ON THE IRREDUCIBILITY OF THE DIRICHLET POLYNOMIAL OF AN ALTERNATING GROUP

MASSIMILIANO PATASSINI

ABSTRACT. Given a finite group G the Dirichlet polynomial of G is

$$P_G(s) = \sum_{H \leq G} \frac{\mu_G(H)}{|G:H|^s},$$

where μ_G is the Möbius function of the subgroup lattice of G . This object is a member of the factorial domain of finite Dirichlet series. In this paper we prove that if G is an alternating group of degree k and $k \leq 4.2 \cdot 10^{16}$ or $k \geq (e^{e^{15}} + 2)^3$, then $P_G(s)$ is irreducible. Moreover, assuming the Riemman Hypothesis, we prove that $P_G(s)$ is irreducible in the remaining cases.

An important object in number theory is the Riemann Zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

In the literature there are many generalizations of this object, also connected with the Euler expansion associated to $\zeta(s)$, i.e.

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right).$$

A generalization was introduced by Mann ([Man96]). Given a finitely generated group G , Mann defined the *Dirichlet series of G* in the following way:

$$P_G(s) = \sum_{k \geq 1} \frac{a_k(G)}{k^s}, \text{ where } a_k(G) = \sum_{H \leq G, |G:H|=k} \mu_G(H),$$

where μ_G is the Möbius function of the subgroup lattice of G , which is defined inductively by $\mu_G(G) = 1$, $\mu_G(H) = -\sum_{K > H} \mu_G(K)$. In particular we have that

$$P_{\mathbb{Z}}(s) = \frac{1}{\zeta(s)} = \prod_{p \text{ prime}} P_{\mathbb{Z}/p\mathbb{Z}}(s).$$

So, the Dirichlet series $1/P_G(s)$ was called the *probabilistic Zeta function* of G . The term *probabilistic* is due to the probabilistic interpretation of the values of the function at the natural numbers (see [Man96] for further references).

Received by the editors May 12, 2011 and, in revised form, June 11, 2011, and June 19, 2011.
 2010 *Mathematics Subject Classification*. Primary 11M41; Secondary 11N05, 20D06, 20E28.

In this paper, we deal with the finite case, so we assume that G is a finite group. Hence $P_G(s)$ is an element of the ring of finite Dirichlet series (also called Dirichlet polynomials) with integer coefficients, i.e.

$$\mathcal{R} = \left\{ \sum_{m \geq 1} \frac{a_m}{m^s} : a_m \in \mathbb{Z}, |\{m : a_m \neq 0\}| < \infty \right\},$$

which is a factorial domain (see [DLM04]). Hence it is interesting to understand the factorization of $P_G(s)$ in $\mathcal{R}$. Indeed, we wish to find some information about the structure of the group (e.g. the chief factors) from the knowledge of the factorization of $P_G(s)$.

The irreducible factors of $P_G(s)$ are usually called the *generalized Euler factors*, and their analysis could be the beginning for a number theoretical interpretation of these Dirichlet polynomials.

There are some results on the factorization of $P_G(s)$. Let $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_k = G$ be a chief series of G . The Dirichlet polynomial of G factors in the following way:

$$(\dagger) \quad P_G(s) = \prod_{i=0}^{k-1} P_{G/G_i, G_{i+1}/G_i}(s),$$

where, for a normal subgroup N of G , we have

$$P_{G,N}(s) = \sum_{k \geq 1} \frac{a_k(G, N)}{k^s}, \quad \text{where } a_k(G, N) = \sum_{\substack{H \leq G, |G:H| = k, \\ NH = G}} \mu_G(H).$$

By [DL03], the factors of $(\dagger)$ are independent on the choice of the chief series. Nevertheless, in general, these factors are not irreducible. It is easy to see that if $P_G(s)$ is irreducible, then $G/\text{Frat}(G)$ is simple (see for example [DLM04, Corollary 7]; $\text{Frat}(G)$ denotes the Frattini subgroup of G). But the converse is not true: for instance, $P_{\text{PSL}_2(7)}(s)$ is reducible.

The question of the irreducibility of the Dirichlet polynomial of a simple group was proposed by N. Boston (see [Bos96]), and the first significant results were obtained in [DLM04], where it was proved that if p is a prime number greater than 3, then $P_{\text{Alt}_p}(s)$ is irreducible, and $P_{\text{PSL}_2(p)}(s)$ is reducible if and only if $p = 2^e - 1$ with $e \equiv 3 \pmod{4}$. In a recent paper ([Pat09]), we proved that given a simple group G of Lie type, $P_G(s)$ is reducible if and only if $G = \text{PSL}_2(p)$ with $p = 2^e - 1$ a prime number and $e \equiv 3 \pmod{4}$.

In this paper we study the case when G is an alternating simple group, i.e. $G = \text{Alt}_k$ and $k \geq 5$, and we prove the following.

Theorem 1. *Assume that $k \geq 5$.*

If $k \leq 4.2 \cdot 10^{16}$ or $k \geq (e^{15} + 2)^3$, then $P_{\text{Alt}_k}(s)$ is irreducible in the ring of Dirichlet finite series.

Assuming the Riemann Hypothesis, $P_{\text{Alt}_k}(s)$ is irreducible in the ring of Dirichlet finite series.

As explained in Section 1, the ring $\mathcal{R}$ is isomorphic to a ring of multivariate polynomials with integer coefficients. Essentially, we study a problem of reducibility of a multivariate polynomial. Using some knowledge of group theory, we reduce the problem to a number theoretical conjecture, which is true conditionally.

The factors of the factorization $(\dagger)$ can be described in terms of Dirichlet polynomials of monolithic primitive groups. We recall that a group G is monolithic primitive if there exists a maximal subgroup M such that $\bigcap_{g \in G} M^g = 1$ and G has a unique minimal normal subgroup. Let G be a finite group, and let N be a minimal normal subgroup of G . It is possible to define a monolithic primitive group associated with N in the following way (see [DL03]):

$$L_N = \begin{cases} G/C_G(N) & \text{if } N \text{ is not abelian,} \\ G/C_G(N) \ltimes N & \text{otherwise.} \end{cases}$$

Clearly N is (isomorphic to) the unique minimal normal subgroup of L , so $N = \text{soc}(L_N)$ (i.e. N is the socle of L_N). Under these assumptions, there exists an integer α such that

$$P_{G,N}(s) = P_{L_N, \text{soc}(L_N)}(s) + \frac{\alpha}{|N|^s}.$$

The above remark shows that the study of the irreducibility of the factors in $(\dagger)$ is strictly connected with the study of the irreducibility of $P_{G, \text{soc}(G)}(s)$, where G is a monolithic primitive group.

Thanks to a result of [Ser08], we obtain the following.

Theorem 2. *Assume that $k \geq 5$. Let G be a primitive monolithic group with a simple component isomorphic to Alt_k . Moreover assume that if $k = 6$, then $N_G(S)/C_G(S)$ is contained in the Symmetric group Sym_k .*

If $k \leq 4.2 \cdot 10^{16}$ or $k \geq (e^{e^{15}} + 2)^3$, then $P_{G, \text{soc}(G)}(s)$ is irreducible in the ring of Dirichlet finite series.

If the Riemann Hypothesis holds, then $P_{G, \text{soc}(G)}(s)$ is irreducible in the ring of Dirichlet finite series.

Using our techniques, when $k = 6$ and $N_G(S)/C_G(S) \not\cong \text{Alt}_6$ (that is, $N_G(S)/C_G(S) \cong \text{PGL}_2(9), M_{10}$ or $\text{Aut}(\text{Alt}_6)$), we have not enough information to decide whether $P_{G, \text{soc}(G)}(s)$ is irreducible or not. However, direct computations show that $P_{G, \text{soc}(G)}(s)$ is irreducible if $G \cong \text{PGL}_2(9), M_{10}$ or $\text{Aut}(\text{Alt}_6)$.

The result we obtain has an interesting consequence, as the following theorem shows.

Theorem 3. *Let G and H be two finite groups such that $P_G(s) = P_H(s)$. Assume that the composition factors of G and H are alternating groups which satisfy the hypothesis of Theorem 2. Then G and H have the same non-Frattini chief factors.*

In Section 1, we outline the strategy of the proof and we introduce notation and lemmas that we shall use throughout the paper.

In Section 2, we state and attack a number theoretical conjecture, which we believe to be true. This conjecture is crucial in order to prove our main result.

Section 3 is devoted to the proof of Theorem 2.

In Section 4 we prove that it is possible to recognize the number of factors of the socle of a primitive monolithic group L from the knowledge of the Dirichlet polynomial $P_{L, \text{soc}(L)}(s)$.

Finally, in Section 5, we prove Theorem 3.

1. STRATEGY OF THE PROOF, SOME DEFINITIONS AND LEMMAS

In the sequel, we outline the strategy of the proof in the simple case, i.e. $G = \text{Alt}_k, k \geq 5$. Let π be a set of primes, and let π' be the set of prime numbers that does not lie in π .

The first ingredient is the ring homomorphism given by

$$\eta_\pi : \begin{array}{ccc} \mathcal{R} & \rightarrow & \mathcal{R}_{\pi'} \\ f(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} & \mapsto & f^{(\pi)}(s) = \sum_{n \in \mathbb{N}} \frac{b_n}{n^s}, \end{array}$$

where

$$b_n = \begin{cases} a_n & \text{if } n \text{ is a } \pi' \text{ number,} \\ 0 & \text{otherwise,} \end{cases}$$

π is a set of prime numbers, and $\mathcal{R}_{\pi'}$ is the image of $\mathcal{R}$ through η_π .

By [DLM04, Lemma 2], in order to prove that a Dirichlet polynomial $f(s)$ is irreducible, it is enough to find a prime number r such that

- (1) $f^{(r)}(s)$ is irreducible;
- (2) if $r|n$ and $a_n(f(s)) \neq 0$, then $|n|_r = |f(s)|_r$ (where $|f(s)|_r = \max\{|m|_r : a_m \neq 0\}$, for $f(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s}$);
- (3) $(f(s), f^{(r)}(s)) = 1$.

It turns out that the most difficult condition to check is the third. However, if $f^{(r)}(s)$ is irreducible, it is easy to see that the following condition,

- (3') the inequality

$$\frac{M(f^{(r)}(s))}{m(f^{(r)}(s))} > \frac{M(f(s) - f^{(r)}(s))}{m(f(s) - f^{(r)}(s))}$$

holds, where $m(g(s)) = \min\{k : a_k \neq 0\}$ and $M(g(s)) = \max\{k : a_k \neq 0\}$ for a Dirichlet polynomial $g(s) = \sum_{k \geq 1} \frac{a_k}{k^s}$,

implies (3).

The second ingredient is a remark. Let r be a prime number such that $\frac{k}{2} < r < k - 2$. Then the index of a non-zero coefficient of $P_G^{(r)}(s)$ is of the form

$$(*) \quad \frac{k!}{i_1! \cdots i_j!(k - i_1 - \cdots - i_j)!},$$

for some $j \in \mathbb{N}$ and $i_1, \dots, i_j \in \mathbb{N}$ such that $i_1 + \cdots + i_j \leq \frac{k-1}{2}$ (see Lemma 19 and Proposition 20 for a precise justification). Let l be a positive integer. We say that a Dirichlet polynomial is of *type* T_l if the index of the non-zero coefficient are of the form $(*)$ and the maximum value of a sum $i_1 + \cdots + i_j$ is l . It turns out that $P_G^{(r)}(s)$ is of type T_{k-r} .

Now, assume that there exists two prime numbers p_1, p_2 such that $k/2 < p_2 < p_1 < k - 2$. We construct two finite sequences $\{r_i\}_{i \leq \iota}$ (of prime numbers) and $\{a_i\}_{i \leq \iota}$ (of natural number) such that $r_i|k - a_i + 1$, $r_\iota = p_2$ and $a_1 \in \{3, 4\}$. This determines a sequence $\{T_{a_i-1}(s)\}_{i \leq \iota}$ of Dirichlet polynomials of type (respectively) $T_{a_1-1}, \dots, T_{a_\iota-1}$ such that $T_{a_i-1}(s) = P_G^{(r_i, \dots, r_\iota)}(s)$ for $1 \leq i \leq \iota - 1$ (so the sequence $\{a_i\}_{i \leq \iota}$ is increasing). In particular, note that $T_{a_{i+1}-1}^{(r_i)}(s) = T_{a_i-1}(s)$ for $1 \leq i \leq \iota - 1$.

Our aim is to prove that $T_{a_\iota-1}(s) = P_G^{(r_\iota)}(s) = P_G^{(p_2)}(s)$ is irreducible. Arguing by induction, we want to show that $T_{a_i-1}(s)$ is irreducible for $1 \leq i \leq \iota$. We try to verify that the conditions (1), (2), and (3)' hold. With some exceptions,

the polynomial $T_{a_i-1}(s)$ is irreducible. By induction, since $T_{a_{i+1}-1}^{(r_i)}(s) = T_{a_i-1}(s)$ is irreducible, condition (1) is fulfilled (with $r = r_i$ and $f(s) = T_{a_{i+1}-1}(s)$). To satisfy condition (2), we ask that $r_i \geq a_i$ and $2a_i \geq a_{i+1}$. Finally, condition (3)' is equivalent to

$$\frac{M(T_{a_i-1}(s))}{m(T_{a_i-1}(s))} > \frac{M(T_{a_{i+1}-1}(s) - T_{a_i-1}(s))}{m(T_{a_{i+1}-1}(s) - T_{a_i-1}(s))}.$$

Some information on the alternating groups show that the above inequality is equivalent to

$$\binom{k}{a_i} > \frac{(k - a_i + 1)!}{(k - a_{i+1} + 1)!}.$$

By the above discussion, in order to construct the sequences $\{r_i\}_{i \leq \iota}$ and $\{a_i\}_{i \leq \iota}$, it is sufficient to find for each pair (k, a) of natural number, with $3 \leq a \leq k - p_2$, a positive integer $h \leq a$ such that $\binom{k}{a} > \frac{(k-a+1)!}{(k-a-h+1)!}$, and the largest prime divisor of $\frac{(k-a)!}{(k-a-h)!}$ is greater than $a + h$. This is the number theoretical problem we try to attack in Section 2 (see Conjecture 7). For the precise argument see the proof of Theorem 23.

When we know that $P_G^{(p_2)}(s)$ is irreducible, the proof of the irreducibility of $P_G(s)$ can be easily completed (see the end of the proof of Theorem 23).

Now, we introduce some notation we shall use throughout the paper.

- Let

$$\mathcal{R}' = \left\{ \sum_{m \geq 1} \frac{a_m}{m^s} : a_m \in m\mathbb{Z}, |\{m : a_m \neq 0\}| < \infty \right\},$$

which is a subring of $\mathcal{R}$. Let $\mathcal{R}'_\pi = \mathcal{R}' \cap \mathcal{R}_\pi$. We define the map

$$\Psi : \mathcal{R}'_\pi \rightarrow \mathbb{Z}[\mathbf{X}_\pi]$$

given by $\Psi(p^{1-s}) = x_p$ for each $p \in \pi$, where $\mathbf{X}_\pi$ is the set of commuting indeterminates $\{x_r : r \in \pi\}$. Clearly, Ψ is a ring isomorphism, hence $\mathcal{R}'_\pi$ and $\mathcal{R}'$ are factorial domains.

- Let k be a positive integer. We let $p(k)$ be the set of prime numbers that divides k . If H is a finite group, we let $p(H) = p(|H|)$. If p is a prime number, then we let $|k|_p$ be the p -part of k , i.e. $|k|_p = p^i$ where p^i divides k but p^{i+1} does not divide k .
- Let $f(s) = \sum_{k \geq 1} \frac{a_k}{k^s}$ be an element of $\mathcal{R}$, and let r be a prime number. We let $a_k(f(s)) = a_k$. Moreover, $|f(s)|_r = \max\{|k|_r : a_k(f(s)) \neq 0\}$. Finally $p(f(s)) = p(\text{lcm}\{k : a_k(f(s)) \neq 0\})$.

We will use repeatedly, often without mention, the following results on the Möbius function of the subgroup lattice of a finite group G .

Lemma 4 (See [Hal36]). *Let G be a finite group and H a subgroup of G . If $\mu_G(H) \neq 0$, then H is intersection of maximal subgroups of G .*

Lemma 5 ([HIÖ89], Theorem 4.5). *Let G be a finite group and H a subgroup of G . The index $|N_G(H) : H|$ divides $\mu_G(H)|G : HG'|$.*

Note that if X is an almost simple group with socle S , then clearly $S \leq G'$, and so, for a subgroup $H \leq G$ such that $HS = G$, we have that $|G : H|$ divides $\mu_G(H)|G : N_G(H)|$. This implies that k divides $a_k(G, S)$ for each $k \in \mathbb{N} - \{0\}$.

Finally, another useful result is the following, which shows a connection between the Dirichlet polynomial of a primitive monolithic group and the Dirichlet polynomial of an almost simple group.

Theorem 6 (See [Ser08], Theorem 5). *Let G be a primitive monolithic group with a simple component S . Let $X = N_G(S)/C_G(S)$ and assume that $S \leq X$. Let $n = |G : N_G(S)|$. We have that*

$$P_{G, \text{soc}(G)}^{(r)}(s) = P_{X, S}^{(r)}(ns - n + 1)$$

for each prime divisor r of the order of S .

2. A NUMBER THEORETICAL RESULT

This section is devoted to the attack on Conjecture 7. Indeed we will prove it in many cases.

For the rest of the paper, let k be a positive integer, let p_1 be the largest prime number less than or equal to $k - 3$, and let p_2 be the largest prime number less than p_1 . Moreover assume that $\frac{k}{2} < p_2 < p_1 < k - 2$. This implies that $k \geq 16$. (Using [Nag52], there exists a prime number in the interval $[x, \frac{6}{5}x]$ for $x \geq 25$; so there are at least two prime numbers in the interval $[\frac{k}{2} + 1, k - 3]$ for $k \geq 48$ and we can check directly the result for $k < 48$.)

We say that a pair (k, a) is *admissible* if $k \geq 16$, $(k, a) \notin \{(19, 3), (20, 4), (21, 5), (55, 5), (84, 3)\}$ and $3 \leq a \leq k - p_2$. In particular, since $p_2 > k/2$, we have that $a < k/2$.

Now we can state the conjecture we want to attack.

Conjecture 7. *Let (k, a) be an admissible pair. Then there exists a positive integer $h \leq a$ such that*

$$(C1) \quad \binom{k}{a} > \frac{(k-a+1)!}{(k-a-h+1)!},$$

$$(C2) \quad \max p\left(\frac{(k-a)!}{(k-a-h)!}\right) > a+h, \text{ i.e. the largest prime divisor of } \frac{(k-a)!}{(k-a-h)!} \text{ is greater than } a+h.$$

Let us introduce some definitions. If (k, a) is an admissible pair, we say that (k, a, h) *satisfies* (C1) (resp. (C2)) if the numbers k, a, h satisfy (C1) (resp. (C2)). If (k, a) is an admissible pair, we say that (k, a) *satisfies* (C1) (resp. (C2)) if there exists $1 \leq h \leq a$ such that (k, a, h) satisfies (C1) (resp. (C2)). We say that k *satisfies* (C1) (resp. (C2)) if for each $a \in \{3, \dots, k - p_2\}$ such that (k, a) is admissible, the pair (k, a) satisfies (C1) (resp. (C2)). If k (resp. (k, a)) satisfies both (C1) and (C2), we say that k (resp. (k, a)) *satisfies* (C).

In the next two lemmas we prove some sufficient conditions for (k, a, h) to satisfy (C1) and (C2). Our aim is to obtain a result of this type: suppose that (k_0, a_0) satisfies (C) under some conditions, then also (k, a_0) satisfies (C) if $k \geq k_0$ (see Corollary 10 and Proposition 11).

Lemma 8. *Let k and a be positive integers and let $h(k, a) = \frac{\log \binom{k}{a}}{\log(k-a+1)}$. If h is a positive integer such that $h \leq h(k, a)$, then (k, a, h) satisfies (C1). Moreover we have $h(k, a) \leq h(k+1, a)$ for each $k \in \mathbb{N}$ such that $1 \leq a < k$.*

Proof. Note that $k - a + 1 < \binom{k}{a}$ for $1 < a < k$. Clearly $(k, a, 1)$ satisfies (C1). Let $2 \leq h \leq \frac{\log \binom{k}{a}}{\log(k-a+1)}$ be an integer. Then

$$\frac{(k-a+1)!}{(k-a-h+1)!} < (k-a+1)^h \leq \binom{k}{a},$$

hence (k, a, h) satisfies (C1).

Note that $h(k, a) \leq h(k+1, a)$ is equivalent to

$$\frac{\log \binom{k+1}{a}}{\log \binom{k}{a}} \geq \frac{\log(k-a+2)}{\log(k-a+1)}.$$

Since $\binom{k+1}{a} = \binom{k}{a} \frac{k+1}{k-a+1}$, the previous inequality is equivalent to

$$(8.1) \quad \log \binom{k}{a} \leq \frac{\log \left(\frac{k+1}{k-a+1} \right) \log(k-a+1)}{\log \left(\frac{k-a+2}{k-a+1} \right)}$$

Now we prove (8.1) by induction on k . If $k = 2$, then $a = 1$ and the claim is clear. Assume that $k \geq 3$. By induction, we have that

$$\begin{aligned} \log \binom{k+1}{a} &= \log \left(\binom{k}{a} \frac{k+1}{k-a+1} \right) \leq \frac{\log \left(\frac{k+1}{k-a+1} \right) \log(k-a+1)}{\log \left(\frac{k-a+2}{k-a+1} \right)} \\ &\quad + \log \left(\frac{k+1}{k-a+1} \right) = \frac{\log \left(\frac{k+1}{k-a+1} \right) \log(k-a+2)}{\log \left(\frac{k-a+2}{k-a+1} \right)}. \end{aligned}$$

In order to complete the proof, it is enough to show that the function

$$f(k) = \frac{\log \left(\frac{k}{k-a} \right)}{\log \left(\frac{k-a+1}{k-a} \right)}$$

is increasing if $k > a$. Note that $f'(k) \geq 0$ if and only if

$$\left(1 + \frac{a}{k-a} \right)^k \geq \left(1 + \frac{1}{k-a} \right)^{a(k-a+1)},$$

i.e.

$$\left(1 + \frac{1}{\frac{k-a}{a}} \right)^{\frac{k-a}{a}+1} \geq \left(1 + \frac{1}{k-a} \right)^{k-a+1},$$

which holds since the map $y \mapsto (1 + \frac{1}{y})^{y+1}$ is decreasing for $y > 0$. $\square$

Lemma 9. Let k, a, h be positive integers such that $k > 2a \geq 2h$, and let $f(k, a, h) = \binom{k-a}{h} - (k-a)^{\pi(a+h)}$, where $\pi: [0, \infty[\rightarrow \mathbb{N}$ is the prime counting function, defined by $\pi(x) = |\{p : p \leq x, p \text{ prime number}\}|$.

- (1) If $f(k, a, h) > 0$, then $\max p \left(\frac{(k-a)!}{(k-a-h)!} \right) > a+h$ (i.e. (k, a, h) satisfies (C2)).
- (2) If $h \geq \pi(a+h)$ and $f(k, a, h) \geq 0$, then $f(k+1, a, h) \geq f(k, a, h)$.

Proof. Let us prove (1). Assume that $\max p(\frac{(k-a)!}{(k-a-h)!}) \leq a+h$. Then also $\max p(\binom{k-a}{h}) \leq a+h$. By [Erd34], we have that $|\binom{n}{i}|_p \leq n$ for p a prime number, n and i positive integer, $n \geq i$. Thus we obtain

$$\binom{k-a}{h} = \prod_{p \text{ prime}, p \leq a+h} \left| \binom{k-a}{h} \right|_p \leq (k-a)^{\pi(a+h)},$$

hence we have (1).

Now we prove (2). The inequality $f(k+1, a, h) \geq f(k, a, h)$ is equivalent to

$$(9.1) \quad \binom{k-a}{h-1} \geq (k+1-a)^{\pi(a+h)} - (k-a)^{\pi(a+h)}.$$

Since $f(k, a, h) \geq 0$, we have that

$$\binom{k-a}{h-1} = \binom{k-a}{h} \frac{h}{k-a-h+1} \geq (k-a)^{\pi(a+h)} \frac{h}{k-a-h+1}.$$

In order to prove (9.1) it is enough to show that

$$(k-a)^{\pi(a+h)} \frac{h}{k-a-h+1} \geq (k+1-a)^{\pi(a+h)} - (k-a)^{\pi(a+h)},$$

which is equivalent to

$$(9.2) \quad \left(1 - \frac{1}{k-a+1}\right)^{\pi(a+h)} \geq 1 - \frac{h}{k-a+1}.$$

By the Bernoulli inequality, we obtain

$$\left(1 - \frac{1}{k-a+1}\right)^{\pi(a+h)} \geq 1 - \frac{\pi(a+h)}{k-a+1}.$$

Moreover, since $h \geq \pi(a+h)$, we get

$$1 - \frac{\pi(a+h)}{k-a+1} \geq 1 - \frac{h}{k-a+1},$$

so (9.2) holds. $\square$

Corollary 10. Let (k, a) be an admissible pair, and let $h(k, a) = \left\lfloor \frac{\log \binom{k}{a}}{\log(k-a+1)} \right\rfloor$ and $f(k, a, h) = \binom{k-a}{h} - (k-a)^{\pi(a+h)}$. Let $k_0, a_0 \in \mathbb{N}, k_0 > a_0 \geq 3$ be such that

- (1) $h(k_0, a_0) \geq \pi(h(k_0, a_0) + a_0)$,
- (2) $f(k_0, a_0, h(k_0, a_0)) > 0$.

Then each element of the set $\{(k, a_0, h(k_0, a_0)) : k \geq k_0, k \in \mathbb{N}\}$ satisfies (C).

Proof. By Lemma 8, we have that $(k, a_0, h(k_0, a_0))$ satisfies (C1) for $k \geq k_0$.

By Lemma 9(2), since $h(k_0, a_0) \geq \pi(h(k_0, a_0) + a_0)$ and $f(k_0, a_0, h(k_0, a_0)) > 0$, we get $f(k, a_0, h(k_0, a_0)) > 0$ for each $k \geq k_0, k \in \mathbb{N}$. Thus, by Lemma 9(1), we have that $(k, a_0, h(k_0, a_0))$ satisfies (C2) for $k \geq k_0$. $\square$

Proposition 11. Let (k, a) be an admissible pair, and let $\theta_0 \in (0, 1)$ be the solution of the equation $\frac{e^{\frac{2-\theta}{(1-\theta)^2} + 1.12}}{2-\theta} = \frac{1}{\theta}$. Let $\theta_0 < \theta < 1$, and let $\frac{e^{\frac{2-\theta}{(1-\theta)^2} + 1.12}}{2-\theta} < a \leq k^{\theta - \frac{1}{a}}$. If (k', a) is an admissible pair such that $k' \geq k$, then (k', a) satisfies (C).

Proof. Note that the function $x \mapsto \frac{e^{\frac{2-x}{(1-x)^2} + 1.12}}{2-x}$ is increasing for $0 < x < 1$. So, it turns out that $\theta_0 \approx 0.068$. Moreover we have $a(1-\theta) > 1$, since

$$a(1-\theta) > \frac{(1-\theta)e^{\frac{2-\theta}{(1-\theta)^2} + 1.12}}{2-\theta} = \frac{e^{1.12}}{1-\theta} \frac{e^{\frac{2-\theta}{(1-\theta)^2}}}{\frac{2-\theta}{1-\theta}} > \frac{e^{1.12}}{1-\theta} > 1.$$

Moreover, using the definition of θ_0 , we get $\theta - \frac{1}{a} > 0$.

We prove that (k, a, h) satisfies (C1) for $h = \lfloor a(1-\theta) + 1 \rfloor$. Since $a(1-\theta) \leq h \leq a(1-\theta) + 1$, there exists $\theta' \in [\theta - \frac{1}{a}, \theta]$ such that $h = a(1-\theta')$. Since $a \leq k^{\theta - \frac{1}{a}} \leq k^{\theta'}$, we have that

$$k^{\frac{1}{1-\theta'}} \geq a^{\frac{1}{1-\theta'}} k \geq a^{\frac{1}{1-\theta'}} (k - a + 1),$$

and it follows that

$$a(1-\theta') \leq \frac{\log\left(\frac{k}{a}\right)^a}{\log(k-a+1)} \leq \frac{\log\left(\frac{k}{a}\right)}{\log(k-a+1)}$$

since $\left(\frac{k}{a}\right)^a \leq \left(\frac{k}{a}\right)$. Thus, by Lemma 8, (k', a, h) satisfies (C1) for $h = a(1-\theta')$ and $k' \geq k$.

Now, let us prove that (k, a, h) satisfies (C2) for $h = a(1-\theta')$. We have

$$\left(\frac{k-a}{a(1-\theta')}\right) \geq \left(\frac{k-a}{a(1-\theta')}\right)^{a(1-\theta')}.$$

Note that the map $x \mapsto (1-(1-x)^{\frac{1}{x}})^{\frac{1}{x-1}}$ is increasing for $0 < x < 1$, so we get that $(1-(1-\theta')^{\frac{1}{\theta'}})^{\frac{1}{\theta'-1}} \leq e \leq k$. Hence $(k-k^{\theta'})^{\theta'} \geq k^{\theta'}(1-\theta')$, thus $(k-a)^{\theta'} \geq a(1-\theta')$. It follows that

$$\left(\frac{k-a}{a(1-\theta')}\right)^{a(1-\theta')} \geq (k-a)^{a(1-\theta')^2}.$$

By Lemma 9, in order to prove that (k, a, h) satisfies (C2), it is enough to show that $f(k, a, a(1-\theta')) > 0$. So it remains to show that

$$(11.1) \quad a(1-\theta')^2 > \pi(a(2-\theta')).$$

Since we assumed that $a > \frac{e^{\frac{2-\theta}{(1-\theta)^2} + 1.12}}{2-\theta}$ and the map $x \mapsto \frac{e^{\frac{2-x}{(1-x)^2} + 1.12}}{2-x}$ is increasing for $0 < x < 1$, we get $a > \frac{e^{\frac{2-\theta'}{(1-\theta')^2} + 1.12}}{2-\theta'}$, i.e.

$$(11.2) \quad a(1-\theta')^2 > \frac{a(2-\theta')}{\log(a(2-\theta')) - 1.12}.$$

By [Dus10] and some direct computation, we have that

$$\pi(x) \leq \frac{x}{\log(x) - 1.12}$$

for $x \geq 4$. Since $a(2-\theta') > 4$ we get

$$\pi(a(2-\theta')) \leq \frac{a(2-\theta')}{\log(a(2-\theta')) - 1.12}.$$

Combining the last inequality with (11.2) we obtain (11.1).

Note that the inequality (11.1) implies that $h \geq \pi(a+h)$, hence $f(k', a, h) \geq f(k, a, h) > 0$, thus (k', a, h) satisfies (C2) for $k' \geq k$. $\square$

When a is small, we can check the conjecture directly.

Proposition 12. *Let $3 \leq a \leq 13$. If (k, a) is admissible, then (k, a) satisfies (C).*

Proof. Note that $4 \leq a + h \leq 26$, since $1 \leq h \leq a$. For a contradiction, assume that (k, a) does not satisfy (C2). Then we have that $p((k - a) \cdots (k - a - h + 1)) \subseteq p((a + h)!)$. In [Leh64, Table 1A], an explicit list of $n \in \mathbb{N}$ such that $p(n(n - 1)) \subseteq p(m!)$ for $m \leq 42$ is given. A direct computation shows that the result holds. $\square$

The next three results concern the distribution of prime numbers in short intervals.

Lemma 13. *Let $1500 \leq x \leq 4.2 \cdot 10^{16}$. Then $[x - \sqrt{x}, x]$ contains at least two prime numbers.*

Proof. The maximal prime gap up to a certain number n is the maximum of the difference between two consecutive prime numbers less than or equal to n . By [NN03], the maximal prime gap for $x \leq 4.2 \cdot 10^{16}$ is less than 1184. Thus, there exist at least two primes in the interval $[x - \sqrt{x}, x]$ if $x \geq (2 \cdot 1184)^2 = 5607424$ and $x \leq 4.2 \cdot 10^{16}$. By [LP67] the maximal prime gap for $x \leq 10^7$ is less than 180. Thus, there exist at least two primes in the interval $[x - \sqrt{x}, x]$ if $x \geq (2 \cdot 180)^2 = 129600$ and $x \leq 10^7$. By [LP67] the maximal prime gap for $x \leq 155000$ is less than 86. Thus, there exist at least two primes in the interval $[x - \sqrt{x}, x]$ if $x \geq (2 \cdot 86)^2 = 29584$ and $x \leq 10^5$. A direct computation shows that the result holds if $1500 \leq x \leq 29584$. $\square$

Theorem 14. *The interval $[x^3, (x + 1)^3]$ contains a prime number for $x \geq e^{e^{15}}$ (see [Che10]).*

Under the Riemann Hypothesis, we have that there exists a prime number in the interval $[x - \frac{\sqrt{x} \log^2 x}{4\pi}, x]$ for $x \geq 599$ (see [Sch76, Theorem 10]).

Proposition 15. *There exists a constant x_0 such that the interval $[x - x^{\frac{2}{3} + 0.001}, x]$ contains at least two prime numbers when $x \geq x_0$. The (best) constant x_0 is at most $(e^{e^{15}} + 2)^3$. Under the Riemann Hypothesis, the (best) constant x_0 is less than 100.*

Proof. Let us prove the first statement. By Theorem 14, the interval $[x^3, (x + 2)^3]$ contains at least two prime numbers for $x \geq e^{e^{15}}$. Let $y = (x + 2)^3$. We have that the interval $[y - (6\sqrt[3]{y^2} - 12\sqrt[3]{y} + 8), y]$ contains at least two prime numbers for $y \geq (e^{e^{15}} + 2)^3$. Now, we have that $[y - (6\sqrt[3]{y^2} - 12\sqrt[3]{y} + 8), y] \subseteq [y - y^{\frac{2}{3} + 0.001}, y]$, hence $[y - y^{\frac{2}{3} + 0.001}, y]$ contains at least two prime numbers for $y \geq (e^{e^{15}} + 2)^3$.

Assume the Riemann Hypothesis. Let $g(x) = \frac{\sqrt{x} \log^2 x}{4\pi}$. By Theorem 14, $[x - g(x) - g(x - g(x)), x]$ contains at least two prime numbers for $x - g(x) \geq 599$, hence for $x \geq 615$. Since $g(x) + g(x - g(x)) \leq \frac{g(x)}{2}$, we get that there are at least two prime numbers in the interval $[x - \frac{\sqrt{x} \log^2 x}{4\pi}, x]$ for $x \geq 615$. Since $\frac{\log^2 x}{2\pi} \leq x^{\frac{1}{6} + 0.001}$ for $x \geq 4.2 \cdot 10^{16}$, we have that there are at least two primes in $[x - x^{\frac{2}{3} + 0.001}, x]$ for $x \geq 4.2 \cdot 10^{16}$. By Lemma 13 and a direct computation, we have the claim. $\square$

Now we are ready to prove the main proposition.

Proposition 16. *Assume that (k, a) is an admissible pair. If $k \geq (e^{e^{15}} + 2)^3$ or $k \leq 4.2 \cdot 10^{16}$, then k satisfies (C).*

Assuming the Riemann Hypothesis, we have that k satisfies (C) for any k , i.e. the Riemann Hypothesis implies Conjecture 7.

Proof. A direct computation shows that the result holds for $k \leq 10000$.

Let p_1 and p_2 be as in the beginning of the section. By Proposition 15, we have that there exists a constant k_0 such that $p_2 \in [k-3-(k-3)^{\frac{2}{3}+0.001}, k-3]$ for $k \geq k_0$. Thus $a \leq k-p_2 \leq (k-3)^{\frac{2}{3}+0.001}+3 \leq k^{0.67}$ if $k \geq 10000$. Let $\theta = 0.671$ and assume that $a \geq \frac{e^{\frac{2-\theta}{(1-\theta)^2}+1.12}}{2-\theta} \approx 5 \cdot 10^5$. By Proposition 11 we have that (k, a) satisfies (C). Thus, if (k, a) is an admissible pair, $a \geq 5 \cdot 10^5$ and $k \geq \max\{k_0, 10000\}$, then with (k, a) satisfies (C).

By Lemma 13 and the same argument as above, we have that $a \leq k-p_2 \leq (k-3)^{0.5}+3 \leq k^{0.509}$ for $10000 \leq k \leq 4.2 \cdot 10^{16}$. Let $\theta = 0.51$ and assume that $a \geq \frac{e^{\frac{2-\theta}{(1-\theta)^2}+1.12}}{2-\theta} \approx 1020$. By Proposition 11 we have that (k, a) satisfies (C). Thus, if (k, a) is an admissible pair, $1020 \leq a \leq (4.2 \cdot 10^{16})^{0.51}$ and $k \geq 10000$, then with (k, a) satisfies (C).

Now, let us prove the conjecture for $a \leq 1020$. By Proposition 12, we may assume that $a \geq 14$. For each $a \in \{14, \dots, 1020\}$, we find the least positive integer $k(a)$ such that $f(k(a), a, h(k(a), a)) > 0$ and $h(k(a), a) \geq \pi(h(k(a), a) + a)$. By Corollary 10, we have that the admissible pair (k, a) satisfies (C) if $k \geq k(a)$. Recall that since $a \leq k-p_2$, we have that $a \leq k^{0.509}$ for $1000 \leq k \leq 4.2 \cdot 10^{16}$. A direct computation shows that $k(a) \leq 10^6$ and $a > k(a)^{0.509}$ for $77 \leq a \leq 1020$. Thus if $k \leq k(a)$, we have that $a > k-p_2$, hence we get the result for $77 \leq a \leq 1020$. The other cases can be checked directly since $k(a) \leq 10000$ for $14 \leq a \leq 76$.

By Proposition 15, we have that $k_0 \leq (e^{e^{15}} + 2)^3$. Assuming the Riemann Hypothesis, we have that $k_0 < 10000$, hence we obtain the claim. $\square$

3. THE IRREDUCIBILITY OF THE DIRICHLET POLYNOMIAL

In this section we assume that n is a fixed positive integer. Let G be a primitive monolithic group with a simple component isomorphic to Alt_k and let $\text{soc}(G) \cong \text{Alt}_k^n$.

Definition. Let $f(s)$ be a Dirichlet polynomial. We define

$$m(f(s)) = \min\{k : a_k \neq 0\} \quad \text{and} \quad M(f(s)) = \max\{k : a_k \neq 0\}.$$

Proposition 17. Let $f(x) = a + bx^l$ be a polynomial in $D[x]$, where $D = \mathbb{Z}[x_1, \dots, x_k]$ and $a, b \in D - \{0\}$, $l \geq 1$. Assume $f(x)$ is reducible and $\text{GCD}(a, b) = 1$. Then there exists a prime divisor p of l such that $a^2, b^2 \in D^{2p}$.

Proof. This is left to the reader: just apply [Lan02, Chap. VI, Theorem 9.1]. $\square$

Proposition 18. Let $f(s)$ be a Dirichlet polynomial in $\mathcal{R}'$. Let r be a prime number and l a positive integer such that

$$f(s) = g(s) + r^{l(1-s)}h(s),$$

with $g^{(r)}(s) = g(s)$ and $h^{(r)}(s) = h(s)$. Assume that

- (1) $g(s)$ is irreducible,
- (2) $\frac{M(g(s))}{m(g(s))} > \frac{M(h(s))}{m(h(s))}$.

Then $f(s)$ is irreducible.

Proof. Let $m = \text{lcm}\{k : a_k(f(s)) \neq 0\}$, and let $\pi = p(m)$. Clearly,

$$\Psi(f(s)) = a + bx_r^l$$

is an element of $\mathbb{Z}[X_\pi]$ for some $a, b \in \mathbb{Z}[X_{\pi-\{r\}}]$ and $x_r = \Psi(r^{1-s})$. We claim that $\text{GCD}(a, b) = 1$. Let $c \in \mathbb{Z}[X_{\pi-\{r\}}]$ such that c divides $\text{GCD}(a, b)$. By (1), the polynomial a is irreducible, so either $c = \pm a$ or $c = \pm 1$. If $c = \pm a$, then a divides b , hence $g(s)$ divides $h(s)$. Thus, there exists $k(s) \in \mathbb{Z}[X_{\pi-\{r\}}]$ such that $h(s) = g(s)k(s)$. In particular, $m(h(s)) = m(g(s))m(k(s))$ and $M(h(s)) = M(g(s))M(k(s))$, so

$$\frac{M(h(s))}{m(h(s))} = \frac{M(g(s))}{m(g(s))} \frac{M(k(s))}{m(k(s))} \geq \frac{M(g(s))}{m(g(s))},$$

against (2). So we conclude that $\text{GCD}(a, b) = 1$.

Now we prove the claim of the proposition. For a contradiction, assume that $f(s)$ is reducible. Since $\text{GCD}(a, b) = 1$, by Proposition 17, we have that there exists a prime divisor p of l such that $a^2, b^2 \in D^{2p}$. But this is not possible, since a is irreducible. Hence $f(s)$ is irreducible. $\square$

Lemma 19. *Let X be an almost simple group with socle Alt_k . Let r be a prime number such that $k/2 < r < k - 2$. If K is a proper subgroup of X containing a Sylow r -subgroup of X and $KS = X$, then K is an intransitive subgroup of X .*

Proof. Assume that K is a maximal primitive subgroup of X . Note that K must contain a cycle of length r . By a result of Jordan (see [Jor57, Note C]), we have that K is at least $(k - r + 1)$ -transitive. By [Jor73, Théorème I], since $k - r + 1 > 2$, we have that K must contain Alt_{k-r} , hence K is an intransitive maximal subgroup of X .

It is easy to see that if K is an imprimitive maximal subgroup of X , then r divides $|X : K|$. $\square$

Proposition 20. *Let X be Sym_k or Alt_k . An intransitive subgroup N_Δ of X , which is intersection of maximal subgroups of X , is isomorphic to $X \cap \prod_{i=1}^l \text{Sym}_{|\Delta_i|}$, where $\Delta = \{\Delta_1, \dots, \Delta_l\}$ be a partition of $\{1, \dots, k\}$ for some positive integer $l \leq k$. In particular we have that*

- (1) $\text{Alt}_k N_\Delta = X$ and
- (2) $\mu_{\text{Sym}_k}(N_\Delta) = (-1)^{l-1}(l-1)!$

Proof. This is well known; for the second statement, see [Sta97, p.128].

Let m be a natural number, and let Λ_m be the set

$$\left\{ \frac{m!}{i_1! \cdots i_j!} : \exists j \in \mathbb{N}, i_1, \dots, i_j \in \mathbb{N}, i_1 + \cdots + i_j = m \right\}.$$

Let $1 \leq l \leq \frac{k-1}{2}$. Let π be a set of prime numbers satisfying the following:

- $p(k(k-1) \cdots (k-l+1)) \cap \pi = \emptyset$,
- $p\binom{k}{i} \cap \pi \neq \emptyset$ for $l+1 \leq i \leq k/2$,
- there exists $r \in \pi$ such that $k/2 < r < k - 2$.

Let c be an $(l+1) \times l!$ matrix with integer coefficients such that $c_{0,1} = 1$ and $c_{m,1} = -1$ for $m \in \{1, \dots, l\}$. We define the Dirichlet polynomial (which is an element of $\mathcal{R}'$)

$$T_{l,c}(s) = \sum_{m=0}^l \sum_{i \in \Lambda_m} c_{m,i} \left(i \binom{k}{m} \right)^{n(1-s)}.$$

Clearly, $T_{l,c}(s)$ also depends on n , but throughout this section we assume that n is a fixed constant.

By Lemma 5, Theorem 6, Lemma 19, and Proposition 20, there exists a matrix c such that

$$P_{G, \text{soc}(G)}^{(\pi)}(s) = T_{l,c}(s).$$

Lemma 21. *Let $k \geq 6$ and $p(k) \neq \{2\}$. The Dirichlet polynomial $T_{2,c}(s)$ is irreducible for each c .*

Proof. Let h and m be natural numbers such that $k = 2^m h$ and $h > 1$ is odd. We get $T_{2,c}(s) = 1 - g(s)h^{n(1-s)}$, where

$$g(s) = 2^{mn(1-s)} + \left(\frac{2^m(2^m h - 1)}{2} \right)^{n(1-s)} - c_{2,2}(2^m(2^m h - 1))^{n(1-s)}.$$

For a contradiction, assume that $T_{2,c}(s)$ is reducible. Since $h > 1$ and $(2^m h - 1, h) = 1$, by Proposition 17, we have that $g(s)$ is a non-trivial power. This is impossible. $\square$

Lemma 22. *Let $k \geq 8$. The Dirichlet polynomial $T_{3,c}(s)$ is irreducible for each c .*

Proof. Let h and m be natural numbers such that $k = 2^m h$ and h is odd.

Assume that $h > 1$. We get $T_{3,c}(s) = 1 - g(s)h^{n(1-s)}$, where

$$\begin{aligned} g(s) = & 2^{mn(1-s)} + \left(\frac{2^m(k-1)}{2} \right)^{n(1-s)} + \left(\frac{2^m(k-1)(k-2)}{6} \right)^{n(1-s)} \\ & - c_{2,2}(2^m(k-1))^{n(1-s)} + -c_{3,3} \left(\frac{2^m(k-1)(k-2)}{2} \right)^{n(1-s)} \\ & - c_{3,6}(2^m(k-1)(k-2))^{n(1-s)}. \end{aligned}$$

For a contradiction, assume that $T_{3,c}(s)$ is reducible. Since $h > 1$ and $((k-1)(k-2), h) = 1$, by Proposition 17, we have that $g(s)$ is a non-trivial power. This is impossible.

Assume that $h = 1$. We get $f(s) = 1 - 2^{mn(1-s)} + g(s) \left(\frac{k-1}{|k-1|_3} \right)^{n(1-s)}$, where

$$\begin{aligned} g(s) = & (2^{(m-1)}|k-1|_3)^{n(1-s)} + \left(\frac{2^{m-1}|k-1|_3(k-2)}{6} \right)^{n(1-s)} \\ & - c_{2,2}(2^m|k-1|_3)^{n(1-s)} + -c_{3,3}(2^{m-1}|k-1|_3(k-2))^{n(1-s)} \\ & - c_{3,6}(2^m|k-1|_3(k-2))^{n(1-s)}. \end{aligned}$$

For a contradiction, assume that $f(s)$ is reducible. Note that $k-1 > |k-1|_3$, since the equation $k-1 = 2^m - 1 = 3^l$ has no solution for $k \geq 5$. Since $(3k(k-2), \frac{k-1}{|k-1|_3}) = 1$ and $(g(s), 1 - 2^{mn(1-s)}) = 1$, by Proposition 18, we have that $g(s)$ is a non-trivial power. This is impossible. $\square$

Theorem 23. *Let k be an integer such that (k, a) is an admissible pair which satisfies (C) for each a (see the definitions after Conjecture 7). Let G be a primitive monolithic group with a simple component isomorphic to Alt_k . Then $P_{G, \text{soc}(G)}(s)$ is irreducible.*

Proof. Using the notation of Conjecture 7, for each $a \in \{3, \dots, k - p_2 + 1\}$, let $\gamma(k, a)$ be the minimum h such that (k, a, h) satisfies (C).

We construct a finite sequence $\{r_i\}_{1 \leq i \leq \iota}$ of prime numbers and a finite sequence $\{a_i\}_{1 \leq i \leq \iota}$ of positive integers, for some $\iota \in \mathbb{N} - \{0\}$.

Let r_1 be a prime number greater than 4 such that $r_1 \in p((k-2)(k-3))$ (it is easy to see that such r_1 exists provided that $k \neq 5, 6, 11$). In particular, if $k = |k|_2$, then take $r_1 \in p(k-3)$ such that $r_1 > 4$ (it always exists). Let $a_1 = j+1$ where j is the unique number such that $j \in \{2, 3\}$ and r_1 divides $k-j$ (in particular, if $|k|_2 = k$, then $a_1 = 4$). Note that $2 < a_1 \leq 4$, the prime r_1 does not divide $\frac{k!}{(k-a_1+1)!}$, and $|\frac{k!}{(k-a_1-j_1)!}|_{r_1} = |\frac{k!}{(k-a_1-j_2)!}|_{r_1}$ for each $0 \leq j_1, j_2 \leq 4$.

Assume that $i \geq 2$. If $a_{i-1} > k - p_2$, then we set $\iota = i - 1$. If $a_{i-1} \leq k - p_2$, then we let r_i be a prime number greater than $a_{i-1} + \gamma(k, a_{i-1})$ such that $r_i \in p(\frac{(k-a_{i-1})!}{(k-a_{i-1}-\gamma(k, a_{i-1}))!})$ (it exists since $(k, a_{i-1}, \gamma(k, a_{i-1}))$ satisfies (C2)). Set $a_i = j+1$ where j is the unique number $a_{i-1} \leq j \leq a_{i-1} + \gamma(k, a_{i-1}) - 1$ such that r_i divides $k-j$. Note that:

- (I) $a_{i-1} < a_i \leq a_{i-1} + \gamma(k, a_{i-1})$,
- (II) the prime r_i does not divide $\frac{k!}{(k-a_i+1)!}$,
- (III) $|\frac{k!}{(k-a_i-j_1)!}|_{r_i} = |\frac{k!}{(k-a_i-j_2)!}|_{r_i}$ for each $0 \leq j_1, j_2 \leq a_{i-1} + \gamma(k, a_{i-1})$.

Without loss of generality, we may assume that $r_\iota = p_2$, hence $a_\iota = k - p_2 + 1$.

Recall that $k/2 < p_2 < k - 3$. Let $P(s) = P_{G, \text{soc}(G)}(s)$. By Lemma 19, if M is a subgroup of X such that M is intersection of maximal subgroups, $M\text{Alt}_k = X$ and $|X : M|_{p_2} = 1$, then M is an intransitive subgroup. In particular, $|X : M| = \frac{k!}{c_1! \dots c_\iota!}$ for some $1 \leq c_1 \leq c_2 \leq \dots \leq c_\iota$ such that $\sum_{i=1}^\iota c_i = k$. We claim that $a_{\binom{k}{k-a_i}}(X, S) \neq 0$ for $1 \leq i \leq \iota - 1$ and $a_{\frac{k!}{(k-a_i+1)!}}(X, S) \neq 0$ for each $1 \leq i \leq \iota$. Now, let M be a subgroup of index $\frac{k!}{(k-a_i+1)!}$. We claim that $M \cong \text{Sym}_{k-a_i+1} \cap X$. As above, we have $\frac{k!}{(k-a_i+1)!} = |X : M| = \frac{k!}{c_1! \dots c_\iota!}$. Since the prime numbers $r_i, \dots, r_\iota$ do not divide $\frac{k!}{(k-a_i+1)!}$, it is easy to see that $c_\iota \geq k - a_i + 1$ and the claim follows. By Proposition 20, we conclude that $a_{\frac{k!}{(k-a_i+1)!}}(X, S) \neq 0$. Similarly we have that $a_{\binom{k}{k-a_i}}(X, S) \neq 0$.

By the above argument and Theorem 6, we have that

$$\begin{aligned} P^{(r_i, \dots, r_\iota)}(s) &= b_0(s) + b_1(s)r_1^{\alpha_1(1-s)} + b_2(s)r_2^{\alpha_2(1-s)} + \dots + b_{i-1}(s)r_{i-1}^{\alpha_{i-1}(1-s)} \\ &= T_{a_i-1, c_i}(s) \end{aligned}$$

and $P^{(r_1, \dots, r_\iota)}(s) = b_0(s) = T_{a_1-1, c}(s)$, for some $\alpha_i \in \mathbb{Z}$, some matrices c_i and c , and some $b_i(s) \in \mathcal{R}$ such that $b_{i-1}^{(r_{i-1})}(s) = b_{i-1}(s)$ for $2 \leq i \leq \iota$. Moreover, $m(b_i(s)r_i^{\alpha_i(1-s)}) = \binom{k}{k-a_i}^n$ and $M(b_i(s)r_i^{\alpha_i(1-s)}) = \left(\frac{k!}{(k-a_{i+1}+1)!}\right)^n$ for $1 \leq i \leq \iota-1$, $m(b_0(s)) = 1$, $M(b_0(s)) = \left(\frac{k!}{(k-a_1+1)!}\right)^n$.

Note that by Lemma 21 and Lemma 22, we have that $b_0(s)$ is irreducible (in fact, by Lemma 5, Lemma 19, and Proposition 20, if $a_1 = 3$, then $b_0(s) = f(s)$ in Lemma 21, if $a_1 = 4$, then $b_0(s) = f(s)$ in Lemma 22; moreover note that if $|k|_2 = k$, then $a_1 = 4$). We claim that $P^{(r_i, \dots, r_\iota)}(s)$ is irreducible for each $1 \leq i \leq \iota$. We argue by induction. For $i = 1$ the result is clear, since $P^{(r_1, \dots, r_\iota)}(s) = b_0(s)$.

Now we assume that $P^{(r_i, \dots, r_\iota)}(s)$ is irreducible for some $i \in \{1, \dots, \iota-1\}$ and we claim that $P^{(r_{i+1}, \dots, r_\iota)}(s)$ is irreducible. We have that

$$P^{(r_{i+1}, \dots, r_\iota)}(s) = P^{(r_i, \dots, r_\iota)}(s) + b_i(s)r_i^{\alpha_i(1-s)}.$$

We apply Proposition 18. Clearly, $b_i^{(r_i)}(s) = b_i(s)$ and $P^{(r_i, r_i, \dots, r_\ell)}(s) = P^{(r_i, \dots, r_\ell)}(s)$. Moreover, $P^{(r_i, \dots, r_\ell)}(s)$ is irreducible by inductions, so condition (1) is satisfied. Now, we check that condition (2) is satisfied. Since $(k, a_i, \gamma(k, a_i))$ satisfies (C1), we have that

$$\binom{k}{a_i} > \frac{(k - a_i + 1)!}{(k - a_i - \gamma(k, a_i) + 1)!}.$$

Since $a_{i+1} \leq a_i + \gamma(k, a_i)$ (see (I) above), we obtain

$$\binom{k}{a_i} > \frac{(k - a_i + 1)!}{(k - a_{i+1} + 1)!},$$

which is equivalent to $\frac{M(P^{(r_i, \dots, r_\ell)}(s))}{m(P^{(r_i, \dots, r_\ell)}(s))} > \frac{M(b_{i+1}(s)r_{i+1}^{\alpha_{i+1}(1-s)})}{m(b_{i+1}(s)r_{i+1}^{\alpha_{i+1}(1-s)})} = \frac{M(b_{i+1}(s))}{m(b_{i+1}(s))}$. Thus condition (2) of Proposition 18 holds. Hence we conclude that $P^{(r_i, \dots, r_\ell)}(s)$ is irreducible.

In particular, we have that $P^{(r_\ell)}(s) = P^{(p_2)}(s)$ is irreducible.

Now, assume that $P(s) = f(s)g(s)$ for some Dirichlet polynomials $f(s)$ and $g(s)$. Since $P^{(p_2)}(s) = f^{(p_2)}(s)g^{(p_2)}(s)$ is irreducible, without loss of generality, we may assume that $P^{(p_2)}(s) = f^{(p_2)}(s)$ and $g^{(p_2)}(s) = 1$. Now, $|P^{(p_2)}(s)|_{p_1} = |P(s)|_{p_1}$, hence $|f(s)|_{p_1} = |P(s)|_{p_1}$ and $|g(s)|_{p_1} = 1$. It follows that $g^{(p_1)}(s) = g(s)$. By Lemma 19, we have that $P^{(p_1)}(s) = P^{(p_1, p_2)}(s)$. Thus $g^{(p_1)}(s) = g^{(p_1, p_2)}(s) = 1$, so $g(s) = 1$. $\square$

Theorem 24. *Let $5 \leq k \leq 15$ or $k \in \{19, 20, 21, 55, 84\}$. Let G be a primitive monolithic group with a simple component isomorphic to Alt_k . Moreover, if $k = 6$, then assume that $N_G(s)/C_G(s)$ is isomorphic to a subgroup of Sym_6 . Then $P_{G, \text{soc}(G)}(s)$ is irreducible.*

Proof. We let $P(s) = P_{G, \text{soc}(G)}(s)$. Assume that $P(s) = f(s)g(s)$.

TABLE 1. Some cases

k	r	π
5	3	$\{2, 5\}$
6	2	$\{3, 5\}$
7	3	$\{5, 7\}$
10	2	$\{5, 7\}$
11	5	$\{7, 11\}$
12	2	$\{3, 11\}$

Case $k = 5, 6, 7, 10, 11, 12$. Let r and π be as in Table 1. Using [GAP], we have that $P^{(r)}(s)$ is irreducible. So we may assume that $f^{(r)}(s) = P^{(r)}(s)$ and $g^{(r)}(s) = 1$. Note that $|P^{(r)}(s)|_p = |P(s)|_r$ for $p \in \pi$, hence $|f(s)|_p = |P(s)|_p$, $|g(s)|_p = 1$ and $g^{(p)}(s) = g(s)$ for $p \in \pi$. In particular, $g^{(\pi)}(s) = g(s)$, but $P^{(\pi)}(s) = 1$. This concludes the proof.

Case $k = 8$. Using [GAP], we have that $P^{(2)}(s)$ and $P^{(5)}(s)$ are irreducible. So we may assume that $f^{(2)}(s) = P^{(2)}(s)$ and $g^{(2)}(s) = 1$. Note that $|P^{(2)}(s)|_r = |P(s)|_r$ for $r \in \{3, 5\}$, hence $|f(s)|_r = |P(s)|_r$, $|g(s)|_r = 1$ and $g^{(r)}(s) = g(s)$ for $r \in \{3, 5\}$.

Since $|P^{(5)}(s)|_3 > 1$ and $P^{(5)}(s)$ is irreducible, we have that $P^{(5)}(s) = f^{(5)}(s)$. Thus $g^{(5)}(s) = 1$ and so $g(s) = g^{(5)}(s) = 1$. This concludes the proof.

Case $k = 9$. Using [GAP], we have that $P^{(5)}(s)$ is irreducible. So we may assume that $f^{(5)}(s) = P^{(5)}(s)$ and $g^{(5)}(s) = 1$. Note that $|P^{(5)}(s)|_7 = |P(s)|_7$, hence $|f(s)|_7 = |P(s)|_7$, $|g(s)|_7 = 1$. Since

$$P^{(3)}(s) = 1 - 280^{n(1-s)},$$

and $|g(s)|_7 = 1$ we have that $f^{(3)}(s) = P^{(3)}(s)$. In particular, $|P^{(3)}(s)|_5 = |P(s)|_5$, hence $|f(s)|_5 = |P(s)|_5$ and $|g(s)|_5 = 1$. So $g(s) = g^{(5)}(s) = 1$. This concludes the proof.

Case $k = 13, 14$. By Lemma 21 and 22 we have that $P^{(11)}(s)$ is irreducible. So we may assume that $f^{(11)}(s) = P^{(11)}(s)$ and $g^{(11)}(s) = 1$. Now, $|P(s)|_{13} = |P^{(11)}(s)|_{13}$, hence $|f(s)|_{13} = |P(s)|_{13}$, thus $|g(s)|_{13} = 1$. By Lemma 19 for $k = 13$ and by [LPS90, Theorem p. 130] we have that $P^{(7,13)}(s) = 1$ and $|P^{(7)}(s)|_{11} = |P(s)|_{11}$. Since $g^{(7)}(s)$ divides $P^{(7)}(s)$ and $|g(s)|_{13} = 1$, we have that $g^{(7)}(s) = 1$. This implies that $f^{(7)}(s) = P^{(7)}(s)$, hence $|P(s)|_{11} = |P^{(7)}(s)|_{11} = |f(s)|_{11}$. Hence we obtain $|g(s)|_{11} = 1$, so $g^{(11)}(s) = g(s)$. But $g^{(11)}(s) = 1$, thus $g(s) = 1$ and we get the claim.

Case $k = 15, 19, 21$. Let $(p_1, p_2) = (13, 11)$ if case $k = 15$ holds, $(p_1, p_2) = (19, 17)$ otherwise. By Lemma 22 we have that $P^{(p_2)}(s)$ is irreducible for $k = 19$. Arguing as in Lemma 22 we have that $P^{(p_2)}(s) = T_4(s)$ is irreducible for $k = 15, 21$. So we may assume that $f^{(p_2)}(s) = P^{(p_2)}(s)$ and $g^{(p_2)}(s) = 1$. Now, $|P(s)|_{p_1} = |P^{(p_2)}(s)|_{p_1}$, hence $|f(s)|_{p_1} = |P(s)|_{p_1}$, thus $|g(s)|_{p_1} = 1$ and $g^{(p_1)}(s) = g(s)$. But $P^{(p_1)}(s) = P^{(p_2, p_1)}(s) = f^{(p_1)}(s)$, hence $g^{(p_1)}(s) = 1$. So $g(s) = g^{(p_1)}(s) = 1$ and we get the claim.

Case $k = 55$. By Lemma 22 the Dirichlet polynomial $P^{(13,47)}(s)$ is irreducible. So we may assume that $P^{(13,47)}(s) = f^{(13,47)}(s)$. Moreover we have $|P(s)|_{53} = |P^{(13,47)}(s)|_{53}$, hence $|f(s)|_{53} = |P(s)|_{53}$, so $|g(s)|_{53} = 1$ and $g^{(53)}(s) = g(s)$. Now, $P^{(53)}(s) = P^{(53,13,47)}(s)$, hence $f^{(53)}(s) = P^{(53)}(s)$ and $g^{(53)}(s) = 1$. This proves the claim.

Case $k = 20, 84$. Let $(\pi, p_1, p_2, p_3) = (\{17\}, 19, 13, 17)$ if case $k = 20$ holds, $(\pi, p_1, p_2, p_3) = (\{5, 79\}, 83, 73, 79)$ otherwise. By Lemma 22 the Dirichlet polynomial $P^{(\pi)}(s)$ is irreducible. So we may assume that $P^{(\pi)}(s) = f^{(\pi)}(s)$. Moreover we have $|P(s)|_{p_1} = |P^{(\pi)}(s)|_{p_1}$, hence $|f(s)|_{p_1} = |P(s)|_{p_1}$, so $|g(s)|_{p_1} = 1$ and $g^{(p_1)}(s) = g(s)$. Now, by [Mas07, Lemma 2] we have that

$$P^{(p_1)}(s) \text{ divides } (1 - k^{n(1-s)})(1 - ((k-3)!)^{n(1-s)}).$$

Let $h(s) = \text{GCD}(g^{(p_1)}(s), 1 - k^{n(1-s)})$. It is easy to see that

$$h^{(\pi)}(s) = \text{GCD}(g^{(\pi \cup \{p_1\})}(s), 1 - k^{n(1-s)}) = 1$$

since $g^{(\pi)}(s) = 1$. Thus $g(s) = g^{(p_1)}(s)$ must divide $1 - ((k-3)!)^{n(1-s)}$. This implies that $P^{(p_2)}(s) = f^{(p_2)}(s)$, but $|P^{(p_2)}(s)|_{p_3} = |P(s)|_{p_3}$, hence $|f(s)|_{p_3} = |P(s)|_{p_3}$ and $|g(s)|_{p_3} = 1$. Since $g(s)$ divides $1 - ((k-3)!)^{n(1-s)}$, it must be $g(s) = 1$. $\square$

4. RECOGNIZING THE NUMBER OF FACTORS OF THE SOCLE OF L FROM $P_{L,N}(s)$

In this section let L be a monolithic primitive group with non-abelian socle N isomorphic to a direct product of n copies of a simple group S . Our aim is to recognize the number n from the knowledge of the Dirichlet polynomials $\{P_{L,N}^{(r)}(s) : r \in p(S)\}$ (see Proposition 27).

Lemma 25. *Let p be a prime number, and let m be a positive integer.*

- (1) *If $p^m + 1$ is a non-trivial power of an integer, then $(p, m) = (2, 3)$ or $(p, m) = (2^k - 1, 1)$ for some $k \in \mathbb{N}$.*
- (2) *If $p \not\equiv 2 \pmod{3}$, then $p^{2m} + p^m + 1$ is not a non-trivial power.*
- (3) *If $p = 2^k - 1$ and $k \geq 5$, then $\frac{p^{2+1}}{2}$ is not a non-trivial power.*

Proof. Let us prove (1). Assume that $p^m + 1 = a^l$ for some positive integers a and l , such that $l \geq 2$. Assume that $a = 2$. It is easy to see that in this case $p^m = p$ is a Mersenne prime. Assume that $a > 2$. We have that $p^m = a^l - 1$ and p divides $a - 1$, hence there are no Zsigmondy primes for $\langle a, l \rangle$. The claim follows.

Now we prove (2). Assume that $p^{2m} + p^m + 1 = a^l$ for some positive integers a and l , such that $l \geq 2$. Without loss of generality, we may assume that l is a prime number. Suppose that $p = 3$. We have that $3^m(3^m + 1) = a^l - 1$. Clearly 3 does not divide a . Assume that $a \equiv 2 \pmod{3}$. In this case l must be even, hence $l = 2$. Therefore 3^m divides $a - 1$ or $a + 1$, which are even numbers, hence $a \geq 2 \cdot 3^m - 1$. So $3^{2m} + 3^m + 1 = a^2 \geq 4 \cdot 3^{2m} - 4 \cdot 3^m + 1$, that implies $m \leq 0$, a contradiction. Assume that $a \equiv 1 \pmod{3}$. Suppose that 3 does not divide l . Since $(a - 1, a^{l-1} + \dots + 1)$ divides l , we have that 3^m divides $a - 1$, hence $a \geq 2 \cdot 3^m + 1$, which is impossible. So we have that $l = 3$. In this case $3^m(3^m + 1) = (a - 1)(a^2 + a + 1)$. Note that $|a^2 + a + 1|_3 = 3$ since $a^2 + a + 1 \equiv 3 \pmod{9}$ when $a \equiv 1 \pmod{3}$. Hence $2 \cdot 3^{m-1}$ divides $a - 1$, so $a \geq 2 \cdot 3^{m-1} + 1$. It follows that $3^{2m} + 3^m + 1 = a^3 \geq (2 \cdot 3^{m-1} + 1)^3$ which does not occur. Finally, suppose that $p \neq 3$. In this case, we have that $p \equiv 1 \pmod{3}$, hence $|p^2 + p + 1|_3 = 3$, as we have seen above. This implies that $l = 1$.

Finally we prove (3). Assume that $p^2 + 1 = 2a^l$ for some positive integers a and l , such that $l \geq 2$. Without loss of generality, we may assume that l is a prime number. We have that $2^k(2^{k-1} - 1) = a^l - 1$. If l is odd, then 2^k divides $a - 1$, hence $a \geq 2^k + 1$ and we get a contradiction. If $l = 2$, then 2^{k-1} divides $a - 1$ or $a + 1$. It is easy to see that if $a - 1$ or $a + 1$ equals 2^{k-1} , then we have a contradiction. Hence $a \geq 3 \cdot 2^{k-1} - 1$, so $2^{2k-1} - 2^k + 1 = a^2 \geq 9 \cdot 2^{2k-2} - 6 \cdot 2^{k-1} + 1$, which does not occur. $\square$

We recall a useful result.

Theorem 26 ([DL07, Theorem 1.1]). *Let X be an almost simple group with socle S . For any $p \in p(S)$, there exists a positive integer m with the following properties:*

- p divides m ;
- there exists a prime which divides $|S|$ but does not divide m ;
- $a_m(X, S) \neq 0$.

Proposition 27. *Let L be a monolithic primitive group with socle $N \cong S^n$, where S is a non-abelian simple group and $n = |L : N_L(S)|$. Then*

$$n = \max\{a \in \mathbb{N} : \forall k \in \mathbb{N} \text{ (if } a_k(L, N) \neq 0 \text{ and } p(k) \neq p(S), \text{ then } k \in \mathbb{Z}^a)\}.$$

FIGURE 1. Dynkin diagrams

So, if we know the Dirichlet polynomials $\{P_{L,N}^{(r)}(s) : r \in \mathfrak{p}(S)\}$, then n is uniquely determined.

Proof. By Theorem 6, if $a_k(L, N) \neq 0$ and $\mathfrak{p}(k) \neq \mathfrak{p}(S)$, then $k = l^n$ and $a_l(X, S) \neq 0$, where $X = N_L(S)/C_L(S)$. So it suffices to prove that if X is an almost simple group with socle S , then there exists a positive integer m such that m is not a non-trivial power, $a_m(X, S) \neq 0$ and $\mathfrak{p}(m) \neq \mathfrak{p}(S)$.

Assume that there exists a prime number r such that $|S|_r = r$. By Theorem 26, we have that there exists a positive integer m such that r divides m , $\mathfrak{p}(m) \neq \mathfrak{p}(S)$ and $a_m(X, S) \neq 0$. In particular, m is not a non-trivial power. Thus we have the claim.

Let m_1 and m_2 be two integers such that m_2 divide m_1 , $\mathfrak{p}(m_i) \neq \mathfrak{p}(S)$ and $a_{m_i}(X, S) \neq 0$. Note that to obtain the claim, it suffices to show that $\frac{m_1}{m_2}$ is not a non-trivial power. So, in order to prove our proposition, we show that

- (a) there exists a prime number r such that $|S|_r = r$, or
- (b) there exist $k \geq 1$ and some subgroups $H, H_1, \dots, H_k$ of X such that $X = HS$, $a_{|X:H_i|}(X, S) \neq 0$, $a_{|X:H|}(X, S) \neq 0$, $\mathfrak{p}(|X:H|) \neq \mathfrak{p}(S)$ and $\frac{|X:H|}{|X:H_i|}$ is not a non-trivial power of an integer for some $i \in \{1, \dots, k\}$.

If S is sporadic, then (a) occurs. Moreover, if S is alternating then by the Bertrand postulate there exists a prime number in the open interval $(n/2, n)$ for $n \geq 5$. So (a) occurs.

From now on, assume that S is a group of Lie type over a field of characteristic p . Let B be a Borel subgroup of S . Let ρ be the symmetry of the Dynkin diagram which defines S : if S is untwisted, then ρ is trivial, otherwise ρ has order 2 or 3. In Table 2, given a simple group of Lie type S , we describe two ρ -invariant sets of roots K_1 and K_2 , referring to Figure 1. In the second column we give some conditions, in particular referring to the graph automorphisms contained in X : when we write *no graph*, we mean that X does not contain a non-trivial graph automorphism; when we write *graph*, we mean that X contains a non-trivial graph automorphism. Sometimes we specify the order of the graph automorphism (for instance, we write *3-graph* in the case when $S = D_4(q)$ and X contains a graph automorphism of order 3). If P is a parabolic subgroup of S , we denote by L_P the Levi factor of P . For $i \in \{1, 2\}$, we let K'_i be the set of the ρ -orbits which are in K_i . Using the notation of [Pat09, Section 2], we associate to each set of ρ -orbits K'_i a parabolic subgroup $P_{K'_i}$ of S . Now, let K_i be as in Table 2. We have that $N_X(P_{K'_i})$ is subgroup of X such that $N_X(P_{K'_i})S = X$, and if Q is another subgroup of X such that $|X:Q| = |X:N_X(P_{K'_i})|$ and $QS = X$, then $Q \cap S$ and $P_{K'_i}$ have the same

Levi factor $L_{P_{K'_i}}$. By [Pat09, Theorem 17], we conclude that $a_{|S:P_{K'_i}|}(X, S) \neq 0$ and $a_{|S:B|}(X, S) \neq 0$. Moreover, p does not divide $|S : B|$.

We want to prove that (b) holds whenever (a) does not hold. Let $H = N_X(B)$ and $H_i = N_X(P_{K'_i})$. Note that it remains to prove that $\frac{|X:H|}{|X:H_i|} = \frac{|S:B|}{|S:P_{K'_i}|}$ is not a non-trivial power for some $i \in \{1, 2\}$, whenever (a) does not hold. In Table 3, we report the index $|P : B|$, where P is a parabolic subgroup, given the Levi factor L_P (see [Pat09, Section 3]).

TABLE 2. Some Levi factors of a simple group of Lie type.

S	Conditions	K_1	$L_{P_{K'_1}}$	K_2	$L_{P_{K'_2}}$
$A_1(q)$		$\{r_1\}$	$A_1(q)$		
$A_l(q)$	$l \geq 2$, no graph	$\{r_1, r_2\}$	$A_2(q)$		
$A_l(q)$	l even, $l \geq 2$, graph	$\{r_{\frac{l}{2}}, r_{\frac{l+2}{2}}\}$	$A_2(q)$		
$A_l(q)$	l odd, $l \geq 3$, graph	$\{r_{\frac{l+1}{2}}\}$	$A_1(q)$	$\{r_{\frac{l-1}{2}}, r_{\frac{l+1}{2}}, r_{\frac{l+3}{2}}\}$	$A_3(q)$
$B_l(q), C_l(q)$	$l \geq 3$	$\{r_1, r_2\}$	$A_2(q)$		
$C_2(q)$	No graph	$\{r_1\}$	$A_1(q)$	$\{r_1, r_2\}$	$C_2(q)$
$C_2(q)$	q even, graph	$\{r_1, r_2\}$	$C_2(q)$		
$D_l(q)$	$l \geq 4$, no 3-graph	$\{r_1, r_2\}$	$A_2(q)$		
$D_4(q)$	3-graph	$\{r_2\}$	$A_1(q)$	$\{r_1, r_2, r_3, r_4\}$	$D_4(q)$
$E_l(q)$	$l \in \{6, 7, 8\}$	$\{r_1, r_4\}$	$A_2(q)$		
$F_4(q)$	No graph	$\{r_1, r_2\}$	$A_2(q)$		
$F_4(q)$	q even, graph	$\{r_2, r_3\}$	$C_2(q)$		
$G_2(q)$	No graph	$\{r_1\}$	$A_1(q)$	$\{r_1, r_2\}$	$G_2(q)$
$G_2(q)$	$p = 3$, graph	$\{r_1, r_2\}$	$G_2(q)$		
${}^2A_l(q)$		$\{r_1, r_l\}$	$A_1(q^2)$		
${}^2B_2(q)$		$\{r_1, r_2\}$	${}^2B_2(q)$		
${}^2D_l(q)$	$l \geq 4$	$\{r_1, r_2\}$	$A_2(q)$		
${}^3D_4(q)$		$\{r_1, r_3, r_4\}$	$A_1(q^3)$		
${}^2E_6(q)$		$\{r_1, r_4\}$	$A_2(q)$		
${}^2F_4(q)$		$\{r_1, r_4\}$	$A_1(q^2)$		
${}^2G_2(q)$		$\{r_1, r_2\}$	${}^2G_2(q)$		

In the sequel we repeatedly use Lemma 25 to show that a certain expression is not a non-trivial power.

Assume that $L_{P_{K_1}} = A_2(q)$. Since $(q+1)(q^2+q+1) = |P_{K_1} : B| = \frac{|S:B|}{|S:P_{K_1}|}$ is not a non-trivial power, we have the claim.

Assume that $L_{P_{K_1}} = A_1(q^2)$ or ${}^2B_2(q)$. Since $q^2+1 = |P_{K_1} : B| = \frac{|S:B|}{|S:P_{K_1}|}$ is not a non-trivial power, we have the claim.

Assume that $L_{P_{K_1}} = {}^2G_2(q)$ for $p = 3$. Since $q^3+1 = |P_{K_1} : B| = \frac{|S:B|}{|S:P_{K_1}|}$ is not a non-trivial power, we have the claim.

Assume that $S = {}^3D_4(q)$, so $L_{P_{K_1}} = A_1(q^3)$. If $q^3+1 = |P_{K_1} : B| = \frac{|S:B|}{|S:P_{K_1}|}$ is a non-trivial power, then $q = 2$. In this case, $|{}^3D_4(2)|_{13} = 13$, hence we have the claim.

TABLE 3. Index of the Borel subgroup in some parabolic subgroups P

L_P	$ P : B $
$A_1(q)$	$q + 1$
$A_1(q^2)$	$q^2 + 1$
$A_1(q^3)$	$q^3 + 1$
$A_2(q)$	$(q^2 + q + 1)(q + 1)$
$A_3(q)$	$(q^2 + 1)(q^2 + q + 1)(q + 1)^2$
${}^2B_2(q)$	$q^2 + 1$
$C_2(q)$	$(q^2 + 1)(q + 1)^2$
$D_4(q)$	$(q + 1)^4(q^2 + 1)^2(q^2 + q + 1)(q^2 - q + 1)$
${}^3D_4(q)$	$(q + 1)^2(q^2 - q + 1)(q^8 + q^4 + 1)$
$F_4(q)$	$(q^{12} + 1)(q^8 + 1)(q^2 - q + 1)(q^2 + q + 1)$
$G_2(q)$	$(q + 1)^2(q^2 + q + 1)(q^2 - q + 1)$
${}^2G_2(q)$	$q^3 + 1$

Assume that $L_{P_{K_1}} = A_1(q)$. If $q + 1 = |P_{K_1} : B| = \frac{|S:B|}{|S:P_{K_1}|}$ is a non-trivial power, then $q = 8$ or $q = 2^k - 1$. So, assume that $q = 8$ or $q = 2^k - 1$. We consider some cases.

- Let $S \cong A_1(q)$. There exists a prime number r such that $|A_1(q)|_r = r$.
- Let $S \cong C_2(q)$ and assume that X does not contain a non-trivial graph automorphism. We have that $(q^2 + 1)(q + 1)^2 = |P_{K_2} : B| = \frac{|S:B|}{|S:P_{K_2}|}$ is not a non-trivial power when $q = 8$ (by direct computation) and when $q = 2^k - 1$.
- In the remaining cases, we have that $|P_{K_2} : B| = \frac{|S:B|}{|S:P_{K_2}|}$ is not a non-trivial power when $q \in \{8, 2^k - 1\}$, since $q^2 + q + 1$ is not a non-trivial power.

Assume that $L_{P_{K_1}} = C_2(q)$, q is even and X contains a non-trivial graph automorphism. The number $(q + 1)^2(q^2 + 1) = |P_{K_1} : B| = \frac{|S:B|}{|S:P_{K_1}|}$ is not a non-trivial power since $q^2 + 1$ is a non-trivial power.

Assume that $S = G_2(q)$, $p = 3$ and X contains a non-trivial graph automorphism. The number $(q + 1)^2(q^2 + q + 1)(q^2 - q + 1) = |P_{K_1} : B| = \frac{|S:B|}{|S:P_{K_1}|}$ is not a non-trivial power. $\square$

5. RECOGNIZING THE CHIEF FACTORS FROM THE DIRICHLET POLYNOMIAL

Proposition 28. *Let $k \geq 5$ and assume that X is an almost simple group with socle Alt_k . Let Z be an almost simple group such that $P_{X, \text{Alt}_k}^{(r)}(s) = P_{Z, \text{soc}(Z)}^{(r)}(s)$ for each $r \in \mathfrak{p}(k!)$. Then $\text{soc}(Z) \cong \text{Alt}_k$.*

Proof. Let m be the minimal index of a proper subgroup of X which is supplemented by Alt_k . We have that $m = 10$ if $X \not\leq \text{Sym}_6$ and $k = 6$, otherwise $m = k$. Let $\pi = \mathfrak{p}(k!) - \mathfrak{p}(m)$ and note that $\pi \neq \emptyset$. We get $P_{X, \text{Alt}_k}^{(\pi)}(s) = P_{Z, \text{soc}(Z)}^{(\pi)}(s)$ hence $a_m(X, \text{Alt}_k) = a_m(Z, \text{soc}(Z)) \neq 0$. So m is the minimal index of a proper subgroup of Z . Moreover, since Z is an almost simple group, we have that Z is a primitive group of degree m .

Let $k = 6$ and assume that $X \not\leq \text{Sym}_6$. By Theorem 26 we have that $\{2, 3, 5\} \subseteq \text{p}(Z)$. By [GAP], if Z is a primitive group of degree k such that $\{2, 3, 5\} \subseteq \text{p}(Z)$ and the minimal index of a proper subgroup is 10, then $\text{soc}(Z) \cong \text{Alt}_6$.

Assume that if $k = 6$, then $X \leq \text{Sym}_6$. Let $k \in \{5, 6, 7\}$. By Theorem 26 we have that $\text{p}(k!) \subseteq \text{p}(Z)$. By [GAP], if Z is a primitive group of degree k such that $\text{p}(k!) \subseteq \text{p}(Z)$ and the minimal index of a proper subgroup is k , then $\text{soc}(Z) \cong \text{Alt}_k$.

Assume that $k \geq 8$. By [Nag52], we have that there exists a prime number p such that $k/2 < p < k - 2$. By Theorem 26, we have that p divides $|Z|$, hence Z contains a cycle of order p . Since Z is a primitive group of degree k containing a cycle of order p , by [Jor73, Théorème I] we have that $Z \geq \text{Alt}_k$ or $k \leq p + 2$. Since $p < k - 2$ we get that Z contains Alt_k and the proof is complete. $\square$

Theorem 29. *Let G and H be two finite groups such that $P_G(s) = P_H(s)$. Assume that the composition factors of G and H are alternating groups which satisfy the hypothesis of Theorem 2. Then G and H have the same non-Frattini chief factors.*

Proof. By [Pat10, Theorem 1] we have that G and H have the same non-Frattini abelian chief factors. So, we may assume that G and H have no non-Frattini abelian chief factors.

Let $\mathcal{CF}(G)$ be the set of non-Frattini chief factors of G . For $A \in \mathcal{CF}(G)$, let L_A be the primitive monolithic group associated with A , defined by $L_A = G/C_G(A)$ (note that $A \cong \text{soc}(L_A)$, for reference see [DL03, Definition 8] [Pat10, Section 2]).

By [DL03, Theorem 18] we have that

$$P_G(s) = \prod_{A \in \mathcal{CF}(G)} \tilde{P}_{L_A, A}(s),$$

where $\tilde{P}_{L_A, A}(s)$ are suitable polynomial such that $\tilde{P}_{L_A, A}^{(r)}(s) = P_{L_A, A}^{(r)}(s)$ for each $r \in \text{p}(A)$. So, $\tilde{P}_{L_A, A}(s)$ is irreducible since in Theorem 23 and Proposition 24 we proved that a Dirichlet polynomial $P(s)$ such that $\text{p}(A) = \{r : \exists k(a_k(P(s)) \neq 0, r \mid k)\}$ and $P^{(r)}(s) = P_{L_A, A}^{(r)}(s)$ for each $r \in \text{p}(A)$ is irreducible.

Thus there exists a bijection between the sets $\mathcal{CF}(G)$ and $\mathcal{CF}(H)$ such that $A \in \mathcal{CF}(G)$ and $B \in \mathcal{CF}(H)$ are associated if and only if $\tilde{P}_{L_A, A}(s) = \tilde{P}_{L_B, B}(s)$. We claim that $A \cong B$. Since $P_{L_A, A}^{(r)}(s) = P_{L_B, B}^{(r)}(s)$ for each $r \in \text{p}(A)$, by Proposition 27 we have that $n = |L_A : N_{L_A}(S_A)| = |L_B : N_{L_B}(S_B)|$, hence we get $P_{X_A, S_A}^{(r)}(s) = P_{X_B, S_B}^{(r)}(s)$ for each $r \in \text{p}(S_A)$. By Proposition 28, this implies that $S_A \cong S_B$, hence $A \cong S_A^n \cong S_B^n \cong B$. $\square$

REFERENCES

- [Bos96] N. Boston. A probabilistic generalization of the Riemann zeta function. *Analytic Number Theory*, 1:155–162, 1996. MR1399336 (97e:11106)
- [Che10] Yuan-You Fu-Rui Cheng. Explicit estimate on primes between consecutive cubes. *Rocky Mountain J. Math.*, 40(1):117–153, 2010. MR2607111 (2011f:11120)
- [DL03] E. Detomi and A. Lucchini. Crowns and factorization of the probabilistic zeta function of a finite group. *J. Algebra*, 265:651–668, 2003. MR1987022 (2004e:20119)
- [DL07] E. Damian and A. Lucchini. Finite groups with p-multiplicative probabilistic zeta function. *Comm. Algebra*, 35(11):3451–3472, 2007. MR2362665 (2008k:20045)
- [DLM04] E. Damian, A. Lucchini, and F. Morini. Some properties of the probabilistic zeta function of finite simple groups. *Pacific J. Math.*, 215:3–14, 2004. MR2060491 (2005b:20042)
- [Dus10] P. Dusart. Estimates of some function over primes without R.H. 2010.

- [Erd34] P. Erdős. A theorem of Sylvester and Schur. *J. London Math. Soc.*, S1-9 no. 4, 282–288, 1934.
- [GAP] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.12*.
- [Hal36] P. Hall. The Eulerian functions of a group. *Quart. J. Math.*, 7:134–151, 1936.
- [HIÖ89] T. Hawkes, M. Isaacs, and M. Özaydin. On the Möbius function of a finite group. *Rocky Mountain Journal*, 19:1003–1034, 1989. MR1039540 (90k:20046)
- [Jor57] C. Jordan. *Traité des substitutions et des équations algébriques - Nouveau tirage de l'édition de 1870*. Gauthier-Villars, A. Blanchard, Paris, 1957. MR0091260 (19:937c)
- [Jor73] C. Jordan. Sur la limite de transitivité des groupes non alternés. *Bull. Soc. Math. France*, 40–71, 1872–73. MR1503635
- [Lan02] S. Lang. *Algebra*, volume 211. Graduate Texts in Mathematics, New York, 2002. MR1878556 (2003e:00003)
- [Leh64] D. H. Lehmer. On a problem of Störmer. *Ill. J. Math.*, 8:57–79, 1964. MR0158849 (28:2072)
- [LP67] L. J. Lander and T. R. Parkin. On the first appearance of prime differences. *Math. Comp.*, 21:483–488, 1967. MR0230677 (37:6237)
- [LPS90] M. Liebeck, C. Praeger, and J. Saxl. *The maximal factorization of the finite simple groups and their automorphism groups*. Mem. Amer. Math. Soc. 86 (1990), no. 432, Providence, Rhode Island. MR1016353 (90k:20048)
- [Man96] A. Mann. Positively finitely generated groups. *Forum Math.*, 8:429–459, 1996. MR1393323 (97j:20029)
- [Mas07] M. Massa. The probabilistic zeta function of the alternating group $\text{Alt}(p+1)$. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.*, 8:581–591, 2007. MR2351530 (2009c:20038)
- [Nag52] J. Nagura. On the interval containing at least one prime number. *Proc. Japan Acad.*, 28:177–181, 1952. MR0050615 (14:355b)
- [NN03] Bertil Nyman and Thomas R. Nicely. New prime gaps between 10^{15} and 5×10^{16} . *J. Integer Seq.*, 6:Article 03.3.1, 2003. MR1997838 (2004e:11143)
- [Pat09] M. Patassini. On the irreducibility of the Dirichlet polynomial of a simple group of Lie type. Accepted by *Israel J. Math.*
- [Pat10] M. Patassini. Recognizing the non-Frattini abelian chief factors from the Probabilistic Zeta function of a finite group. In preparation, 2010.
- [Sch76] L. Schoenfeld. Sharper bounds for the Chebyshev functions $\varphi(x)$ ii. *Math. Comput.*, 30(134):337–360, 1976. MR0457374 (56:15581b)
- [Ser08] P. Jiménez Seral. Coefficient of the Probabilistic Zeta function of a monolithic group. *Glasgow J. Math.*, 50:75–81, 2008. MR2381734 (2008j:20066)
- [Sta97] R. P. Stanley. *Enumerative Combinatorics, Vol. 1*. Cambridge University Press, Cambridge, 1997. MR1442260 (98a:05001)

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PADOVA, VIA TRIESTE, 63 - 35121 PADOVA, ITALIA

E-mail address: frapmass@gmail.com