ON THE COMPOSITION SERIES OF THE STANDARD WHITTAKER $(\mathfrak{g}, K)$ -MODULES

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ABSTRACT. For a real reductive linear Lie group G , the space of Whittaker functions is the representation space induced from a non-degenerate unitary character of the Iwasawa nilpotent subgroup. Defined are the standard Whittaker $(\mathfrak{g}, K)$ -modules, which are K -admissible submodules of the space of Whittaker functions. We first determine the structures of them when the infinitesimal characters characterizing them are generic. As an example of the integral case, we determine the composition series of the standard Whittaker $(\mathfrak{g}, K)$ -module when G is the group $U(n, 1)$ and the infinitesimal character is regular integral.

1. INTRODUCTION

One of the most basic problems in representation theory is to study the composition series of a standard representation. In the category of highest weight modules, Verma modules play the role of standard representations, and in the category of Harish-Chandra modules, principal series representations do. The composition series problem is called the Kazhdan-Lusztig conjecture.

In this paper, the author proposes a Whittaker version of standard $(\mathfrak{g}, K)$ -modules and studies their composition series problem.

Let G be a real reductive linear Lie group in the sense of [14] and $G = KAN$ be an Iwasawa decomposition of it. Let $\eta : N \rightarrow \mathbb{C}^\times$ be a unitary character of N and denote the differential representation $\mathfrak{n}_0 \rightarrow \sqrt{-1}\mathbb{R}$ of it by the same letter η . We assume η is non-degenerate; i.e. it is non-trivial on every root space corresponding to a simple root of $\Delta^+(\mathfrak{g}_0, \mathfrak{a}_0)$. Define

$$(1.1) \quad C^\infty(G/N; \eta) := \{f : G \xrightarrow{C^\infty} \mathbb{C} \mid f(gn) = \eta(n)^{-1}f(g), \ g \in G, n \in N\}$$

and call it the space of Whittaker functions on G . This is a representation space of G by the left translation, which is denoted by L . Let $C^\infty(G/N; \eta)_K$ be the subspace of $C^\infty(G/N; \eta)$ consisting of K -finite vectors. As for the subrepresentations of this space, there are many deep and interesting results, called the theory of Whittaker models. On the other hand, it is not too much to say that the structure of the whole space is not known at all. Though our ultimate goal is to determine the structure of $C^\infty(G/N; \eta)$, this space is too large to analyze. So we need to cut off a submodule of suitable size from it.

As usual, let M be the centralizer of A in K , and let

$$M^\eta := \{m \in M \mid \eta(m^{-1}nm) = \eta(n), n \in N\}$$

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be the stabilizer of η in M . This subgroup acts naturally on $C^\infty(G/N; \eta)_K$ by right translation. Consider the subspace of $C^\infty(G/N; \eta)_K$ consisting of those functions f which satisfy the following conditions:

- (1) f is a joint eigenfunction of $Z(\mathfrak{g})$ (the center of the universal enveloping algebra $U(\mathfrak{g})$) with eigenvalue χ_Λ : $L(z)f = \chi_\Lambda(z)f$, $z \in Z(\mathfrak{g})$.
- (2) For an irreducible representation $(\sigma, V_\sigma^{M^\eta})$ of M^η , f is in the σ^* -isotypic subspace (σ^* is the dual of σ) with respect to the right action of M^η .
- (3) f grows moderately at infinity ([15]).

Denote by $I_{\eta, \Lambda}^\circ$ the subspace consisting of $f \in C^\infty(G/N; \eta)_K$ satisfying (1). Then

$$I_{\eta, \Lambda}^\circ \simeq \bigoplus_{\sigma \in \widehat{M^\eta}} \text{Hom}_{M^\eta}(V_{\sigma^*}^{M^\eta}, I_{\eta, \Lambda}^\circ) \otimes V_{\sigma^*}^{M^\eta},$$

and the space $\text{Hom}_{M^\eta}(V_{\sigma^*}^{M^\eta}, I_{\eta, \Lambda}^\circ)$ is isomorphic to

$$\begin{aligned} I_{\eta, \Lambda, \sigma}^\circ &= C^\infty(G/M^\eta N; \sigma \otimes \eta)_{K, \Lambda} \\ &:= \{f : G \xrightarrow{C^\infty} V_\sigma^{M^\eta} \mid f(gmn) = \eta(n)^{-1} \sigma(m)^{-1} f(g), \quad g \in G, m \in M^\eta, n \in N; \\ &\quad L(z)f = \chi_\Lambda(z)f, \quad z \in Z(\mathfrak{g}); \text{ left } K\text{-finite}\}. \end{aligned}$$

Therefore, the space of functions f satisfying the above conditions (1)–(3) is isomorphic to

$$I_{\eta, \Lambda, \sigma} := \{f \in I_{\eta, \Lambda, \sigma}^\circ \mid f \text{ grows moderately at infinity}\}.$$

We call these the *standard Whittaker* $(\mathfrak{g}, K)$ -modules. Note that these are *not* the “standard Whittaker module” defined in [6]. It is easy to show that these are K -admissible and then have finite length (Corollary 2.4).

Though the composition series problem of standard Whittaker $(\mathfrak{g}, K)$ -modules is an interesting problem by itself, we may hope to apply the result of it to the analysis of principal series representations. $I_{\eta, \Lambda, \sigma}$ is induced from $M^\eta N$, and the behavior of $f \in I_{\eta, \Lambda, \sigma}$ on A is controlled by the infinitesimal character and the asymptotic behavior, so we may think this module is near to the principal series representation. Therefore, it is significant to compare the structure of this module and that of a principal series representation. According to the theory of Whittaker models, an irreducible Harish-Chandra module π can be a submodule of $I_{\eta, \Lambda, \sigma}$ only if the Gelfand-Kirillov dimension of π is equal to $\dim N$ ([10]). On the other hand, any irreducible Harish-Chandra module can be a submodule of some principal series representation. This difference comes from the difference of the structures of $I_{\eta, \Lambda, \sigma}$ and principal series. So if you understand the common features and the different points of these modules, then new insights of Whittaker models and principal series are expected to be obtained.

For example, assume $G = SL(2, \mathbb{R})$ and the infinitesimal character Λ is regular dominant integral. In this case $M = M^\eta \simeq \{\pm 1\}$, so we identify an irreducible representation of M and that of M^η , which is denoted by σ . There are two irreducible representations of $\{\pm 1\}$, one is trivial, denoted by 1, and the other is the signature representation, denoted by -1 . Let $\rho_A = \frac{1}{2} \text{tr}(\text{ad}_{\mathfrak{a}}|_{\text{Lie}(N)}) \in \text{Lie}(A)^*$. This notation ρ_A will be used for other groups. Then the principal series representation $\text{Ind}_{MAN}^G(\sigma \otimes e^{\Lambda + \rho_A})$ is reducible if and only if $\Lambda \equiv \sigma + 1 \pmod{2}$. There are four equivalence classes of irreducible Harish-Chandra modules with the infinitesimal character Λ . The irreducible principal series is denoted by $\overline{\pi}_{01}$, the irreducible

finite-dimensional representation by $\bar{\pi}_{01}^+$ and two discrete series are denoted by π_0, π_1 . It is well known that the composition series of reducible principal series are

$$\mathrm{Ind}_{MAN}^G(\sigma \otimes e^{\Lambda+\rho_A}) \simeq \begin{array}{c} \bar{\pi}_{01}^+ \\ \swarrow \searrow \\ \pi_0 \quad \pi_1 \end{array} \quad \mathrm{Ind}_{MAN}^G(\sigma \otimes e^{-\Lambda+\rho_A}) \simeq \begin{array}{c} \pi_0 \quad \pi_1 \\ \searrow \swarrow \\ \bar{\pi}_{01}^+ \end{array}.$$

For the meaning of these diagrams, see Definition 3.3. On the other hand, if $\sigma \in \widehat{M^\eta} = \widehat{M}$ corresponds to the reducible (resp. irreducible) principal series, then the composition series of the standard Whittaker $(\mathfrak{g}, K)$ -modules are

$$I_{\eta, \Lambda, \sigma} \simeq \begin{array}{c} \pi_0 \\ \downarrow \\ \bar{\pi}_{01}^+ \\ \downarrow \\ \pi_1 \end{array} q, \quad I_{-\eta, \Lambda, \sigma} \simeq \begin{array}{c} \pi_1 \\ \downarrow \\ \bar{\pi}_{01}^+ \\ \downarrow \\ \pi_0 \end{array} \quad (\text{resp.} \quad I_{\eta, \Lambda, \sigma} \simeq \bar{\pi}_{01}^-)$$

for an appropriately chosen η . This result can be obtained by direct computation.

In this paper, we first determine the structure of $I_{\eta, \Lambda, \sigma}$ when Λ is regular and generic. Let $X_P(\delta, \nu)$ be the Harish-Chandra module of the C^∞ -induced principal series representation $C^\infty\text{-Ind}_P^G(\delta \otimes e^{\nu+\rho_A})$. Here δ is an irreducible representation of M . The Weyl group of $\mathfrak{g}$ is denoted by W . Let $\mathcal{H}_\Lambda$ be the set of equivalence classes of irreducible Harish-Chandra modules with the infinitesimal character Λ . We call Λ generic if every principal series representation with the infinitesimal character Λ is irreducible. The main theorem on the generic case is

Theorem 1.1 (Theorem 2.2). *Let G be a real reductive linear Lie group. Suppose Λ is regular and generic, and σ is an irreducible representation of M^η . Then $I_{\eta, \Lambda}^\circ$ is completely reducible. Moreover, the irreducible decomposition of $I_{\eta, \Lambda, \sigma}$ is given by*

$$(1.2) \quad I_{\eta, \Lambda, \sigma} \simeq \bigoplus_{X_P(\delta, \nu) \in \mathcal{H}_\Lambda} m_\delta(\sigma) X_P(\delta, \nu), \quad m_\delta(\sigma) = \dim \mathrm{Hom}_{M^\eta}(\delta|_{M^\eta}, \sigma).$$

For the non-generic case, there is little result that can be applied to general groups. Therefore, we examine the case $G = U(n, 1)$ in the second half of this paper so that it becomes a springboard to the study of general cases. Let $\bar{\pi}_{i,j}$ be the irreducible Harish-Chandra module of $U(n, 1)$ defined in §3.2. The main result of this case is

Theorem 1.2 (Theorem 5.16). *Suppose $G = U(n, 1)$ and the infinitesimal character Λ is regular integral. If the highest weight of $\sigma \in \widehat{M^\eta} \simeq \widehat{U(n-2)} \times \widehat{U(1)}$ satisfies (4.1) for some $i = 1, \dots, n-1, j = 2, \dots, n+1-i$, then the composition series of $I_{\eta, \Lambda, \sigma}$ is*

$$I_{\eta, \Lambda, \sigma} \simeq \begin{array}{ccccccc} & \bar{\pi}_{i-1, j+1} & \bar{\pi}_{i-1, j-1} & \bar{\pi}_{i+1, j+1} & \bar{\pi}_{i+1, j-1} & & \\ & \downarrow & \searrow & \swarrow & \downarrow & & \\ & \bar{\pi}_{i-1, j} & \bar{\pi}_{i, j+1} & \bar{\pi}_{i, j-1} & \bar{\pi}_{i+1, j} & & \\ & & \searrow & \swarrow & \searrow & \swarrow & \\ & & & \bar{\pi}_{i, j} & & & \end{array}$$

Here, if $i+j = n$ or $n+1$, the modules $\bar{\pi}_{a,b}$, $a+b > n+1$, are regarded to be zero and the arrows starting from or ending at such modules are omitted.

This paper is organized as follows. The generic case is treated in §2. The main result of this section is Theorem 2.2. For §3 and later, we put $G = U(n, 1)$ and examine the composition series of $I_{\eta, \Lambda, \sigma}$ when the infinitesimal character is regular integral. §3 recalls the structure of $U(n, 1)$ and the classification of irreducible Harish-Chandra modules of it. In §4, we first show that $I_{\eta, \Lambda, \sigma}$ has a unique irreducible submodule if it is non-zero. Also determined are the possible irreducible modules appearing in the composition series of it. In §5, the composition series of $I_{\eta, \Lambda, \sigma}$ is completely determined. For this step, we use the explicit form of K -type shift operators and the central elements of the universal enveloping algebra. The key lemmas for our calculation are Lemmas 5.7 and 5.11, and the main theorem of the latter half of this paper is Theorem 5.16. In §6, another formulation of our problem is discussed.

Before going ahead, we introduce notation used in this paper. For a real Lie group L , the Lie algebra of it is denoted by $\mathfrak{l}_0$ and its complexification by $\mathfrak{l} = \mathfrak{l}_0 \otimes_{\mathbb{R}} \mathbb{C}$. This notation will be applied to groups denoted by other Roman letters in the same way without comment. For a compact Lie group L , the set of equivalence classes of irreducible representations of L is denoted by $\widehat{L}$. The representation space of $\pi \in \widehat{L}$ is denoted by V_{π}^L . When L is connected and π is the irreducible representation whose highest weight is λ , we also denote it by V_{λ}^L . For $\pi \in \widehat{L}$, the contragredient representation is denoted by π^* , and if λ is the highest weight of π , then the highest weight of π^* is denoted by λ^* .

Suppose that K is a maximal compact subgroup of a real reductive group G . For a $(\mathfrak{g}, K)$ -module π , the K -spectrum $\{\tau \in \widehat{K} \mid \tau \subset \pi|_K\}$ is denoted by $\widehat{K}(\pi)$.

For a numerical vector $\mathbf{a} = (a_1, \dots, a_{\ell}) \in \mathbb{C}^{\ell}$ or $\mathbb{R}^{\ell}$, write $|\mathbf{a}| := \sum_{i=1}^{\ell} a_i$. This notation will be applied for an element of the dual of a Cartan subalgebra when this space is identified with a numerical vector space by using some fixed basis.

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2. THE GENERIC CASE

In this section, we first write the differential equations characterizing $I_{\eta, \Lambda, \sigma}^{\circ}$. After that, we determine the structure of the standard Whittaker $(\mathfrak{g}, K)$ -modules when Λ is regular and generic. This is the first main theorem of this paper. As a corollary to the proof of this theorem, the K -admissibility of any standard Whittaker $(\mathfrak{g}, K)$ -module is obtained.

The K -type decomposition of $I_{\eta, \Lambda, \sigma}^{\circ}$ is given by

$$I_{\eta, \Lambda, \sigma}^{\circ} \simeq \bigoplus_{\tau \in \widehat{K}} \mathrm{Hom}_K(V_{\tau}^K, I_{\eta, \Lambda, \sigma}^{\circ}) \otimes V_{\tau}^K.$$

By Iwasawa decomposition, an element of $\mathrm{Hom}_K(V_{\tau}^K, I_{\eta, \Lambda, \sigma}^{\circ})$ is determined by its restriction to A . For $\phi_1 \in \mathrm{Hom}_K(V_{\tau}^K, I_{\eta, \Lambda, \sigma}^{\circ})$, $a \in A$, $m \in M^{\eta}$ and $v \in V_{\tau}^K$,

$$\phi_1(\tau(m)v)(a) = L(m)(\phi_1(v))(a) = \phi_1(v)(m^{-1}a) = \phi_1(v)(am^{-1}) = \sigma(m)\phi_1(v)(a).$$

Therefore, we may identify ϕ_1 with an element ϕ_2 of $C^{\infty}(A \rightarrow \mathrm{Hom}_{M^{\eta}}(V_{\tau}^K, V_{\sigma}^{M^{\eta}}))$ by $\phi_1(v)(a) = \phi_2(a)(v)$, $v \in V_{\tau}^K$, $a \in A$. The $\mathfrak{g}$ action on ϕ_1 can be transferred to

ϕ_2 , which we denote by $X \cdot \phi_2$: $(X \cdot \phi_2)(a)(v) = L(X)(\phi_1(v))(a)$. It follows that $\text{Hom}_K(V_\tau^K, I_{\eta, \Lambda, \sigma}^\circ)$ is isomorphic to

(2.1)

$$I_{\eta, \Lambda, \sigma}^\circ(\tau) := \{\phi_2 \in C^\infty(A \rightarrow \text{Hom}_{M^\eta}(V_\tau^K, V_\sigma^{M^\eta})) \mid z \cdot \phi_2 = \chi_\Lambda(z)\phi_2, z \in Z(\mathfrak{g})\},$$

and the subspace $I_{\eta, \Lambda, \sigma}(\tau)$ of $I_{\eta, \Lambda, \sigma}^\circ(\tau)$ consisting of functions which grow moderately at infinity is isomorphic to $\text{Hom}_K(V_\tau^K, I_{\eta, \Lambda, \sigma})$.

We write the action of $z \in Z(\mathfrak{g})$ on ϕ_2 . First, the $U(\mathfrak{k})$ action is

$$(u \cdot \phi_2)(a)(v) = (L(u)\phi_1(v))(a) = \phi_1(\tau(u)v)(a) = \phi_2(a)(\tau(u)v), \quad u \in U(\mathfrak{k}).$$

Secondly, consider the action of $U(\mathfrak{a})$. Denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the set of simple roots of $\Delta^+(\mathfrak{g}_0, \mathfrak{a}_0)$. Let $H_1, \dots, H_l$ be the basis of $\mathfrak{a}_0$ dual to $\alpha_1, \dots, \alpha_l$: $\alpha_i(H_j) = \delta_{ij}$. For $t = (t_1, \dots, t_l) \in (\mathbb{R}_{>0})^l$, define

$$a_t = \exp \left(- \sum_{i=1}^l (\log t_i) H_i \right) \in A.$$

For $u = \sum_{\mathbf{n}} c_{\mathbf{n}} H_1^{n_1} \cdots H_l^{n_l} \in U(\mathfrak{a})$, $\mathbf{n} = (n_1, \dots, n_l) \in (\mathbb{Z}_{\geq 0})^l$, $c_{\mathbf{n}} \in \mathbb{C}$, let

$$(2.2) \quad \partial_t(u) = \sum_{\mathbf{n}} c_{\mathbf{n}} \partial_1^{n_1} \cdots \partial_l^{n_l}, \quad \partial_i := t_i \frac{\partial}{\partial t_i}.$$

Then

$$(u \cdot \phi_2(a_t))(v) = (\partial_t(u)\phi_2(a_t))(v), \quad u \in U(\mathfrak{a}).$$

Lastly, we write down the action of $U(\mathfrak{n})$. Denote by $(\mathfrak{g}_0)_\alpha$ the root space corresponding to a root α . Let $\{N_{\alpha, j} \mid \alpha \in \Delta^+(\mathfrak{g}_0, \mathfrak{a}_0), 1 \leq j \leq \dim(\mathfrak{g}_0)_\alpha\}$ be a basis of $\mathfrak{n}_0$ such that it satisfies $\eta(N_{\alpha, j}) \neq 0$ if $\alpha \in \Pi$ and $j = 1$, and $\eta(N_{\alpha, j}) = 0$ otherwise. We define

$$(2.3) \quad \eta_t(N_{\alpha, j}) = \begin{cases} t_i \eta(N_{\alpha, 1}) & \text{if } \alpha = \alpha_i \in \Pi, j = 1, \\ 0 & \text{otherwise} \end{cases}$$

and extend it to an algebra homomorphism $U(\mathfrak{n}) \rightarrow \mathbb{C}[t_1, \dots, t_l]$. Then

$$(u \cdot \phi_2(a_t))(v) = (L(u)\phi_1(v))(a_t) = \eta_t(u)\phi_1(v)(a_t) = \eta_t(u)\phi_2(a_t)(v),$$

for $u \in U(\mathfrak{n})$. Therefore, $U(\mathfrak{g})$ acts on $C^\infty(A \rightarrow \text{Hom}_{M^\eta}(V_\tau^K, V_\sigma^{M^\eta}))$ by

$$(2.4) \quad ((u_{\mathfrak{n}} u_{\mathfrak{a}} u_{\mathfrak{k}}) \cdot \phi_2)(a_t)(v) = \eta_t(u_{\mathfrak{n}}) \partial(u_{\mathfrak{a}}) \phi_2(a_t)(\tau(u_{\mathfrak{k}})v),$$

$$u_{\mathfrak{n}} \in U(\mathfrak{n}), u_{\mathfrak{a}} \in U(\mathfrak{a}), u_{\mathfrak{k}} \in U(\mathfrak{k}).$$

Choose a Cartan subalgebra $\mathfrak{t}_{\mathfrak{m}}$ of $\mathfrak{m}$. Fix a positive system $\Delta^+(\mathfrak{m}, \mathfrak{t}_{\mathfrak{m}})$ of the root system $\Delta(\mathfrak{m}, \mathfrak{t}_{\mathfrak{m}})$. Denote by $\rho_{\mathfrak{m}}$ half the sum of the roots in $\Delta^+(\mathfrak{m}, \mathfrak{t}_{\mathfrak{m}})$. Let $\delta \in \widehat{M}$. Its highest weight with respect to $\Delta^+(\mathfrak{m}, \mathfrak{t}_{\mathfrak{m}})$ is denoted by μ_δ . Note that since we assume every Cartan subgroup of G is commutative [14, (0.1.2) f)], the highest weight μ_δ of the restriction of δ to the identity component of M is well defined even if M is not connected. Let $P = MAN$ be the minimal parabolic subgroup of G corresponding to our Iwasawa N . For $\delta \in \widehat{M}$ and $\nu \in \mathfrak{a}^*$, let $X_P(\delta, \nu)$ be the Harish-Chandra module of the smooth principal series representation $C^\infty\text{-Ind}_P^G(\delta \otimes e^{\nu + \rho_A})$.

Definition 2.1. An infinitesimal character Λ is called *generic* if every principal series representation $X_P(\delta, \nu)$ which admits the infinitesimal character Λ is irreducible.

Choose a Cartan subalgebra $\mathfrak{h} := \mathfrak{t}_{\mathfrak{m}} + \mathfrak{a}$ of $\mathfrak{g}$. Let $W = W(\mathfrak{g}, \mathfrak{h})$ and $W_{\mathfrak{m}} = W(\mathfrak{m}, \mathfrak{t}_{\mathfrak{m}})$ be the Weyl groups of $\mathfrak{g}$ and $\mathfrak{m}$, respectively. The little Weyl group is denoted by $W(G, A)$. It is well known that the infinitesimal character of $X_P(\delta, \nu)$ is $\Lambda \in \mathfrak{h}^*$ if and only if $(\mu_{\delta} + \rho_{\mathfrak{m}}, \nu)$ is in the orbit $W \cdot \Lambda$. It is also well known that two principal series representations $X_P(\delta, \nu)$ and $X_P(\delta', \nu')$ have the same composition factors if and only if there exists $w \in W(G, A)$ such that $(\delta', \nu') = (w \cdot \delta, w \cdot \nu)$. We denote by $\mathcal{A}_{\Lambda}$ the set of $(\delta, \nu) \in \widehat{M} \times \mathfrak{a}^*$ satisfying $(\mu_{\delta} + \rho_{\mathfrak{m}}, \nu) \in W \cdot \Lambda$. The set of equivalence classes of irreducible Harish-Chandra modules with the infinitesimal character Λ is denoted by $\mathcal{H}_{\Lambda}$. Note that, if Λ is generic, every member of $\mathcal{H}_{\Lambda}$ is a principal series representation. Therefore, $\mathcal{H}_{\Lambda}$ is parametrized by $W(G, A) \backslash \mathcal{A}_{\Lambda}$, the set of $W(G, A)$ -orbits in $\mathcal{A}_{\Lambda}$.

The first main result of this paper is the following theorem.

Theorem 2.2. *Suppose Λ is regular and generic, and σ is an irreducible representation of M^{η} . Then $I_{\eta, \Lambda}^{\circ}$ is completely reducible. Moreover, the irreducible decomposition of $I_{\eta, \Lambda, \sigma}$ is given by*

$$(2.5) \quad I_{\eta, \Lambda, \sigma} \simeq \bigoplus_{X_P(\delta, \nu) \in \mathcal{H}_{\Lambda}} m_{\delta}(\sigma) X_P(\delta, \nu), \quad m_{\delta}(\sigma) = \dim \operatorname{Hom}_{M^{\eta}}(\delta|_{M^{\eta}}, \sigma).$$

Proof. We first count the dimension of $I_{\eta, \Lambda, \sigma}^{\circ}(\tau)$.

Let $\bar{\mathfrak{n}}$ be the nilpotent subalgebra opposite to $\mathfrak{n}$. We denote by $\mathfrak{u}_{\mathfrak{m}}$ and $\bar{\mathfrak{u}}_{\mathfrak{m}}$ the nilpotent subalgebras in $\mathfrak{m}$ corresponding to $\Delta^+(\mathfrak{m}, \mathfrak{t}_{\mathfrak{m}})$ and $-\Delta^+(\mathfrak{m}, \mathfrak{t}_{\mathfrak{m}})$, respectively. Then $\mathfrak{u} := \mathfrak{u}_{\mathfrak{m}} + \bar{\mathfrak{n}}$ is the nilradical of a Borel subalgebra $\mathfrak{h} + \mathfrak{u}$. We define non-shifted Harish-Chandra maps γ'_1 , γ'_2 and γ' by

$$\begin{aligned} \gamma'_1 : U(\mathfrak{g}) &= U(\mathfrak{m} + \mathfrak{a}) \oplus (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\bar{\mathfrak{n}}) \rightarrow U(\mathfrak{m} + \mathfrak{a}), \\ \gamma'_2 : U(\mathfrak{m} + \mathfrak{a}) &= U(\mathfrak{h}) \oplus (\bar{\mathfrak{u}}_{\mathfrak{m}}U(\mathfrak{m} + \mathfrak{a}) + U(\mathfrak{m} + \mathfrak{a})\mathfrak{u}_{\mathfrak{m}}) \rightarrow U(\mathfrak{h}), \\ \gamma' &= \gamma'_2 \circ \gamma'_1 : U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\bar{\mathfrak{u}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{u}) \rightarrow U(\mathfrak{h}), \end{aligned}$$

respectively. Note that, on $Z(\mathfrak{g})$, the map γ'_1 is given by the projection

$$(2.6) \quad \gamma'_1 : Z(\mathfrak{g}) \subset (U(\mathfrak{a}) \otimes Z(\mathfrak{m})) \oplus \mathfrak{n}U(\mathfrak{g}) \rightarrow U(\mathfrak{a}) \otimes Z(\mathfrak{m}).$$

The shifted Harish-Chandra maps are given by

$$\begin{aligned} \gamma_1 &= \tau_1 \circ \gamma'_1 : Z(\mathfrak{g}) \rightarrow U(\mathfrak{a}) \otimes Z(\mathfrak{m}), \quad \tau_1(H) = H + \rho_A(H), H \in \mathfrak{a}, \\ \gamma_2 &= \tau_2 \circ \gamma'_2 : U(\mathfrak{m} + \mathfrak{n}) \rightarrow U(\mathfrak{h}), \quad \tau_2(H) = H - \rho_{\mathfrak{m}}(H), H \in \mathfrak{h}, \\ \gamma &= \gamma_2 \circ \gamma_1 : Z(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{h})^W, \end{aligned}$$

respectively. The infinitesimal character χ_{Λ} is, of course, defined by $\chi_{\Lambda}(z) = \gamma(z)(\Lambda)$, $z \in Z(\mathfrak{g})$.

Choose a K -type $\tau \in \widehat{K}(I_{\eta, \Lambda, \sigma}^{\circ})$. We shall write the differential equations which characterize $\phi_2 \in I_{\eta, \Lambda, \sigma}^{\circ}(\tau)$. By (2.6), an element $z \in Z(\mathfrak{g})$ is expressed as

$$(2.7) \quad z = \gamma'_1(z) + \sum_q \tilde{u}_{\mathfrak{n}}^{(q)} \tilde{u}_{\mathfrak{a}}^{(q)} \tilde{u}_{\mathfrak{k}}^{(q)}, \quad \gamma'_1(z) = \sum_p u_{\mathfrak{a}}^{(p)} u_{\mathfrak{k}}^{(p)} \in U(\mathfrak{a}) \otimes Z(\mathfrak{m}),$$

$$\tilde{u}_{\mathfrak{n}}^{(q)} \in \mathfrak{n}U(\mathfrak{n}), \tilde{u}_{\mathfrak{a}}^{(q)} \in U(\mathfrak{a}), \tilde{u}_{\mathfrak{k}}^{(q)} \in U(\mathfrak{k}).$$

The action of this element on ϕ_2 is given by (2.4):

$$(2.8) \quad \sum_p \partial(u_{\mathfrak{a}}^{(p)}) \phi_2^0(a_t) (\tau(u_{\mathfrak{k}}^{(p)})v) + \sum_q \eta_t(\tilde{u}_{\mathfrak{n}}^{(q)}) \partial(\tilde{u}_{\mathfrak{a}}^{(q)}) \phi_2(a_t) (\tau(\tilde{u}_{\mathfrak{k}}^{(q)})v) = \chi_{\Lambda}(z) \phi_2(a_t)(v),$$

where $a_t \in A$, $v \in V_{\tau}^K$. By the definitions (2.2), (2.3) of $\partial_t(u)$ and η_t , we know that the system of partial differential equations $z \cdot \phi_2 = \chi_{\Lambda}(z) \phi_2$, $z \in Z(\mathfrak{g})$, has regular singularity at $t = 0$. (In [9], Matumoto studies such systems when G is a real split semisimple group.) For the systems of partial differential equations with regular singularities, see [12] or [5, Appendix B] for example. Suppose there exists a non-zero solution ϕ_2 of this system. Since $\tilde{u}_{\mathfrak{n}}^{(q)} \in \mathfrak{n}U(\mathfrak{n})$ implies $\eta_0(\tilde{u}_{\mathfrak{n}}^{(q)}) = 0$ (see the definition (2.3) of η_t), the leading term ϕ_2^0 of ϕ_2 satisfies the system of differential equations

$$(2.9) \quad \sum_p \partial(u_{\mathfrak{a}}^{(p)}) \phi_2^0(a_t) (\tau(u_{\mathfrak{k}}^{(p)})v) = \chi_{\Lambda}(z) \phi_2^0(a_t)(v), \quad \text{for } z \text{ as in (2.7).}$$

Suppose $\delta \in \widehat{M}$ satisfies $\sigma \subset \delta|_{M^{\eta}}$ and $\delta \subset \tau|_M$. Choose a non-zero vector v in the δ -isotypic subspace of V_{τ}^K . Since $u_{\mathfrak{k}}^{(p)} \in Z(\mathfrak{m})$ by (2.7), $\tau(u_{\mathfrak{k}}^{(p)})v = \delta(u_{\mathfrak{k}}^{(p)})v = (\mu_{\delta} + \rho_{\mathfrak{m}})(\gamma_2(u_{\mathfrak{k}}^{(p)}))v$. Let $\nu + \rho_A \in \mathfrak{a}^* \simeq \mathbb{C}^l$ be a characteristic exponent of a solution ϕ_2 to (2.8). Here, we identified $\mathfrak{a}^*$ with $\mathbb{C}^l$ by $\sum_{i=1}^l c_i \alpha_i \leftrightarrow (c_1, \dots, c_l)$. Equation (2.9) says that μ_{δ} and ν satisfy

$$\sum_p (\nu + \rho_A)(u_{\mathfrak{a}}^{(p)}) (\mu_{\delta} + \rho_{\mathfrak{m}})(\gamma_2(u_{\mathfrak{k}}^{(p)})) = \chi_{\Lambda}(z).$$

But since $\sum_p u_{\mathfrak{a}}^{(p)} u_{\mathfrak{k}}^{(p)} = \gamma_1'(z)$, this means that

$$(2.10) \quad \chi_{(\mu_{\delta} + \rho_{\mathfrak{m}}, \nu)}(z) = \chi_{\Lambda}(z) \Leftrightarrow (\mu_{\delta} + \rho_{\mathfrak{m}}, \nu) \in W \cdot \Lambda.$$

Since Λ is regular, $w\Lambda$ ($w \in W$) are all different. Therefore, every solution of equation (2.9) is a linear combination of

$$(2.11) \quad \begin{aligned} \phi_2^0(\delta, \nu, \psi_1, \psi_2; a_t)(v) &:= t^{\nu + \rho_A} \psi_2 \circ \psi_1(v), \\ \psi_1 &\in \text{Hom}_M(\tau|_M, \delta), \quad \psi_2 \in \text{Hom}_{M^{\eta}}(\delta|_{M^{\eta}}, \sigma), \quad (\mu_{\delta} + \rho_{\mathfrak{m}}, \nu) \in W \cdot \Lambda. \end{aligned}$$

Suppose ϕ_2, ϕ_2' are two solutions of (2.8). If all the coefficients of $\phi_2^0(\delta, \nu, \psi_1, \psi_2; a_t)$, $\delta \in \widehat{M}$, $\nu \in \mathfrak{a}^*$, in the power series expansions of ϕ_2, ϕ_2' are identical, then $\phi_2 = \phi_2'$. It follows that the dimension of the solution space of (2.8), i.e., $\dim I_{\eta, \Lambda, \sigma}^{\circ}(\tau)$, is estimated as

$$(2.12) \quad \dim I_{\eta, \Lambda, \sigma}^{\circ}(\tau) \leq \sum_{(\delta, \nu) \in \mathcal{A}_{\Lambda}} \dim \text{Hom}_M(\tau|_M, \delta) \dim \text{Hom}_{M^{\eta}}(\delta|_{M^{\eta}}, \sigma).$$

Let $I_{\eta, \Lambda}^{\circ}(\tau) = \text{Hom}_K(V_{\tau}^K, I_{\eta, \Lambda}^{\circ})$. By (2.12), we have

$$(2.13) \quad \begin{aligned} \dim I_{\eta, \Lambda}^{\circ}(\tau) &= \sum_{\sigma \in \widehat{M^{\eta}}} \dim I_{\eta, \Lambda, \sigma}^{\circ}(\tau) \dim \sigma \\ &\leq \sum_{\sigma \in \widehat{M^{\eta}}} \sum_{(\delta, \nu) \in \mathcal{A}_{\Lambda}} \dim \text{Hom}_M(\tau|_M, \delta) \dim \text{Hom}_{M^{\eta}}(\delta|_{M^{\eta}}, \sigma) \dim \sigma \\ &= \sum_{(\delta, \nu) \in \mathcal{A}_{\Lambda}} \dim \text{Hom}_M(\tau|_M, \delta) \dim \delta. \end{aligned}$$

In the fundamental paper [8], Lynch gives the dimension of the space of dual Whittaker vectors of a principal series representation. His result, together with Theorem C in [10], says that, if $(\mu_\delta + \rho_m, \nu) \in W \cdot \Lambda$, then

$$\dim \operatorname{Hom}_{\mathfrak{g}, K}(X_P(\delta, \nu), I_{\eta, \Lambda}^\circ) = \#W(G, A) \dim \delta.$$

Since Λ is generic, (i) every non-zero element in $\operatorname{Hom}_{\mathfrak{g}, K}(X_P(\delta, \nu), I_{\eta, \Lambda}^\circ)$ is injective, and (ii) if $X_P(\delta_1, \nu_1)$ and $X_P(\delta_2, \nu_2)$ are not isomorphic, then for any $\Phi_i \in \operatorname{Hom}_{\mathfrak{g}, K}(X_P(\delta_i, \nu_i), I_{\eta, \Lambda}^\circ)$, $i = 1, 2$, $\operatorname{Image} \Phi_1 \cap \operatorname{Image} \Phi_2 = 0$. Then we have

$$\begin{aligned} \dim I_{\eta, \Lambda}^\circ(\tau) &\geq \sum_{X_P(\delta, \nu) \in \mathcal{H}_\Lambda} \dim \operatorname{Hom}_{\mathfrak{g}, K}(X_P(\delta, \nu), I_{\eta, \Lambda}^\circ) \dim \operatorname{Hom}_K(\tau, X_P(\delta, \nu)) \\ &= \sum_{[(\delta, \nu)] \in W(G, A) \backslash \mathcal{A}_\Lambda} \#W(G, A) \dim \delta \times \dim \operatorname{Hom}_M(\tau|_M, \delta) \\ &\stackrel{(2.13)}{\geq} \dim I_{\eta, \Lambda}^\circ(\tau). \end{aligned}$$

Here, we first used the Frobenius reciprocity $\operatorname{Hom}_K(\tau, X_P(\delta, \nu)) \simeq \operatorname{Hom}_M(\tau|_M, \delta)$, and used the fact that every $W(G, A)$ -orbit in $\mathcal{A}_\Lambda$ consists of $\#W(G, A)$ elements, since Λ is regular. It follows that every composition factor of $I_{\eta, \Lambda}^\circ$ is a submodule of it. In other words, the modules $I_{\eta, \Lambda}^\circ$, $I_{\eta, \Lambda, \sigma}^\circ$ and $I_{\eta, \Lambda, \sigma}$ are completely reducible. It also follows that the equality in (2.12) holds.

In order to complete the proof, we recall a result of Wallach's ([16]). Let dn be the Haar measure of N and w_0 be the longest element of $W(G, A)$ (with respect to N). Recall the Jacquet integral

$$(2.14) \quad J_\nu : C^\infty\text{-Ind}_P^G(\delta \otimes e^{\nu + \rho_A}) \rightarrow C^\infty(G/M^\eta N; (\delta|_{M^\eta}) \otimes \eta),$$

$$J_\nu(f)(g) = \int_N f(gnw_0)\eta(n)dn.$$

Note that J_ν is right M^η -equivariant, since we may choose w_0 to commute with M^η . Let $\operatorname{Ind}_P^G(\delta \otimes e^{\nu + \rho_A})'$ be the continuous dual space of $C^\infty\text{-Ind}_P^G(\delta \otimes e^{\nu + \rho_A})$, and $\operatorname{Wh}_\eta^{-\infty}(X_P(\delta, \nu))$ be the space of Whittaker vectors in it. Let $C^\infty\text{-Ind}_M^K(\delta)$ be the C^∞ -induced representation of K . As a K -representation, this is isomorphic to $C^\infty\text{-Ind}_P^G(\delta \otimes e^{\nu + \rho_A})|_K$. Let $(\operatorname{Ind}_M^K(\delta))'$ be the space of all continuous functionals on $C^\infty\text{-Ind}_M^K(\delta)$, which is endowed with the C^∞ -topology.

Theorem 2.3 ([16]). *Let $v^* \in (V_\delta^M)^*$. Then $\nu \mapsto \langle v^*, J_\nu(\cdot)(e) \rangle_\delta$ extends to a weakly holomorphic map of $\mathfrak{a}^*$ into $(\operatorname{Ind}_M^K(\delta))'$. Moreover, for any $\nu \in \mathfrak{a}^*$,*

$$(2.15) \quad (V_\delta^M)^* \ni v^* \mapsto (f \mapsto \langle v^*, J_\nu(f)(e) \rangle_\delta) \in \operatorname{Wh}_\eta^{-\infty}(X_P(\delta, \nu))$$

is an isomorphism of vector spaces. Here, $\langle \cdot, \cdot \rangle_\delta$ is the pairing of V_δ^M and its dual.

The image of a continuous dual Whittaker vector is characterized by the moderate growth condition ([15]). From this theorem and the map (2.14), we know that there are $m_\delta(\sigma) = \dim \operatorname{Hom}_{M^\eta}(\delta|_{M^\eta}, \sigma)$ copies of $X_P(\delta, \nu)$ in the socle of $I_{\eta, \Lambda, \sigma}$ if $X_P(\delta, \nu) \in \mathcal{H}_\Lambda$. As we noted before the theorem, every member of $\mathcal{H}_\Lambda$ is a principal series. So the socle of $I_{\eta, \Lambda, \sigma}$ is the right-hand side of (2.5). Since $I_{\eta, \Lambda, \sigma}$ is completely reducible, the theorem is shown. $\square$

Corollary 2.4. *The standard Whittaker $(\mathfrak{g}, K)$ -modules are K -admissible and they have finite length.*

Proof. If Λ is regular, the multiplicity of each K -type is finite because of (2.12). Up to (2.10), the discussion of the proof of Theorem 2.2 is valid for a non-regular infinitesimal character Λ . If Λ is not regular, the solution of (2.9) is a finite linear combination of

$$t^{\nu+\rho_A}(\log t_1)^{m_1} \cdots (\log t_l)^{m_l} \psi_2 \circ \psi_1(v), \quad m_i \in \mathbb{Z}_{\geq 0},$$

where ψ_1, ψ_2, ν are as in (2.11). Therefore, the multiplicity of each K -type is also finite for non-regular Λ . The second assertion is clear since these modules admit an infinitesimal character and are K -admissible. $\square$

3. THE GROUP $U(n, 1)$ AND ITS IRREDUCIBLE HARISH-CHANDRA MODULES

Up to now very little is known about the properties of $I_{\eta, \Lambda, \sigma}$ with non-generic Λ , so no smart technique can be used for the analysis of it. Therefore, we will choose a group G such that the structure of Harish-Chandra modules of it is well known and simple (for example K -multiplicity free), and we determine the $(\mathfrak{g}, K)$ -module structure of $I_{\eta, \Lambda, \sigma}$ for non-generic Λ by direct calculation. Such an example is expected to be a good guide to general cases.

For such reasons, we assume $G = U(n, 1)$ and Λ is regular integral hereafter.

3.1. Structure of $U(n, 1)$. Denote by E_{ij} the standard generators of $\mathfrak{gl}_{n+1}(\mathbb{C})$ and define $I_{n,1} = \sum_{p=1}^n E_{pp} - E_{n+1,n+1}$. Let $G = U(n, 1)$ be the subgroup of $GL(n+1, \mathbb{C})$ consisting of the matrices g satisfying ${}^t \bar{g} I_{n,1} g = I_{n,1}$. Here, ${}^t g$ is the transpose of the matrix g , and $\bar{g}$ is the complex conjugate of g with respect to the real form $GL(n+1, \mathbb{R})$ of $GL(n+1, \mathbb{C})$. The Lie algebra $\mathfrak{g}_0 = \mathfrak{u}(n, 1)$ consists of those matrices $X \in \mathfrak{gl}_{n+1}(\mathbb{C})$ which satisfy ${}^t \bar{X} I_{n,1} + I_{n,1} X = O$. Let $\theta g = I_{n,1} g I_{n,1}$ be a Cartan involution of G . The corresponding maximal compact subgroup K of G is

$$K = \left\{ \begin{pmatrix} k & 0 \\ 0 & k_{n+1} \end{pmatrix} \middle| k \in U(n), k_{n+1} \in U(1) \right\}.$$

Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ be the corresponding Cartan decomposition of $\mathfrak{g}_0$. Then

$$\{E_{i,n+1} + E_{n+1,i}, \sqrt{-1}(E_{i,n+1} - E_{n+1,i}) \mid 1 \leq i \leq n\}$$

is a basis of $\mathfrak{s}_0$.

Let

$$h := E_{n,n+1} + E_{n+1,n}, \quad \mathfrak{a}_0 := \mathbb{R}h,$$

and define $f \in \mathfrak{a}_0^*$ by $f(h) = 1$. Then $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{s}_0$. The restricted root system $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ is

$$\Delta(\mathfrak{g}_0, \mathfrak{a}_0) = \{\pm f, \pm 2f\}.$$

Choose a positive system

$$\Delta^+(\mathfrak{g}_0, \mathfrak{a}_0) = \{f, 2f\},$$

and denote the corresponding nilpotent subalgebra $\sum_{\alpha \in \Delta^+(\mathfrak{g}_0, \mathfrak{a}_0)} (\mathfrak{g}_0)_\alpha$ by $\mathfrak{n}_0$. One obtains an Iwasawa decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0, \quad G = KAN,$$

where $A = \exp \mathfrak{a}_0$ and $N = \exp \mathfrak{n}_0$. Let

$$(3.1) \quad \begin{aligned} X_i &:= E_{in} - E_{ni} - E_{i,n+1} - E_{n+1,i} & (1 \leq i \leq n-1), \\ Y_i &:= \sqrt{-1}(E_{in} + E_{ni} - E_{i,n+1} + E_{n+1,i}) & (1 \leq i \leq n-1), \\ Z &:= \sqrt{-1}(E_{nn} - E_{n+1,n+1} - E_{n,n+1} + E_{n+1,n}). \end{aligned}$$

Then $\{X_i, Y_i \mid 1 \leq i \leq n-1\}$ is a basis of $(\mathfrak{g}_0)_f$, and $\{Z\}$ is a basis of $(\mathfrak{g}_0)_{2f}$.

In our $U(n, 1)$ case, M is isomorphic to $U(n-1) \times U(1)$. It acts on the space of non-degenerate unitary characters of N by $\eta \mapsto \eta^m(n) := \eta(m^{-1}nm)$, $m \in M$. Therefore, we may choose a manageable unitary character when we calculate Whittaker modules. We use the non-degenerate character η defined by

$$(3.2) \quad \begin{aligned} \eta(X_i) &= 0, & i = 1, \dots, n-1, & \quad \eta(Y_i) = 0, & i = 1, \dots, n-2, \\ \eta(Y_{n-1}) &= \sqrt{-1}\xi, & \xi > 0, & \quad \eta(Z) = 0. \end{aligned}$$

It is easy to see that M^n is isomorphic to $U(n-2) \times U(1)$.

3.2. Classification of irreducible Harish-Chandra modules. We review the classification of irreducible Harish-Chandra modules of $G = U(n, 1)$ with regular integral infinitesimal character. For details, see [2], [7], for example. We use the notation $\pi_{i,j}$, $\bar{\pi}_{i,j}$, etc. in [2].

There are two conjugacy classes of Cartan subgroups in G ; one is compact and the other is maximally split. Let H_c be the compact Cartan subgroup consisting of diagonal matrices and let $\mathfrak{h}_c$ be the complexification of its Lie algebra. Define a basis $\{\epsilon_i \mid i = 1, \dots, n+1\}$ of $\mathfrak{h}_c^*$ by $\epsilon_i(E_{jj}) = \delta_{ij}$. Choose a maximally split Cartan subgroup $H_s := (H_c \cap M)A$. The complexified Lie algebra $\mathfrak{h}_s$ of it is the linear span of E_{ii} ($i = 1, \dots, n-1$), $E_{nn} + E_{n+1,n+1}$ and h . Define $\tilde{\epsilon}_n \in (\mathfrak{h}_c \cap \mathfrak{m})^*$ by $\tilde{\epsilon}_n(E_{jj}) = 0$ for $j = 1, \dots, n-1$ and $\tilde{\epsilon}_n(E_{nn} + E_{n+1,n+1}) = 1$. Then ϵ_i , $i = 1, \dots, n-1$, $\tilde{\epsilon}_n$ and f is a basis of $\mathfrak{h}_s^*$.

Consider the irreducible Harish-Chandra modules with the regular integral infinitesimal character Λ , which is conjugate to

$$(3.3) \quad \sum_{p=1}^{n+1} \Lambda_p \epsilon_p \in \mathfrak{h}_c^*, \quad \Lambda_p \in \mathbb{Z} + n/2, \quad \Lambda_1 > \Lambda_2 > \dots > \Lambda_{n+1}.$$

There are $n+1$ inequivalent discrete series representations π_i , $i = 0, \dots, n$, whose Harish-Chandra parameters are

$$\sum_{p=1}^i \Lambda_p \epsilon_p + \sum_{p=i+1}^n \Lambda_{p+1} \epsilon_p + \Lambda_{i+1} \epsilon_{n+1},$$

respectively. π_i is also denoted by $\bar{\pi}_{i,n+1-i}$.

For $i = 0, \dots, n-1$ and $j = 1, \dots, n-i$, define $\mu_{i,j} \in (\mathfrak{h}_c \cap \mathfrak{m})^*$ and $\nu_{i,j} \in \mathfrak{a}^*$ by

$$(3.4) \quad \begin{aligned} \mu_{i,j} &:= \sum_{p=1}^i \Lambda_p \epsilon_p + \sum_{p=i+1}^{n-j} \Lambda_{p+1} \epsilon_p + \sum_{p=n-j+1}^{n-1} \Lambda_{p+2} \epsilon_p - \rho_m \\ &\quad + (\Lambda_{i+1} + \Lambda_{n-j+2}) \tilde{\epsilon}_n, \\ \nu_{i,j} &:= (\Lambda_{i+1} - \Lambda_{n-j+2}) f, \end{aligned}$$

where $\rho_m := \frac{1}{2} \sum_{p=1}^{n-1} (n-2p)\epsilon_p$. Let $\delta_{i,j}$ be the irreducible representation of M with the highest weight $\mu_{i,j}$, and let $\pi_{i,j} := X_P(\delta_{i,j}, \nu_{i,j})$. Then $\pi_{i,j}$ has the unique irreducible quotient, which we denote by $\bar{\pi}_{i,j}$.

Theorem 3.1. *The irreducible Harish-Chandra modules of $U(n, 1)$ with the regular integral infinitesimal character Λ are parametrized, up to K -conjugacy, by the set $\{\bar{\pi}_{i,j} \mid i = 0, \dots, n, j = 1, \dots, n+1-i\}$.*

The K -type structure of $\bar{\pi}_{i,j}$ is explicitly known. To state the theorem, let $\Lambda_0 := \infty$ and $\Lambda_{n+2} := -\infty$. As is explained in §1, write $|v| := \sum_{i=1}^{n+1} v_i$ for an element $v = \sum_{i=1}^{n+1} v_i \epsilon_i \in \mathfrak{h}_c^*$.

Theorem 3.2 ([7]). *For $i = 0, \dots, n$ and $j = 1, \dots, n+1-i$, the K -spectrum $\widehat{K}(\bar{\pi}_{i,j})$ is*

(3.5)

$$\begin{aligned} \{(\tau_\lambda, V_\lambda^K) \mid & \Lambda_{p-1} - n/2 + p - 1 \geq \lambda_p \geq \Lambda_p - n/2 + p, \quad p = 1, \dots, i; \\ & \Lambda_p - n/2 + p - 1 \geq \lambda_p \geq \Lambda_{p+1} - n/2 + p, \quad p = i+1, \dots, n-j+1; \\ & \Lambda_{p+1} - n/2 + p - 1 \geq \lambda_p \geq \Lambda_{p+2} - n/2 + p, \quad p = n-j+2, \dots, n; \\ & |\lambda| = |\Lambda|\}, \end{aligned}$$

and each K -type occurs in $\bar{\pi}_{i,j}$ with multiplicity one.

In order to state the composition series, we use diagrammatic expression.

Definition 3.3. Suppose A_1, A_2 are distinct composition factors of a $(\mathfrak{g}, K)$ -module V . If there exist elements $\{v_i\} \subset A_1$ and $\{X_i\} \subset \mathfrak{g}$ such that $\sum_i X_i v_i$ is non-zero and contained in A_2 , then we connect A_1 and A_2 by the arrow $A_1 \rightarrow A_2$.

Theorem 3.4 ([7], [2]). *The composition series of $\pi_{i,j}$, $i = 0, \dots, n-1$, $j = 1, \dots, n-i$, is*

$$\pi_{i,j} = X_P(\delta_{i,j}, \nu_{i,j}) \simeq \begin{array}{ccc} & \bar{\pi}_{i,j} & \\ \swarrow & & \searrow \\ \bar{\pi}_{i,j+1} & & \bar{\pi}_{i+1,j} \\ \swarrow & & \searrow \\ & \bar{\pi}_{i+1,j+1} & \end{array}$$

If $i+j = n$, $i = 0, \dots, n-1$, then

$$\pi_{i,n-i} = X_P(\delta_{i,n-i}, \nu_{i,n-i}) \simeq \begin{array}{ccc} & \bar{\pi}_{i,n-i} & \\ \swarrow & & \searrow \\ \bar{\pi}_{i,n-i+1} & & \bar{\pi}_{i+1,n-i} \end{array} = \begin{array}{ccc} & \bar{\pi}_{i,n-i} & \\ \swarrow & & \searrow \\ \pi_i & & \pi_{i+1} \end{array}.$$

4. COMPOSITION FACTORS OF $I_{\eta,\Lambda,\sigma}$

In this section we first determine the submodules of $I_{\eta,\Lambda,\sigma}$. For this purpose, we need some results on the Whittaker models.

4.1. Whittaker models. Let (π, V) be an irreducible Harish-Chandra module. A realization of (π, V) as a submodule of $C^\infty(G/N; \eta)_K$ is called a Whittaker model of (π, V) . For a Harish-Chandra module V , let V_∞ be its C^∞ -globalization. As in §2, let $\text{Wh}_\eta^-(V)$ be the space of Whittaker vectors in the continuous dual space of V_∞ . Note that the image of an element of $\text{Wh}_\eta^-(V)$ is characterized by the moderate growth condition. The next theorem tells us which irreducible $(\mathfrak{g}, K)$ -module can be a submodule of $C^\infty(G/N; \eta)_K$.

Theorem 4.1 ([10], [11]). *Let V be a Harish-Chandra module.*

- (1) *An irreducible Harish-Chandra module V has a non-trivial Whittaker model if and only if the Gelfand-Kirillov dimension $\text{Dim} V$ of V is equal to $\dim N$.*
- (2) *(Casselman) $V \rightarrow \text{Wh}_\eta^{-\infty}(V)$ is an exact functor.*

The Gelfand-Kirillov dimensions of the irreducible modules $\bar{\pi}_{i,j}$ are

$$\begin{aligned} \text{Dim} \bar{\pi}_{0,1} &= 0, \\ \text{Dim} \bar{\pi}_{i,1} &= \text{Dim} \bar{\pi}_{0,j} = n, & i = 1, \dots, n, \quad j = 2, \dots, n+1, \\ \text{Dim} \bar{\pi}_{i,j} &= 2n-1 = \dim N, & i = 1, \dots, n-1, \quad j = 2, \dots, n+1-i. \end{aligned}$$

See [2] for example. Therefore, an irreducible submodule of $I_{\eta,\Lambda,\sigma}$ is isomorphic to one of $\bar{\pi}_{i,j}$, $i = 1, \dots, n-1$, $j = 2, \dots, n+1-i$.

4.2. Unique simple submodule. Let (π, V) be an irreducible Harish-Chandra module with $\text{Dim} V = \dim N$. Suppose that it is a composition factor of some principal series representation $X_P(\delta, \nu)$. By Theorem 4.1(2) and Theorem 2.3, every continuous embedding of V_∞ into $C^\infty(G/N; \eta)$ is a composition of (i) a realization of V_∞ as a subquotient of $C^\infty\text{-Ind}_P^G(\delta \otimes e^{\nu+\rho_A})$ and (ii) a Jacquet integral. Since a Jacquet integral is right M^η -equivariant, V can be a submodule of $I_{\eta,\Lambda,\sigma}$ only if $\sigma \subset \delta|_{M^\eta}$. Let $\{X_P(\delta_p, \nu_p) \mid p = 1, \dots, k\}$ be the set of principal series representations which contain (π, V) as a subquotient. If (π, V) is a submodule of $I_{\eta,\Lambda,\sigma}$, then, by the above discussion, σ is a submodule of $\delta_p|_{M^\eta}$ for every $p = 1, \dots, k$.

Conversely, for $\sigma \in \widehat{M^\eta}$, suppose that there exists a principal series $X_P(\delta, \nu) \in \mathcal{H}_\Lambda$ such that $\sigma \subset \delta|_{M^\eta}$. Then by Theorem 2.3, the intersection of the image of the Jacquet integral (2.14) and $I_{\eta,\Lambda,\sigma}$ is non-zero. Especially, $I_{\eta,\Lambda,\sigma}$ is non-zero.

Proposition 4.2. *Suppose the regular infinitesimal character Λ is integral. The irreducible module $\bar{\pi}_{i,j}$, $i = 1, \dots, n-1$, $j = 2, \dots, n+1-i$, is a submodule of $I_{\eta,\Lambda,\sigma}$ if and only if the highest weight $\gamma = (\gamma_1, \dots, \gamma_{n-2}; \gamma_{n-1})$ of the irreducible representation σ of $M^\eta \simeq U(n-2) \times U(1)$ satisfies*

(4.1)

$$\begin{cases} \Lambda_p - n/2 + p \geq \gamma_p \geq \Lambda_{p+1} - n/2 + p + 1, & p = 1, \dots, i-1, \\ \Lambda_{p+1} - n/2 + p \geq \gamma_p \geq \Lambda_{p+2} - n/2 + p + 1, & p = i, \dots, n-j, \\ \Lambda_{p+2} - n/2 + p \geq \gamma_p \geq \Lambda_{p+3} - n/2 + p + 1, & p = n-j+1, \dots, n-2, \\ \gamma_{n-1} = |\Lambda| - \sum_{p=1}^{n-2} \gamma_p. \end{cases}$$

Especially, $I_{\eta,\Lambda,\sigma}$ is non-zero if and only if the highest weight of σ satisfies the condition (4.1) for some i, j . In this case, $\bar{\pi}_{i,j}$ is the unique simple submodule of $I_{\eta,\Lambda,\sigma}$.

Proof. By Theorem 3.4, $\bar{\pi}_{i,j}$, $i = 1, \dots, n-1$, $j = 2, \dots, n+1-i$, is a composition factor of the principal series $\pi_{k,l}$ if and only if $(k, l) = (i, j)$ (only when $i+j \leq n$), $(i, j-1)$, $(i-1, j)$ or $(i-1, j-1)$. Therefore, if $\bar{\pi}_{i,j}$ is a submodule of $I_{\eta,\Lambda,\sigma}$, then σ is a submodule of $\delta_{a,b}|_{M^\eta}$, $(a, b) = (i, j), (i, j-1), (i-1, j)$ and $(i-1, j-1)$. Conversely, if σ satisfies this condition, then $I_{\eta,\Lambda,\sigma}$ is non-zero, as is stated before this proposition.

Recall the branching rule for the restriction $U(m)$ to $U(m-1)$. For an irreducible representation δ_μ of $U(m)$ with the highest weight $\mu = (\mu_1, \dots, \mu_m)$, the restriction

$\delta_\mu|_{U(m-1)}$ is a direct sum of $\sigma_\gamma \in U(m-1)$, with

$$\gamma = (\gamma_1, \dots, \gamma_{m-1}), \quad \mu_p \geq \gamma_p \geq \mu_{p+1}, \quad p = 1, \dots, m-1, \quad \gamma_p \in \mathbb{Z}.$$

It follows that the restriction $\delta_{k,l} \in \widehat{M} \simeq \widehat{U(n-1)} \times \widehat{U(1)}$ to $M^\eta \simeq U(n-2) \times U(1)$ is a direct sum of $\sigma_\gamma \in \widehat{U(n-2)} \times \widehat{U(1)}$, whose highest weight $\gamma = (\gamma_1, \dots, \gamma_{n-2}; \gamma_{n-1})$ satisfies

$$\begin{cases} \Lambda_p - n/2 + p \geq \gamma_p \geq \Lambda_{p+1} - n/2 + p + 1, & p = 1, \dots, k-1, \\ \Lambda_k - n/2 + k \geq \gamma_k \geq \Lambda_{k+2} - n/2 + k + 1, \\ \Lambda_{p+1} - n/2 + p \geq \gamma_p \geq \Lambda_{p+2} - n/2 + p + 1, & p = k+1, \dots, n-l-1, \\ \Lambda_{n-l+1} + n/2 - l \geq \gamma_{n-l} \geq \Lambda_{n-l+3} + n/2 - l + 1, \\ \Lambda_{p+2} - n/2 + p \geq \gamma_p \geq \Lambda_{p+3} - n/2 + p + 1, & p = n-l+1, \dots, n-2, \\ \gamma_{n-1} = |\Lambda| - \sum_{p=1}^{n-2} \gamma_p \end{cases}$$

if $k+l \leq n-1$. The last condition for γ_{n-1} is obtained from the action of the center of G . Therefore, if $i+j \leq n$, then $\sigma \in \widehat{M}^\eta$ is a submodule of $\delta_{a,b}|_{M^\eta}$, $(a,b) = (i,j), (i,j-1), (i-1,j), (i-1,j-1)$ if and only if the highest weight γ of σ satisfies (4.1). This proves the “only if” part of the proposition for the case $i+j \leq n$. The case $i+j = n+1$ is shown analogously.

We will show that the condition is sufficient and that the multiplicity in the socle is one.

Let $(\widehat{M}^\eta)_{i,j}$ be the set of $\sigma \in \widehat{M}^\eta$ whose highest weight γ satisfies the condition (4.1). Then it is easy to see that $(\widehat{M}^\eta)_{i,j} \cap (\widehat{M}^\eta)_{k,l} = \emptyset$ if $(i,j) \neq (k,l)$. It follows that if $\sigma \in (\widehat{M}^\eta)_{i,j}$, then every irreducible factor in the socle of $I_{\eta,\Lambda,\sigma}$ is isomorphic to $\pi_{i,j}$. Let m_σ be the multiplicity of such factors. Then $\dim \text{Wh}_\eta^{-\infty}(\pi_{i,j}) = \sum_{\sigma \in (\widehat{M}^\eta)_{i,j}} m_\sigma \dim \sigma$. By Theorems 3.4 and 4.1, we have

$$\dim \text{Wh}_\eta^{-\infty}(\pi_{i,j}) = \sum_{a,b=0,1} \dim \text{Wh}_\eta^{-\infty}(\pi_{i+a,j+b}) = \sum_{a,b=0,1} \sum_{\sigma \in (\widehat{M}^\eta)_{i+a,j+b}} m_\sigma \dim \sigma.$$

On the other hand, it is easy to see from Theorem 2.3 and (4.1) that

$$\dim \text{Wh}_\eta^{-\infty}(\pi_{i,j}) = \dim \delta_{i,j} = \sum_{a,b=0,1} \sum_{\sigma \in (\widehat{M}^\eta)_{i+a,j+b}} \dim \sigma.$$

Since $m_\sigma \geq 1$ for any $\sigma \in \bigcup_{a,b=0,1} (\widehat{M}^\eta)_{i+a,j+b}$, they are all one. This completes the proof of the proposition. $\square$

4.3. Composition factors. Hereafter, we denote $I_{\eta,\Lambda,\sigma}$ by $I_{\eta,\Lambda,\gamma}$ if the highest weight of σ is γ . We also denote by σ_γ the irreducible representation of M^η whose highest weight is γ . We first determine the irreducible representations appearing in the composition series of $I_{\eta,\Lambda,\gamma}$.

Proposition 4.3. *Suppose that Λ is regular integral and that γ satisfies (4.1), so $\pi_{i,j}$ is the unique simple submodule of $I_{\eta,\Lambda,\gamma}$. In this case, an irreducible module $\pi_{k,l}$ is a composition factor of $I_{\eta,\Lambda,\gamma}$ only if $(k,l) = (i+a,j+b)$ with $a = 0, \pm 1$ and $b = 0, \pm 1$.*

Proof. We have seen in §2 that, if $\bar{\pi}_{k,l}$ is a composition factor of $I_{\eta,\Lambda,\gamma}$, each K -type of it must contain the representation σ_γ . By (3.5) and (4.1), this is possible if and only if $(k,l) = (i+a, j+b)$ with $a = 0, \pm 1$ and $b = 0, \pm 1$. $\square$

Proposition 4.4. *Suppose that γ satisfies (4.1) and $(i', j') = (i+a, j+b)$, $a, b = 0, \pm 1$. Then the multiplicity of $\bar{\pi}_{i',j'}$ in $I_{\eta,\Lambda,\gamma}$ is at least one.*

Proof. By Theorem 3.4, $\bar{\pi}_{i',j'}$ is a composition factor of $\pi_{k,l} = X_P(\delta_{k,l}, \nu_{k,l})$, with $(k,l) = (i+a, j+b)$, $a, b = 0, -1$. Consider the principal series representation $X_P(\delta_{k,l}, \nu)$ with $\nu \in \mathfrak{a}^*$. If ν is generic, then $I_{\eta,(\mu_{k,l}+\rho_{\mathfrak{m}},\nu),\gamma}$ is isomorphic to $X_P(\delta_{k,l}, \nu)$ by Theorem 2.2. Here we used the fact that $m_{\delta_{k,l}}(\sigma_\gamma) = 1$. Choose a K -type τ of $\bar{\pi}_{i',j'}$. This is also a K -type of $X_P(\delta_{k,l}, \nu)$. By Frobenius reciprocity, the multiplicity of τ in $X_P(\delta_{k,l}, \nu)$ is one. Therefore, the space of moderately growing solutions of (2.8), with Λ replaced by $(\mu_{k,l}+\rho_{\mathfrak{m}}, \nu)$ and σ by σ_γ , is one-dimensional. Let f_ν be a non-zero moderately growing solution. Then by Theorem 2.3, this function is expressed by the Jacquet integral and it is holomorphic in ν . Suppose the order of zero of f_ν at $\nu = \nu_{k,l}$ is m . Let $g_\nu := f_\nu/(\nu - \nu_{k,l})^m$. Then $g_{\nu_{k,l}}$ is non-zero. It satisfies equation (2.8) (with σ replaced by σ_γ) and grows moderately at infinity, so it is an element of $I_{\eta,\Lambda,\gamma}$.

We have proved that, for every K -type τ of $\bar{\pi}_{i',j'}$, the multiplicity of τ in $I_{\eta,\Lambda,\gamma}$ is at least one. Since $\widehat{K}(\bar{\pi}_{a,b}) \cap \widehat{K}(\bar{\pi}_{a',b'}) = \emptyset$ if $(a,b) \neq (a',b')$, the multiplicity of $\bar{\pi}_{i',j'}$ in $I_{\eta,\Lambda,\gamma}$ is at least one. $\square$

5. DETERMINATION OF THE COMPOSITION SERIES

In this section, we determine the composition series of $I_{\eta,\Lambda,\gamma}$ in the case when Λ is integral. For this purpose, we need to write the actions of $Z(\mathfrak{g})$ and $\mathfrak{s}$ on this space explicitly. The former is achieved by the determinant type central element of $U(\mathfrak{gl}_{n+1})$, and the latter by the K -type shift operators.

5.1. Shift operators. We review the K -type shift operators briefly. Choose a K -type $(\tau_\lambda, V_\lambda^K)$ of $I_{\eta,\Lambda,\sigma}$, whose highest weight is λ . Let ψ be an element of $\text{Hom}_K(V_\lambda^K, I_{\eta,\Lambda,\sigma})$. For $v \in V_\lambda^K$ and $X \in \mathfrak{s}$, define

$$\tilde{\psi}(v \otimes X)(g) := L_X(\psi(v))(g).$$

Then it is easy to see that $\tilde{\psi}$ is an element of $\text{Hom}_K(V_\lambda^K \otimes \mathfrak{s}, I_{\eta,\Lambda,\sigma})$. Here, we regard $\mathfrak{s}$ as a representation of K by the adjoint action Ad . Denote by $\Delta_{\mathfrak{s}}$ the set of weights on $\mathfrak{s}$ with respect to a fixed Cartan subalgebra of $\mathfrak{k}$. In our case, the irreducible decomposition of $V_\lambda^K \otimes \mathfrak{s}$ is $\bigoplus_{\alpha \in \Delta_{\mathfrak{s}}} m(\alpha) V_{\lambda+\alpha}^K$, $m(\alpha) = 0$ or 1 . When $m(\alpha) = 1$, let ι_α be the embedding of $V_{\lambda+\alpha}^K$ into $V_\lambda^K \otimes \mathfrak{s}$. Define

$$\tilde{\psi}_\alpha(v_\alpha)(g) := \tilde{\psi}(\iota_\alpha(v_\alpha))(g), \quad v_\alpha \in V_{\lambda+\alpha}^K.$$

Then $\tilde{\psi}_\alpha$ is an element of $\text{Hom}_K(V_{\lambda+\alpha}^K, I_{\eta,\Lambda,\sigma})$, and the correspondence $\psi \mapsto \tilde{\psi}_\alpha$ is a K -type shift in $I_{\eta,\Lambda,\sigma}$ coming from the $\mathfrak{s}$ -action.

5.2. Gelfand-Tsetlin basis. In order to write the K -type shift operators explicitly, we realize the space $\text{Hom}_{M^\eta}(V_\tau^K, V_\sigma^{M^\eta})$ by using the Gelfand-Tsetlin basis ([3]).

Definition 5.1. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a dominant integral weight of $U(n)$. A (λ) -Gelfand-Tsetlin pattern is a set of vectors $Q = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ such that

- (1) $\mathbf{q}_i = (q_{1,i}, q_{2,i}, \dots, q_{i,i})$.
- (2) The numbers $q_{i,j}$ are all integers.
- (3) $q_{i,j+1} \geq q_{i,j} \geq q_{i+1,j+1}$, for any $i = 1, \dots, j$.
- (4) $q_{i,n} = \lambda_i$, $i = 1, \dots, n$.

The set of all λ -Gelfand-Tsetlin patterns is denoted by $GT(\lambda)$.

Theorem 5.2 ([3]). For a dominant integral weight λ of $U(n)$, let $(\tau_\lambda, V_\lambda^{U(n)})$ be the irreducible representation of $U(n)$ with the highest weight λ . Then $GT(\lambda)$ is identified with a basis of $(\tau_\lambda, V_\lambda^{U(n)})$.

The action of elements $E_{ij} \in \mathfrak{gl}(n, \mathbb{C})$ is expressed as follows. Let $l_{i,j} := q_{i,j} - i$ and $|\mathbf{q}_j| := \sum_{i=1}^j q_{i,j}$. Let $\sigma_{i,j}^\pm$ be the shift operators on $GT(\lambda)$, sending $\mathbf{q}_j$ to $\mathbf{q}_j + (0, \dots, \pm 1, 0, \dots, 0)$. Define $a_{i,j}(Q)$ and $b_{i,j}(Q)$ by

$$(5.1) \quad a_{i,j}(Q) = \left| \frac{\prod_{k=1}^{j+1} (l_{k,j+1} - l_{i,j}) \prod_{k=1}^{j-1} (l_{k,j-1} - l_{i,j} - 1)}{\prod_{\substack{k=1 \\ k \neq i}}^j (l_{k,j} - l_{i,j})(l_{k,j} - l_{i,j} - 1)} \right|^{1/2}, \quad b_{i,j}(Q) = a_{i,j}(\sigma_{i,j}^- Q).$$

Theorem 5.3 ([3]). For $Q \in GT(\lambda)$, the action of the Lie algebra is given by

$$\begin{aligned} \tau_\lambda(E_{j,j+1})Q &= \sum_{i=1}^j a_{i,j}(Q) \sigma_{i,j}^+ Q, & \tau_\lambda(E_{j+1,j})Q &= \sum_{i=1}^j b_{i,j}(Q) \sigma_{i,j}^- Q, \\ \tau_\lambda(E_{jj})Q &= (|\mathbf{q}_j| - |\mathbf{q}_{j-1}|)Q. \end{aligned}$$

Remark 5.4. The Gelfand-Tsetlin basis is compatible with the restriction to smaller unitary groups $U(k)$, $k = 1, \dots, n-1$. More precisely, the restriction of τ_λ to $U(n-1)$ is multiplicity free, and the highest weights of the irreducible representation appearing in $\tau_\lambda|_{U(n-1)}$ are the above $\mathbf{q}_{n-1}$'s.

Remark 5.5. The highest weight λ^* of the contragredient representation $(\tau_{\lambda^*}, V_{\lambda^*}^K)$ of $(\tau_\lambda, V_\lambda^K)$ is $\lambda^* = (-\lambda_n, \dots, -\lambda_1)$. In this case, $Q^* := (\mathbf{q}_1^*, \dots, \mathbf{q}_n^*) \in GT(\lambda^*)$ is dual to $Q \in GT(\lambda)$, where $\mathbf{q}_i^* := (-q_{i,i}, \dots, -q_{1,i})$.

5.3. Explicit formulas of shift operators. In §2, we identified an element $\phi_1 \in \text{Hom}_K(V_\lambda^K, I_{\eta, \Lambda, \gamma}^\circ)$ with a function $\phi_2 \in I_{\eta, \Lambda, \gamma}^\circ(\tau_\lambda)$ (for the definition of this space, see (2.1)). The space $\text{Hom}_{M^\eta}(V_\lambda^K, V_\gamma^{M^\eta})$ is isomorphic to $(V_{\lambda^*}^K \otimes V_\gamma^{M^\eta})^{M^\eta}$, the space of M^η -invariants in $V_{\lambda^*}^K \otimes V_\gamma^{M^\eta}$. By Remark 5.4, a basis of this space is identified with the “partial Gelfand-Tsetlin patterns”

$$(5.2) \quad GT((\lambda/\gamma)^*) := \{Q = (\mathbf{q}_{n-2}, \mathbf{q}_{n-1}, \mathbf{q}_n) \mid \mathbf{q}_{n-2} = \gamma^*, \mathbf{q}_n = \lambda^*; \mathbf{q}_{n-1} \text{ satisfies Definition 5.1 (1)–(3)}\}.$$

The correspondence is given by

$$(5.3) \quad \begin{aligned} GT((\lambda/\gamma)^*) \ni Q &\mapsto \langle \langle *, Q \rangle \rangle_\lambda \in \text{Hom}_{M^\eta}(V_\lambda^K, V_\gamma^{M^\eta}), \text{ where} \\ \langle \langle Q', Q \rangle \rangle_\lambda &= \begin{cases} 0 & \text{if } \mathbf{q}'_{n-1} \neq \mathbf{q}_{n-1}^* \text{ or } \mathbf{q}'_{n-2} \neq \gamma, \\ (\mathbf{q}'_1, \dots, \mathbf{q}'_{n-2}) \in GT(\gamma) & \text{if } \mathbf{q}'_{n-1} = \mathbf{q}_{n-1}^* \text{ and } \mathbf{q}'_{n-2} = \gamma, \end{cases} \\ \text{for } Q' &= (\mathbf{q}'_1, \dots, \mathbf{q}'_n) \in GT(\lambda). \end{aligned}$$

The action of $\mathfrak{k}$ on $GT((\lambda/\gamma)^*)$ is given by $\langle \langle Q', \tau_{\lambda^*}(\cdot)Q \rangle \rangle_\lambda = -\langle \langle \tau_\lambda(\cdot)Q', Q \rangle \rangle_\lambda$.

Let $V_{(\lambda/\gamma)^*}^{K/M^\eta}$ be the vector space spanned by $GT((\lambda/\gamma)^*)$. Then we can identify $\phi(a) \in C^\infty(A \rightarrow V_{(\lambda/\gamma)^*}^{K/M^\eta})$ with $\phi_2 \in I_{\eta, \Lambda, \gamma}^\circ(\tau_\lambda)$ via

$$\phi_2(a)(v) = \langle \langle v, \phi(a) \rangle \rangle_\lambda, \quad v \in V_\lambda^K, a \in A.$$

We introduce a coordinate system on A defined by

$$(5.4) \quad \mathbb{R}_{>0} \ni t \mapsto a_t := \exp(\log(\xi/t)h) \in A.$$

Here, ξ is the positive number appearing in (3.2). Then the action of $\mathfrak{g}$ on ϕ is given by

$$(5.5) \quad \begin{aligned} h \cdot \phi(a_t) &= \theta \phi(a_t), \quad \theta := t \frac{d}{dt}, \quad Y_{n-1} \cdot \phi(a_t) = \sqrt{-1}t \phi(a_t), \\ W \cdot \phi(a_t) &= 0 \quad \text{for other basis vectors } W = X_i, Y_i, Z \text{ of } \mathfrak{n}_0, \\ W \cdot \phi(a_t) &= -\tau_{\lambda^*}(W)\phi(a_t) \quad \text{for } W \in \mathfrak{k}. \end{aligned}$$

Here, we used the definition (3.2) of the non-degenerate character η .

Fix a non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_0$ and choose an orthonormal basis $\{W_i\}$ of $\mathfrak{s}_0$. Let pr_{α^*} be the natural projection from $V_{\lambda^*}^K \otimes \mathfrak{s} \simeq \bigoplus_{\alpha \in \Delta_{\mathfrak{s}}} m(\alpha) V_{(\lambda+\alpha)^*}^K$ to $V_{(\lambda+\alpha)^*}^K$. Then the K -type shift $\psi \mapsto \tilde{\psi}_\alpha$ which is explained in §5.1 is transferred to the following operator:

$$\begin{aligned} P_\alpha : C^\infty(A \rightarrow V_{(\lambda/\gamma)^*}^{K/M^\eta}) &\rightarrow C^\infty(A \rightarrow V_{(\lambda+\alpha/\gamma)^*}^{K/M^\eta}), \\ P_\alpha \phi(a_t) &:= \text{pr}_{\alpha^*} \circ \nabla \phi(a_t), \quad \nabla \phi(a_t) := \sum_i W_i \cdot \phi(a_t) \otimes W_i. \end{aligned}$$

Actually, if we write $\iota_\alpha(v_\alpha) = \sum_i v_\alpha^{(i)} \otimes W_i$ for $v_\alpha \in V_{\lambda+\alpha}^K$, then

$$\begin{aligned} \tilde{\psi}_\alpha(v_\alpha)(a_t) &= \tilde{\psi}(\iota_\alpha(v_\alpha))(a_t) = \tilde{\psi}(\sum_i v_\alpha^{(i)} \otimes W_i)(a_t) = \sum_i L_{W_i} \psi(v_\alpha^{(i)})(a_t) \\ &= \sum_i \langle \langle v_\alpha^{(i)}, W_i \cdot \phi(a_t) \rangle \rangle_\lambda = \langle \langle \sum_i v_\alpha^{(i)} \otimes W_i, \sum_j W_j \cdot \phi(a_t) \otimes W_j \rangle \rangle'_\lambda \\ &= \langle \langle \iota_\alpha(v_\alpha), \nabla \phi(a_t) \rangle \rangle'_\lambda \\ &= \langle \langle v_\alpha, P_\alpha \phi(a_t) \rangle \rangle_{\lambda+\alpha}. \end{aligned}$$

Here, $\langle \langle Q' \otimes W_i, Q \otimes W_j \rangle \rangle'_\lambda := \langle \langle Q', Q \rangle \rangle_\lambda \times \langle W_i, W_j \rangle$.

In our $G = U(n, 1)$ case, $\Delta_{\mathfrak{s}} = \{\pm(\epsilon_k - \epsilon_{n+1}) \mid k = 1, \dots, n\}$ and $(\lambda \pm (\epsilon_{n+1-k} - \epsilon_{n+1}))^* = \lambda^* \mp (\epsilon_k - \epsilon_{n+1})$. We write $P_k^\pm$ instead of $P_{\pm(\epsilon_{n+1-k} - \epsilon_{n+1})}$, for simplicity. These operators are calculated in [13]. (The expression is slightly different because of the change of notation and setting.)

Proposition 5.6. Suppose $\phi(a_t) = \sum_{Q \in GT((\lambda/\gamma)^*)} c(Q; t) Q \in C^\infty(A \rightarrow V_{(\lambda/\gamma)^*}^{K/M^n})$. Then the K -type shift operators $P_k^\pm$, $k = 1, \dots, n$ are given by the following formulas:

$$(5.6) \quad P_k^+ \phi(a_t) = \sum_{Q \in GT((\lambda/\gamma)^*)} b_{k,n}(Q)(\theta - |\Lambda| - |\mathbf{q}_{n-1}| - 2l_{k,n} - 2n) c(Q; t) \sigma_{k,n}^- Q \\ - t \sum_{i=1}^{n-1} \sum_{\sigma_{i,n-1}^+ Q \in GT((\lambda/\gamma)^*)} \frac{b_{k,n}(Q) a_{i,n-1}(Q)}{l_{k,n} - l_{i,n-1}} c(\sigma_{i,n-1}^+ Q; t) \sigma_{k,n}^- Q,$$

$$(5.7) \quad P_k^- \phi(a_t) = \sum_{Q \in GT((\lambda/\gamma)^*)} a_{k,n}(Q)(\theta + |\Lambda| + |\mathbf{q}_{n-1}| + 2l_{k,n} + 2) c(Q; t) \sigma_{k,n}^+ Q \\ + t \sum_{i=1}^{n-1} \sum_{\sigma_{i,n-1}^- Q \in GT((\lambda/\gamma)^*)} \frac{a_{k,n}(Q) b_{i,n-1}(Q)}{l_{k,n} - l_{i,n-1} + 1} c(\sigma_{i,n-1}^- Q; t) \sigma_{k,n}^+ Q.$$

In order to state the next lemma, let $q_{0,n-2} := \infty$ and $q_{n-1,n-2} := -\infty$.

Lemma 5.7. Let ϕ be an element of $C^\infty(A \rightarrow V_{(\lambda/\gamma)^*}^{K/M^n})$.

- (1) If $\gamma_{k-1}^* < \lambda_k^*$ and $P_k^+ \phi = 0$ for $k \in \{2, \dots, n\}$, then $\phi = 0$.
- (2) If $\gamma_{k-1}^* > \lambda_k^*$ and $P_k^- \phi = 0$ for $k \in \{1, \dots, n-1\}$, then $\phi = 0$.

Proof. For any number $*$ depending on $Q \in GT((\lambda/\gamma)^*)$, we denote it by $*(Q)$ if we need to specify Q . For example, $q_{i,j}(Q)$ is the $q_{i,j}$ part of $Q \in GT((\lambda/\gamma)^*)$.

Since the proofs of these two are analogous, we show (2) only.

We will show $c(Q; t) = 0$ by induction on $\lambda_k^* - q_{k,n-1}(Q)$.

Let Q_0 be an element of $GT((\lambda/\gamma)^*)$ which satisfies $q_{k,n-1}(Q_0) = \lambda_k^*$, and let $Q_1 := \sigma_{k,n-1}^+ Q_0$. Then Q_1 is not in $GT((\lambda/\gamma)^*)$, but $\sigma_{k,n-1}^- Q_1 = Q_0 \in GT((\lambda/\gamma)^*)$ and $\sigma_{k,n}^+ Q_1 \in GT(\lambda^* + (\epsilon_k - \epsilon_{n+1}))$, because $\gamma_{k-1}^* > \lambda_k^*$ implies that $\sigma_{k,n}^+ Q_1$ satisfies the conditions in Definition 5.1 (3): $q_{k-1,n-1}(Q_1) \geq \gamma_{k-1}^* \geq \lambda_k^* + 1 = q_{k,n}(\sigma_{k,n}^+ Q_1) = q_{k,n-1}(\sigma_{k,n}^+ Q_1)$. Therefore, the term $\sigma_{k,n}^+ Q_1$ appears in (5.6), and its coefficient in (5.6) is

$$\frac{a_{k,n}(Q_1) b_{k,n-1}(Q_1)}{l_{k,n}(Q_1) - l_{k,n-1}(Q_1) + 1} c(\sigma_{k,n-1}^- Q_1; t) = \frac{a_{k,n}(Q_0) b_{k,n-1}(\sigma_{k,n}^+ Q_1)}{l_{k,n}(Q_1) - l_{k,n-1}(Q_1) + 2} c(Q_0; t).$$

Here, we used the definition (5.1) of $a_{i,j}(Q)$ and $b_{i,j}(Q)$. Since the coefficient of the right-hand side is not zero, $c(Q_0; t) = 0$ when $P_k^- \phi = 0$. We have shown that, if $\lambda_k^* - q_{k,n-1}(Q) = 0$, $c(Q; t)$ is zero.

Assume that $c(Q; t) = 0$ is proved for those Q 's which satisfy $\lambda_k^* - q_{k,n-1}(Q) = p$. Let Q_2 be an element of $GT((\lambda/\gamma)^*)$ which satisfies $\lambda_k^* - q_{k,n-1}(Q_2) = p + 1$. Set $Q_3 := \sigma_{k,n-1}^+ Q_2$. This Q_3 is an element of $GT((\lambda/\gamma)^*)$ and it satisfies $\lambda_k^* - q_{k,n-1}(Q_3) = p$ and $\lambda_k^* - q_{k,n-1}(\sigma_{i,n-1}^- Q_3) = p$ if $i \neq k$. Then by the hypothesis of induction, $c(Q_3; t) = 0$ and $c(\sigma_{i,n-1}^- Q_3; t) = 0$ for $i \neq k$. Consider the right-hand side of (5.6) for $Q = Q_3$. The terms other than $c(Q_2; t) = c(\sigma_{k,n-1}^- Q_3; t)$ are zero. We can easily see that its coefficient $a_{k,n}(Q_3) b_{k,n-1}(Q_3) / (l_{k,n}(Q_3) - l_{k,n-1}(Q_3) + 1)$ is non-zero. Therefore, if $P_k^- \phi(a_t) = 0$, then $c(Q_2; t) = 0$. This completes the proof. $\square$

5.4. Central elements of $U(\mathfrak{gl}_{n+1})$. In order to show Lemma 5.11 below, we use the explicit forms of the elements in $Z(\mathfrak{gl}_{n+1})$. One of the most useful forms of the central elements of $U(\mathfrak{gl}_{n+1})$ is the determinant type one ([1]). For the standard generator E_{ij} of $\mathfrak{gl}_{n+1}$ and a parameter $u \in \mathbb{C}$, let $E_{ij}(u) := E_{ij} + u\delta_{ij}$ (Kronecker's delta). We define

$$C_{n+1}(u) := \sum_{\sigma \in \mathfrak{S}_{n+1}} \text{sgn}(\sigma) E_{\sigma(n+1), n+1}(u+n) E_{\sigma(n), n}(u+n-1) \cdots E_{\sigma(1), 1}(u).$$

Then $C_{n+1}(u)$ is an element of $Z(\mathfrak{gl}_{n+1})$ for any u , and we obtain all the generators of $Z(\mathfrak{gl}_{n+1})$ by specializing u . Since $C_{n+1}(u) \equiv \prod_{p=1}^{n+1} (E_{pp} + u + p - 1)$ modulo the left ideal generated by strictly lower triangular matrices, the infinitesimal character is $\chi_\Lambda(C_{n+1}(u)) = \prod_{p=1}^{n+1} (\Lambda_p + (\rho_{\mathfrak{g}})_p + u + p - 1) = \prod_{p=1}^{n+1} (u + \Lambda_p + n/2)$. Here, $\rho_{\mathfrak{g}} = \sum_{p=1}^{n+1} (\rho_{\mathfrak{g}})_p \epsilon_p := \frac{1}{2} \sum_{p=1}^{n+1} (n+2-2p) \epsilon_p$.

Lemma 5.8. $C_{n+1}(u)$ acts on $I_{\eta, \Lambda, \gamma}$ by the scalar $\prod_{p=1}^{n+1} (u + \Lambda_p + n/2)$.

The exterior calculus is very useful for the manipulation of non-commutative determinants. We use the method developed in [4].

The exterior algebra $\wedge \mathbb{C}^{2(n+1)}$ is an associative algebra generated by $2(n+1)$ elements $e_1, \dots, e_{n+1}, e'_1, \dots, e'_{n+1}$ subject to the relations $e_i e_j + e_j e_i = 0$, $e'_i e'_j + e'_j e'_i = 0$ and $e'_i e'_j + e'_j e'_i = 0$. We will work in the algebra $\wedge \mathbb{C}^{2(n+1)} \otimes U(\mathfrak{gl}_{n+1})$, where the subalgebras $\wedge \mathbb{C}^{2(n+1)}$ and $U(\mathfrak{gl}_{n+1})$ commute with each other. Consider the following elements:

$$\begin{aligned} \eta_j(u) &= \sum_{p=1}^{n+1} e_p E_{pj}(u), & \eta'_i(u) &= \sum_{q=1}^{n+1} e'_q E_{iq}(u), \\ \Xi(u) &= \sum_{p,q=1}^{n+1} e_p e'_q E_{pq}(u) = \sum_{j=1}^{n+1} \eta_j(u) e'_j = \sum_{i=1}^{n+1} e_i \eta'_i(u). \end{aligned}$$

Lemma 5.9 ([4]). (1) For $i, j = 1, \dots, n+1$,

$$\eta_i(u+1)\eta_j(u) + \eta_j(u+1)\eta_i(u) = 0, \quad \eta'_i(u)\eta'_j(u+1) + \eta'_j(u)\eta'_i(u+1) = 0,$$

$$(2) \text{ For } i, j = 1, \dots, n+1, [E_{ij}, \Xi(u)] = e_j \eta'_i(u) - \eta_j(u) e'_i.$$

$$(3) \text{ For any } u, v \in \mathbb{C}, \Xi(u) \text{ and } \Xi(v) \text{ are commutative.}$$

For $k \geq 0$, we consider the element

$$\Xi^{(k)}(u) = \Xi(u) \Xi(u-1) \cdots \Xi(u-k+1) = \Xi(u-k+1) \Xi(u-k+2) \cdots \Xi(u).$$

By Lemma 5.9, it is not hard to see that

$$(5.8) \quad \begin{aligned} \Xi^{(k)}(u) &= k \eta_j(u) e'_j \Xi^{(k-1)}(u-1) \\ &\quad + (\Xi(u) - \eta_j(u) e'_j) \cdots (\Xi(u-k+1) - \eta_j(u-k+1) e'_j) \end{aligned}$$

$$(5.9) \quad \begin{aligned} &= k e_i \eta'_i(u-k+1) \Xi^{(k-1)}(u) \\ &\quad + (\Xi(u-k+1) - e_i \eta'_i(u-k+1)) \cdots (\Xi(u) - e_i \eta'_i(u)), \end{aligned}$$

$$(5.10) \quad 0 = \eta_i(u+n) e'_j \Xi^{(n)}(u+n-1) \quad \text{if } i \neq j,$$

$$(5.11) \quad 0 = e_i \eta'_j(u) e_k e'_{n+1} \Xi^{(n-1)}(u+n-1) \quad \text{if } j \neq i, k.$$

From (5.9), we obtain

$$(5.12) \quad e_j e'_{n+1} \Xi^{(n)}(u+n-1) = n e_j e_i \eta'_i(u) e'_{n+1} \Xi^{(n-1)}(u+n-1).$$

By Proposition 2.2 in [4],

$$(5.13) \quad \Xi^{(n+1)}(u+n) = (n+1)! C_{n+1}(u) \wedge^{\text{top}},$$

where $\wedge^{\text{top}} := e_1 e'_1 e_2 e'_2 \cdots e_{n+1} e'_{n+1}$. We need the cofactor expansion of $C_{n+1}(u)$ along the $(n+1)$ -st row and column.

By (3.1), we have

$$(5.14) \quad \eta_{n+1}(u+n) - \eta_n(u+n) = \sum_{p=1}^{n-1} \frac{1}{2} (-X_p + \sqrt{-1} Y_p) e_p + \frac{\sqrt{-1}}{2} Z(e_n + e_{n+1})$$

$$+ \frac{1}{2} (h - E_{nn} - E_{n+1,n+1} - 2u - 2n)(e_n - e_{n+1}),$$

$$(5.15) \quad \eta'_{n+1}(u) + \eta'_n(u) = \sum_{p=1}^{n-1} \frac{1}{2} (-X_p - \sqrt{-1} Y_p) e'_p + \frac{\sqrt{-1}}{2} Z(-e_n + e_{n+1})$$

$$+ \frac{1}{2} (h + E_{nn} + E_{n+1,n+1} + 2u)(e'_n + e'_{n+1}).$$

For two elements $x, y \in U(\mathfrak{g})$, $x \equiv y$ means that they are equivalent modulo the right ideal generated by $X - \eta_t(X)$, $X \in \mathfrak{n}_0$. Since the actions of elements in $\mathfrak{g}_0$ on $C^\infty(A \rightarrow V_{(\lambda/\gamma)^*}^{K/M^\eta})$ are given by (5.5), we have

$$\begin{aligned} & \frac{4}{(n+1)n} \Xi^{(n+1)}(u+n) \\ &= \frac{4}{n} (\eta_{n+1}(u+n) - \eta_n(u+n)) e'_{n+1} \Xi^{(n)}(u+n-1) \\ &\equiv \frac{2}{n} \{-te_{n-1} + (h - E_{nn} - E_{n+1,n+1} - 2u - 2n)(e_n - e_{n+1})\} e'_{n+1} \Xi^{(n)}(u+n-1) \\ &= 2\{te_{n-1} - (h - E_{nn} - E_{n+1,n+1} - 2u - 2n)e_n\} \\ &\quad \times (\eta'_{n+1}(u) + \eta'_n(u)) e_{n+1} e'_{n+1} \Xi^{(n-1)}(u+n-1) \\ &\equiv \{t^2 e_{n-1} e'_{n-1} + t(h + E_{nn} + E_{n+1,n+1} + 2u) e_{n-1} e'_n \\ &\quad - t(h - E_{nn} - E_{n+1,n+1} - 2u - 2n) e_n e'_{n-1} \\ &\quad - (h - E_{nn} - E_{n+1,n+1} - 2u - 2n)(h + E_{nn} + E_{n+1,n+1} + 2u) e_n e'_n\} \\ &\quad \times e_{n+1} e'_{n+1} \Xi^{(n-1)}(u+n-1). \end{aligned}$$

Here, we used (5.8), (5.10) for the first equality, (5.14) for the second equivalence, (5.11), (5.12) for the third equality, and (5.15) for the last equivalence.

By (5.13), we have

$$e_n e'_n e_{n+1} e'_{n+1} \Xi^{(n-1)}(u+n-1) = (n-1)! C_{n-1}(u+1) \wedge^{\text{top}}.$$

From this equation, we see that

$$\begin{aligned} e_n e'_{n-1} e_{n+1} e'_{n+1} \Xi^{(n-1)}(u+n-1) &= (n-1)! \operatorname{ad}(E_{n-1,n}) C_{n-1}(u+1) \wedge^{\operatorname{top}}, \\ e_{n-1} e'_n e_{n+1} e'_{n+1} \Xi^{(n-1)}(u+n-1) &= -(n-1)! \operatorname{ad}(E_{n,n-1}) C_{n-1}(u+1) \wedge^{\operatorname{top}}, \\ e_{n-1} e'_{n-1} e_{n+1} e'_{n+1} \Xi^{(n-1)}(u+n-1) \\ &= (n-1)! \{1 - \operatorname{ad}(E_{n-1,n}) \operatorname{ad}(E_{n,n-1})\} C_{n-1}(u+1) \wedge^{\operatorname{top}}. \end{aligned}$$

We shall write the action of $C_{n+1}(u)$ on $\phi(a_t) = \sum_{Q \in GT((\lambda/\gamma)^*)} c(Q; t) Q \in C^\infty(A \rightarrow \operatorname{Hom}_{M^\eta}(V_\lambda^K, V_\gamma^{M^\eta}))$.

Let $\bar{\mathfrak{u}}_{\mathfrak{m}}$ be the nilpotent subalgebra of $\mathfrak{m}$ consisting of upper triangular matrices. Since the $\mathbf{q}_{n-1}$ part of $Q = (\mathbf{q}_{n-2}, \mathbf{q}_{n-1}, \mathbf{q}_n) \in GT((\lambda/\gamma)^*)$ is a highest weight of $V_{\lambda^*}^K|_{U(n-1)}$, an element $z \in Z(\mathfrak{m})$ acts on Q by $\tau_{\lambda^*}({}^t z)Q = \gamma'_2(z)(-\mathbf{q}_{n-1})Q$. (γ'_2 is the non-shifted Harish-Chandra map defined in the proof of Theorem 2.2.) Here, we used (5.5) and defined ${}^t(Z_1 \cdots Z_l) := (-Z_l) \cdots (-Z_1)$ for $Z_p \in \mathfrak{g}$. Since $\gamma'_2(C_{n-1}(u+1)) = E_{n-1,n-1}(u+n-1) \cdots E_{11}(u+1)$, it acts on Q by the scalar $\prod_{p=1}^{n-1} (u+p-q_{p,n-1}) = \prod_{p=1}^{n-1} (u-l_{p,n-1}) =: S(Q)$.

For $z \in Z(\mathfrak{gl}_{n+1})$, let $\mathcal{D}_{\lambda,\gamma}(z)$ be the differential operator on $C^\infty(A \rightarrow V_{(\lambda/\gamma)^*}^{K/M^\eta})$ defined by $z \cdot \phi = \mathcal{D}_{\lambda,\gamma}(z)\phi$. Bringing the above results together, we get the following formula.

Proposition 5.10. *The action of $C_{n+1}(u)$ on $\phi(a_t) = \sum_{Q \in GT((\lambda/\gamma)^*)} c(Q; t) Q \in C^\infty(A \rightarrow V_{(\lambda/\gamma)^*}^{K/M^\eta})$ is expressed as follows:*

$$\begin{aligned} &-4\mathcal{D}_{\lambda,\gamma}(C_{n+1}(u))\phi(a_t) \\ &= \sum_{Q \in GT((\lambda/\gamma)^*)} S(Q) \left[\{(\theta-n)^2 - (|\Lambda| + |\mathbf{q}_{n-1}| + 2u+n)^2 - A(Q)t^2\} c(Q; t) \right. \\ (5.16) \quad &\quad - t \sum_{p=1}^{n-1} \frac{a_{p,n-1}(Q)}{u-l_{p,n-1}} (\theta + |\Lambda| + |\mathbf{q}_{n-1}| + 1 + 2u) c(\sigma_{p,n-1}^+ Q; t) \\ &\quad + t \sum_{p=1}^{n-1} \frac{b_{p,n-1}(Q)}{u-l_{p,n-1}} (\theta - |\Lambda| - |\mathbf{q}_{n-1}| + 1 - 2n - 2u) c(\sigma_{p,n-1}^- Q; t) \\ &\quad \left. - t^2 \sum_{\substack{p,r=1 \\ p \neq r}}^{n-1} \frac{b_{p,n-1}(Q) a_{r,n-1}(\sigma_{p,n-1}^- Q)}{(u-l_{p,n-1})(u-l_{r,n-1})} c(\sigma_{p,n-1}^- \sigma_{r,n-1}^+ Q; t) \right] Q, \\ &A(Q) := 1 - \sum_{p=1}^{n-1} \frac{a_{p,n-1}(Q)^2 - b_{p,n-1}(Q)^2}{u-l_{p,n-1}}. \end{aligned}$$

Lemma 5.11. *Let τ_λ be a K -type of $I_{\eta,\Lambda,\gamma}$. On the space $C^\infty(A \rightarrow V_{(\lambda/\gamma)^*}^{K/M^\eta})$, the operators $P_k^- \circ P_k^+$ and $P_k^+ \circ P_k^-$, $k \in \{1, \dots, n\}$, are central, namely*

$$(5.17) \quad P_k^- \circ P_k^+ = \mathcal{D}_{\lambda,\gamma}(C_{n+1}(l_{k,n})), \quad P_k^+ \circ P_k^- = \mathcal{D}_{\lambda,\gamma}(C_{n+1}(l_{k,n} + 1)).$$

Proof. By Propositions 5.6, 5.10, we know that we may show the identity

$$1 = \sum_{p=1}^{n-1} \frac{a_{p,n-1}(Q)^2}{l_{k,n} - l_{p,n-1}} - \sum_{p=1}^{n-1} \frac{b_{p,n-1}(Q)^2}{l_{k,n} - l_{p,n-1} + 1}.$$

This identity is obtained by comparing the coefficients of $\sigma_{k,n}^+ Q$ in the identity $\tau_{\lambda^*}(E_{n,n+1})Q = \tau_{\lambda^*}([E_{n,n-1}, [E_{n-1,n}, E_{n,n+1}]]Q$. $\square$

5.5. Determination of composition series. In this subsection, we determine the composition series of $I_{\eta,\Lambda,\gamma}$. When something concerning the irreducible modules $\pi_{a,b}$ is described, the statement concerning them is assumed to be excluded if there is no such module $\pi_{a,b}$, i.e. if $a + b > n + 1$.

Lemma 5.12. *Suppose that γ is given by (4.1). If a pair V_1 and V_2 of composition factors in $I_{\eta,\Lambda,\gamma}$ satisfies one of the following conditions, then there is no non-zero $\mathfrak{g}$ -action in $I_{\eta,\Lambda,\gamma}$ which sends V_1 to V_2 :*

- (1) $V_1 \simeq \pi_{i,j}$, $V_2 \simeq \pi_{i+a,j}$, $\pi_{i,j+b}$, $a, b = \pm 1$,
- (2) $V_1 \simeq \pi_{i+a,j}$, $V_2 \simeq \pi_{i+a,j+b}$, $a, b = \pm 1$,
- (3) $V_1 \simeq \pi_{i,j+b}$, $V_2 \simeq \pi_{i+a,j+b}$, $a, b = \pm 1$.

(Any double signs are allowed.)

Proof. If there is a $\mathfrak{g}$ -action sending an element of $\pi_{a,b}$ to $\pi_{a',b'}$, then the K -spectra $\widehat{K}(\pi_{a,b})$ and $\widehat{K}(\pi_{a',b'})$ should be adjacent; i.e., there should be K -types $\tau_{\lambda} \in \widehat{K}(\pi_{a,b})$ and $\tau_{\lambda'} \in \widehat{K}(\pi_{a',b'})$ such that $\lambda - \lambda'$ is a weight of $\mathfrak{s}$.

Assume that V_1 and V_2 are isomorphic to $\pi_{i,j}$ and $\pi_{i-1,j}$, respectively, and $V_1 \rightarrow V_2$. Two K -types $\lambda \in \widehat{K}(\pi_{i,j})$ and $\lambda' \in \widehat{K}(\pi_{i-1,j})$ are adjacent if and only if $\lambda_i = \Lambda_i - n/2 + i$, $\lambda'_i = \Lambda_i - n/2 + i - 1$ and $\lambda_p = \lambda'_p$ for $p \neq i$. Recall the discussion in §5.3. The $\mathfrak{s}$ -action which sends an element of $V_{\lambda}^K \subset V_1$ to $V_{\lambda'}^K \subset V_2$ is realized by the shift operator $P_{n+1-i}^- \phi$.

Consider the shift $P_{n+1-i}^+ \circ P_{n+1-i}^- \phi$. Lemma 5.11 asserts that this is equal to $\mathcal{D}_{\lambda,\gamma}(C_{n+1}(l_{n+1-i,n} + 1))\phi = \chi_{\Lambda}(C_{n+1}(l_{n+1-i,n} + 1))\phi$. By Lemma 5.8, we know that $\chi_{\Lambda}(C_{n+1}(l_{n+1-i,n} + 1)) = 0$, since $l_{n+1-i,n} = \lambda_{n+1-i}^* - (n+1-i) = -\lambda_i - (n+1-i) = -(\Lambda_i - n/2 + i) - (n+1-i) = -\Lambda_i - n/2 - 1$. Therefore, $P_{n+1-i}^+ P_{n+1-i}^- \phi = 0$. Now, $\lambda'_i = \Lambda_i - n/2 + i - 1 < \gamma_{i-1}$ implies $(\lambda')_{n+1-i}^* > \gamma_{n-i}^*$, and the condition $1 \leq i \leq n-1$ of Proposition 4.2 is paraphrased as $2 \leq n+1-i \leq n$. Under these conditions, P_{n+1-i}^+ is injective by Lemma 5.7, so $P_{n+1-i}^- \phi = 0$. But this contradicts $V_1 \rightarrow V_2$. Therefore, $V_1 \not\rightarrow V_2$. Other cases can be shown analogously. $\square$

Corollary 5.13. *The multiplicity of $\pi_{i,j}$ in $I_{\eta,\Lambda,\gamma}$ is one.*

Proof. We know that the socle of $I_{\eta,\Lambda,\gamma}$ is isomorphic to $\pi_{i,j}$ (Proposition 4.3). Assume that there exists a composition factor V_1 which is isomorphic to $\pi_{i,j}$ but is not in the socle. By Proposition 4.3, $\pi_{a,b}$ is a composition factor of $I_{\eta,\Lambda,\gamma}$ only if $a = i, i \pm 1$ and $b = j, j \pm 1$. By Theorem 3.2, it is adjacent to V_1 if and only if $|a - a'| + |b - b'| = 1$. Therefore, there exists a composition factor V_2 which is isomorphic to one of $\pi_{i \pm 1, j}$, $\pi_{i, j \pm 1}$ such that $V_1 \rightarrow V_2$. But we have shown in Lemma 5.12 that this is impossible. $\square$

Lemma 5.14. *Suppose that γ is given by (4.1).*

- (1) *The socle of $I_{\eta,\Lambda,\gamma}/\pi_{i,j}$ is $\pi_{i-1,j} \oplus \pi_{i,j+1} \oplus \pi_{i,j-1} \oplus \pi_{i+1,j}$.*
- (2) *The multiplicities of $\pi_{i \pm 1, j}$, $\pi_{i, j \pm 1}$ in $I_{\eta,\Lambda,\gamma}$ are all one.*

Proof. (1) As we stated in the proof of the previous corollary, a composition factor V is adjacent to $\bar{\pi}_{i,j}$ only if V is isomorphic to one of $\bar{\pi}_{i\pm 1,j}$, $\bar{\pi}_{i,j\pm 1}$. So only $\bar{\pi}_{i\pm 1,j}$, $\bar{\pi}_{i,j\pm 1}$ can be a simple submodule of $I_{\eta,\Lambda,\gamma}/\bar{\pi}_{i,j}$. We know from Proposition 4.4 that the multiplicity of each of them in $I_{\eta,\Lambda,\gamma}$ is at least one.

Choose a composition factor V isomorphic to, say, $\bar{\pi}_{i-1,j}$. Other cases can be shown analogously. Recall the proof of Lemma 5.12. Let ϕ be the function which characterizes the non-zero K -type λ' of $V \simeq \bar{\pi}_{i-1,j}$. Assume that there is no non-zero $\mathfrak{s}$ -action from V to the unique simple submodule which is isomorphic to $\bar{\pi}_{i,j}$. Since the multiplicity of $\bar{\pi}_{i,j}$ in $I_{\eta,\Lambda,\gamma}$ is one, $P_{n+1-i}^+\phi = 0$. But we have seen in the proof of Lemma 5.12 that this implies $\phi = 0$. This is a contradiction, so $V \rightarrow \bar{\pi}_{i,j}$.

Assume that there are two composition factors V_1, V_2 in the socle of $I_{\eta,\Lambda,\gamma}/\bar{\pi}_{i,j}$, both of which are isomorphic to $\bar{\pi}_{i-1,j}$. Let ϕ_k , $k = 1, 2$, be the functions which characterize the non-zero K -type λ' of V_k , respectively. Then both of $P_{n+1-i}^+\phi_k$ characterize the same K -type λ of the unique simple submodule, and the multiplicity of this K -type is one, so there are constants c_k such that $c_1 P_{n+1-i}^+\phi_1 = c_2 P_{n+1-i}^+\phi_2$. Since P_{n+1-i}^+ is injective in this case, this implies that $c_1\phi_1 + c_2\phi_2 = 0$. This contradicts the fact that ϕ_k , $k = 1, 2$, characterize the K -types λ' of different factors V_k , $k = 1, 2$. Therefore, the multiplicity of $\bar{\pi}_{i-1,j}$ in the socle of $I_{\eta,\Lambda,\gamma}/\bar{\pi}_{i,j}$ is one.

(2) Assume that there is a composition factor V_1 isomorphic to, say, $\bar{\pi}_{i-1,j}$ but not in the second floor of $I_{\eta,\Lambda,\gamma}$. The irreducible modules which are adjacent to $\bar{\pi}_{i-1,j}$ are $\bar{\pi}_{i,j}$ and $\bar{\pi}_{i-1,j\pm 1}$. Therefore, there exists a composition factor V_2 in the third or higher floor such that it is isomorphic to one of the above and $V_1 \rightarrow V_2$. But this is impossible since (i) the multiplicity of $\bar{\pi}_{i,j}$ is one and it is located in the bottom, and (ii) $\bar{\pi}_{i-1,j} \not\rightarrow \bar{\pi}_{i-1,j\pm 1}$ by Lemma 5.12(2). $\square$

Lemma 5.15. *Suppose that γ is given by (4.1).*

- (1) *The socle of $(I_{\eta,\Lambda,\gamma}/\bar{\pi}_{i,j})/(\bar{\pi}_{i-1,j} \oplus \bar{\pi}_{i,j+1} \oplus \bar{\pi}_{i,j-1} \oplus \bar{\pi}_{i+1,j}/\bar{\pi}_{i,j})$ is $\bar{\pi}_{i-1,j-1} \oplus \bar{\pi}_{i-1,j+1} \oplus \bar{\pi}_{i+1,j-1} \oplus \bar{\pi}_{i+1,j+1}$. Moreover, the non-zero $\mathfrak{s}$ -actions from the third floor to the second are $\bar{\pi}_{a,j\pm 1} \rightarrow \bar{\pi}_{a,j}$ and $\bar{\pi}_{i\pm 1,b} \rightarrow \bar{\pi}_{i,b}$, $a = i \pm 1$, $b = j \pm 1$.*
- (2) *The multiplicities of $\bar{\pi}_{i\pm 1,j\pm 1}$ in $I_{\eta,\Lambda,\gamma}$ are all one.*

Proof. The proof is almost the same as that of Lemma 5.14.

Since (i) the multiplicities of $\bar{\pi}_{i,j\pm 1}$, $\bar{\pi}_{i\pm 1,j}$ and $\bar{\pi}_{i,j}$ are one, and (ii) they are in the first or second floor, the third floor is a direct sum of $\bar{\pi}_{i\pm 1,j\pm 1}$'s, and there is no higher floor in $I_{\eta,\Lambda,\gamma}$.

For each $\bar{\pi}_{i\pm 1,j\pm 1}$, there exists at least one factor isomorphic to it in $I_{\eta,\Lambda,\gamma}$. Suppose, say, $V_1 \simeq \bar{\pi}_{i-1,j+1}$ and $V_2 \simeq \bar{\pi}_{i,j+1}$, where the latter is in the second floor. Let ϕ be the function which characterizes a K -type of V_1 adjacent to $V_2 \simeq \bar{\pi}_{i,j+1}$. The shift operator sending ϕ to a K -type of V_2 is P_{n+1-i}^+ . The proof of Lemma 5.12 says that this is injective. Therefore, $V_1 \rightarrow V_2$. The uniqueness of the factor isomorphic to $\bar{\pi}_{i-1,j+1}$ is shown in the same way as in the proof of the previous lemma. $\square$

We have obtained the following second main theorem of this paper.

Theorem 5.16. *Suppose that $G = U(n, 1)$ and the infinitesimal character Λ is regular integral. If the highest weight of $\sigma \in \widehat{M}^n \simeq \widehat{U(n-2)} \times \widehat{U(1)}$ satisfies (4.1)*

for some $i = 1, \dots, n-1$, $j = 2, \dots, n+1-i$, then the composition series of $I_{\eta, \Lambda, \gamma}$ is

$$(5.18) \quad I_{\eta, \Lambda, \gamma} \simeq \begin{array}{ccccccc} & & \bar{\pi}_{i-1, j+1} & \bar{\pi}_{i-1, j-1} & \bar{\pi}_{i+1, j+1} & \bar{\pi}_{i+1, j-1} & \\ & & \downarrow & \searrow & \swarrow & \downarrow & \\ & \bar{\pi}_{i-1, j} & & \bar{\pi}_{i, j+1} & & \bar{\pi}_{i, j-1} & \bar{\pi}_{i+1, j} \\ & & \searrow & \swarrow & \swarrow & \searrow & \\ & & & \pi_{i, j} & & & \end{array}$$

Here, if $i+j = n$ or $n+1$, the modules $\bar{\pi}_{a,b}$, $a+b > n+1$, are regarded to be zero and the arrows starting from or ending at such modules are omitted.

6. ENDING REMARK

In this paper, we characterized the module $I_{\eta, \Lambda, \sigma}$ by the conditions (1)–(3) in §1. Condition (1) is imposed to make the modules $I_{\eta, \Lambda}^\circ$, $I_{\eta, \Lambda, \sigma}^\circ$ and $I_{\eta, \Lambda, \sigma}$ suitably small, i.e., K -admissible. The author thinks that it is interesting to investigate the structure of modules which are characterized by other conditions that make the modules in question K -admissible. For example, if the real rank of G is one, we may replace the condition “ f is a joint eigenfunction of $Z(\mathfrak{g})$ ” in (1) with “ f is an eigenfunction of the Casimir operator and admits a generalized infinitesimal character”. Under the latter condition, the module is still K -admissible and has finite length. For $G = U(n, 1)$, the composition series of such a module (with the trivial generalized infinitesimal character) is

$$\begin{array}{ccccccc} & & & \bar{\pi}_{i, j} & & & \\ & & \swarrow & \downarrow & \searrow & & \\ \bar{\pi}_{i-1, j} & \bar{\pi}_{i, j+1} & & \bar{\pi}_{i, j-1} & & \bar{\pi}_{i+1, j} & \\ \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \searrow \\ \bar{\pi}_{i-1, j+1} & \bar{\pi}_{i-1, j-1} & \bar{\pi}_{i, j} \oplus \bar{\pi}_{i, j} & \bar{\pi}_{i+1, j+1} & \bar{\pi}_{i+1, j-1} & & \\ \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \searrow \\ \bar{\pi}_{i-1, j} & \bar{\pi}_{i, j+1} & & \bar{\pi}_{i, j-1} & & \bar{\pi}_{i+1, j} & \\ \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \searrow \\ & & \pi_{i, j} & & & & \end{array}$$

This structure is more symmetric than that of $I_{\eta, \Lambda, \sigma}$.

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