THE AUTOMORPHISM GROUP OF A SIMPLE $\mathcal{Z}$ -STABLE C^* -ALGEBRA

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ABSTRACT. We study the automorphism group of a simple, unital, $\mathcal{Z}$ -stable C^* -algebra. We show that $\overline{\text{Inn}}_0(\mathfrak{A})$ is a simple topological group and $\frac{\overline{\text{Inn}}(\mathfrak{A})}{\overline{\text{Inn}}_0(\mathfrak{A})}$ is isomorphic (as topological groups) to the inverse limit of quotient groups of $K_1(\mathfrak{A})$, where $\mathfrak{A}$ is a $\mathcal{Z}$ -stable C^* -algebra satisfying the following property: for every UHF algebra $\mathfrak{B}$, $\mathfrak{A} \otimes \mathfrak{B}$ is a nuclear, separable, simple, tracially AI algebra satisfying the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet. By the recent results of Lin and Winter, ordered K -theory, traces, and the class of the unit is a complete isomorphism invariant for this class of C^* -algebra.

1. INTRODUCTION

Denote the group of automorphisms of $\mathfrak{A}$ equipped with the topology of pointwise convergence by $\text{Aut}(\mathfrak{A})$, denote the closure of the group of inner automorphisms of $\mathfrak{A}$ by $\overline{\text{Inn}}(\mathfrak{A})$, and denote the closure of the group of inner automorphisms of $\mathfrak{A}$ whose implementing unitaries are connected to $1_{\mathfrak{A}}$ via a norm continuous path of unitaries by $\overline{\text{Inn}}_0(\mathfrak{A})$. We then have that $\text{Aut}(\mathfrak{A})$ decomposes into the following series of closed normal subgroups:

$$\overline{\text{Inn}}_0(\mathfrak{A}) \triangleleft \overline{\text{Inn}}(\mathfrak{A}) \triangleleft \text{Aut}(\mathfrak{A}).$$

In [7], Elliott and Rørdam showed that for a simple, unital C^* -algebra $\mathfrak{A}$ that either is real rank zero, stable rank one and weakly unperforated, or is purely infinite, $\overline{\text{Inn}}_0(\mathfrak{A})$ is a simple topological group (no nontrivial closed normal subgroup). They also showed that $\frac{\overline{\text{Inn}}(\mathfrak{A})}{\overline{\text{Inn}}_0(\mathfrak{A})}$ is totally disconnected when $\mathfrak{A}$ is a simple AT algebra with real rank zero. Hence, when $\mathfrak{A}$ is a simple AT algebra with real rank zero, the results of Elliott and Rørdam give a structure theorem for $\text{Aut}(\mathfrak{A})$ since $\text{Aut}(\mathfrak{A})$ fits into the following exact sequence:

$$\{1\} \rightarrow \overline{\text{Inn}}(\mathfrak{A}) \rightarrow \text{Aut}(\mathfrak{A}) \rightarrow \text{Aut}(K_*(\mathfrak{A}))_{+,1} \rightarrow \{1\}.$$

In their paper, they asked if $\overline{\text{Inn}}_0(\mathfrak{A})$ is a simple topological group and if $\frac{\overline{\text{Inn}}(\mathfrak{A})}{\overline{\text{Inn}}_0(\mathfrak{A})}$ is totally disconnected for every simple, unital C^* -algebra $\mathfrak{A}$.

Recently, the authors in [21] and [22] proved $\overline{\text{Inn}}_0(\mathfrak{A})$ is a simple topological group and $\frac{\overline{\text{Inn}}(\mathfrak{A})}{\overline{\text{Inn}}_0(\mathfrak{A})}$ is totally disconnected for all nuclear, separable, simple, tracially AI algebras satisfying the UCT and for all nuclear, purely infinite, separable, simple C^* -algebras satisfying the UCT. In this paper, we generalize the results of

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[21] and [22]. We show that $\overline{\text{Inn}(\mathfrak{A})}_{\text{Inn}_0(\mathfrak{A})}$ is isomorphic to an inverse limit of discrete abelian groups (similar to the one used by Elliott and Rørdam in [7]) for a nuclear, separable, simple $\mathcal{Z}$ -stable C^* -algebra $\mathfrak{A}$ such that for each supernatural number $\mathfrak{p}$ of infinite type, $\mathfrak{A} \otimes M_{\mathfrak{p}}$ is a tracially AI algebra that satisfies the UCT. (Here, $M_{\mathfrak{p}}$ is the UHF algebra with supernatural number $\mathfrak{p}$.) In fact, we show that $\overline{\text{Inn}(\mathfrak{A})}_{\text{Inn}_0(\mathfrak{A})}$ is isomorphic (as topological groups) to the inverse limit of quotient groups of $K_1(\mathfrak{A})$, where the quotient groups are given the discrete topology and the inverse limit is given the inverse limit topology. Consequently, $\overline{\text{Inn}(\mathfrak{A})}_{\text{Inn}_0(\mathfrak{A})}$ is totally disconnected. Moreover, we show that $\overline{\text{Inn}_0(\mathfrak{A})}$ is a simple topological group for any separable, simple, unital, $\mathcal{Z}$ -stable C^* -algebra that is either nuclear and quasidiagonal or is exact and has a unique tracial state.

It turns out that a large class of simple C^* -algebras satisfies the above condition. We give examples of C^* -algebras in this class.

- (1) Nuclear, separable, simple, unital, tracially AI algebras which satisfy the UCT.
- (2) Simple, unital, $\mathcal{Z}$ -stable AH algebras (see Corollary 11.12 of [16]).
- (3) The Jiang-Su algebra $\mathcal{Z}$ ([9]).
- (4) Simple, unital $\mathcal{Z}$ -stable C^* -algebras which are locally type I with unique tracial state (see Corollary 5.6 of [17]; see also Corollary 8.2 of [33]).
- (5) Simple, unital $\mathcal{Z}$ -stable, ASH-algebras $\mathfrak{A}$ such that $T(\mathfrak{A}) = S[1](K_0(\mathfrak{A}))$, where $S[1](K_0(\mathfrak{A}))$ is the state space of $K_0(\mathfrak{A})$ (see Corollary 5.5 of [17]; also see Corollaries 6.3 and 8.3 of [33]).
- (6) Simple, unital ATD algebras (an ATD algebra is an inductive limit of dimension drop circle algebras; see Definitions 2.3 and 2.4 and Theorem 4.2 of [18]).

The paper is organized as follows: In Section 2, we show that for a separable, simple, unital, $\mathcal{Z}$ -stable C^* -algebra $\mathfrak{A}$ that is either nuclear and quasidiagonal or is exact and has a unique tracial state, $\overline{\text{Inn}_0(\mathfrak{A})}$ is a simple topological group. In Section 3, we introduce the Bott Maps which are defined by Lin in [13] and we present some technical results. See Theorem 3.16, Corollary 3.17, and Theorem 3.22. In Section 4, we show that the topological group $\overline{\text{Inn}(\mathfrak{A})}_{\text{Inn}_0(\mathfrak{A})}$ is isomorphic to the inverse limit of discrete abelian groups, where the inverse limit is given the inverse limit topology. We also provide a partial structure theorem for $\text{Aut}(\mathfrak{A})$.

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2. SIMPLICITY OF $\overline{\text{Inn}_0(\mathfrak{A})}$

First we start with some notation that will be used throughout the paper. Let $\mathfrak{A}$ be a C^* -algebra and let p and q be projections in $\mathfrak{A}$.

- (1) If p and q are *Murray-von Neumann equivalent*, i.e., there exists $v \in \mathfrak{A}$ such that $v^*v = p$ and $vv^* = q$, we write $p \sim q$.
- (2) If there exists $v \in \mathfrak{A}$ such that $v^*v = p$ and $vv^* \leq q$, we write $p \precsim q$.
- (3) If there exists $v \in \mathfrak{A}$ such that $v^*v = p$, $vv^* \leq q$, and $vv^* \neq q$, we write $p \prec q$.
- (4) For $a, b \in \mathfrak{A}$, $(a, b) = aba^*b^*$.

- (5) If $\mathfrak{A}$ is unital, then denote the norm closure of the commutator subgroup of $U(\mathfrak{A})$ by $CU(\mathfrak{A})$ and denote the norm closure of the commutator subgroup of $U(\mathfrak{A})_0$ by $CU(\mathfrak{A})_0$.

Definition 2.1. Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers. Set

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} = \{f \in C([0, 1], M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}) : f(0) \in M_{\mathfrak{p}} \otimes 1_{M_{\mathfrak{q}}} \text{ and } f(1) \in 1_{M_{\mathfrak{p}}} \otimes M_{\mathfrak{q}}\}.$$

(Here, $M_{\mathfrak{p}}$ is the UHF algebra with supernatural number $\mathfrak{p}$; similar for $M_{\mathfrak{q}}$.)

We shall regard $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ (and any tensor product with it) as a $C([0, 1])$ -algebra with the obvious central embedding of $C([0, 1])$.

Lemma 2.2. *Let $\mathfrak{A}$ be a separable, simple, unital C^* -algebra and let $\mathfrak{C}$ be a UHF algebra. Then $\mathfrak{A} \otimes \mathfrak{C}$ is $\mathcal{Z}$ -stable and hence either purely infinite or stably finite. Moreover, if $\mathfrak{A} \otimes \mathfrak{C}$ is (stably) finite, then it has the following properties:*

- (1) *Stable rank one.*
- (2) *Cancellation of projections.*
- (3) *Strict comparison of positive elements when $\mathfrak{A}$ is, additionally, exact.*
- (4) *Weak unperforation.*
- (5) *K_1 -injectivity.*
- (6) *The (SP) property.*
- (7) *For every nonzero projection $p \in \mathfrak{A} \otimes \mathfrak{C}$, for every $n \geq 2$, $p(\mathfrak{A} \otimes \mathfrak{C})p$ contains a unital sub- C^* -algebra which is isomorphic to $M_n \oplus M_{n+1}$.*
- (8) *If p, q are nonzero projections in $\mathfrak{A} \otimes \mathfrak{C}$, then there exist nonzero projections p', q' in $p(\mathfrak{A} \otimes \mathfrak{C})p$ and $q(\mathfrak{A} \otimes \mathfrak{C})q$, respectively, such that $p' \sim q'$.*

Proof. These results are contained in [2], [8], [9], [25], [26], [27] and [24]. $\square$

Lemma 2.3. *Let $\mathfrak{A}$ be a unital $\mathcal{Z}$ -stable C^* -algebra. Then $\mathbb{T} \subseteq CU(\mathfrak{A})_0$, i.e., $CU(\mathfrak{A})_0$ contains all scalar unitaries.*

Proof. Since $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{Z}$, it is enough to show that $\mathbb{T} \subseteq CU(\mathfrak{A} \otimes \mathcal{Z})_0$. Clearly, $CU(1_{\mathfrak{A}} \otimes \mathcal{Z}) = CU(1_{\mathfrak{A}} \otimes \mathcal{Z})_0 \subseteq CU(\mathfrak{A} \otimes \mathcal{Z})_0$. By [20], $CU(1_{\mathfrak{A}} \otimes \mathcal{Z})_0$ contains all scalar unitaries. Therefore, $\mathbb{T} \subseteq CU(\mathfrak{A} \otimes \mathcal{Z})_0$. $\square$

The next lemma can be proven using a spectral theory argument.

Lemma 2.4. *For every $\epsilon > 0$, there exists $\delta > 0$ such that for any unital C^* -algebra $\mathfrak{A}$, if*

- (1) $p_1, p_2, \dots, p_n$ are pairwise orthogonal projections in $\mathfrak{A}$,
- (2) $q_1, q_2, \dots, q_n$ are pairwise orthogonal projections in $\mathfrak{A}$,
- (3) $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars (complex numbers) with norm one,
- (4) $|\alpha_i - \alpha_j| \geq \epsilon$ for $i \neq j$, and
- (5) $\|(\alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n) - (\alpha_1 q_1 + \alpha_2 q_2 + \dots + \alpha_n q_n)\| < \delta$,

then $\|p_i - q_i\| < \epsilon$ and $p_i \sim q_i$ in $\mathfrak{A}$ for $1 \leq i \leq n$.

Lemma 2.5. *There exists a $*$ -isomorphism $\Phi : \mathcal{Z} \rightarrow \mathcal{Z} \otimes \mathcal{Z}$ and there exists a sequence of unitaries $\{u_n\}_{n=1}^{\infty}$ in $U(\mathcal{Z} \otimes \mathcal{Z})_0$ such that for all $a \in \mathcal{Z}$,*

$$\lim_{n \rightarrow \infty} \|\Phi(a) - u_n(a \otimes 1_{\mathcal{Z}})u_n^*\| = 0.$$

Proof. The result follows from Theorems 7.6 and 8.7 of [9]. $\square$

Lemma 2.6. *Let $\mathfrak{B}$ be a simple, separable, unital C^* -algebra and let G be a closed normal subgroup of $U(\mathfrak{B} \otimes \mathcal{Z} \otimes \mathcal{Z})_0$ that properly contains $\mathbb{T}$. Then G contains $CU(1_{\mathfrak{B} \otimes \mathcal{Z} \otimes \mathcal{Z}})_0$.*

Proof. By Theorem 3 of [8], $\mathfrak{B} \otimes \mathcal{Z} \otimes \mathcal{Z} (\cong \mathfrak{B} \otimes \mathcal{Z})$ is either purely infinite or stably finite. If $\mathfrak{B} \otimes \mathcal{Z} \otimes \mathcal{Z}$ is purely infinite, then by Theorem 2.4 of [7], $G = U(\mathfrak{B} \otimes \mathcal{Z} \otimes \mathcal{Z})_0$; thus, G contains $CU(1_{\mathfrak{B} \otimes \mathcal{Z} \otimes \mathcal{Z}})_0$. Hence, we may assume that $\mathfrak{B} \otimes \mathcal{Z} \otimes \mathcal{Z}$ (and hence $\mathfrak{B}$; see Lemma 3.3 of [8]) is stably finite.

Set $\mathfrak{A} = \mathfrak{B} \otimes \mathcal{Z}$. Let u be an element of $G \setminus \mathbb{T}$ and $\Phi : \mathcal{Z} \rightarrow \mathcal{Z} \otimes \mathcal{Z}$ be the $*$ -isomorphism given in Lemma 2.5. Then by Lemma 2.5, there exist $w \in \mathfrak{A}$ and a sequence of unitaries $\{u_n\}_{n=1}^\infty \subseteq U(\mathfrak{A} \otimes \mathcal{Z})_0$ such that $(\text{id}_{\mathfrak{B}} \otimes \Phi)(w) = u$ and

$$\lim_{n \rightarrow \infty} \|u_n^*(\text{id}_{\mathfrak{B}} \otimes \Phi)(w)u_n - w \otimes 1_{\mathcal{Z}}\| = 0.$$

Since $\text{id}_{\mathfrak{B}} \otimes \Phi$ is an isomorphism and since $u \in U(\mathfrak{A} \otimes \mathcal{Z})_0 \setminus \mathbb{T}$, $w \in U(\mathfrak{A})_0 \setminus \mathbb{T}$. Since G is a closed normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z})_0$, $u_n^* u u_n = u_n^*(\text{id}_{\mathfrak{B}} \otimes \Phi)(w)u_n \in G$, which implies that $w \otimes 1_{\mathcal{Z}} \in G$. Hence, G contains a unitary of the form $x = w \otimes 1_{\mathcal{Z}}$, where $w \in U(\mathfrak{A})_0 \setminus \mathbb{T}$.

Let $\mathfrak{p}, \mathfrak{q}$ be relatively prime supernatural numbers of infinite type. By Theorem 3.4 of [28], $\mathcal{Z}$ is a C^* -inductive limit $\mathcal{Z} = \overline{\bigcup_{n=1}^\infty \mathcal{Z}_n}$, where $\mathcal{Z}_n \cong \mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}$ for all $n \geq 1$ and where the connecting maps are unital and injective. Consider an arbitrary building block $\mathcal{Z}_N$. We will prove that G contains a nonscalar unitary in $CU(1_{\mathfrak{A} \otimes \mathcal{Z}_N}) = CU(1_{\mathfrak{A} \otimes \mathcal{Z}_N})_0$.

Now $x \in U(\mathfrak{A} \otimes \mathcal{Z}_N)_0$; in particular, $x = w \otimes 1_{\mathcal{Z}_N} \in \mathfrak{A} \otimes \mathcal{Z}_N \cong \mathfrak{A} \otimes \mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}$. Since $\mathbb{T} \subseteq G$, multiplying x by a scalar if necessary, we may assume that 1 is in the spectrum of w . Hence, 1 will be in the spectrum of x . Since w is not in $\mathbb{T}$, the spectrum of w and x contain a point other than 1.

Case 1. Suppose that the spectrum of w contains a point $\alpha \neq -1, 1, i, -i$. To proceed, recall the following matrix computation:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Let $\{u(t)\}_{t \in [0,1]}$ be the continuous path of unitaries in M_2 given by

$$(2.1) \quad u(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\pi t) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

for all $t \in [0, 1]$. Hence, $u(0) = 1_{M_2}$ and $u(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also, for $\beta, \gamma \in \mathbb{T}$, if $\{v(t)\}_{t \in [0,1]}$ is the continuous path of unitaries in M_2 that is given by

$$(2.2) \quad v(t) = u(t) \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix} u(t)^* \begin{bmatrix} \bar{\beta} & 0 \\ 0 & \bar{\gamma} \end{bmatrix}$$

for all $t \in [0, 1]$, then $v(0) = 1_{M_2}$ and $v(1) = \begin{bmatrix} \bar{\beta}\gamma & 0 \\ 0 & \beta\bar{\gamma} \end{bmatrix}$. Moreover, by a direct computation, we get that if $\beta \neq \pm\gamma$, then for all $t \in (0, 1)$, the eigenvalues of $v(t)$ are distinct and are complex conjugates of each other. Also, for $0 < s, t < 1$ with $s \neq t$, the set of eigenvalues of $v(s)$ is different from the set of eigenvalues of $v(t)$.

Let $h_1, h_2 : [0, 1] \rightarrow \mathbb{T}$ be the unique continuous functions such that $h_1(0) = h_2(0) = 1, h_1(1) = \alpha^2, h_2(1) = \bar{\alpha}^2$, and $\{h_1, h_2\}$ are the eigenfunctions of the

continuous path of unitaries $\{u(t)\text{diag}(\alpha, \bar{\alpha})u(t)^*\text{diag}(\bar{\alpha}, \alpha)\}_{t \in [0,1]}$, i.e., for all $t \in [0, 1]$, $\{h_1(t), h_2(t)\}$ is the set of (distinct when $0 < t < 1$) eigenvalues of

$$u(t)\text{diag}(\alpha, \bar{\alpha})u(t)^*\text{diag}(\bar{\alpha}, \alpha).$$

Define two continuous functions $g_1, g_2 : [0, 1] \rightarrow \mathbb{R}$ by

$$g_1(t) = \begin{cases} \alpha^2 & \text{if } t \in [\frac{2}{5}, \frac{3}{5}], \\ 1 & \text{if } t \in [0, \frac{1}{5}] \cup [\frac{4}{5}, 1], \\ h_1(5t - 1) & \text{if } t \in [\frac{1}{5}, \frac{2}{5}], \\ h_1(-5t + 4) & \text{if } t \in [\frac{3}{5}, \frac{4}{5}], \end{cases}$$

$$g_2(t) = \begin{cases} \bar{\alpha}^2 & \text{if } t \in [\frac{2}{5}, \frac{3}{5}], \\ 1 & \text{if } t \in [0, \frac{1}{5}] \cup [\frac{4}{5}, 1], \\ h_2(5t - 1) & \text{if } t \in [\frac{1}{5}, \frac{2}{5}], \\ h_2(-5t + 4) & \text{if } t \in [\frac{3}{5}, \frac{4}{5}]. \end{cases}$$

Claim 1. For every $\epsilon > 0$, there exist pairwise orthogonal nonzero projections $r_1, r_2 \in C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ such that if p_1, p_2 are nonzero projections in $r_1(C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q)r_1, r_2(C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q)r_2$, respectively, with $p_1 \sim p_2$, then there exists a unitary $w' \in G$ such that the following hold:

- (a) $w'(1 - (p_1 + p_2)) = (1 - (p_1 + p_2))w' = 1 - (p_1 + p_2)$ in $C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ and
- (b) $\|w'(t) - (g_1(t)p_1(t) + g_2(t)p_2(t) + 1 - (p_1(t) + p_2(t)))\| < \epsilon$ for all $t \in [0, 1]$.

Towards proving Claim 1, let $\epsilon > 0$ be given. For simplicity, we may assume that $\epsilon < 1/2$. Plug $\min \left\{ \frac{|1-\alpha|}{2}, \frac{|1-\bar{\alpha}|}{2}, \frac{|\alpha-\bar{\alpha}|}{2}, \frac{\epsilon}{100} \right\}$ into Lemma 2.4 to get a positive real number δ' . Let $\delta = \min \left\{ \frac{\epsilon}{1000}, \frac{\delta'}{1000}, \frac{|1-\alpha|}{1000}, \frac{|1-\bar{\alpha}|}{1000}, \frac{|\alpha-\bar{\alpha}|}{1000} \right\}$. Contracting $\delta > 0$ if necessary, we may assume that for all $\gamma_1, \gamma_2 \in \mathbb{T}$, if $|\gamma_1 - \gamma_2| < 100\delta$, then $|\gamma_1^2 - \gamma_2^2| < \frac{\epsilon}{100}$. By Lemma 2.2, there exist nonzero projections r'_1, s'_1 in $\mathfrak{A} \otimes M_p \otimes M_q$ such that the following hold:

- (1) r'_1 is contained in the hereditary sub- C^* -algebra of $\mathfrak{A} \otimes M_p \otimes M_q$ generated by $f(w \otimes 1_{M_p \otimes M_q})$, where f is a continuous real-valued function on $\mathbb{C}$ with $0 \leq f \leq 1$, $f(1) = 1$ and f vanishes outside of a small precompact open neighbourhood O_1 of 1.
- (2) s'_1 is contained in the hereditary sub- C^* -algebra of $\mathfrak{A} \otimes M_p \otimes M_q$ that is generated by $g(w \otimes 1_{M_p \otimes M_q})$, where g is a continuous real-valued function on $\mathbb{C}$ with $0 \leq g \leq 1$, $g(\alpha) = 1$ and g vanishes outside of a small precompact neighbourhood O_2 of α .
- (3) $\overline{O_1} \cap \overline{O_2} = \emptyset$.
- (4) $r'_1 \sim s'_1$ in $\mathfrak{A} \otimes M_p \otimes M_q$.
- (5) For all $s, t \in \{r'_1(w \otimes 1_{M_p \otimes M_q}), (w \otimes 1_{M_p \otimes M_q})r'_1, r'_1(w \otimes 1_{M_p \otimes M_q})r'_1, r'_1\}$,

$$\|s - t\| < \delta.$$
- (6) For all $s, t \in \{s'_1(w \otimes 1_{M_p \otimes M_q}), (w \otimes 1_{M_p \otimes M_q})s'_1, s'_1(w \otimes 1_{M_p \otimes M_q})s'_1, \alpha s'_1\}$,

$$\|s - t\| < \delta.$$

Define $r_1, s_1 \in C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ by $r_1 = r'_1 \otimes 1_{C[0,1]}$ and $s_1 = s'_1 \otimes 1_{C[0,1]}$. Let p_1, q_1 be projections in $\mathfrak{A} \otimes M_p \otimes M_q$ that satisfy the hypotheses of the statement

of Claim 1. From the definition of r_1, s_1 and the properties of r'_1, s'_1 , we know that the following must hold for p_1, q_1 :

- (7) p_1 is contained in the hereditary sub- C^* -algebra of $C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ generated by $f(x)$, and q_1 is contained in the hereditary sub- C^* -algebra of $C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ generated by $g(x)$, where f, g are the functions in the definition of r'_1, s'_1 .

- (8) For all $s, t \in \{p_1x, xp_1, p_1xp_1, p_1\}$,

$$\|s - t\| < \delta.$$

- (9) For all $s, t \in \{q_1x, xq_1, q_1xq_1, \alpha q_1\}$,

$$\|s - t\| < \delta.$$

Since p_1 and q_1 are orthogonal projections and since $p_1 \sim q_1$, there exists $v_1 \in U(C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q)_0$ such that $v_1p_1v_1^* = v_1^*p_1v_1 = q_1$ and $v_1(1 - (p_1 + q_1)) = (1 - (p_1 + q_1))v_1 = 1 - (p_1 + q_1)$. We may assume that $p_1v_1 = v_1^*q_1$.

By (7)–(9) and the definition of v_1 , we get $\|x^*v_1xv_1^*p_1 - \alpha p_1\| < 2\delta$, $\|x^*v_1xv_1^*q_1 - \overline{\alpha}q_1\| < 2\delta$, and

$$\|x^*v_1xv_1^*(1 - (p_1 + q_1)) - (1 - (p_1 + q_1))\| < 6\delta.$$

The above inequalities imply that

$$(2.3) \quad \|x^*v_1xv_1^* - (\alpha p_1 + \overline{\alpha}q_1 + 1 - (p_1 + q_1))\| < 10\delta.$$

Recall that if a, b are elements in a unital C^* -algebra such that a is invertible and $\|a - b\| < \frac{1}{\|a^{-1}\|}$, then b is also invertible in the C^* -algebra. From this, (2.3) and the definition of δ , it follows that the spectrum of $x^*v_1xv_1^*$ is contained in three pairwise disjoint open balls with centres $1, \alpha$ and $\overline{\alpha}$. Since the spectrum is a compact set, we may assume that the closures of the three open balls are also pairwise disjoint. In particular, we can take each open ball to have radius 10δ . Hence, there exist pairwise disjoint self-adjoint partial isometries $x_1, y_1, z_1 \in C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ and there exist pairwise disjoint projections $c_1, d_1, e_1 \in C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ such that the following hold:

$$(10) \quad x^*v_1xv_1^* = x_1 + y_1 + z_1.$$

$$(11) \quad x_1, y_1, z_1 \text{ are elements of the open balls about } \alpha, \overline{\alpha}, 1 \text{ (respectively), all with radius } 10\delta. \text{ (Of course, we are really applying the continuous functional calculus to } x^*v_1xv_1^*.)$$

$$(12) \quad x_1^*x_1 = x_1x_1^* = c_1, y_1^*y_1 = y_1y_1^* = d_1 \text{ and } z_1^*z_1 = z_1z_1^* = e_1.$$

$$(13) \quad c_1 + d_1 + e_1 = 1_{C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q}.$$

By (11) and (12), we have that

$$(2.4) \quad \|x^*v_1xv_1^* - (\alpha c_1 + \overline{\alpha}d_1 + e_1)\| < 10\delta.$$

Together with (2.3), we get

$$(2.5) \quad \|\alpha p_1 + \overline{\alpha}q_1 + 1 - (p_1 + q_1) - (\alpha c_1 + \overline{\alpha}d_1 + e_1)\| < 20\delta.$$

Hence, by (2.5), the definition of δ and Lemma 2.4, we have that c_1, d_1, e_1 is Murray-von Neumann equivalent and close to $p_1, q_1, 1 - (p_1 + q_1)$, respectively.

Let $w_1 \in U(C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q)_0$ be a unitary such that $w_1c_1w_1^* = p_1$, $w_1d_1w_1^* = q_1$ and $w_1e_1w_1^* = 1 - (p_1 + q_1)$. We can choose w_1 to be close to 1. Then, by (2.4),

$$(2.6) \quad \|w_1x^*v_1xv_1^*w_1^* - (\alpha p_1 + \overline{\alpha}q_1 + 1 - (p_1 + q_1))\| < 10\delta.$$

Let x_1 be the unitary in $U(C[0, 1] \otimes \mathfrak{A} \otimes \mathbf{M}_p \otimes \mathbf{M}_q)_0$ that is given by

$$x_1 = (w_1 x^* v_1 x v_1^* w_1^*) v_1 (w_1 x^* v_1 x v_1^* w_1^*)^* v_1^*.$$

By (10)–(13), (2.6), and the definitions of v_1 , w_1 , x_1 and δ , the following hold:

- (14) $x_1 p_1 = p_1 x_1 = p_1 x_1 p_1$ and $\|x_1 p_1 - \alpha^2 p_1\| < 20\delta$.
- (15) $x_1 q_1 = q_1 x_1 = q_1 x_1 q_1$ and $\|x_1 q_1 - \overline{\alpha}^2 q_1\| < 20\delta$.
- (16) $x_1(1 - (p_1 + q_1)) = (1 - (p_1 + q_1))x_1 = 1 - (p_1 + q_1)$.

We now define elements $v_2, w_2, v_3 \in U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$. To simplify notation, assume that $p_1, q_1, v_1, w_1 \in 1_{C[0,1]} \otimes \mathfrak{A} \otimes \mathbf{M}_p \otimes \mathbf{M}_q$. The general case is similar. To define v_2 , we first consider the norm-continuous path of unitaries $\{u(t)\}_{t \in [0,1]}$ in $(p_1 + q_1)(\mathfrak{A} \otimes \mathbf{M}_p \otimes \mathbf{M}_q)(p_1 + q_1)$ given by (2.1) with the standard system of matrix units for $\mathbf{M}_2$ as $e_{1,1} = p_1$, $e_{2,2} = q_1$, $e_{1,2} = p_1 v_1 = v_1^* q_1$. The last equality follows from the definition of v_1 . Define v_2 by

$$v_2(t) = \begin{cases} v_1 & \text{if } t \in [\frac{2}{5}, \frac{3}{5}], \\ 1 & \text{if } t \in [0, \frac{1}{5}] \cup [\frac{4}{5}, 1], \\ u(5t - 1) + 1 - (p_1 + q_1) & \text{if } t \in [\frac{1}{5}, \frac{2}{5}], \\ u(-5t + 4) + 1 - (p_1 + q_1) & \text{if } t \in [\frac{3}{5}, \frac{4}{5}]. \end{cases}$$

Note that $v_2 \in U(\mathfrak{A} \otimes \mathcal{Z})_0$. Let $w_2, v_3 \in U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$ be unitaries chosen so that $w_2(t) = w_1$ for $t \in [\frac{1}{5}, \frac{4}{5}]$, $w_2(0) = w_2(1) = 1$, $v_3(t) = v_1$ for $t \in [\frac{1}{5}, \frac{4}{5}]$, and $v_3(0) = v_3(1) = 1$.

Since $x \in G$ and G is a normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$,

$$w_3 = (w_2 x^* v_3 x v_3^* w_2^*) v_2 (w_2 x^* v_3 x v_3^* w_2^*)^* v_2^*$$

is an element of G . Hence, it follows, by the definitions above, that the following hold:

- (17) $w_3(1 - (p_1 + q_1)) = (1 - (p_1 + q_1))w_3 = 1 - (p_1 + q_1)$.
- (18) $\|w_3 - (\alpha p_1 + \overline{\alpha} q_1 + 1 - (p_1 + q_1))v_2(\alpha p_1 + \overline{\alpha} q_1 + 1 - (p_1 + q_1))^* v_2^*\| < 20\delta$.
- (19) The eigenvalues of $(\alpha p_1 + \overline{\alpha} q_1 + 1 - (p_1 + q_1))v_2(t)(\alpha p_1 + \overline{\alpha} q_1 + 1 - (p_1 + q_1))^* v_2^*(t)$ are given by $\{g_1(t), g_2(t)\}$ for all $t \in [0, 1]$.

Since $g_1(t) \neq g_2(t)$ for $t \in [\frac{1}{5}, \frac{4}{5}]$ and $g_1(t) = g_2(t) = 1$ for $t \in [0, \frac{1}{5}] \cup [\frac{4}{5}, 1]$, there exist two projections $p_2, q_2 \in C[0, 1] \otimes \mathfrak{A} \otimes \mathbf{M}_p \otimes \mathbf{M}_q$ such that

$$(\alpha p_1 + \overline{\alpha} q_1 + 1 - (p_1 + q_1))v_2(\alpha p_1 + \overline{\alpha} q_1 + 1 - (p_1 + q_1))^* v_2^* = g_1 p_2 + g_2 q_2 + 1 - (p_1 + q_1).$$

Indeed, p_2, q_2 can be chosen to be projections inside the copy of $\mathbf{M}_2$ that is generated by $\{p_1, q_1, v_1 p_1\}$, and they can be chosen to satisfy $p_1 \sim p_2$, $q_1 \sim q_2$. Hence, there exists a unitary $v_4 \in U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$ such that

$$v_4(\alpha p_1 + \overline{\alpha} q_1 + 1 - (p_1 + q_1))v_2(\alpha p_1 + \overline{\alpha} q_1 + 1 - (p_1 + q_1))^* v_2^* v_4^* = g_1 p_1 + g_2 q_1 + 1 - (p_1 + q_1)$$

and

$$v_4(1 - (p_1 + q_1)) = (1 - (p_1 + q_1))v_4 = 1 - (p_1 + q_1).$$

Set $w' = v_4 w_3 v_4^*$. Then $w' \in G$ is a unitary that satisfies the statement of Claim 1. This completes the proof of Claim 1.

Claim 2. Suppose that r, s are pairwise orthogonal projections in $C[0, 1] \otimes \mathfrak{A} \otimes \mathbf{M}_p \otimes \mathbf{M}_q$ which are Murray-von Neumann equivalent in $C[0, 1] \otimes \mathfrak{A} \otimes \mathbf{M}_p \otimes \mathbf{M}_q$. Then

$$g_1 r + g_2 s + 1 - (r + s) \in G.$$

We will show that $g_1r + g_2s + 1 - (r + s)$ can be approximated arbitrarily close by elements of G . Let $\epsilon > 0$ be given. Plug this ϵ into Claim 1 to get nonzero orthogonal projections $r_1, r_2 \in C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$. Decompose r, s into pairwise orthogonal projections

$$r = r_{1,1} + r_{1,2} + \cdots + r_{1,n}$$

and

$$s = s_{1,1} + s_{1,2} + \cdots + s_{1,n}$$

so that the following hold:

- (i) $r_{1,j}$ and $s_{1,j}$ are projections in $C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ for all $j \geq 1$.
- (ii) $r_{1,j} \sim s_{1,j}$ in $C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ for all $j \geq 1$.
- (iii) $r_{1,j} \precsim r_1 \sim r_2$ in $C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ for all $j \geq 1$.

Since $C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ has cancellation of projections and since G is a normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z})_0$, it follows, by Claim 1, that for $j \geq 1$, there exists a unitary $w_{1,j} \in G$ such that the following conditions hold:

- (iv) $w_{1,j}(1 - (r_{1,j} + s_{1,j})) = (1 - (r_{1,j} + s_{1,j}))w_{1,j} = 1 - (r_{1,j} + s_{1,j})$ for all $j \geq 1$ and
- (v) $\|w_{1,j} - (g_1r_{1,j} + g_2s_{1,j} + 1 - (r_{1,j} + s_{1,j}))\| < \epsilon$ for all $j \geq 1$.

Set $w_4 = w_{1,1}w_{1,2} \cdots w_{1,n}$. Then

$$\|w_4 - (g_1r + g_2s + 1 - (r + s))\| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary and since G is closed,

$$g_1r + g_2s + 1 - (r + s) \in G,$$

as required. This completes the proof of Claim 2.

We now complete the proof that $CU(1_{\mathfrak{A}} \otimes \mathcal{Z})_0 \subseteq G$ for Case 1. Choose two nonzero orthogonal projections $r, s \in C[0, 1] \otimes 1_{\mathfrak{A}} \otimes M_p \otimes M_q$ such that $r \sim s$ in $C[0, 1] \otimes 1_{\mathfrak{A}} \otimes M_p \otimes M_q$. By Claim 2, $w_5 = g_1r + g_2s + 1 - (r + s) \in G$. Note that w_5 is also a unitary in $CU(1_{\mathfrak{A}} \otimes \mathcal{Z})$ which is not a scalar multiple of the identity. By [20], $CU(\mathcal{Z})/\mathbb{T}$ is a simple topological group. Hence, $CU(1_{\mathfrak{A}} \otimes \mathcal{Z}) \subseteq G$, as required.

Case 2. Suppose that the spectrum of w (and hence the spectrum of x) is contained in $\{-1, 1, i, -i\}$. Recall that w is not in $\mathbb{T}$ and its spectrum contains 1. For simplicity, let us assume that the spectrum of w is $\{1, \alpha\}$, where $\alpha \in \{-1, i, -i\}$. (The proofs for the other cases are similar.) Hence, there exist nonzero orthogonal projections $p, q \in \mathfrak{A}$ such that $w = p + \alpha q$. Therefore, $x = p \otimes 1_{\mathcal{Z}_p, q} + \alpha q \otimes 1_{\mathcal{Z}_p, q}$.

Let $r, s \in 1_{\mathfrak{A}} \otimes M_p \otimes M_q$ be nonzero orthogonal projections such that r is orthogonal to p , s is orthogonal to q , $r \sim s \precsim p \otimes 1_{M_p \otimes M_q}$ and $s \precsim q \otimes 1_{M_p \otimes M_q}$ in $\mathfrak{A} \otimes M_p \otimes M_q$. Then there exist projections $r', s' \in \mathfrak{A} \otimes M_p \otimes M_q$ and $v \in \mathfrak{A} \otimes M_p \otimes M_q$ such that $r \sim r' \precsim p \otimes 1_{M_p \otimes M_q}$, $s \sim s' \precsim q \otimes 1_{M_p \otimes M_q}$, and $v^*v = r'$ and $vv^* = s'$.

Let $\{u(t)\}_{t \in [0, 1]}$ be the norm-continuous path of unitaries in M_2 as in (2.1) except that the canonical system of matrix units is taken to be $e_{1,1} = r'$, $e_{2,2} = s'$ and $e_{2,1} = v$. Define the norm-continuous path of unitaries $\{w(t)\}_{t \in [0, 1]}$ in $U(\mathfrak{A} \otimes M_p \otimes M_q)$ by

$$w(t) = \begin{cases} u(2t) + 1 - (r' + s') \otimes 1_{C[0, 1]} & \text{if } t \in [0, \frac{1}{2}], \\ u(-2t + 2) + 1 - (r' + s') \otimes 1_{C[0, 1]} & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Note that $w \in U(\mathfrak{A} \otimes \mathcal{Z}_{p, q})_0$. Since G is a closed normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z})_0$, $w' = wxw^*x^*$ is an element of $G \cap U(\mathfrak{A} \otimes \mathcal{Z}_{p, q})_0$. Moreover, we have that w' is not a

scalar multiple of the identity, $w'(1 - (r' + s') \otimes 1_{C[0,1]}) = (1 - (r' + s') \otimes 1_{C[0,1]})w' = 1 - (r' + s') \otimes 1_{C[0,1]}$ in $C[0,1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$ and $w'(0) = w'(1) = 1$.

Let $f_3, g_3 : [0,1] \rightarrow \mathbb{T}$ be two continuous functions such that for all $t \in [0,1]$, $\{f_3(t), g_3(t)\}$ are the eigenvalues of $(r' + s')w'(t)(r' + s')$. Note that $f_3(0) = f_3(1) = g_3(0) = g_3(1) = 1$. Set $w'' = f_3r' + g_3s' + 1 - (r' + s') \otimes 1_{C[0,1]}$. Then $w'' \in U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})$. Note that $f_3r' + g_3s' \in C[0,1] \otimes C^*(r', s', v) \cong C[0,1] \otimes M_2$. Hence, by [30], w' and w'' are approximately unitarily equivalent in $C[0,1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$. Moreover, since $w'(0) = w'(1) = w''(0) = w''(1) = 1$, we can choose the implementing unitaries to also have unit value at the endpoints and be elements of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$. Hence, since G is a closed normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z})_0$, $w'' \in G$.

Since $f_3(t) = f_3(1 - t)$, $g_3(t) = g_3(1 - t)$ for all $t \in [0,1]$, $w''' = f_3r + g_3s + 1 - (r + s) \otimes 1_{C[0,1]} \in U(1_{\mathfrak{A}} \otimes \mathcal{Z}_{p,q})_0$. Note also that w''' is not a scalar multiple of the identity. Since $r \sim r'$, $s \sim s'$, r is orthogonal to r' , s is orthogonal to s' , and $w'''(0) = w''(0) = w'''(1) = w''(1) = 1$, we have that w'' and w''' are approximately unitarily equivalent, where we can choose the unitaries to have unit value at endpoints and be elements in $U(\mathfrak{A} \otimes \mathcal{Z})_0$. Hence, since G is a closed normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z})_0$, we must have that $w''' \in G$, i.e., G contains an element of $U(1_{\mathfrak{A}} \otimes \mathcal{Z})_0$ which is not a scalar multiple of the identity. By [20], $U(1_{\mathfrak{A}} \otimes \mathcal{Z})_0/\mathbb{T}$ is a simple topological group. Hence, $U(1_{\mathfrak{A}} \otimes \mathcal{Z})_0 \subseteq G$, as required. This completes the proof for Case 2 and, hence, all the cases. $\square$

Recall that for a C^* -algebra $\mathfrak{C}$ and for elements $a, b \in \mathfrak{C}$, (a, b) is defined to be $(a, b) = aba^*b^*$.

Lemma 2.7. *Consider the supernatural numbers $p = 2^\infty$ and $q = 3^\infty$. Let $\mathfrak{A}$ be a simple unital C^* -algebra. Let G be a closed normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$ that contains $CU(1_{\mathfrak{A}} \otimes \mathcal{Z}_{p,q})_0$ and let $u_i, v_i : [0,1] \rightarrow U(\mathfrak{A} \otimes 1_{\mathcal{Z}_{p,q}})_0$ ($1 \leq i \leq n$) be norm-continuous paths. Define w by*

$$w = \prod_{i=1}^n (u_i, v_i) = (u_1, v_1)(u_2, v_2) \cdots (u_n, v_n).$$

Note that $w \in CU(C[0,1] \otimes \mathfrak{A} \otimes 1_{M_p \otimes M_q})_0 \subseteq CU(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$. If $w(0) = 1$, then $w \in G$.

Proof. Since G is a closed subset of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$, we may assume that there exists $\delta > 0$ ($\delta < 1$) such that for all $t \in [0, 2\delta)$, $w(t) = 1$. By the definition of q ,

$$M_q = \overline{\bigcup_{j=1}^{\infty} M_{3^j}}$$

with connecting maps defined by $a \mapsto \text{diag}(a, a, a)$.

As an intermediate step, we will work inside $C[0,1] \otimes \mathfrak{A} \otimes M_3$. Let $x_{1,2,i}, x_{3,i}$ and $x_{4,i}$ be elements of $C[0,1] \otimes \mathfrak{A} \otimes M_3$ defined as follows:

$$x_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad x_{2,i} = \begin{bmatrix} u_i & 0 & 0 \\ 0 & v_i^* & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$x_{3,i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & v_i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad x_{4,i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u_i & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, in $C[0, 1] \otimes \mathfrak{A} \otimes \mathbf{M}_3$, we have that

$$\begin{bmatrix} u_i v_i & 0 & 0 \\ 0 & u_i^* v_i^* & 0 \\ 0 & 0 & 1 \end{bmatrix} = x_{3,i} x_{2,i} x_1 x_{2,i}^* x_1^* x_{3,i}^* = x_{3,i} (x_{2,i}, x_1) x_{3,i}^*$$

and

$$\begin{bmatrix} u_i^* v_i^* & 0 & 0 \\ 0 & u_i v_i & 0 \\ 0 & 0 & 1 \end{bmatrix} = x_{3,i}^* x_{2,i}^* x_1 x_{2,i} x_1^* x_{3,i} = x_{3,i}^* (x_{2,i}^*, x_1) x_{3,i}.$$

Hence,

$$(2.7) \quad \begin{bmatrix} (u_i, v_i) & 0 & 0 \\ 0 & (u_i, v_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} = x_{3,i} x_{4,i} [x_{3,i} (x_{2,i}, x_1) x_{3,i}^*] [x_{3,i}^* (x_{2,i}^*, x_1) x_{3,i}] x_{4,i}^* x_{3,i}^* \\ = x_{3,i} x_{4,i} x_{3,i} (x_{2,i}, x_1) x_{3,i}^{*2} (x_{2,i}^*, x_1) x_{3,i} x_{4,i}^* x_{3,i}^*.$$

Let $y_1, y_{2,i}, y_{3,i}, y_{4,i} \in C[0, 1] \otimes \mathfrak{A} \otimes \mathbf{M}_3$ be given as follows:

$$y_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad y_{2,i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u_i & 0 \\ 0 & 0 & v_i^* \end{bmatrix}, \\ y_{3,i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v_i \end{bmatrix}, \quad y_{4,i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u_i \end{bmatrix}.$$

Hence, in $C[0, 1] \otimes \mathfrak{A} \otimes \mathbf{M}_3$, we have that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & v_i u_i & 0 \\ 0 & 0 & u_i^* v_i^* \end{bmatrix} = y_1 y_{2,i}^* y_1^* y_{2,i} = (y_1, y_{2,i}^*)$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & u_i^* v_i^* & 0 \\ 0 & 0 & u_i v_i \end{bmatrix} = y_{3,i}^* y_{2,i}^* y_1 y_{2,i} y_1^* y_{3,i} = y_{3,i}^* (y_{2,i}^*, y_1) y_{3,i}.$$

Hence,

$$(2.8) \quad \begin{bmatrix} (u_i, v_i) & 0 & 0 \\ 0 & (u_i, v_i) & 0 \\ 0 & 0 & (u_i, v_i) \end{bmatrix} \\ = y_{3,i} y_{4,i} \begin{bmatrix} (u_i, v_i) & 0 & 0 \\ 0 & u_i v_i u_i^* v_i^* & 0 \\ 0 & 0 & 1 \end{bmatrix} (y_1, y_{2,i}^*) [y_{3,i}^* (y_{2,i}^*, y_1) y_{3,i}] y_{4,i}^* y_{3,i}^*.$$

From this and (2.7), we have in $C[0, 1] \otimes \mathfrak{A} \otimes \mathbf{M}_3$ that

$$(2.9) \quad w = \prod_{i=1}^n \begin{bmatrix} (u_i, v_i) & 0 & 0 \\ 0 & (u_i, v_i) & 0 \\ 0 & 0 & (u_i, v_i) \end{bmatrix} \\ = \prod_{i=1}^n y_{3,i} y_{4,i} \begin{bmatrix} (u_i, v_i) & 0 & 0 \\ 0 & (u_i, v_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} (y_1, y_{2,i}^*) [y_{3,i}^* (y_{2,i}^*, y_1) y_{3,i}] y_{4,i}^* y_{3,i}^*.$$

We now use the above to manufacture elements in $CU(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$. Let $h : [0, 1] \rightarrow [0, \delta]$ be the continuous function that is given by

$$h(t) = \begin{cases} t & \text{for } t \in [0, \delta], \\ \delta & \text{for } t \in [\delta, 1]. \end{cases}$$

For $1 \leq i \leq n$, $2 \leq j \leq 4$, let $x'_1, y'_1, x'_{j,i}, y'_{j,i} : [0, 1] \rightarrow U(\mathfrak{A} \otimes M_3)_0$ be continuous functions such that the following hold:

- (i) $x'_1(t) = x_1(t)$, $y'_1(t) = y_1(t)$, $x'_{j,i}(t) = x_{j,i}(t)$ and $y'_{j,i}(t) = y_{j,i}(t)$ for $t \in [\delta, 1]$.
- (ii) $x'_1(0) = y'_1(0) = x'_{j,i}(0) = y'_{j,i}(0) = 1$.
- (iii) $x'_1(t), y'_1(t), x'_{j,i}(t), y'_{j,i}(t) \in \mathfrak{A} \otimes M_3$ for $t \in (0, \delta]$.
- (iv) $x'_1, y'_1, x'_{j,i}, y'_{j,i} \in U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$.
- (v) $x'_1, y'_1, x'_1 \circ h, y'_1 \circ h \in CU(1_{\mathfrak{A}} \otimes \mathcal{Z}_{p,q})_0$.

(Note that the determinants of x_1 and y_1 are both one.) Next, we will replace $x_1, y_1, x_{j,i}, y_{j,i}$ with $x'_1, y'_1, x'_{j,i}, y'_{j,i}$, respectively, in the expressions in (2.7) and (2.8) – except at one occurrence of y_1 . We now proceed with the details.

By (i)–(v), we have that $x'_1 \in CU(1_{\mathfrak{A}} \otimes \mathcal{Z}_{p,q})_0 \subseteq G$. Since G is a normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$,

$$z_i = x'_{3,i} x'_{4,i} x'_{3,i} (x'_{2,i}, x'_1) x'^{*}_{3,i} (x'^{*}_{2,i}, x'_1) x'_{3,i} x'^{*}_{4,i} x'^{*}_{3,i} \in G.$$

Note also by (2.7) and (i)–(v) that

$$\begin{bmatrix} (u_i(t), v_i(t)) & 0 & 0 \\ 0 & (u_i(t), v_i(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix} = z_i(t)$$

for $t \in [\delta, 1]$. Also by (i)–(v), since $y'_1 \in G$ and since G is a normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$,

$$z'_i = y'_{3,i} y'_{4,i} z_i (y'_1, y'^{*}_{2,i}) [y'^{*}_{3,i} (y'^{*}_{2,i}, y'_1) y'_{3,i}] y'^{*}_{4,i} y'^{*}_{3,i} \in G.$$

Again, by (2.8) and (i)–(v),

$$\begin{bmatrix} (u_i(t), v_i(t)) & 0 & 0 \\ 0 & (u_i(t), v_i(t)) & 0 \\ 0 & 0 & (u_i(t), v_i(t)) \end{bmatrix} = z'_i(t)$$

for $t \in [\delta, 1]$.

Now, recall that we have the condition

$$1 = w(t) = \prod_{i=1}^n (u_i(t), v_i(t))$$

for $t \in [0, \delta]$. To obtain this, we replace the last occurrence of y'_1 in z'_n by another element $y''_1 \in G$. We define y''_1 as follows:

$$y''_1(t) = y_1(t)$$

for $t \in [\delta, 1]$, and for $t \in [0, \delta]$, set $y_1''(t)$ to be

$$y_{2,n}'^*(t)y_1'^*(t)y_{2,n}'(t)y_{3,n}'(t)(y_1'(t), y_{2,n}'^*(t))^*z_n(t)^*y_{4,n}'^*(t)y_{3,n}'^*(t) \left[\prod_{i=1}^{n-1} z_i'(t) \right]^* y_{3,n}'(t)y_{4,n}'(t)y_{3,n}'^*(t)^*.$$

(The complicated definition over $[0, \delta]$ is to ensure that

$$\left[\prod_{i=1}^{n-1} z_i'(t) \right] y_{3,n}'(t)y_{4,n}'(t)z_n(t)(y_1'(t), y_{2,n}'^*(t))[y_{3,n}'^*(t)y_{2,n}'^*(t)y_1'(t)y_{2,n}'(t)y_1''(t)]y_{3,n}'(t)y_{4,n}'^*(t)y_{3,n}'^*(t) = 1$$

for $t \in [0, \delta]$.) From (2.8) and (i)–(v), we have that $y_1'' \in C[0, 1] \otimes \mathfrak{A} \otimes M_p \otimes M_q$, $y_1''(t) \in \mathfrak{A} \otimes M_3$ for $t \in (0, 1]$ and $y_1''(0) = 1$. Thus, $y_1'' \in U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$.

We now prove that $y_1'' \in G$. Note that by (i)–(v) and by an argument similar to that used to show that $z_i, z_i' \in G$, we have that $z_i \circ h, z_i' \circ h, x_1' \circ h, y_1' \circ h \in G$. Note that y_1'' is equal to

$$(y_{2,n}'^* \circ h)(y_1'^* \circ h)(y_{2,n}' \circ h)(y_{3,n}' \circ h)(y_1' \circ h, y_{2,n}'^* \circ h)^*(z_n^* \circ h)(y_{4,n}'^* \circ h)(y_{3,n}'^* \circ h) \left[\prod_{i=1}^{n-1} (z_i' \circ h) \right]^* (y_{3,n}' \circ h)(y_{4,n}' \circ h)(y_{3,n}'^* \circ h).$$

Since G is a normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$, we have that $y_1'' \in G$. Since G is a normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$ and

$$w = \left[\prod_{i=1}^{n-1} z_i' \right] y_{3,n}'y_{4,n}'z_n(y_1', y_{2,n}'^*)[y_{3,n}'^*[y_{2,n}'^*y_1'y_{2,n}'y_1'']y_{3,n}']y_{4,n}'^*y_{3,n}'^*,$$

we have that $w \in G$, as required. $\square$

Lemma 2.8. *Consider the supernatural numbers $p = 2^\infty$ and $q = 3^\infty$. Let $\mathfrak{A}$ be a simple unital C^* -algebra and let G be a closed normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$ that contains $CU(1_{\mathfrak{A}} \otimes \mathcal{Z}_{p,q})_0$. Let $u_i, v_i : [0, 1] \rightarrow U(\mathfrak{A} \otimes 1_{\mathcal{Z}_{p,q}})_0$ ($1 \leq i \leq n$) be norm-continuous paths and let*

$$w = \prod_{i=1}^n (u_i, v_i) = (u_1, v_1)(u_2, v_2) \cdots (u_n, v_n).$$

Note that $w \in CU(C[0, 1] \otimes \mathfrak{A} \otimes 1_{M_p \otimes M_q})_0 \subseteq CU(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$. If $w(1) = 1$, then $w \in G$.

Proof. The proof is similar (actually slightly easier) to that of Lemma 2.7. $\square$

Lemma 2.9. *Consider the supernatural numbers $p = 2^\infty$ and $q = 3^\infty$. Let $\mathfrak{A}$ be a simple unital C^* -algebra and let G be a closed normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$ that contains $CU(1_{\mathfrak{A}} \otimes \mathcal{Z}_{p,q})_0$. Let $u_i, v_i : [0, 1] \rightarrow U(\mathfrak{A} \otimes 1_{\mathcal{Z}_{p,q}})_0$ ($1 \leq i \leq n$) be norm-continuous paths and let*

$$w = \prod_{i=1}^n (u_i, v_i) = (u_1, v_1)(u_2, v_2) \cdots (u_n, v_n).$$

Note that $w \in CU(C[0, 1] \otimes \mathfrak{A} \otimes 1_{M_p \otimes M_q})_0 \subseteq CU(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0$. Then $w \in G$.

Proof. Decompose w into a product of unitaries $w = w'w''$ such that w' satisfies the hypotheses of Lemma 2.7 and w'' satisfies the hypotheses of Lemma 2.8. (Sketch

of argument: For all i , since $u_i(1) \in U(\mathfrak{A} \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})_0$, there exists a norm-continuous path $u'_i : [0, 1] \rightarrow U(\mathfrak{A} \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})_0$ such that $u'_i(0) = 1$ and $u'_i(1) = u_i(1)$. Let $w' = \prod_{i=1}^n (u'_i, v_i)$ and $w'' = w'^* w$.) Now apply Lemmas 2.7 and 2.8. $\square$

Lemma 2.10. *Consider the supernatural numbers $\mathfrak{p} = 2^\infty$ and $\mathfrak{q} = 3^\infty$. Let $\mathfrak{A}$ be a simple unital C^* -algebra and let G be a closed normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})_0$ that contains $CU(1_{\mathfrak{A}} \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})_0$. Then $CU(\mathfrak{A} \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})_0 \subseteq G$.*

Proof. Say that $u, v \in U(\mathfrak{A} \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})_0$. Thus, u, v are constant functions from $[0, 1]$ to $U(\mathfrak{A} \otimes 1_{M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}})_0$ (always taking the constant values $u(0), v(0)$, respectively). It follows from Lemma 2.9 that $(u, v) \in G$. Thus, since G is a group, the commutator subgroup of $U(\mathfrak{A} \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})_0$ is contained in G . Therefore, since G is closed, $CU(\mathfrak{A} \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})_0 \subseteq G$. $\square$

Theorem 2.11. *Let $\mathfrak{A}$ be an exact, separable, simple, unital, $\mathcal{Z}$ -stable C^* -algebra. Suppose that G is a closed normal subgroup of $U(\mathfrak{A})_0$ that properly contains $\mathbb{T}$. Then $CU(\mathfrak{A})_0 \subseteq G$.*

Proof. Since $\mathfrak{A}$ is $\mathcal{Z}$ -stable, $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{Z} \cong \mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z}$. Hence, it is enough to prove the theorem with $\mathfrak{A}$ replaced by $\mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z}$. By Lemma 2.6, we have that $CU(1_{\mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z}})_0 \subseteq G$. It suffices to prove the following: Let $u, v \in U(\mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z})_0$ be given. Then $(u, v) \in G$.

By Lemma 2.5, there is a $*$ -isomorphism $\Phi : \mathfrak{A} \otimes \mathcal{Z} \rightarrow \mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z}$ such that Φ is approximately unitarily equivalent to the map $\mathfrak{A} \otimes \mathcal{Z} \rightarrow \mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z} : a \mapsto a \otimes 1_{\mathcal{Z}}$, where the unitaries come from $U(\mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z})_0$. Let $u', v' \in U(\mathfrak{A} \otimes \mathcal{Z})_0$ be unitaries such that $\Phi(u') = u$ and $\Phi(v') = v$. Hence, $u' \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}, v' \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}$ are approximately unitarily equivalent to u, v , respectively, with implementing unitaries in $U(\mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z})_0$.

Let $\mathfrak{p}, \mathfrak{q}$ be the supernatural numbers that are given by $\mathfrak{p} = 2^\infty$ and $\mathfrak{q} = 3^\infty$. By Theorem 3.4 of [28], $\mathcal{Z}$ is a C^* -inductive limit $\mathcal{Z} = \bigcup_{n=1}^\infty \mathcal{Z}_n$, where $\mathcal{Z}_n \cong \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ for all $n \geq 1$ and where the connecting maps are unital and injective. By Lemma 2.10, we have that $(u' \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}, v' \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}) \in G$. Since $(u' \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}, v' \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})$ is approximately unitarily equivalent to (u, v) , with unitaries coming from $U(\mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z})_0$ and since G is a closed normal subgroup of $U(\mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z})_0$, $(u, v) \in G$, as required. $\square$

Lemma 2.12. *Let $\mathfrak{A}$ be an exact, unital, stably finite, $\mathcal{Z}$ -stable C^* -algebra. For every $\epsilon > 0$, there exists $\delta > 0$ such that for every self-adjoint element $a \in \mathfrak{A}$ such that $|\tau(a)| < \delta$ for all $\tau \in T(\mathfrak{A})$,*

$$\text{dist}(e^{ia}, CU(\mathfrak{A})_0) = \inf\{\|e^{ia} - u\| : u \in CU(\mathfrak{A})_0\} < \epsilon.$$

Proof. This follows from [31], which gives a topological group isomorphism:

$$\Delta : U(\mathfrak{A})_0 / CU(\mathfrak{A})_0 \rightarrow \text{Aff}(T(\mathfrak{A})) / \overline{K_0(\mathfrak{A})},$$

where Δ is the determinant map. Note that for a self-adjoint $a \in \mathfrak{A}$, $\Delta([e^{i2\pi a}]) = a + \overline{K_0(\mathfrak{A})}$. $\square$

Lemma 2.13. *Let $\mathfrak{A}$ be an exact, simple, unital, stably finite C^* -algebra and let $\mathfrak{C}$ be a UHF-algebra. Then for every $\epsilon > 0$, for every $N \geq 1$, for every nonzero*

self-adjoint element $a \in \mathfrak{A} \otimes \mathfrak{C}$ and nonzero projection $p \in \mathfrak{A} \otimes \mathfrak{C}$, there exists a self-adjoint element $c \in p\mathfrak{A}p$ such that

$$|\tau(a) - N\tau(c)| < \epsilon$$

for all $\tau \in T(\mathfrak{A} \otimes \mathfrak{C})$.

Proof. We have that $\mathfrak{C}$ can be realized as an inductive limit

$$\mathfrak{C} = \overline{\bigcup_{k=1}^{\infty} M_{n_k}},$$

where the connecting maps are diagonal maps

$$M_{n_k} \rightarrow M_{n_{k+1}} : c \mapsto \bigoplus^{\frac{n_{k+1}}{n_k}} c.$$

Note that $n_k \rightarrow \infty$. Hence, $\mathfrak{A} \otimes \mathfrak{C}$ can be realized as an inductive limit

$$\mathfrak{A} \otimes \mathfrak{C} = \overline{\bigcup_{k=1}^{\infty} M_{n_k}(\mathfrak{A})},$$

where the connecting maps are diagonal maps

$$M_{n_k}(\mathfrak{A}) \rightarrow M_{n_{k+1}}(\mathfrak{A}) : c \mapsto \bigoplus^{\frac{n_{k+1}}{n_k}} c.$$

Note that the connecting map divides c up into $\frac{n_{k+1}}{n_k}$ pairwise orthogonal Cuntz equivalent pieces, each with $\frac{n_k}{n_{k+1}}$ th trace of c for any tracial state on $\mathfrak{A} \otimes \mathfrak{C}$.

The result follows from the nature of the connecting maps for the inductive limit decomposition of $\mathfrak{A} \otimes \mathfrak{C}$ and since $\mathfrak{A} \otimes \mathfrak{C}$ has strict comparison for positive elements (see Lemma 2.2). $\square$

Lemma 2.14. *Let $\mathfrak{A}$ be a unital C^* -algebra and let $a, b \in \mathfrak{A}$ be self-adjoint elements. Then*

$$e^{ia}e^{ib}e^{-i(a+b)} \in CU(\mathfrak{A})_0.$$

Proof. This follows immediately from the formula

$$\exp(ia)\exp(ib)\exp(-i(a+b)) = \lim_{n \rightarrow \infty} \exp(ia)\exp(ib) \left[\exp\left(\frac{-ia}{n}\right) \exp\left(\frac{-ib}{n}\right) \right]^n.$$

$\square$

The next lemma is motivated by the main result of [23]. However, we note that unlike [23], we need to assume that our C^* -algebra is nuclear. In fact, it is not clear to us whether if $\mathfrak{A}$ is an arbitrary unital simple separable quasidiagonal C^* -algebra and $\mathfrak{C}$ is a UHF-algebra, then $\mathfrak{A} \otimes \mathfrak{C}$ has the “sufficiently many projections” condition of Popa. (See [23] (1.1), (2.1), (2.1’) and (2.1’’).)

Lemma 2.15. *Let $\mathfrak{A}$ be a nuclear, separable, simple, unital C^* -algebra, and let $\mathfrak{C}$ be a UHF-algebra. Then $\mathfrak{A}$ is quasidiagonal if and only if $\mathfrak{A} \otimes \mathfrak{C}$ has the Popa property; i.e., for every $\epsilon > 0$ and for every finite subset $\mathcal{F} \subseteq \mathfrak{A} \otimes \mathfrak{C}$, there exists a nonzero finite dimensional sub- C^* -algebra $\mathfrak{D}$ of $\mathfrak{A} \otimes \mathfrak{C}$ with unit $p = 1_{\mathfrak{D}}$ such that*

for every $a \in \mathcal{F}$, the following hold:

- i. $\|pa - ap\| < \epsilon$ and
- ii. pap is within ϵ of an element of $\mathfrak{D}$.

Proof. By Theorem 1 of [32], if $\mathfrak{A} \otimes \mathfrak{C}$ has the Popa property, then $\mathfrak{A} \otimes \mathfrak{C}$ is quasidiagonal (see, for example, [3], the argument after Theorem 12.1) and, hence, $\mathfrak{A}$ is quasidiagonal. This completes the proof of the “if” direction.

We now prove the “only if” direction. Suppose that $\mathfrak{A}$ is quasidiagonal. Since $\mathfrak{C}$ is a UHF-algebra, it can be expressed as an inductive limit

$$\mathfrak{C} = \overline{\bigcup_{k=1}^{\infty} M_{n_k}},$$

where the connecting maps are defined by $c \mapsto \text{diag}(c, c, \dots, c)$ (each c being repeated $\frac{n_{k+1}}{n_k}$ times). Hence, $\mathfrak{A} \otimes \mathfrak{C}$ is an inductive limit

$$\mathfrak{A} \otimes \mathfrak{C} = \overline{\bigcup_{k=1}^{\infty} M_{n_k}(\mathfrak{A})},$$

where the connecting maps are diagonal maps. It suffices to prove the Popa property for finite subsets of the building blocks $M_{n_k}(\mathfrak{A})$. Let $K \geq 1$ be given. Let $\epsilon > 0$ be given and let $\mathcal{F} \subseteq M_{n_K}(\mathfrak{A})$ be a finite set. We may assume that the elements of $\mathcal{F}$ have norm less than or equal to one. Let $\mathcal{M}(M_{n_K}(\mathfrak{A}) \otimes \mathbb{K})$ be the multiplier algebra of the stabilization of $M_{n_K}(\mathfrak{A})$. (Note that $M_{n_K}(\mathfrak{A}) \otimes \mathbb{K} \cong \mathfrak{A} \otimes \mathbb{K}$.) Define $\phi : M_{n_K}(\mathfrak{A}) \rightarrow \mathcal{M}(M_{n_K}(\mathfrak{A}) \otimes \mathbb{K})$ by $\phi(a) = a \otimes 1_{\mathcal{M}(\mathbb{K})}$ and let $\psi' : M_{n_K}(\mathfrak{A}) \rightarrow \mathcal{M}(\mathbb{K}) = \mathbb{B}(\mathcal{H})$ be any unital (and hence essential) $*$ -homomorphism. Set $\psi = 1_{M_{n_K}(\mathfrak{A})} \otimes \psi' : M_{n_K}(\mathfrak{A}) \rightarrow \mathcal{M}(M_{n_K}(\mathfrak{A}) \otimes \mathbb{K})$. Hence, $\phi, \psi : M_{n_K} \otimes \mathfrak{A} \rightarrow \mathcal{M}(M_{n_K}(\mathfrak{A}) \otimes \mathbb{K})$ are injective, unital $*$ -homomorphisms. Note that ϕ and ψ are both full $*$ -homomorphisms. Hence, since $\mathfrak{A}$ is nuclear, it follows, by [6], that there exists a unitary $u \in \mathcal{M}(M_{n_K}(\mathfrak{A}) \otimes \mathbb{K})$ such that for every $a \in M_{n_K} \otimes \mathfrak{A}$,

$$(2.10) \quad \phi(a) - u\psi(a)u^* \in M_{n_K}(\mathfrak{A}) \otimes \mathbb{K}.$$

Since $M_{n_K}(\mathfrak{A})$ is quasidiagonal, $\psi'(M_{n_K}(\mathfrak{A}))$ is a quasidiagonal collection of operators on $\mathcal{H}$; i.e., there exists an increasing sequence $\{p_n\}_{n=1}^{\infty}$ of finite rank operators on $\mathcal{H}$ such that

- i. $p_n \rightarrow 1_{\mathbb{B}(\mathcal{H})}$ in the strong operator topology and
- ii. $\|p_n\psi'(a) - \psi'(a)p_n\| \rightarrow 0$ for all $a \in M_{n_K} \otimes \mathfrak{A}$.

This and (2.10) imply that there exists an integer $N \geq 1$ such that

- (1) $\|u(p_m - p_n)u^*\phi(a) - \phi(a)u(p_m - p_n)u^*\| < \frac{\epsilon}{100}$ for all $m > n \geq N$ and for all $a \in \mathcal{F}$,
- (2) $u(p_m - p_n)u^*\phi(a)u(p_m - p_n)u^*$ is within $\frac{\epsilon}{100}$ of an element of the finite dimensional C^* -algebra $u(p_m - p_n)\mathbb{B}(\mathcal{H})(p_m - p_n)u^*$ for all $a \in \mathcal{F}$ and all $m > n \geq N$.

Now let $\{e_{i,j} : 1 \leq i, j < \infty\}$ be a system of matrix units for $\mathbb{K}$. Note that $1_{\mathcal{M}(M_{n_K}(\mathfrak{A}) \otimes \mathbb{K})} = \sum_{i=1}^{\infty} 1 \otimes e_{i,i}$, where the sum converges in the strict topology on $\mathcal{M}(M_{n_K}(\mathfrak{A}) \otimes \mathbb{K})$. Hence, for all $a \in M_{n_K}(\mathfrak{A})$,

$$\phi(a) = a \otimes 1 = \sum_{i=1}^{\infty} a \otimes e_{i,i},$$

where the sum converges in the strict topology on $\mathcal{M}(\mathbf{M}_{n_K}(\mathfrak{A}) \otimes \mathbb{K})$. Also, note that for every $n \geq 1$,

$$\lim_{M \rightarrow \infty} \left(\sum_{i=1}^M 1 \otimes e_{i,i} \right) u p_n u^* = u p_n u^*.$$

Hence, (1)–(2) and the definition of ϕ imply that there exists an $M \geq 1$ and there exists a nonzero projection $r \in \mathbf{M}_M(\mathbf{M}_{n_K}(\mathfrak{A}))$ (which, inside $\mathcal{M}(\mathbf{M}_{n_K}(\mathfrak{A}) \otimes \mathbb{K})$, is close to $u(p_m - p_n)u^*$ for some integers $m > n$) such that the following hold:

- (i) $M = \frac{n_L}{n_K}$ for some $L \geq K$.
- (ii) $\|(\bigoplus^M a)r - r(\bigoplus^M a)\| < \epsilon$ for all $a \in \mathcal{F}$.
- (iii) There exists a finite dimensional sub- C^* -algebra $\mathfrak{D} \subseteq \mathbf{M}_M(\mathbf{M}_{n_K}(\mathfrak{A}))$ such that the unit of $\mathfrak{D}$ is $1_{\mathfrak{D}} = r$. ($\mathfrak{D}$ will be “close to” $u(p_m - p_n)\mathbb{B}(\mathcal{H})(p_m - p_n)u^*$ for some positive integers $m > n$.)
- (iv) For every $a \in \mathcal{F}$, $r(\bigoplus^M a)r$ is within ϵ of an element of $\mathfrak{D}$.

Now since, in the inductive limit decomposition of $\mathfrak{A} \otimes \mathfrak{C}$, the connecting map $\mathbf{M}_{n_K}(\mathfrak{A}) \rightarrow \mathbf{M}_{n_L}(\mathfrak{A})$ has the form $c \mapsto \bigoplus^M c$ (a diagonal map), we are done. $\square$

Lemma 2.16. *Let $\mathfrak{A}$ be a quasidiagonal, nuclear, separable, simple, unital C^* -algebra. Let $a \in \mathfrak{A}$ be a self-adjoint element and $\mathcal{F} \subseteq \mathfrak{A}$ be a finite set. Then for every $\epsilon > 0$, there exists a unitary $u \in CU(\mathfrak{A} \otimes \mathcal{Z})_0$ such that*

$$\|(e^{ia} \otimes 1_{\mathcal{Z}})(b \otimes 1_{\mathcal{Z}})(e^{-ia} \otimes 1_{\mathcal{Z}}) - u(b \otimes 1_{\mathcal{Z}})u^*\| < \epsilon$$

for all $b \in \mathcal{F}$.

Proof. We may assume that every element of $\mathcal{F}$ has norm less than or equal to one. Let $\mathfrak{p}, \mathfrak{q}$ be the relatively prime supernatural numbers given by $\mathfrak{p} = 2^\infty$ and $\mathfrak{q} = 3^\infty$. By Theorem 3.4 of [28], $\mathcal{Z}$ is a C^* -inductive limit $\mathcal{Z} = \overline{\bigcup_{n=1}^\infty \mathcal{Z}_n}$, where $\mathcal{Z}_n \cong \mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}$ for all $n \geq 1$ and the connecting maps are unital and injective.

By Lemma 2.15, both $\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}}$ and $\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{q}}$ have the Popa property. Hence, let $\mathfrak{D}_1 \subseteq \mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}}$ and $\mathfrak{D}_2 \subseteq \mathfrak{A} \otimes \mathbf{M}_{\mathfrak{q}}$ be nonzero finite dimensional simple sub- C^* -algebras with units $e_1 = 1_{\mathfrak{D}_1}$ and $e_2 = 1_{\mathfrak{D}_2}$ such that the following statements hold:

- (1) $\|ce_1 - e_1c\| < \frac{\epsilon}{100}$ for all $c \in (\mathcal{F} \otimes 1_{\mathbf{M}_{\mathfrak{p}}}) \cup \{a \otimes 1_{\mathbf{M}_{\mathfrak{p}}}\} \cup \{e^{ita} \otimes 1_{\mathbf{M}_{\mathfrak{p}}} : t \in [0, 1]\}$.
- (2) e_1ce_1 is within $\frac{\epsilon}{100}$ of an element of $\mathfrak{D}_1$ for all $c \in (\mathcal{F} \otimes 1_{\mathbf{M}_{\mathfrak{p}}}) \cup \{a \otimes 1_{\mathbf{M}_{\mathfrak{p}}}\} \cup \{e^{ita} \otimes 1_{\mathbf{M}_{\mathfrak{p}}} : t \in [0, 1]\}$.
- (3) $\|ce_2 - e_2c\| < \frac{\epsilon}{100}$ for all $c \in (\mathcal{F} \otimes 1_{\mathbf{M}_{\mathfrak{q}}}) \cup \{a \otimes 1_{\mathbf{M}_{\mathfrak{q}}}\} \cup \{e^{ita} \otimes 1_{\mathbf{M}_{\mathfrak{q}}} : t \in [0, 1]\}$.
- (4) e_2ce_2 is within $\frac{\epsilon}{100}$ of an element of $\mathfrak{D}_2$ for all $c \in (\mathcal{F} \otimes 1_{\mathbf{M}_{\mathfrak{q}}}) \cup \{a \otimes 1_{\mathbf{M}_{\mathfrak{q}}}\} \cup \{e^{ita} \otimes 1_{\mathbf{M}_{\mathfrak{q}}} : t \in [0, 1]\}$.

Plug $\frac{\epsilon}{100}$ into Lemma 2.12 to get a positive real number $\delta > 0$. Now let $\{e_{i,j}\}_{1 \leq i,j \leq m}$ be a system of matrix units for $\mathfrak{D}_1$. By Lemma 2.13, let $d \in e_{1,1}(\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}})e_{1,1}$ be a self-adjoint element such that

$$|\tau(a) - m\tau(d)| < \frac{\delta}{100}$$

for all $\tau \in T(\mathfrak{A} \otimes \mathcal{Z})$. Consider the element $d_1 \in e_1(\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}})e_1$ given by $d_1 = \bigoplus^m d$. (The i th copy of d sits inside $e_{i,i}$.) Then d_1 is a self-adjoint element of $\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}}$ such

that the following statements hold:

- (a) d_1 commutes with every element of $\mathfrak{D}_1$ and e^{-isd_1} commutes with every element of $\mathfrak{D}_1 \oplus (1 - e_1)(\mathfrak{A} \otimes \mathfrak{M}_p)(1 - e_1)$ for all $s \in [0, 1]$. Hence, by (1)–(4), we have that $\|e^{-isd_1}c - ce^{-isd_1}\| < \frac{6\epsilon}{100}$ for all $s \in [0, 1]$ and for all $c \in (\mathcal{F} \otimes 1_{\mathfrak{M}_p}) \cup \{e^{ita} \otimes 1_{\mathfrak{M}_p} : t \in [0, 1]\}$.
- (b) $|\tau(a) - \tau(d_1)| < \frac{\delta}{100}$ for all $\tau \in T(\mathfrak{A})$. Hence, $|\tau((1-t)a) - \tau((1-t)d_1)| < \frac{\delta}{100}$ for all $\tau \in T(\mathfrak{A})$ and for all $t \in [0, 1]$. Hence, by our choice of δ , by Lemma 2.14 and Lemma 2.12, the map $[0, 1] \rightarrow U(\mathfrak{A} \otimes \mathcal{Z})_0$ defined by $t \mapsto (e^{i(1-t)a} \otimes 1)e^{-i(1-t)d_1}$ is an element of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0 \subseteq U(\mathfrak{A} \otimes \mathcal{Z})_0$ which is within $\frac{\epsilon}{100}$ of an element u_1 of $CU(\mathfrak{A} \otimes \mathcal{Z})_0$.

By a similar argument, we can find a self-adjoint element $d_2 \in e_2(\mathfrak{A} \otimes \mathfrak{M}_q)e_2$ satisfying the following statements:

- (i) $\|e^{-isd_2}c - ce^{-isd_2}\| < \frac{6\epsilon}{100}$ for all $s \in [0, 1]$ and for all $c \in (\mathcal{F} \otimes 1_{\mathfrak{M}_q}) \cup \{e^{ita} \otimes 1_{\mathfrak{M}_q} : t \in [0, 1]\}$.
- (ii) The map $[0, 1] \rightarrow U(\mathfrak{A} \otimes \mathcal{Z})_0$ defined by $t \mapsto (e^{ita} \otimes 1)e^{-itd_2}$ is an element of $U(\mathfrak{A} \otimes \mathcal{Z}_{p,q})_0 \subseteq U(\mathfrak{A} \otimes \mathcal{Z})_0$ which is within $\frac{\epsilon}{100}$ of an element u_2 of $CU(\mathfrak{A} \otimes \mathcal{Z})_0$.

By (a)–(b) and (i)–(ii), we have that

$$\begin{aligned}
 & \| (e^{ia} \otimes 1_{\mathcal{Z}})(b \otimes 1_{\mathcal{Z}})(e^{-ia} \otimes 1_{\mathcal{Z}}) - u_1 u_2 (b \otimes 1_{\mathcal{Z}}) u_2^* u_1^* \| \\
 &= \| (e^{i(1-t)a} e^{ita} \otimes 1_{\mathcal{Z}})(e^{-i(1-t)d_1} e^{-itd_2} e^{itd_2} e^{i(1-t)d_1})(b \otimes 1_{\mathcal{Z}})(e^{-ita} e^{-i(1-t)a} \otimes 1_{\mathcal{Z}}) \\
 &\quad - u_1 u_2 (b \otimes 1_{\mathcal{Z}}) u_2^* u_1^* \| \\
 &\leq \| (e^{i(1-t)a} e^{ita} \otimes 1_{\mathcal{Z}})(e^{-i(1-t)d_1} e^{-itd_2} e^{itd_2} e^{i(1-t)d_1})(b \otimes 1_{\mathcal{Z}})(e^{-ita} e^{-i(1-t)a} \otimes 1_{\mathcal{Z}}) \\
 &\quad - (e^{i(1-t)a} \otimes 1)e^{-i(1-t)d_1}(e^{ita} \otimes 1)e^{-itd_2}(b \otimes 1_{\mathcal{Z}})e^{itd_2}(e^{-ita} \otimes 1)e^{i(1-t)d_1} \\
 &\quad (e^{-i(1-t)a} \otimes 1) \| \\
 &\quad + \| (e^{i(1-t)a} \otimes 1)e^{-i(1-t)d_1}(e^{ita} \otimes 1)e^{-itd_2}(b \otimes 1_{\mathcal{Z}})e^{itd_2}(e^{-ita} \otimes 1)e^{i(1-t)d_1} \\
 &\quad (e^{-i(1-t)a} \otimes 1) \\
 &\quad - u_1 u_2 (b \otimes 1_{\mathcal{Z}}) u_2^* u_1^* \| \\
 &< 24\epsilon/100 + 4\epsilon/100 \\
 &< \epsilon
 \end{aligned}$$

for all $b \in \mathcal{F}$. Hence, taking $u = u_1 u_2$, we are done. $\square$

Lemma 2.17. *Let $\mathfrak{A}$ be a quasidiagonal, nuclear, separable, simple, unital C^* -algebra. Let v be a unitary in $U(\mathfrak{A})_0$ and let $\mathcal{F} \subseteq \mathfrak{A}$ be a finite set. Then for every $\epsilon > 0$, there exists a unitary $u \in CU(\mathfrak{A} \otimes \mathcal{Z})_0$ such that*

$$\|(v \otimes 1_{\mathcal{Z}})(b \otimes 1_{\mathcal{Z}})(v^* \otimes 1_{\mathcal{Z}}) - u(b \otimes 1_{\mathcal{Z}})u^*\| < \epsilon$$

for all $b \in \mathcal{F}$.

Proof. Since $v \in U(\mathfrak{A})_0$, v has the form $v = e^{ia_1} e^{ia_2} \cdots e^{ia_n}$, where $a_1, a_2, \dots, a_n$ are self-adjoint elements of $\mathfrak{A}$. The rest of the proof is the same as that of Lemma 2.16. $\square$

Lemma 2.18. *Let $\mathfrak{A}$ be a quasidiagonal, nuclear, separable, simple, unital $\mathcal{Z}$ -stable C^* -algebra. Let v be a unitary in $U(\mathfrak{A})_0$ and let $\mathcal{F} \subseteq \mathfrak{A}$ be a finite set. Then for*

every $\epsilon > 0$, there exists a unitary $u \in CU(\mathfrak{A})_0$ such that

$$\|vcv^* - ucu^*\| < \epsilon$$

for all $c \in \mathcal{F}$.

Proof. Since $\mathfrak{A}$ is $\mathcal{Z}$ -stable, $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{Z} \cong \mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z}$. Hence, it is enough to prove the theorem with $\mathfrak{A}$ replaced by $\mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z}$. By Lemma 2.5 and Lemma 4.1 of [20], there exists a $*$ -isomorphism $\Phi : \mathfrak{A} \otimes \mathcal{Z} \rightarrow \mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z}$ which is approximately unitarily equivalent to the natural inclusion map $\mathfrak{A} \otimes \mathcal{Z} \rightarrow \mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z}$ defined by $b \mapsto b \otimes 1_{\mathcal{Z}}$, where we can choose the unitaries to be in $CU(\mathfrak{A} \otimes \mathcal{Z} \otimes \mathcal{Z})_0$. Hence, we may assume that the elements of $\mathcal{F} \cup \{v\}$ are all inside $\mathfrak{A} \otimes \mathcal{Z} \otimes 1_{\mathcal{Z}}$. The result then follows from Lemma 2.17. $\square$

Lemma 2.19. *Let $\mathfrak{A}$ be an exact, separable, simple, unital, $\mathcal{Z}$ -stable C^* -algebra with unique tracial state. Let v be a unitary in $U(\mathfrak{A})_0$. Then there exists a unitary $u \in CU(\mathfrak{A})_0$ such that*

$$vav^* = uau^*$$

for all $a \in \mathfrak{A}$.

Proof. Let τ be the unique tracial state of $\mathfrak{A}$. Since $v \in U(\mathfrak{A})_0$, v has the form $v = e^{ia_1} e^{ia_2} \cdots e^{ia_n}$, where $a_1, a_2, \dots, a_n \in \mathfrak{A}$ are self-adjoint elements. By Lemma 2.12 and Lemma 2.14, and since $e^{i\tau(a_1+a_2+\cdots+a_n)}1$ (being scalar) commutes with every element of $\mathfrak{A}$, the unitary

$$u = e^{-i\tau(a_1+a_2+\cdots+a_n)}v$$

satisfies the requirements of the lemma. $\square$

Theorem 2.20. *Let $\mathfrak{A}$ be an exact, separable, simple, unital $\mathcal{Z}$ -stable C^* -algebra. Suppose that either*

- (1) $\mathfrak{A}$ is nuclear and quasidiagonal or
- (2) $\mathfrak{A}$ has unique tracial state.

Then we have the following:

- (a) $CU(\mathfrak{A})_0/\mathbb{T}$ is a simple topological group.
- (b) Every automorphism in $\overline{\text{Inn}}_0(\mathfrak{A})$ can be realized using unitaries in $CU(\mathfrak{A})_0$.
- (c) $\overline{\text{Inn}}_0(\mathfrak{A})$ is a simple topological group.

Proof. (a) and (b) follow from Lemmas 2.18, 2.19 and Theorem 2.11.

(c) follows from (a) and (b), arguing as in the proof of Theorem 3.2(b) of [22] (which is a modification of the argument of Corollary 2.5 in [7]). For the convenience of the reader, we provide the (short) argument: Let G be a nontrivial closed normal subgroup of $\overline{\text{Inn}}_0(\mathfrak{A})$. Let $H = \{u \in CU(\mathfrak{A})_0 : \text{Ad}(u) \in G\}$. Then H is a closed normal subgroup of $CU(\mathfrak{A})_0$ such that H contains all scalar unitaries. Since G is nontrivial, let $\beta \in G$ be different from the identity automorphism. Hence, there exists $v \in CU(\mathfrak{A})_0$ such that $v^*\beta(v) \notin \mathbb{C}1$. Since $(\text{Ad}(v))^{-1}\beta\text{Ad}(v)\beta^{-1} = \text{Ad}(v^*\beta(v))$, it follows that $v^*\beta(v) \in H$. Therefore, by (a), $H = CU(\mathfrak{A})_0$. Hence, by (b), $G = \overline{\text{Inn}}_0(\mathfrak{A})$. $\square$

3. BOTT MAPS AND A CONTINUOUS PATH OF UNITARIES

3.1. K -theory with $\mathbb{Z}_n$ coefficient. For $n \in \mathbb{N}$, define $\theta_n : C(S^1) \rightarrow C(S^1)$ by $\theta_n(x) = x^n$. By identifying $C_0(0, 1)$ with

$$\{f \in C(S^1) : f(1) = 0\} \subseteq C(S^1),$$

$\theta_n|_{C_0(0,1)}$ is a homomorphism from $C_0(0, 1)$ to $C_0(0, 1)$. Denote by C_n the mapping cone of $\theta_n|_{C_0(0,1)}$. Set $C_0 = \mathbb{C}$. For a C^* -algebra $\mathfrak{A}$, define $\underline{K}(\mathfrak{A})$ by

$$\underline{K}(\mathfrak{A}) = \bigoplus_{i=0}^1 \bigoplus_{n=0}^{\infty} K_i(\mathfrak{A}; \mathbb{Z}_n),$$

where $K_i(\mathfrak{A}; \mathbb{Z}_n) = K_i(\mathfrak{A} \otimes C_n)$. For C^* -algebras $\mathfrak{A}$ and $\mathfrak{B}$, a homomorphism from $\underline{K}(\mathfrak{A})$ to $\underline{K}(\mathfrak{B})$ is a collection of group homomorphisms $(\phi_n^0, \phi_n^1)_{n=0}^{\infty}$, where $\phi_n^i : K_i(\mathfrak{A}; \mathbb{Z}_n) \rightarrow K_i(\mathfrak{B}; \mathbb{Z}_n)$, satisfying the Bockstein operations of [5].

3.2. Embeddings.

- (1) For each $n \in \mathbb{N}$, let $j_n : \mathfrak{A} \otimes C_0((0, 1)^2) \rightarrow \mathfrak{A} \otimes C_n$ be the natural inclusion.
- (2) For a C^* -algebra $\mathfrak{A}$, let $j_{\mathfrak{A}} : \mathfrak{A} \otimes C_0(0, 1) \rightarrow \mathfrak{A} \otimes C(S^1)$ be the canonical embedding that sends $a \otimes f$ to $a \otimes f$, using the identification of $C_0(0, 1)$ as a sub- C^* -algebra of $C(S^1)$.
- (3) Let $\mathfrak{A}$ be a C^* -algebra. Define $\iota_{\mathfrak{A}} : \mathfrak{A} \rightarrow \mathfrak{A} \otimes C(S^1)$ by $\iota_{\mathfrak{A}}(a) = a \otimes 1_{C(S^1)}$.
- (4) Let $m, n \in \mathbb{N}$ with $0 < m \leq n \in \mathbb{N}$, let $\mathfrak{A}_1, \dots, \mathfrak{A}_m, \mathfrak{B}_1, \dots, \mathfrak{B}_n$ be unital C^* -algebras, let $\iota_1, \dots, \iota_m \in \{1, \dots, n\}$ be pairwise distinct numbers, and let $\alpha_j : \mathfrak{A}_j \rightarrow \mathfrak{B}_{\iota_j}$ be a $*$ -homomorphism for each j . Then these $*$ -homomorphisms induce a $*$ -homomorphism

$$\alpha = \alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_m : \mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_m \rightarrow \mathfrak{B}_{\iota_1} \otimes \dots \otimes \mathfrak{B}_{\iota_m}.$$

The composition of this map with the canonical unital embedding

$$\iota : \mathfrak{B}_{\iota_1} \otimes \dots \otimes \mathfrak{B}_{\iota_m} \rightarrow \mathfrak{B}_1 \otimes \dots \otimes \mathfrak{B}_n$$

will be denoted by $\alpha^{[\iota_1 \dots \iota_m]}$. Note that ι may be expressed as $\text{id}^{[\iota_1 \dots \iota_m]}$.

3.3. Partial maps on $\underline{K}(\mathfrak{A})$. Let $\mathfrak{A}$ be a C^* -algebra. Let $\mathfrak{A}^\dagger$ be $\mathfrak{A}$ if $\mathfrak{A}$ is unital and the unitization of $\mathfrak{A}$ if $\mathfrak{A}$ is not unital. We will denote the set of projections and unitaries in $\bigcup_{m=1}^{\infty} (\mathfrak{A} \otimes M_m)^\dagger$ by $\mathbb{P}_0(\mathfrak{A})$, and we will denote the set of projections and unitaries in

$$\bigcup_{m=1}^{\infty} \bigcup_{n=0}^{\infty} (\mathfrak{A} \otimes M_m \otimes C_n)^\dagger$$

by $\mathbb{P}(\mathfrak{A})$.

Let $\mathcal{P}$ be a subset of $\mathbb{P}(\mathfrak{A})$ and $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ be a contractive, completely positive linear map. We now define $\underline{K}(\psi)|_{\mathcal{P}}$ as in Section 2.3 of [15]. We use $\underline{K}(\psi)|_{\mathcal{P}}$ instead of the notation $[\psi]|_{\mathcal{P}}$ used in [15].

Definition 3.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^* -algebras and let $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ be a linear map. Let $\epsilon > 0$ and $\mathcal{S} \subseteq \mathfrak{A}$. Then ψ is $\mathcal{S}$ - ϵ -multiplicative if

$$\|\psi(ab) - \psi(a)\psi(b)\| < \epsilon$$

for all $a, b \in \mathcal{S}$.

The following lemmas are well known and can be found in Section 2.5 of [10].

Lemma 3.2. *For each $\epsilon > 0$, there exists $\delta > 0$ such that the following holds: Suppose $\mathfrak{A}$ is a unital C^* -algebra.*

- (1) *If $x \in \mathfrak{A}$ with $\|x^*x - 1_{\mathfrak{A}}\| < \delta$ and $\|xx^* - 1_{\mathfrak{A}}\| < \delta$, then there exists a unitary $u \in \mathfrak{A}$ such that $\|x - u\| < \epsilon$. Moreover, if $\epsilon < \frac{1}{2}$ and if u_1 and u_2 are unitaries such that $\|x - u_i\| < \epsilon$, then $[u_1] = [u_2]$ in $K_1(\mathfrak{A})$.*
- (2) *If $x \in \mathfrak{A}$ is a self-adjoint element with $\|x^2 - x\| < \delta$, then there exists a projection p in $\mathfrak{A}$ such that $\|x - p\| < \epsilon$. Moreover, if $\epsilon < \frac{1}{2}$ and if p_1 and p_2 are projections such that $\|x - p_i\| < \epsilon$, then $[p_1] = [p_2]$ in $K_0(\mathfrak{A})$.*

Lemma 3.3. *Let $\mathfrak{A}$ be a C^* -algebra and let $\mathfrak{C}$ be a nuclear C^* -algebra. Let $\epsilon > 0$ and let $\mathcal{F}$ be a finite subset of $\mathfrak{A} \otimes \mathfrak{C}$. Then there exist $\delta > 0$ and a finite subset $\mathcal{G}$ of $\mathfrak{A}$ such that if $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive, linear map that is $\mathcal{G}$ - δ -multiplicative, then $\psi \otimes \text{id}_{\mathfrak{C}} : \mathfrak{A} \otimes \mathfrak{C} \rightarrow \mathfrak{B} \otimes \mathfrak{C}$ is $\mathcal{F}$ - ϵ -multiplicative.*

Proof. Set $m_1 = \max\{\|x\| : x \in \mathcal{F}\}$. Note that each element of $\mathfrak{A} \otimes \mathfrak{C}$ can be approximated by a finite linear combination of elementary tensors, $a \otimes c$, where $a \in \mathfrak{A}$ and $c \in \mathfrak{C}$. Thus, there exists $N \in \mathbb{N}$ and there exist $a_1, \dots, a_N \in \mathfrak{A}$ and $c_1, \dots, c_N \in \mathfrak{C}$ such that for each $x \in \mathcal{F}$, there exist a finite subset $\mathcal{S}_x \subseteq \{k \in \mathbb{N} : k \leq N\}$ and functions $k_x, n_x : \mathcal{S} \rightarrow \{k \in \mathbb{N} : k \leq N\}$ such that

$$\left\| x - \sum_{i \in \mathcal{S}_x} a_{k_x(i)} \otimes c_{n_x(i)} \right\| < \min \left\{ \frac{\epsilon}{5(m_1 + 1)}, 1 \right\}.$$

Note that $\left\| \sum_{i \in \mathcal{S}_x} a_{k_x(i)} \otimes c_{n_x(i)} \right\| \leq 1 + m_1$.

Define $M, m_2 \in \mathbb{R}$ and $\mathcal{G} \subseteq \mathcal{A}$ as follows:

$$\begin{aligned} M &= \max\{|\mathcal{S}_x| : x \in \mathcal{F}\}, \\ m_2 &= \{\|c_i\| : i \in \mathbb{N}, 1 \leq i \leq N\}, \\ \mathcal{G} &= \{a_i : i \in \mathbb{N}, 1 \leq i \leq N\}. \end{aligned}$$

Set $\delta = \frac{\epsilon}{5M^2(m_2+1)^2}$. Suppose $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive, linear map such that ψ is $\mathcal{G}$ - δ -multiplicative. Suppose $x, y \in \{a_i : i \in \mathbb{N}, 1 \leq i \leq N\}$ and $s, t \in \{c_i : i \in \mathbb{N}, 1 \leq i \leq n\}$. Then

$$\begin{aligned} &\|(\psi \otimes \text{id}_{\mathfrak{C}})(x \otimes s)(\psi \otimes \text{id}_{\mathfrak{C}})(y \otimes t) - (\psi \otimes \text{id}_{\mathfrak{C}})(xy \otimes st)\| \\ &= \|\psi(x)\psi(y) \otimes st - \psi(xy) \otimes st\| \\ &\leq \|\psi(x)\psi(y) - \psi(xy)\| m_2^2 \\ &\leq \delta m_2^2. \end{aligned}$$

Let S_1, S_2 be subsets of $\{i \in \mathbb{N} : i \leq N\}$ and $k_i, n_i : \mathcal{S}_i \rightarrow \{i \in \mathbb{N} : i \leq N\}$ be functions. Set $y_1 = \sum_{i \in \mathcal{S}_1} a_{k_1(i)} \otimes c_{n_1(i)}$ and $y_2 = \sum_{i \in \mathcal{S}_2} a_{k_2(i)} \otimes c_{n_2(i)}$:

$$\begin{aligned} \|(\psi \otimes \text{id}_{\mathfrak{C}})(y_1)(\psi \otimes \text{id}_{\mathfrak{C}})(y_2) - (\psi \otimes \text{id}_{\mathfrak{C}})(y_1 y_2)\| &< \sum_{i \in \mathcal{S}_1, j \in \mathcal{S}_2} \delta m_2^2 \\ &< \frac{\epsilon}{5}. \end{aligned}$$

Let $x_1, x_2 \in \mathcal{F}$. Then there exist subsets S_1, S_2 of $\{i \in \mathbb{N} : i \leq N\}$ and functions $k_i, n_i : \mathcal{S}_i \rightarrow \{i \in \mathbb{N} : i \leq N\}$ such that

$$\|x_i - y_i\| < \min \left\{ \frac{\epsilon}{5(m_1 + 1)}, 1 \right\},$$

where $y_j = \sum_{i \in \mathcal{S}_j} a_{k_j(i)} \otimes c_{n_j(i)}$. Therefore,

$$\begin{aligned}
 & \|(\psi \otimes \text{id}_{\mathfrak{C}})(x_1)(\psi \otimes \text{id}_{\mathfrak{C}})(x_2) - (\psi \otimes \text{id}_{\mathfrak{C}})(x_1 x_2)\| \\
 & \leq \|(\psi \otimes \text{id}_{\mathfrak{C}})(x_1 - y_1)\| \|(\psi \otimes \text{id}_{\mathfrak{C}})(x_2)\| + \|(\psi \otimes \text{id}_{\mathfrak{C}})(y_1)\| \|(\psi \otimes \text{id}_{\mathfrak{C}})(x_2 - y_2)\| \\
 & \quad + \|(\psi \otimes \text{id}_{\mathfrak{C}})(y_1)(\psi \otimes \text{id}_{\mathfrak{C}})(y_2) - (\psi \otimes \text{id}_{\mathfrak{C}})(y_1 y_2)\| + \|(\psi \otimes \text{id}_{\mathfrak{C}})(y_1(y_2 - x_2))\| \\
 & \quad + \|(\psi \otimes \text{id}_{\mathfrak{C}})((y_1 - x_1)x_2)\| \\
 & \leq \|(\psi \otimes \text{id}_{\mathfrak{C}})(y_1)(\psi \otimes \text{id}_{\mathfrak{C}})(y_2) - (\psi \otimes \text{id}_{\mathfrak{C}})(y_1 y_2)\| \\
 & \quad + \|x_1 - y_1\| \|x_2\| + \|y_1\| \|x_2 - y_2\| \\
 & \quad + \|y_1\| \|y_2 - x_2\| + \|y_1 - x_1\| \|x_2\| \\
 & < \epsilon.
 \end{aligned}$$

□

Using Lemma 3.3 and arguing as in Remark 4.5.1 and 6.1.1 of [10], we get the following lemma. See also Section 2.3 of [15].

Lemma 3.4. *Let $\mathfrak{A}$ be a C^* -algebra. Let $\mathcal{P}_0$ be a finite set of projections in $\bigcup_{m=1}^{\infty} (\mathfrak{A} \otimes \mathbf{M}_m)^{\dagger}$, and let $\mathcal{P}_1$ be a finite set of unitaries in $\bigcup_{m=1}^{\infty} (\mathfrak{A} \otimes \mathbf{M}_m)^{\dagger}$. Then there exist $\delta > 0$ and a finite subset $\mathcal{F}$ of $\mathfrak{A}$ such that the following holds: Suppose $\mathfrak{B}$ is a C^* -algebra and $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive, completely positive, linear map such that ψ is $\mathcal{F}$ - δ -multiplicative. Then there exist a finite set $\mathcal{G}_0$ of projections in $\bigcup_{m=1}^{\infty} (\mathfrak{B} \otimes \mathbf{M}_m)^{\dagger}$ and a finite set of unitaries $\mathcal{G}_1$ of $\bigcup_{m=1}^{\infty} (\mathfrak{B} \otimes \mathbf{M}_m)^{\dagger}$ such that*

- (1) *For each $p \in \mathcal{P}_0$, there exists $e(p) \in \mathcal{G}_0$ such that*

$$\|\psi(p) - e(p)\| < \frac{1}{2},$$

and if $p_1, p_2 \in \mathcal{P}_0$ such that $[p_1] = [p_2]$ in $K_0(\mathfrak{A})$, then $[e(p_1)] = [e(p_2)]$ in $K_0(\mathfrak{B})$, and if $p_1, p_2, p_1 \oplus p_2 \in \mathcal{P}_0$, then $[e(p_1 \oplus p_2)] = [e(p_1)] + [e(p_2)]$ in $K_0(\mathfrak{B})$.

- (2) *For each $u \in \mathcal{P}_1$, there exists $v(u) \in \mathcal{G}_1$ such that*

$$\|\psi(u) - v(u)\| < \frac{1}{2},$$

and if $u_1, u_2 \in \mathcal{P}_1$ such that $[u_1] = [u_2]$ in $K_1(\mathfrak{A})$, then $[v(u_1)] = [v(u_2)]$ in $K_1(\mathfrak{B})$, and if $u_1, u_2, u_1 \oplus u_2 \in \mathcal{P}_1$, then $[v(u_1 \oplus u_2)] = [v(u_1)] + [v(u_2)]$ in $K_1(\mathfrak{B})$.

Definition 3.5. Let $\mathfrak{A}$ be a C^* -algebra and let $\mathcal{P}$ be a finite subset of $\mathbb{P}(\mathfrak{A})$. By Lemma 3.4, there exist a finite subset $\mathcal{F}$ of $\mathfrak{A}$ and $\delta > 0$ such that the following holds: Suppose $\mathfrak{B}$ is a C^* -algebra and $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive, complete positive, linear map such that ψ is $\mathcal{F}$ - δ -multiplicative. Then there exists a finite subset $\mathcal{Q}$ of $\mathbb{P}(\mathfrak{A})$ such that for each projection $p \in \mathcal{P} \cap (\mathfrak{A} \otimes \mathbf{M}_m \otimes C_n)^{\dagger}$ and unitary $u \in \mathcal{P} \cap (\mathfrak{A} \otimes \mathbf{M}_m \otimes C_n)^{\dagger}$, there exist a projection $e(p) \in \mathcal{Q} \cap (\mathfrak{B} \otimes \mathbf{M}_m \otimes C_n)^{\dagger}$ and a unitary in $v(u) \in \mathcal{Q} \cap (\mathfrak{A} \otimes \mathbf{M}_m \otimes C_n)^{\dagger}$ such that

$$\|(\psi \otimes \text{id}_{\mathbf{M}_m} \otimes \text{id}_{C_n})(p) - e(p)\| < \frac{1}{2} \quad \text{and} \quad \|(\psi \otimes \text{id}_{\mathbf{M}_m} \otimes \text{id}_{C_n})(u) - v(u)\| < \frac{1}{2}.$$

Moreover,

- (1) *If p_1, p_2 are projections in $\mathcal{P}$ and $[p_1] = [p_2]$ in $\underline{K}(\mathfrak{A})$, then $[e(p_1)] = [e(p_2)]$ in $\underline{K}(\mathfrak{B})$.*

- (2) If u_1, u_2 are unitaries in $\mathcal{P}$ and $[u_1] = [u_2]$ in $\underline{K}(\mathfrak{A})$, then $[v(u_1)] = [v(u_2)]$ in $\underline{K}(\mathfrak{B})$.
- (3) If $p_1, p_2, p_1 \oplus p_2$ are projections in $\mathcal{P}$, then $[e(p_1 \oplus p_2)] = [e(p_1)] + [e(p_2)]$ in $\underline{K}(\mathfrak{B})$.
- (4) If $u_1, u_2, u_1 \oplus u_2$ are unitaries in $\mathcal{P}$, then $[v(u_1 \oplus u_2)] = [v(u_1)] + [v(u_2)]$ in $\underline{K}(\mathfrak{B})$.

Let $\overline{\mathcal{P}}$ be the image of $\mathcal{P}$ in $\underline{K}(\mathfrak{A})$. Define the *partial map on $\underline{K}(\mathfrak{A})$ induced by ψ* , $\underline{K}(\psi)|_{\overline{\mathcal{P}}} : \overline{\mathcal{P}} \rightarrow \underline{K}(\mathfrak{B})$, by

$$\underline{K}(\psi)|_{\overline{\mathcal{P}}}(x) = \begin{cases} [e(p)], & \text{if } x = [p] \text{ for some projection } p \text{ in } \mathcal{P}, \\ [v(u)], & \text{if } x = [u] \text{ for some unitary } u \text{ in } \mathcal{P}. \end{cases}$$

By Lemma 3.4 and the above remarks, we have that $\underline{K}(\psi)|_{\overline{\mathcal{P}}}$ is a well-defined map on $\overline{\mathcal{P}}$. Moreover, as in Section 6.1.1 of [10], by enlarging $\mathcal{F}$ and decreasing the size of δ if necessary, $\underline{K}(\psi)|_{\overline{\mathcal{P}}}$ can be extended to a well-defined group homomorphism from the sub-group $G(\overline{\mathcal{P}})$ of $\underline{K}(\mathfrak{A})$ generated by $\overline{\mathcal{P}}$.

Throughout the paper, we will abuse notation and denote this extension by $\underline{K}(\psi)|_{\mathcal{P}}$.

Remark 3.6. (1) Note that if $p \in \mathcal{P}$ and $[p] \in K_i(\mathfrak{A}; \mathbb{Z}_n)$, then $\underline{K}(\psi)|_{\mathcal{P}}([p])$ is an element of $K_i(\mathfrak{B}; \mathbb{Z}_n)$. Hence, $\underline{K}(\psi)|_{\mathcal{P}}$ is group homomorphism from $G(\overline{\mathcal{P}}) \cap K_*(\mathfrak{A}; \mathbb{Z}_n)$ to $K_*(\mathfrak{B}; \mathbb{Z}_n)$. We will denote this homomorphism by $K_*(\psi; \mathbb{Z}_n)|_{\mathcal{P}}$. When $n = 0$, we simply write $K_*(\psi)|_{\mathcal{P}}$.

- (2) Throughout the paper, for $\mathcal{P} \subseteq \mathbb{P}(\mathfrak{A})$, when we write $\underline{K}(\psi)|_{\mathcal{P}}$ we mean that $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive, complete positive, linear map such that ψ is $\mathcal{F}$ - δ -multiplicative for some $\delta > 0$ and some finite subset $\mathcal{F}$ of $\mathbb{P}(\mathfrak{A})$, so that we have a well-defined homomorphism from $G(\overline{\mathcal{P}})$ to $\underline{K}(\mathfrak{B})$.

Lemma 3.7. *Let $\mathfrak{A}$ be a nuclear C^* -algebra and let $\mathcal{P}$ be a finite subset of $\mathbb{P}(\mathfrak{A})$. Then there exist a finite subset $\mathcal{F}$ of $\mathfrak{A}$ and $\delta > 0$ such that if $\mathfrak{B}$ is a C^* -algebra and $\psi_1, \psi_2 : \mathfrak{A} \rightarrow \mathfrak{B}$ are contractive, completely positive, linear maps such that ψ_i is $\mathcal{F}$ - δ -multiplicative and*

$$\|\psi_1(a) - \psi_2(a)\| < \delta$$

for all $a \in \mathcal{F}$, then $\underline{K}(\psi_i)|_{\mathcal{P}}$ is well defined and

$$\underline{K}(\psi_1)|_{\mathcal{P}} = \underline{K}(\psi_2)|_{\mathcal{P}}.$$

Proof. Choose a finite subset $\mathcal{F}_1$ of $\mathfrak{A}$ and $\delta_1 > 0$ such that for any contractive, completely positive, linear map $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ that is also $\mathcal{F}_1$ - δ_1 -multiplicative, then $\underline{K}(\psi)|_{\mathcal{P}}$ is well defined. Note that we can choose δ small enough with $\delta < \delta_1$ and a finite subset $\mathcal{F}$ of $\mathfrak{A}$ large enough with $\mathcal{F}_1 \subseteq \mathcal{F}$ such that if $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive, completely positive, linear map that is $\mathcal{F}$ - δ -multiplicative, then for each $p \in \mathcal{P}$,

$$\|(\psi \otimes \text{id}_{\mathbf{M}_m} \otimes \text{id}_{C_n})(p) - x(p)\| < \frac{1}{2},$$

where $x(p)$ is the element of $\mathbb{P}(\mathfrak{B})$ given in Lemma 3.4.

Let $\psi_1, \psi_2 : \mathfrak{A} \rightarrow \mathfrak{B}$ be contractive, completely positive, linear maps such that ψ_i is $\mathcal{F}$ - δ -multiplicative. Let $p \in \mathcal{P}$. Let $x_i(p)$ be the element in $\mathbb{P}(\mathfrak{B})$ given by

Lemma 3.4. By the definition of $\underline{K}(\psi_i)|_{\mathcal{P}}$, $\underline{K}(\psi_i)|_{\mathcal{P}}([p]) = [x_i(p)]$. Since

$$\|(\psi_i \otimes \text{id}_{\mathbf{M}_m} \otimes \text{id}_{C_n})(p) - x_i(p)\| < \frac{1}{2},$$

we have that

$$\|x_1(p) - x_2(p)\| < 1.$$

Therefore, $[x_1(p)] = [x_2(p)]$ in $\underline{K}(\mathfrak{B})$. Hence, $\underline{K}(\psi_1)|_{\mathcal{P}}([p]) = \underline{K}(\psi_2)|_{\mathcal{P}}([p])$. $\square$

3.4. Bott maps. We now define the Bott maps as in [16]. Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A} \otimes C(S^1)$. Then there exist $\delta > 0$ and a finite subset $\mathcal{G}$ of $\mathfrak{A}$ such that the following holds: if $h : \mathfrak{A} \rightarrow \mathfrak{B}$ is a $*$ -homomorphism and u is a unitary in $\mathfrak{B}$ such that

$$\|h(a)u - uh(a)\| < \delta$$

for all $a \in \mathcal{G}$, then the contractive, completely positive, linear map $\varphi_{h,u} : \mathfrak{A} \otimes C(S^1) \rightarrow \mathfrak{B}$ defined by $\varphi_{h,u}(a \otimes f) = h(a)f(u)$ is $\mathcal{F}$ - ϵ -multiplicative.

Note that for every finite subset $\mathcal{F}$ of $\mathfrak{C}$ and $\epsilon > 0$, there exist $\delta > 0$ and a finite subset $\mathcal{G}$ of $\mathfrak{A}$ such that the following holds: if $\psi : \mathfrak{C} \rightarrow \mathfrak{A} \otimes C(S^1)$ and $h : \mathfrak{A} \rightarrow \mathfrak{B}$ are homomorphisms and u is unitary in $\mathfrak{B}$ such that

$$\|h(a)u - uh(a)\| < \delta$$

for all $a \in \mathcal{G}$, then the contractive, completely positive, linear map $\varphi_{h,u} \circ \psi : \mathfrak{C} \rightarrow \mathfrak{B}$ is $\mathcal{F}$ - ϵ -multiplicative.

Definition 3.8. Note that by the Künneth Formula [29], the embedding $j_{\mathfrak{A}} : \mathfrak{A} \otimes C_0(0, 1) \rightarrow \mathfrak{A} \otimes C(S^1)$ induces injective group homomorphisms

$$\begin{aligned} K_0(j_{\mathfrak{A}}) : K_0(\mathfrak{A} \otimes C_0(0, 1)) &\rightarrow K_0(\mathfrak{A} \otimes C(S^1)), \\ K_1(j_{\mathfrak{A}}) : K_1(\mathfrak{A} \otimes C_0(0, 1)) &\rightarrow K_1(\mathfrak{A} \otimes C(S^1)). \end{aligned}$$

Using Bott periodicity to identify $K_0(\mathfrak{A} \otimes C_0(0, 1))$ with $K_1(\mathfrak{A})$ and $K_1(\mathfrak{A} \otimes C_0(0, 1))$ with $K_0(\mathfrak{A})$, we obtain injective group homomorphisms

$$\begin{aligned} \beta_{\mathfrak{A}}^{(0)} : K_0(\mathfrak{A}) &\rightarrow K_1(\mathfrak{A} \otimes C(S^1)), \\ \beta_{\mathfrak{A}}^{(1)} : K_1(\mathfrak{A}) &\rightarrow K_0(\mathfrak{A} \otimes C(S^1)). \end{aligned}$$

Using Bott periodicity again, we obtain injective group homomorphisms

$$\begin{aligned} \beta_{\mathfrak{A},k}^{(0)} : K_0(\mathfrak{A}; \mathbb{Z}_k) &\rightarrow K_1(\mathfrak{A} \otimes C(S^1); \mathbb{Z}_k), \\ \beta_{\mathfrak{A},k}^{(1)} : K_1(\mathfrak{A}; \mathbb{Z}_k) &\rightarrow K_0(\mathfrak{A} \otimes C(S^1); \mathbb{Z}_k). \end{aligned}$$

Let $\mathcal{P}$ be a finite subset of $\mathbb{P}(\mathfrak{A})$. By the above remarks and Section 3.3, there exist $\delta > 0$ and a finite subset $\mathcal{F}$ of $\mathfrak{A}$ such that if $h : \mathfrak{A} \rightarrow \mathfrak{B}$ is a $*$ -homomorphism and u is a unitary in $\mathfrak{B}$ with

$$\|h(a)u - uh(a)\| < \delta$$

for all $a \in \mathcal{F}$, then $\underline{K}(\varphi_{h,u} \circ j_{\mathfrak{A}})|_{\mathcal{P}}$ is well defined, which we will denote as

$$\text{Bott}(h, u)|_{\mathcal{P}}.$$

In particular, $\text{bott}_1(h, v)|_{\mathcal{P}} = K_0(\varphi_{h,u}) \circ \beta_{\mathfrak{A}}^{(1)}|_{\mathcal{P}}$ and $\text{bott}_0(h, v)|_{\mathcal{P}} = K_1(\varphi_{h,u}) \circ \beta_{\mathfrak{A}}^{(0)}|_{\mathcal{P}}$. If $uh(a) = h(a)u$ for all $a \in \mathfrak{A}$, then $\text{Bott}(h, u)$ is well defined on $\underline{K}(\mathfrak{A})$.

Lemma 3.9. *Let $\mathfrak{A}$ be a unital C^* -algebra and let $\mathcal{P}$ be a finite subset of $\mathbb{P}(\mathfrak{A})$. Then there exist $\delta > 0$ and a finite subset $\mathcal{F}$ of $\mathfrak{A}$ such that the following holds: if $h : \mathfrak{A} \rightarrow \mathfrak{B}$ is a $*$ -homomorphism and $u_1, \dots, u_n$ are unitaries in $\mathfrak{B}$ with*

$$\|h(a)u_i - u_i h(a)\| < \frac{\delta}{n}$$

for all $a \in \mathcal{F}$ and for all i , then $\text{Bott}(h, u_i)|_{\mathcal{P}}$ and $\text{Bott}(h, u_1 \cdots u_n)|_{\mathcal{P}}$ are well defined and

$$\text{Bott}(h, u_1 \cdots u_n)|_{\mathcal{P}} = \sum_{i=1}^n \text{Bott}(h, u_i)|_{\mathcal{P}}.$$

Moreover, if $\text{Bott}(h, u_1)|_{\mathcal{P}} = \text{Bott}(h, u_2)|_{\mathcal{P}}$, then $\text{Bott}(h, u_1 u_2^)|_{\mathcal{P}} = 0$.*

Proof. Let $\delta > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}$ such that if $h : \mathfrak{A} \rightarrow \mathfrak{B}$ is a $*$ -homomorphism and u is a unitary in $\mathfrak{B}$ such that

$$\|h(a)u - uh(a)\| < \delta$$

for all $a \in \mathcal{F}$, then $\text{Bott}(h, u)|_{\mathcal{P}}$ is well defined.

Suppose h is a $*$ -homomorphism and $u_1, \dots, u_n$ are unitaries in $\mathfrak{B}$ such that $\|h(a)u_i - u_i h(a)\| < \frac{\delta}{n}$ for all $a \in \mathcal{F}$ and for all i . Then

$$\begin{aligned} & \|h(a)u_1 \cdots u_n - u_1 \cdots u_n h(a)\| \\ & \leq \|h(a)u_1 \cdots u_n - u_1 h(a)u_2 \cdots u_n\| \\ & \quad + \sum_{i=2}^{n-1} \|u_1 u_2 \cdots u_{i-1} h(a)u_i u_{i+1} \cdots u_n - u_1 \cdots u_i h(a)u_{i+1} \cdots u_n\| \\ & \quad + \|u_1 \cdots u_{n-1} h(a)u_n - u_1 \cdots u_n h(a)\| \\ & = \sum_{i=1}^n \|h(a)u_i - u_i h(a)\| \\ & < \delta \end{aligned}$$

for all $a \in \mathcal{F}$. Hence, by the choice of δ and $\mathcal{F}$, $\text{Bott}(h, u_i)|_{\mathcal{P}}$ and $\text{Bott}(h, u_1 \cdots u_n)|_{\mathcal{P}}$ are well defined. Moreover, enlarging $\mathcal{F}$ and decreasing the size of δ if necessary, we get that

$$\text{Bott}(h, u_1 \cdots u_n)|_{\mathcal{P}} = \sum_{i=1}^n \text{Bott}(h, u_i)|_{\mathcal{P}}.$$

□

3.5. Some technical results.

Lemma 3.10. *Let $\mathfrak{A}$ be a unital C^* -algebra. Let $\epsilon > 0$, $\mathcal{F}_1$ be a finite subset of $\mathfrak{A}$, and $\mathcal{F}_2$ be a finite subset of $\mathfrak{A} \otimes C(S^1)$. Then there exist $\delta > 0$, a finite subset $\mathcal{G}_1$ of $\mathfrak{A}$, and a finite subset $\mathcal{G}_2$ of $\mathfrak{A} \otimes C(S^1)$ such that the following holds: if $\psi : \mathfrak{A} \otimes C(S^1) \rightarrow \mathfrak{A}$ is a contractive, completely positive, linear map and u is a unitary in $\mathfrak{A}$ such that*

- (1) ψ is $\mathcal{G}_2$ - δ -multiplicative,
- (2) $\|\psi(a \otimes 1_{C(S^1)}) - a\| < \delta$ for all $a \in \mathcal{G}_1$, and
- (3) $\|\psi(1_{\mathfrak{A}} \otimes z) - u\| < \delta$,

where z is the function on the circle that sends ξ to ξ , then

- (i) $\|ua - au\| < \epsilon$ for all $a \in \mathcal{F}_1$ and
- (ii) $\|\psi(x) - \varphi_{\text{id}_{\mathfrak{A}}, u}(x)\| < \epsilon$ for all $x \in \mathcal{F}_2$.

Proof. Set $m_1 = \max\{\|a\| : a \in \mathcal{F}_1\}$ and set $\mathcal{G}_1 = \mathcal{F}_1 \cup \{1_{\mathfrak{A}}\}$. Note that $\mathfrak{A} \otimes C(S^1)$ is generated by $\{a \otimes 1_{C(S^1)} : a \in \mathfrak{A}\}$ and $\{1_{\mathfrak{A}} \otimes z^k : k \in \mathbb{Z}\}$. Therefore, there exist $N \in \mathbb{N}$ and a finite subset of $\mathfrak{A}$, $\{a_i \in \mathfrak{A} : i \in \mathbb{Z}, |i| \leq N\}$, such that for each $x \in \mathcal{F}_2$, there exist a finite subset $\mathcal{S}_x$ of $\mathbb{Z}$ and functions $k_x, m_x : \mathcal{S}_x \rightarrow \{k \in \mathbb{Z} : |k| \leq N\}$ such that

$$\left\| x - \sum_{i \in \mathcal{S}_x} a_{k_x(i)} \otimes z^{m_x(i)} \right\| < \frac{\epsilon}{4}.$$

Define M and m_2 as follows:

$$M = \max\{|S_x| : x \in \mathcal{F}_2\} \quad \text{and} \quad m_2 = \max\{\|a_i\| : i \in \mathbb{Z}, |i| \leq N\}.$$

Set $\delta = \min\left\{\frac{\epsilon}{2m_1+4}, \frac{\epsilon}{2M((2N-1)m_2+2)}\right\}$. Let

$$\begin{aligned} G_2 = & \left(\{a \otimes 1_{C(S^1)} : a \in \mathcal{F}_1\} \cup \{a_i \otimes 1_{C(S^1)} : i \in \mathbb{Z}, |i| \leq N\} \right) \\ & \cup \left(\{a_i \otimes z^j : i, j \in \mathbb{Z}, |i|, |j| \leq N\} \cup \{1_{\mathfrak{A}} \otimes z^k : k \in \mathbb{Z}, |k| \leq N\} \right). \end{aligned}$$

Note that for each $a \in \mathcal{F}_1$,

$$\begin{aligned} \|ua - au\| & \leq \|(u - \psi(1_{\mathfrak{A}} \otimes z))a\| \\ & \quad + \|\psi(1_{\mathfrak{A}} \otimes z)(a - \psi(a \otimes 1_{C(S^1)}))\| \\ & \quad + \|\psi(1_{\mathfrak{A}} \otimes z)\psi(a \otimes 1_{C(S^1)}) - \psi(a \otimes z)\| \\ & \quad + \|\psi(a \otimes z) - \psi(a \otimes 1_{C(S^1)})\psi(1_{\mathfrak{A}} \otimes z)\| \\ & \quad + \|\psi(a \otimes 1_{C(S^1)})(\psi(1_{\mathfrak{A}} \otimes z) - u)\| \\ & \quad + \|(\psi(a \otimes 1_{C(S^1)}) - a)u\| \\ & < (2m_1 + 4)\delta \\ & < \epsilon. \end{aligned} \tag{3.1}$$

Since for each $k \in \mathbb{Z} \setminus \{0\}$ with $|k| \leq N$,

$$\|\psi(1_{\mathfrak{A}} \otimes z^k) - u^k\| \leq (2N - 1)\delta \tag{3.2}$$

and

$$\begin{aligned} \|\psi(1_{\mathfrak{A}} \otimes z^0) - u^0\| & = \|\psi(1_{\mathfrak{A}} \otimes 1_{C(S^1)}) - 1_{\mathfrak{A}}\| \\ & < \delta, \end{aligned} \tag{3.3}$$

we have that for each $i, k \in \mathbb{Z}$ with $|i| \leq N$ and $|k| \leq N$,

$$\begin{aligned} \|\psi(a_i \otimes z^k) - \varphi_{\text{id}_{\mathfrak{A}}, u}(a_i \otimes z^k)\| & = \|\psi(a_i \otimes z^k) - a_i u^k\| \\ & \leq \|\psi(a_i \otimes z^k) - \psi(a_i \otimes 1_{C(S^1)})\psi(1_{\mathfrak{A}} \otimes z^k)\| \\ & \quad + \|\psi(a_i \otimes 1_{C(S^1)})(\psi(1_{\mathfrak{A}} \otimes z^k) - u^k)\| \\ & \quad + \|(\psi(a_i \otimes 1_{C(S^1)}) - a_i)u^k\| \\ & < ((2N - 1)m_2 + 2)\delta. \end{aligned} \tag{3.4}$$

Therefore, for each $x \in \mathcal{F}_2$,

$$\begin{aligned} \|\psi(x) - \varphi_{\text{id}_{\mathfrak{A}}, u}(x)\| &\leq \left\| \psi(x) - \psi \left(\sum_{i \in S_x} a_{k_x(i)} \otimes z^{m_x(i)} \right) \right\| \\ &\quad + \left\| \psi \left(\sum_{i \in S_x} a_{k_x(i)} \otimes z^{m_x(i)} \right) - \varphi_{\text{id}_{\mathfrak{A}}, u} \left(\sum_{i \in S_x} a_{k_x(i)} \otimes z^{m_x(i)} \right) \right\| \\ &\quad + \left\| \varphi_{\text{id}_{\mathfrak{A}}, u} \left(\sum_{i \in S_x} a_{m_x(i)} \otimes z^{k_x(i)} \right) - \varphi_{\text{id}_{\mathfrak{A}}, u}(x) \right\| \\ &\leq \frac{\epsilon}{2} + M((2N-1)m_2 + 2)\delta \\ &< \epsilon. \end{aligned}$$

□

The following lemma is the result of the proof of Lemma 4 of [21]. In [21] we did not state the lemma as below, since we were not concerned with the bott maps.

Lemma 3.11. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^* -algebras satisfying the UCT. Suppose $\varphi : \underline{K}(\mathfrak{A}) \rightarrow \underline{K}(\mathfrak{B})$ is a homomorphism such that $\varphi|_{K_0(\mathfrak{A})}$ is positive. Suppose $\gamma : K_0(\mathfrak{A}) \rightarrow K_1(\mathfrak{B})$ is a group homomorphism. Then there exists $\alpha : \underline{K}(\mathfrak{A} \otimes C(S^1)) \rightarrow \underline{K}(\mathfrak{B})$ such that*

- (1) $\alpha|_{K_0(\mathfrak{A} \otimes C(S^1))}$ is positive,
- (2) $\alpha \circ \underline{K}(\iota_{\mathfrak{A}}) = \varphi$,
- (3) $\alpha \circ \beta_{\mathfrak{A}}^{(1)} = 0$, and
- (4) $\alpha \circ \beta_{\mathfrak{A}}^{(0)} = \gamma$.

Lemma 3.12. *Let $\mathfrak{A}$ be an infinite dimensional, simple, unital C^* -algebra satisfying the UCT. Suppose $\mathfrak{A}$ is a tracially AI algebra. Then for each $\epsilon > 0$, for each finite subset $\mathcal{P}$ of $\mathbb{P}(\mathfrak{A})$, and for each finite subset $\mathcal{F}$ of $\mathfrak{A} \otimes C(S^1)$ there exists a finitely generated subgroup $\mathcal{G}$ of $K_0(\mathfrak{A})$ containing $[1_{\mathfrak{A}}]$ such that the following holds: for every unital, simple, tracially AI algebra $\mathfrak{B}$, if $\gamma : \mathcal{G} \rightarrow K_1(\mathfrak{B})$ is a homomorphism and $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a unital $*$ -homomorphism, then there exists a contractive, completely positive, linear map $\psi : \mathfrak{A} \otimes C(S^1) \rightarrow \mathfrak{B}$ such that*

- (1) ψ is $\mathcal{F}$ - ϵ -multiplicative,
- (2) $\underline{K}(\psi \circ \iota_{\mathfrak{A}})|_{\mathcal{P}} = \underline{K}(\varphi)|_{\mathcal{P}}$,
- (3) $K_0(\psi) \circ \beta_{\mathfrak{A}}^{(1)}|_{\mathcal{P}} = 0$, and
- (4) $K_1(\psi) \circ \beta_{\mathfrak{A}}^{(0)}|_{\mathcal{P}} = \gamma|_{\mathcal{P}}$.

Proof. Since $\mathfrak{A}$ is an infinite dimensional tracially AI algebra satisfying the UCT, we may assume that

$$\mathfrak{A} = \varinjlim (\mathfrak{A}_n, \varphi_{n,n+1}),$$

where $\mathfrak{A}_n = \bigoplus_{i=1}^{k(n)} P_{[n,i]} M_{[n,i]}(C(X_{[n,i]})) P_{[n,i]}$, each $X_{[n,i]}$ is a connected finite CW-complex, and $\varphi_{n,n+1}$ is a unital $*$ -monomorphism.

Choose $n_0 \in \mathbb{N}$ large enough and choose finite subsets, $\mathcal{F}_{n_0}$ of $\mathfrak{A}_{n_0}$ and $\mathcal{P}_{n_0}$ of $\mathbb{P}(\mathfrak{A}_{n_0})$ such that every element of $\mathcal{F}$ is within $\frac{\epsilon}{20}$ to an element of $(\varphi_{n_0} \otimes \text{id}_{C(S^1)})(\mathcal{F}_{n_0})$, and for every $p \in \mathcal{P}$, there exists $e_p \in \mathcal{P}_{n_0}$ such that

$$[p] = \underline{K}(\varphi_{n_0, \infty})([e_p]).$$

Set $\mathcal{G} = K_0(\varphi_{n_0, \infty})(K_0(\mathfrak{A}_{n_0}))$. Let $\mathfrak{B}$ be a simple, unital, tracially AI algebra. Suppose $\gamma : \mathcal{G} \rightarrow K_1(\mathfrak{B})$ is a homomorphism and suppose $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a unital $*$ -monomorphism. Then, by Lemma 3.12, there exists $\alpha \in \text{Hom}_\Lambda(\underline{K}(\mathfrak{A}_{n_0} \otimes C(S^1)), \underline{K}(\mathfrak{B}))$ such that $\alpha|_{K_0(\mathfrak{A}_{n_0} \otimes C(S^1))}$ is a positive homomorphism, $\alpha \circ \underline{K}(\iota_{\mathfrak{A}_{n_0}}) = \underline{K}(\varphi \circ \varphi_{n_0, \infty})$, $\alpha \circ \beta_{\mathfrak{A}_{n_0}}^{(1)}|_{K_1(\mathfrak{A}_{n_0})} = 0$ and $\alpha \circ \beta_{\mathfrak{A}_{n_0}}^{(0)}|_{K_0(\mathfrak{A}_{n_0})} = \gamma \circ K_0(\varphi_{n_0})$.

By Theorem 9.12 of [12] and Theorem 5.4 of [16], there exists a nuclear, separable, simple, unital, tracially AF algebra $\mathfrak{B}'$ and an embedding $\varphi : \mathfrak{B}' \rightarrow \mathfrak{B}$ such that $\underline{K}(\varphi)$ is an isomorphism. Composing the maps obtained from Proposition 9.10 of [12] and Theorem 6.2.9 of [10] with φ , we get a sequence of unital, contractive, completely positive, linear maps $\{L_{n_0, k} : \mathfrak{A}_{n_0} \otimes C(S^1) \rightarrow \mathfrak{B}\}_{k=1}^\infty$ such that

$$\lim_{k \rightarrow \infty} \|L_{n_0, k}(xy) - L_{n_0, k}(x)L_{n_0, k}(y)\| = 0$$

for all $x, y \in \mathfrak{A}_{n_0} \otimes C(S^1)$,

$$\underline{K}(L_{n_0, k} \circ \iota_{\mathfrak{A}_{n_0}})|_{\mathcal{P}_{n_0}} = \alpha \circ \underline{K}(\iota_{\mathfrak{A}_{n_0}})|_{\mathcal{P}_{n_0}},$$

$K_0(L_{n_0, k}) = \alpha|_{K_0(\mathfrak{A}_{n_0} \otimes C(S^1))}$, and $K_1(L_{n_0, k}) = \alpha|_{K_1(\mathfrak{A}_{n_0} \otimes C(S^1))}$.

Since $\mathfrak{A}_{n_0} \otimes C(S^1)$ is nuclear, there exists a sequence of contractive, completely positive, linear maps, $\{\psi_{n_0, k} : \mathfrak{A} \otimes C(S^1) \rightarrow \mathfrak{A}_{n_0} \otimes C(S^1)\}_{k=1}^\infty$ such that

$$\lim_{k \rightarrow \infty} \|(\psi_{n_0, k} \circ (\varphi_{n_0, \infty} \otimes \text{id}_{C(S^1)}))(x) - x\| = 0$$

for all $x \in \mathfrak{A}_{n_0} \otimes C(S^1)$. Set $\beta_{n_0, k} = L_{n_0, k} \circ \psi_{n_0, k}$ for large enough k ; $\beta_{n_0, k}$ satisfies the desired property. Hence, set $\psi = \beta_{n_0, k}$. $\square$

Definition 3.13. Let $\mathfrak{A}$ be a unital C^* -algebra such that $T(\mathfrak{A}) \neq \emptyset$. Denote the state space of $K_0(\mathfrak{A})$ by $S(K_0(\mathfrak{A}))$. The canonical map from $T(\mathfrak{A})$ to $S(K_0(\mathfrak{A}))$ which sends τ to the function $\{[p] \mapsto \tau(p)\}$ will be denoted $r_{\mathfrak{A}}(\tau)([p]) = \tau(p)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^* -algebras such that $T(\mathfrak{A})$ and $T(\mathfrak{B})$ are nonempty sets. Let $\kappa \in \text{Hom}_\Lambda(\underline{K}(\mathfrak{A}), \underline{K}(\mathfrak{B}))$ such that $\kappa([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$ in $K_0(\mathfrak{B})$. Then an affine map

$$\Delta : T(\mathfrak{B}) \rightarrow T(\mathfrak{A})$$

is said to be **compatible to** κ if the diagram

$$\begin{array}{ccc} T(\mathfrak{B}) & \xrightarrow{r_{\mathfrak{B}}} & S(K_0(\mathfrak{B})) \\ \Delta \downarrow & & \downarrow \kappa_0^S \\ T(\mathfrak{A}) & \xrightarrow{r_{\mathfrak{A}}} & S(K_0(\mathfrak{A})) \end{array}$$

is commutative, where κ_0 is the homomorphism from $K_0(\mathfrak{A})$ to $K_0(\mathfrak{B})$ induced by κ and $\kappa_0^S(f) = f \circ \kappa_0$.

Lemma 3.14. Let $\mathfrak{C} = M_n(C(X)) \otimes C(S^1)$, where X is either $[0, 1]$ or a space with one point, and let $\mathfrak{A}$ be a simple, tracially AI algebra. Suppose κ is an element of $\text{Hom}_\Lambda(\underline{K}(\mathfrak{C}), \underline{K}(\mathfrak{A}))$ such that $\kappa(K_0(\mathfrak{C})_+ \setminus \{0\}) \subseteq K_0(\mathfrak{A})_+ \setminus \{0\}$ and $\kappa([1_{\mathfrak{C}}]) = [1_{\mathfrak{A}}]$, and $\gamma : T(\mathfrak{A}) \rightarrow T(\mathfrak{C})$ is a continuous affine map that is compatible to κ .

Let $\epsilon > 0$ and $\mathcal{H}$ be a finite subset of $\mathfrak{C}_{s.a.}$. Then there exists a unital $*$ -homomorphism $h : \mathfrak{C} \rightarrow \mathfrak{A}$ such that

- (1) $\underline{K}(h) = \kappa$ and
- (2) $\sup \{|\tau \circ h(x) - \gamma(\tau)(x)| : \tau \in T(\mathfrak{A})\} < \epsilon$ for all $x \in \mathcal{H}$.

Proof. By Theorem 9.12 of [12] and Theorem 5.4 of [16], there exists a unital, separable, nuclear, simple, tracially AF algebra $\mathfrak{B}$ and an embedding $\varphi : \mathfrak{B} \rightarrow \mathfrak{A}$ such that $\underline{K}(\varphi)$ is an isomorphism. Hence, there exists $\kappa_0 \in \text{Hom}_\Lambda(\underline{K}(\mathfrak{C}), \underline{K}(\mathfrak{B}))$ such that $\kappa_0(K_0(\mathfrak{C})_+ \setminus \{0\}) \subseteq K_0(\mathfrak{A})_+ \setminus \{0\}$, $\kappa_0([1_{\mathfrak{C}}]) = [1_{\mathfrak{B}}]$, and $\kappa = \underline{K}(\varphi) \circ \kappa_0$.

Using Lemma 6.2 of [16] and applying φ , there exist a projection p_0 in $\mathfrak{A}$, a finite dimensional sub- C^* -algebra $\mathfrak{F}$ of $\mathfrak{A}$ with $1_{\mathfrak{F}} = 1_{\mathfrak{A}} - p_0$ and unital $*$ -homomorphisms $h_0 : \mathfrak{C} \rightarrow p_0 \mathfrak{A} p_0$ and $h_1 : \mathfrak{C} \rightarrow \mathfrak{F}$ such that

$$\underline{K}(h_0 + h_1) = \kappa \quad \text{and} \quad \tau(p_0) < \frac{\epsilon}{3}$$

for all $\tau \in T(\mathfrak{A})$.

Since $K_1(h_1) = 0$, by Lemma 9.5 of [12], there is a sub- C^* -algebra $\mathfrak{B}_0$ of $(1_{\mathfrak{A}} - p_0)\mathfrak{A}(1_{\mathfrak{A}} - p_0)$, where $\mathfrak{B}_0$ is a finite direct sum of C^* -algebras of the form M_k and $M_n(C([0, 1]))$, and there exists a unital $*$ -homomorphism $h_2 : \mathfrak{C} \rightarrow \mathfrak{B}_0$ such that

$$K_0(h_2) = K_0(h_1) \quad \text{and} \quad |\tau \circ h_2(f) - \tau(1_{\mathfrak{A}} - p_0)\gamma(\tau(f))| < \frac{\epsilon}{3}$$

for all $f \in \mathcal{H}$ and for all $\tau \in T(\mathfrak{A})$.

Set $h = h_0 + h_2$. Then $h : \mathfrak{C} \rightarrow \mathfrak{A}$ is a unital $*$ -homomorphism such that

$$\underline{K}(h) = \underline{K}(h_0 + h_1) = \kappa$$

and

$$\begin{aligned} |\tau \circ h(f) - \gamma(\tau)(f)| &< |\tau \circ h(f) - \tau(1_{\mathfrak{A}} - p_0)\gamma(\tau)(f)| + \frac{\epsilon}{3} \\ &< \frac{\epsilon}{3} + |\tau \circ h_2(f) - \tau(1_{\mathfrak{A}} - p_0)\gamma(\tau)(f)| + \frac{\epsilon}{3} \\ &< \epsilon \end{aligned}$$

for all $\tau \in T(\mathfrak{A})$ and for all $f \in \mathcal{H}$. □

Lemma 3.15. *Let $\mathfrak{A}$ be as in Lemma 3.12. For each $\epsilon > 0$ and for each finite subset $\mathcal{P}$, $\mathcal{F}_1$, and $\mathcal{F}_2$ of $\mathbb{P}(\mathfrak{A})$, $\mathfrak{A}_{s.a.}$, and $\mathfrak{A} \otimes C(S^1)$, respectively, there exists a finitely generated subgroup $\mathcal{G}$ of $K_0(\mathfrak{A})$ containing $[1_{\mathfrak{A}}]$ such that the following holds: for every homomorphism $\gamma : \mathcal{G} \rightarrow K_1(\mathfrak{A})$ there exists a unital, contractive, completely positive, linear map $\psi : \mathfrak{A} \otimes C(S^1) \rightarrow \mathfrak{A}$ such that*

- (1) ψ is $\mathcal{F}_2$ - ϵ -multiplicative,
- (2) $\underline{K}(\psi) \circ \underline{K}(\iota_{\mathfrak{A}})|_{\mathcal{P}} = \underline{K}(\text{id}_{\mathfrak{A}})|_{\mathcal{P}}$,
- (3) $\sup\{|\tau \circ \psi \circ \iota_{\mathfrak{A}}(a) - \tau(a)| : \tau \in T(\mathfrak{A})\} < \epsilon$ for all $a \in \mathcal{F}_1$,
- (4) $K_0(\psi) \circ \beta_{\mathfrak{A}}^{(1)}|_{\mathcal{P}} = 0$, and
- (5) $K_1(\psi) \circ \beta_{\mathfrak{A}}^{(0)}|_{\mathcal{P}} = \gamma|_{\mathcal{P}}$.

Proof. Let $\{\mathcal{H}_n\}_{n=1}^\infty$ be an increasing sequence of finite subsets of $\mathfrak{A}$ whose union is dense in $\mathfrak{A}$. Now, for each $n \in \mathbb{N}$, there exist a projection $p_n \in \mathfrak{A}$, a sub- C^* -algebra $\mathfrak{D}_n = \bigoplus_{i=1}^{k(n)} M_{m(i,n)}(C(X_{[i,n]}))$ of $\mathfrak{A}$, where $X_{[i,n]}$ is either $[0, 1]$ or a space with one point with $1_{\mathfrak{D}_n} = p_n$, and a sequence of contractive, completely positive, linear maps $\{L_n : \mathfrak{A} \rightarrow \mathfrak{D}_n\}_{n=1}^\infty$ such that

- (a) $\|p_n x - x p_n\| < \frac{1}{2^n}$ for all $x \in \mathcal{H}_n$,
- (b) $\|p_n x p_n - L_n(x)\| < \frac{1}{2^n}$ for all $x \in \mathcal{H}_n$,
- (c) $\|x - (1_{\mathfrak{A}} - p_n)x(1_{\mathfrak{A}} - p_n) - L_n(x)\| < \frac{1}{2^n}$ for all $x \in \mathcal{H}_n$ with $\|x\| \leq 1$, and
- (d) $\tau(1_{\mathfrak{A}} - p_n) < \frac{1}{2^n}$ for all $\tau \in T(\mathfrak{A})$.

Note that

$$\lim_{n \rightarrow \infty} \|L_n(xy) - L_n(x)L_n(y)\| = 0$$

for all $x, y \in \mathfrak{A}$.

Denote the i th summand of $\mathfrak{D}_n$ by $\mathfrak{D}_{[n,i]}$, and let $d_{[n,i]} = 1_{\mathfrak{D}_{[n,i]}}$. Choose n large enough such that $\frac{1}{2^n} < \frac{\epsilon}{6}$. Let $\mathcal{P}_1$ be a finite subset of $\mathbb{P}(\mathfrak{A})$ such that $\mathcal{P}_1$ contains $\mathcal{P}$, $d_{[n,i]}$, p_n . Choose a finite subset $\mathcal{F}_3$ of $\mathfrak{A} \otimes C(S^1)$ such that $\mathcal{F}_3$ contains $\iota_{\mathfrak{A}}(\mathcal{F}_1) \cup \mathcal{F}_2$ and the set

$$\{((1_{\mathfrak{A}} - p_n) \otimes 1_{C(S^1)})x((1_{\mathfrak{A}} - p_n) \otimes 1_{C(S^1)}) : x \in \mathcal{F}\}.$$

Let $\mathcal{G}$ be the finitely generated subgroup of $K_0(\mathfrak{A})$ in Lemma 3.12 which corresponds to $\frac{1}{2^n}$, $\mathcal{P}_1$, and $\mathcal{F}_3$.

Suppose $\gamma : \mathcal{G} \rightarrow K_1(\mathfrak{A})$ is a homomorphism. Then, by Lemma 3.12, there exists a contractive, completely positive, linear map $L : \mathfrak{A} \otimes C(S^1) \rightarrow \mathfrak{A}$ such that

- (a) L is $\mathcal{F}_3$ - $\frac{1}{2^n}$ -multiplicative,
- (b) $\underline{K}(L \circ \iota_{\mathfrak{A}})|_{\mathcal{P}_1} = \underline{K}(\text{id}_{\mathfrak{A}})|_{\mathcal{P}_1}$,
- (c) $K_0(L) \circ \beta_{\mathfrak{A}}^{(1)}|_{\mathcal{P}} = 0$, and
- (d) $K_1(L) \circ \beta_{\mathfrak{A}}^{(0)}|_{\mathcal{P}} = \gamma|_{\mathcal{P}}$.

Choose a projection $q_n \in \mathfrak{A}$ such that $[q_n] = K_0(L) \left(\sum_{i=1}^{k(n)} [d_{[n,i]} \otimes 1_{C(S^1)}] \right)$. Let $\mathcal{G}_n$ be a finite subset of $\mathfrak{D}_n$ such that $\mathcal{G}_n$ contains the generators of $\mathfrak{D}_n$. Define $\eta : T(\mathfrak{A}) \rightarrow T(\mathfrak{D}_n \otimes C(S^1))$ by

$$\eta(\tau) = \frac{1}{\tau(p_n)} \tau \circ (\text{id}_{\mathfrak{A}} \otimes \text{ev})|_{\mathfrak{D}_n \otimes C(S^1)}.$$

Since $K_0(L) = K_0(\text{id}_{\mathfrak{A}} \otimes \text{ev})$, we have that $K_0(L|_{\mathfrak{D}_n \otimes C(S^1)})$ and γ are compatible. Also, note that $K_0(\mathfrak{D}_n \otimes C(S^1)) = K_0(\iota_{\mathfrak{D}_n})(K_0(\mathfrak{D}_n))$. Therefore $K_0(L|_{\mathfrak{D}_n \otimes C(S^1)})$ sends $K_0(\mathfrak{D}_n \otimes C(S^1))_+ \setminus \{0\}$ to $K_0(q_n \mathfrak{A} q_n)_+ \setminus \{0\}$. Hence, by Lemma 3.14, there exists a unital $*$ -homomorphism $h : \mathfrak{D}_n \otimes C(S^1) \rightarrow q_n \mathfrak{A} q_n$ such that $\underline{K}(h) = \underline{K}(L|_{\mathfrak{D}_n \otimes C(S^1)})$ and

$$\sup \{ |(\tau \circ h)(g) - \eta(\tau)(g)| : \tau \in T(\mathfrak{A}) \} < \frac{1}{2^n}$$

for all $h \in \mathcal{G}_n$.

Define $\psi : \mathfrak{A} \otimes C(S^1) \rightarrow \mathfrak{A}$ by

$$\psi(x) = L([(1_{\mathfrak{A}} - p_n) \otimes 1_{C(S^1)}]x[(1_{\mathfrak{A}} - p_n) \otimes 1_{C(S^1)}]) + (h \circ (L_n \otimes \text{id}_{C(S^1)}))(x).$$

By construction, ψ satisfies the desired properties of the lemma. $\square$

Theorem 3.16. *Let $\mathfrak{A}$ be as in Lemma 3.12. For each $\epsilon > 0$ and for each finite subset $\mathcal{P}$, $\mathcal{F}_1$, and $\mathcal{F}_2$ of $\mathbb{P}(\mathfrak{A})$, $\mathfrak{A}$, and $\mathfrak{A} \otimes C(S^1)$, respectively, there exists a finitely generated subgroup $\mathcal{G}$ of $K_0(\mathfrak{A})$ containing $[1_{\mathfrak{A}}]$ such that the following holds: for every homomorphism $\gamma : \mathcal{G} \rightarrow K_1(\mathfrak{A})$ there exists a unital, contractive, completely positive, linear map $\psi : \mathfrak{A} \otimes C(S^1) \rightarrow \mathfrak{A}$ such that*

- (1) ψ is $\mathcal{F}_2$ - ϵ -multiplicative,
- (2) $\underline{K}(\psi) \circ \underline{K}(\iota_{\mathfrak{A}})|_{\mathcal{P}} = \underline{K}(\text{id}_{\mathfrak{A}})|_{\mathcal{P}}$,
- (3) $\|(\psi \circ \iota_{\mathfrak{A}})(a) - a\| < \epsilon$ for all $a \in \mathcal{F}_1$,
- (4) $K_0(\psi) \circ \beta_{\mathfrak{A}}^{(1)}|_{\mathcal{P}} = 0$, and
- (5) $K_1(\psi) \circ \beta_{\mathfrak{A}}^{(0)}|_{\mathcal{P}} = \gamma|_{\mathcal{P}}$.

Proof. Arguing as in the proof of Theorem 4: The tracially AI case (p. 439 of [21]) and using Lemma 3.15 instead of Lemma 7 of [21], we get the desired result. $\square$

Corollary 3.17. *Let $\mathfrak{A}$ be as in Lemma 3.12. Let $\epsilon > 0$ and let $\mathcal{F}$ and $\mathcal{P}$ be finite subsets of $\mathfrak{A}$ and $\mathbb{P}(\mathfrak{A})$, respectively. Then there exists a finitely generated subgroup $\mathcal{G}$ of $K_0(\mathfrak{A})$ containing $[1_{\mathfrak{A}}]$ such that the following holds: if $\gamma : \mathcal{G} \rightarrow K_1(\mathfrak{A})$, then there exists a unitary $w \in \mathfrak{A}$ such that*

- (1) $\|aw - wa\| < \epsilon$ for all $a \in \mathcal{F}$,
- (2) $\text{bott}_1(\text{id}_{\mathfrak{A}}, w)|_{\mathcal{P}} = 0$ and $\text{bott}_0(\text{id}_{\mathfrak{A}}, w)|_{\mathcal{P}} = \gamma|_{\mathcal{P}}$, and
- (3) $\gamma([1_{\mathfrak{A}}]) = [w]$.

Proof. Let $\epsilon > 0$, $\mathcal{P}$ be a finite subset of $\mathbb{P}(\mathfrak{A})$, and $\mathcal{F}$ be a finite subset of $\mathfrak{A}$. Set $\mathcal{Q} = j_{\mathfrak{A}}(\mathcal{P})$. By Lemma 3.7, there exist $\delta_1 > 0$ and a finite subset $\mathcal{H}$ of $\mathfrak{A} \otimes C(S^1)$ corresponding to $\mathcal{Q}$. By Lemma 3.10, there exist $\delta_2 > 0$, a finite subset $\mathcal{G}_1$ of $\mathfrak{A}$, and the finite subset $\mathcal{G}_2$ of $\mathfrak{A} \otimes C(S^1)$ corresponding to $\min\{\delta_1, \epsilon\}$, $\mathcal{H}$, and $\mathcal{F}$. By Lemma 3.2, there exists δ_3 corresponding $\min\{\delta_1, \delta_2\}$. By Theorem 3.16, there exists a finitely generated subgroup $\mathcal{G}$ of $K_0(\mathfrak{A})$ containing $[1_{\mathfrak{A}}]$ corresponding to $\mathcal{P}$, $\mathcal{G}_1 \cup \mathcal{F}$, $\mathcal{G}_2 \cup \mathcal{H}$, and $\delta = \min\{\epsilon, \delta_1, \delta_2, \delta_3\}$.

Suppose $\gamma : \mathcal{G} \rightarrow K_1(\mathfrak{A})$ is a group homomorphism. Then there exists a unital, contractive, completely positive, linear map $\psi : \mathfrak{A} \otimes C(S^1) \rightarrow \mathfrak{A}$ such that

- (a) ψ is $(\mathcal{G}_2 \cup \mathcal{H})$ - δ -multiplicative,
- (b) $\|\psi(a \otimes 1_{C(S^1)}) - a\| < \delta$ for all $a \in \mathcal{G}_1 \cup \mathcal{F}$,
- (c) $K_0(\psi) \circ \beta_{\mathfrak{A}}^{(1)}|_{\mathcal{P}} = 0$, and
- (d) $K_1(\psi) \circ \beta_{\mathfrak{A}}^{(0)}|_{\mathcal{P}} = \gamma|_{\mathcal{P}}$.

Since $\|L(1_{\mathfrak{A}} \otimes z)^* L(1_{\mathfrak{A}} \otimes z) - 1_{\mathfrak{A}}\| < \delta < \delta_3$ and $\|L(1_{\mathfrak{A}} \otimes z)L(1_{\mathfrak{A}} \otimes z)^* - 1_{\mathfrak{A}}\| < \delta < \delta_3$, there exists a unitary $w \in \mathfrak{A}$ such that

$$\|\psi(1_{\mathfrak{A}} \otimes z) - w\| < \delta_2.$$

Therefore, ψ is $(\mathcal{G}_2 \cup \mathcal{H})$ - δ_2 -multiplicative with

$$\|\psi(a \otimes 1_{C(S^1)}) - a\| < \delta_2 \quad \text{and} \quad \|\psi(1_{\mathfrak{A}} \otimes z) - w\| < \delta_2$$

for all $a \in \mathcal{G}_1 \cup \mathcal{F}$. Hence, $\|\psi(x) - \varphi_{\text{id}_{\mathfrak{A}}, w}(x)\| < \min\{\epsilon, \delta_1\}$ for all $x \in \mathcal{H}$ and $\|wa - aw\| < \min\{\epsilon, \delta_1\}$ for all $a \in \mathcal{F}$. Thus, $\underline{K}(\psi)|_{\mathcal{Q}} = \underline{K}(\varphi_{\text{id}_{\mathfrak{A}}, w})|_{\mathcal{Q}}$. So

$$\text{bott}_1(\text{id}_{\mathfrak{A}}, w)|_{\mathcal{P}} = K_0(\varphi_{\text{id}_{\mathfrak{A}}, w}) \circ \beta_{\mathfrak{A}}^{(1)}|_{\mathcal{P}} = K_0(\psi) \circ \beta_{\mathfrak{A}}^{(1)}|_{\mathcal{P}} = 0$$

and

$$\text{bott}_0(\text{id}_{\mathfrak{A}}, w)|_{\mathcal{P}} = K_1(\varphi_{\text{id}_{\mathfrak{A}}, w}) \circ \beta_{\mathfrak{A}}^{(0)}|_{\mathcal{P}} = K_1(\psi) \circ \beta_{\mathfrak{A}}^{(0)}|_{\mathcal{P}} = \gamma|_{\mathcal{P}}.$$

Moreover,

$$\gamma([1_{\mathfrak{A}}]) = K_1(\psi) \circ \beta_{\mathfrak{A}}^{(0)}([1_{\mathfrak{A}}]) = K_1(\psi)([1_{\mathfrak{A}} \otimes z]) = [w].$$

$\square$

Definition 3.18. Let $\mathcal{A}$ denote the class of all nuclear, separable, simple, unital C^* -algebras $\mathfrak{A}$ such that $\mathfrak{A} \otimes M_{\mathfrak{p}}$ is a tracially AI algebra that satisfies the UCT for all supernatural numbers $\mathfrak{p}$ of infinite type. By Theorem 2.11 of [18], $\mathfrak{A} \otimes M_{\mathfrak{p}}$ is a tracially AI algebra that satisfies the UCT for all supernatural numbers $\mathfrak{p}$ of infinite type if and only if $\mathfrak{A} \otimes M_{\mathfrak{p}}$ is a tracially AI algebra that satisfies the UCT for some supernatural number $\mathfrak{p}$ of infinite type.

Let $\mathcal{A}_{\mathcal{Z}}$ denote the class of C^* -algebras $\mathfrak{A}$ in $\mathcal{A}$ such that $\mathfrak{A}$ is $\mathcal{Z}$ -stable. Note that if $\mathfrak{A} \in \mathcal{A}_{\mathcal{Z}}$, then $\mathfrak{A} \otimes M_p \in \mathcal{A}_{\mathcal{Z}}$ for all supernatural numbers p .

Notation 3.19. Let $\mathcal{Q}$ be the UHF algebra such that $K_0(\mathcal{Q}) = \mathbb{Q}$ and $[1_{\mathcal{Q}}] = 1$.

Definition 3.20. Let $\mathfrak{A}$ be a unital C^* -algebra and let $\mathcal{G}$ be a subgroup of $K_0(\mathfrak{A})$. Then $H_{[1_{\mathfrak{A}}]}(\mathcal{G}, K_1(\mathfrak{A}))$ denotes the subgroup of all $x \in K_1(\mathfrak{A})$ such that there exists a homomorphism α from $\mathcal{G}$ to $K_1(\mathfrak{A})$ with $\alpha([1_{\mathfrak{A}}]) = x$.

Lemma 3.21. *Let $\mathfrak{A}$ be a nuclear, separable C^* -algebra satisfying the UCT such that $K_*(\mathfrak{A})$ is torsion free. Let $\mathcal{P}$ be a finite subset of $\mathbb{P}(\mathfrak{A})$. Then there exist a finite subset $\mathcal{Q}$ of $\mathbb{P}(\mathfrak{A})$, $\delta > 0$ and a finite subset of $\mathcal{F}$ such that if $\mathfrak{B}$ is a nuclear, simple C^* -algebra satisfying the UCT and $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive, completely positive, linear map with ψ a $\mathcal{F}$ - δ -multiplicative map, then $K_*(\psi)|_{\mathcal{Q}} = 0$ implies that $\underline{K}(\psi)|_{\mathcal{P}} = 0$.*

Proof. Let $\alpha : K_*(\mathfrak{A}) \rightarrow \underline{K}(\mathfrak{A})$ be the natural homomorphism given in [5]. Since $K_*(\mathfrak{A})$ is torsion free, by [5], there exists a finite subset $\mathcal{P}_0$ of $\mathbb{P}_0(\mathfrak{A})$ such that for each $p \in \mathcal{P}$ there exists $q \in \mathcal{P}_0$ such that $\alpha([q]) = [p]$ in $\underline{K}(\mathfrak{A})$.

Set $\mathcal{Q} = \mathcal{P}_0 \cup \mathcal{P}$. Choose $\delta > 0$ and a finite subset $\mathcal{F}$ of $\mathfrak{A}$ such that if $\mathfrak{B}$ is a nuclear, simple C^* -algebra satisfying the UCT and $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive, completely positive, linear map with ψ being a $\mathcal{F}$ - δ -multiplicative map, then $\underline{K}(\psi)|_{\mathcal{P}}$ and $\underline{K}(\psi)|_{\mathcal{Q}}$ are well defined.

Suppose $\mathfrak{B}$ is a nuclear, simple C^* -algebra satisfying the UCT and $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a contractive, completely positive, linear map with ψ being a $\mathcal{F}$ - δ -multiplicative map. Define $\iota_1 : \mathfrak{A} \rightarrow \mathfrak{A} \otimes \mathcal{O}_{\infty}$ by $\iota_1(a) = a \otimes 1_{\mathcal{O}_{\infty}}$ and $\iota_2 : \mathfrak{B} \rightarrow \mathfrak{B} \otimes \mathcal{O}_{\infty}$ by $\iota_2(b) = b \otimes 1_{\mathcal{O}_{\infty}}$. By the Künneth Formula [29], $\underline{K}(\iota_i)$ is an isomorphism. Note that $\iota_2 \circ \psi = (\psi \otimes \text{id}_{\mathcal{O}_{\infty}}) \circ \iota_1$ is a contractive, completely positive, linear map that is $\mathcal{F}$ - δ -multiplicative.

By Theorem 6.7 of [11], there exists a homomorphism $\varphi : \mathfrak{A} \otimes \mathcal{O}_{\infty} \rightarrow \mathfrak{B} \otimes \mathcal{O}_{\infty}$ such that

$$\underline{K}(\varphi) \circ \underline{K}(\iota_1)|_{\mathcal{Q}} = \underline{K}((\psi \otimes \text{id}_{\mathcal{O}_{\infty}}) \circ \iota_1)|_{\mathcal{Q}} = \underline{K}(\iota_2) \circ \underline{K}(\psi)|_{\mathcal{Q}}.$$

Let $\beta_1 : K_*(\mathfrak{A} \otimes \mathcal{O}_{\infty}) \rightarrow \underline{K}(\mathfrak{A} \otimes \mathcal{O}_{\infty})$ and $\beta_2 : K_*(\mathfrak{B} \otimes \mathcal{O}_{\infty}) \rightarrow \underline{K}(\mathfrak{B} \otimes \mathcal{O}_{\infty})$ be the natural homomorphisms given in [5]. Suppose $K_*(\psi)|_{\mathcal{Q}} = 0$. Then $K_*(\varphi) \circ K_*(\iota_1)|_{\mathcal{Q}} = 0$. Let $p \in \mathcal{P}$. Then there exists $q \in \mathcal{P}_0$ such that $\alpha([q]) = [p]$ in $\underline{K}(\mathfrak{A})$. Then

$$\begin{aligned} \underline{K}(\iota_2)(\underline{K}(\psi)|_{\mathcal{P}}([p])) &= \underline{K}(\varphi \circ \iota_1)([p]) \\ &= \underline{K}(\varphi \circ \iota_1)(\alpha([q])) \\ &= \beta_2(K_*(\varphi \circ \iota_1)([q])) \\ &= 0. \end{aligned}$$

Since $\underline{K}(\iota_2)$ is an isomorphism, $\underline{K}(\psi)|_{\mathcal{P}}([p]) = 0$. □

Theorem 3.22. *Let $\mathfrak{A}$ be in $\mathcal{A}_{\mathcal{Z}}$. Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}$. Then there exists a finitely generated subgroup $\mathcal{G}$ of $K_0(\mathfrak{A})$ such that the following holds: if $[u] \in H_{[1_{\mathfrak{A}}]}(\mathcal{G}, K_1(\mathfrak{A}))$, then there exists a continuous path of unitaries $w(t)$ in $\mathfrak{A} \otimes M_p \otimes M_q$ such that*

- (1) $w(0) \in \mathfrak{A} \otimes M_p \otimes 1_{M_q}$ and $w(1) \in \mathfrak{A} \otimes 1_{M_p} \otimes M_q$,
- (2) $[w(0)] = [u \otimes 1_{M_p} \otimes 1_{M_q}]$ in $K_1(\mathfrak{A} \otimes M_p \otimes 1_{M_q})$ and $[w(1)] = [u \otimes 1_{M_p} \otimes 1_{M_q}]$ in $K_1(\mathfrak{A} \otimes 1_{M_p} \otimes M_q)$, and

$$(3) \quad \|w(t)(a \otimes 1_{M_p} \otimes 1_{M_q}) - (a \otimes 1_{M_p} \otimes 1_{M_q})w(t)\| < \epsilon \text{ for all } a \in \mathcal{F} \text{ and } t \in [0, 1],$$

where p and q are supernatural numbers of infinite type with $M_p \otimes M_q$ isomorphic $\mathcal{Q}$.

Proof. Let $\{\mathcal{F}_{1,n}\}_{n=1}^\infty$ be an increasing sequence of finite subsets of $\mathfrak{A}$ such that $\bigcup_{n=1}^\infty \mathcal{F}_{1,n}$ is dense in $\mathfrak{A}$, let $\{\mathcal{F}_{2,n}\}_{n=1}^\infty$ be an increasing sequence of finite subsets of M_p such that $\bigcup_{n=1}^\infty \mathcal{F}_{2,n}$ is dense in M_p , and let $\{\mathcal{F}_{3,n}\}_{n=1}^\infty$ be an increasing sequence of finite subsets of M_q such that $\bigcup_{n=1}^\infty \mathcal{F}_{3,n}$ is dense in M_q . Let $\{\mathcal{P}_{1,n}\}_{n=1}^\infty$ be an increasing sequence of finite subsets of $\mathbb{P}(\mathfrak{A})$, $\{\mathcal{P}_{2,n}\}_{n=1}^\infty$ be an increasing sequence of finite subsets of $\mathbb{P}(M_p)$, and $\{\mathcal{P}_{3,n}\}_{n=1}^\infty$ be an increasing sequence of finite subsets of $\mathbb{P}(M_q)$ such that

$$\bigcup_{n=1}^\infty \mathcal{P}_{1,n} = \mathbb{P}(\mathfrak{A}), \quad \bigcup_{n=1}^\infty \mathcal{P}_{2,n} = \mathbb{P}(M_p), \quad \text{and} \quad \bigcup_{n=1}^\infty \mathcal{P}_{3,n} = \mathbb{P}(M_q).$$

For each n , let $\mathcal{G}_{1,n}$ be the finitely generated subgroup of $K_0(\mathfrak{A} \otimes M_p)$ provided by Corollary 3.17 that corresponds to $\mathcal{F}_{1,n} \otimes \mathcal{F}_{2,n}$, $\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n}$, and $\frac{1}{2^n}$, and let $\mathcal{G}_{2,n}$ be the finitely generated subgroup of $K_0(\mathfrak{A} \otimes M_q)$ provided by Corollary 3.17 that corresponds to $\mathcal{F}_{1,n} \otimes \mathcal{F}_{3,n}$, $\mathcal{P}_{1,n} \otimes \mathcal{P}_{3,n}$, and $\frac{1}{2^n}$. Note that we may assume that $\mathcal{G}_{1,n} \subseteq \mathcal{G}_{1,n+1}$, $\mathcal{G}_{2,n} \subseteq \mathcal{G}_{2,n+1}$, $\bigcup_{n=1}^\infty \mathcal{G}_{1,n} = K_0(\mathfrak{A} \otimes M_p)$, and $\bigcup_{n=1}^\infty \mathcal{G}_{2,n} = K_0(\mathfrak{A} \otimes M_q)$.

Let $\{\mathcal{G}_n\}_{n=1}^\infty$ be an increasing sequence of finitely generated subgroups of $K_0(\mathfrak{A})$ containing $[1_{\mathfrak{A}}]$ such that if $\sum_{i=1}^n x_i \otimes y_i$ is a generator of $\mathcal{G}_{1,n}$, then $x_i \in \mathcal{G}_n$, and if $\sum_{i=1}^n x_i \otimes y_i$ is a generator of $\mathcal{G}_{2,n}$, then $x_i \in \mathcal{G}_n$. (Note that we are identifying $K_0(\mathfrak{A} \otimes M_p)$ with $K_0(\mathfrak{A}) \otimes K_0(M_p)$ and $K_0(\mathfrak{A} \otimes M_q)$ with $K_0(\mathfrak{A}) \otimes K_0(M_q)$.)

Since $K_0(\mathfrak{A} \otimes M_p \otimes M_q)$ and $K_1(\mathfrak{A} \otimes M_p \otimes M_q)$ are torsion free groups, by Theorem 8.4 of [14] and Lemma 3.21, there exist $\delta > 0$, a finite subset $\mathcal{G}$ of $\mathfrak{A} \otimes M_p \otimes M_q$, and a finite subset $\mathcal{H}$ of $\mathbb{P}(\mathfrak{A} \otimes M_p \otimes M_q)$ such that if v is a unitary in $\mathfrak{A} \otimes M_p \otimes M_q$ with

$$\|xv - vx\| < \delta \text{ for all } x \in \mathcal{G}$$

and

$$\text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, v)|_{\mathcal{H}} = 0 \text{ and } \text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, v)|_{\mathcal{H}} = 0,$$

then there exists a continuous path of unitaries $v(t)$ in $\mathfrak{A} \otimes M_p \otimes M_q$ such that $v(0) = 1_{\mathfrak{A} \otimes M_p \otimes M_q}$, $v(1) = v$, and

$$\|xv(t) - v(t)x\| < \frac{\epsilon}{2}$$

for all $x \in \mathcal{F} \otimes 1_{M_p} \otimes 1_{M_q}$ and $t \in [0, 1]$.

Choose n large enough such that $\frac{1}{2^{n-1}} < \frac{\epsilon}{2}$, and if v is a unitary in $\mathfrak{A} \otimes M_p \otimes M_q$ with

$$\|vx - xv\| < \frac{1}{2^{n-1}}$$

for all $x \in \mathcal{F}_{1,n} \otimes \mathcal{F}_{2,n} \otimes \mathcal{F}_{3,n}$, then

$$\|vx - xv\| < \frac{\delta}{2}$$

for all $x \in \mathcal{G}$, and if

$$\text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, v)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}} = 0$$

and

$$\text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, v)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}} = 0,$$

then

$$\text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, v)|_{\mathcal{H}} = 0 \quad \text{and} \quad \text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, v)|_{\mathcal{H}} = 0.$$

Note that the last part of the statement can be done since we are identifying $K_i(\mathfrak{A} \otimes M_p \otimes M_q)$ with $K_i(\mathfrak{A}) \otimes K_0(M_p) \otimes K_0(M_q)$.

Let $[u] \in H_{[1, \mathfrak{A}]}(\mathcal{G}_n, K_1(\mathfrak{A}))$. Then there exists $\gamma : \mathcal{G}_n \rightarrow K_1(\mathfrak{A})$ such that $\gamma([1_{\mathfrak{A}}]) = [u]$. By Corollary 3.17, there exists a unitary $v_0 \in \mathfrak{A} \otimes M_p$ such that

$$\|v_0(x_1 \otimes x_2) - (x_1 \otimes x_2)v_0\| < \frac{1}{2^n}$$

for all $x_1 \in \mathcal{F}_{1,n}$ and $x_2 \in \mathcal{F}_{2,n}$,

$$\text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_p}, v_0)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n}} = 0,$$

and

$$\text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_p}, v_0)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n}} = \gamma \otimes K_0(\text{id}_{M_p})|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n}}.$$

By Corollary 3.17, there exists a unitary $v_1 \in \mathfrak{A} \otimes M_q$ such that

$$\|v_1(x_1 \otimes x_3) - (x_1 \otimes x_3)v_1\| < \frac{1}{2^n}$$

for all $x_1 \in \mathcal{F}_{1,n}$ and $x_3 \in \mathcal{F}_{3,n}$,

$$\text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_q}, v_1)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{3,n}} = 0,$$

and

$$\text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_q}, v_1)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{3,n}} = \gamma \otimes K_0(\text{id}_{M_q})|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{3,n}}.$$

Set $w_0 = v_0 \otimes 1_{M_q} \in \mathfrak{A} \otimes M_p \otimes 1_{M_q}$ and set $w_1 = \text{id}^{[1,3,2]}(v_1 \otimes 1_{M_p}) \in \mathfrak{A} \otimes 1_{M_p} \otimes M_q$. Then

$$\|xw_i - w_ix\| < \frac{1}{2^n}$$

for all $x \in \mathcal{F}_{1,n} \otimes \mathcal{F}_{2,n} \otimes \mathcal{F}_{3,n}$, and for each $i = 0, 1$,

$$\text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_i)|_{\mathcal{P}_{1,n} \otimes \mathcal{S}_{2,n} \otimes \mathcal{S}_{3,n}} = 0$$

and

$$\text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_i)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}} = \gamma \otimes K_0(\text{id}_{M_p}) \otimes K_0(\text{id}_{M_q})|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}}.$$

Note that

$$\|xw_0^*w_1 - w_0^*w_1x\| < \frac{1}{2^{n-1}}$$

for all $x \in \mathcal{F}_{1,n} \otimes \mathcal{F}_{2,n} \otimes \mathcal{F}_{3,n}$. By Lemma 3.11,

$$\begin{aligned} & \text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_0^*w_1)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}} \\ &= -\text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_0)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}} \\ & \quad + \text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_1)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} & \text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_0^*w_1)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}} \\ &= -\text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_0)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}} \\ & \quad + \text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_1)|_{\mathcal{P}_{1,n} \otimes \mathcal{P}_{2,n} \otimes \mathcal{P}_{3,n}} \\ &= 0. \end{aligned}$$

Therefore,

$$\|xw_0^*w_1 - w_0^*w_1x\| < \frac{\delta}{2}$$

for all $x \in \mathcal{G}$,

$$\text{bott}_0(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_0^*w_1)|_{\mathcal{H}} = 0,$$

and

$$\text{bott}_1(\text{id}_{\mathfrak{A} \otimes M_p \otimes M_q}, w_0^*w_1)|_{\mathcal{H}} = 0.$$

Hence, there exists a continuous path of unitaries $v(t)$ in $\mathfrak{A} \otimes M_p \otimes M_q$ such that

$$\|v(t)(a \otimes 1_{M_p} \otimes 1_{M_q}) - (a \otimes 1_{M_p} \otimes 1_{M_q})v(t)\| < \frac{\epsilon}{2}$$

for all $a \in \mathcal{F}$, and for all $t \in [0, 1]$, $v(0) = 1_{\mathfrak{A}} \otimes 1_{M_p} \otimes 1_{M_q}$ and $v(1) = w_0^*w_1$.

Set $w(t) = w_0v(t)$. Then $w(t)$ is a continuous path of unitaries in $\mathfrak{A} \otimes M_p \otimes M_q$ such that $w(0) = w_0$, $w(1) = w_1$, and

$$\|w(t)(a \otimes 1_{M_p} \otimes 1_{M_q}) - (a \otimes 1_{M_p} \otimes 1_{M_q})w(t)\| < \epsilon$$

for all $a \in \mathcal{F}$.

Note that $[v_0] = \gamma \otimes K_0(\text{id}_{M_p})([1_{\mathfrak{A}} \otimes 1_{M_p}]) = \gamma([1_{\mathfrak{A}}]) \otimes [1_{M_p}] = [u \otimes 1_{M_p}]$ in $K_1(\mathfrak{A} \otimes M_p)$ and $[v_1] = \gamma \otimes K_1(\text{id}_{M_q})([1_{\mathfrak{A}} \otimes 1_{M_q}]) = \gamma([1_{\mathfrak{A}}]) \otimes [1_{M_q}] = [u \otimes 1_{M_q}]$ in $K_0(\mathfrak{A} \otimes M_q)$. Hence, $[w(0)] = [u \otimes 1_{M_p} \otimes 1_{M_q}]$ in $K_1(\mathfrak{A} \otimes M_p \otimes 1_{M_q})$ and $[w(1)] = [u \otimes 1_{M_p} \otimes 1_{M_q}]$ in $K_1(\mathfrak{A} \otimes 1_{M_p} \otimes M_q)$. $\square$

4. THE AUTOMORPHISM GROUP OF A $\mathcal{Z}$ -STABLE C^* -ALGEBRA

Let $\mathfrak{A}$ be a separable, unital C^* -algebra and let $\{G_n\}_{n=1}^\infty$ be an increasing sequence of finitely generated subgroups of $K_0(\mathfrak{A})$ containing $[1_{\mathfrak{A}}]$. Note that there exists a sequence of surjective homomorphisms

$$\frac{K_1(\mathfrak{A})}{H_{[1_{\mathfrak{A}}]}(G_1, K_1(\mathfrak{A}))} \leftarrow \frac{K_1(\mathfrak{A})}{H_{[1_{\mathfrak{A}}]}(G_2, K_1(\mathfrak{A}))} \leftarrow \cdots$$

which gives an inverse limit group $\lim_{\leftarrow} \frac{K_1(\mathfrak{A})}{H_{[1_{\mathfrak{A}}]}(G_n, K_1(\mathfrak{A}))}$. For each $n \in \mathbb{N}$, equip

$\frac{K_1(\mathfrak{A})}{H_{[1_{\mathfrak{A}}]}(G_n, K_1(\mathfrak{A}))}$ with the discrete topology and give the inverse limit the inverse limit topology.

For each $g \in K_1(\mathfrak{A})$, denote the image of g in the inverse limit by $\check{g}$. For each $g \in U(\mathfrak{A})$, denote the image of g in $\frac{\overline{\text{Inn}}(\mathfrak{A})}{\overline{\text{Inn}}_0(\mathfrak{A})}$ by $\hat{g} = \text{Ad}(g) + \overline{\text{Inn}}_0(\mathfrak{A})$. The authors in [21] showed that the map $\mu : \hat{g} \mapsto \check{g}$ extends uniquely to a continuous group homomorphism from $\frac{\overline{\text{Inn}}(\mathfrak{A})}{\overline{\text{Inn}}_0(\mathfrak{A})}$ to $\lim_{\leftarrow} \frac{K_1(\mathfrak{A})}{H_{[1_{\mathfrak{A}}]}(G_n, K_1(\mathfrak{A}))}$. With an abuse of notation, we denote this unique extension by μ . Moreover, by Theorem 2 of [21], if $\mathfrak{A}$ satisfies Property (C) (see Definition 4.1), then μ is surjective, and if, in addition, the natural map from $U(\mathfrak{A})/U_0(\mathfrak{A})$ to $K_1(\mathfrak{A})$ is an isomorphism, then μ is an isomorphism of topological abelian groups. Consequently, $\frac{\overline{\text{Inn}}(\mathfrak{A})}{\overline{\text{Inn}}_0(\mathfrak{A})}$ is a totally disconnected group. Moreover, the authors showed that if $\mathfrak{A}$ is a unital C^* -algebra satisfying the UCT and if $\mathfrak{A}$ is either a Kirchberg algebra or simple tracially AI algebra, then $\mathfrak{A}$ satisfies Property (C). We use the results from the previous section to show that every element in $\mathcal{A}_{\mathcal{Z}}$ satisfies Property (C).

4.1. $\mathcal{Z}$ -stable C^* -algebras and Property (C).

Definition 4.1. Let $\mathfrak{A}$ be a unital C^* -algebra. $\mathfrak{A}$ is said to satisfy **Property (C)** if for each $\epsilon > 0$ and for each finite subset $\mathcal{F}$ of $\mathfrak{A}$, there exists a finitely generated subgroup $\mathcal{G}$ of $K_0(\mathfrak{A})$ containing $[1_{\mathfrak{A}}]$ such that the following holds: for every $u \in U(\mathfrak{A})$ with $[u] \in H_{[1_{\mathfrak{A}}]}(\mathcal{G}, K_1(\mathfrak{A}))$, there exists $w \in U(\mathfrak{A})$ such that

$$\|wa - aw\| < \epsilon$$

for all $a \in \mathcal{F}$ and $[w] = [u]$ in $U(\mathfrak{A})/U(\mathfrak{A})_0$.

The following is an equivalent definition of Property (C) which will be easier to check in our context.

Lemma 4.2. *Let $\mathfrak{A}$ be a unital C^* -algebra. Then the following are equivalent:*

- (1) $\mathfrak{A}$ satisfies Property (C).
- (2) For each $\epsilon > 0$ and for each finite subset $\mathcal{F}$ of $\mathfrak{A}$, there exists a finitely generated subgroup $\mathcal{G}$ of $K_0(\mathfrak{A})$ containing $[1_{\mathfrak{A}}]$ such that the following holds: for every $u \in U(\mathfrak{A})$ with $[u] \in H_{[1_{\mathfrak{A}}]}(\mathcal{G}, K_1(\mathfrak{A}))$, there exists $w \in U(\mathfrak{A})_0$ such that

$$\|uau^* - waw^*\| < \epsilon$$

for all $a \in \mathcal{F}$.

Proof. (1) $\implies$ (2) : Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}$. Let $\mathcal{G}$ be the finitely generated subgroup of $K_0(\mathfrak{A})$ given in Property (C) that corresponds to ϵ and $\mathcal{F}$. Let $u \in U(\mathfrak{A})$ with $[u] \in H_{[1_{\mathfrak{A}}]}(\mathcal{G}, K_1(\mathfrak{A}))$. Then, by our assumption, there exists $v \in U(\mathfrak{A})$ such that

$$\|va - av\| < \epsilon$$

for all $a \in \mathcal{F}$ and $uv^* \in U(\mathfrak{A})_0$. Set $w = uv^*$. Then $w \in U(\mathfrak{A})_0$ and

$$\|uau^* - waw^*\| = \|a - v^*av\| = \|va - av\| < \epsilon$$

for all $a \in \mathcal{F}$.

(2) $\implies$ (1) : Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}$. Let $\mathcal{G}$ be the finitely generated subgroup of $K_0(\mathfrak{A})$ given in (2) that corresponds to ϵ and $\mathcal{F}$. Suppose $u \in U(\mathfrak{A})$ such that $[u] \in H_{[1_{\mathfrak{A}}]}(\mathcal{G}, K_1(\mathfrak{A}))$. Then, by our assumption, there exists $v \in U(\mathfrak{A})_0$ such that

$$\|uau^* - vav^*\| < \epsilon$$

for all $a \in \mathcal{F}$. Set $w = v^*u$. Since $[v] \in U(\mathfrak{A})_0$, $[w] = [u]$ in $U(\mathfrak{A})/U(\mathfrak{A})_0$. Moreover,

$$\|wa - aw\| = \|v^*ua - av^*u\| = \|uau^* - vav^*\| < \epsilon$$

for all $a \in \mathcal{F}$. □

Lemma 4.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^* -algebras, with $\mathfrak{B}$ unital and $\mathcal{Z}$ -stable, and let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers which are relatively prime. Suppose $\varphi_1, \varphi_2 : \mathfrak{A} \rightarrow \mathfrak{B}$ are $*$ -homomorphisms such that*

$$\varphi_1 \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}, \varphi_2 \otimes 1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}} : \mathfrak{A} \rightarrow \mathfrak{B} \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$$

are approximately unitarily equivalent via unitaries $\{w_n\}_{n \in \mathbb{N}}$ with $[w_n] = 0$ in $K_1(\mathfrak{B} \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$. Then φ_1, φ_2 are approximately unitarily equivalent via unitaries $\{v_n\}_{n \in \mathbb{N}}$ in $\mathfrak{B}$ such that $[v_n] = 0$ in $K_1(\mathfrak{B})$.

Consequently, for every $\epsilon > 0$ and for every finite subset $\mathcal{F}$ of $\mathfrak{A}$, there exists $\delta > 0$ such that if w is a unitary in $\mathfrak{B} \otimes \mathcal{Z}_{\mathfrak{p},q}$ such that

$$\|(\varphi_1 \otimes 1_{\mathcal{Z}_{\mathfrak{p},q}})(a) - w(\varphi_2 \otimes 1_{\mathcal{Z}_{\mathfrak{p},q}})(a)w^*\| < \delta$$

for all $a \in \mathcal{F}$ and $[w] = 0$ in $K_1(\mathfrak{B} \otimes \mathcal{Z}_{\mathfrak{p},q})$, then there exists a unitary v in $\mathfrak{B}$ with $[v] = 0$ in $K_1(\mathfrak{B})$ such that

$$\|\varphi_1(a) - v\varphi_2(a)v^*\| < \epsilon$$

for all $a \in \mathcal{F}$.

Proof. Since $\mathfrak{B} \cong \mathfrak{B} \otimes \mathcal{Z} \cong \mathfrak{B} \otimes \mathcal{Z} \otimes \mathcal{Z}$, by Lemma 2.5 there exist a $*$ -isomorphism $\nu_{\mathfrak{B}} : \mathfrak{B} \rightarrow \mathfrak{B} \otimes \mathcal{Z}$ and a sequence of unitaries $\{u_n\}_{n \in \mathbb{N}}$ in $\mathfrak{B} \otimes \mathcal{Z}$ such that

$$\lim_{n \rightarrow \infty} \|\nu_{\mathfrak{B}}(b) - u_n(b \otimes 1_{\mathcal{Z}})u_n^*\| = 0$$

for all $b \in \mathfrak{B}$. Let $\bar{\sigma}_{\mathfrak{p},q} : \mathcal{Z}_{\mathfrak{p},q} \rightarrow \mathcal{Z}$ be the unital embedding in Proposition 3.4 of [33]. Note that

$$\lim_{n \rightarrow \infty} \|\varphi_i(a) - \nu_{\mathfrak{B}}^{-1}(u_n)\nu_{\mathfrak{B}}^{-1}(\varphi_i(a) \otimes 1_{\mathcal{Z}})\nu_{\mathfrak{B}}^{-1}(u_n^*)\| = 0$$

and

$$\begin{aligned} \varphi_1(a) \otimes 1_{\mathcal{Z}} &= \varphi_1(a) \otimes \bar{\sigma}_{\mathfrak{p},q}(1_{\mathcal{Z}_{\mathfrak{p},q}}) \\ &= (\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(\varphi_1(a) \otimes 1_{\mathcal{Z}_{\mathfrak{p},q}}) \\ \varphi_2(a) \otimes 1_{\mathcal{Z}} &= \varphi_2(a) \otimes \bar{\sigma}_{\mathfrak{p},q}(1_{\mathcal{Z}_{\mathfrak{p},q}}) \\ &= (\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(\varphi_2(a) \otimes 1_{\mathcal{Z}_{\mathfrak{p},q}}) \end{aligned}$$

for all $a \in \mathfrak{A}$. Therefore,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \|\varphi_1(a) \otimes 1_{\mathcal{Z}} - (\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(w_n)(\varphi_2(a) \otimes 1_{\mathcal{Z}})(\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(w_n^*)\| \\ &= \lim_{n \rightarrow \infty} \|(\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(\varphi_1(a) \otimes 1_{\mathcal{Z}_{\mathfrak{p},q}}) - (\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(w_n(\varphi_2(a) \otimes 1_{\mathcal{Z}_{\mathfrak{p},q}})w_n^*)\| \\ &= 0 \end{aligned}$$

for all $a \in \mathfrak{A}$. Set $v_n = \nu_{\mathfrak{B}}^{-1}(u_n [\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q}](w_n)u_n^*)$. Let $a \in \mathfrak{A}$. Then

$$\begin{aligned} &\|\varphi_1(a) - v_n\varphi_2(a)v_n^*\| \\ &\leq \|\varphi_1(a) - \nu_{\mathfrak{B}}^{-1}(u_n)\nu_{\mathfrak{B}}^{-1}(\varphi_1(a) \otimes 1_{\mathcal{Z}})\nu_{\mathfrak{B}}^{-1}(u_n^*)\| \\ &\quad + \|\nu_{\mathfrak{B}}^{-1}(u_n)\nu_{\mathfrak{B}}^{-1}(\varphi_1(a) \otimes 1_{\mathcal{Z}})\nu_{\mathfrak{B}}^{-1}(u_n^*) \\ &\quad - \nu_{\mathfrak{B}}^{-1}(u_n(\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(w_n))\nu_{\mathfrak{B}}^{-1}(\varphi_2(a) \otimes 1_{\mathcal{Z}})\nu_{\mathfrak{B}}^{-1}((\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(w_n^*)u_n^*)\| \\ &\quad + \|\nu_{\mathfrak{B}}^{-1}(u_n(\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(w_n))\nu_{\mathfrak{B}}^{-1}(\varphi_2(a) \otimes 1_{\mathcal{Z}})\nu_{\mathfrak{B}}^{-1}((\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(w_n^*)u_n^*) - v_n\varphi_2(a)v_n^*\| \\ &\leq \|\varphi_1(a) - \nu_{\mathfrak{B}}^{-1}(u_n)\nu_{\mathfrak{B}}^{-1}(\varphi_1(a) \otimes 1_{\mathcal{Z}})\nu_{\mathfrak{B}}^{-1}(u_n^*)\| \\ &\quad + \|\varphi_1(a) \otimes 1_{\mathcal{Z}} - (\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(w_n)(\varphi_2(a) \otimes 1_{\mathcal{Z}})(\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q})(w_n^*)\| \\ &\quad + \|\varphi_2(a) - \nu_{\mathfrak{B}}^{-1}(u_n)\nu_{\mathfrak{B}}^{-1}(\varphi_2(a) \otimes 1_{\mathcal{Z}})\nu_{\mathfrak{B}}^{-1}(u_n^*)\|. \end{aligned}$$

Therefore, for each $a \in \mathfrak{A}$,

$$\lim_{n \rightarrow \infty} \|\varphi_1(a) - v_n\varphi_2(a)v_n^*\| = 0.$$

Since $[w_n] = 0$ in $K_1(\mathfrak{B} \otimes \mathcal{Z}_{\mathfrak{p},q})$,

$$\begin{aligned} [v_n] &= (K_1(\nu_{\mathfrak{B}}^{-1}) \circ K_1(\text{Ad}(u_n)) \circ K_1(\text{id}_{\mathfrak{B}} \otimes \bar{\sigma}_{\mathfrak{p},q}))([w_n]) \\ &= 0 \end{aligned}$$

in $K_1(\mathfrak{B})$. $\square$

Lemma 4.4. *Let $\mathfrak{A}$ be in $\mathcal{A}_{\mathcal{Z}}$. The canonical homomorphism from $U(\mathfrak{A})/U(\mathfrak{A})_0$ to $K_1(\mathfrak{A})$ is then an isomorphism.*

Proof. Let $\mathfrak{p}$ be a supernatural number of infinite type. Then $\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}}$ is a tracially AI algebra. Thus $\mathfrak{A}$ is a finite $\mathcal{Z}$ -stable C^* -algebra. Hence, by [27], $\mathfrak{A}$ has stable rank one. Since $\mathfrak{A}$ is a simple unital C^* -algebra with stable rank one, by Theorem 10.12 of [24] and Corollary 7.14 of [1], the canonical homomorphism from $U(\mathfrak{A})/U(\mathfrak{A})_0$ to $K_1(\mathfrak{A})$ is an isomorphism. $\square$

Theorem 4.5. *Let $\mathfrak{A}$ be in $\mathcal{A}_{\mathcal{Z}}$. Then $\mathfrak{A}$ satisfies Property (C).*

Proof. We prove that $\mathcal{A}$ satisfies Property (2) of Lemma 4.2. Let $\epsilon > 0$ and let $\mathcal{F}$ be a finite subset of $\mathfrak{A}$. Let $\delta > 0$ be the quantity given in Lemma 4.3 corresponding to $\mathcal{F}$ and ϵ . Let $\mathcal{G}$ be the finitely generated subgroup of $K_0(\mathfrak{A})$ given in Theorem 3.22 corresponding to $\mathcal{F}$ and $\frac{\delta}{2}$.

Let $u \in U(\mathfrak{A})$ with $[u] \in H_{[1, \mathfrak{A}]}(\mathcal{G}, K_1(\mathfrak{A}))$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers of infinite type such that $\mathbf{M}_{\mathfrak{p}} \otimes \mathbf{M}_{\mathfrak{q}}$ is isomorphic to $\mathcal{Q}$. By Theorem 3.22, there exists a continuous path of unitaries $w(t)$ in $\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}} \otimes \mathbf{M}_{\mathfrak{q}}$ such that

- (1) $w(0) \in \mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}}$ and $w(1) \in \mathfrak{A} \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes \mathbf{M}_{\mathfrak{q}}$,
- (2) $[w(0)] = [u \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}}]$ in $K_1(\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}})$ and $[w(1)] = [u \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}}]$ in $K_1(\mathfrak{A} \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes \mathbf{M}_{\mathfrak{q}})$, and
- (3) $\|w(t)(a \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}}) - (a \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}})w(t)\| < \frac{\delta}{2}$ for all $a \in \mathcal{F}$ and $t \in [0, 1]$.

Since

$$\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}} = \{f \in C([0, 1], \mathbf{M}_{\mathfrak{p}} \otimes \mathbf{M}_{\mathfrak{q}}) : f(0) \in \mathbf{M}_{\mathfrak{p}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}} \text{ and } f(1) \in 1_{\mathbf{M}_{\mathfrak{p}}} \otimes \mathbf{M}_{\mathfrak{q}}\},$$

$\mathfrak{A} \otimes \mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}$ can be identified with the C^* -algebra

$$\{f \in C([0, 1], \mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}} \otimes \mathbf{M}_{\mathfrak{q}}) : f(0) \in \mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}} \text{ and } f(1) \in \mathfrak{A} \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes \mathbf{M}_{\mathfrak{q}}\}.$$

With this identification, w is an element of $U(\mathfrak{A} \otimes \mathcal{Z}_{\mathfrak{p}, \mathfrak{q}})$ such that

$$\|w(a \otimes 1_{\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}}) - (a \otimes 1_{\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}})w\| < \delta$$

for all $a \in \mathcal{F}$. Since $[w(0)] = [u \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}}]$ in $K_1(\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}})$ and $[w(1)] = [u \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}}]$ in $K_1(\mathfrak{A} \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes \mathbf{M}_{\mathfrak{q}})$, we have that

$$[(u \otimes 1_{\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}})w^*](0) = 0$$

in $K_1(\mathfrak{A} \otimes \mathbf{M}_{\mathfrak{p}} \otimes 1_{\mathbf{M}_{\mathfrak{q}}})$ and

$$[(u \otimes 1_{\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}})w^*](1) = 0$$

in $K_1(\mathfrak{A} \otimes 1_{\mathbf{M}_{\mathfrak{p}}} \otimes \mathbf{M}_{\mathfrak{q}})$. By Proposition 5.2 of [33], $[(u \otimes 1_{\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}})w^*] = 0$ in $K_1(\mathfrak{A} \otimes \mathcal{Z}_{\mathfrak{p}, \mathfrak{q}})$. Let $a \in \mathcal{F}$. Then

$$\|(\text{Ad}(u) \otimes 1_{\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}})(a) - \text{Ad}((u \otimes 1_{\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}})w^*)(a \otimes 1_{\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}})\| < \delta.$$

Hence, by Lemma 4.3, there exists a unitary $v \in U(\mathfrak{A})$ such that $[v] = 0$ in $K_1(\mathfrak{A})$ and

$$\|uau^* - vav^*\| < \epsilon$$

for all $a \in \mathcal{F}$. By Lemma 4.4, $v \in U(\mathfrak{A})_0$. We have just shown that (2) of Lemma 4.2 holds. Therefore, $\mathfrak{A}$ satisfies Property (C). $\square$

4.2. The structure of the automorphism group of a C^* -algebra in $\mathcal{A}_{\mathcal{Z}}$. Recall from [5] that there is a natural order structure on $\underline{K}(\mathfrak{A})$. Let $\text{Aut}(\underline{K}(\mathfrak{A}))$ be the topological group of all ordered group isomorphisms of $\underline{K}(\mathfrak{A})$, which preserve the grading and the Bockstein operations in [5]. For a unital C^* -algebra $\mathfrak{A}$, let $\text{Aut}(\underline{K}(\mathfrak{A}))_1$ be the subgroup of $\text{Aut}(\underline{K}(\mathfrak{A}))$ which sends $[1_{\mathfrak{A}}]$ to $[1_{\mathfrak{A}}]$. Set

$$J(\mathfrak{A}) = (\underline{K}(\mathfrak{A}), T(\mathfrak{A}), U(\mathfrak{A})/CU(\mathfrak{A})).$$

Then an element of $\text{Aut}(J(\mathfrak{A}))_1$ is an ordered tuple $(\alpha, \lambda_T, \lambda_U)$, where α is an element of $\text{Aut}(\underline{K}(\mathfrak{A}))_1$, $\lambda_T : T(\mathfrak{A}) \rightarrow T(\mathfrak{A})$ is an affine homeomorphism, and $\lambda_U : U(\mathfrak{A})/CU(\mathfrak{A}) \rightarrow U(\mathfrak{A})/CU(\mathfrak{A})$ is a homeomorphism such that α , λ_T , and λ_U are compatible in the sense of Section 2, p. 5 of [19].

Theorem 4.6. *Let $\mathfrak{A}$ be a C^* -algebra in $\mathcal{A}_{\mathcal{Z}}$ satisfying the UCT. Then*

- (a) $\overline{\text{Inn}}_0(\mathfrak{A})$ is a simple topological group,
- (b) $\frac{\overline{\text{Inn}}(\mathfrak{A})}{\overline{\text{Inn}}_0(\mathfrak{A})}$ is totally disconnected, and
- (c) the sequence

$$0 \rightarrow \overline{\text{Inn}}(\mathfrak{A}) \rightarrow \text{Aut}(\mathfrak{A}) \rightarrow \text{Aut}(J(\mathfrak{A}))_1$$

is exact.

In addition, if $K_1(\mathfrak{A})$ is torsion free, then the sequence in (c) becomes a short exact sequence

$$0 \rightarrow \overline{\text{Inn}}(\mathfrak{A}) \rightarrow \text{Aut}(\mathfrak{A}) \rightarrow \text{Aut}(J(\mathfrak{A}))_1 \rightarrow 0.$$

Proof. (a) follows from Theorem 2.20. (b) follows from Theorem 2 of [21] and Theorem 4.5 and Lemma 4.4. (c) follows from Theorem 4.8 of [19].

Suppose $K_1(\mathfrak{A})$ is torsion free. Let $(\alpha, \lambda_T, \lambda_U)$ be an element of $\text{Aut}(J(\mathfrak{A}))_1$. By Theorem 5.9 of [19], there exist unital homomorphisms $\phi, \psi : \mathfrak{A} \rightarrow \mathfrak{A}$ such that

- (i) $(\alpha, \lambda_T, \lambda_U)$ is induced by ϕ and
- (ii) $(\alpha^{-1}, \lambda_T^{-1}, \lambda_U^{-1})$ is induced by ψ .

Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be an increase sequence of finite subsets of $\mathfrak{A}$ whose union is dense in $\mathfrak{A}$. Using Theorem 4.8 of [19], we get a sequence of unital homomorphisms $\phi_n, \psi_n : \mathfrak{A} \rightarrow \mathfrak{A}$ such that

$$\|(\psi_n \circ \phi_n)(x) - x\| < \frac{1}{2^n} \quad \text{and} \quad \|(\phi_{n+1} \circ \psi_n)(a) - a\| < \frac{1}{2^n}$$

for all $a \in \mathcal{F}_n$, and ϕ_n is unitarily equivalent to ϕ for each $n \in \mathbb{N}$ and ψ_n is unitarily equivalent to ψ for each $n \in \mathbb{N}$. Therefore, we get an isomorphism $\beta : \mathfrak{A} \rightarrow \mathfrak{A}$ such that β and ϕ induce the same element in $\text{Aut}(J(\mathfrak{A}))_1$, i.e., β induces $(\alpha, \lambda_T, \lambda_U)$. $\square$

Remark 4.7. Note that if $\mathfrak{A}$ is a nuclear, separable, unital, tracially AI algebra satisfying the UCT, then by [12], the sequence

$$0 \rightarrow \overline{\text{Inn}}(\mathfrak{A}) \rightarrow \text{Aut}(\mathfrak{A}) \rightarrow \text{Aut}(J(\mathfrak{A}))_1 \rightarrow 0$$

is exact even when $K_1(\mathfrak{A})$ has an element with finite order. Moreover, $\mathfrak{A}$ is an element of $\mathcal{A}_{\mathcal{Z}}$. The authors believe that the above sequence should be exact for an arbitrary $\mathfrak{A}$ in $\mathcal{A}_{\mathcal{Z}}$ satisfying the UCT, but the proof has eluded them.

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