FRAMED MODULI AND GRASSMANNIANS OF SUBMODULES

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ABSTRACT. In this work we study a realization of moduli spaces of framed quiver representations as Grassmannians of submodules devised by Markus Reineke. Obtained is a generalization of this construction to finite dimensional associative algebras and for quivers with oriented cycles over an arbitrary infinite field. As an application we get an explicit realization of fibers for the moduli space bundle over the categorical quotient for the quiver $A_{n-1}^{(1)}$ and the ground fields $\mathbb{C}$ and $\mathbb{R}$.

1. INTRODUCTION

A quiver Q is a diagram of arrows determined by two finite sets Q_0 (the set of “vertices”) and Q_1 (the set of “arrows”) with two maps $h, t : Q_1 \rightarrow Q_0$ which indicate the vertices at the head and tail of each arrow. A representation (W, φ) of Q consists of a collection of finite dimensional $\mathbb{k}$ -vector spaces W_i , for each $i \in Q_0$, together with linear maps $\varphi_a : W_{ta} \rightarrow W_{ha}$, for each $a \in Q_1$. The dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ of such a representation is given by $\alpha_i = \dim_{\mathbb{k}} W_i$. A morphism $f : (W_i, \varphi_a) \rightarrow (U_i, \psi_a)$ of representations consists of linear maps $f_i : W_i \rightarrow U_i$, for each $i \in Q_0$ such that $f_{ha}\varphi_a = \psi_a f_{ta}$, for each $a \in Q_1$. Evidently, it is an isomorphism if and only if each f_i is as well. Thus, isomorphism classes of representations of Q with dimension vector α coincide with orbits of the action of $GL(\alpha) = \prod_{i \in Q_0} GL_i(\mathbb{k})$ on the representation space $\text{Rep}(Q, \alpha)$.

In studying quiver representations standard approaches of invariant theory often fail because the algebra of invariants is poor or even trivial, as in the case of the quiver without oriented cycles, and so the categorical quotient $\text{Rep}(Q, \alpha) // GL(\alpha) := \text{Spec } \mathbb{k}[\text{Rep}(Q, \alpha)]^{GL(\alpha)}$ is a point. Constructions of geometric invariant theory may help to compensate this defect. Indeed, one can consider the trivial linearization twisted by a character χ of $GL(\alpha)$, which restricts our attention to an open subset of $\text{Rep}(Q, \alpha)$, consisting of χ -semistable representations. Within the open set there are more closed orbits, and the corresponding algebraic quotient is more interesting. In his paper [9] A. D. King showed that the notions of semistability and stability, that arise from geometric invariant theory, coincide with more algebraic notions, expressed in the language of abelian categories. Namely, he devised a link between this concept and the so-called θ -stability. All the characters of $GL(\alpha)$ are given by $\chi_\theta(g) = \prod_{i \in Q_0} \det(g_i)^{\theta_i}$ for $\theta \in \mathbb{Z}^{Q_0}$, and a representation W is a χ_θ -(semi)stable point of the space $\text{Rep}(Q, \alpha)$ if and only if it is θ -(semi)stable as an element of the abelian category $\text{Rep}(Q)$ (see [9, Section 2]). King proved the existence of

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coarse moduli space for semistable representations of quivers and guaranteed that for stable representations there is a fine moduli space. This technique allowed a straightforward and convenient adaptation to the study of representations of finite dimensional algebras [9, Section 4]. The approach used by King was generalized and reformulated in a more flexible form by A. N. Rudakov [14, §3].

An alternative approach to the problem considered was introduced by B. Huisgen-Zimmermann [7]. Let A be a finite dimensional associative algebra with unity. Fix a semisimple A -module T with a projective cover P and a positive integer m . Denote by $\mathbf{Grass}_m^T$ the Grassmannian of all $(\dim P - m)$ -dimensional submodules of the radical $\text{rad} P$. There is then a bijection between $\text{Aut}_A P$ -orbits in $\mathbf{Grass}_m^T$ and isomorphism classes of m -dimensional A -modules with top isomorphic to T , sending $\text{Aut}_A P \cdot C$ to the isomorphism class of P/C . Investigated in [7] are such triples (A, T, m) , where $\mathbf{Grass}_m^T$ itself provides a moduli space classifying the d -dimensional A -modules with top T , up to isomorphism. It is also proved there that $\mathbf{Grass}_m^T$ admits an open covering by representation-theoretically defined affine charts. For further generalization and a systematical treatment of the whole hierarchy of moduli-parametrizing Grassmannian varieties, see [8].

Another possible way of applying invariant theory to the study of quiver representations is to consider framed representations, thus achieving better precision at the expense of extending the representation space. They first appeared in [12] as one of the steps in the construction of Nakajima varieties.

Let Q be a quiver and α be a dimension vector. Fix an additional dimension vector ζ and consider the space $\text{Rep}(Q, \alpha, \zeta) := \text{Rep}(Q, \alpha) \oplus \bigoplus_{i \in Q_0} \text{Hom}_{\mathbb{k}}(\mathbb{k}^{\alpha_i}, \mathbb{k}^{\zeta_i})$. Its elements are said to be *framed representations* of Q . Define a $GL(\alpha)$ -action on $\text{Rep}(Q, \alpha, \zeta)$ by $g \cdot (M, (f_i)_{i=1}^n) = (g \cdot M, (f_i g_i^{-1})_{i=1}^n)$. A framed representation (M, f) is called *stable* if there is no nonzero subrepresentation N of M which is contained in $\ker f$. Denote by $\text{Rep}^s(Q, \alpha, \zeta)$ the space of stable framed representations. One can show that the quotient of $\text{Rep}^s(Q, \alpha, \zeta)$ is more efficient in discriminating orbits than the standard categorical quotient. Furthermore, it can be shown that it is in a sense reducible to King's construction and thus enjoys all its properties.

Grassmannians of submodules of injective modules arise independently in the course of this approach. In [13] M. Reineke obtained for acyclic quivers a realization of framed moduli space as a Grassmannian of subrepresentations of an injective representation depending only on dimension vectors α and ζ . He further investigated its cohomology (see also [3]) and applications to quantum groups.

The aim of this work is to explore possible generalizations of Reineke's construction. So far, there are two of them. First, we can adapt the construction for quivers with relations, i.e. at least for finite dimensional algebras. We may also try to eliminate the condition of having no oriented cycles. Both possibilities are discussed below.

Section 2 is devoted to giving some basic results on the connection between finite dimensional algebras and quivers. In Subsection 3.1 we introduce the notion of a framed representation space for finite dimensional algebras.

A straightforward generalization of Reineke's ideas to finite dimensional algebras is presented in Subsection 3.2. We prove that for a finite dimensional algebra A and two dimension vectors α and ζ the quotient space $\text{Rep}^s(A, \alpha, \zeta) // GL(\alpha)$ is isomorphic to the Grassmannian of submodules of a certain injective A -module J .

In Sections 4 to 6 we investigate a possible way to generalize this result to quivers with oriented cycles. It is crucial for Reineke's construction that A is finite dimensional, since otherwise we are no longer able to embed all stable framed representations in a finite dimensional module. In general, we thus come to consider Grassmannians of submodules in infinite dimensional spaces, which makes it impossible to apply the usual techniques. So, having established a bijection between such a Grassmannian and the moduli space we provide the former with a structure of algebraic variety, but this gives us no information about the moduli space itself. Hence we are forced to restrict our attention to fibers of the moduli space bundle over the categorical quotient.

In their work [6] J. Engel and M. Reineke, using Luna's stratification, described fibers of the bundle for $\mathbb{k} = \mathbb{C}$ and for arbitrary quivers as nilpotent parts of the framed moduli space for some new quiver Q' and dimension vectors α' and ζ' (see [6, Theorem 4.1]). Moreover, it follows from their results that fibers over points of a Luna stratum are isomorphic to each other. Theorem 3.5 together with Proposition 3.10 thus imply that each fiber of π_s is isomorphic to the Grassmannian of submodules of a module over some finite dimensional algebra with a certain dimension vector.

We try to make our approach work over arbitrary infinite fields. Without the assumption on the ground field the results of Engel and Reineke are not true in the way in which they are stated. At least for $\mathbb{k} = \mathbb{R}$ we give such an example in Section 5. Moreover, one should be careful when trying to apply constructions of invariant theory. For this reason we prefer to say that we construct a moduli space of framed representations with given values of basic invariant functions.

In Sections 4 and 5 we study the quiver $A_{n-1}^{(1)}$ with cyclic orientation. We obtain an embedding of $\text{Rep}^s(A_{n-1}^{(1)}, \alpha, \zeta)$ into an infinite dimensional representation J . Recall that by Procesi-Razmyslov's Theorem [11, Theorem 1] (see also [4] and [15] for generalizations to arbitrary infinite fields) the algebra of invariant polynomial functions on $\text{Rep}(Q, \alpha, \zeta)$ is generated by traces of oriented cycles in Q . Since in $Q = A_{n-1}^{(1)}$ there is only one cycle τ_i of minimal length starting at each vertex $i \in Q_0$, traces of τ_i^k are polynomials in coefficients of characteristic polynomials χ_i of τ_i . So, we consider a map $\pi : \text{Rep}^s(Q, \alpha, \zeta) \rightarrow (\mathbb{C}[t])^n$, taking a pair (M, f) to a tuple of characteristic polynomials of τ_i , and investigate its fibers. Let $\bar{\chi} = (\chi_1, \dots, \chi_n)$, and also let $h_0(x) = x, h_1(x), \dots, h_N(x)$ be all different irreducible factors of $\chi_1, \dots, \chi_n$ and r_{ij} be the multiplicity of h_j as a factor of h_i (we allow that $r_{i0} = 0$). For each $j = 0, 1, \dots, N$ consider a new dimension vector $\underline{r}_j$ with $(\underline{r}_j)_i = r_{ij}$. It turns out that images of representations from $\pi^{-1}(\bar{\chi})$ lie in the direct sum of finite dimensional submodules $J(h_j, \underline{r}_j) \subseteq J$. Moreover, it may be shown that the moduli space $\pi^{-1}(\bar{\chi})//GL(\alpha)$ is isomorphic to the product of the Grassmannians of their subrepresentations. The details are given in Section 4. In Section 5 we obtain an explicit presentation by equations in projective space of fibers of the moduli space bundle over the categorical quotient for ground field $\mathbb{C}$ or $\mathbb{R}$. Section 6 is devoted to applying this technique to any quiver Q where all oriented cycles pairwise commute. The main result here is that for every $y \in \text{Spec } \mathbb{k}[\text{Rep}(Q, \alpha, \zeta)]^{GL(\alpha)}$ and two dimension vectors α and ζ there is a quiver $Q^\blacklozenge$, a dimension vector $\tilde{\alpha} \in (\mathbb{Z}_{\geq 0})^{Q_0^\blacklozenge}$, and a finite dimensional representation $W^\blacklozenge$ of

$Q^\blacklozenge$ such that the moduli space of representations belonging to the fiber $\pi^{-1}(y)$ is isomorphic to the Grassmannian of $\tilde{\alpha}$ -dimensional submodules of $W^\blacklozenge$.

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2. BACKGROUND INFORMATION

First, recall some general facts concerning finite dimensional algebras and their connection with quivers.

Let A be an associative finite dimensional algebra over an arbitrary field $\mathbb{k}$. An element $e \in A$ is called *idempotent* if $e^2 = e$. We say that two idempotents e_1 and e_2 are *orthogonal* if $e_1e_2 = e_2e_1 = 0$. An idempotent e is *primitive* if it is not a sum of two nonzero orthogonal idempotents. It is well known that for an algebra A with unity there always exists a decomposition $1 = e_1 + \dots + e_n$, where e_i are primitive orthogonal idempotents. Note that this decomposition induces a decomposition $A = Ae_1 \oplus \dots \oplus Ae_n$ of the regular A -module called *Peirce decomposition*.

An algebra is said to be *splitting* if the quotient $A/\mathfrak{r}$, where $\mathfrak{r}$ stands for the radical of A , is isomorphic to a direct product of matrix algebras over the ground field. Note that all algebras over an algebraically closed field are splitting. An algebra is called *basic* if $A/\mathfrak{r}$ is isomorphic to a direct product of division rings. This condition is equivalent to the following: in the decomposition $A = P_1 \oplus \dots \oplus P_k$ of the regular module, where all P_i are indecomposable projective modules, all the summands are pairwise nonisomorphic [5, Theorem 3.5.4].

Fix a decomposition $1 = e_1 + \dots + e_n$ of the unity $1 \in A$, where all e_i are primitive orthogonal idempotents. It is not hard to see that every A -module M as a $\mathbb{k}$ -vector space may be decomposed as $M = \bigoplus_{i=1}^n M_i$, where $M_i = e_iM$. The *dimension vector* of M is the vector $\alpha = \underline{\dim} M$ with $\alpha_i = \dim M_i$. Decomposing the ideals Ae_i of A (which are submodules of the regular module) in this way, we obtain the *two-sided Peirce decomposition* of A : $A = \bigoplus_{i,j} e_j Ae_i$. The components $e_j Ae_i$ are neither left nor right ideals, but they provide a convenient matrix interpretation of the elements of A (see [5, Chapter 1, §7]).

The set of all A -modules with dimension vector α will be denoted by $\text{Rep}(A, \alpha)$. The group $GL(\alpha) = \prod_{i=1}^n GL_{\alpha_i}(\mathbb{k})$ acts on this set, each factor acting by base change in M_i . Namely, for an element $a \in e_j Ae_i$ the corresponding operator $\varphi(a)$ of the representation $A \rightarrow L(M)$ maps M_i to M_j and all the rest of the components to zero. Thus we may define the action $g = (g_t)_{t=1}^n \in \prod_{i=1}^n GL(M_i) = GL(\alpha)$ as follows: $(g \cdot \varphi(a))(m) = (g_j \varphi(a) g_i^{-1})(m)$, for all $g \in G$, $m \in M_i$. Since A admits the two-sided Peirce decomposition $A = \bigoplus_{i,j} e_j Ae_i$, the action is well defined.

Now we recall the connection between algebras and quivers. Let $\mathbb{k}$ be a field. For a quiver Q one defines a *path algebra* $\mathbb{k}Q$. As a linear space it is the span of all paths in Q , including those of length 0, which we identify with vertices of Q . Multiplication in $\mathbb{k}Q$ is defined by

$$\sigma \cdot \tau = \begin{cases} \sigma\tau, & \text{if it is a path in } Q, \\ 0, & \text{otherwise,} \end{cases}$$

for two paths σ and τ in Q .

A *relation* in Q is a $\mathbb{k}$ -linear combination of paths in Q of length not less than 2 with the same source and target. For a set of relations ρ denote by $\langle \rho \rangle$ the ideal of the algebra $\mathbb{k}Q$ generated by these relations.

Proposition 2.1 ([1, Proposition II.2.5]). *For a finite dimensional algebra A the category of finitely generated A -modules is equivalent to the category of finitely generated Γ -modules for some basic algebra Γ .*

Thus the problem of classifying the representations of arbitrary finite dimensional algebras can be in a sense reduced to the case of basic algebras. So we will be considering only basic algebras.

Let us briefly recall the procedure of assigning a quiver $Q(A)$ to a splitting basic algebra A . Let $\mathfrak{r}$ be the radical of A , $1 = \bar{e}_1 + \dots + \bar{e}_n$ be a decomposition of unity in $A/\mathfrak{r}$, and $1 = e_1 + \dots + e_n$ the corresponding decomposition of unity in A ; further, let $W = (\mathfrak{r}/\mathfrak{r}^2)$. Denote $t_{ij} = \dim_{\mathbb{k}} e_j W e_i$. Now set $Q(A) = (Q_0, Q_1)$ with $Q_0 = \{1, \dots, n\}$ and t_{ij} arrows from the i -th vertex to the j -th one.

Let $(\mathbb{k}Q)_{\geq 1}$ be the ideal generated by the arrows in Q . An ideal $I \triangleleft \mathbb{k}Q$ is said to be *regular* if $(\mathbb{k}Q)_{\geq 1}^2 \supseteq I \supseteq (\mathbb{k}Q)_{\geq 1}^t$, for some $t \geq 2$.

Theorem 2.2 ([1, Theorem III.1.9]). *Every splitting basic finite dimensional algebra with quiver Q is isomorphic to a factor algebra $\mathbb{k}Q/I$, where I is a regular ideal.*

Corollary 2.3. *For a splitting basic finite dimensional algebra A there is a set of relations ρ such that $A \cong \mathbb{k}Q(A)/\langle \rho \rangle$.*

We now use Corollary 2.3 to get a concept of representation space that would be convenient for our purposes.

Let A be a splitting (but not necessarily basic) finite dimensional algebra and B be a maximal semisimple subalgebra in A supplemented by the Jacobson radical $\mathfrak{r}$. Since $\mathbb{k}$ is algebraically closed, B is isomorphic to $\prod_{i=1}^s \text{Mat}_{n_i \times n_i}(\mathbb{k})$. We denote by $e_{jk}^{(i)}$, $i = 1, \dots, s$, $j, k = 1, \dots, n_i$, the elements of the canonical basis of B . We also denote by $a_1, \dots, a_q$ the elements of $\bigcup_{i,j} e_{11}^{(i)} \mathfrak{r} e_{11}^{(j)}$ chosen in such a way that their residue classes give a basis for $\bigoplus_{i,j} e_{11}^{(i)} \mathfrak{r} / \mathfrak{r}^2 e_{11}^{(j)}$. It is known that elements $e_{jk}^{(i)}$ and a_i generate A as an algebra.

Now fix a positive integer d . Then the set $\text{Rep}(A, d)$ of all d -dimensional representations of A is the set of all algebra homomorphisms $A \rightarrow \text{Mat}_{d \times d}(\mathbb{k})$. Such a homomorphism is defined by the images of $e_{jk}^{(i)}$ and a_l . Thus $\text{Rep}(A, d)$ may be identified with a Zariski closed subvariety in $\text{Mat}_{d \times d}(\mathbb{k})^{(\sum_i n_i^2) + l}$, where all the relations from I are satisfied. The structure and properties of $\text{Rep}(A, \alpha)$ as an algebraic variety are investigated in [2].

Observe that $e_{jj}^{(i)}$ are primitive pairwise orthogonal idempotents and $1 = \sum e_{jj}^{(i)}$. So, we may define dimension vectors of A -modules according to this decomposition. For a module M we set $(\underline{\dim}(M))_{i,j} := \dim_{\mathbb{k}}(e_{jj}^{(i)} M)$. This identifies the representation space $\text{Rep}(A, \alpha)$ with the subset in $\text{Rep}(A, \sum_{i,j} \alpha_{i,j})$, where each $e_{jj}^{(i)}$ is a projection onto an $\alpha_{i,j}$ -dimensional subspace. On the representation space we have a natural action of GL_d , where $d = \sum_{i,j} \alpha_{i,j}$. It is easy to see that isomorphism classes of α -dimensional A -modules coincide with GL_d -orbits in $\text{Rep}(A, \alpha)$.

Example 2.4. For a quiver Q and a dimension vector α consider two representation spaces $\text{Rep}(Q, \alpha)$ and $\text{Rep}(\mathbb{k}Q, \alpha)$. The latter is larger because when we define $\text{Rep}(Q, \alpha)$ we fix the projections e_i , $i \in Q_0$, and only vary the maps $a \in Q_1$.

In fact the information contained in $\text{Rep}(A, \alpha)$ is excessive for the purposes of invariant theory. It is useful to consider the smaller subvariety that plays the same role as $\text{Rep}(Q, \alpha)$ for $\text{Rep}(\mathbb{k}Q, \alpha)$. Fix a decomposition $\mathbb{k}^{\sum_{i,j} \alpha_{i,j}} = \bigoplus_{i,j} V_{i,j}$, where the dimension of each $V_{i,j}$ equals $\alpha_{i,j}$. Define

$$\underline{\text{Rep}}(A, \alpha) = \left\{ (x_{j,k}^{(i)}, y_l)_{i,j,k,l} \in \text{Rep}(A, \alpha) \mid \text{Im}(x_{jj}^{(i)}) = V_{i,j}, \text{ for all } i, j \right\}.$$

Observe that each GL_d -orbit in $\text{Rep}(Q, \alpha)$ intersects $\underline{\text{Rep}}(A, \alpha)$ and moreover that the intersection of each GL_d -orbit with $\underline{\text{Rep}}(A, \alpha)$ is a single $GL(\alpha)$ -orbit, where $GL(\alpha) := \prod_{i,j} GL(V_{i,j})$. This implies that the moduli problem for GL_d -action on $\text{Rep}(A, \alpha)$ may be reduced to the one for $GL(\alpha)$ -action on $\underline{\text{Rep}}(A, \alpha)$.

If A is the quotient algebra $\mathbb{k}Q/\langle \rho \rangle$, the set $\underline{\text{Rep}}(A, \alpha)$ embeds into $\text{Rep}(Q, \alpha)$ as a Zariski closed subvariety, namely as the subset where $X_p \equiv 0$ for all $p \in \langle \rho \rangle$. Here for a representation $X \in \text{Rep}(Q, \alpha)$ and an element

$$\lambda_1 a_{i_{11}} \dots a_{i_{1k(1)}} + \dots + \lambda_s a_{i_{s1}} \dots a_{i_{sk(s)}}$$

we denote by X_p the linear transformation

$$\lambda_1 X_{a_{i_{11}}} \dots X_{a_{i_{1k(1)}}} + \dots + \lambda_s X_{a_{i_{s1}}} \dots X_{a_{i_{sk(s)}}}.$$

3. FRAMED REPRESENTATIONS OF FINITE DIMENSIONAL ALGEBRAS

3.1. Framed representations. Let A be a (splitting basic) finite dimensional algebra with $|Q(A)_0| = n$. Fix two dimension vectors $\alpha, \zeta \in (\mathbb{Z}_{\geq 0})^n$ and a $Q(A)_0$ -graded vector space $V = V_1 \oplus \dots \oplus V_n$ such that $\dim V_i = \zeta_i$, for $i = 1, \dots, n$. Consider the extended representation space $\text{Rep}(A, \alpha, \zeta)$ consisting of pairs $(M, \tilde{f})$ with $M \in \text{Rep}(A, \alpha)$ and $\tilde{f} = (f_i : M \rightarrow V_i)_{i=1}^n$ as a tuple of linear maps such that $\tilde{f}_i e_i = \tilde{f}_i$, for all $i \in Q_0$ and $\tilde{f}_i e_j = 0$, for $j \neq i$. The latter two conditions mean that each $\tilde{f}_i$ factors through the projection of M onto $e_i M$ along $\bigoplus_{i \neq j} e_j M$. Elements of $\text{Rep}(A, \alpha, \zeta)$ will be called *framed representations* of the algebra A . Define the action of $GL(\alpha)$ on this variety by $g \cdot (M, (f_i)_{i=1}^n) = (g \cdot M, (f_i g_i^{-1})_{i=1}^n)$.

When passing from $\text{Rep}(A, \alpha)$ to $\underline{\text{Rep}}(A, \alpha)$ we fix the spaces $e_i M$, so we may actually replace $\tilde{f}_i$ by $f_i = \tilde{f}_i|_{e_i M}$. Thus we obtain the simplified version of $\text{Rep}(A, \alpha, \zeta)$ as $\underline{\text{Rep}}(A, \alpha, \zeta) = \{(M, f) \mid M \in \underline{\text{Rep}}(A, \alpha), f = (f_i)_{i=1}^n : M_1 \oplus \dots \oplus M_n \rightarrow V_1 \oplus \dots \oplus V_n \text{ is a graded linear map}\} \cong \underline{\text{Rep}}(A, \alpha) \oplus \bigoplus_{i=1}^n \text{Hom}_{\mathbb{k}}(\mathbb{k}^{\alpha_i}, V_i)$, where $\dim V_i = \zeta_i$. The GL_d -action on $\text{Rep}(A, \alpha, \zeta)$ restricts to the following action of $GL(\alpha)$ on $\underline{\text{Rep}}(A, \alpha, \zeta)$: $g \cdot (M, (f_i)_{i=1}^n) = (g \cdot M, (f_i g_i^{-1})_{i=1}^n)$.

Definition 3.1. A pair $(M, f) \in \text{Rep}(A, \alpha, \zeta)$ is *stable* if no nonzero submodule of M is contained in $\ker f$. The set consisting of such pairs will be denoted by $\text{Rep}^s(A, \alpha, \zeta)$. We will also denote by $\underline{\text{Rep}}^s(A, \alpha, \zeta)$ the subset of all stable pairs in $\underline{\text{Rep}}(A, \alpha, \zeta)$.

It is easy to see that the thus constructed framed representation space $\underline{\text{Rep}}(A, \alpha, \zeta)$ embeds in $\text{Rep}(Q, \alpha, \zeta)$ as a Zariski closed $GL(\alpha)$ -invariant subvariety. It is a general fact of invariant theory (see [10, §II.3.2]) that in this case the restriction π_A of the quotient map $\pi : \text{Rep}^s(Q, \alpha, \zeta) \rightarrow \mathcal{M}(Q, \alpha, \zeta)$ to $\underline{\text{Rep}}^s(A, \alpha, \zeta)$

is again a quotient map. Moreover, the restriction of the natural morphism $\mathcal{M}^s(Q, \alpha, \zeta) \rightarrow \mathcal{M}(Q, \alpha, \zeta) = \mathcal{M}(Q, \alpha)$ to $\mathcal{M}^s(A, \alpha, \zeta)$ gives us a projective morphism $\mathcal{M}^s(A, \alpha, \zeta) \rightarrow \mathcal{M}(A, \alpha) = \{pt\}$. Collecting these observations and the well-known properties of moduli spaces for quivers, we obtain the following.

Proposition 3.2. *The quotient $\pi_A : \underline{\text{Rep}}^s(A, \alpha, \zeta) \rightarrow \mathcal{M}^s(A, \alpha, \zeta)$ is geometric. Moreover, if $\text{Rep}^s(A, \alpha, \zeta)$ is nonempty, then $\mathcal{M}^s(A, \alpha, \zeta)$ is a smooth projective variety.*

3.2. The construction of the quotient space. From now on for $\sigma \in \mathbb{k}Q$ we will denote by $\bar{\sigma}$ its image $\sigma + \langle \rho \rangle$ in $\mathbb{k}Q / \langle \rho \rangle \cong A$. It is easy to see that in $\mathbb{k}Q / \langle \rho \rangle$ there is a (finite) $\mathbb{k}$ -basis Ξ consisting of images of paths in $\mathbb{k}Q$. Its elements will be referred to as paths in the algebra A . Denote by I_i the injective A -module associated with the i -th vertex of the quiver. Recall that the corresponding representation from $\underline{\text{Rep}}(Q, \rho)$ may be described as follows: $(I_i)_j = \text{span} \{ \bar{\tau} : j \rightsquigarrow i \}^*$, where “ $\bar{\tau} : j \rightsquigarrow i$ ” means that $\bar{\tau}$ is the image of a path τ starting in the j -th vertex and ending in the i -th one. In this case $((I_i)_{a:k \rightarrow l} f)(\bar{\tau}) = f(\bar{\tau} \bar{a})$, where $\bar{\tau} : l \rightsquigarrow i$. Consider the injective module $J := \bigoplus_{i \in Q_0} I_i \otimes_{\mathbb{k}} V_i$. Observe that as a $\mathbb{k}$ -linear space

$$J_i = e_i J \cong \bigoplus_{j \in Q_0} (I_j)_i \otimes_{\mathbb{k}} V_j \cong \bigoplus_{j \in Q_0} \bigoplus_{\Xi \ni \bar{\tau} : i \rightsquigarrow j} V_j \cong \bigoplus_{\Xi \ni \bar{\tau} : i \rightsquigarrow j} V_j.$$

Now, given a point $(M, f) \in \underline{\text{Rep}}(A, \alpha, \zeta)$, define a map $\Phi_{(M, f)} = (\varphi_i)_{i \in Q_0} : M \rightarrow J$ by the following rule:

$$(2) \quad \varphi_i = \bigoplus_{\Xi \ni \bar{\tau} : i \rightsquigarrow j} f_j \bar{\tau} : M_i \rightarrow \bigoplus_{\Xi \ni \bar{\tau} : i \rightsquigarrow j} V_j.$$

Here we view $\bar{\tau}$ as an element of A ; i.e. $\bar{\tau}(m) = \bar{\tau} \cdot m$.

Proposition 3.3. *The subspace $\ker \Phi_{(M, f)} = \bigoplus_{i \in Q_0} \ker \varphi_i$ is the maximal A -submodule of M contained in $\ker f$.*

Proof. It is a straightforward computation to check that $\Phi_{(M, f)}$ is a homomorphism of A -modules. Hence $\ker \Phi_{(M, f)}$ is an A -submodule of M . Now let U be an A -submodule of M contained in $\ker f$. For each $\Xi \ni \bar{\tau} : i \rightsquigarrow j$ we then have $\bar{\tau} U_i = \bar{\tau} e_i U = \bar{\tau} U = e_j \bar{\tau} U \subseteq U_j$. This implies that $f_j(\bar{\tau} \cdot x) = 0$, for all $x \in U, j \in Q_0, \bar{\tau} \in \Xi$, i.e. $U \subseteq \ker \Phi_{(M, f)}$. $\square$

Corollary 3.4. *The map $\Phi_{(M, f)} : M \rightarrow J$ is injective if and only if the pair (M, f) is stable.*

Now introduce the notion of the Grassmannian of submodules. Let $N = \bigoplus_{i \in Q_0} N_i$ be an A -module; the Grassmannian of A -submodules of N with dimension vector γ is a set $\text{Gr}_{\gamma}^A(N)$ of all Q_0 -graded subspaces $U = \bigoplus_{i \in Q_0} U_i \subseteq N$ with $\dim U_i = \gamma_i$ that are A -submodules. Note that $\text{Gr}_{\gamma}^A(N)$ is a closed variety in the product of classical Grassmannians $\text{Gr}_{\gamma}(N) = \prod_{i \in Q_0} \text{Gr}_{\gamma_i}(N_i)$ defined by $\bar{\tau}(U_i) \subseteq U_j$, for all $\Xi \ni \bar{\tau} : i \rightsquigarrow j$.

The following result is a generalization of [13, Proposition 3.9].

Theorem 3.5. *The moduli space $\mathcal{M}^s(A, \alpha, \zeta)$ is isomorphic to the Grassmannian of submodules $\text{Gr}_{\alpha}^A(J)$.*

Proof. Denote by $\mathrm{IHom}_\alpha(V)$ the set of all injective graded vector space homomorphisms from a space with dimension vector α to V . It is easy to see that $\mathrm{Gr}_\alpha(J)$ is a quotient of $\prod_{i \in Q_0} \mathrm{IHom}_\mathbb{k}(M_i, J_i)$ by the natural action of $GL(\alpha)$ (note that $GL(\alpha_i)$ acts on M_i by base change). Denote the inverse image of $\mathrm{Gr}_\alpha^A(J) \subseteq \mathrm{Gr}_\alpha(J)$ by $\mathrm{IHom}_\alpha^A(J)$. Now to prove the theorem it is sufficient to construct a $GL(\alpha)$ -invariant isomorphism $\Phi : \underline{\mathrm{Rep}}(A, \alpha, \zeta) \xrightarrow{\sim} \mathrm{IHom}_\alpha^A(J)$. We set $\Phi : (M, f) \mapsto \Phi_{(M, f)}$. This map is $GL(\alpha)$ -invariant. Indeed, let $g = (g_i)_{i \in Q_0} \in GL(\alpha)$. Then for $m \in M$ we have $\Phi(g \cdot (M, f))_i(m_i) = (\Phi_{g \cdot (M, f)})_i(m) = \bigoplus_{\Xi \ni \bar{\tau}: i \rightsquigarrow j} f_j(g_j \bar{\tau} m) = (g \cdot \Phi_{(M, f)})(m)$.

From Corollary 3.4 it follows that $\Phi(\underline{\mathrm{Rep}}^s(A, \alpha, \zeta)) \subseteq \mathrm{IHom}_\alpha^A(J)$. Let us show that given an injection $\{F : M \hookrightarrow J\} \in \mathrm{IHom}_\alpha^A(J)$, the pair (M, f) may be recovered. But f is obtained as a composition

$$f : M_i \xrightarrow{\varphi_i} \bigoplus_{\Xi \ni \bar{\tau}: i \rightsquigarrow j} V_j \rightarrow V_i,$$

where the last map is a projection on a summand associated to $\bar{\tau} = e_i$. As for the module M , the following lemma gives the possibility to recover it.

Lemma 3.6. *Let K be a finite dimensional submodule in J and $\underline{\mathrm{Rep}}^s(A, \alpha, K) \subseteq \underline{\mathrm{Rep}}^s(A, \alpha)$ be the inverse image of $\mathrm{IHom}_\alpha^A(K)$ under Φ . Then there is a morphism $s : \mathrm{IHom}_\alpha^A(K) \rightarrow \underline{\mathrm{Rep}}^s(A, \alpha, K)$ such that $s \circ \Phi = \mathrm{id}_{\underline{\mathrm{Rep}}^s(A, \alpha, K)}$.*

Proof. To recover $M \in \underline{\mathrm{Rep}}^s(A, \alpha, K)$ means to define the action of A on the vector space $\bigoplus_{i \in Q_0} M_i$, i.e. of the elements $A \ni \bar{a} : i \rightarrow j$ for $a \in Q_1$. Furthermore, for each of them we have the following commutative diagram:

$$\begin{array}{ccc} K_i & \xrightarrow{K_a} & K_j \\ \varphi_i \uparrow & & \uparrow \varphi_j \\ M_i & \xrightarrow{M_a} & M_j \end{array}$$

where $M_a = \bar{a}|_{M_i}$. This is because for each pair (M, f) the map $\Phi_{(M, f)}$ is an A -homomorphism. So, each M_a satisfies the equations $K_a \varphi_i = \varphi_j M_a$, which may be considered as matrix equations. Analogously, φ_j may be regarded as a matrix of dimension $\dim K_j \times \dim M_j$. Its rank is maximal and equals $\dim M_j$, since φ_j are injections. Therefore, $\mathrm{IHom}_\alpha^A(K)$ may be covered by open subsets, where various minors of the matrix of φ_j do not vanish, and M_a are recovered from matrix elements of φ_i , φ_j and K_a using Cramer's Theorem. $\square$

Together with the above described way of recovering f , the morphism

$$s : \mathrm{IHom}_\alpha^A(J) \rightarrow \underline{\mathrm{Rep}}^s(A, \alpha)$$

gives a morphism that is inverse to Φ . Consequently, Φ is an isomorphism and, being $GL(\alpha)$ -invariant, descends to quotients implying that $\Phi/GL(\alpha) : \mathcal{M}^s(A, \alpha, \zeta) \cong \underline{\mathrm{Rep}}^s(A, \alpha, \zeta)/GL(\alpha) \xrightarrow{\sim} \mathrm{IHom}_\alpha^A(J)/GL(\alpha) \cong \mathrm{Gr}_\alpha^A(J)$. Theorem 3.5 is proved. $\square$

Theorem 3.5 describes a variety that may serve as a substitute of a moduli space of A -modules with dimension vector α whenever $\underline{\mathrm{Rep}}^s(A, \alpha, \zeta)$ is nonempty. So, it is important to have a criterion of existence of a stable pair. For quivers M . Reineke proved that $\underline{\mathrm{Rep}}^s(A, \alpha, \zeta) \neq \emptyset$ if and only if $\zeta_i \geq (\mathbf{i}, \alpha_i)_Q$, for all $i \in Q_0$, where $\mathbf{i}$ stands for the vector with all coordinates zero except 1 on the i -th place, and

$(\cdot, \cdot)_Q$ is the Euler form, i.e. $(\mathbf{i}, \mathbf{j})_Q = \delta_{ij} - (\text{number of arrows from } i \text{ to } j)$; see [13, Proposition 4.3]. For arbitrary finite dimensional algebras we do not have such a result. However, we can state a weaker proposition. Recall that a *socle* of an A -module M is the sum of all its simple submodules.

Proposition 3.7. *For an A -module M with dimension vector α there is a map $f : M \rightarrow V$ making the pair $(M, f) \in \underline{\text{Rep}}(A, \alpha, \zeta)$ stable if and only if the socle $\text{soc } M$ may be embedded in $\text{soc } J$.*

Proof. The “only if” part is trivial. We now prove sufficiency. First, note that $(\text{soc } J)_i = V_i^{e_i}$. Consider a decomposition $M = \text{soc } M \oplus W$ of M as a vector space. Let f be the composition of the projection of M onto $\text{soc } M$ along W and an inclusion $\text{soc } M \hookrightarrow \text{soc } J = \bigoplus_{i \in Q_0} V_i$. Then $\ker f = W$. Now, if W contains a submodule N , then $W \supseteq \text{soc } N$, which is a contradiction since $\text{soc } N \subseteq \text{soc } M$. $\square$

Taking the dimensions of $(\text{soc } J)_i$ and $(\text{soc } M)_i$, his condition may be reformulated as: for $M \in \underline{\text{Rep}}(Q, \alpha)$ there is a map $f : M \rightarrow V$ making the pair $(M, f) \in \underline{\text{Rep}}(A, \alpha, \zeta)$ stable if and only if $\zeta_i \geq \dim(\text{soc } M)_i$. Note that $(\text{soc } M)_i$ is the multiplicity in $\text{soc } M$ of the simple module $S(i) = \mathbb{k}e_i$ corresponding to the i -th vertex. For quivers this obviously coincides with [13, Lemma 4.1]. When using this criterion it is also convenient to have in mind that $(\text{soc } M)_i = \bigcap_{\bar{a}: i \rightarrow j} \ker M_{\bar{a}}$.

Now, we need a way of determining, for a point of the classical Grassmannian $\text{Gr}_\alpha(J) = \prod_{i \in Q_0} \text{Gr}_{\alpha_i}(J_i)$, whether it lies in the Grassmannian of submodules. To formulate the next proposition, we should recall that the summands V_s of J_i are indexed by paths in A . Furthermore, since an arrow $\bar{a} : i \rightarrow s$ induces, for each $V_k^{(\bar{\tau})} \subseteq J_i$ with $\bar{\tau} = \bar{\tau}'\bar{a} : i \rightsquigarrow k$, an isomorphism $V_k^{(\bar{\tau})} \xrightarrow{\sim} V_k^{(\bar{\tau}')}$, there is an injection $\dagger_{\bar{a}} : \widetilde{J}_s^{(\bar{a})} \hookrightarrow J_i$, where $\widetilde{J}_s^{(\bar{a})}$ is a sum of all $V_k^{(\bar{\sigma})} \subseteq J_s$ such that $\bar{\sigma}\bar{a} \neq 0$. This injection acts as a simple index change, and, moreover, $\bar{a}\dagger_{\bar{a}} = \text{id}_{\widetilde{J}_s^{(\bar{a})}}$. Note that for hereditary algebras, i.e. for quivers with no relations, $\widetilde{J}$ coincides with J , so J_s is embedded in J_i whenever there is an arrow $i \rightarrow s$.

Proposition 3.8. *A point $(U_i \subseteq \bigoplus_{\Xi \ni \bar{\tau}: i \rightsquigarrow j} V_j)_{i \in Q_0} \in \prod_{i \in Q_0} \text{Gr}_{\alpha_i}(\bigoplus_{\Xi \ni \bar{\tau}: i \rightsquigarrow j} V_j)$ lies in the image of $\Phi//GL(\alpha)$ if and only if*

$$(3) \quad U_i \subseteq V_i^{(e_i)} \oplus \bigoplus_{T(\bar{a}) \ni \bar{\tau}: k \rightsquigarrow l} V_l^{(\bar{\tau})} \oplus \dagger_{\bar{a}}(U_j \cap \widetilde{J}_j^{(\bar{a})}), \quad \forall i \in Q_0, \bar{a} : i \rightarrow j,$$

where $T(\bar{a})$ is the set of all $\Xi \ni \bar{\tau} : k \rightsquigarrow l$ such that there is an arrow $\bar{a} \neq \bar{b} : i \rightarrow k$ and there is no path σ in Q with $\bar{\tau}\bar{b} = \bar{\sigma}\bar{a}$.

Proof. As it was shown above, $(U_i \subseteq \bigoplus_{\Xi \ni \bar{\tau}: i \rightsquigarrow j} V_j)_{i \in Q_0} \in \prod_{i \in Q_0} \text{Gr}_{\alpha_i}(\bigoplus_{\Xi \ni \bar{\tau}: i \rightsquigarrow j} V_j)$ belongs to the image of $\Phi//GL(\alpha)$ if and only if it is an A -submodule of J , which means that $\bar{\tau}(U_i) \subseteq U_j$, $\forall \Xi \ni \bar{\tau} : i \rightarrow j$. It is straightforward to check that these conditions are equivalent to (3). $\square$

On $\mathcal{M}^s(A, \alpha, \zeta)$ one may construct the universal family $\mathcal{C} = ((\mathcal{C}_i)_{i \in Q_0}, (\mathcal{C}_a)_{a \in Q_0})$. For a point $y = (U_i \subseteq J_i)_{i \in Q_0}$, we take $(\mathcal{C}_i)_y = \text{pr}_i(U_i)$, where pr_i is the projection of $\bigoplus_{\bar{\tau}: i \rightsquigarrow j} V_j^{(\bar{\tau})}$ onto $V_i^{(e_i)}$, and $(\mathcal{C}_a)_y : (\mathcal{C}_{ta})_y \rightarrow (\mathcal{C}_{ha})_y$ acts as J_a . It is not hard to check that the maps $\mathcal{C}_a$ are well defined.

Now let Q be a quiver without oriented cycles and $i_0 \in Q_0$ be a source. Also let Q' be a full subquiver of Q with support $Q_0 \setminus \{i_0\}$. Denote by α' and ζ' the

restrictions of α and ζ respectively to Q' . Notice that $I = \text{span}\{\bar{\tau} : i_0 \rightsquigarrow j\}$ is a two-sided ideal in A , so we may consider a factor algebra $A' = A/I$. Clearly it may be associated with a subalgebra of A generated by e_i , for $i \neq i_0$, and $a \in Q_1$ with $ta \neq i_0$. Then the subspace $J' = \bigoplus_{i \in Q'_0} J_i \subseteq J$ is naturally an injective A' -module. So, $\text{Gr}_\alpha^A(J)$ is a subvariety in $\text{Gr}_{\alpha'}^{A'}(J') \times \text{Gr}_{\alpha_{i_0}}(\bigoplus_{\bar{\tau}: i \rightsquigarrow j} V_j^{(\tau)})$ consisting of pairs $((U_i)_{i \in Q'_0}, U)$ satisfying conditions from Proposition 3.8. If there are no relations starting from i , by [13, Lemma 4.9] the projection $\text{Gr}_\alpha^A(J) \rightarrow \text{Gr}_{\alpha'}^{A'}(J')$ is isomorphic to the Grassmann bundle $\text{Gr}_{\alpha_0}(V_i \oplus \bigoplus_{a: i \rightarrow j} C_j)$. Therefore, in this case a reduction similar to the one introduced in [13, Section 4.3] may be performed.

Example 3.9. Let A be the algebra with quiver $Q : 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3$ and relations $\rho = \{a_2 a_1\}$. So, A is a Nakayama algebra with admissible sequence $(1, 1)$; see [1, Section IV.2]. Consider two dimension vectors α and ζ and a vector space V_i for each ζ_i . Then, as we have proved, $\mathcal{M}^s(A, \alpha, \zeta) \cong \text{Gr}_\alpha^A(J)$, where $J_1 = V_1 \oplus V_2$, $J_2 = V_2 \oplus V_3$, and $J_3 = V_3$. Using Proposition 3.8 we obtain that $\mathcal{M}^s(A, \alpha, \zeta) \cong \{(U_i \subseteq J_i)_{i=1}^3 \mid U_3 \subseteq V_3, U_2 \subseteq V_2 \oplus U_3, U_1 \subseteq V_1 \oplus (U_2 \cap V_2)\}$. Note that in this example we do not need to write the index (e_i) over V_i , since there is only one summand V_i in J_i .

Example 3.10. Let A be the algebra with quiver

and a single relation $a_2 a_1 - b_2 b_1$. Then A is a $\mathbb{k}$ -linear span of the path images $\bar{e}_i$, $i = 1, \dots, 4$, $\bar{a}_i$, $\bar{b}_i$ and $\bar{a}_2 \bar{a}_1$. The components of J are $J_1 = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, $J_2 = V_2 \oplus V_4$, $J_3 = V_3 \oplus V_4$ and $J_4 = V_4$ with the arrow images acting as $\bar{a}_1 = 0 \oplus \text{id} \oplus 0 \oplus \text{id}$, $\bar{b}_1 = 0 \oplus 0 \oplus \text{id} \oplus \text{id}$, $\bar{a}_2 = 0 \oplus \text{id}$, $\bar{a}_4 = 0 \oplus \text{id}$. Hence $\tilde{J}_i = J_i$, for all i , and all $\dagger$'s are natural inclusions. That is why we venture to omit them as well as the indexes (e_i) in the final formula. So, $\mathcal{M}^s(A, \alpha, \zeta) \cong \{(U_i \subseteq J_i)_{i=1}^4 \mid U_1 \subseteq V_1 \oplus V_3 \oplus U_2, U_1 \subseteq V_1 \oplus V_2 \oplus U_3, U_2 \subseteq V_2 \oplus U_4, U_3 \subseteq V_3 \oplus U_4\}$.

Example 3.11. Consider the algebra $A = \mathbb{k}[x]/(x)^n$. It is also a Nakayama algebra with admissible sequence (n) , but now its quiver contains a loop. In fact, it is a Jordan quiver with a single vertex and a single loop a , and the only relation is a^n . Let $\alpha = (m)$, $\zeta = (k)$, and V be a k -dimensional vector space. Then $J = J_1 = V^{(e)} \oplus V^{(a)} \oplus \dots \oplus V^{(a^{n-1})}$. Observe that $\bar{a}$ acts on J as follows:

$$\begin{array}{ccccccc}
 J & : & V^{(e)} & \oplus & V^{(a)} & \oplus & V^{(a^2)} \oplus \dots \oplus V^{(a^{n-1})} \\
 \bar{a} \downarrow & & 0 \downarrow & & \text{id} \downarrow & & \text{id} \downarrow & & \text{id} \downarrow \\
 J & : & 0 & \oplus & V^{(e)} & \oplus & V^{(a)} \oplus \dots \oplus V^{(a^{n-2})} \oplus V^{(a^{n-1})}.
 \end{array}$$

So, $\tilde{J} = V^{(e)} \oplus V^{(a)} \oplus \dots \oplus V^{(a^{n-2})}$, and $\dagger_{\bar{a}}$ maps $V^{(a^t)}$, $t = 0, \dots, n-2$, to $V^{(a^{t+1})}$ isomorphically.

It is easy to check that J can be viewed as $\mathbb{k}[x]/(x^n) \otimes V$ with $\bar{a}$ acting as $\frac{d}{dt}$. Then $\tilde{J}$ becomes $\mathbb{k}[x]/(x^{n-1}) \otimes V$, $\dagger_{\bar{a}} : f \otimes v \mapsto xf \otimes v$, and the moduli space is isomorphic to the Grassmannian of m -dimensional $\frac{d}{dt}$ -invariant subspaces in J . This interpretation is in fact rather fruitful and will be further explored in the next sections.

Denote by $\mathbb{k}Q^{(1)}$ an ideal in the path algebra generated by all oriented cycles in Q . We also use the notation $\mathbb{k}Q^{(t)} := (\mathbb{k}Q^{(1)})^t$.

Let G be a reductive group acting on a vector space X . Recall that the *null cone* of this action is the set $\mathcal{N} = \{x \in X \mid f(x) = 0, \text{ for all nonconstant homogeneous } G\text{-invariant functions on } X\}$. By Hilbert-Mumford's Criterion [10, Theorem III.2.4] this is equivalent, for $x \in X$, to the existence of a one-parameter subgroup $\lambda : \mathbb{k} \setminus \{0\} \rightarrow G$ with $\lim_{t \rightarrow 0} \lambda(t)x = 0$.

One of the possible applications of the above construction is the study of the null cone of $\text{Rep}(Q, \alpha, \zeta)$ in the case when Q is a quiver with oriented cycles. This is made possible by the following proposition.

Proposition 3.12. *Let Q be a quiver with oriented cycles. Then the null cone of the variety $\text{Rep}(Q, \alpha, \zeta)$ is isomorphic to $\underline{\text{Rep}}(\mathbb{k}Q/(\mathbb{k}Q)^{(\max_i \alpha_i)}, \alpha, \zeta)$.*

Proof. First of all, recall what is the categorical quotient of a framed representation space. There is a $GL(\alpha)$ -invariant isomorphism $\psi : \text{Rep}(Q, \alpha, \zeta) \cong \text{Rep}(\tilde{Q}, \tilde{\alpha})$ for a quiver $\tilde{Q}$ and an extended dimension vector $\tilde{\alpha}$ (see [13]). We have

$$\text{Rep}(Q, \alpha, \zeta) // GL(\alpha) \cong \text{Rep}(\tilde{Q}, \tilde{\alpha}) // GL(\alpha)$$

or, equivalently, $\psi^* : \mathbb{k}[\text{Rep}(\tilde{Q}, \tilde{\alpha})]^{GL(\alpha)} \xrightarrow{\sim} \mathbb{k}[\text{Rep}(Q, \alpha, \zeta)]^{GL(\alpha)}$. But the algebra $\mathbb{k}[\text{Rep}(\tilde{Q}, \tilde{\alpha})]^{GL(\alpha)}$ is generated by the traces of oriented cycles in $\tilde{Q}$. On the other hand, oriented cycles in $\tilde{Q}$ are those of Q , and hence the algebra $\mathbb{k}[\text{Rep}(Q, \alpha, \zeta)]^{GL(\alpha)}$ is also generated by traces of the oriented cycles in Q .

Thus the null cone consists of pairs (M, f) , where M are such representations on which all the oriented cycles in Q act as nilpotent operators. Since $A^k = 0$ for a nilpotent operator A in a k -dimensional space, all the oriented cycles as operators in M vanish in the $(\max_i \alpha_i)$ -th power. But we state that the stronger property holds: the product of any $\max_i \alpha_i$ oriented cycles is zero. If σ_1 and σ_2 are two such cycles, then by our conditions $\sigma_1 \sigma_2$ and $\sigma_2 \sigma_1$, as well as any of their product, are nilpotent. Moreover, all of them vanish in the same power. This implies that the commutator $[\sigma_1, \sigma_2]$ is also nilpotent. Indeed, any of its power equals a sum of products of σ_i . Since these products are nilpotent, their traces are all zero, providing that the trace of any power of the operator $[\sigma_1, \sigma_2]$ is zero. Therefore, the commutator $[\sigma_1, \sigma_2]$ is nilpotent.

Now, using Engel's Theorem, we can conclude that there is such a basis in the representation space where the matrices of both σ_1 and σ_2 are upper-triangular with zeroes on the diagonal. Applying induction, we get a basis in which the matrices of all oriented cycles starting in a given vertex are upper-niltriangular. As a product of oriented cycles starting in different vertices is zero by definition, the product of any $\max_i \alpha_i$ such operators is zero. $\square$

We briefly discuss an application of this construction. Let $\mathbb{k}$ be field $\mathbb{C}$ of complex numbers. Since $\text{Rep}^s(Q, \alpha, \zeta) // GL(\alpha)$ is projective over $\text{Rep}(Q, \alpha, \zeta) // GL(\alpha) = \text{Rep}(Q, \alpha) // GL(\alpha)$ (see [9]), there is a natural projection $\pi_s : \text{Rep}^s(Q, \alpha, \zeta) // GL(\alpha)$

$\rightarrow \text{Rep}(Q, \alpha) // GL(\alpha)$. From [6, Theorem 4.1] we know that for each point $y \in \text{Rep}(Q, \alpha) // GL(\alpha)$ there is a quiver $\tilde{Q}$ and a pair of dimension vectors $\bar{\alpha}, \bar{\zeta} \in \mathbb{Z}_{\geq 0}^{\tilde{Q}_0}$ such that $\pi_s^{-1}(y) \cong \tilde{\pi}_s^{-1}(0)$, where $\tilde{\pi}_s$ is the natural projection $\text{Rep}^s(\tilde{Q}, \bar{\alpha}, \bar{\zeta}) // GL(\bar{\alpha}) \rightarrow \text{Rep}(\tilde{Q}, \bar{\alpha}) // GL(\bar{\alpha})$. As a consequence, we obtain

Corollary 3.13. *If $\mathbb{k} = \mathbb{C}$, then for every point $y \in \text{Rep}(Q, \alpha) // GL(\alpha)$ there is a finite dimensional algebra A , an injective A -module J and a dimension vector $\beta \in \mathbb{Z}^{Q(A)_0}$ such that $\pi_s^{-1}(y) \cong \text{Gr}_\beta^A(J)$.*

As an example we work out the cyclic quiver Q of type $A_{n-1}^{(1)}$:

It is well known that points of $\text{Rep}(Q, \alpha) // GL(\alpha)$ are in one-to-one correspondence with isomorphism classes of semisimple representations of Q . For the quiver, the considered simple representations are the ones from the following list:

- S_i , for $i \in Q_0$, with $\dim(S_i)_j = \delta_{ij}$ and
- E_λ , for $\lambda \in \mathbb{k} \setminus \{0\}$, with dimension vector $(1, \dots, 1)$ having $a_1 = \dots = a_{n-1} = 1$, and $a_n = \lambda$.

Choose an isomorphism class $y := [\bigoplus_{i=1}^n m_i S_i \oplus \bigoplus_{\lambda \in \Lambda} m_\lambda E_\lambda]$. The new quiver from [6, Theorem 4.1] is then the disjoint union of Q (if some of m_i is nonzero) and $|\Lambda|$ copies of $A_0^{(1)}$. Moreover, let $\hat{\pi}_{m_\lambda}$, $\lambda \in \Lambda$, be the projection

$$\text{Rep}^s(A_0^{(1)}, m_\lambda, \sum_{i \in Q_0} \zeta_i) // GL(\alpha) \rightarrow \text{Rep}(A^{(1)}, m_\lambda) // GL(\alpha)$$

and $\hat{\pi}_s$ be the projection $\text{Rep}^s(Q, \underline{m}, \zeta) // GL(\alpha) \rightarrow \text{Rep}(Q, \underline{m}) // GL(\alpha)$, where $\underline{m} = (m_1, \dots, m_n)$. Then we have $\pi_s^{-1}(y) \cong \hat{\pi}_s^{-1}(0) \times \prod_{\lambda \in \Lambda} \hat{\pi}_{m_\lambda}^{-1}(0)$ (the first factor would be absent if all $m_i = 0$). For $\hat{\pi}_{m_\lambda}^{-1}(0)$ we may now apply considerations outlined in Example 3.11. In Section 5 we will give an explicit realization of the fibers by equations in projective space along with its version for $\mathbb{k} = \mathbb{R}$. In Section 4 we will also give a generalization of this to arbitrary infinite fields.

4. FRAMED MODULI SPACES FOR QUIVER $A_{n-1}^{(1)}$

In the three remaining sections we work over the arbitrary infinite field $\mathbb{k}$.

We will see that in attempts to apply Reineke's construction to quivers with oriented cycles we have to tackle Grassmannians in infinite dimensional spaces that do not carry a natural structure of algebraic variety. However, over an algebraically closed field the quotient $\text{Rep}^s(Q, \alpha, \zeta) // GL(\alpha)$ is projective over $\text{Rep}(Q, \alpha, \zeta) // GL(\alpha)$; see [9]. So, we can consider the natural projection

$$\pi_s : \text{Rep}^s(Q, \alpha, \zeta) // GL(\alpha) \rightarrow \text{Rep}(Q, \alpha, \zeta) // GL(\alpha)$$

and investigate its fibers. This also gives an idea of what to do when we do not possess a total framed moduli space.

Let Q be a cyclic quiver of type $A_{n-1}^{(1)}$:

This notation for vertices and arrows of the quiver will be used throughout Sections 4 and 5. There is a map $\pi : \text{Rep}(Q, \alpha, \zeta) \rightarrow \mathbb{k}[x]_{\alpha_1} \times \dots \times \mathbb{k}[x]_{\alpha_n}$, where $\mathbb{k}[x]_r$ is a space of polynomials of degree not higher than $r + 1$, sending a representation to the tuple of characteristic polynomials of τ_i . In this section our aim is to describe, for each $y \in \mathbb{k}[x]_{\alpha_1} \times \dots \times \mathbb{k}[x]_{\alpha_n}$, the moduli space of representations belonging to $\pi^{-1}(y)$ for a cyclic quiver Q of type $A_{n-1}^{(1)}$:

To each path σ in Q we associate a linear function $\sigma^* \in (\mathbb{k}Q)^*$, taking values

$$\sigma^*(\sigma') = \begin{cases} 1, & \text{for } \sigma' = \sigma, \\ 0, & \text{otherwise,} \end{cases}$$

for a path σ' . Further, for $\sigma : i \rightsquigarrow j$ denote $V_i^{(\sigma)} := \mathbb{k}\sigma^* \subseteq (\mathbb{k}Q)^*$ and set $\tau_i = a_{i-1}a_{i-2} \dots a_{i+1}a_i$, the shortest path starting in i . Consider the $\mathbb{k}Q$ -module J with

$$(4) \quad (J)_i := \prod_{\sigma: i \rightsquigarrow k} V_k^{(\sigma)} = \prod_{j=0}^{\infty} V_i^{(\tau_i^j)} \times \prod_{j=0}^{\infty} V_{i+1}^{(\tau_{i+1}^j a_i)} \times \dots \times \prod_{j=0}^{\infty} V_{i-1}^{(\tau_{i-1}^j a_{i-2} \dots a_{i+1} a_i)}.$$

Here we identify τ_i^0 with e_i . Furthermore, we use the symbol of direct product instead of direct sum, since we allow our tuples to contain infinitely many nonzero terms. It should be explained how $a_i \in \mathbb{k}Q$ acts on J . Considering the restrictions of a_i to the summands of (4) we have

$$a_i : J_i \supseteq \prod_{j=0}^{\infty} V_k^{(\tau_k^j a_{k-1} \dots a_{i+1} a_i)} \xrightarrow{\sim} \prod_{j=0}^{\infty} V_k^{(\tau_k^j a_{k-1} \dots a_{i+1})} \subseteq J_{i+1},$$

for $k \neq j$, where a_i acts as the componentwise isomorphism $V_k^{(\tau_k^j a_{k-1} \dots a_{i+1} a_i)} \xrightarrow{\text{id}} V_k^{(\tau_k^j a_{k-1} \dots a_{i+1})}$. In cases where $k = i$ a shift $J_i \supseteq V_i^{(\tau_i^j)} \xrightarrow{\text{id}} V_i^{(\tau_i^{j-1} a_{i-1} \dots a_{i+1})} \subseteq J_{i+1}$ takes place and, in particular, $a_i : J_i \supseteq V_i^{(e_i)} \rightarrow 0$.

We will now try to make our notation more convenient. Since each $\prod_{j=0}^{\infty} V_k^{(\tau_k^j a_{k-1} \dots a_{i+1} a_i)}$ is isomorphic as a vector space to $c_{\infty} \otimes V_k$, where c_{∞} is the space $\mathbb{k}^{\mathbb{Z}_{\geq 0}}$, we may rewrite J_i as $J_i = (c_{\infty} \otimes V_1) \oplus (c_{\infty} \otimes V_2) \oplus \dots \oplus (c_{\infty} \otimes V_n)$, for $i = 1, \dots, n$. The $\mathbb{k}Q$ -module structure will be translated as $a_i \cdot (G_1, \dots, G_n) = (G_1, \dots, G_{i-1}, \delta G_i, G_{i+1}, \dots, G_n)$ (elements of $c_{\infty} \otimes V_i$ can be regarded as vectors $G = (g_1, \dots, g_{\zeta_i})$, $g_j \in c_{\infty}$; in this situation we define δ as a componentwise left shift).

For a pair $(M, f) \in \text{Rep}(Q, \alpha, \zeta)$ define as before the map $\Phi_{(M, f)} = (\varphi_i)_{i=1}^n : M \rightarrow J$ by $\varphi_i(m) = (f_j(\tau m))_{\tau: i \rightsquigarrow j}$. Here we mean that the component of $\varphi(m)$ that corresponds to $V_k^{(\tau)} \subseteq J_i$ equals $f(\tau m)$. According to the above decomposition we get

$$\varphi_i(m) = (f_i(\text{exp}(\tau_i)m), f_{i+1}(\text{exp}(\tau_{i+1})a_i m), \dots, f_{i-1}(\text{exp}(\tau_{i-1})a_{i-2} \dots a_{i+1} a_i m)),$$

where $\mathbf{exp}(\tau_r) : V_p \rightarrow c_\infty \otimes V_r$ are linear operators defined by $v \mapsto (v, \tau_r v, \tau_r^2 v, \dots, \tau_r^s r v, \dots)$ and f_r act as componentwise multiplications.

Now it will be shown that the above constructed map shares some of the useful properties of its finite dimensional analog.

- Lemma 4.1.** (i) *The map $\Phi_{(M,f)}$ is a $\mathbb{k}Q$ -homomorphism.*
(ii) *The subspace $\ker \Phi_{(M,f)}$ is the maximal $\mathbb{k}Q$ -submodule of M contained in $\ker f$.*
(iii) *A pair (M, f) is stable if and only if $\Phi_{(M,f)}$ is injective.*

Proof. (i) Let a_i be an arrow in Q . Denote by φ_i the components of $\Phi_{(M,f)}$. It is sufficient to show that $(a_i)|_{J_i} \circ \phi_i = \phi_{i+1} \circ (a_i)|_{M_i}$.

Observe that, for all r , we have $\delta(f_r \mathbf{exp}(\tau_r)) = f_r \mathbf{exp}(\tau_r) \tau_r$. Hence,

$$\begin{aligned} a_i \cdot \varphi_i(m) &= (\delta f_i(\mathbf{exp}(\tau_i t)m), f_{i+1}(\mathbf{exp}(\tau_{i+1} t)a_i m), \dots, f_{i-1}(\mathbf{exp}(\tau_{i-1} t)a_{i-2} \dots a_{i+1} a_i m)) \\ &= (f_i(\mathbf{exp}(\tau_i t)\tau_i m), f_{i+1}(\mathbf{exp}(\tau_{i+1} t)a_i m), \dots, f_{i-1}(\mathbf{exp}(\tau_{i-1} t)a_{i-2} \dots a_{i+1} a_i m)) \\ &= (f_i(\mathbf{exp}(\tau_i t)a_{i-1} \dots a_{i+1} \cdot a_i m), f_{i+1}(\mathbf{exp}(\tau_{i+1} t) \cdot a_i m), \dots, \\ &\quad f_{i-1}(\mathbf{exp}(\tau_{i-1} t)a_{i-2} \dots a_{i+1} \cdot a_i m)) \\ &= \phi_{i+1}(a_i m). \end{aligned}$$

Thus, $\Phi_{(M,f)}$ is a $\mathbb{k}Q$ -homomorphism.

(ii) It follows from (i) that $\ker \Phi_{(M,f)}$ is a $\mathbb{k}Q$ -submodule of M . Let $N \subseteq M$ be a submodule contained in $\ker f$. Then for a path τ we have $\tau(N) \subseteq N$ and, therefore, $\Phi_{(M,f)}(N) = 0$. Consequently, $N \subseteq \ker \Phi_{(M,f)}$.

(iii) It is a direct consequence of (ii). $\square$

In the finite dimensional case, to each element $(M, f) \in \text{Rep}^s(Q, \alpha, \zeta)$ there is associated a submodule of J with dimension vector α that is a point in the Grassmannian of submodules $\text{Gr}_\alpha^{\mathbb{k}Q}(J)$. Our next aim is to prove that the same holds in the current setting.

Let $U \subseteq J$ be a $\mathbb{k}Q$ -submodule with dimension vector α . For each $i = 1, \dots, n$ we have an inclusion $a_i(U_i) \subseteq U_{i+1}$. This implies that $\tau_i(U_i) \subseteq U_i$. We know that $\tau_i : J_i = \bigoplus_{j=1}^n (c_\infty \otimes V_j) \rightarrow \bigoplus_{j=1}^n (c_\infty \otimes V_j) = J_{i+1}$, $(G_1, \dots, G_n) \mapsto (\delta G_1, \dots, \delta G_n)$. Hence if U is a submodule, then it is preserved by componentwise left shifts. This means that U_i are δ -invariant subspaces in J_i (note that $\bigoplus_{j=1}^n (c_\infty \otimes V_j) \cong c_\infty \otimes (\bigoplus_{j=1}^n V_j)$, and so its elements may be considered as $\sum_{j=1}^n \zeta_j$ -tuples of sequences). Further, let $D^{(i)} = (d_{pq}^{(i)})$ be the matrix of the restricted operator $\delta|_{U_i}$ and set $k = \alpha_i$, $m = \sum_{j=1}^n \zeta_j$. Fix a base $\bar{g}_1, \dots, \bar{g}_k$ in U_i (here we consider $\bar{g}_j$ as an m -tuple of sequences: $\bar{g}_j = (g_{jl})_{l=1}^n$). Then, for all j , we get

$$\delta \bar{g}_j = d_{1j}^{(i)} \bar{g}_1 + d_{2j}^{(i)} \bar{g}_2 + \dots + d_{kj}^{(i)} \bar{g}_k.$$

For a vector sequence $\bar{h}$ denote $\bar{h}(0) := (\bar{h})_0$. It is an easy calculation to check that

$$(5) \quad (\bar{g}_1, \dots, \bar{g}_k) = (\bar{g}_1(0), \dots, \bar{g}_k(0)) \mathbf{exp}(D^{(i)}).$$

Now Cayley-Hamilton's Theorem implies that $\chi_i(D^{(i)}) = 0$ in U_i , where χ_i is a characteristic polynomial of $D^{(i)}$. Thus each vector sequence $\bar{g} \in U_i$ satisfies the equation $\chi_i(\delta) \bar{g} = 0$.

We can finally show that the above constructed correspondence between stable pairs and points in Grassmannian is one-to-one.

Proposition 4.2. *The map $\Phi : \text{Rep}^s(Q, \alpha, \zeta) \rightarrow \text{IHom}_{\alpha}^{\mathbb{k}Q}(J)$, $(M, f) \mapsto \Phi_{(M, f)}$ is a bijection.*

Proof. We need to show that by having an inclusion $\Phi \in \text{IHom}_{\alpha}^{\mathbb{k}Q}(J)$ we can uniquely recover a pair (M, f) . First of all note that since for each j, r we have $\tau_j^r a_{j-1} \dots a_i = a_{j-1} \dots a_i \tau_i^r$, the map φ_i may be written as

$$(6) \quad \begin{aligned} \varphi_i(m) &= (f_i(\mathbf{exp}(\tau_i t)m), f_{i+1}(a_i \mathbf{exp}(\tau_i t)m), \dots, \\ &\quad f_{i-1}(a_{i-2} \dots a_{i+1} a_i \mathbf{exp}(\tau_i t)m)). \end{aligned}$$

Now let $U = (U_i)_{i=1}^n \subseteq J$ be a submodule with dimension vector α . Also let $U_i = \text{span} \{ \bar{g}_1^{(i)}, \dots, \bar{g}_{\alpha_i}^{(i)} \}$.

Now recall equality (5), setting $\bar{g}_r = \bar{g}_r^{(i)}$. The matrices $(\bar{g}_1^{(i)}, \dots, \bar{g}_{\alpha_i}^{(i)})$ and $(\bar{g}_1^{(i)}(0), \dots, \bar{g}_{\alpha_i}^{(i)}(0))$ may be divided into horizontal blocks of size $\zeta_j \times \alpha_i$, those blocks corresponding to the natural projections $P_j : J_i \rightarrow c_{\infty} \otimes V_j$. Thus, for each j , we obtain

$$(7) \quad (P_j \bar{g}_1^{(i)}, \dots, P_j \bar{g}_{\alpha_i}^{(i)}) = (P_j \bar{g}_1^{(i)}(0), \dots, P_j \bar{g}_{\alpha_i}^{(i)}(0)) \mathbf{exp} D^{(i)} t \cdot E_{\alpha_i},$$

where E_{α_i} is the identity matrix of size $\alpha_i \times \alpha_i$. These equalities can be interpreted as follows: there is a map $\Psi = (\Psi_{ij})_{i,j=1}^n$ with

$$\Psi_{ij} = (P_j \bar{g}_1^{(i)}(0), \dots, P_j \bar{g}_{\alpha_i}^{(i)}(0)) \mathbf{exp} D^{(i)} t : M_i \rightarrow c_{\infty} \otimes V_j \subseteq J_i,$$

where M is a graded vector space with dimension vector α ; one may easily establish that Ψ is bijective. We need to show that $\Psi = \Phi_{(M, f)}$ for a map $f : M \rightarrow V$ and a certain $\mathbb{k}Q$ -module structure on M .

As for the module structure, it is quite clear: viewing (6) as formulas defining the natural inclusion $U \hookrightarrow J$, we set $M = U$. Further, $f = (f_i)_{i=1}^n$ is defined as a tuple of compositions $f_i : M_i \hookrightarrow J_i \twoheadrightarrow V_i$, where V_i should be regarded as the component of J_i associated to e_i (we mean $V_i^{(e_i)} \subseteq V_i^{(e_i)} \oplus \bigoplus_{j=1}^{\infty} V_i^{(\tau_i^j)} = (c_{\infty} \otimes V_i)$). Hence,

$$f_i = \Psi_{ii}|_{t=0} : m \mapsto (P_i \bar{g}_1^{(i)}(0), \dots, P_i \bar{g}_{\alpha_i}^{(i)}(0)) m.$$

Now it is left to prove that (7) defines a map Ψ coinciding with $\Phi_{(M, f)}$ for the above M and f . From (6) it follows that it is sufficient to show that

$$(8) \quad (P_j \bar{g}_1^{(i)}(0), \dots, P_j \bar{g}_{\alpha_i}^{(i)}(0)) \mathbf{exp} D^{(i)} t = f_j \cdot a_{j-1} \dots a_i \mathbf{exp}(\tau_i t),$$

for $j \neq i$. But $D^{(i)}$ is just a notation for the matrix of τ_i , so $\mathbf{exp}(D^{(i)} t) \equiv \mathbf{exp}(\tau_i t)$. Canceling this exponent, we turn (8) into

$$(P_j \bar{g}_1^{(i)}(0), \dots, P_j \bar{g}_{\alpha_i}^{(i)}(0)) = (P_j \bar{g}_1^{(j)}(0), \dots, P_j \bar{g}_{\alpha_j}^{(j)}(0)) \cdot a_{j-1} \dots a_i.$$

We are going to prove this as a matrix equality, and so we can immediately write

$$(a_{j-1} \dots a_i \bar{g}_1^{(i)}, \dots, a_{j-1} \dots a_i \bar{g}_{\alpha_i}^{(i)}) = (\bar{g}_1^{(j)}, \dots, \bar{g}_{\alpha_j}^{(j)}) \cdot a_{j-1} \dots a_i.$$

Descending to the level of projections, we obtain

$$\left(P_j a_{j-1} \dots a_i \bar{g}_1^{(i)}, \dots, P_j a_{j-1} \dots a_i \bar{g}_{\alpha_i}^{(i)}\right) = \left(P_j \bar{g}_1^{(j)}, \dots, P_j \bar{g}_{\alpha_j}^{(j)}\right) \cdot a_{j-1} \dots a_i,$$

but a_r acts on J_i as $(\text{id}, \dots, \delta, \dots, \text{id})$, where the shift takes place only at the r -th position. Consequently, the product $a_{j-1} \dots a_i$ does not change the j -th projection of $\bar{g}^{(i)}$, implying that

$$\left(P_j a_{j-1} \dots a_i \bar{g}_1^{(i)}, \dots, P_j a_{j-1} \dots a_i \bar{g}_{\alpha_i}^{(i)}\right) = \left(P_j \bar{g}_1^{(i)}, \dots, P_j \bar{g}_{\alpha_i}^{(i)}\right),$$

and the required equality follows. The proposition is proved. $\square$

This proposition cannot be used to describe (as was done in Section 3) the total framed moduli space. So, we restrict our attention to fibers of π .

Lemma 4.3. *Let $\bar{\chi} = (\chi_1, \dots, \chi_n)$ be an admissible tuple of polynomials and $h_0 = x, h_1, \dots, h_N$ be all different irreducible factors of $\chi_1, \dots, \chi_n$ in $\mathbb{k}[x]$. Also let r_{ij} be the multiplicity of h_j as a factor of χ_i . If $j \neq 0$, then $r_{1j} = \dots = r_{nj}$.*

Proof. Let $M \in \pi^{-1}(\bar{\chi})$. Consider the block decomposition of M_i with respect to τ_i . Then $r_{ij} \cdot \deg(h_j)$ is the dimension of the subspace $M_i^{h_j} := \{m \in M_i \mid h_j^{r_{ij}} m = 0\}$. If $h_j(x) \neq x$, the restriction of $\tau_i = a_{i-1} \dots a_i$ on $M_i^{h_j}$ is nondegenerate. Therefore, $a_{p-1} \dots a_i(M_i^{h_j}) \cap \ker a_p = 0$, for all $p = 1, \dots, n$. As $\tau_p a_{p-1} \dots a_i = a_{p-1} \dots a_i \tau_i$, all $\tau_p|_{a_{p-1} \dots a_i(M_i^{h_j})}$ are conjugate and therefore have the same characteristic polynomials. $\square$

Fix a tuple $\bar{\chi}$. Let h_j and r_{ij} be as in Lemma 4.3. Define

$$\underline{r}_j = (r_{1j} \deg(h_j), \dots, r_{nj} \deg(h_j)),$$

for $j = 0, \dots, N$. Consider the submodules $J(h_j, \underline{r}_j)$ of J , where $J(h_j, \underline{r}_j)_i$ is the subspace in $c_\infty \otimes (\bigoplus_{j=1}^n V_j)$ generated by all solutions of the equation $(h_j(\delta))^{\max_i r_{ij}} \bar{g} = 0$. It is not hard to see that these submodules are finite dimensional.

Lemma 4.4. *For $M \in \pi^{-1}(\bar{\chi})$, the image $\Phi_{(M,f)}(M)$ lies in $\bigoplus_{j=0}^N J(h_j, \underline{r}_j)$.*

Proof. As was shown before, each vector function in $(\Phi_{(M,f)}(M))_i$ satisfies the equation $\chi_i(\delta) \cdot \bar{g} = 0$, where χ_i is a characteristic polynomial of τ_i as an operator in M_i . If $\chi_i(x) = h_0(x)^{r_{i0}} h_1(x)^{r_{i1}} \dots h_N(x)^{r_{iN}}$, then clearly $(\Phi_{(M,f)}(M))_i \subseteq J(h_0, \underline{r}_0)_i \oplus J(h_1, \underline{r}_1)_i \oplus \dots \oplus J(h_N, \underline{r}_N)_i$. This yields the required inclusion. $\square$

Now let $W = \bigoplus_{j=0}^N J(h_j, \underline{r}_j)$ and $\Phi_{(M,f)}^j$ be $\Phi_{(M,f)}$, followed by the projection on $J(h_j, \underline{r}_j)$ along $\bigoplus_{p \neq j} J(h_p, \underline{r}_p)$.

Lemma 4.5. *A stable pair (M, f) is in $\pi_{GL(\alpha)}^{-1}(\bar{\chi})$ if and only if the dimension vector of each $\Phi_{(M,f)}^j(M) \subseteq J(h_j, \underline{r}_j)$, $j = 0, \dots, N$, equals $\underline{r}_{ij}$.*

Proof. The “only if” part is straightforward. Conversely, if (M, f) is a stable pair, we have $\Phi_{(M,f)}^j(M) = \bigoplus_{j=0}^N W(j)$, where $W(j) = W \cap J(h_j, \underline{r}_j)$, so that $W(j)_i = W_i^{h_j}$. Since $\Phi_{(M,f)}$ is a $\mathbb{k}Q$ -homomorphism, $\dim M_i^{h_j} = \dim W_i^{h_j} = r_{ij}$. This implies that the characteristic polynomial of τ_i acting on M_i is $\prod_{j=0}^N h_j(x)^{r_{ij}} = \chi_i(x)$. So, $(M, f) \in \pi_{GL(\alpha)}^{-1}(\bar{\chi})$. $\square$

This lemma ensures that $\Phi : \text{Rep}^s(Q, \alpha, \zeta) \rightarrow \text{IHom}_{\alpha}^{\mathbb{k}Q}(J), (M, f) \mapsto \Phi_{(M, f)}$ restricted to $\pi^{-1}(\overline{\chi})$ induces a bijection

$$\text{Rep}^s(Q, \alpha, \zeta) \cap \pi^{-1}(\overline{\chi}) \rightarrow \prod_{j=0}^N \text{IHom}_{r_j}^{\mathbb{k}Q}(J(h_j, \underline{r}_j)).$$

As $J(h_j, \underline{r}_j)$ are finite dimensional, we can use Lemma 3.6 to prove that $\pi^{-1}(\overline{\chi})//GL(\alpha)$ is isomorphic (this time as an algebraic variety) to $\prod_{j=0}^N \text{Gr}_{r_j}^{\mathbb{k}Q}(J(h_j, \underline{r}_j))$.

The last ingredient we need is

Lemma 4.6. *For $j > 0$ we have $\text{Gr}_{r_j}^{\mathbb{k}Q}(J(h_j, \underline{r}_j)) \cong \text{Gr}_{r_{1j} \deg(h_j)}^{\mathbb{k}A_0^{(1)}}(J(h_j, r_{1j} \deg(h_j)))$.*

Proof. As a vector space we may identify $J(h_j, r_{1j} \deg(h_j))$ with $J(h_j, \underline{r}_j)_1$. Moreover, the $\mathbb{k}Q$ -module structure on $J(h_j, r_{1j})$ induces a $\mathbb{k}A_0^{(1)}$ -module structure on $J(h_j, \underline{r}_j)_1$. Namely, the loop acts as τ_1 . There is an obvious morphism $\text{Gr}_{r_j}^{\mathbb{k}Q}(J(h_j, \underline{r}_j)) \rightarrow \text{Gr}_{r_{1j} \deg(h_j)}^{\mathbb{k}A_0^{(1)}}(J(h_j, r_{1j} \deg(h_j)))$, $(U_i)_{i=1}^n \mapsto U_1$. Its inverse may be defined as

$$U \mapsto (a_{i-1} \dots a_1(U))_{i=1}^n.$$

□

Collecting the results obtained, we can state the following.

Theorem 4.7. (1) *Points of the quotient $\mathcal{M}^s(Q, \alpha, \zeta)$ are in one-to-one correspondence with points of the Grassmannian of submodules $\text{Gr}_{\alpha}^{\mathbb{k}Q}(J)$.*
 (2) *Let $\overline{\chi} = (\chi_1, \dots, \chi_n)$ be an admissible tuple of polynomials, $h_0 = x, h_1, \dots, h_N$ be all different irreducible factors of $\chi_1, \dots, \chi_n$, and r_{ij} be the multiplicity of h_j as a factor of χ_i . Then*

$$\pi^{-1}(\overline{\chi})//GL(\alpha) \cong \text{Gr}_{r_0}^{\mathbb{k}Q}(J(h_0, \underline{r}_0)) \times \prod_{j=1}^N \text{Gr}_{r_{1j} \deg(h_j)}^{\mathbb{k}A_0^{(1)}}(J(h_j, r_{1j} \deg(h_j))),$$

where $\underline{r}_0 = (r_{10}, \dots, r_{n0})$.

If $\mathbb{k}$ is one of the fields $\mathbb{C}$ or $\mathbb{R}$, the modules $J(h_j, \underline{r}_j)$ allow the following convenient interpretation. Consider the space $J^! = \bigoplus_{i=1}^n J_i^!$ with $J_i^! = (\mathbb{k}[[t]] \otimes V_1) \oplus (\mathbb{k}[[t]] \otimes V_2) \oplus \dots \oplus (\mathbb{k}[[t]] \otimes V_n)$, for $i = 1, \dots, n$, with the following $\mathbb{k}Q$ -module structure: $a_i \cdot (G_1, \dots, G_n) = (G_1, \dots, G_{i-1}, \frac{d}{dt}G_i, G_{i+1}, \dots, G_n)$ (elements of $\mathbb{k}[[t]] \otimes V_i$ can be regarded as vectors $G = (g_1, \dots, g_{\zeta_i})$, $g_j \in \mathbb{k}[[t]]$; in this situation $\frac{d}{dt}$ is a componentwise formal differentiation). Further, define a map $T = (T_i)_{i=1}^n : J \rightarrow J^!$ by

$$\begin{aligned} T_i : J_i &= (c_{\infty} \otimes V_i) \oplus (c_{\infty} \otimes V_{i+1}) \oplus (c_{\infty} \otimes V_{i+2}) \oplus \dots \oplus (c_{\infty} \otimes V_{i-1}) \\ \rightarrow J_i^! &= (\mathbb{k}[[t]] \otimes V_i) \oplus (\mathbb{k}[[t]] \otimes V_{i+1}) \oplus (\mathbb{k}[[t]] \otimes V_{i+2}) \oplus \dots \oplus (\mathbb{k}[[t]] \otimes V_{i-1}). \end{aligned}$$

This map is componentwise, and its components $T_{ik} : (c_{\infty} \otimes V_k) \rightarrow (\mathbb{k}[[t]] \otimes V_k)$ are

$$T_{ij}(\overline{c}_0, \overline{c}_1, \dots, \overline{c}_k, \dots) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \overline{c}_k.$$

It is easy to check that this map is a $\mathbb{k}Q$ -isomorphism.

Furthermore, we consider the map $\Phi^!_{(M,f)} = (\varphi^!_i)_{i=1}^n := T \circ \Phi_{M,f}$. It is not hard to see that

$$\varphi^!_i(m) = (f_i(\exp(\tau_i t)m), f_{i+1}(\exp(\tau_{i+1} t)a_i m), f_{i+2}(\exp(\tau_{i+2} t)a_{i+1} a_i m), \dots, f_{i-1}(\exp(\tau_{i-1} t)a_{i-2} \dots a_{i+1} a_i m)).$$

This map obviously enjoys the properties of its precursor. Now denote by $\mathcal{A}[t]$ the subspace in $\mathbb{k}[[t]]$ consisting of power series converging for all t . For $\mathbb{k} = \mathbb{C}$, this conditions means that the series is a Taylor series of a holomorphic function, and hence the uniqueness theorem implies that $\mathcal{A}[t]$ is the ring of entire functions $\mathcal{O}(\mathbb{C})$. For $\mathbb{k} = \mathbb{R}$ it may also be viewed as a space of functions that coincide with their Taylor series in 0 in any point. So, we replace $J^!$ by $J^{!!}$ with $J^{!!}_i = (\mathcal{A}[t] \otimes V_i) \oplus (\mathcal{A}[t] \otimes V_{i+1}) \oplus (\mathcal{A}[t] \otimes V_{i+2}) \oplus \dots \oplus (\mathcal{A}[t] \otimes V_{i-1})$ and the same action of ${}^{\mathbb{k}}Q$. It is again possible to prove that there is a bijection between $\mathcal{M}(Q, \alpha, \zeta)$ and $\text{Gr}^{{}^{\mathbb{k}}Q}_{\alpha}(J^{!!})$. Moreover, $J(h_j, \underline{r}_j)_i$ is now the subspace in $\mathcal{A}[t] \otimes (\bigoplus_{j=1}^n V_j)$ generated by all solutions of the differential equation $h_j(\frac{d}{dt})^{\max_i r_{ij}} \cdot \bar{g} = 0$. For $\mathbb{k} = \mathbb{C}$ all the possible $h_j(x)$ are of the form $(x - \lambda_j)$, for some $\lambda_j \in \mathbb{C}$, and for $\mathbb{k} = \mathbb{R}$ polynomials of degree 2 with negative discriminant may also occur. In both cases one can easily write a basis in $J(h_j, \underline{r}_j)_i$.

5. AN EXPLICIT REALIZATION OF FIBERS

In this section the ground field $\mathbb{k}$ is either $\mathbb{C}$ or $\mathbb{R}$. This allows us to use the language of functions introduced at the end of Section 4.

Let Q be a Jordan quiver consisting of a single vertex and a single loop (both the loop and the corresponding operator in a representation will be denoted by a). Also set $\alpha = (m)$, $\zeta = (q)$. This is the case when our construction becomes as clear as possible.

It is evident that the standard categorical quotient for the action $GL(m) : \text{Rep}(Q, m)$ is isomorphic to $\mathbb{A}^m$: points of the quotient are tuples of characteristic polynomial coefficients of the operator corresponding to the arrow a . Having this in mind we will further assume that $\mathcal{M}(Q, m)$ is embedded in $\mathbb{k}[t]_m$.

We have $J^! = J^!_1 = \mathcal{A} \otimes V_1$, and the map $\Phi^!_{(M,f)}$ becomes $\varphi(m) = f_1(\exp(at)m)$. Further, for a subspace $U \in \text{Gr}_m(J^!)$ the equivalence $U \in \text{Im}(\Phi^!) \Leftrightarrow \frac{d}{dt}(U) \subseteq U$ holds. Thus, the fiber over $\chi \in \mathbb{k}[x]$ is precisely $\text{Gr}^{\frac{d}{dt}}_m(J^!_{\chi})$ (the Grassmannian of m -dimensional $\frac{d}{dt}$ -invariant subspaces in $J^!_{\chi}$).

Imagine $q = 1$. We then have ordinary functions instead of vector ones, and the dimension of the solution space of the differential equation $\chi(\frac{d}{dt})g = 0$ equals m . So each Grassmannian $\text{Gr}^{\frac{d}{dt}}_m(J^!_{\chi})$ consists in this case of a single subspace. In other words, in each fiber of the projection $\pi_s : \mathcal{M}^s(Q, m, 1) \rightarrow \mathcal{M}(Q, m, 1)$ there is at most one point. Such a result is rather upsetting, though the situation will be more favorable for $q > 1$. We may even guarantee that for $q = m$ each ${}^{\mathbb{k}}Q$ -module M arises as a member of a stable pair $(M, f) \in \text{Rep}^s(Q, m, q)$ (it is, for instance, (M, id)).

Proposition 5.1. *For all $\lambda \in \mathbb{k}$ we have $\text{Gr}^{\frac{d}{dt}}_m(J(x - \lambda, m)) \cong \text{Gr}^{\frac{d}{dt}}_m(J(x, m))$.*

Proof. For $\mathbb{k} = \mathbb{C}$ it may be deduced from [6, Theorem 4.1]. However, this simple statement is true over an arbitrary field. Indeed, both in $J(x - \lambda, m)$ and in $J(x, m)$ the operator $\frac{d}{dt}$ can be brought to the Jordan normal form. Moreover,

we may arrange the bases in such a way that these normal forms will only differ by diagonal coefficients. Now, consider a linear map F sending basic vectors of $J(x - \lambda, m)$ to the corresponding ones in $J(x, m)$. Clearly F is an isomorphism of linear spaces, and both it and its inverse send $\frac{d}{dt}$ -stable subspaces to $\frac{d}{dt}$ -stable subspaces. $\square$

Notice that for $\mathbb{k} = \mathbb{C}$ this together with Theorem 4.7 gives the first statement of [6, Theorem 4.1] for cyclic quivers of type $A_n^{(1)}$, i.e. the presentation recalled at the end of Section 3.

Proposition 5.2. *If $\mathbb{k} = \mathbb{R}$ and h_1, h_2 are irreducible polynomials of degree 2, then $\text{Gr}_m^{\frac{d}{dt}}(J(h_1, m)) \cong \text{Gr}_m^{\frac{d}{dt}}(J(h_2, m))$ for all even m .*

Proof. Again we can use the fact that in both $J(h_1, m)$ and $J(h_2, m)$ the operator $\frac{d}{dt}$ can be brought to an analog of Jordan normal form and establish an isomorphism that preserves $\frac{d}{dt}$ -stable subspaces. $\square$

By Theorem 4.7, to describe the fibers of π_s it is sufficient to tackle with Grassmannians $\text{Gr}_{r \deg(h)}^{\frac{d}{dt}}(J(h, r \deg(h)))$, where h is irreducible over $\mathbb{k}$.

Theorem 5.3. *Let m, q be positive integers.*

- (a) *If $m = 1$ and $\lambda \in \mathbb{k}$, then $\text{Gr}_m^{\frac{d}{dt}}(J(x - \lambda, m)) \cong \mathbb{P}^{q-1}$.*
- (b) *If $m > 1$ and $\lambda \in \mathbb{k}$, then $\text{Gr}_m^{\frac{d}{dt}}(J(x - \lambda, m))$ is isomorphic to the subvariety in $\mathbb{P}(\bigwedge^m \mathbb{k}^{qm})$ given by the following equations:*
 - (i) *The Plücker relations.*
 - (ii) *For each tuple $0 \leq i_1 < \dots < i_m$:*

$$\sum_{\substack{\epsilon_1, \dots, \epsilon_m \in \{0; 1\}^m \\ \epsilon_1^2 + \dots + \epsilon_m^2 \neq 0}} p_{i_1 + q\epsilon_1, \dots, i_m + q\epsilon_m} = 0.$$

- (c) *For $\mathbb{k} = \mathbb{R}$ and h an irreducible degree 2 polynomial we number coordinates in $\mathbb{k}^{2qm}$ by residue classes modulo $2mq$. Then $\text{Gr}_{2m}^{\frac{d}{dt}}(J(h, 2m))$ is isomorphic to the subvariety in $\mathbb{P}(\bigwedge^{2m} \mathbb{k}^{2qm})$ given by the following equations:*
 - (i) *The Plücker relations.*
 - (ii) *For each tuple $0 \leq i_1 < \dots < i_{2m}$:*

$$p_{ql_1 + \nu_1, \dots, ql_{2m} + \nu_m} = \sum_{(\epsilon, \delta) \in \Delta} (-1)^{\sum_{i: \delta_i \neq 0} \theta(l_i)} p_{q(l_1 + \epsilon_1 + m\delta_1) + \nu_1, \dots, q(l_{2m} + \epsilon_{2m} + m\delta_{2m}) + \nu_m},$$

where

$$\theta(l) = \begin{cases} 1, & \text{if } m \leq l < 2m, \\ 0, & \text{otherwise,} \end{cases}$$

and Δ is the set of the pair of tuples

$$((\epsilon_1, \dots, \epsilon_{2m}), (\delta_1, \dots, \delta_{2m})) \in \{0; 1\}^{2m} \times \{0; 1\}^{2m}$$

satisfying conditions $\sum_i \epsilon_i^2 + \sum_i \delta_i^2 \neq 0$ and $\epsilon_i + \delta_i = 1$, for all i .

Proof. (a) For $m = 1$, we have $\chi(x) = (x - \lambda)$, where $\lambda \in \mathbb{k}$. But each one-dimensional $\frac{d}{dt}$ -invariant subspace in $J(x - \lambda, 1)$ is generated by a vector function $\bar{g}$ satisfying $\frac{d}{dt}\bar{g} = \lambda\bar{g}$; that is, by $(\alpha_1 e^{\lambda t}, \dots, \alpha_q e^{\lambda t})$, where $\alpha_i \in \mathbb{k}$. Hence, $\text{Gr}_1^{\frac{d}{dt}}(J(x - \lambda, 1))$ is a projectivization of the linear span of such functions, i.e. it is isomorphic to $\mathbb{P}^{q-1}$.

(b) From Proposition 5.1 it follows that $\pi^{-1}(\chi)$ is in this case isomorphic to $\text{Gr}_m^{\frac{d}{dt}}(J(x - 1, m))$. So, we will be working only with the latter Grassmannian.

It is necessary to fix some notation. For $i = 1, \dots, q$, $j = 0, \dots, m - 1$, set $e_{ij} = (0, \dots, 0, \frac{1}{j!} t^j e^t, 0, \dots, 0)$, where the only nonzero component is the i -th one.

Now let $U \in \text{Gr}_m^{\frac{d}{dt}}(J(x - 1, m))$. Then $U = \text{span}_{\mathbb{k}} \left\{ \sum_{i,j} \alpha_{ij}^{(k)} e_{ij} \mid k = 1, \dots, m \right\}$.

In $\mathbb{P}(\bigwedge^m J(x - 1, m))$ to the subspace U is associated a line spanned by $\omega_U = (\sum_{i,j} \alpha_{ij}^{(1)} e_{ij}) \wedge \dots \wedge (\sum_{i,j} \alpha_{ij}^{(m)} e_{ij})$. It is easy to see that coefficients in the decomposition $\omega_U = \sum_{ql_1+\nu_1 < \dots < ql_m+\nu_m} p_{ql_1+\nu_1, \dots, ql_m+\nu_m} e_{\nu_1 l_1} \wedge \dots \wedge e_{\nu_m l_m}$ satisfy the relations (ii).

Now let $\omega_U = (\sum_{i,j} \alpha_{ij}^{(1)} e_{ij}) \wedge \dots \wedge (\sum_{i,j} \alpha_{ij}^{(m)} e_{ij}) = \sum_{L=(ql_1+\nu_1 < \dots < ql_m+\nu_m)} p_L e_L$ be a tensor corresponding to a subspace U . One can easily check that U is $\frac{d}{dt}$ -invariant if and only if ω_U is a relative $\frac{d}{dt}$ -invariant. On the other hand, $\frac{d}{dt} \cdot \omega_U = (\sum_{i,j} \alpha_{ij}^{(1)} \frac{d}{dt} e_{ij}) \wedge \dots \wedge (\sum_{i,j} \alpha_{ij}^{(m)} \frac{d}{dt} e_{ij}) = (\sum_i (\alpha_{i0}^{(1)} e_{i0} + \alpha_{i1}^{(1)} (e_{i1} + e_{i0}) + \dots + \alpha_{i,m-1}^{(1)} (e_{i,m-1} + e_{i,m-2}))) \wedge \dots \wedge (\sum_i (\alpha_{i0}^{(m)} e_{i0} + \alpha_{i1}^{(m)} (e_{i1} + e_{i0}) + \dots + \alpha_{i,m-1}^{(m)} (e_{i,m-1} + e_{i,m-2}))) = \omega_U + \Upsilon_U$. The last term denoted by Υ_U is to be investigated. Its summands are obtained when $e_{i,j-1}$ are taken instead of e_{ij} in the above wedge product. Therefore, the coefficient of $e_{\nu_1 l_1} \wedge \dots \wedge e_{\nu_m l_m}$ in this term equals

$$\sum_{\substack{\epsilon_1, \dots, \epsilon_m \in \{0;1\}^m \\ \epsilon_1^2 + \dots + \epsilon_m^2 \neq 0}} p_{ql_1+\nu_1+q\epsilon_1, \dots, ql_m+\nu_m+q\epsilon_m}.$$

But let $l_1^*, \dots, l_m^*$ be such a tuple that $p_{qr_1+\eta_1, \dots, qr_m+\eta_m} = 0$ or not defined for all vectors $(qr_1+\eta_1, \dots, qr_m+\eta_m)$ with $r_i > l_i^* \forall i$, and, moreover, $p_{qr_1+\mu_1, \dots, qr_m+\mu_m} \neq 0$ for some $\mu_1, \dots, \mu_m$. Then the coefficient $e_{\mu_1 r_1} \wedge \dots \wedge e_{\mu_m r_m}$ of $\frac{d}{dt} \cdot \omega_U$ equals $p_{qr_1+\mu_1, \dots, qr_m+\mu_m}$, and thus if ω_U is a relative $\frac{d}{dt}$ -invariant, we have $\frac{d}{dt} \cdot \omega_U = \omega_U$. Consequently, U is $\frac{d}{dt}$ -invariant if and only if Υ_U is zero. But we have already shown that, rewritten in terms of Plücker coordinates of ω_U , this condition becomes (ii).

(c) By Proposition 5.2 it is sufficient to get a realization for $\text{Gr}_{2m}^{\frac{d}{dt}}(J(x^2 + 1, 2m))$.

For $i = 1, \dots, q$, $j = 0, \dots, m - 1$, set $e_{ij} = (0, \dots, 0, \frac{1}{j!} t^j \cos t, 0, \dots, 0)$ and $e_{i,m+j} = (0, \dots, 0, \frac{1}{j!} t^j \sin t, 0, \dots, 0)$, where each time the only nonzero component is the i -th one. Now let $U \in \text{Gr}_m^{\frac{d}{dt}}(J(x^2 + 1, m))$. Then

$$U = \text{span}_{\mathbb{k}} \left\{ \sum_{i,j} \alpha_{ij}^{(k)} e_{ij} \mid k = 1, \dots, 2m \right\}.$$

In $\mathbb{P}(\bigwedge^{2m} J(x^2 + 1, m))$ to the subspace U is associated a line spanned by $\omega_U = (\sum_{i,j} \alpha_{ij}^{(1)} e_{ij}) \wedge \dots \wedge (\sum_{i,j} \alpha_{ij}^{(2m)} e_{ij})$. It is easy to see that coefficients in the decomposition $\omega_U = \sum_{ql_1+\nu_1 < \dots < ql_{2m}+\nu_{2m}} p_{ql_1+\nu_1, \dots, ql_{2m}+\nu_{2m}} e_{\nu_1 l_1} \wedge \dots \wedge e_{\nu_{2m} l_{2m}}$ satisfy the relations (ii).

Now let $\omega_U = (\sum_{i,j} \alpha_{ij}^{(1)} e_{ij}) \wedge \dots \wedge (\sum_{i,j} \alpha_{ij}^{(2m)} e_{ij}) = \sum_{L=(q l_1 + \nu_1 < \dots < q l_{2m} + \nu_{2m})} p_L e_L$ be a tensor corresponding to a subspace U . Again U is $\frac{d}{dt}$ -invariant if and only if ω_U is a relative $\frac{d}{dt}$ -invariant. On the other hand, $\frac{d}{dt} \cdot \omega_U = (\sum_{i,j} \alpha_{ij}^{(1)} \frac{d}{dt} e_{ij}) \wedge \dots \wedge (\sum_{i,j} \alpha_{ij}^{(2m)} \frac{d}{dt} e_{ij}) = (\sum_i (-\alpha_{i0}^{(1)} e_{im} + \alpha_{i1}^{(1)} (-e_{i,m+1} + e_{i0}) + \dots + \alpha_{im}^{(1)} (-e_{i,2m} + e_{i,m-1})) + \alpha_{i,m}^{(1)} e_{i0} + \alpha_{i,m+1}^{(1)} (e_{i,1} + e_{im}) + \dots + \alpha_{i,2m-1}^{(1)} (e_{i,m-1} + e_{i,2m-2})) \wedge \dots \wedge (\sum_i (-\alpha_{i0}^{(2m)} e_{im} + \alpha_{i1}^{(2m)} (-e_{i,m+1} + e_{i0}) + \dots + \alpha_{im}^{(2m)} (-e_{i,2m} + e_{i,m-1})) + \alpha_{i,m}^{(2m)} e_{i0} + \alpha_{i,m+1}^{(2m)} (e_{i,1} + e_{im}) + \dots + \alpha_{i,2m-1}^{(2m)} (e_{i,m-1} + e_{i,2m-2})).$ Computing the coefficient of $e_{\nu_1 l_1} \wedge \dots \wedge e_{\nu_{2m} l_{2m}}$ in the difference $\omega_U - \frac{d}{dt} \cdot \omega_U$, we come to the equations (ii). This completes the proof of the theorem. $\square$

Example 5.4. For $m = q = 2$ these relations are very simple. It is a straightforward computation to check that in this case, for $\mu \neq \lambda$,

$$\pi_s^{-1}((x - \lambda)(x - \mu)) \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

which is a nondegenerate quadric. When the eigenvalues coincide, the fiber is

$$\pi_s^{-1}((x - \lambda)^2) \cong \left\{ \omega = \sum_{k < l} p_{kl} e_k \wedge e_l \in \mathbb{P} \left(\bigwedge^2 \mathbb{k}^4 \right) \mid \begin{array}{l} p_{14}^2 - p_{13} p_{24} = 0, \\ p_{34} = 0, p_{14} = p_{23} \end{array} \right\},$$

i.e. a degenerate quadric. Now, if $\mathbb{k} = \mathbb{R}$ and h is an irreducible quadratic polynomial, then

$$\pi_s^{-1}(h) \cong \left\{ \omega = \sum_{k < l} p_{kl} e_k \wedge e_l \in \mathbb{P} \left(\bigwedge^2 \mathbb{k}^4 \right) \mid \begin{array}{l} p_{13}^2 + p_{14}^2 - p_{12} p_{34} = 0, \\ p_{24} = p_{13}, p_{23} = -p_{14} \end{array} \right\}.$$

This real quadric is nondegenerate, but it is not isomorphic to $\mathbb{RP}^1 \times \mathbb{RP}^1$.

Now let Q be an arbitrary quiver of type $A_n^{(1)}$ (we use the notation from the previous section for its vertices and arrows). As was shown before, the fiber of $\mathcal{M}^s(Q, \alpha, \zeta)$ over a point $\bar{\chi} = (\chi_1, \dots, \chi_n)$ of the standard categorical quotient is isomorphic to $\text{Gr}_{\mathbb{L}_0}^{\mathbb{k}Q}(J(h_0, \mathbb{L}_0)) \times \prod_{j=1}^N \text{Gr}_{r_{1j} \deg(h_j)}^{\mathbb{k}A_0^{(1)}}(J(h_j, r_{1j} \deg(h_j)))$, where $\mathbb{L}_0 = (r_{10}, \dots, r_{n0})$, where the notation is as in Theorem 4.7. For $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$ we have already described all the factors of the product except the first one. It is thus left to give a realization of the null fiber $\pi_s^{-1}(0)$.

Fix a dimension vector α . Recall that

$$\text{Gr}_{\alpha}^{\mathbb{k}Q}(J(h_0, \alpha)) = \left\{ \left(N_i \subseteq \mathcal{A}[t] \otimes \bigoplus_{j=1}^m V_j \right)_{i=1}^n \mid \begin{array}{l} \frac{d}{dt}(N_i) \subseteq N_i \\ a_i(N_i) \subseteq N_{i+1} \\ (\frac{d}{dt})^{\max_i \alpha_i} \alpha_i|_{N_i} \equiv 0 \end{array} \right\},$$

where $a_i : \mathcal{A}[t] \otimes \bigoplus_{j=1}^m V_j \rightarrow \mathcal{A}[t] \otimes \bigoplus_{j=1}^m V_j$ acts as $\frac{d}{dt}$ on $\mathcal{A}[t] \otimes V_i$ and trivially on all other components.

It is convenient to fix the following basis in all $J(h_0, \beta)_i$: $e_{qrs}^{(i)} = (0, \dots, t^s, \dots, 0) \in \mathcal{A}[t] \otimes V_q$, where the only nonzero component is the r -th one. Furthermore, in $\text{Gr}_{\alpha_i}(J(h_0, \alpha)_i)$ we will be considering Plucker's coordinates corresponding to this basis. They will be denoted by $p_{k_1, \dots, k_{\alpha_i}}^{(i)}$, where k_h are in fact triples of indices (q_h, r_h, s_h) .

Theorem 5.5. *Let Q be a cyclic quiver of type $A_n^{(1)}$. The Grassmannian $\mathrm{Gr}_\alpha^{\mathbb{k}Q}(J(h_0, \alpha))$ is isomorphic to the subvariety of $\prod_{i=1}^n \mathbb{P}(\bigwedge^{\alpha_i}(J(h_0, \alpha))_i)$ given by:*

- (a) *The equations from Theorem 5.3(b) for each component $\mathbb{P}(\bigwedge^{\alpha_i}(J(h_0, \alpha))_i)$;*
- (b) $\sum_{l=0}^{\alpha_{i+1}} (-1)^l p_{j'_1 \dots j'_{\alpha_i-1} k'_l}^{(i)} p_{k_0 \dots \widehat{k_l} \dots k_{\alpha_{i+1}}}^{(i+1)} = 0$, *for all tuples $j_1, \dots, j_{\alpha_i-1}$ (the last component in these triples has to be less than α_i) and $k_1, \dots, k_{\alpha_{i+1}}$, where $k'_l = (q, r, s + 1)$ if $k_l = (q, r, s)$ with $q = i$, and $k'_l = k_l$ otherwise.*

Proof. At first, all M_i are $\frac{d}{dt}$ -invariant, so the statement of Theorem 5.3(b) holds. In addition, there is a condition $a_i(M_i) \subseteq M_{i+1}$. Let $\omega_{M_i} \in P(\bigwedge^{\alpha_i}(J(h_0, \alpha))_i)$ and $\omega_{M_{i+1}} \in P(\bigwedge^{\alpha_{i+1}}(J(h_0, \alpha))_{i+1})$ be the tensors corresponding to M_i and M_j ; also let $\omega'_{M_i} = a_i \cdot \omega_{M_i}$. Consider the basis $\xi_{qr s}^{(i+1)}$ in $\bigwedge^{\alpha_i}(J(h_0, \alpha))_i$ that is adjoined to $e_{q, r, s}^{(i)}$. Then $a_i(M_i) \subseteq M_{i+1}$ if and only if for each tuple $j_1, \dots, j_{\alpha_i-1}$ we have $\omega'_{M_i}(\xi_{j_1}^{(i+1)}, \dots, \xi_{j_{\alpha_i-1}}^{(i+1)}, \cdot) \wedge \omega_{M_{i+1}} = 0$. But, on the other hand, $\langle a_i(e_k^{(i)}), \xi_j^{(i+1)} \rangle \neq 0$ if and only if $k = j'$ (the meaning of prime was defined in the statement of the theorem), so $\omega'_{M_i}(\xi_{j_1}^{(i+1)}, \dots, \xi_{j_{\alpha_i-1}}^{(i+1)}, \cdot) = \sum_l p_{j'_1 \dots j'_{\alpha_i-1} l'}^{(i)} e_l^{(i+1)}$. Consequently, $\omega'_{M_i}(\xi_{j_1}^{(i+1)}, \dots, \xi_{j_{\alpha_i-1}}^{(i+1)}, \cdot) \wedge \omega_{M_{i+1}} = (\sum_l p_{j'_1 \dots j'_{\alpha_i-1} l'}^{(i)} e_l^{(i+1)}) \wedge (\sum p_{h_1 \dots h_{\alpha_{i+1}}}^{(i+1)} e_{h_1}^{(i+1)} \wedge \dots \wedge e_{h_{\alpha_{i+1}}}^{(i+1)})$. Having opened the brackets, we obtain that the coefficient of $e_{k_0}^{(i+1)} \wedge \dots \wedge e_{k_{\alpha_{i+1}}}^{(i+1)}$ equals $\pm \sum_{l=0}^{\alpha_{i+1}} (-1)^l p_{j'_1 \dots j'_{\alpha_i-1} k'_l}^{(i)} p_{k_0 \dots \widehat{k_l} \dots k_{\alpha_{i+1}}}^{(i+1)}$. But as was mentioned before, the inclusion $a_i(M_i) \subseteq M_{i+1}$ is equivalent to the fact that all $\omega'_{M_i}(\xi_{j_1}^{(i+1)}, \dots, \xi_{j_{\alpha_i-1}}^{(i+1)}, \cdot) \wedge \omega_{M_{i+1}}$ are zero, which means that their coefficients are zero. Thus we come to (b). $\square$

6. QUIVERS WITH SUCCESSIVE CYCLES

A quiver Q will be called a *quiver with successive cycles* if whenever two oriented cycles in Q have a common vertex, they are both powers of a certain cycle. It is easy to see that all such quivers may be constructed through the following procedure (that justifies our choice of terminology). Take a quiver without oriented cycles and replace some of its vertices by oriented cycles so that the arrows that used to start from the replaced vertex may now start from any chosen vertex of the pasted cycle. Here is an example of a quiver obtained through such a transformation:

Using our description of the quotient for the case of one oriented cycle, we can generalize the technique we possess to such quivers. As before, we denote by π the natural map $\mathrm{Rep}^s(Q, \alpha, \zeta) \rightarrow \mathrm{Spec} \mathbb{k}[\mathrm{Rep}(Q, \alpha, \zeta)]^{GL(\alpha)}$.

Theorem 6.1. *Let Q be a quiver with successive cycles. Also let α and ζ be two dimension vectors and y be a point in $\mathrm{Spec} \mathbb{k}[\mathrm{Rep}(Q, \alpha, \zeta)]^{GL(\alpha)}$. There exists a quiver $Q^\spadesuit$, a dimension vector $\tilde{\alpha} \in (\mathbb{Z}_{\geq 0})^{Q^\spadesuit_0}$, and a finite dimensional representation $W^\spadesuit$ of $Q^\spadesuit$ such that $\pi^{-1}(y) // GL(\alpha) \cong \mathrm{Gr}_{\tilde{\alpha}}^{\mathbb{k}Q^\spadesuit}(W^\spadesuit)$.*

Proof. We begin the proof by giving a construction of a module J such that points in $\mathrm{Gr}_\alpha^{\mathbb{k}Q}(N)$ are in one-to-one correspondence with points in the orbit space $\mathcal{M}^s(Q, \alpha, \zeta)$.

Let $\widehat{Q}$ be the quiver without oriented cycles, from which Q may be obtained through the above procedure, and Ξ_i be the set of all paths in $\widehat{Q}$ starting at i . For the vertices of the oriented cycle that we place instead of the i -th vertex of $\widehat{Q}$, we set the following notation:

$$\begin{array}{ccccccc} i^{(0)} & \xrightarrow{c_{i,1}} & i^{(1)} & \xrightarrow{c_{i,2}} & i^{(2)} & \xrightarrow{c_{i,3}} & \dots & \xrightarrow{c_{i,n_i-1}} & i^{(n_i-1)} \\ & & & & & & & & \searrow \\ & & & & & & & & i^{(n_i)} \\ & & & & & & & & \swarrow \\ & & & & & & & & i^{(0)} \end{array}$$

Also let $\tau_{i,j}$ be the cycle of minimal nonzero length starting at $i^{(j)}$, if there is any, or $e_i = e_{i^{(0)}}$ otherwise.

As before, we consider the $\mathbb{k}Q$ -module J with $J_i = \prod_{\sigma: i \rightsquigarrow j} V_j^\sigma$. Every path σ in Q is of the form $\sigma = B_l \tau_{i_l, j_l}^{k_l} a_l \dots B_2 \tau_{i_2, j_2}^{k_2} a_2 B_1 \tau_{i_1, j_1}^{k_1} a_1 B_0 \tau_{i_0, j_0}^{k_0}$, where $l = l(\sigma)$ is the length of σ , a_j are arrows of $\widehat{Q}$, and B_t are segments of τ_{i_t, j_t} , that are not oriented cycles. Let $a_0 = i$ and $u_t = ha_t$. We then have that, for $r = 0, \dots, n_{u_l}$,

$$J_{i^{(r)}} \cong \prod_{\Xi_i \ni \rho = a_l \dots a_2 a_1}^{\max(n_{u_l}-1, 0)} \bigoplus_{t=0}^{\max(n_{u_l}-1, 0)} K_l[x_{\rho, l}] \otimes \dots \otimes K_0[x_{\rho, 0}] \otimes V_{u_l^{(t)}},$$

where

$$K_q[x] := \begin{cases} \mathbb{k}[[x]], & \text{if } l(\tau_{i_q, j_q}) > 0, \\ \mathbb{k}, & \text{otherwise.} \end{cases}$$

Here we use $\mathbb{k}[[x]]$ instead of c_∞ to be able to distinguish components corresponding to different cycles without using additional multiple indices. All the operations on power series are interpreted as direct translations of those on sequences. For instance, the left shift corresponds to the left shift and not to formal differentiation.

Let $c_1 = c_{u_l, 1}$ be the arrow belonging to τ_{i_l, j_l} that starts at u_l . For a pair $(M, f) \in \mathrm{Rep}(Q, \alpha, \zeta)$ the map $\Phi_{(M, f)} : M \rightarrow J^{\mathbb{N}}$ acts as

$$(9) \quad \varphi_{i^{(r)}} = \bigoplus_{\Xi_i \ni \rho = a_l \dots a_2 a_1}^{\max(n_{u_l}-1, 0)} \bigoplus_{t=0}^{\max(n_{u_l}-1, 0)} f_{i_l^{t+1}} c_t \dots c_1 \mathrm{Exp}(\tau_{i_l, j_l} x_{\rho, l}) \dots a_1 B_0 \mathrm{Exp}(\tau_{i_0, r} x_{\rho, 0}),$$

where

$$\mathrm{Exp}(\tau_{i, j} y) := \begin{cases} \mathrm{Exp}(\tau_{i, j} y), & \text{if } l(\tau_{i, j}) > 0, \\ \mathrm{id}, & \text{otherwise.} \end{cases}$$

Combining the proofs of Proposition 3.3 and Lemma 4.1 we see that the thus defined $\Phi_{(M, f)}$ enjoys its usual properties, i.e. that the statement of Lemma 4.1 holds in this situation.

Proposition 6.2. *The map $\Phi : \mathrm{Rep}^s(Q, \alpha, \zeta) \rightarrow \mathrm{IHom}_\alpha^{\mathbb{k}Q}(J)$, $(M, f) \mapsto \Phi_{(M, f)}$ is a bijection.*

Proof. Let $U \subseteq J$ be a subrepresentation with dimension vector α . We need a pair (M, f) such that $\mathrm{Im}(\Phi_{(M, f)}) = U$. Take $M = U$. The maps $f_{i^{(j)}}$ are then

reconstructed as compositions of the projections $U_{i(j)} \rightarrow \mathcal{K}_0[x_{e_i,0}] \otimes V_{i(j)}$ with an evaluation at 0. It is now only left to show that $\Phi_{(U,f)}$ is the natural inclusion $U \hookrightarrow J$.

The proof is by “downward induction over $\widehat{Q}$ ”. If $i \in \widehat{Q}$ is a sink, we apply Proposition 4.2. For an arbitrary vertex i , fix a basis $G_1, \dots, G_{d_j}$ in each $U_{i(j)}$ and observe that, as in (5),

$$(G_1(x_{e_i,0}), \dots, G_{d_j}(x_{e_i,0})) = (G_1(0), \dots, G_{d_j}(0)) \mathbf{exp}(\tau_{i,j}x_{e_i,0}),$$

where $G_p(x_{e_i,0})$ are in fact functions in $x_{\rho,t}$, for $\rho \neq e_i$. We need to prove that the right hand side of this equality coincides with (9). Recall that $\bigoplus_{a:i \rightarrow p} aB_{0,a}$, where $B_{0,a}$ is the shortest segment of $\tau_{i,j}$ linking i^j with ta , acts as an evaluation at $x_{e_i,0} = 0$. So, $(G_1(0), \dots, G_{d_j}(0))$ consists of horizontal blocks representing the bases of $aB_{0,a}(U_{ij})$, where a are arrows starting at vertices of $\tau_{i,j}$. By the induction hypothesis all these blocks are of the form (9). Thus the claim follows. $\square$

This, however, characterizes neither $\mathrm{Gr}_\alpha^{\mathbb{k}Q}(J)$ nor the orbit space. As in Section 4, we restrict our attention to the fibers of π .

Recall that the algebra $\mathbb{k}[\mathrm{Rep}(Q, \alpha, \zeta)]$ is generated by coefficients of characteristic polynomials of all oriented cycles in Q . So we can consider y as a tuple $\{\widehat{\chi}_{i,j} \mid i \in \widehat{Q}_0, j = 0, \dots, n_i\}$, where

$$\widehat{\chi}_{i,j} := \begin{cases} \text{the characteristic polynomial of } \tau_{i,j}, & \text{if } l(\tau(i,j)) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We are now ready to prove the theorem. Introduce the module W with

$$W_{i(r)} := \bigoplus_{\Xi_i \ni \rho = a_l \dots a_2 a_1} \left(\bigotimes_{t=0}^l \mathcal{K}_t[x_{\rho,t}] \otimes \left(\bigoplus_{t=0}^{\max(n_{u_l}-1,0)} V_{u_l^{(t)}} \right) \right)_{\widehat{\chi}_{i_t,j_t}(\delta_{x_{\rho,t}}), t=0,\dots,l},$$

where $j_0 = r$. It is clearly finite dimensional. We claim that for a stable pair $(M, f) \in \pi^{-1}(y)$ the image of $\varphi_{(M,f)}$ lies in W .

We prove this by “downward induction over $\widehat{Q}$ ”. Let i be a sink in $\widehat{Q}$. If this vertex is replaced by an oriented cycle, we may use the results of Section 4; otherwise the claim is trivial.

Now, consider an arbitrary vertex i . If it does not belong to an oriented cycle, recall that by Proposition 3.8 we have, for every submodule U of J , $U_{i(0)} \subseteq V_{i(0)} \oplus \bigoplus_{a:i(0) \rightarrow j(r)} U_{j(r)} \subseteq V_{i(0)} \oplus \bigoplus_{a:i(0) \rightarrow j(r)} W_{j(r)}$, but by definition of $\widehat{\chi}_{i,0}$ this clearly lies in $W_{i(0)}$. Otherwise, we use (9) with $\mathbf{exp}$ the usual matrix exponents to understand that all $\widehat{\chi}_{i_t,j_t}(\delta_{x_{\rho,t}})$ annihilate $W_{i(r)}$, for $r = 0, \dots, n_i$.

Although $\Phi : (M, f) \mapsto \Phi^{(M,f)}$ induces an inclusion $\pi_{GL(\alpha)}^{-1}(y) \cap \mathrm{Rep}^s(Q, \alpha, \zeta) \hookrightarrow \mathrm{IHom}_\alpha^{\mathbb{k}Q}(W)$, in general it is not surjective. For example, let Q be a Jordan quiver consisting of a single loop a . Take $\alpha = \zeta = 2$ and $\chi_a(x) = (x-1)(x-2)$. Then the corresponding W contains a subrepresentation U spanned by $\begin{pmatrix} e_t \\ e_0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e_t \end{pmatrix}$, which is not in $\pi_{GL(2)}^{-1}(\chi_a)$. Hence we need to refine our construction.

Let $h_0^{(i)} = x_{e_i,0}, h_1^{(i)}, \dots, h_{N_i}^{(i)}$ be all different irreducible factors of $\widehat{\chi}_{i,j}$, for $j = 1, \dots, n_i$. Also let $r_l^{(i,j)}$ be the multiplicity of $h_l^{(i)}$ as a factor of $\widehat{\chi}_{i,j}$ and $\underline{r}_l^{(i)} = (r_l^{(i,0)} \deg(h_e^{(i)}), \dots, r_l^{(i,n_i)} \deg(h_e^{(i)}))$. If $\widehat{\chi}_{i,j}$ is a zero polynomial, we set $N_i = 1$,

$h_1^{(i)} = 0$, and $r_1^{(i,0)} = \alpha_{i(j)}$. For $j = 0, \dots, n_i$, define $J(h_l^{(i)}, \underline{r}_l^{(i)})_{i(j)}$ as the subspace of $W_{i(j)}$ generated by all solutions of the equation $\left(h_l^{(i)}(\delta_{x_{e_i,0}})\right)^{\max_q r_l^{(i,q)}} \bar{g} = 0$. Note that usually these subspaces do not form a subrepresentation, as they used to in Section 4. Since $W_{i(j)} = \bigoplus_{l=1}^{N_i} \left(W_{i(j)} \cap J(h_l^{(i)}, \underline{r}_l^{(i)})_{i(j)}\right)$, there are natural projections $\mathbf{p}_l^{(i,j)} : W_{i(j)} \rightarrow W_{i(j)} \cap J(h_l^{(i)}, \underline{r}_l^{(i)})_{i(j)}$ along $\bigoplus_{l' \neq l}^{N_i} \left(W_{i(j)} \cap J(h_{l'}^{(i)}, \underline{r}_{l'}^{(i)})_{i(j)}\right)$. The same arguments as we used to prove Lemma 4.5 show that a stable pair (M, f) lies in $\pi_{GL(\alpha)}^{-1}(y)$ if and only if the dimensions of $\mathbf{p}_l^{(i,j)}(\Phi_{(M,f)}(M)_{i(j)})$ equal $r_l^{(i,j)} \deg(h_e^{(i)})$, for all i, j and l . Hence, Φ , induces an inclusion of $\pi_{GL(\alpha)}^{-1}(y) \cap \text{Rep}^s(Q, \alpha, \zeta)$ into the subvariety $\widetilde{\text{IH}}(W)$ in

$$\prod_{\substack{i^{(j)} \in \mathbb{k}Q_0, \\ l=1, \dots, N_i}} \text{IHom}_{r_l^{(i,j)}} \deg(h_e^{(i)}) \left(W_{i(j)} \cap J(h_l^{(i)}, \underline{r}_l^{(i)})_{i(j)}\right),$$

consisting of those tuples of injections whose images give a Q -subrepresentation of W . Using Lemma 3.6 one may deduce that this inclusion is in fact a $GL(\alpha)$ -invariant isomorphism of algebraic varieties. Therefore, $\pi^{-1}(y) // GL(\alpha) \cong \widetilde{\text{IH}}(W) // GL(\alpha)$. However, the latter does not yet look like a Grassmannian of submodules.

We now construct a quiver $Q^\spadesuit$, setting

$$Q_0^\spadesuit = \left\{ i^{(j,l)} \mid i^{(j)} \in Q_0, l = 1, \dots, N_i \right\}$$

and

$$\begin{aligned} Q_1^\spadesuit &= \left\{ c_{i,j,l} : i^{(j,l)} \rightarrow i^{(j+1,l)} \mid j = 0, \dots, n_i; l = 1, \dots, N \right\} \\ &\cup \left\{ a^{(l_1, l_2)} : i_1^{(j_1, l_1)} \rightarrow i_2^{(j_2, l_2)} \mid a : i_1^{j_1} \rightarrow i_2^{j_2} \right\}. \end{aligned}$$

Observe that W may be considered as a representation $W^\spadesuit$ of $Q^\spadesuit$ with $W_{i^{(j,l)}}^\spadesuit = W_{i(j)} \cap J(h_l^{(i)}, \underline{r}_l^{(i)})_{i(j)}$. The dimension vector of $W^\spadesuit$ is $\tilde{\alpha}$ with $\tilde{\alpha}_{i^{(j,l)}} = r_l^{(i,j)} \deg(h_e^{(i)})$. It is now straightforward to check that $\widetilde{\text{IH}}(W) // GL(\alpha)$ is isomorphic to the Grassmannian $\text{Gr}_{\tilde{\alpha}}^{\mathbb{k}Q^\spadesuit}(W^\spadesuit)$. This finishes the proof of Theorem 6.1. $\square$

Remark 6.3. For $Q = A_{n-1}^{(1)}$ the quiver $Q^\spadesuit$ is nonconnected. In fact, it is a union of $(N+1)$ copies of Q , where N is as in Theorem 4.7. So, a $\mathbb{k}Q^\spadesuit$ -module is an N -tuple of representations of Q . In particular, $W^\spadesuit$ splits as $\bigoplus_j J(h_j, \underline{r}_j)$. So, $\text{Gr}_{\tilde{\alpha}}^{\mathbb{k}Q^\spadesuit}(W^\spadesuit) \cong \prod_j \text{Gr}_{\underline{r}_j}^{\mathbb{k}Q}(J(h_j, \underline{r}_j))$.

Example 6.4. Let $\mathbb{k} = \mathbb{C}$ and $\widehat{Q} = 1 \xrightleftharpoons[a_1'']{a_1'} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} 4 \xrightarrow{a_4} 5$. Replace

its second and fourth vertices by loops b and c respectively, and denote the quiver obtained by Q .

Let $(\chi_b, \chi_c) \in \mathbb{k}[x]_{\alpha_2} \oplus \mathbb{k}[x]_{\alpha_4}$. Also let $\chi_b = \prod_{i=0}^{N_b} (x - \lambda_i^{(b)})^{r_i^{(b)}}$ and $\chi_c = \prod_{i=0}^{N_c} (x - \lambda_i^{(c)})^{r_i^{(c)}}$, where $\lambda_0^{(b)} = \lambda_0^{(c)} = 0$ and $\lambda_i^{(b)} \neq \lambda_j^{(b)}$, $\lambda_i^{(c)} \neq \lambda_j^{(c)}$, for $i \neq j$. Then the quiver $Q^\spadesuit$ has vertices $Q_0^\spadesuit = \{1, 2_0, 2_1, \dots, 2_{N_b}, 3, 4_0, 4_1, \dots, 4_{N_c}, 5\}$ and

arrows $a'_{1j}, a'_{1j'} : 1 \rightarrow 2_j, b_j : 2_j \rightarrow 2_j, a_{2j} : 2_j \rightarrow 3, a_{3l} : 3 \rightarrow 4_l, c_l : 4_l \rightarrow 4_l$, and $a_{4l} : 4_l \rightarrow 5$, for $j = 0, 1, \dots, N_b, l = 0, 1, \dots, N_c$. Thus, $Q^\spadesuit$ is

and $\tilde{\alpha}$ is the dimension vector with $\tilde{\alpha}_i = \alpha_i$, for $i \in \{1, 3, 5\}$, $\tilde{\alpha}_{2_j} = r_j^{(b)}$, and $\tilde{\alpha}_{4_l} = r_l^{(c)}$.

On the other side, the semisimple representation lying in the orbit closure of a representation with the above characteristic polynomials (χ_b, χ_c) is

$$T := S_1 \oplus S_3 \oplus S_5 \oplus r_0^{(b)} S_2 \oplus r_0^{(c)} S_4 \oplus \bigoplus_{i=1}^{N_b} r_i^{(b)} E_{\lambda_i^{(b)}}^{(b)} \oplus \bigoplus_{i=1}^{N_c} r_i^{(c)} E_{\lambda_i^{(c)}}^{(c)},$$

where $E_{\lambda}^{(b)}$ and $E_{\lambda}^{(c)}$ are simple representations E_{λ} of the Jordan quiver inserted at the second and the fourth vertices respectively. Then the new quiver Q_T and the dimension vector α_T from [6, Theorem 4.1] are as follows. The vertices of Q_T are indexed by pairwise nonisomorphic simple summands in T , i.e. $(Q_T)_0 = \{1, 3, 5, 2_0, 2_1, \dots, 2_{N_b}, 4_0, 4_1, \dots, 4_{N_c}\}$. Moreover, vertices i and j corresponding to simple modules S and S' in Q_T are joined by $\delta_{ij} - \langle \dim S, \dim S' \rangle_Q$ arrows $i \rightarrow j$, where $\langle \cdot, \cdot \rangle_Q$ is the Euler form of Q . Straightforward computation now shows that Q_T coincides with $Q^\spadesuit$. Components of the dimension vector α_T are by definition multiplicities of simple summands of T , so it coincides with $\tilde{\alpha}$. Moreover, the same technique as we used in the proof of Proposition 5.1 allows us to show that if $Q^\spadesuit$ is fixed, the Grassmannian $\text{Gr}_{\tilde{\alpha}}^{\mathbb{k}Q^\spadesuit}(W^\spadesuit)$ is up to isomorphism independent of particular values of λ_i . So, in the construction of $W^\spadesuit$ we may assume that all $\lambda_i = 0$. We thus obtain the first statement of [6, Theorem 4.1] for the quiver Q .

One may check that this is the general rule over $\mathbb{C}$. For any quiver Q with successive cycles we have $Q^\spadesuit = Q_T$, $\tilde{\alpha} = \alpha_T$ and $W^\spadesuit$ is the module J from Theorem 3.5 for the algebra $\mathbb{k}Q/(\mathbb{k}Q)^{(\max_i \alpha_i)}$ and a special choice of framing $\tilde{\zeta}$.

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