad &\quad \left. + \left(\int_{B_R} |g(x) - g_R| dx \right)^{1/q'} \right] \cdot \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx.
 \end{aligned}$$

Again with Corollary 3.3, we can estimate the second term in ω_0 as follows:

$$\begin{aligned}
 R^{2-m} \int_{B_R} |Dv|^2 dx &\leq c(R^{\rho_2(2R)-m} \int_{B_R} |Dv|^{\rho_2(2R)} dx)^{2/\rho_2(2R)} \\
 &\leq c(R^{\rho_2(2R)-m} \int_{B_R} (1 + |Dv|^2)^{(1+\omega_1(2R))p(x)/2} dx)^{2/\rho_2(2R)} \\
 &\leq c(R^{\rho_2(2R)-m} \int_{B_R} (1 + |Du|^2)^{(1+\omega_1(2R))p(x)/2} dx)^{2/\rho_2(2R)} \\
 &\leq c(R^{\rho_2(2R)-m} R^{-\omega_1(2R)m} \left\{ \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx \right\}^{1+\omega_1(2R)})^{2/\rho_2(2R)}.
 \end{aligned}$$

Since $R^{-\omega_1(2R)}$ and $\int |Du|^{p(x)} dx$ are bounded, from the above estimate, we obtain

$$\begin{aligned}
 R^{2-m} \int_{B_R} |Dv|^2 dx &\leq c(R^{\rho_2(2R)-m} \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx)^{2/\rho_2(2R)} \\
 (4.29) \quad &\leq c(R^{\rho_2(2R)-m} \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx)^{2/\rho_2(2R)}
 \end{aligned}$$

and from (4.28) and (4.29) we obtain

$$\begin{aligned}
 III' &\leq c \left[\omega_0^{1/q'} (c \{ R^{\rho_2(2R)-m} \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx \})^{2/\rho_2(2R)} \right. \\
 (4.30) \quad &\quad \left. + \eta^{1/q'}(g, 2R) \right] \cdot \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx.
 \end{aligned}$$

Here, for the definition of $\eta(g, 2R)$, see Definition 2.3. Combining (4.20), (4.25), (4.26), (4.27) and (4.30), we arrive at

$$(4.31) \quad \begin{aligned} & \int_{B_R} |Du - Dv|^{\rho_2(2R)} dx \\ & \leq c [R^{\sigma-m\delta} + \omega_0^{1/q'} (c_1 \{R^{\rho_2(2R)-m} \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx\}^{2/\rho_2(2R)} \\ & \quad + \eta^{1/q'}(g, 2R))] \cdot \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx. \end{aligned}$$

Now, from (4.9) and (4.31), putting $r = 2R$ and changing the constants if necessary, we obtain for any $0 < s < r/4$ with $B_r \subseteq B(x_0, R_1)$,

$$(4.32) \quad \begin{aligned} & \int_{B_s} (1 + |Du|^2)^{\rho_2(r)/2} dx \\ & \leq c_0 \left[\left(\frac{s}{r} \right)^{m-\beta} + r^{\sigma-m\delta} + \omega_0^{1/q'} (c_1 \{r^{\rho_2(r)-m} \int_{B_r} (1 + |Du|^2)^{\rho_2(r)/2} dx\}^{2/\rho_2(r)} \right. \\ & \quad \left. + \eta^{1/q'}(g, r) \right] \cdot \int_{B_r} (1 + |Du|^2)^{\rho_2(r)/2} dx + c_2 s^{m-\beta}. \end{aligned}$$

In the rest of the proof we shall show that the above estimate implies a certain decay estimate on $(\int_{B_s} (1 + |Du|^2)^{\rho_2(s)/2} dx)^{1/\rho_2(s)}$.

For $\tau \in (0, 1)$, let $s = \tau r$ in (4.32). Then we have

$$(4.33) \quad \begin{aligned} & \int_{B_{\tau r}} (1 + |Du|^2)^{\rho_2(r)/2} dx \\ & \leq c_0 [\tau^{m-\beta} + r^{\sigma-m\delta} + \omega_0^{1/q'} + \eta^{1/q'}(g, r)] \cdot \int_{B_r} (1 + |Du|^2)^{\rho_2(r)/2} dx \\ & \quad + c_2 (\tau r)^{m-\beta}. \end{aligned}$$

Multiplying both sides by $(\tau r)^{\rho_2(r)-m}$, we get

$$(4.34) \quad \begin{aligned} & (\tau r)^{\rho_2(r)-m} \int_{B_{\tau r}} (1 + |Du|^2)^{\rho_2(r)/2} dx \\ & \leq c_0 [\tau^{\rho_2(r)-\beta} + \tau^{\rho_2(r)-m} r^{\sigma-m\delta} + \tau^{\rho_2(r)-m} \omega_0^{1/q'} + \tau^{\rho_2(r)-m} \eta^{1/q'}] \\ & \quad \cdot r^{\rho_2(r)-m} \int_{B_r} (1 + |Du|^2)^{\rho_2(r)/2} dx + c_2 (\tau r)^{\rho_2(r)-\beta} \\ & = c_0 \tau^{\rho_2(r)-\beta} [1 + \tau^{\beta-m} r^{\sigma-m\delta} + \tau^{\beta-m} \omega_0^{1/q'} + \tau^{\beta-m} \eta^{1/q'}] \\ & \quad \cdot r^{\rho_2(r)-m} \int_{B_r} (1 + |Du|^2)^{\rho_2(r)/2} dx + c_2 (\tau r)^{\rho_2(r)-\beta}. \end{aligned}$$

Let us put

$$\begin{aligned} \Phi(r, p) &:= \left(r^p \int_{B_r} (1 + |Du|^2)^{p/2} dx \right)^{1/p} \\ &= \omega_m^{1/p} \left(r^{p-m} \int_{B_r} (1 + |Du|^2)^{p/2} dx \right)^{1/p} \end{aligned}$$

and

$$\Psi(r) := \Phi(r, \rho_2(r)).$$

Here ω_m denotes the volume of the m -dimensional unit ball. From (4.34) we have

$$\begin{aligned} (4.35) \quad & \Phi^{\rho_2(r)}(\tau r, \rho_2(r)) \\ & \leq c_0 \tau^{\rho_2(r)-\beta} [1 + \tau^{\beta-m} \{r^{\sigma-m\delta} + \omega_0^{1/q'}(c_1 \omega_m^{1/\rho_2(r)} \Psi^2(r)) + \eta^{1/q'}(g, r)\}] \\ & \quad \cdot \Psi^{\rho_2(r)}(r) + c_2(\tau r)^{\rho_2(r)-\beta}. \end{aligned}$$

Mentioning that

$$\Phi(r, p) \leq \Phi(r, q) \text{ for } p \leq q,$$

we obtain from (4.35),

$$\begin{aligned} (4.36) \quad & \Psi(\tau r) = \Phi(\tau r, \rho_2(\tau r)) \leq \Phi(\tau r, \rho_2(r)) \\ & \leq c_3 \tau^{1-(\beta/\rho_2(r))} [1 + \tau^{(\beta-m)/\rho_2(r)} \{r^{(\sigma-m\delta)/\rho_2(r)} + \omega_0^{1/(q'\rho_2(r))} \\ & \quad + \eta^{1/(q'\rho_2(r))}(g, r)\}] \cdot \Psi(r) + c_4(\tau r)^{1-(\beta/\rho_2(r))}. \end{aligned}$$

Since $0 < \beta < 1$, $m > 2$, $\gamma_1 \leq \rho_2(r) \leq \gamma_2$, and $\tau < 1$, we have

$$(4.37) \quad \tau^{(\beta-m)/\rho_2(r)} \leq \tau^{(\beta-m)/\gamma_1}.$$

Without loss of generality we can assume that $0 < r < 1$, so we see that

$$(4.38) \quad r^{(\sigma-m\delta)/\rho_2(r)} \leq r^{(\sigma-m\delta)/\gamma_2}, \quad (\tau r)^{1-(\beta/\rho_2(r))} \leq (\tau r)^{1-(\beta/\gamma_1)}.$$

In the following, we consider the case that ω_0 and $\eta(g, r)$ are sufficiently small, so we can assume that $\omega_0, \eta(g, r) < 1$. So, we have

$$(4.39) \quad \omega_0^{1/q'\rho_2(r)} \leq \omega_0^{1/q'\gamma_2}, \quad \eta(g, r)^{1/q'\rho_2(r)} \leq \eta(g, r)^{1/q'\gamma_2}.$$

For the sake of simplicity, let us put

$$\gamma := 1 - \frac{\beta}{\gamma_1}, \quad \tilde{\omega}_0 := \omega_0^{1/(q'\gamma_2)}, \quad \tilde{\eta}(g, r) := \eta(g, r)^{1/(q'\gamma_2)}, \quad c_5 := c_1 \omega_m^{1/\gamma_1}.$$

Then, from (4.36), we get

$$\begin{aligned} \Psi(\tau r) & \leq c_3 \tau^\gamma [1 + \tau^{(\beta-m)/\gamma_1} \{r^{\sigma-m\delta}/\gamma_2 + \tilde{\omega}_0(c_5 \Psi(r)) + \tilde{\eta}(g, r)\}] \Psi(r) \\ & \quad + c_4(\tau r)^\gamma. \end{aligned}$$

For a technical reason we replace the exponent γ of the last term by $\alpha \in (0, \gamma)$, and obtain

$$(4.40) \quad \begin{aligned} \Psi(\tau r) & \leq c_3 \tau^\gamma [1 + \tau^{(\beta-m)/\gamma_1} \{r^{\sigma-m\delta}/\gamma_2 + \tilde{\omega}_0(c_7 \Psi(r)) + \tilde{\eta}(g, r)\}] \Psi(r) \\ & \quad + c_4(\tau r)^\alpha. \end{aligned}$$

Fix $\nu \in (\alpha, \gamma)$ and choose $\tau \in (0, 1)$ so that $c_3 \tau^\gamma < \tau^\nu/5$. Take $\varepsilon_0 > 0$ such that

$$(4.41) \quad \tilde{\omega}_0(c_5 \varepsilon_0) < \tau^{(m-\beta)/\gamma_1}.$$

Finally, let $r_0 > 0$ be a sufficiently small constant for which

$$(4.42) \quad \tau^{(\beta-m)/\gamma_1} r_0^{(\sigma-m\delta)/\gamma_2} \leq 1, \quad \tau^{(\beta-m)/\gamma_1} \tilde{\eta}(g, r_0) < 1 \text{ and } c_4 r_0^\alpha \leq \frac{1}{5} \varepsilon_0$$

hold.

Now, assume that $\Psi(r) < \varepsilon_0$ holds for some $r \in (0, r_0)$. Then, mentioning that $c_6\tau^\alpha < c_6$, we get from (4.40) that

$$\begin{aligned} \Psi(\tau r) &\leq \frac{\tau^\nu}{5}[1 + 1 + 1 + 1]\Psi(r) + c_4r^\alpha \\ &= \frac{4}{5}\tau^\nu\Psi(r) + c_4r^\alpha \\ (4.43) \quad &\leq \frac{4}{5}\tau^\nu\varepsilon_0 + \frac{1}{5}\varepsilon_0 < \varepsilon_0. \end{aligned}$$

By virtue of (4.43), we can repeat the above estimate replacing r by $\tau r, \tau^2 r, \tau^3 r, \dots$ and get

$$\begin{aligned} \Psi(\tau^{k+1}r) &\leq (\tau^{k+1})^\nu\Psi(r) + c_4r^\alpha\tau^{k\alpha}\sum_{j=0}^k\tau^{j(\nu-\alpha)} \\ (4.44) \quad &\leq (\tau^{k+1})^\nu\Psi(r) + c_6(\tau^k r)^\alpha. \end{aligned}$$

For any $s \in (0, r)$ there exists k with $\tau^{k+1}r < s \leq \tau^k r$, and we have

$$\begin{aligned} \Psi(s) &\leq \tau^{1-(1/\rho_2(s))}\Phi(\tau^k r, \rho_2(s)) \leq \tau^{1-(1/\gamma_1)}\Psi(\tau^k r) \\ &\leq \tau^{1-(1/\gamma_1)}\left[\tau^{-\nu}\left(\frac{s}{r}\right)^\nu\Psi(r) + c_6\tau^{-\alpha}s^\alpha\right] \\ (4.45) \quad &\leq c_7s^\alpha. \end{aligned}$$

From the above estimate for the integral of $|Du|^{\gamma_1}$ with $\gamma_1 = \inf_\Omega p(x)$ we obtain

$$\begin{aligned} s^{-m+\gamma_1-\alpha\gamma_1}\int_{B(x_1,s)}|Du|^{\gamma_1}dx &\leq \omega_m(s^{-\alpha}\Phi(s, \gamma_1))^{\gamma_1} \\ (4.46) \quad &\leq \omega_m(s^{-\alpha}\Psi(s))^{\gamma_1} \leq \omega_m c_7^{\gamma_1}. \end{aligned}$$

Now, let

$$\begin{aligned} D_0 &:= \{y \in B(x_0, R_1) ; \left(r^{p_2-m}\int_{B(y,r)}|Du|^{p_2}dx\right)^{1/p_2} \leq \varepsilon_0 \\ &\text{for some } r \text{ with } B(y, r) \in B(x_0, R_1)\}. \end{aligned}$$

Then, by virtue of the continuity of the integral, we see that D_0 is an open set. Moreover, for every $x_1 \in D_0$ the estimate (4.46) holds, and hence by virtue of Morrey's Dirichlet growth theorem, we have that $u \in C^{0,\alpha}(D_0)$. On the other hand, since $|Du|^{p_2} \in L^{1+(\delta/2)}(B(x_0, R_1))$, by a well-known property of the Hausdorff measures we have that

$$\dim^{\mathcal{H}}(B(x_0, R_1) \setminus D_0) < m - p_2,$$

where $\dim^{\mathcal{H}}$ denotes the Hausdorff dimension. Now, covering Ω by balls with the property which we considered on $B(x_0, R_1)$, we conclude that $u \in C^{0,\alpha}(\Omega_0)$, for some open subset $\Omega_0 \subset \Omega$. Moreover, since $\gamma_1 \leq p_2$ on every $B(x_0, R_1)$, we have $\dim^{\mathcal{H}}(\Omega \setminus \Omega_0) < m - \gamma_1$.

In order to obtain the Hölder continuity of Du on Ω_0 , let us assume that $g_{\alpha\beta}$ and h^{ij} are Hölder continuous. Namely, in the following we suppose that for some $\theta, \theta' \in (0, 1)$,

$$(4.47) \quad \omega_0(t) \leq ct^\theta, \quad \eta(g, t) \leq ct^{\theta'}.$$

Let $B_{2R} = B(y, 2R) \subseteq \Omega_0$ be a ball with sufficiently small $R > 0$, and let v be a minimizer of $\mathcal{F}_1$ as before. For $\rho < R/4$ we estimate $\int_{B_\rho} |Du - Dv|^{\rho_2(2R)} dx$ as follows:

$$(4.48) \quad \begin{aligned} & \int_{B_\rho} |Du - (Du)_\rho|^{\rho_2(2R)} dx \\ & \leq c \int_{B_\rho} |Dv - (Dv)_\rho|^{\rho_2(2R)} dx + c \int_{B_\rho} |Du - Dv|^{\rho_2(2R)} dx. \end{aligned}$$

By (3.26) of Proposition 3.5, Proposition 3.1 and Corollary 3.3, we can estimate the first term as follows for some $\beta' \in (0, 1)$:

$$(4.49) \quad \begin{aligned} & \int_{B_\rho} |Dv - (Dv)_\rho|^{\rho_2(2R)} dx \\ & \leq c \left(\frac{\rho}{R} \right)^{\beta' \rho_2(2R) + m} \int_{B_{R/2}} |Dv|^{\rho_2(2R)} dx + c R^\sigma \int_{B_R} (1 + |Dv|^2)^{\rho_2(2R)/2} dx \\ & \leq c \left\{ \left(\frac{\rho}{R} \right)^{\beta' \rho_2(2R) + m} + R^\sigma \right\} \int_{B_R} (1 + |Dv|^2)^{(1+\omega_1(2R))p(x)/2} dx \\ & \leq c \left\{ \left(\frac{\rho}{R} \right)^{\beta' \rho_2(2R) + m} + R^\sigma \right\} R^{-m\omega_1(2R)} \int_{B_R} (1 + |Dv|^2)^{p(x)/2} dx \\ & \leq c \left\{ \left(\frac{\rho}{R} \right)^{\beta' \rho_2(2R) + m} + R^\sigma \right\} \int_{B_{2R}} (1 + |Du|^2)^{p(x)/2} dx. \end{aligned}$$

Here, we used the boundedness of $R^{-m\omega_1(2R)}$ and $\int_{B_R} |Dv|^{p(x)} dx$ (that is lower than $\int_{B_R} |Du|^{p(x)} dx$) also.

About the second term of the right-hand side of (4.48), from (4.31), we see that

$$(4.50) \quad \begin{aligned} & \int_{B_R} |Du - Dv|^{\rho_2(2R)/2} dx \\ & \leq c \left[R^{\sigma-m\delta} + \omega_0^{1/q'}(c_5 \Psi^2(2R)) \right. \\ & \quad \left. + \eta^{1/q'}(g, 2R) \right] \cdot \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx. \end{aligned}$$

Since $B_{2R} \subset \Omega_0$, we have

$$\Psi(2R) \leq cR^\alpha.$$

So, by the assumption that (4.47) holds, we obtain

$$(4.51) \quad \omega_0(c_5 \Psi(2R)) \leq cR^{2\alpha\theta}.$$

Here, we mention that we can take $\alpha \in (0, 1)$ arbitrarily near to 1. We obtain

$$\begin{aligned} & \int_{B_R} |Du - Dv|^{\rho_2(2R)/2} dx \\ & \leq c \left[R^{\sigma-m\delta} + R^{2\alpha\theta/q'} + R^{\theta'/q'} \right] \cdot \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx. \end{aligned}$$

Putting

$$\zeta := \min\{\sigma - m\delta, 2\alpha\theta/q', \theta'/q'\},$$

we have

$$(4.52) \quad \int_{B_R} |Du - Dv|^{\rho_2(2R)/2} dx \leq cR^\zeta \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx.$$

Combining (4.48), (4.49) and (4.52), and putting $\mu := \min\{\zeta, \sigma\}$, we see that

$$\begin{aligned} & \int_{B_\rho} |Du - (Du)_\rho|^{\rho_2(2R)} dx \\ & \leq c \left\{ \left(\frac{\rho}{R} \right)^{\beta' \rho_2(2R) + m} + R^\zeta + R^\sigma \right\} \int_{B_{2R}} (1 + |Du|^2)^{\rho_2(2R)/2} dx \\ & \leq c \left\{ \left(\frac{\rho}{R} \right)^{\beta' \rho_2(2R) + m} + R^\mu R^{m - \rho_2(2R)} \right\} (\Psi(2R))^{\rho_2(2R)} \\ & \leq c \left\{ \left(\frac{\rho}{R} \right)^{\beta' \rho_2(2R) + m} + R^\mu \right\} R^{m - \rho_2(2R) + \alpha \rho_2(2R)} \\ (4.53) \quad & \leq c \left(\frac{\rho}{R} \right)^{\beta' \gamma_1 + m} R^{m - (1 - \alpha)\gamma_2} + cR^{m - (1 - \alpha)\gamma_2 + \mu}. \end{aligned}$$

Let us take $\rho = R^{1+a}/4$ with $a > 0$ to be specified later. Then from (4.53) we get

$$(4.54) \quad \begin{aligned} & \int_{B_\rho} |Du - (Du)_\rho|^{\rho_2} dx \\ & \leq c \left(R^{m - (1 - \alpha)\gamma_2 + (m + \beta' \gamma_1)a} + R^{m - (1 - \alpha)\gamma_2 + \mu} \right). \end{aligned}$$

Take $a = \mu/(m + \beta' \gamma_1)$. Then the right-hand side will be

$$cR^{m - (1 - \alpha)\gamma_2 + \mu} = c\rho^{m + (a\beta' \gamma_1 - (1 - \alpha)\gamma_2)/(1 + a)}.$$

Now, let us choose $\alpha \in (0, 1)$ sufficiently near to 1, then

$$\alpha' := \frac{a\beta' \gamma_1 - (1 - \alpha)\gamma_2}{1 + a} > 0,$$

and we obtain

$$\begin{aligned} \left(\int_{B_\rho} |Du - (Du)_\rho|^{\gamma_1} dx \right)^{1/\gamma_1} & \leq \left(\int_{B_\rho} |Du - (Du)_\rho|^{\rho_2(2R)} dx \right)^{1/\rho_2(2R)} \\ & \leq c\rho^{\alpha'/\rho_2(2R)} \leq c\rho^{\alpha'/\gamma_2}. \end{aligned}$$

From the above estimate, by Campanato's theorem (see [17, Theorem 5.4]), we conclude that

$$Du \in C^{1, \alpha' \gamma_1 / \gamma_2}(\Omega_0).$$

□

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REFERENCES

1. Emilio Acerbi and Giuseppe Mingione, *Regularity results for a class of quasiconvex functionals with nonstandard growth*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **30** (2001), no. 2, 311–339. MR1895714 (2003b:49037)
2. Lucio Boccardo, Paolo Marcellini, and Carlo Sbordone, *L^∞ -regularity for variational problems with sharp nonstandard growth conditions*, Boll. Un. Mat. Ital. A (7) **4** (1990), no. 2, 219–225. MR1066774 (91g:35040)
3. Luciano Carbone and Carlo Sbordone, *Some properties of Γ -limits of integral functionals*, Ann. Mat. Pura Appl. (4) **122** (1979), 1–60. MR565062 (81f:49034)
4. Valeria Chiadò Piat and Alessandra Coscia, *Hölder continuity of minimizers of functionals with variable growth exponent*, Manuscripta Math. **93** (1997), no. 3, 283–299. MR1457729 (98e:49091)
5. Alessandra Coscia and Giuseppe Mingione, *Hölder continuity of the gradient of $p(x)$ -harmonic mappings*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 4, 363–368. MR1675954 (2000a:49071)
6. Alessandra Coscia and Domenico Mucci, *Integral representation and Γ -convergence of variational integrals with $P(X)$ -growth*, ESAIM Control Optim. Calc. Var. **7** (2002), 495–519 (electronic). MR1925039 (2003f:49023)
7. Giovanni Cupini, Paolo Marcellini, and Elvira Mascolo, *Regularity under sharp anisotropic general growth conditions*, Discrete Contin. Dyn. Syst. Ser. B **11** (2009), no. 1, 66–86. MR2461809 (2009h:49063)
8. Gianni Dal Maso and Luciano Modica, *A general theory of variational functionals*, Topics in functional analysis, 1980–81, Quaderni, Scuola Norm. Sup. Pisa, Pisa, 1981, pp. 149–221. MR671757 (84b:49017)
9. Michela Eleuteri, *Hölder continuity results for a class of functionals with non-standard growth*, Bollettino Union Mat. Ital. (2004), 129–157. MR2044264 (2005a:49073)
10. Xianling Fan and Dun Zhao, *A class of De Giorgi type and Hölder continuity*, Nonlinear Anal. **36** (1999), no. 3, Ser. A: Theory Methods, 295–318. MR1688232 (2000a:49072)
11. ———, *Regularity of quasi-minimizers of integral functionals with discontinuous $p(x)$ -growth conditions*, Nonlinear Anal. **65** (2006), no. 8, 1521–1531. MR2246352 (2007c:49047)
12. Nicola Fusco, *On the convergence of integral functionals depending on vector-valued functions*, Ricerche Mat. **32** (1983), no. 2, 321–339. MR766684 (86h:49039)
13. Nicola Fusco and John Hutchinson, *Partial regularity for minimisers of certain functionals having nonquadratic growth*, Ann. Mat. Pura Appl. (4) **155** (1989), 1–24. MR1042826 (91g:49019)
14. Mariano Giaquinta, *Growth conditions and regularity, a counterexample*, Manuscripta Math. **59** (1987), no. 2, 245–248. MR905200 (88h:49034)
15. Mariano Giaquinta and Enrico Giusti, *On the regularity of the minima of variational integrals*, Acta Math. **148** (1982), 31–46. MR666107 (84b:58034)
16. ———, *Differentiability of minima of non-differentiable functionals*, Invent. Math. **72** (1983), 285–298. MR700772 (84f:58037)
17. Mariano Giaquinta and Luca Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 2, Edizioni della Normale, Pisa, 2005. MR2192611 (2007b:35003)
18. Mariano Giaquinta and Giuseppe Modica, *Regularity results for some classes of higher order nonlinear elliptic systems*, J. für reine und angew. Math. **311/312** (1979), 145–169. MR549962 (81a:35035)
19. Mariano Giaquinta and Giuseppe Modica, *Remarks on the regularity of the minimizers of certain degenerate functionals*, Manuscripta Math. **57** (1986), no. 1, 55–99. MR866406 (87m:49036)
20. Mariano Giaquinta, Giuseppe Modica, and Jiří Souček, *Cartesian currents in the calculus of variations. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 38, Springer-Verlag, Berlin, 1998, Variational integrals. MR1645082 (2000b:49001b)

21. Fritz John and Louis Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426. MR0131498 (24:A1348)
22. Jürgen Jost and Michael Meier, *Boundary regularity for minima of certain quadratic functionals*, Math. Ann. **262** (1983), no. 4, 549–561. MR696525 (84i:35051)
23. Paolo Marcellini, *Un esempio di soluzione discontinua d'un problème variationnel dans le cas scalaire, report 11*, Istituto Matematico U. Dini, Università di Firenze (1987), no. 11, 267–284.
24. ———, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), no. 3, 267–284. MR969900 (90a:49017)
25. ———, *Regularity and existence of solutions of elliptic equations with p, q -growth conditions*, J. Differential Equations **90** (1991), no. 1, 1–30. MR1094446 (92e:35061)
26. ———, *Regularity for elliptic equations with general growth conditions*, J. Differential Equations **105** (1993), no. 2, 296–333. MR1240398 (95h:35069)
27. ———, *Everywhere regularity for a class of elliptic systems without growth conditions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **23** (1996), no. 1, 1–25. MR1401415 (97h:35048)
28. ———, *Regularity for some scalar variational problems under general growth conditions*, J. Optim. Theory Appl. **90** (1996), no. 1, 161–181. MR1397651 (97g:49002)
29. Charles B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Die Grundlehren der mathematischen Wissenschaften, Band 130, Springer-Verlag New York, Inc., New York, 1966. MR0202511 (34:2380)
30. Maria Alessandra Ragusa and Atsushi Tachikawa, *Partial regularity of the minimizers of quadratic functionals with VMO coefficients*, J. London Math. Soc. (2) **72** (2005), no. 3, 609–620. MR2190327 (2006h:49063)
31. ———, *Regularity of minimizers of some variational integrals with discontinuity*, Z. Anal. Anwend. **27** (2008), no. 4, 469–482. MR2448746 (2009k:49078)
32. Donald Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405. MR0377518 (51:13690)
33. Karen Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), 219–240. MR0474389 (57:14031)
34. Vasilii V. Zhikov, *On some variational problems*, Russian J. Math. Phys. **5** (1997), 105–116. MR1486765 (98k:49004)

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI CATANIA, VIALE ANDREA DORIA, 6-95128 CATANIA, ITALY

E-mail address: maragusa@dmf.unict.it

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, TOKYO UNIVERSITY OF SCIENCE, NODA, CHIBA, 278-8510, JAPAN

E-mail address: tachikawa_atsushi@ma.noda.tus.ac.jp

KASA AI 103, 1-20 MUKAIHARA-CHO, KASHIWA, CHIBA 277-0851, JAPAN

E-mail address: h.takaba119@hotmail.co.jp