EXTENSIONS OF FINITE CYCLIC GROUP ACTIONS ON NON-ORIENTABLE SURFACES

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ABSTRACT. Conditions are derived for the extension of an action of a cyclic group on a compact non-orientable surface to the faithful action of some larger group on the same surface. It is shown that if such a cyclic action is realised by means of a non-maximal NEC signature, then the action always extends. The special case where the full automorphism group is cyclic of the largest possible order (for given genus) is also considered. This extends previous work by the authors for group actions on orientable surfaces. In addition, the smallest algebraic genus of a non-orientable surface on which a given cyclic group acts as the full automorphism group is determined.

INTRODUCTION

A natural extension of the definition of a compact Riemann surface, which is orientable, is to allow dianalytic transition functions — that is, functions which are either analytic or the composite of complex conjugation with an analytic function. This applies to compact non-orientable surfaces, which have been called non-orientable Riemann surfaces (by Singerman in [14]), or non-orientable Klein surfaces without boundary (using the terminology introduced by Alling and Greenleaf in [1]). Under the well-known correspondence between compact Klein surfaces and real algebraic curves explained in [1], the non-orientable surfaces we will deal with correspond to the so-called *purely imaginary real algebraic curves* — that is, real algebraic curves with no real points.

An automorphism of a non-orientable surface S is a homeomorphism $v : S \rightarrow S$ which is dianalytic in local coordinates. The group of all such homeomorphisms is called the automorphism group of S , and denoted by $\text{Aut}(S)$. A group G is said to act on S if it is a subgroup of $\text{Aut}(S)$. Here we consider the question of whether a given cyclic group acting on S is the full group of all automorphisms of S . The same question for Riemann surfaces was considered in [5].

A compact non-orientable surface S of algebraic genus $g \geq 2$ can be represented as the quotient H/Λ of the hyperbolic plane H under the action of a fixed point-free non-Euclidean crystallographic (NEC) group Λ . A finite group G then acts as a group of automorphisms of S if and only if G is isomorphic to Γ/Λ for some NEC group Γ containing Λ as a normal subgroup. If $G \neq \text{Aut}(S)$, then Γ is properly contained with finite index in some other NEC group Γ' which also normalises Λ .

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The converse holds as well, and accordingly, the above question is closely related to the finite-index extendability of NEC groups.

The extendability of Γ depends mainly on the geometry of a fundamental region for Γ . In particular, although Γ could be contained in an NEC group Γ' normalising Λ , the group Γ might be abstractly isomorphic to a *maximal* NEC group — that is, to a group which is not contained as a subgroup of finite index in any other NEC group. If this happens and $f : \Gamma \rightarrow f(\Gamma)$ is such an isomorphism, then $f(\Gamma)/f(\Lambda)$ is the group of all automorphisms of the surface $S = H/f(\Lambda)$. For some signatures, however, it can happen that every NEC group Γ with signature σ is properly contained in another NEC group Γ' with finite index, and the dimensions of the Teichmüller spaces of Γ and Γ' coincide. Such a signature σ is called *non-maximal*, and the pair (σ, σ') of signatures of Γ and Γ' is called a *normal pair* if Γ is normal in Γ' , and *non-normal* otherwise.

The same question for Fuchsian signatures was analysed by Greenberg in [9] and answered completely by Singerman in [15]. Using the list of non-maximal Fuchsian signatures produced by Singerman, the first author produced a complete list of normal pairs of NEC signatures in [3], and subsequently Estévez and Izquierdo gave the list of the non-normal ones in [8]. These lists play a key role in this paper, since it follows from the above remarks that if a finite group G can be written as Γ/Λ and the signature of Γ does not appear in either list, then G is the group of all automorphisms of some surface homeomorphic to H/Λ . If, on the other hand, the signature of Γ is non-maximal, then the action of G in any surface homeomorphic to H/Λ could possibly be extended. The main result of this paper shows that such an extension always occurs when the group G is cyclic.

The paper is organised as follows. In Section 1 we recall basic facts and introduce notation for NEC groups and non-orientable surfaces to be used in the paper. In Section 2 we analyse the 15 normal pairs of NEC signatures (from Table 1 in Section 1) in terms of the possible extension of the action of a cyclic group. The analysis yields our main result, namely Theorem 2.1. In Section 3 we describe some applications concerning non-orientable surfaces whose full automorphism group is cyclic of largest possible order. A further application is considered in Section 4, where we determine for each n the smallest algebraic genus of a non-orientable surface on which the cyclic group of order n acts as the full automorphism group of the surface. We call this the *full cross-cap genus* of the cyclic group, following the definition by May [13] of the symmetric cross-cap number as the smallest topological genus of a non-orientable surface on which the cyclic group acts effectively. We also show that for each integer $n > 1$, there exists some g_0 such that C_n is the full automorphism group of some non-orientable surface of algebraic genus g for every $g \geq g_0$. Finally, we give a table showing the spectrum of all algebraic genera of the non-orientable surfaces on which C_n acts as the full automorphism group, for $2 \leq n \leq 10$.

1. PRELIMINARIES

We will use combinatorial theory of non-Euclidean crystallographic groups (which are known as *NEC groups*, for short). We refer the reader to [6, Section 0.2] for a general account of this topic, but we briefly recall the main facts we need and fix the notation we will use here. An NEC group is a cocompact discrete subgroup of the group of orientation-preserving or -reversing isometries of the hyperbolic plane

H. The *signature* of an NEC group Γ , as introduced by Macbeath in [12], is a collection of symbols and non-negative integers, of the form

$$(1.1) \quad \sigma(\Gamma) = (\gamma; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}).$$

The integers $m_1, \dots, m_r$ are called *proper periods*; each bracket $(n_{i1}, \dots, n_{is_i})$ is a *period cycle* and the integers n_{ij} are called *link periods*. An empty set of proper periods (where $r = 0$) will be denoted by $[-]$, an empty period-cycle (where $s_i = 0$) by $(-)$, and the fact that σ has no period-cycles (that is, where $k = 0$) by $\{-\}$.

The signature of Γ gives some topological features of the projection $H \rightarrow H/\Gamma$, and also provides a presentation of Γ . The generators are as follows:

- elliptic elements x_i , for $1 \leq i \leq r$;
- reflections $c_{i0}, \dots, c_{is_i}$, for $1 \leq i \leq k$;
- orientation-preserving elements e_i , for $1 \leq i \leq k$;
- hyperbolic elements a_i and b_i , for $1 \leq i \leq \gamma$, if the sign is $+$;
- glide reflections d_i , for $1 \leq i \leq \gamma$, if the sign is $-$.

The defining relations are as follows:

- $x_i^{m_i} = 1$ for $1 \leq i \leq r$;
- $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1$ for $1 \leq j \leq s_i$, for $1 \leq i \leq k$;
- $c_{is_i} = e_i c_{i0} e_i^{-1}$ for $1 \leq i \leq k$;
- $x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_\gamma b_\gamma a_\gamma^{-1} b_\gamma^{-1} = 1$ if the sign is $+$;
- $x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_\gamma^2 = 1$ if the sign is $-$.

The orientation-preserving elements of Γ are those expressible as words (on the generators) in which the total number of occurrences of the reflections c_{ij} and glide reflections d_i is even. These elements constitute the canonical Fuchsian subgroup Γ^+ , which has index 1 or 2 in Γ , with the former case occurring precisely when Γ itself is a Fuchsian group.

The area of a fundamental region for an NEC group Γ with signature (1.1) is $2\pi\mu(\Gamma)$, given by

$$\mu(\Gamma) = \alpha\gamma + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right),$$

where $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise. If Γ' is a subgroup of finite index in Γ , then Γ' is also an NEC group, and its area is given by the *Riemann-Hurwitz formula*:

$$\mu(\Gamma') = [\Gamma : \Gamma'] \cdot \mu(\Gamma).$$

The *algebraic genus* g of a compact surface of topological genus γ with k boundary components, is defined as

$$g = \begin{cases} 2\gamma + k - 1 & \text{if the surface is orientable,} \\ \gamma + k - 1 & \text{otherwise.} \end{cases}$$

It follows from the Uniformisation Theorem that any non-orientable surface without boundary of algebraic genus $g \geq 2$ is of the form H/Λ , for some torsion-free proper NEC group Λ with signature $(g+1; -; [-]; \{-\})$; see [14]. A finite group G acts (faithfully) as a group of automorphisms of such a surface if and only if there exist an NEC group Γ and an epimorphism $\theta: \Gamma \rightarrow G$ whose kernel is Λ . Epimorphisms whose kernel is a torsion-free proper NEC group are called *smooth epimorphisms*, for short.

Non-orientability requires that the canonical Fuchsian subgroup Γ^+ is taken by the smooth epimorphism θ to G , rather than to some subgroup of index 2 in G . In particular, this implies that G is generated by elements expressible as words in which the total number of occurrences of the images of the reflections c_{ij} and glide reflections d_i is even.

The following proposition shows that there is a strong restriction in the signature of Γ when G is cyclic.

Proposition 1.1. *Let Γ be an NEC group admitting a smooth epimorphism onto a cyclic group. Then the signature of Γ has no link periods.*

Proof. Assume the contrary, namely that $\sigma(\Gamma)$ has a non-trivial link period $n_{ij} > 1$. Let c and c' be the canonical reflections associated with it, and let $\theta: \Gamma \rightarrow C_n = \langle v \mid v^n = 1 \rangle$ be a smooth epimorphism. Clearly n has to be even since c and c' have order 2, and $\ker \theta$ is torsion-free. Moreover, $\theta(c) = \theta(c') = v^{n/2}$, since this is the unique element of order 2 in C_n . But then $\theta(cc') = v^{n/2}v^{n/2} = 1$, which is a contradiction, since the elliptic element cc' (of order $n_{ij} > 1$) cannot lie in $\ker \theta$. $\square$

Proposition 1.1 rules out all of the non-normal pairs of NEC signatures, and 13 of the 36 normal pairs, namely the pairs 5, 7, 12, 13, 15, 16, 17, 18, 22, 23, 27, 31 and 33 listed in [8, Table 3]. (Incidentally, that table has typographical errors in its 3rd and 26th entries.) The pairs 24, 25, 26, 28, 32, 34, 35 and 36 are also ruled out, because in those cases, Γ is not a proper NEC group (but is a Fuchsian group). This leaves the 15 normal pairs in Table 1 to be analysed.

TABLE 1. Normal pairs of NEC signatures to be analysed.

	Signature $\sigma = \sigma(\Gamma)$	Signature $\sigma' = \sigma(\Gamma')$	$ \Gamma' : \Gamma $
Case 1	$(3; -, [-]; \{-\})$	$(0; +; [2, 2, 2]; \{(-)\})$	2
Case 2	$(2; -, [t]; \{-\})$	$(0; +; [2, 2]; \{(t)\})$	2
Case 3	$(2; -, [-]; \{(-)\})$	$(0; +; [2, 2]; \{(2, 2)\})$	2
Case 4	$(1; +; [-]; \{(-)\})$	$(0; +; [2, 2, 2]; \{(-)\})$	2
Case 5	$(1; -, [t]; \{(-)\})$	$(0; +; [2]; \{(2, 2, t)\})$	2
Case 6	$(1; -, [-]; \{(-), (-)\})$	$(0; +; [2]; \{(2, 2, 2, 2)\})$	2
Case 7	$(1; -, [t, u]; \{-\}), \max(t, u) \geq 3$	$(0; +; [2]; \{(t, u)\})$	2
Case 8	$(1; -, [t, t]; \{-\}), t \geq 3$	$(0; +; [2, t]; \{(-)\})$	2
Case 9	$(0; +; [-]; \{(-), (-), (-)\})$	$(0; +; [-]; \{(2, 2, 2, 2, 2, 2)\})$	2
Case 10	$(0; +; [t]; \{(-), (-)\})$	$(0; +; [-]; \{(2, 2, 2, 2, t)\})$	2
Case 11	$(0; +; [t, u]; \{(-)\}), \max(t, u) \geq 3$	$(0; +; [-]; \{(2, 2, t, u)\})$	2
Case 12	$(0; +; [t, t]; \{(-)\}), t \geq 3$	$(0; +; [t]; \{(2, 2)\})$	2
Case 13	$(0; +; [t, t]; \{(-)\}), t \geq 3$	$(0; +; [t, 2]; \{(-)\})$	2
Case 14	$(1; -, [t, t]; \{-\}), t \geq 3$	$(0; +; [-]; \{(2, 2, 2, t)\})$	4
Case 15	$(0; +; [t, t]; \{(-)\}), t \geq 3$	$(0; +; [-]; \{(2, 2, 2, t)\})$	4

Necessary and sufficient conditions for the existence of a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ can be found from Theorems 3.5, 3.6 and 3.7 of [4] and Theorems 3.1.3, 3.1.6 and 3.1.8 of [6]. These also yield some arithmetic conditions involving the

periods of Γ and n , which will be described in each case in the next section; for instance, in all but Cases 1, 2, 7, 8 and 14, the group Γ contains reflections, which implies that n has to be even.

2. ANALYSIS OF CASES

For each normal pair (σ, σ') in Table 1, we consider the possibility of some extension of a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ to a smooth epimorphism $\theta': \Gamma' \rightarrow G'$, where Γ is a proper NEC group with signature σ , and Γ' is an NEC group with signature σ' , such that Γ can be embedded as a subgroup of Γ' with finite index m , and G' is a finite group of order mn containing C_n as a subgroup of index m . Two embeddings $i_1, i_2: \Gamma \rightarrow \Gamma'$ are said to be *equivalent* if there exists an automorphism $\beta \in \text{Aut}(\Gamma')$ such that $\beta i_1 = i_2$. In the cases that follow, inequivalent embeddings were found and analysed with the help of the MAGMA system [2].

We will use the following standard notation: $|H:K|$ denotes the index of a subgroup K in a group H , and $[a, b]$ denotes the commutator $a^{-1}b^{-1}ab$ of any pair (a, b) of elements of a group, while a^x denotes the conjugate $x^{-1}ax$ of a by the element x (or the image of a under the automorphism x , depending on the context). Note that the index $|\Gamma':\Gamma|$ is 2 in all cases other than cases 14 and 15, where it is 4.

Case 1: $\sigma = (3; -; [-]; \{-\})$, $\sigma' = (0; +; [2, 2, 2]; \{(-)\})$. In this case Γ is generated by elements d_1, d_2 and d_3 such that $d_1^2 d_2^2 d_3^2 = 1$, while Γ' is generated by involutions x_1, x_2, x_3 and c such that $[x_1 x_2 x_3, c] = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists for all values of n . Up to equivalence there is a unique embedding of Γ as a subgroup of index 2 in Γ' , given by $d_1 = x_1 c$, $d_2 = c x_2$ and $d_3 = x_2 c x_3 x_2$; see [3, Proposition 4.8]. Conjugation by c is an involutory automorphism of the embedded copy of Γ with the property that $d_1^c = d_1^{-1}$, and $d_2^c = d_2^{-1}$, and $d_3^c = d_1^{-2} d_3^{-1} d_1^2$. Hence in any extension of $\theta: \Gamma \rightarrow C_n$ to a smooth epimorphism $\theta': \Gamma' \rightarrow G'$, with $|G':C_n| = 2$, conjugation by the image of c must invert the images of all the d_i , and so G' is the dihedral group D_n . Such an extension is always possible, so the given action of C_n is never maximal.

Case 2: $\sigma = (2; -; [t]; \{-\})$, $\sigma' = (0; +; [2, 2]; \{(t)\})$. The analysis of this case is similar to Case 1. Here Γ is generated by elements d_1, d_2 and x such that $d_1^2 d_2^2 x = 1$ and x has order t , while Γ' is generated by involutions x_1, x_2 and c such that $[x_1 x_2, c]$ has order t . An embedding of Γ into Γ' is given by taking $d_1 = x_1 c$, $d_2 = c x_2$ and $x = x_2 c x_2 x_1 c x_1$, because the latter has order t (being conjugate to $[x_1 x_2, c]$). This embedding is unique up to equivalence, since Γ' has only one subgroup of index 2 with the same abelianisation (first homology group) $\mathbb{Z} \oplus \mathbb{Z}_{2t}$ as Γ . Let us write $n = 2^e n'$ and $t = 2^{e'} t'$, where n' and t' are odd. It follows from the results of [4, 6] that there exists a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ if and only if t is a divisor of n , and $e' = e - 1$ if n is even. Conjugation by c is an involutory automorphism with the property that $d_1^c = d_1^{-1}$, and $d_2^c = d_2^{-1}$, and $x^c = d_2^2 d_1^2 = d_2^2 x^{-1} d_2^{-2}$. Hence an extension of $\theta: \Gamma \rightarrow C_n$ is always possible, to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Case 3: $\sigma = (2; -; [-]; \{(-)\})$, $\sigma' = (0; +; [2, 2]; \{(2, 2)\})$. Here Γ is generated by elements d_1, d_2 and c such that $c^2 = [d_1^2 d_2^2, c] = 1$, while Γ' is generated by involutions x_1, x_2, c_0 and c_1 such that $(c_0 c_1)^2 = (c_1 (x_1 x_2)^{-1} c_0 (x_1 x_2))^2 = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if n is even. Up to equivalence

(including the outer automorphism of Γ' that interchanges x_1 with x_2 and c_0 with c_1), there is a unique embedding of Γ into Γ' , given by $d_1 = c_1x_1$, $d_2 = x_1c_1x_2x_1$ and $c = c_0$; this is equivalent to the one given in [3, Proposition 4.4]. Conjugation by x_1 is an involutory automorphism with the property that $d_1^{x_1} = d_1^{-1}$ and $d_2^{x_1} = d_1d_2^{-1}d_1^{-1}$ while $c^{x_1} = d_1^{-1}cd_1 = d_1^{-1}c^{-1}d_1$. Hence an extension of $\theta: \Gamma \rightarrow C_n$ is always possible, to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Case 4: $\sigma = (1, +; [-]; \{(-)\})$, $\sigma' = (0; +; [2, 2, 2]; \{(-)\})$. In this case Γ is generated by elements a, b and c such that c has order 2 and $[[a, b], c] = 1$, while Γ' is generated by involutions x_1, x_2, x_3 and c_0 such that $[x_1x_2x_3, c_0] = 1$. As above, $\theta: \Gamma \rightarrow C_n$ exists if and only if n is even. Up to equivalence there is a unique embedding of Γ into Γ' , given by $a = x_2x_1$, $b = x_1x_3$ and $c = c_0$; see also [3, Proposition 4.5]. Conjugation by x_1 is an involutory automorphism with the property that $a^{x_1} = a^{-1}$ and $b^{x_1} = b^{-1}$ while $c^{x_1} = bac^{-1}b^{-1} = bac^{-1}(ba)^{-1}$. Once again it follows that an extension of $\theta: \Gamma \rightarrow C_n$ is always possible, to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Case 5: $\sigma = (1; -; [t]; \{(-)\})$, $\sigma' = (0; +; [2]; \{(2, 2, t)\})$. The analysis of this case is similar to Case 2. The group Γ is generated by elements d, x and c such that $[d^2x, c] = x^t = c^2 = 1$, while Γ' is generated by involutions x_1, c_0, c_1 and c_2 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2x_1c_0x_1)^t = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if n is divisible by both 2 and t . An embedding of Γ into Γ' is given by $d = c_0x_1$, $x = x_1c_0x_1c_2$ and $c = c_1$, and this is unique up to equivalence for all $t \geq 2$ (allowing for an outer automorphism when $t = 2$). Conjugation by c_0 is an involutory automorphism with the property that $d^{c_0} = d^{-1}$ and $x^{c_0} = d^2x^{-1}d^{-2}$ while $c^{c_0} = c = c^{-1}$, so again $\theta: \Gamma \rightarrow C_n$ always extends to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Case 6: $\sigma = (1; -; [-]; \{(-), (-)\})$, $\sigma' = (0; +; [2]; \{(2, 2, 2, 2)\})$. The analysis of this case is similar to Case 3. The group Γ is generated by elements d, e_1, e_2, c_{10} and c_{20} such that $d^2e_1e_2 = c_{10}^2 = c_{20}^2 = [e_1, c_{10}] = [e_2, c_{20}] = 1$, while Γ' is generated by involutions x, c_0, c_1, c_2 and c_3 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3xc_0x)^2 = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if n is even. Up to equivalence (including the outer automorphism of Γ' that centralises x and interchanges c_0 with c_3 and c_1 with c_2), there is a unique embedding of Γ into Γ' , given by $d = xc_0$, $e_1 = c_0c_2$, $e_2 = c_2xc_0x$, $c_{10} = c_1$ and $c_{20} = c_3$; see also [3, Proposition 4.2]. Conjugation by c_0 is an involutory automorphism with the property that $d^{c_0} = d^{-1}$ and $e_1^{c_0} = e_1^{-1}$ and $e_2^{c_0} = e_1e_2^{-1}e_1^{-1}$, while $c_{10}^{c_0} = c_{10} = c_{10}^{-1}$ and $c_{20}^{c_0} = e_1c_{20}^{-1}e_1^{-1}$. Therefore once again, an extension of $\theta: \Gamma \rightarrow C_n$ is always possible, to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Case 7: $\sigma = (1; -; [t, u]; \{-\})$, $\sigma' = (0; +; [2]; \{(t, u)\})$, $\max(t, u) \geq 3$. The analysis of this case is again similar to that in earlier cases. The group Γ is generated by elements d, x and y such that $d^2xy = x^t = y^u = 1$, while Γ' is generated by involutions x_1, c_0 and c_1 such that $(c_0c_1)^t = (c_1x_1c_0x_1)^u = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if n is the least common multiple of t and u , and n/t and n/u are both odd. Up to equivalence there is a unique embedding of Γ into Γ' , given by $d = x_1c_0$, $x = c_0c_1$ and $y = c_1x_1c_0x_1$. Conjugation by c_0 is an involutory automorphism with the property that $d^{c_0} = d^{-1}$ and $x^{c_0} = x^{-1}$ and $y^{c_0} = xy^{-1}x^{-1}$, so an extension of $\theta: \Gamma \rightarrow C_n$ is always possible to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Before proceeding with other cases, we consider what happens when C_n acts with signature $\sigma = (1; -; [t, t]; \{-\})$. According to Case 7 with $t = u$, this action always extends to one of a larger group, namely a dihedral group acting with signature $\sigma' = (0; +; [2]; \{(t, t)\})$. The possibility of other extensions with different signatures σ' will be encountered below, in Case 8 with $\sigma' = (0; +; [2, t]; \{(-)\})$, and Case 14 with $\sigma' = (0; +; [-]; \{(2, 2, 2, t)\})$. We will see the same kind of thing occurring in Cases 12, 13 and 15, compared with Case 11.

Case 8: $\sigma = (1; -; [t, t]; \{-\})$, $\sigma' = (0; +; [2, t]; \{(-)\})$, $t \geq 3$. Here the group Γ is generated by elements d, x and y such that $d^2xy = x^t = y^t = 1$, while Γ' is generated by elements x_1, x_2 and c such that $x_1^2 = x_2^t = c^2 = [x_1x_2, c] = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if $t = n$. Up to equivalence there is a unique embedding of Γ into Γ' , given by $d = x_1cx_2^{-1}$, $x = x_2$ and $y = x_1x_2x_1$. (Note: when t is odd, there are two subgroups of index 2 in Γ' with the same abelianisation $\mathbb{Z}_{2t} \oplus \mathbb{Z}_t$ as Γ , but one of them is isomorphic to the NEC group with signature $(0; +; [t, t]; \{(-)\})$, and so we must take the other one.) Conjugation by x_1 is an involutory automorphism with the property that $d^{x_1} = x^{-1}dx$ while $x^{x_1} = y$ and $y^{x_1} = x$. Hence the epimorphism $\theta: \Gamma \rightarrow C_n$ can be extended if and only if there exists an involutory automorphism of C_n that interchanges $\theta(x)$ and $\theta(y)$ while centralising $\theta(d)$.

Now this is possible in many cases. For example, if $\theta(y) = \theta(x)^{-1}$ while $\theta(d) = 1$ (or $\theta(d) = \theta(x)^{n/2}$ when n is even), then θ extends to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$. In other cases where such an automorphism exists, θ will extend to the action of a dicyclic group $C_n:C_2$ (of order $2n$). But there are also many cases where this is not possible, such as when $n > 6$ and either $(\theta(y), \theta(d)) = (\theta(x)^2, \theta(x)^{\frac{n-3}{2}})$ for n odd, or $(\theta(x)^3, \theta(x)^{\frac{n-4}{2}})$ for n even. Hence $\theta: \Gamma \rightarrow C_n$ does not always extend in this way.

Case 9: $\sigma = (0; +; [-]; \{(-), (-), (-)\})$, $\sigma' = (0; +; [-]; \{(2, 2, 2, 2, 2, 2)\})$. In this case, which is similar to Case 6, the group Γ is generated by elements e_1, e_2, e_3 and involutions c_{10}, c_{20} and c_{30} such that $e_1e_2e_3 = [e_1, c_{10}] = [e_2, c_{20}] = [e_3, c_{30}] = 1$, while Γ' is generated by involutions c_0, c_1, c_2, c_3, c_4 and c_5 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_4)^2 = (c_4c_5)^2 = (c_5c_0)^2 = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if n is even. Up to equivalence there is just one embedding of Γ into Γ' , given by $e_1 = c_0c_2$, $e_2 = c_2c_4$, $e_3 = c_4c_0$, $c_{10} = c_1$, $c_{20} = c_3$, and $c_{30} = c_5$; see also [3, Proposition 4.1]. Conjugation by c_0 is an involutory automorphism with the property that $e_1^{c_0} = e_1^{-1}$, and $e_2^{c_0} = e_1e_2^{-1}e_1^{-1}$ and $e_3^{c_0} = e_3^{-1}$, while $c_{10}^{c_0} = c_{10} = c_{10}^{-1}$, and $c_{20}^{c_0} = e_1c_{20}^{-1}e_1^{-1}$ and $c_{30}^{c_0} = c_{30} = c_{30}^{-1}$. Once more, an extension of $\theta: \Gamma \rightarrow C_n$ is always possible, to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Case 10: $\sigma = (0; +; [t]; \{(-), (-)\})$, $\sigma' = (0; +; [-]; \{(2, 2, 2, 2, t)\})$. The analysis here is again similar to earlier cases. The group Γ is generated by elements x, e_1, e_2, c_{10} and c_{20} such that $xe_1e_2 = x^t = c_{10}^2 = c_{20}^2 = [e_1, c_{10}] = [e_2, c_{20}] = 1$, while Γ' is generated by involutions c_0, c_1, c_2, c_3 and c_4 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_4)^2 = (c_4c_0)^t = 1$. As in Case 5, a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if t divides n and n is even. Up to equivalence (including an outer automorphism when $t = 2$), there is just one embedding of Γ into Γ' , given by $x = c_4c_0$, $e_1 = c_0c_2$, $e_2 = c_2c_4$, $c_{10} = c_1$ and $c_{20} = c_3$. Conjugation by c_0 is an involutory automorphism such that $x^{c_0} = x^{-1}$, and $e_1^{c_0} = e_1^{-1}$, and $e_2^{c_0} = xe_2^{-1}x^{-1}$,

while $c_{10}^{c_0} = c_{10} = c_{10}^{-1}$ and $c_{20}^{c_0} = xc_{20}^{-1}x^{-1}$. Hence $\theta: \Gamma \rightarrow C_n$ is always extendable to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Case 11: $\sigma = (0; +; [t, u]; \{(-)\})$, $\sigma' = (0; +; [-]; \{(2, 2, t, u)\})$, $\max(t, u) \geq 3$. Here the analysis is similar to Case 7. The group Γ is generated by elements x, y and c such that $x^t = y^u = c^2 = [xy, c] = 1$, while Γ' is generated by involutions c_0, c_1, c_2 and c_3 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^t = (c_3c_0)^u = 1$. Up to equivalence (including an outer automorphism when $t = 2$ or $u = 2$) there is a unique embedding of Γ into Γ' , given by $x = c_2c_3$, $y = c_3c_0$ and $c = c_1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if n is even and $n = \text{lcm}(t, u)$. Conjugation by c_0 is an involutory automorphism such that $x^{c_0} = y^{-1}x^{-1}y$ and $y^{c_0} = y^{-1}$ while $c_1^{c_0} = c_1 = c_1^{-1}$, so $\theta: \Gamma \rightarrow C_n$ is always extendable to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$.

Case 12: $\sigma = (0; +; [t, t]; \{(-)\})$, $\sigma' = (0; +; [t]; \{(2, 2)\})$, $t \geq 3$. Here the group Γ is generated by elements x, y and c such that $x^t = y^t = c^2 = [xy, c] = 1$, while Γ' is generated by elements x_1, c_0 and c_1 such that $x_1^t = c_0^2 = c_1^2 = (c_0c_1)^2 = (c_1x_1^{-1}c_0x_1)^2 = 1$. A smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if n is even and $t = n$. Up to equivalence (including an outer automorphism of Γ' that inverts x_1 and interchanges c_0 with c_1), there is a unique embedding of Γ into Γ' , given by $x = x_1^{-1}$, $y = c_0x_1c_0$ and $c = c_1$. Conjugation by c_0 is an involutory automorphism such that $x^{c_0} = y^{-1}$ (and $y^{c_0} = x^{-1}$) while $c^{c_0} = c$.

Hence the epimorphism $\theta: \Gamma \rightarrow C_n$ can be extended if and only if there exists an involutory automorphism of C_n that interchanges $\theta(x)$ and $\theta(y)^{-1}$ while centralising $\theta(c)$. As in Case 8, this can happen sometimes but not always. For example, if $\theta(y) = \theta(x)$ while $\theta(d) = 1$ or $\theta(x)^{n/2}$, then θ extends to a smooth epimorphism $\theta': \Gamma' \rightarrow D_n$. In other cases where such an automorphism exists, θ will extend to the action of a dicyclic group $C_n : C_2$ (of order $2n$). But there are clearly many other examples where this is not possible.

Case 13: $\sigma = (0; +; [t, t]; \{(-)\})$, $\sigma' = (0; +; [t, 2]; \{(-)\})$, $t \geq 3$. In this case the analysis is very similar to Case 12, except that Γ' is generated by elements x_1, x_2 and c_0 such that $x_1^t = x_2^2 = c_0^2 = [x_1x_2, c_0] = 1$. Again we require n to be even and $t = n$. Up to equivalence there is a unique embedding of Γ into Γ' , given by $x = x_1$, $y = x_2x_1x_2$ and $c = c_0$. Conjugation by x_2 is an involutory automorphism with the property that $x^{x_2} = y$ (and $y^{x_2} = x$) while $c^{x_2} = ycy^{-1}$. Hence the epimorphism $\theta: \Gamma \rightarrow C_n$ can be extended if and only if there exists an involutory automorphism of C_n that interchanges $\theta(x)$ and $\theta(y)$ while centralising $\theta(c)$. When such an extension occurs, it gives an action of the dihedral group D_n or some other dicyclic group $C_n : C_2$ (of order $2n$), but there are clearly many cases where the extension is impossible.

Case 14: $\sigma = (1; -; [t, t]; \{-\})$, $\sigma' = (0; +; [-]; \{(2, 2, 2, t)\})$, $t \geq 3$. In this case the group Γ is generated by elements d, x and y such that $d^2xy = x^t = y^t = 1$, while Γ' is generated by involutions c_0, c_1, c_2 and c_3 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^t = 1$, but (in contrast to the previous cases) the index $|\Gamma': \Gamma|$ is 4. As in Case 8, a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if $t = n$. There are three subgroups of index 4 in Γ' with the same abelianisation $\mathbb{Z}_{2t} \oplus \mathbb{Z}_t$ as Γ , but two of them contain an element of order 2 lying outside the subgroup generated by the elements of order t , and so we must take the third one. This gives an embedding of Γ into Γ' via $d = c_2c_1c_0$, $x = c_0c_3$ and $y = c_2c_3c_0c_2$, which is

unique up to equivalence. Conjugation by c_0 is an involutory automorphism with the property that $d^{c_0} = d^{-1}$ and $x^{c_0} = x^{-1}$ and $y^{c_0} = xy^{-1}x^{-1}$, and conjugation by c_2 is another involutory automorphism with the property that $d^{c_2} = d^{-1}$ and $x^{c_2} = y^{-1}$ and $y^{c_2} = x^{-1}$. (Also conjugation by $c_0c_2y^{-1}$ is the same automorphism as considered in Case 8.) Now θ can always be extended to a smooth epimorphism from $\langle \Gamma, c_0 \rangle$ to the dihedral group D_n , but the full extension to an epimorphism $\theta': \Gamma' \rightarrow C_n : (C_2 \times C_2)$ depends on the existence of an involutory automorphism of C_n interchanging $\theta(x)$ and $\theta(y)^{-1}$ while inverting $\theta(d)$. Such a full extension is possible frequently, but not always.

Case 15: $\sigma = (0; +; [t, t]; \{(-)\})$, $\sigma' = (0; +; [-]; \{(2, 2, 2, t)\})$, $t \geq 3$. Here the group Γ is generated by elements x, y and c such that $x^t = y^t = c^2 = [xy, c] = 1$, while Γ' is generated by involutions c_0, c_1, c_2 and c_3 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^2 = 1$, and again the index $|\Gamma' : \Gamma|$ is 4. As in Case 12, a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ exists if and only if n is even and $t = n$. Up to equivalence there is just one embedding of Γ into Γ' , given by $x = c_0c_3$, $y = c_2c_3c_0c_2$, and $c = c_1$. Conjugation by c_0 is an involutory automorphism such that $x^{c_0} = x^{-1}$ and $y^{c_0} = xy^{-1}x^{-1}$ while $c^{c_0} = c = c^{-1}$, and conjugation by c_2 is another involutory automorphism such that $x^{c_2} = y^{-1}$ and $y^{c_2} = x^{-1}$ while $c^{c_2} = c = c^{-1}$. Hence θ can always be extended to a smooth epimorphism from $\langle \Gamma, c_0 \rangle$ to D_n , but the full extension to an epimorphism $\theta': \Gamma' \rightarrow C_n : (C_2 \times C_2)$ depends on the existence of an involutory automorphism of C_n interchanging $\theta(x)$ and $\theta(y)^{-1}$ while centralising $\theta(c)$. Such a full extension is possible frequently, but not always.

As a consequence of the analysis of the above 15 cases we have the following.

Theorem 2.1. *Let C_n be a cyclic group acting with non-maximal signature on a non-orientable surface S . Then the action of C_n always extends to the action of a larger group on S .*

Proof. The only cases above in which the expected extension does not always occur are Cases 8 and 14 (where Γ has signature $(1; -; [t, t]; \{-\})$) and Cases 12, 13 and 15 (where Γ has signature $(0; +; [t, t]; \{(-)\})$). The actions in those cases, however, extend in the way described in Cases 7 and 11, when we take $u = t$. $\square$

Remark 2.2. Observe that in all cases, the action of C_n extends at least to an action of the dihedral group D_n on the same surface.

Remark 2.3. It is worth mentioning that Theorem 2.1 contrasts with the analogous theorem for cyclic actions on orientable surfaces. For in the orientable case, a cyclic group acting with a non-maximal Fuchsian signature does not always extend; this was shown by the first and third authors of the present paper in [5, Theorem 4.1].

3. APPLICATIONS

It has been known for some time that the maximum order of a cyclic group of automorphisms acting on a non-orientable surface of algebraic genus g is $2g + 2$ when g is even, and $2g$ when g is odd. This was shown by Wendy Hall in [10], and again by the first author of the present paper in [4, Corollary 4.4]. Moreover, if $\theta: \Gamma \rightarrow C_{2g+2}$ or C_{2g} is a smooth epimorphism realising such a maximal cyclic

action, then the signature of Γ is

- $(0; +; [2, g+1]; \{(-)\})$ if g is even, or
- $(0; +; [2, 2g]; \{(-)\})$ or $(1; -; [2, 2g]; \{-\})$ if g is odd.

This is an easy consequence of the Riemann-Hurwitz formula and the fact that there is no smooth epimorphism from an NEC group with signature $(1; -; [2, g+1]; \{-\})$ to C_{2g+2} .

It is also known that such a maximal cyclic action on a non-orientable surface always extends to a larger group of automorphisms; see [7]. We can now give a direct proof of this result, from our analysis of the cases in Section 2.

Corollary 3.1. *Let S be a compact non-orientable surface of algebraic genus $g \geq 2$ admitting an automorphism v of maximum possible order (namely, $2g+2$ if g is even, or $2g$ if g is odd). Then the full automorphism group of S properly contains $\langle v \rangle$.*

Proof. This is a consequence of the fact that such a maximal cyclic group $\langle v \rangle$ acts with signature of the form $(0; +; [t, u]; \{(-)\})$ or $(1; -; [t, u]; \{-\})$, and all such actions were shown to extend in Cases 7 and 11 of Section 2. $\square$

Next, it is a natural question to ask what is the largest order of a cyclic group that acts as the full group of automorphisms of some non-orientable surface S of given genus.

Theorem 3.2. *For every integer $g \geq 2$, the order of the largest cyclic group C_n that acts as the full group of automorphisms of a compact non-orientable surface of algebraic genus g is given in Table 2, together with all possibilities for the corresponding signature.*

TABLE 2. Largest C_n as the full automorphism group of a non-orientable surface of genus g .

Genus	Largest n	Possible signatures
$g \equiv 1 \pmod{4}$	$g+1$	$(0; +; [2, 2, \frac{g+1}{2}]; \{(-)\})$ or $(1; -; [2, 2, \frac{g+1}{2}]; \{-\})$
g even	g	$(0; +; [2, 2, g]; \{(-)\})$, or $(0; +; [2, 3, 3]; \{(-)\})$ when $g = 6$, or $(0; +; [2, 3, 4]; \{(-)\})$ when $g = 12$, or $(0; +; [2, 3, 5]; \{(-)\})$ when $g = 30$
$g \equiv 3 \pmod{4}$	$g-1$	$(0; +; [2, 2]; \{(-), (-)\})$, $(1; -; [2, 2]; \{(-)\})$ or $(2; -; [2, 2]; \{-\})$; or $(0; +; [2, 2, 2, 2]; \{(-)\})$ or $(1; -; [2, 2, 2, 2]; \{-\})$ when $g = 3$; or $(0; +; [2, 3, 6]; \{(-)\})$ or $(1; -; [2, 3, 6]; \{-\})$ when $g = 7$

Proof. First, suppose $\theta: \Gamma \rightarrow C_n = \langle v \mid v^n = 1 \rangle$ is a smooth epimorphism such that $\ker \theta$ has signature $(g+1; -; [-]; \{-\})$. Then by Proposition 1.1, we know that Γ has signature $\sigma = (\gamma; \pm; [m_1, \dots, m_r]; \{(-), \dots, (-)\})$ for a suitable choice

of non-negative integers γ , r and k and integers $m_1, \dots, m_r \geq 2$. Moreover, by the Riemann-Hurwitz formula these must satisfy

$$\alpha\gamma + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) = \frac{g-1}{n},$$

where $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise.

Now let us consider all possible signatures σ for which $n \geq g-1$. Since $1 - \frac{1}{m_i} \geq \frac{1}{2}$ for all i , we require

$$\alpha\gamma + k + \frac{r}{2} \leq 3.$$

Also for the action to be non-orientable, we require either $\gamma > 0$ or $k > 0$, with $\alpha = 1$ (for sign $-$) if $k = 0$, and of course $\alpha = 2$ (for sign $+$) if $\gamma = 0$.

A straightforward calculation shows there are 17 different types of NEC signatures that satisfy these conditions, namely the following:

$$\begin{array}{lll} (0; +; [2, 2, 2, 2]; \{(-)\}), & (1; -; [2, 2, 2, 2]; \{-\}), & (0; +; [m_1, m_2, m_3]; \{(-)\}), \\ (1; -; [m_1, m_2, m_3]; \{-\}), & (0; +; [2, 2]; \{(-), (-)\}), & (0; +; [m_1, m_2]; \{(-)\}), \\ (1; -; [2, 2]; \{(-)\}), & (1; -; [m_1, m_2]; \{-\}), & (2; -; [2, 2]; \{-\}), \\ (0; +; [m_1]; \{(-), (-)\}), & (1; -; [m_1]; \{(-)\}), & (2; -; [m_1]; \{-\}), \\ (0; +; [-]; \{(-), (-), (-)\}), & (1; -; [-]; \{(-), (-)\}), & (1; +; [-]; \{(-)\}), \\ (2; -; [-]; \{(-)\}), & (3; -; [-]; \{-\}). \end{array}$$

Among these, 10 give non-maximal NEC signatures, in which case Theorem 2.1 shows that the cyclic action is never the full group of automorphisms of the surface. We deal with the remaining 7 signature types below. Since each is a maximal NEC signature, we may choose a maximal NEC group Γ with such signature, so that if $\theta: \Gamma \rightarrow C_n$ is a smooth epimorphism with non-orientable kernel, then the maximality of Γ prevents θ from being extended to a larger group Γ' . Accordingly, this guarantees that C_n acts as the full group of automorphisms of the non-orientable surface $H/\ker \theta$.

If $\sigma = (0; +; [2, 2, 2, 2]; \{(-)\})$, then $n = g - 1$, and the group Γ is generated by elements x_1, x_2, x_3, x_4 and c such that $x_1^2 = x_2^2 = x_3^2 = x_4^2 = c^2 = [x_1 x_2 x_3 x_4, c] = 1$. Here θ must take each generator to the unique involution of C_n , and therefore $n = 2$ (so $g = 3$).

Similarly, if $\sigma = (1; -; [2, 2, 2, 2]; \{-\})$, then $n = g - 1$, and Γ is generated by elements d, x_1, x_2, x_3 and x_4 such that $x_1^2 = x_2^2 = x_3^2 = x_4^2 = d^2 x_1 x_2 x_3 x_4 = 1$. Again θ must take each x_i to the unique involution of C_n , and it also follows that the θ -image of d has order at most 2, giving $n = 2$ (and $g = 3$).

Next, if $\sigma = (0; +; [m_1, m_2, m_3]; \{(-)\})$, we have

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 2 - \frac{g-1}{n} \geq 1,$$

and hence (m_1, m_2, m_3) are the parameters of a spherical or Euclidean triangle group. Also Γ is generated by elements x_1, x_2, x_3 and c such that $x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = c^2 = [x_1 x_2 x_3, c] = 1$, and so for a non-orientable action of C_n we need $\text{lcm}(m_1, m_2, m_3) = n$. Again by the Riemann-Hurwitz formula, it follows that we

have the following possibilities:

$$\begin{aligned} [m_1, m_2, m_3] &= [3, 3, 3], \quad n = 3, \quad g = 4; & [m_1, m_2, m_3] &= [2, 4, 4], \quad n = 4, \quad g = 5; \\ [m_1, m_2, m_3] &= [2, 3, 3], \quad n = g = 6; & [m_1, m_2, m_3] &= [2, 3, 4], \quad n = g = 12; \\ [m_1, m_2, m_3] &= [2, 3, 5], \quad n = g = 30; & [m_1, m_2, m_3] &= [2, 3, 6], \quad n = 6, \quad g = 7; \\ [m_1, m_2, m_3] &= [2, 2, g], \quad n = g, \text{ even}; & [m_1, m_2, m_3] &= [2, 2, \frac{g+1}{2}], \\ & & n &= g + 1 \equiv 2 \pmod{4}. \end{aligned}$$

The first two give smaller values of n for the given g than we find in other cases, so we ignore them. The next four give sporadic examples for the cases where g is even and $g \equiv 3 \pmod{4}$, and it is an easy exercise to show that these four are all achievable. The seventh one is achievable when $\theta(x_1) = \theta(x_2) = \theta(c) = v^{\frac{g}{2}}$ and $\theta(x_3) = v$, and the eighth when $\theta(x_1) = \theta(x_2) = \theta(c) = v^{\frac{g+1}{2}}$ and $\theta(x_3) = v^2$.

The same possibilities for $[m_1, m_2, m_3]$ arise when $\sigma = (1; -; [m_1, m_2, m_3]; \{-\})$. In that case, however, Γ is generated by elements d, x_1, x_2 and x_3 such that $x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = d^2 x_1 x_2 x_3 = 1$, and so for a non-orientable action of C_n we also need the θ -image of $x_1 x_2 x_3$ to be a square in C_n . This restriction leaves the only possibilities of interest as $[m_1, m_2, m_3] = [2, 3, 6]$ and $[2, 2, \frac{g+1}{2}]$, with $n = g - 1 = 6$ and $n = g + 1$ respectively. Both are achievable — the latter when $\theta(d) = v$, $\theta(x_1) = \theta(x_2) = v^{\frac{g+1}{2}}$ and $\theta(x_3) = v^{-2}$.

If $\sigma = (0; +; [2, 2]; \{(-), (-)\})$, then $n = g - 1$, and Γ is generated by elements x_1, x_2, e_1, c_1 and c_2 such that $x_1^2 = x_2^2 = c_1^2 = c_2^2 = [e_1, c_1] = [x_1 x_2 e_1, c_2] = 1$. This is achievable for $g \equiv 3 \pmod{4}$ when $\theta(x_1) = \theta(x_2) = \theta(c_1) = \theta(c_2) = v^{\frac{g-1}{2}}$ and $\theta(e_1) = v$.

Similarly, if $\sigma = (1; -; [2, 2]; \{(-)\})$, then $n = g - 1$, and Γ is generated by elements d, x_1, x_2 and c such that $x_1^2 = x_2^2 = c^2 = [d^2 x_1 x_2, c] = 1$. This time we can take $\theta(d) = v$ and $\theta(x_1) = \theta(x_2) = \theta(c) = v^{\frac{g-1}{2}}$ whenever $g \equiv 3 \pmod{4}$.

Finally, if $\sigma = (2; -; [2, 2]; \{-\})$, then $n = g - 1$, and Γ is generated by elements d_1, d_2, x_1 and x_2 such that $x_1^2 = x_2^2 = d_1^2 d_2^2 x_1 x_2 = 1$. Here we can take $\theta(d_1) = v$, $\theta(d_2) = v^{\frac{g-3}{2}}$ and $\theta(x_1) = \theta(x_2) = v^{\frac{g-1}{2}}$ whenever $g \equiv 3 \pmod{4}$. $\square$

As a consequence of Remark 2.2 and the proof of Theorem 3.2 we also have the following.

Corollary 3.3. *If the cyclic group C_n acts faithfully on a non-orientable surface S of algebraic genus $g \geq 2$, and either $n > g + 1$ where $g \equiv 1 \pmod{4}$, or $n > g$ where g is even, or $n > g - 1$ where $g \equiv 3 \pmod{4}$, then also the dihedral group D_n acts faithfully on S .*

Next, it is well-known that every compact Riemann surface of genus g whose full automorphism group is cyclic of the largest possible order, namely $4g + 2$, is hyperelliptic (see [11]). For non-orientable surfaces, however, the situation is slightly different, as we show below.

First we recall the definition of q -hyperellipticity. A surface S of algebraic genus $g \geq 2$ is said to be q -hyperelliptic if it admits an involutory automorphism ϕ such that the quotient surface $S/\langle\phi\rangle$ has algebraic genus q . If $q = 0$, then S is hyperelliptic, while 1-hyperelliptic surfaces are usually called *elliptic-hyperelliptic*.

Proposition 3.4. *Let S be a compact non-orientable surface of algebraic genus $g \geq 2$ whose full automorphism group is cyclic of the largest possible order for its genus g (as given in Table 2), but $g \neq 3, 6, 7, 12, 30$.*

- (1) If $g \not\equiv 3 \pmod{4}$, then S is hyperelliptic.
- (2) If $g \equiv 3 \pmod{4}$, then S is elliptic-hyperelliptic but not hyperelliptic.

Proof. Let $S = H/\Lambda$ where Λ is a torsion-free proper NEC group, and $\text{Aut } S = \Gamma/\Lambda$, where Γ is an NEC group whose signature is one of those given in Table 2. Now $\text{Aut } S \cong C_n = \langle v \mid v^n = 1 \rangle$, where n is even. Let $\phi (= v^{n/2})$ be the unique involution in $\text{Aut } S$, and let $\langle \phi \rangle = \Lambda_1/\Lambda$ where Λ_1 is an NEC group contained in Γ . Then the algebraic genus of the quotient surface $S/\langle \phi \rangle = H/\Lambda_1$ can be read from the signature of Λ_1 .

Observe that $\sigma(\Lambda_1)$ has no link periods (by Proposition 1.1), and that its proper periods are all equal to 2 (since Λ is torsion-free and has index 2 in Λ_1). It follows that

$$\sigma(\Lambda_1) = (\gamma'; \pm; [2, r', 2]; \{(-), k', (-)\})$$

for non-negative integers γ', r' and k' . The algebraic genus q of $S/\langle \phi \rangle$ is then $\alpha'\gamma' + k' - 1$, where $\alpha' = 1$ or 2 according to the sign of $\sigma(\Lambda_1)$. Applying the Riemann-Hurwitz formula to $\Lambda_1 \subseteq \Gamma$, we obtain

$$(3.1) \quad q - 1 + \frac{r'}{2} = \mu(\Lambda_1) = |\Gamma : \Lambda_1| \cdot \mu(\Gamma) = \frac{n}{2} \cdot \mu(\Gamma),$$

and thus everything reduces to computing the number r' of proper periods in $\sigma(\Lambda_1)$.

To do that, we apply Theorems 2.2.3 and 2.2.4 of [6], which reveal the relationship between r' and the proper periods of Γ .

If $g \equiv 1 \pmod{4}$, then $n = g + 1$, and the proper periods of Γ are $[2, 2, \frac{g+1}{2}]$, with $\frac{g+1}{2}$ odd. Moreover, the Riemann-Hurwitz formula gives $\mu(\Gamma) = \frac{g-1}{g+1}$ for each of the two possible signatures of Γ in this case (as given in Table 2). Letting x_i (for $i = 1, 2, 3$) be the canonical elliptic generators of Γ that have these orders, we find that the orders of the cosets $\Lambda_1 x_i \in \Gamma/\Lambda_1$ are $1, 1, \frac{g+1}{2}$, respectively. A direct application of [6, Theorem 2.2.3] then shows that $r' = g + 1 = n$, and we deduce from (3.1) above that $q = 0$; in other words, S is hyperelliptic.

A similar argument also shows that if g is even, then $q = 0$, so S is hyperelliptic. On the other hand, if $g \equiv 3 \pmod{4}$, then $q = 1$, so S is elliptic-hyperelliptic. $\square$

4. THE FULL CROSS-CAP GENUS OF A CYCLIC GROUP

The *symmetric cross-cap number* of a finite group G is the minimum topological genus of any compact non-orientable surface S (with empty boundary) on which G acts effectively as a group of automorphisms; see [13] (and also [16] for the original concept). A variant of this number was determined for all cyclic G by the first author in [4], namely the smallest such genus greater than 2. For instance, if $n \equiv 2 \pmod{4}$, then this number is $n/2 + 1$, attainable via a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ where Γ has signature $(0; +; [2, n/2]; \{(-)\})$. This signature was investigated in Case 11 in Section 2, where we found that the smooth epimorphism θ always extends to a larger group action. Hence for all such n , whenever C_n acts on a non-orientable surface of the minimum genus, the full automorphism group of the surface is strictly larger than C_n . In this section we completely answer the following question: *What is the minimum algebraic genus of a non-orientable surface S on which C_n acts effectively as the full automorphism group of S ?* We call this number the *full cross-cap genus* of the group.

Suppose the cyclic group G acts on a non-orientable surface H/Λ of algebraic genus g , and let $\theta: \Gamma \rightarrow G$ be the corresponding epimorphism, with $\ker \theta = \Lambda$. Since the signature of Λ is $(g+1; -; [-]; \{-\})$, the Riemann-Hurwitz formula $\mu(\Lambda) = |G|\mu(\Gamma)$ gives

$$g = 1 + |G|\mu(\Gamma).$$

In order to find the full cross-cap genus of G , we have to minimize the area of Γ among all NEC groups admitting a smooth epimorphism onto G which cannot be extended to a larger NEC group Γ' .

For $G = C_n$, Theorem 2.1 shows that Γ cannot have any of the signatures σ occurring in the second column of Table 1. In the following lemma, we find the only signatures for which the full cross-cap genus can be attained.

Lemma 4.1. *If the full cross-cap genus of C_n is attained by means of a smooth epimorphism $\theta: \Gamma \rightarrow C_n$, then the signature of Γ is of the form*

$$(0; +; [m_1, m_2, m_3]; \{(-)\}) \text{ or } (1; -; [m_1, m_2, m_3]; \{-\}),$$

where $2 \leq m_1 \leq m_2 \leq m_3$.

Proof. We first show that the area of Γ is bounded above by

$$(4.1) \quad \mu(\Gamma) \leq 2 - \frac{2}{p} - \frac{1}{n}, \quad \text{where } p \text{ is the smallest prime divisor of } n.$$

To see this, note that for n even, we may choose a maximal NEC group Δ with signature $(0; +; [2, 2, n]; \{(-)\})$ and define $\theta: \Delta \rightarrow C_n = \langle v \rangle$ by

$$\theta(x_1) = \theta(x_2) = v^{n/2}, \quad \theta(x_3) = v, \quad \theta(e_1) = v^{-1}, \quad \theta(c_1) = v^{n/2},$$

while for n odd, we may choose a maximal NEC group Δ with signature $(1; -; [p, p, n]; \{-\})$ and define $\theta: \Delta \rightarrow C_n = \langle v \rangle$ by

$$\theta(d_1) = v^{(n-1)/2-n/p}, \quad \theta(x_1) = \theta(x_2) = v^{n/p}, \quad \theta(x_3) = v.$$

It is easy to check that in both cases, θ is a smooth epimorphism, and then by the maximality of Δ it follows that C_n acts as the full automorphism group of the surface $H/\ker \theta$. Also $\mu(\Delta) = 2 - 2/p - 1/n$ in both cases, with $p = 2$ if n is even. Thus $\mu(\Gamma) \leq \mu(\Delta) = 2 - 2/p - 1/n$, giving inequality (4.1).

Next we consider signatures σ for which C_n acts as the full group and for which the area $\mu(\Gamma)$ satisfies the above inequality. Let $\sigma = (\gamma; \pm; [m_1, \dots, m_r]; \{(-), \cdot^k, (-)\})$.

When n is even, we have $p = 2$, and then since $1/m_i \leq 1/2$ for all i , we find

$$\alpha\gamma + k - 2 + \frac{r}{2} \leq \alpha\gamma + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) = \mu(\Gamma) \leq 1 - 1/n < 1,$$

which gives $\alpha\gamma + k + \frac{r}{2} < 3$. As in the proof of Theorem 3.2, we find there are seven different types of NEC signatures satisfying this condition, namely $(0; +; [m_1, m_2, m_3]; \{(-)\})$, $(1; -; [m_1, m_2, m_3]; \{-\})$, $(0; +; [m_1, m_2]; \{(-)\})$, $(1; -; [m_1, m_2]; \{-\})$, $(0; +; [m_1]; \{(-), (-)\})$, $(1; -; [m_1]; \{(-)\})$ and $(2; -; [m_1]; \{-\})$.

The last five of these, however, give non-maximal NEC signatures, so by Theorem 2.1 we may ignore them. This leaves only the two signature types $(0; +; [m_1, m_2, m_3]; \{(-)\})$ and $(1; -; [m_1, m_2, m_3]; \{-\})$, as required.

When n is odd, the NEC group Γ cannot contain reflections, so $k = 0$, and σ is of the form $(\gamma; -; [m_1, \dots, m_r]; \{-\})$ for some $\gamma > 0$. Then since $m_i \geq p$ for all i , we have

$$2(1 - \frac{1}{p}) > 2 - \frac{2}{p} - \frac{1}{n} \geq \mu(\Gamma) = \gamma - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) \geq \gamma - 2 + r(1 - \frac{1}{p}),$$

and further, since $1 - \frac{1}{p} \geq 1 - \frac{1}{3} = \frac{2}{3}$, this gives $6 > 3\gamma + 2(r-2)$; that is, $3\gamma + 2r < 10$. The latter inequality has just four solutions for which the area $\mu(\Gamma)$ is positive, namely $(\gamma, r) = (3, 0)$, $(2, 1)$, $(1, 2)$ and $(1, 3)$. In the first three cases, we have non-maximal signatures $(3; -; [-]; \{-\})$, $(2; -; [m_1]; \{-\})$ and $(1; -; [m_1, m_2]; \{-\})$, which we can eliminate using Theorem 2.1. This leaves just the fourth case, which gives signature type $(1; -; [m_1, m_2, m_3]; \{-\})$. $\square$

Conditions for an NEC group Γ with given signature to admit a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ were given in [4]. For signature type $(0; +; [m_1, m_2, m_3]; \{(-)\})$, one requires only that n is even and $\text{lcm}(m_1, m_2, m_3) = n$; see [4, Thm. 3.5]. On the other hand, for signature type $(1; -; [m_1, m_2, m_3]; \{-\})$, the condition $\text{lcm}(m_1, m_2, m_3) = n$ is both necessary and sufficient for odd n , but an extra condition is required for even n ; see [4, Thm. 3.7] and [6, Thm. 3.1.8]. Observe, however, that these two signature types give the same area, namely $\mu(\Gamma) = 2 - (1/m_1 + 1/m_2 + 1/m_3)$. Hence all we need to do (for each n , whether even or odd) is find the maximum value of $1/m_1 + 1/m_2 + 1/m_3$ over the set

$$T(n) = \{(m_1, m_2, m_3) : 2 \leq m_1 \leq m_2 \leq m_3 \text{ and } \text{lcm}(m_1, m_2, m_3) = n\}.$$

The next lemma exhibits the triples (m_1, m_2, m_3) for which this maximum is attained.

Lemma 4.2. *Let $n = p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$ be the prime-power decomposition of n , such that $p_1 < p_2 < \dots < p_s$. Then the maximum value of $1/m_1 + 1/m_2 + 1/m_3$ over all triples (m_1, m_2, m_3) in the set $T(n)$ is attained by:*

- (a) (p_1, p_2, p_3) , when n is of the form $p_1 p_2 p_3$ with $3 < p_1 < p_2 < p_3 < \frac{p_1(p_2-1)}{p_2-p_1}$,
- (b) $(p_1, p_1, \frac{n}{p_1})$, when $s > 1$ and $e_1 = 1$ and n is not of the form in (a), and
- (c) (p_1, p_1, n) otherwise.

Proof. First observe that if a, b and c are rational numbers such that $a < c$ and $b > 1$, then $\frac{1}{ab} + \frac{1}{c} < \frac{1}{a} + \frac{1}{bc}$, since the latter is equivalent to $c + ab < bc + a$ and hence to $c - a < b(c - a)$. It follows that in order to maximize the expression $1/m_1 + 1/m_2 + 1/m_3$ over all triples $(m_1, m_2, m_3) \in T(n)$, we require m_1 and m_2 to be prime; moreover, if m_1 and m_2 are equal, then $m_1 = m_2 = p_1$, while if m_1 and m_2 are distinct, then $(m_1, m_2) = (p_1, p_2)$.

Now if $m_1 = m_2 = p_1$, then since $n = \text{lcm}(m_1, m_2, m_3) = \text{lcm}(p_1, m_3)$, we have $m_3 = n$ or n/p_1 , and hence the maximum value for all triples in this case is attained when $m_3 = n/p_1$ or n . When $m_3 = n/p_1$, we need $s > 1$ and $e_1 = 1$ (because $\text{lcm}(m_1, m_2, m_3) = n$ is divisible by $p_1^{e_1}$), and so this possibility is covered by (b). Otherwise $m_3 = n$, as in (c).

On the other hand, if $(m_1, m_2) = (p_1, p_2)$, then m_3 must be divisible by n/p_1p_2 , so $m_3 = n/p_1p_2$ or n/p_2 or n/p_1 or n . We can eliminate the last two possibilities, because in each case a larger value of $1/m_1 + 1/m_2 + 1/m_3$ is obtainable by taking $m_2 = p_1$ (and maintaining the condition $\text{lcm}(m_1, m_2, m_3) = n$).

If $(m_1, m_2, m_3) = (p_1, p_2, n/p_2)$, then $e_2 = 1$ (since $\text{lcm}(m_1, m_2, m_3) = n$ is divisible by $p_2^{e_2}$), so $n = p_1^{e_1}p_2q$ where q is coprime to p_1 and p_2 . Now if the maximum is not also attained by (p_1, p_1, n) , then

$$0 > \left(\frac{1}{p_1} + \frac{1}{p_1} + \frac{1}{n} \right) - \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{p_2}{n} \right) = \frac{n(p_2 - p_1) + p_1p_2(1 - p_2)}{np_1p_2},$$

and therefore $n(p_2 - p_1) < p_1p_2(p_2 - 1)$. Dividing by $p_1p_2(p_2 - p_1)$, we find

$$p_1^{e_1-1}q = \frac{n}{p_1p_2} < \frac{p_2 - 1}{p_2 - p_1} = 1 + \frac{p_1 - 1}{p_2 - p_1} \leq 1 + (p_1 - 1) = p_1,$$

which forces $e_1 = 1$ and $q = 1$, so that $n = p_1p_2$. But then $m_3 = n/p_2 = p_1 < p_2 = m_2$, so this case is also impossible.

Finally, if $(m_1, m_2, m_3) = (p_1, p_2, n/p_1p_2)$, then $e_1 = e_2 = 1$ (since $\text{lcm}(m_1, m_2, m_3) = n$ is divisible by $p_1^{e_1}$ and $p_2^{e_2}$), so $n = p_1p_2q$ where q is coprime to p_1 and p_2 . Moreover, $q = n/p_1p_2 = m_3 > 1$, so $s \geq 3$, and hence $q = m_3 \geq p_3 > p_2 \geq 3$. Now if the maximum is not attained by $(p_1, p_1, n/p_1)$, then

$$0 > \left(\frac{1}{p_1} + \frac{1}{p_1} + \frac{p_1}{n} \right) - \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{p_1p_2}{n} \right) = \frac{n(p_2 - p_1) + p_1^2p_2(1 - p_2)}{np_1p_2},$$

and therefore $n(p_2 - p_1) < p_1^2p_2(p_2 - 1)$. Dividing by $p_1p_2(p_2 - p_1)$, we find

$$q < \frac{p_1(p_2 - 1)}{p_2 - p_1} = p_1 + \frac{p_1(p_1 - 1)}{p_2 - p_1} \leq p_1 + p_1(p_1 - 1) = p_1^2.$$

This shows that q cannot be divisible by the square of a prime, or by the product of two primes, so q itself is a prime. Thus $n = p_1p_2p_3$, where $3 < p_3 < \frac{p_1(p_2-1)}{p_2-p_1}$, as given in (a). $\square$

By Lemma 4.1 and the Riemann-Hurwitz formula, these maxima give the full cross-cap genus of C_n for all n , as follows.

Theorem 4.3. *Let $n = p_1^{e_1}p_2^{e_2} \dots p_s^{e_s}$ be the prime-power decomposition of n , such that $p_1 < p_2 < \dots < p_s$. Then the full cross-cap genus of the cyclic group C_n is:*

- (a) $2p_1p_2p_3 - p_1p_2 - p_1p_3 - p_2p_3 + 1$ when $n = p_1p_2p_3$ with $3 < p_1 < p_2 < p_3 < \frac{p_1(p_2-1)}{p_2-p_1}$,
- (b) $2n - \frac{2n}{p_1} - p_1 + 1$ when $s > 1$ and $e_1 = 1$ and n is not of the form in (a), and
- (c) $2n - \frac{2n}{p_1}$ otherwise.

We can also easily obtain the following theorem.

Theorem 4.4. *For each integer $n > 1$, there exists some g_0 such that C_n is the full automorphism group of some non-orientable surface of algebraic genus g for every $g \geq g_0$.*

Proof. For n even, let Δ be a maximal NEC group with signature $(\gamma; -; [n, .^r., n]; \{(-)\})$ where $\gamma > 0$ and $\gamma + r > 2$, and define $\theta: \Delta \rightarrow C_n = \langle v \rangle$ by

$$\begin{aligned}\theta(d_i) &= v \quad \text{for } 1 \leq i \leq \gamma, \quad \theta(x_j) = v \quad \text{for } 1 \leq j \leq r, \\ \theta(e_1) &= v^{-(2\gamma+r)} \quad \text{and} \quad \theta(c_{10}) = v^{n/2},\end{aligned}$$

while for n odd, let Δ be a maximal NEC group with signature $(\gamma; -; [n, .^r., n]; \{-\})$ where $\gamma > 0$ and $\gamma + r > 3$, and define $\theta: \Delta \rightarrow C_n = \langle v \rangle$ by

$$\theta(d_i) = v^{\alpha_i} \quad \text{for } 1 \leq i \leq \gamma \quad \text{and} \quad \theta(x_j) = v \quad \text{for } 1 \leq j \leq r,$$

where the α_i are chosen such that twice their sum is congruent to $-r \pmod n$. (Note that this is always possible since n is odd.) It is easy to check that in both cases, θ is a smooth epimorphism, and then by the maximality of Δ it follows that C_n acts as the full automorphism group of the non-orientable surface $H/\ker \theta$.

The algebraic genus of this surface is given by

$$\begin{aligned}g &= 1 + |G|\mu(\Gamma) \\ &= \begin{cases} 1 + n(\gamma + 1 - 2 + r(1 - 1/n)) = n(\gamma - 1) + r(n - 1) + 1 & \text{for } n \text{ even,} \\ 1 + n(\gamma - 2 + r(1 - 1/n)) = n(\gamma - 2) + r(n - 1) + 1 & \text{for } n \text{ odd.} \end{cases}\end{aligned}$$

By allowing γ to take values from 1 to $n - 1$ and letting r be arbitrary (but non-negative and subject to $\gamma + r > 2$ or $\gamma + r > 3$ respectively), we can make this genus equal to any integer greater than $(n - 1)^2$. $\square$

Note that g_0 may be taken to be smaller than $(n - 1)^2$ when $n > 3$. Table 3 shows the exact range of genera of non-orientable surfaces on which C_n acts as the full automorphism group, for small n . We leave the verification of details as an exercise for the reader.

TABLE 3. Algebraic genera of non-orientable surfaces on which C_n acts as the full automorphism group, for $2 \leq n \leq 10$.

C_n	Coverage	Signature type	Genera
C_2	All $g \geq 2$	$(0; +; [2, r, 2]; \{(-)\})$	$g = r - 1$, for $r \geq 3$
C_3	All $g \geq 4$	$(1; -; [3, r, 3]; \{-\})$	$g = 2r - 2$ (even), for $r \geq 3$
		$(2; -; [3, r, 3]; \{-\})$	$g = 2r + 1$ (odd), for $r \geq 2$
C_4	All $g \geq 4$	$(0; +; [2, r-1, 2, 4]; \{(-)\})$	$g = 2r - 2$ (even), for $r \geq 3$
		$(0; +; [2, r-2, 2, 4, 4]; \{(-)\})$	$g = 2r - 1$ (odd), for $r \geq 3$
C_5	All $g \geq 8$	$(1; -; [5, r, 5]; \{-\})$	$g = 4r - 4 \equiv 0 \pmod{4}$, for $r \geq 3$
		$(2; -; [5, r, 5]; \{-\})$	$g = 4r + 1 \equiv 1 \pmod{4}$, for $r \geq 2$
		$(3; -; [5, r, 5]; \{-\})$	$g = 4r + 6 \equiv 2 \pmod{4}$, for $r \geq 1$
		$(4; -; [5, r, 5]; \{-\})$	$g = 4r + 11 \equiv 3 \pmod{4}$, for $r \geq 0$
C_6	All $g \geq 5$	$(0; +; [2, r-1, 2, 3]; \{(-)\})$	$g = 3r - 4 \equiv 2 \pmod{3}$, for $r \geq 3$
		$(0; +; [2, r-1, 2, 6]; \{(-)\})$	$g = 3r - 3 \equiv 0 \pmod{3}$, for $r \geq 3$
		$(0; +; [2, r-2, 2, 3, 6]; \{(-)\})$	$g = 3r - 2 \equiv 1 \pmod{3}$, for $r \geq 3$
C_7	All $g \geq 12$, $g \neq 16, 17, 23$	$(1; -; [7, r, 7]; \{-\})$	$g = 6r - 6 \equiv 0 \pmod{6}$, for $r \geq 3$
		$(2; -; [7, r, 7]; \{-\})$	$g = 6r + 1 \equiv 1 \pmod{6}$, for $r \geq 2$
		$(3; -; [7, r, 7]; \{-\})$	$g = 6r + 8 \equiv 2 \pmod{6}$, for $r \geq 1$
		$(4; -; [7, r, 7]; \{-\})$	$g = 6r + 15 \equiv 3 \pmod{6}$, for $r \geq 0$
		$(5; -; [7, r, 7]; \{-\})$	$g = 6r + 22 \equiv 4 \pmod{6}$, for $r \geq 0$
		$(6; -; [7, r, 7]; \{-\})$	$g = 6r + 29 \equiv 5 \pmod{6}$, for $r \geq 0$
C_8	All $g \geq 8$	$(0; +; [2, r-1, 2, 8]; \{(-)\})$	$g = 4r - 4 \equiv 0 \pmod{4}$, for $r \geq 3$
		$(1; -; [2, r, 2]; \{(-)\})$	$g = 4r + 1 \equiv 1 \pmod{4}$, for $r \geq 2$
		$(0; +; [2, r-2, 2, 4, 8]; \{(-)\})$	$g = 4r - 2 \equiv 2 \pmod{4}$, for $r \geq 3$
		$(1; -; [2, r-1, 2, 4]; \{(-)\})$	$g = 4r + 3 \equiv 3 \pmod{4}$, for $r \geq 2$
C_9	All $g \geq 12$	$(1; -; [3, r-1, 3, 9]; \{-\})$	$g = 6r - 6 \equiv 0 \pmod{6}$, for $r \geq 3$
		$(2; -; [3, r, 3]; \{-\})$	$g = 6r + 1 \equiv 1 \pmod{6}$, for $r \geq 2$
		$(1; -; [3, r-2, 3, 9, 9]; \{-\})$	$g = 6r - 4 \equiv 2 \pmod{6}$, for $r \geq 3$
		$(2; -; [3, r-1, 3, 9]; \{-\})$	$g = 6r + 3 \equiv 3 \pmod{6}$, for $r \geq 2$
		$(1; -; [3, r-3, 3, 9, 9, 9]; \{-\})$	$g = 6r - 2 \equiv 4 \pmod{6}$, for $r \geq 3$
		$(2; -; [3, r-2, 3, 9, 9]; \{-\})$	$g = 6r + 5 \equiv 5 \pmod{6}$, for $r \geq 2$
C_{10}	All $g \geq 9$	$(0; +; [2, r-1, 2, 5]; \{(-)\})$	$g = 5r - 6 \equiv 4 \pmod{5}$, for $r \geq 3$
		$(0; +; [2, r-1, 2, 10]; \{(-)\})$	$g = 5r - 5 \equiv 0 \pmod{5}$, for $r \geq 3$
		$(1; -; [2, r, 2]; \{(-)\})$	$g = 5r + 1 \equiv 1 \pmod{5}$, for $r \geq 2$
		$(0; +; [2, r-2, 2, 5, 5]; \{(-)\})$	$g = 5r - 3 \equiv 2 \pmod{5}$, for $r \geq 3$
		$(0; +; [2, r-2, 2, 5, 10]; \{(-)\})$	$g = 5r - 2 \equiv 3 \pmod{5}$, for $r \geq 3$

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