LOWER BOUNDS OF GROWTH OF HOPF ALGEBRAS

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ABSTRACT. Some lower bounds of GK-dimension of Hopf algebras are given.

0. INTRODUCTION

A seminal result of Gromov states that a finitely generated group has polynomial growth or, equivalently, the associated group algebra has finite Gelfand-Kirillov dimension if and only if it has a nilpotent subgroup of finite index [Gr]. Group algebras form a special class of cocommutative Hopf algebras. It is natural to ask

Question 0.1. What are necessary and sufficient conditions on a finitely generated Hopf algebra H such that its Gelfand-Kirillov dimension is finite?

Let k be a base field and everything be over k . Assume that, for simplicity, k is algebraically closed of characteristic zero. It is clear that an affine (i.e., finitely generated) commutative Hopf algebra has a finite GK-dimension (short for Gelfand-Kirillov dimension) which equals its Krull dimension. If H is cocommutative, by a classification result [Mo, Corollary 5.6.4 and Theorem 5.6.5], it is isomorphic to a smash product $U(\mathfrak{g}) \# kG$ for some group G and some Lie algebra $\mathfrak{g}$. Consequently,

$$(I0.1.1) \quad \text{GKdim } H = \text{GKdim } kG + \dim \mathfrak{g},$$

which solves Question 0.1 in terms of conditions on G and $\mathfrak{g}$. Question 0.1 is also answered for several classes of noncommutative and noncocommutative Hopf algebras, including quantum groups $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$; see [BG2, GZ1]. The present paper attempts to study Question 0.1 for a larger class of noncommutative and noncocommutative Hopf algebras by providing three lower bounds of GK-dimension in terms of certain invariants of skew primitive elements.

Let H be a Hopf algebra over k . A nonzero element $y \in H$ is called $(1, g)$ -primitive (or generally skew primitive) if $\Delta(y) = y \otimes 1 + g \otimes y$, and such a g is called the *weight* of y and denoted by $\mu(y)$. Let $G(H)$ denote the group of group-like elements in H and let $C_0 = kG(H)$. Here is the first lower bound theorem.

Theorem 0.2 (First lower bound theorem). *Let $D \supseteq C_0$ be a Hopf subalgebra of H . Let $\{y_i\}_{i=1}^w$ be a set of skew primitive elements such that*

- (a) $\{y_i\}_{i=1}^w$ is linearly independent in H/D ,
- (b) for all $i \leq j$, $y_i \mu(y_j) = \lambda_{ij} \mu(y_j) y_i$ for some $\lambda_{ij} \in k^\times$,
- (c) for each i , λ_{ii} is either 1 or not a root of unity.

Then $\text{GKdim } H \geq \text{GKdim } D + w$.

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In general λ_{ij} in condition (b) may not exist. If that is the case, we have other ways of obtaining lower bounds.

Let W denote the set of weights $\mu(y)$ for all skew primitive elements $y \notin C_0$ and let $W_{\sqrt{}} be the subset of W consisting of weights $\mu(y)$ for all y such that y^n is also a skew primitive for some $n > 1$. (Note that in this paper the term “skew primitive” means “(1, g)-primitive”.) For any subset $\Phi \subset G(H)$, the subgroup of $G(H)$ generated by Φ is denoted by $\langle \Phi \rangle$. Here is the second lower bound theorem.$

Theorem 0.3 (Second lower bound theorem). *Suppose $\langle W \setminus W_{\sqrt{}} \rangle$ is abelian. Then*

$$(I0.3.1) \quad \text{GKdim } H \geq \text{GKdim } C_0 + \#(W \setminus W_{\sqrt{}}).$$

There are examples such that $W = W_{\sqrt{}}$ and $\text{GKdim } H = \text{GKdim } C_0$, but $\#(W_{\sqrt{}})$ is arbitrarily large [Example 2.7]. Therefore $W_{\sqrt{}}$ has to be removed from W when we estimate the GK-dimension of H .

Let y be a skew primitive element not in C_0 . If

$$(I0.3.2) \quad \mu(y)^{-1}y\mu(y) - cy \in C_0$$

for some $c \in k^\times$, then c is called the *commutator* of y (with its weight) and is denoted by $\gamma(y)$. By Lemma 1.6, (I0.3.2) is equivalent to

$$(I0.3.3) \quad \mu(y)^{-1}y\mu(y) - cy = \tau(\mu(y) - 1)$$

for some $\tau \in k$. Define Γ to be the set of $\gamma(y)$ for all skew primitive elements $y \notin C_0$ such that $\gamma(y)$ exists, and let $\Gamma_{\sqrt{}}$ be the subset of Γ consisting of those $\gamma(y)$ which are roots of unity but not 1. If $\gamma(y)$ exists, the pair $(\mu(y), \gamma(y))$ is denoted by $\omega(y)$ and is called the *weight commutator* of y . When (I0.3.3) holds and if $c \neq 1$, y can be replaced by $z := y + (c - 1)^{-1}\tau(\mu(y) - 1)$, which is a skew primitive element with $\omega(z) = \omega(y)$ and satisfies the equation $\mu(z)^{-1}z\mu(z) - \gamma(z)z = 0$.

Define Ω to be the set of $\omega(y)$ for all skew primitive elements $y \notin C_0$ such that $\omega(y)$ exists, and let $\Omega_{\sqrt{}}$ be the subset of Ω consisting of those $\omega(y)$ in which $\gamma(y)$ is a root of unity but not 1. Theorem 0.3 can be improved a little under the same hypothesis:

$$\text{GKdim } H \geq \text{GKdim } C_0 + \#(\Omega \setminus \Omega_{\sqrt{}}).$$

Let y be a skew primitive element not in C_0 with $g = \mu(y)$. Let $T_{g^{-1}}$ be the inverse conjugation by g , namely, $T_{g^{-1}} : a \rightarrow g^{-1}ag$. A scalar c is called a *commutator of y of level n* if n is the least nonnegative integer such that

$$(I0.3.4) \quad (T_{g^{-1}} - cId_H)^n(y) \in C_0.$$

In this case we also write $\gamma(y) = c$. Let Z denote the space spanned by the identity element 1 and all skew primitive elements of H , and let $Y_{\sqrt{}}$ denote the subspace of Z spanned by those y with commutator of finite level and with $\gamma(y)$ being a root of unity but not 1. Here is the third lower bound theorem. Let $W_\times$ be the subset of W consisting of weights $\mu(y)$ such that the commutator of y (as defined in (I0.3.4)) exists and is either 1 or not a root of unity. Note that $W \setminus W_{\sqrt{}} \subseteq W_\times$, and these are often equal [Remark 3.9].

Theorem 0.4 (Third lower bound theorem). *Suppose $\langle W_\times \rangle$ is abelian. Then*

$$(I0.4.1) \quad \text{GKdim } H \geq \text{GKdim } C_0 + \dim Z / (C_0 + Y_{\sqrt{}}).$$

When H is cocommutative, equality holds in Theorem 0.4; see (I0.1.1). There are examples such that $Z = Y_{\sqrt{}} + C_0$ and $\text{GKdim } H = \text{GKdim } C_0$, but $\dim Y_{\sqrt{}}$ is arbitrarily large [Examples 2.7 and 3.13]. Therefore it is sensible to consider the quotient space $Z/(C_0 + Y_{\sqrt{}})$ in the above theorem. This is analogous to removing $W_{\sqrt{}}$ in Theorem 0.3.

If $\langle W_{\times} \rangle$ is abelian, Theorem 0.4 is a generalization of Theorem 0.3 [Lemma 3.12]. After some analysis, Theorem 0.2 (when $D = C_0$) can be viewed as a consequence of Theorem 0.4. These lower bounds provide some evidence that the GK-dimension of H is related to some combinatorial data coming from the skew primitive elements when H is pointed.

The proof of these lower bounds is based on a version of the Poincaré-Birkhoff-Witt (PBW) theorem [Theorem 1.5(b)] which states that under some hypotheses the set of monomials generated by skew primitive elements is linearly independent (over the Hopf subalgebra C_0). Restricted to the universal enveloping algebra of a finite dimensional Lie algebra, Theorem 1.5 implies the original PBW theorem. Theorem 1.5 is in a similar spirit to Kharchenko's quantum analog of the PBW theorem [Kh]. One of the hypotheses in Theorem 1.5 is (I1.2.3) which assumes essentially that the action of the group generated by weights on the space generated by skew primitive elements is locally finite. When $\text{GKdim } H$ is finite, this is a reasonable hypothesis indicated by a result of the third-named author [Zhu, Theorem 1.2] (see also Lemma 2.5).

In general we are far from answering Question 0.1. There are many unsolved questions concerning the growth of Hopf algebras. The hypotheses in Theorems 0.3 and 0.4 could be superfluous, but we don't know how to remove them at this moment. When $\langle W \rangle$ is nonabelian, a possible better lower bound could be obtained by replacing $\#(W \setminus W_{\sqrt{}})$ in Theorem 0.3 by $\text{GKdim } k\langle W \rangle$; see Lemma 2.6(b) for details. It is expected that these lower bounds can (or should) be improved and that possible upper bounds should be found once finer invariants are introduced. The ultimate goal is to find a formula for the GK-dimension of a Hopf algebra which is analogous to Bass' theorem [KL, Theorem 11.14] in the group algebra case, and then eventually to solve Question 0.1.

There are further connections between the growth of Hopf algebras and W and other invariants defined by skew primitive elements. Let rank denote the torsionfree rank of an abelian group.

Proposition 0.5. *Suppose $\langle W \rangle$ is abelian and torsionfree. If $\text{rank}\langle \Gamma \rangle > \text{rank}\langle W \rangle = 1$, then H has exponential growth.*

Note that $\text{rank}\langle \Gamma \setminus \Gamma_{\sqrt{}} \rangle = \text{rank}\langle \Gamma \rangle$ since elements in $\Gamma_{\sqrt{}}$ have finite order. The rank of $\langle W \rangle$ and $\langle \Gamma \rangle$ should be related when $\text{GKdim } H$ is finite.

Question 0.6. Suppose $\text{rank}\langle \Gamma \rangle > \text{rank}\langle W \rangle$. Does H then have exponential growth?

Quite a few families of Hopf algebras of finite GK-dimension have been analyzed extensively by several authors [AA, AS1, AS2, Br1, Br2, BG1, BG2, BZ, GZ1, GZ2, LWZ, WuZ1, WuZ2, Zhu] during the last few years. But the classification of such Hopf algebras is far from complete. These lower bounds are useful for studying pointed Hopf algebras of low GK-dimension. For example, if $\text{GKdim } H = 2$, then there are only three possibilities for $\text{GKdim } C_0$, $\#(W \setminus W_{\sqrt{}})$, $\#(\Omega \setminus \Omega_{\sqrt{}})$ and

$\dim Z/(C_0 + Y_{\sqrt{}})$. This is one of the initial steps in our ongoing project of classifying pointed Hopf algebra domains of GK-dimension two and three.

Definitions and basic properties of GK-dimension can be found in the first three chapters of [KL]. Our reference book for Hopf algebras is [Mo].

1. FIRST LOWER BOUND THEOREM

In this section we prove Theorem 0.2. We need some lemmas.

Lemma 1.1. *Let D be a Hopf subalgebra of H and $0 \neq F \in H$. Suppose that*

- (a) *L is a subcoalgebra of H containing D ,*
- (b) *L is a left D -module via the multiplication, and*
- (c) *there are nonzero-divisors (regular elements) $h, g \in L$ such that $\Delta(F) - F \otimes h - g \otimes F \in L \otimes L$.*

Define $V = \{a \in D \mid aF \in L\}$. Then V is either 0 or D .

Proof. Suppose V is nonzero and let a be a nonzero element in V . Let C be the subcoalgebra of D generated by a . There is a k -linear basis

$$\{a_1, \dots, a_v, a_{v+1}, \dots, a_w\}$$

of C such that $C \cap V$ is spanned by $\{a_1, \dots, a_v\}$. This means that $a_i F \in L$ for all $i \leq v$ and that any nontrivial linear combination of $\{a_{v+1}F, \dots, a_wF\}$ is not in L . Write $\Delta(a) = \sum_{1 \leq i, j \leq w} \xi_{ij} a_i \otimes a_j$ for some $\xi_{ij} \in k$.

For simplicity, we use the symbol ldt_1 for any element in L and use ldt_2 for any element in $L \otimes L$. By the definition of V , we have $aF + ldt_1 = 0$ for some $ldt_1 \in L$, and hence

$$\begin{aligned} 0 &= \Delta(aF + ldt_1) = \Delta(a)\Delta(F) + \Delta(ldt_1) \\ &= \left(\sum_{i,j} \xi_{ij} a_i \otimes a_j\right)(F \otimes h + g \otimes F + ldt_2) + ldt_2 \\ &= \left(\sum_{i,j} \xi_{ij} a_i \otimes a_j\right)(F \otimes h) + \left(\sum_{i,j} \xi_{ij} a_i \otimes a_j\right)(g \otimes F) + ldt_2 \\ &= \left(\sum_{\substack{i > v \\ \text{all } j}} \xi_{ij} a_i \otimes a_j\right)(F \otimes h) + \left(\sum_{\substack{j > v \\ \text{all } i}} \xi_{ij} a_i \otimes a_j\right)(g \otimes F) + ldt_2, \end{aligned}$$

where the last equation uses the fact $a_i F \in L$ for all $i \leq v$. The above equation implies that

$$\left(\sum_{\substack{j > v \\ \text{all } i}} \xi_{ij} a_i \otimes a_j\right)(g \otimes F) = -\left(\sum_{\substack{i > v \\ \text{all } j}} \xi_{ij} a_i \otimes a_j\right)(F \otimes h) + ldt_2 \in H \otimes L$$

or, equivalently, $\sum_{i=1}^w (a_i g) \otimes (\sum_{j > v} \xi_{ij} a_j F) \in H \otimes L$. Since $\{a_i g\}_{i=1}^w$ is linearly independent, we have $\sum_{j > v} \xi_{ij} a_j F \in L$ for all i . By the definition of $\{a_{v+1}, \dots, a_w\}$, we obtain that $\xi_{ij} = 0$ for all $j > v$. Similarly, $\xi_{ij} = 0$ for all $i > v$. Thus $\Delta(a) \in V \otimes V$, and hence V is a subcoalgebra of D . Since V is a subcoalgebra, there is an element $v \in V$ such that $\epsilon(v) = 1$. Then

$$1F = \epsilon(v)F = \sum S(v_1)v_2F \in L$$

since $v_2 \in V$ and $S(v_1) \in D$. This shows that $1 \in V$. Since V is a left ideal of D , $V = D$. $\square$

Remark 1.2. Recall that $(\mathbb{N}^v, +)$, for every $v \geq 1$, is a linearly ordered semigroup with respect to the following ordering. Define $(c_1, \dots, c_v) < (d_1, \dots, d_v)$ if either $\sum_{i=1}^v c_i < \sum_{i=1}^v d_i$ or $\sum_{i=1}^v c_i = \sum_{i=1}^v d_i$ and there is a $p < v$ such that $c_i = d_i$ for all $i \leq p$ and $c_{p+1} < d_{p+1}$.

Let $T := k\langle \{x_j\}_{j \in J} \rangle$ be the free algebra generated by $\{x_j\}_{j \in J}$. Given any family $(f_j)_{j \in J}$ where $f_j \in \mathbb{N}^v$, we can define an $\mathbb{N}^v$ -graded structure on T by setting $\deg x_j = f_j$ for all $j \in J$. Then $T = \bigoplus_{w \in \mathbb{N}^v} T_w$. Since $\mathbb{N}^v$ is linearly ordered, T has a canonical $\mathbb{N}^v$ -filtration defined by $F_w(T) = \sum_{w' \leq w} T_{w'}$. Let B be any factor ring of T . The $\mathbb{N}^v$ -filtration on T induces a unique $\mathbb{N}^v$ -filtration on B , denoted by $\{F_w(B) \mid w \in \mathbb{N}^v\}$. We say that an element $x \in B$ has *filtered multi-degree*

$$\deg x := w = (d_1, \dots, d_v)$$

and *filtered total-degree* $d = \sum d_i$ if $x \in F_w(B) \setminus \sum_{w' < w} F_{w'}(B)$. Note that the filtered total-degree induces an $\mathbb{N}$ -filtration on B .

In applications, we usually start with an algebra A generated by $\{y_1, \dots, y_v\}$ and $G = \{g_i\}_{i \in I}$ for some index set I . By the discussion in the previous paragraph, we can define two filtrations (and corresponding filtered degrees) on A such that if $f = y_{i_1} y_{i_2} \cdots y_{i_s} \in A$, then the filtered total y -degree of f is at most s and the filtered multi- y -degree of f is at most $(n_1, \dots, n_v)$, where n_i is the number of y_i appearing in f , and if $g \in G$, the filtered total- y -degree and the filtered multi- y -degree of g are both 0.

These two filtrations can be extended to the tensor product $A \otimes A$, namely, $F_w(A \otimes A) := \sum_{w' + w'' \leq w} F_{w'}(A) \otimes F_{w''}(A)$ for all $w \in \mathbb{N}^v$ (or $w \in \mathbb{N}$). For simplicity, the words “filtered” and “filtration” might be omitted below.

Assume that $S := \{y_i\}_{i \in I}$ is a set of skew primitive elements of H where I is either $\mathbb{N}$ or $\{1, \dots, v\}$ for some positive integer v . Suppose that D is a Hopf subalgebra of H and that

(I1.2.1) $g_i := \mu(y_i) \in D$ for all $i \in I$,

(I1.2.2) S is linearly independent in the space H/D ,

(I1.2.3) for each pair $i \leq j$, $y_i g_j = \lambda_{ij} g_j y_i + b_{ij}$ for some $\lambda_{ij} \in k^\times$ and $b_{ij} \in D$, and there is a subalgebra $A \subset D$ containing all b_{ij} such that $y_i A \subset A y_i + A$ and $g_i A \subset A g_i + A$ for all i .

In most of the applications D is the coradical C_0 of H and the commutators of the y_i exist. When $b_{ij} = 0$ for all $i \leq j$, we may take $A = k$, and then (I1.2.3) is automatic. For every positive integer d , define

$$S^d := \{y_1^{d_1} \cdots y_n^{d_n} \cdots \mid \sum_s d_s = d\}.$$

The following lemma is known and easy to check by a direct computation.

Lemma 1.3. *Suppose (I1.2.1)-(I1.2.3) hold.*

(a) *For every n ,*

$$\Delta(y_i^n) = \sum_{s=0}^n \binom{n}{s}_{\lambda_{ii}} g_i^s y_i^{n-s} \otimes y_i^s + \sum_{s+s' < n} a_{ss'} y_i^s \otimes y_i^{s'}$$

for some $a_{ss'} \in \sum_{t \geq 0} A g_i^t$. If $b_{ii} = 0$, then $a_{ss'} = 0$ for all s, s' .

(b) Let $\{y_1, y_2, \dots, y_z\}$ be a finite subset of S . Then, for $n_1, \dots, n_z \geq 0$,

$$\Delta(y_1^{n_1} \cdots y_z^{n_z}) = \sum_{s_1, \dots, s_z} \left(\prod_{t=1}^z \binom{n_t}{s_t} \right)_{\lambda_{tt}} c_{(s_t)} g_1^{s_1} \cdots g_z^{s_z} y_1^{n_1-s_1} \cdots y_z^{n_z-s_z} \otimes y_1^{s_1} \cdots y_z^{s_z} + ldt_2,$$

where $c_{(s_t)} = \prod_{i < j} \lambda_{ij}^{s_j(n_i-s_i)} \in k^\times$. Here ldt_2 is a linear combination of elements of the form $f y_1^{a_1} \cdots y_z^{a_z} \otimes y_1^{b_1} \cdots y_z^{b_z}$ with $\sum_i (a_i + b_i) < \sum_i n_i$, where $f \in \sum_{t_1, \dots, t_z \geq 0} A g_1^{t_1} \cdots g_z^{t_z}$. If $b_{ij} = 0$ for all $i \leq j$, then $ldt_2 = 0$.

For $\alpha = (n_1, \dots, n_z, 0, \dots)$, define

$$(I1.3.1) \quad L_\alpha = \sum_G DG,$$

where G runs through elements $y_1^{m_1} \cdots y_w^{m_w}$ such that $(m_1, \dots, m_w, 0, \dots) < \alpha$.

Lemma 1.4. Retain the notation as above and suppose (I1.2.1)-(I1.2.3) hold. Let $\alpha = (n_1, \dots, n_z, 0, \dots)$ and $F = y_1^{n_1} \cdots y_z^{n_z}$. Define

$$V = \{a \in D \mid aF \in L_\alpha\}.$$

Then V is either 0 or D .

Proof. Let L denote L_α in the proof. First we claim that $\Delta(L) \subset L \otimes L$. It suffices to show that $\Delta(G) \in L \otimes L$ for all $G = y_1^{m_1} \cdots y_w^{m_w}$ with $(m_1, \dots, m_w, 0, \dots) < \alpha$. By Lemma 1.3,

$$\Delta(G) = G \otimes 1 + g_1^{m_1} \cdots g_w^{m_w} \otimes G + ldt_2 \in L \otimes L.$$

Thus we proved our claim. It is easy to see that the hypotheses in Lemma 1.1(a, b) hold. For the hypothesis in Lemma 1.1(c), we note that

$$\Delta(F) = F \otimes 1 + g_1^{n_1} \cdots g_z^{n_z} \otimes F + ldt'_2$$

by Lemma 1.3(b), where $ldt'_2 \in L \otimes L$. The assertion follows from Lemma 1.1. $\square$

Here is the main result of this section. Recall that $g_i = \mu(y_i)$ for all i .

Theorem 1.5. Assume that (I1.2.1)-(I1.2.3) hold. Let λ_i denote λ_{ii} for all i .

- (a) Suppose the elements in $\bigcup_{j \geq 0} S^j$ are linearly dependent over D (on the left or on the right). Then there is some $z \in \mathbb{N}$ such that
- (i) λ_z is a primitive p_z -th root of unity for some $p_z > 1$,
 - (ii) there are $a_i, b_j \in k$, $p_j \in \mathbb{N}$ such that $y_z^{p_z} + \sum_i a_i y_i + \sum_{j \neq z} b_j y_j^{p_j} \in D$,
 - (iii) $g_i = g_z^{p_z}$ whenever $a_i \neq 0$ in part (ii), and
 - (iv) $g_j^{p_j} = g_z^{p_z}$ and λ_j is a primitive p_j -th root of unity whenever $b_j \neq 0$ in part (ii).
- (b) Suppose λ_i is either 1 or not a root of unity for every i . Then the elements in $\bigcup_{j \geq 0} S^j$ are linearly independent over D (on the left and on the right). As a consequence,

$$\text{GKdim } H \geq \text{GKdim } D + \#(S).$$

Proof. (a) Suppose that $\bigcup_{j \geq 0} S^j$ is linearly dependent over D on the left. Then there is an $F = y_1^{n_1} \cdots y_z^{n_z} \in S^d$ for some $d \geq 0$ such that

$$(I1.5.1) \quad aF \in L_\alpha, \quad \text{for some } 0 \neq a \in D,$$

where $\alpha = (n_1, \dots, n_z, 0, \dots)$. The definition of L_α is given in (II.3.1). Choose F among all (a, F) satisfying (II.5.1) so that α is minimal with respect to the linear order $<$ defined at the beginning of Remark 1.2. For simplicity let $L = L_\alpha$ for the rest of the proof. Let $V = \{b \in D \mid bF \in L\}$. Then $0 \neq a \in V$. By Lemma 1.4, $1 \in V$ or, equivalently, $F \in L$. So we can write $F = ldt_1$ where ldt_1 denotes any element in L . By the minimality of α , L is a free left D -module with a basis $\{y_1^{m_1} \cdots y_w^{m_w} \mid (m_1, \dots, m_w, 0, \dots) < \alpha\}$. Note that $L \otimes L$ is a free $D \otimes D$ -module with a basis

$$\{y_1^{m_1} \cdots y_w^{m_w} \otimes y_1^{l_1} \cdots y_{w'}^{l_{w'}} \mid (m_1, \dots, m_w, 0, \dots), (l_1, \dots, l_{w'}, 0, \dots) < \alpha\}.$$

We define a multi-degree on L such that, for any nonzero $a \in D$, $\deg(a) = 0$ and $\deg(ay_1^{m_1} \cdots y_w^{m_w}) = (m_1, \dots, m_w, 0, \dots)$ whenever $(m_1, \dots, m_w, 0, \dots) < \alpha$. Notice that under this definition L is a graded D -module (but not an algebra) which can obviously be viewed as a filtered D -module. Extend this multi-grading naturally to $L \otimes L$ by adding the multi-degrees of the tensor components.

Recall that $F = y_1^{n_1} \cdots y_z^{n_z}$. We may assume $n_z > 0$ (if not, delete y_z in the expression of F). Following the last paragraph, there is an $ldt_1 \in L$ such that $F = -ldt_1$, or equivalently, $y_1^{n_1} \cdots y_z^{n_z} + ldt_1 = 0$. By the choice of F , any element in L has multi-degree less than α . Let ldt_2 denote any element in $L \otimes L$ and let lmt_2 denote any element in $L \otimes L$ with multi-degree less than α . Since the multi-degree of ldt_1 is less than α , $\Delta(ldt_1)$ is an lmt_2 by Lemma 1.3. Then, by Lemma 1.3 again, we have

(II.5.2)

$$\begin{aligned} 0 &= \Delta(F + ldt_1) = \Delta(F) + lmt_2 \\ &= \sum_{s_1, \dots, s_z} \left(\prod_{t=1}^z \binom{n_t}{s_t} \right)_{\lambda_t} c_{(s_t)} g_1^{s_1} \cdots g_z^{s_z} y_1^{n_1-s_1} \cdots y_z^{n_z-s_z} \otimes y_1^{s_1} \cdots y_z^{s_z} + lmt_2 \\ &= \sum_{(s_t) \neq (0), (n_t)} \left(\prod_{t=1}^z \binom{n_t}{s_t} \right)_{\lambda_t} c_{(s_t)} g_1^{s_1} \cdots g_z^{s_z} y_1^{n_1-s_1} \cdots y_z^{n_z-s_z} \otimes y_1^{s_1} \cdots y_z^{s_z} \\ &\quad + F \otimes 1 + g_1^{n_1} \cdots g_z^{n_z} \otimes F + lmt_2 \\ &= \sum_{(s_t) \neq (0), (n_t)} \left(\prod_{t=1}^z \binom{n_t}{s_t} \right)_{\lambda_t} c_{(s_t)} g_1^{s_1} \cdots g_z^{s_z} y_1^{n_1-s_1} \cdots y_z^{n_z-s_z} \otimes y_1^{s_1} \cdots y_z^{s_z} + lmt_2, \end{aligned}$$

where lmt_2 represents an element in $L \otimes L$ with multi-degree less than α . The multi-degree of $g_1^{n_1} \cdots g_z^{n_z} y_1^{n_1-s_1} \cdots y_z^{n_z-s_z} \otimes y_1^{s_1} \cdots y_z^{s_z}$ equals α for any $(s_t) \neq (0), (n_t)$. Using the fact that L is a free D -module with basis $\{y_1^{m_1} \cdots y_w^{m_w} \mid (m_1, \dots, m_w, 0, \dots) < \alpha\}$, we obtain that $(\prod_{t=1}^z \binom{n_t}{s_t})_{\lambda_t} c_{(s_t)} = 0$ or $\prod_{t=1}^z \binom{n_t}{s_t}_{\lambda_t} = 0$ for all $(s_t) \neq (0), (n_t)$. If $n_j > 0$ for some $1 \leq j < z$, we take $(s_t) = (0, 0, \dots, 0, n_z)$, then $\prod_{t=1}^z \binom{n_t}{s_t}_{\lambda_t} = 1$, a contradiction. Therefore $n_j = 0$ for all $j < z$, which means that $F = y_z^{n_z}$.

If $n_z = 1$, we have $y_z = \sum_{i < z} b_i y_i + c$ for $c, b_i \in D$. Hence $\sum_{i < z} b_i y_i + c$ is $(1, g_z)$ -primitive. Then applying Δ we obtain that

$$\begin{aligned} \Delta(b_i) &= b_i \otimes 1, \\ \Delta(b_i)(g_i \otimes 1) &= g_z \otimes b_i, \\ \Delta(c) &= c \otimes 1 + g_z \otimes c. \end{aligned}$$

These imply that $b_i \in k$ and $g_i = g_z$ when $b_i \neq 0$. This contradicts (I1.2.2). Therefore $n_z > 1$.

By the last two paragraphs, $n_z > 1$ and $n_i = 0$ for all $i < z$ and $\binom{n_z}{s_z}_{\lambda_z} = 0$ for all $1 \leq s_z \leq n_z - 1$. This can only happen when λ_z is a primitive n_z -th root of unity [GZ2, Lemma 7.5].

Next let us re-name n_z by p_z and write $F = y_z^{p_z}$. Then $y_z^{p_z} + \sum_i b_i G_i + c_0 = 0$, where $b_i, c_0 \in D$ and the G_i are monomials with multi- y -degree less than $(0, \dots, 0, p_z, 0, \dots)$ (where p_z is in the z -th position). Repeating a computation similar to (I1.5.2) (and the induction on the multi- y -degree of G_i) one can show that each G_i (when $b_i \neq 0$) is of the form $y_i^{n_i}$ and each $y_i^{n_i}$ is a skew primitive. If $n_i > 1$, then λ_i is a primitive n_i -th root of unity. In summary, when λ_i is not a root of unity, then $n_i = 1$, and when λ_i is a primitive p_i -th root of unity, then n_i is either 1 or p_i . So we have

$$-y_z^{p_z} = \sum_i a_i y_i + \sum_{j \neq z} b_j y_j^{p_j} + c,$$

where $0 \neq a_i, b_j \in D$ and $c \in D$. Thus $\sum_i a_i y_i + \sum_j b_j y_j^{p_j} + c$ is $(1, g_z^{p_z})$ -primitive. Since L is a free left D -module, each of the nonzero $a_i y_i$, $b_j y_j^{p_j}$ and c is $(1, g_z^{p_z})$ -primitive. The coproduct computation shows that $a_i, b_j \in k$ and $g_i = g_z^{p_z}$ and $g_j^{p_j} = g_z^{p_z}$.

(b) The first assertion is an immediate consequence of part (a). To prove the second assertion, we take W to be a finite dimensional subspace of D and let S be a finite set $\{y_1, \dots, y_z\}$. For a subspace $V \subset H$, let V^n be the linear span of all elements $v_1 \cdots v_n$ for $v_i \in V$. By the first assertion,

$$\begin{aligned} \dim(W + k1 + \sum_{i=1}^z k y_i)^{2n} &\geq \dim W^n (k1 + \sum_{i=1}^z k y_i)^n \\ &\geq (\dim W^n) \# \left(\bigcup_{d=0}^n S^d \right) \geq (\dim W^n) c n^z \end{aligned}$$

for some positive constant c . This implies that $\text{GKdim } H \geq \text{GKdim } D + \#S$. If S is infinite, let S' be any finite subset of S . Then the above argument shows that $\text{GKdim } H \geq \text{GKdim } D + \#S'$ for any S' . Thus $\text{GKdim } H = \infty = \text{GKdim } D + \#S$. $\square$

Proof of Theorem 0.2. Let $S = \{y_1, \dots, y_w\}$. Then (I1.2.1)-(I1.2.3) follow easily from (a) and (b). The hypothesis in Theorem 1.5(b) is the same as that in Theorem 0.2(c). Therefore the assertion follows from Theorem 1.5(b). $\square$

The following easy lemma will be used implicitly later.

Lemma 1.6. *Let P be the set of all skew primitive elements in a Hopf algebra H with weight μ . Then P is a k -subspace of H and $P \cap C_0 = k(\mu - 1)$.*

Proof. It is clear that P is a k -subspace of H . For any element $y \in P \cap C_0$, write $y = \sum_{i=1}^n c_i g_i$ for some $c_i \in k$ and $g_i \in G(H)$. Then the equation $\Delta(y) = y \otimes 1 + \mu \otimes y$ forces the fact that $y \in k(\mu - 1)$. $\square$

2. SECOND LOWER BOUND THEOREM

In this section we prove Theorem 0.3, which is a consequence of Theorem 0.2. A stronger version will be proved in the next section. Lemmas presented here are also needed for the next section and cannot be omitted even if we skip Theorem 0.3. If $\text{GKdim } H = \infty$, then Theorem 0.3 is vacuous. So we may assume that $\text{GKdim } H < \infty$. We refer to Section 0 for the definitions of W, Ω, Γ and $W_{\sqrt{\cdot}}, \Omega_{\sqrt{\cdot}}, \Gamma_{\sqrt{\cdot}}$.

Lemma 2.1. *Let y be a skew primitive element not in C_0 such that $\gamma(y)$ is defined.*

- (a) *If $\gamma(y) \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}$, then y^n is not skew primitive for any $n > 1$.*
- (b) *If $\gamma(y) \in \Gamma_{\sqrt{\cdot}}$, then $\mu(y) \in W_{\sqrt{\cdot}}$.*

Proof. (a) Take S to be the singleton $\{y\}$ and $D = C_0$. Then (I1.2.1)-(I1.2.3) hold for $A = k[\mu(y)^{\pm 1}]$. Since $\gamma(y) \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}$ the hypothesis in Theorem 1.5(b) holds. By Theorem 1.5(b), $\{y^n\}_{n \geq 0}$ is linearly independent over C_0 . Since $\gamma(y)$ is not a root of unity, for any $n > 1$, $\Delta(y^n) \notin H \otimes k + C_0 \otimes H$ by Lemma 1.3(a). Thus y^n is not a skew primitive. The assertion follows.

(b) Suppose $\gamma(y) \in \Gamma_{\sqrt{\cdot}}$. Since $\gamma(y) \neq 1$, replacing y by $y + \alpha(\mu(y) - 1)$ for a suitable $\alpha \in k$, we have $\mu(y)^{-1}y\mu(y) = \gamma(y)y$. Since $\gamma(y)$ is a primitive n -th root of unity for some $n > 1$, Lemma 1.3(a) says that $\Delta(y^n) = y^n \otimes 1 + \mu(y)^n \otimes y^n$, which means that y^n is a skew primitive (could be zero). Therefore $\mu(y) \in W_{\sqrt{\cdot}}$. $\square$

Let $G(H)$ denote the group of all group-like elements in a Hopf algebra H . Recall that $\text{GKdim } H < \infty$ by a general assumption in this section.

Lemma 2.2. *Let y be a skew primitive element not in C_0 and let $x = \mu(y)$. Suppose $\gamma(y)$ exists. Assume that G_0 is a subgroup of $G(H)$ commuting with x . Let $V = k(x - 1) + \sum_{g \in G_0} k(g^{-1}yg)$.*

- (a) *Every $z \in V$ is $(1, x)$ -primitive, and $\omega(z) = \omega(y)$ for all $z \in V \setminus k(x - 1)$.*
- (b) *If $\gamma(y) \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}$, then $\dim V \leq \text{GKdim } H - \text{GKdim } C_0 + 1$.*
- (c) *Suppose that V is finite dimensional and that G_0 is abelian. Then there is $z \in V \setminus k(x - 1)$ such that either*
 - (ci) *for every $g \in G_0$, $g^{-1}zg = \lambda_g z$ for some $\lambda_g \in k^\times$ or*
 - (cii) *for every $g \in G_0$, $g^{-1}zg = z + \tau_g(x - 1)$ for some $\tau_g \in k$.*
- (d) *If $\gamma(y)$ is not a root of unity, then $\mu(y)$ has infinite order.*

Proof. (a) Since $gx = xg$ for all $g \in G_0$, $g^{-1}yg$ is a $(1, x)$ -primitive with $\omega(g^{-1}yg) = \omega(y)$.

(b) Let $S = \{g_i^{-1}yg_i\}_{i=1}^w$ be a finite subset of V which is linearly independent in the space $V/k(x - 1)$. Here $g_i \in G_0$ for all $i = 1, \dots, w$. For different i , we have $\mu(g_i^{-1}yg_i) = x$, and

$$x^{-1}(g_i^{-1}yg_i)x = g_i^{-1}(x^{-1}yx)g_i = \gamma(y)(g_i^{-1}yg_i) + \tau(x - 1)$$

where τ is the same as the one in (I0.3.3). Then the hypotheses (I1.2.1)-(I1.2.3) hold for $A = k[x^{\pm 1}]$ and $D = C_0$. Since $\lambda := \gamma(y)$ is either 1 or not a root of unity, Theorem 1.5(b) says that $\#S \leq \text{GKdim } H - \text{GKdim } C_0$. Clearly $V \cap C_0 = k(x - 1)$. Thus

$$\dim V - 1 = \dim V/(V \cap C_0) = \#S \leq \text{GKdim } H - \text{GKdim } C_0,$$

since S is a basis of $V/(V \cap C_0)$.

(c) First we may assume $x \in G_0$. If not, replace G_0 by the subgroup generated by G_0 and x (replacing G_0 by this larger subgroup does not enlarge V , because of

(I0.3.3)). Then V is a G_0 -module by conjugation action. Since G_0 is abelian and k is algebraically closed, every finite dimensional simple G_0 -module is 1-dimensional. Thus V has a 1-dimensional simple G_0 -submodule kz . If $z \notin k(x-1)$, then kz being a simple G_0 -module is equivalent to (ci). Otherwise, no element $z \in V \setminus k(x-1)$ generates a simple G_0 -submodule. Hence V has a unique simple G_0 -submodule $M_0 := k(x-1)$. Note that $g^{-1}(x-1)g = (x-1)$ for all $g \in G_0$, so M_0 is the trivial G_0 -module. Since G_0 is commutative and V has only one simple submodule, every simple subquotient of V must be isomorphic to the simple M_0 . Pick $z \in V \setminus k(x-1)$ so that the submodule M generated by z is 2-dimensional. Then $M/k(x-1) \cong M_0$, which says that $g^{-1}zg \equiv z$ modulo $k(x-1)$. Hence $g^{-1}zg = z + \tau_g(x-1)$ for some $\tau_g \in k$.

(d) Let $G_0 = \langle \mu(y) \rangle$. It follows from the definition that the existence of $\gamma(y)$ implies that V is finite dimensional. Applying part (ci) to the cyclic group G_0 there is a skew primitive $z \in H \setminus C_0$ such that

$$g^{-1}zg = \lambda(g)z$$

for all $g \in G_0$. It is also clear that $\lambda(\mu(y)) = \gamma(y)$. Since $\gamma(y)$ is not a root of unity, the image of $\lambda : G_0 \rightarrow k^\times$ is infinite. Consequently, G_0 is infinite and $\mu(y)$ has infinite order. $\square$

Lemma 2.3. *Let $\{z_i\}_{i=1}^w$ be a set of skew primitive elements not in C_0 such that $\gamma(z_i)$ exists for each i . If the elements $\omega(z_1), \dots, \omega(z_w)$ are distinct, then $\{z_i\}_{i=1}^w$ is linearly independent in H/C_0 .*

Proof. Suppose $\{z_i\}_{i=1}^w$ is linearly dependent in H/C_0 . Pick a minimal subset, say $\{z_j\}_{j=1}^v$, such that $\sum_{j=1}^v a_j z_j =: c \in C_0$ for some scalars $a_j \in k^\times$. Thus $v > 1$ since $z_i \notin C_0$ for any i . Applying Δ to the equation $\sum_{j=1}^v a_j z_j = c$, we have

$$\Delta(c) = \Delta\left(\sum_{j=1}^v a_j z_j\right) = \sum_{j=1}^v a_j z_j \otimes 1 + \sum_{j=1}^v a_j \mu(z_j) \otimes z_j = c \otimes 1 + \sum_{j=1}^v a_j \mu(z_j) \otimes z_j.$$

Hence $\sum_{j=1}^v \mu(z_j) \otimes a_j z_j \in C_0 \otimes C_0$. By the minimality of v , $\mu(z_j) = \mu(z_{j'})$ for all j, j' .

Set $x = \mu(z_j)$ for all $1 \leq j \leq v$. Applying the conjugation by x to the equation $\sum_{j=1}^v a_j z_j = c$, we obtain $\sum_{j=1}^v \gamma(z_j) a_j z_j = -\sum_{j=1}^v \tau_j(x-1) + x^{-1}cx \in C_0$ for some $\tau_j \in k$. Using the minimality of v , $\gamma(z_j) = \gamma(z_{j'})$ for all j, j' . Thus we obtain a contradiction. The assertion follows. $\square$

Theorem 2.4. *Let $\{y_i\}_{i=1}^w$ be a set of skew primitive elements not in C_0 such that $\omega(y_1), \dots, \omega(y_w)$ are defined and distinct elements in $\Omega \setminus \Omega_{\sqrt{\cdot}}$. If the subgroup G_0 generated by $\{\mu(y_i)\}_{i=1}^w$ is abelian, then $\text{GKdim } H \geq \text{GKdim } C_0 + w$.*

Proof. By Lemma 2.2(a, c), for each i there is a z_i in $k(\mu(y_i) - 1) + \sum_{g \in G_0} kg^{-1}y_i g$ but not in C_0 such that

$$\omega(z_i) = \omega(y_i),$$

and, for every $g \in G_0$,

$$g^{-1}z_i g = \lambda_{ig} z_i + \tau_{ig}(\mu(y_i) - 1)$$

for some $\lambda_{ig} \in k^\times$, $\tau_{ig} \in k$. Let $A = kG_0$ and $D = C_0$. Then (I1.2.1) is clear and (I1.2.2) follows from Lemma 2.3 for the set $\{z_i\}_{i=1}^w$. (I1.2.3) is a consequence of Lemma 2.2(c), as we have already seen. By hypothesis, each $\lambda_i := \gamma(z_i)$ is either 1

or not a root of unity. Therefore $\text{GKdim } H \geq \text{GKdim } C_0 + w$ by applying Theorem 1.5(b) to the set $\{z_i\}_{i=1}^w$. $\square$

The next lemma is a result of [Zhu]. As before, we assume that $\text{GKdim } H < \infty$, which is one of the hypotheses in [Zhu, Theorem 1.2].

Lemma 2.5 ([Zhu, Theorem 1.2]). *Let y be a skew primitive element not in C_0 with $g = \mu(y)$. Then there is a skew primitive element $z = \sum_{i=0}^n b_i g^{-i} y g^i \in H \setminus C_0$, where $b_i \in k$, such that $g^{-1} z g = \lambda z + \tau(g - 1)$ for some $\lambda \in k^\times$ and $\tau \in k$. Further, if $\lambda \neq 1$, then there is $z' = z + \alpha(g - 1)$ for a suitable $\alpha \in k$ such that $g^{-1} z' g = \lambda z'$.*

Proof. In [Zhu, Theorem 1.2] H is assumed to be pointed, but the statement is valid without this hypothesis. The first assertion is equivalent to [Zhu, Theorem 1.2]. If $\lambda \neq 1$, take $\alpha = (\lambda - 1)^{-1} \tau$. Then $z' = z + \alpha(g - 1)$ is a $(1, g)$ -primitive element satisfying $g^{-1} z' g = \lambda z'$. $\square$

Now we are ready to prove Theorem 0.3.

Proof of Theorem 0.3. Pick any finite subset $\{\mu(y_i)\}_{i=1}^w$ of $W \setminus W_{\sqrt{}}$ where each y_i is a skew primitive not in C_0 . By Lemma 2.5, for each i there is a skew primitive y'_i not in C_0 such that $g_i := \mu(y'_i) = \mu(y_i)$ and that $\gamma(y'_i)$ is defined. By Lemma 2.1(b), $\gamma(y'_i)$ is not a root of unity or 1. Hence $\omega(y'_i) \in \Omega \setminus \Omega_{\sqrt{}}$. The assertion follows from Theorem 2.4. $\square$

Theorem 2.4 shows in fact that if $\langle W \setminus W_{\sqrt{}} \rangle$ is abelian, then

$$\text{GKdim } H \geq \text{GKdim } C_0 + \#(\Omega \setminus \Omega_{\sqrt{}}).$$

There is also an inequality

$$\text{GKdim } H \geq \text{GKdim } C_0 + \#(W')$$

for any $W' \subset W \setminus W_{\sqrt{}}$ such that $\langle W' \rangle$ is abelian.

Suppose there is a surjective Hopf algebra morphism $\pi : H \rightarrow C_0$ such that the restriction to C_0 is the identity. Let A be the subalgebra of H generated by all skew primitive elements in $\ker \pi$, and let G_W be the subsemigroup of $G(H)$ generated by $\mu(y)$ for all skew primitive elements $y \in A$. We do not assume that G_W is abelian.

Lemma 2.6. *Suppose there is a surjective Hopf algebra morphism $\pi : H \rightarrow C_0$ such that the restriction to C_0 is the identity. Let A be defined as above.*

- (a) *$H = R \# C_0$, where R is the ring of right coinvariants of π . Then A is a subalgebra of R and*

$$\text{GKdim } H \geq \text{GKdim } R + \text{GKdim } C_0 \geq \text{GKdim } A + \text{GKdim } C_0.$$

- (b) *Assume that A is a domain. Then $\text{GKdim } A \geq \text{GKdim } kG_W$. As a consequence,*

$$\text{GKdim } H \geq \text{GKdim } kG_W + \text{GKdim } C_0.$$

Proof. (a) By [Mo, Theorem 7.2.2], H is isomorphic to a crossed product $R \#_{\sigma} C_0$ as algebras, and by [Mo, Proposition 7.2.3], σ is trivial. Hence $H = R \# C_0$, where R is the ring of right coinvariants of π . It is clear that every skew primitive element in $\ker \pi$ is in R . Therefore $A \subset R$.

Since $H = R \# C_0$, $\text{GKdim } H \geq \text{GKdim } R + \text{GKdim } C_0$. The assertion follows by the fact $A \subset R$.

(b) Define a map $\rho : A \rightarrow C_0 \otimes H$ to be the composition $(\pi \otimes Id_H) \circ \Delta$. Since $\rho(y) \in kG_W \otimes A$ for all skew primitive elements $y \in A$ and since A is generated by these y 's, the image of ρ is in $kG_W \otimes A$. Consequently, (A, ρ) is a left kG_W -comodule algebra. This means that A is a G_W -graded algebra. Let $f : A \rightarrow C_0$ be the map sending any nonzero homogeneous element $h \in A$ to its degree, for example, sending $y_1 \cdots y_n$ to $\mu(y_1) \cdots \mu(y_n)$. Since A is a domain, f is multiplicative.

Pick any finite dimensional space $V = k + \sum_{i=1}^m k\mu(y_i)$ of kG_W where the y_i are skew primitive elements in A , and let $W = \{1\} \cup \{y_i\}_{i=1}^m$. Then $\dim(kW)^n \geq \#(f(W^n)) \geq \#(f(W))^n \geq \dim V^n$ for all n . Hence $\text{GKdim } A \geq \text{GKdim } kG_W$. $\square$

The next example shows why we need to remove $W_{\sqrt{}}$ from W (or remove $\Gamma_{\sqrt{}}$ from Γ) in the lower bound theorems.

Example 2.7. Let B be the Hopf algebra $B(1, 1, p_1, \dots, p_s, q)$ defined in [GZ2, Construction 1.2]. This is a finitely generated, noetherian, pointed Hopf domain of GK-dimension 2. By [GZ2, Construction 1.2] B is generated by $x, x^{-1}, y_1, \dots, y_s$, where x is a group-like element and the y_i 's are skew primitive elements. Let $z = y_1^{p_1}$. Then $z = y_j^{p_j}$ for all j and it is a central skew primitive element. Let $H = B/(z, x^m - 1)$, where $m = \prod_i p_i$. Then H is a finite dimensional pointed Hopf algebra of GK-dimension 0 and $C_0 = k[x, x^{-1}]/(x^m - 1)$ has GK-dimension 0.

By [GZ2, Construction 1.2], $W = W_{\sqrt{}} = \{x^{m_i}\}_{i=1}^s$, where $m_i = m/p_i$, $\Gamma = \Gamma_{\sqrt{}} = \{q^{-m_i^2}\}_{i=1}^s$, $\Omega = \Omega_{\sqrt{}} = \{(x^{m_i}, q^{-m_i^2})\}_{i=1}^s$, and $Z = Y_{\sqrt{}} + C_0$ and $Y_{\sqrt{}} = \sum_{i=1}^s ky_i$. Thus

$$\#(W_{\sqrt{}}) = \#(\Gamma_{\sqrt{}}) = \#(\Omega_{\sqrt{}}) = \dim Y_{\sqrt{}} = s,$$

which can be arbitrarily large.

3. THIRD LOWER BOUND THEOREM

The first half of this section concerns some preliminary analysis of Hopf algebras with exponential growth and the proof of Proposition 0.5. The proof of the third lower bound theorem is given at the end of the section.

Let $(G_0, \times)$ be a multiplicative abelian group and $\Lambda := \{\lambda_1, \dots, \lambda_v\}$ be a list of 1-dimensional group representations of G_0 for some $v > 1$. Note that this list is allowed to have repetitions. When some λ_i is the trivial representation of G_0 (namely, $\lambda_i(g) = 1$ for all $g \in G_0$), then we also need a group homomorphism $\tau_i : (G_0, \times) \rightarrow (k, +)$ (which must be zero if G_0 is torsion, since $\text{char } k = 0$). When λ_i is not trivial, we set $\tau_i = 0$.

Now pick a list of elements $\mu := \{\mu_1, \dots, \mu_v\}$ in G_0 (again allowing repetitions). Let $K := K(\Lambda, \mu)$ be the Hopf algebra generated as an algebra by the elements in the abelian group G_0 and a set of skew primitive elements $y_1, \dots, y_v$ subject to the relations within G_0 and the following additional relations between G_0 and $\{y_i\}_{i=1}^v$:

$$y_i g = \lambda_i(g) g y_i + \tau_i(g) g (\mu_i - 1), \text{ for all } i \text{ and all } g \in G_0.$$

The coalgebra structure of K is determined by

$$\begin{aligned} \Delta(g) &= g \otimes g, \quad \epsilon(g) = 1, \quad \text{for all } g \in G_0, \\ \Delta(y_i) &= y_i \otimes 1 + \mu_i \otimes y_i, \quad \epsilon(y_i) = 0, \quad \text{for all } i = 1, \dots, v. \end{aligned}$$

And the antipode of K is determined by

$$\begin{aligned} S(g) &= g^{-1}, \quad \text{for all } g \in G_0, \\ S(y_i) &= -\mu_i^{-1}y_i, \quad \text{for all } i = 1, \dots, v. \end{aligned}$$

Let $\lambda_{ij} = \lambda_i(\mu_j)$ for all i, j , and let Λ_M be the $v \times v$ -matrix (λ_{ij}) .

By Remark 1.2, the total y -degree and the multi- y -degree are defined for elements in K . For example, the (filtered) multi- y -degree of gy_3y_2 is $(0, 1, 1, 0, \dots, 0) \in \mathbb{N}^v$.

Let F be a nonzero skew primitive element in K with total y -degree $z \geq 2$. Write $F = \sum c_{h,(i_s)} hy_{i_1}y_{i_2} \cdots y_{i_n}$, where $h \in G_0$ and $0 \neq c_{h,(i_s)} \in k$. A term of F means a nonzero monomial $c_{h,(i_s)} hy_{i_1}y_{i_2} \cdots y_{i_n}$ appearing in F .

Lemma 3.1. *Let $K := K(\Lambda, \mu)$ be defined as above.*

(a) *K has a k -linear basis*

$$\{gy_{i_1}y_{i_2} \cdots y_{i_s}\},$$

where $g \in G_0$, $i_1, \dots, i_s \in \{1, \dots, v\}$. As a consequence, K contains a free subalgebra $k\langle y_1, y_2 \rangle$ and has exponential growth.

(b) *The coradical of K is kG_0 .*

(c) *If F is a skew primitive element of total y -degree $z \geq 2$, then for any term of F with multi- y -degree $(N_1, \dots, N_v)$ and $\sum_i N_i = z$,*

$$\prod_{i=1}^v (\lambda_{ii})^{N_i(N_i-1)} \prod_{i < j} (\lambda_{ij}\lambda_{ji})^{N_i N_j} = 1.$$

Proof. (a) The first assertion follows from Bergman's Diamond Lemma [Be, Theorem 1.2]. Consequently, K contains the free algebra of rank 2, $k\langle y_1, y_2 \rangle$. Therefore K has exponential growth.

(b) By definition, Δ is compatible with filtrations defined in Remark 1.2. Hence Δ is a homomorphism of filtered algebras. So every group-like element must have total y -degree 0. The assertion follows.

(c) Let $F = \sum c_{h,(i_s)} hy_{i_1}y_{i_2} \cdots y_{i_n}$ with coefficients $c_{h,(i_s)} \neq 0$. For simplicity, let ldt denote any linear combination of monomials of total y -degree less than z . Then we can write $F = \sum c_{h,(i_s)} hy_{i_1}y_{i_2} \cdots y_{i_z} + ldt$. Since $\Delta(F) = F \otimes 1 + \mu(F) \otimes F$, $h = 1$ for terms with total degree z . Pick any term of y -degree z in F , say $c_{1,(i_s)} y_{i_1}y_{i_2} \cdots y_{i_z}$, and let $(N_1, \dots, N_v)$ be its multi- y -degree.

Since F is skew primitive, $S(F) = -\mu(F)^{-1}F$. Since $S(y_i) = -\mu_i^{-1}y_i$, we have

$$\begin{aligned} S(c_{1,(i_s)} y_{i_1} \cdots y_{i_z}) &= c_{1,(i_s)} (-\mu_{i_z}^{-1}y_{i_z}) \cdots (-\mu_{i_1}^{-1}y_{i_1}) \\ &= c_{1,(i_s)} (-1)^z \prod_{s>t} \lambda_{i_s i_t}^{-1} \mu^{-1}y_{i_z} \cdots y_{i_1} + ldt, \end{aligned}$$

where $\mu = \prod_{s=1}^z \mu_{i_s}$. Since $S(F) = -\mu(F)^{-1}F$, $\mu = \mu(F)$ and F contains a nonzero term of the form $c'_{(i_s)} y_{i_z} \cdots y_{i_1}$. The same computation shows that

$$\begin{aligned} S(c'_{(i_s)} y_{i_z} \cdots y_{i_1}) &= c'_{(i_s)} (-\mu_{i_1}^{-1}y_{i_1}) \cdots (-\mu_{i_z}^{-1}y_{i_z}) \\ &= c'_{(i_s)} (-1)^z \prod_{a<b} \lambda_{i_a i_b}^{-1} \mu^{-1}y_{i_z} \cdots y_{i_1} + ldt. \end{aligned}$$

Comparing the coefficients in the terms $\mu^{-1}y_{i_z} \cdots y_{i_1}$ and $\mu^{-1}y_{i_1} \cdots y_{i_z}$ in the equation $S(F) = -\mu^{-1}F$, we have

$$-c'_{(i_s)} = c_{1,(i_s)}(-1)^z \prod_{s>t} \lambda_{i_s i_t}^{-1}, \quad -c_{1,(i_s)} = c'_{(i_s)}(-1)^z \prod_{a<b} \lambda_{i_a i_b}^{-1}.$$

Since $c_{1,(i_s)}$ and $c'_{(i_s)}$ are nonzero, the above two equations imply

$$\prod_{s>t} (\lambda_{i_s i_t} \lambda_{i_t i_s}) = 1$$

or

$$(I3.1.1) \quad \prod_{\{s \neq t\} \subset \{1, 2, \dots, z\}} (\lambda_{i_s i_t} \lambda_{i_t i_s}) = 1.$$

We know the monomial $y_{i_1} \cdots y_{i_n}$ contains N_i copies of y_i for all $i = 1, \dots, v$. Hence equation (I3.1.1) is in fact

$$\prod_{i=1}^v (\lambda_{ii})^{N_i(N_i-1)} \prod_{i<j} (\lambda_{ij} \lambda_{ji})^{N_i N_j} = 1.$$

□

There is a slight modification of Lemma 3.1. Suppose λ_{11} is a primitive p_1 -th root of unity for some $p_1 > 1$. Recycle most of the notation before Lemma 3.1. Let $L := L(\Lambda, \mu, p_1)$ be the Hopf algebra generated as an algebra by the abelian group G_0 and $y_1, \dots, y_v$ subject to the relations within G_0 and the following additional relations between G_0 and $\{y_i\}_{i=1}^v$:

$$\begin{aligned} y_i g &= \lambda_i(g) g y_i + \tau_i(g) g (\mu_i - 1), \quad \text{for all } i \text{ and all } g \in G_0, \\ y_1^{p_1} &= \beta (\mu_1^{p_1} - 1), \quad \text{for some } \beta \in k. \end{aligned}$$

The coalgebra structure of L is determined by

$$\begin{aligned} \Delta(g) &= g \otimes g, \quad \epsilon(g) = 1, \quad \text{for all } g \in G_0, \\ \Delta(y_i) &= y_i \otimes 1 + \mu_i \otimes y_i, \quad \epsilon(y_i) = 0, \quad \text{for all } i = 1, \dots, v, \end{aligned}$$

and the antipode of L is determined by

$$\begin{aligned} S(g) &= g^{-1}, \quad \text{for all } g \in G_0, \\ S(y_i) &= -\mu_i^{-1} y_i, \quad \text{for all } i = 1, \dots, v. \end{aligned}$$

Define $\Lambda_M := (\lambda_{ij}) = (\lambda_i(\mu_j))$. The total y -degree and the multi- y -degree are defined as before.

Lemma 3.2. *Let $L := L(\Lambda, \mu, p_1)$ be defined as above. Suppose either $\beta = 0$ or $\lambda_1(g)^{p_1} = 1$ for all $g \in G_0$.*

(a) *L has a k -linear basis*

$$\{g y_{i_1} y_{i_2} \cdots y_{i_s}\},$$

where $g \in G_0$, $i_1, \dots, i_s \in \{1, \dots, v\}$ and there is no u such that $i_u = i_{u+1} = \dots = i_{u+p_1-1} = 1$. As a consequence, L contains a free subalgebra $k\langle y_1 y_2, y_1 y_2^2 \rangle$ and has exponential growth.

(b) *The coradical of L is kG_0 .*

- (c) If F is a skew primitive element of total y -degree $z \geq 2$, then for any term of F with multi- y -degree $(N_1, \dots, N_v)$ and $\sum_i N_i = z$,

$$\prod_{i=1}^v (\lambda_{ii})^{N_i(N_i-1)} \prod_{i < j} (\lambda_{ij} \lambda_{ji})^{N_i N_j} = 1.$$

Proposition 3.3. Let H be a Hopf algebra and y_1, y_2 be two skew primitive elements linearly independent in H/C_0 . Suppose that

- (i) there is a group-like element x and $d_1, d_2 \in \mathbb{Z}$ such that $\mu(y_i) = x^{d_i}$ for $i = 1, 2$, and
 - (ii) there are two scalars $q_1, q_2 \in k^\times$ such that $y_i x = q_i x y_i$ for $i = 1, 2$.
- (a) If x has infinite order and H does not contain a free algebra of rank 2, then

$$(I3.3.1) \quad q_1^{d_1(M_1(M_1-1))+d_2(M_1M_2)} q_2^{d_2(M_2(M_2-1))+d_1(M_1M_2)} = 1$$

for some integers $M_1, M_2 \geq 0$ satisfying $M_1 + M_2 \geq 2$.

- (b) Assume that one of the following holds:

- (1) $q_1 = q_2$ is not a root of unity and $d_1 d_2 > 0$;
- (2) $q_1^{d_1} = 1$ and q_2 is not a root of unity and $d_1 d_2 > 0$;
- (3) $q_1^{d_1} \neq 1$ is a root of unity and q_2 is not a root of unity and $d_1 d_2 > 0$;
- (4) the group $\langle q_1, q_2 \rangle \subset k^\times$ is free abelian of rank 2, $d_1 d_2 \neq 0$.

Then H contains a free subalgebra of rank 2. Consequently, H has exponential growth.

Proof. (a) Let $\mu_i = x^{d_i}$ and $\lambda_{ij} = q_i^{d_j}$. Then $y_i \mu_j = \lambda_{ij} \mu_j y_i$ and $\gamma(y_i) = \lambda_{ii}$ for all $i, j \in \{1, 2\}$.

Let H_0 be the Hopf subalgebra generated by x, x^{-1}, y_1, y_2 . Let $G_0 = \langle g \rangle \cong \mathbb{Z}$ and let $\lambda_i(g^n) = q_i^n$ for $i = 1, 2$ and all n . Let $\Lambda = \{\lambda_1, \lambda_2\}$ and $\mu = \{g, g\}$. Then there is a surjective Hopf algebra homomorphism $\phi : K := K(\Lambda, \mu) \rightarrow H_0$ sending $g \mapsto x$ and $y_i \mapsto y_i$ for $i = 1, 2$, where we choose $\tau_i = 0$. By Lemma 3.1(a), K contains a free algebra of rank 2. If H does not contain a free algebra of rank 2, then $K \rightarrow H_0$ is not injective. By [Mo, Theorem 5.3.1], there is a nonzero skew primitive element $F \in K$ such that $\phi(F) = 0$. Since ϕ is injective on skew primitive elements of y -degree ≤ 1 , F has total y -degree $z \geq 2$. By Lemma 3.1(c), for any term of F with multi- y -degree (M_1, M_2) and $M_1 + M_2 = z$, we have the following:

$$(\lambda_{11})^{M_1(M_1-1)} (\lambda_{22})^{M_2(M_2-1)} (\lambda_{12} \lambda_{21})^{M_1 M_2} = 1$$

or, equivalently,

$$(q_1^{d_1})^{M_1(M_1-1)} (q_2^{d_2})^{M_2(M_2-1)} (q_1^{d_2} q_2^{d_1})^{M_1 M_2} = 1.$$

This can be simplified to

$$q_1^{d_1(M_1(M_1-1))+d_2(M_1M_2)} q_2^{d_2(M_2(M_2-1))+d_1(M_1M_2)} = 1,$$

which is (I3.3.1).

(b) Assume H does not contain a free algebra of rank 2, and we will obtain a contradiction. If one of the hypotheses holds, then x has infinite order in $G(H)$ by Lemma 2.2(d). Therefore we can apply part (a).

In case (b)(1), (I3.3.1) implies that

$$d_1(M_1(M_1-1)) + d_2(M_1M_2) + d_2(M_2(M_2-1)) + d_1(M_1M_2) = 0.$$

This is impossible since $d_1 d_2 > 0$ and $M_1 + M_2 \geq 2$. Therefore H contains a free algebra of rank 2.

A similar argument works for case (b)(4).

In case (b)(2), equation (I3.3.1) implies that $(q_2^{d_2(M_2(M_2-1))+d_1(M_1M_2)})^{d_1} = 1$, or

$$d_2(M_2(M_2 - 1)) + d_1(M_1M_2) = 0$$

because q_2 is not a root of unity. Since $d_1 d_2 > 0$, the only solution is $M_2 = 0$ and $M_1 = z \geq 2$. Thus we have $F = cy_1^z + ldt$ for some $c \in k^\times$. By Lemma 1.3 and the fact that $\lambda_{11} = q_1^{d_1} = 1$, F cannot be skew primitive for any $z \geq 2$. So case (b)(2) has been taken care of.

It remains to consider case (b)(3). Suppose $q_1^{d_1}$ is a primitive p_1 -th root of unity. Then $y_1^{p_1}$ is a skew primitive element. If $y_1^{p_1} \notin C_0$, then $\{y_1^{p_1}, y_2\}$ is linearly independent in H/C_0 . Note that if $\alpha y_1^{p_1} + \beta y_2 \in C_0$, then $x^{-1}(\alpha y_1^{p_1} + \beta y_2)x \in C_0$, which would imply $y_1^{p_1}, y_2 \in C_0$ because $q_1^{p_1} \neq q_2$. The assertion follows from case (b)(2) applied to $\{y_1^{p_1}, y_2\}$. If $y_1^{p_1} \in C_0$, then $y_1^{p_1} = \beta(\mu_1^{p_1} - 1)$ for some β . Replacing K by L in the above argument, (I3.3.1) holds again.

Since q_1 is a root of unity, we have

$$(q_2^p)^{d_2(M_2(M_2-1))+d_1(M_1M_2)} = 1$$

for some $p > 1$. Since q_2 is not a root of unity,

$$d_2(M_2(M_2 - 1)) + d_1(M_1M_2) = 0.$$

Since $d_1 d_2 > 0$, the only solution is $M_2 = 0$ and $M_1 = z \geq 2$. Now $F = cy_1^z + ldt$, where $c \in k^\times$ and $z < p_1$. By Lemma 1.3, F cannot be skew primitive. This is a contradiction. $\square$

Corollary 3.4. *Suppose H has subexponential growth. Let y be a skew primitive element not in C_0 such that $\gamma(y)$ is defined and is not a root of unity. Let G_0 be a finitely generated abelian subgroup of $G(H)$ containing $\mu(y)$ (which has infinite order automatically). Then $V := k(\mu(y) - 1) + \sum_{g \in G_0} k(g^{-1}yg)$ is 2-dimensional. As a consequence, there is a group representation $\lambda : G_0 \rightarrow k$ such that*

$$g^{-1}y'g = \lambda(g)y'$$

for all $g \in G_0$, where $y' = y + \alpha(\mu(y) - 1)$ for some $\alpha \in k$.

Proof. Since $\gamma(y)$ is not a root of unity, we may assume that $y\mu(y) = \gamma(y)\mu(y)y$ after replacing y by $y + \alpha(\mu(y) - 1)$ for some $\alpha \in k$. Let $g \in G_0$. Let $y_1 = y$ and $y_2 = g^{-1}yg$. Then $\gamma(y_1) = \gamma(y_2)$, and it is not a root of unity. By Proposition 3.3(b)(1), y_1 and y_2 are not linearly independent in H/C_0 . The assertion that $\dim V = 2$ follows by applying Lemma 1.6. The consequence follows from Lemma 2.2(ci). $\square$

Proof of Proposition 0.5. We prove the assertion by contradiction. So we assume that H has subexponential growth.

Pick a pair of skew primitive elements (y_1, y_2) such that the subgroup of $\langle \Gamma \rangle$ generated by $\{\gamma(y_1), \gamma(y_2)\}$ has rank 2. Let $\lambda_{ii} = \gamma(y_i)$ and $g_i = \mu(y_i)$ for $i = 1, 2$. Since $\langle W \rangle$ is abelian, the subgroup $G_0 := \langle g_1, g_2 \rangle$ is abelian. By Corollary 3.4, we may further assume that, for $i = 1, 2$ and $j = 1, 2$, $g_j^{-1}y_i g_j = \lambda_{ij}y_i$ for some

$\lambda_{ij} \in k^\times$. Since H does not contain a free subalgebra of rank 2, the proofs of Proposition 3.3 and (I3.3.1) show that

$$(I3.4.1) \quad (\lambda_{11})^{M_1(M_1-1)}(\lambda_{22})^{M_2(M_2-1)}(\lambda_{12}\lambda_{21})^{M_1M_2} = 1$$

for some nonnegative M_1, M_2 with $M_1 + M_2 \geq 2$.

Since G_0 is a finitely generated subgroup of $\langle W \rangle$ and since $\langle W \rangle$ is abelian and torsionfree of rank 1, G_0 is isomorphic to $\mathbb{Z}$. Therefore there is an $x \in G_0$ such that $g_1 = x^a$ and $g_2 = x^b$ for some nonzero integers a, b . Consequently, $g_2^a = g_1^b$. Thus the equation $g_j^{-1}y_i g_j = \lambda_{ij}y_i$ implies that $\lambda_{i2}^a = \lambda_{i1}^b$ for all $i = 1, 2$. Then (I3.4.1) implies that

$$(\lambda_{11})^{abM_1(M_1-1)}(\lambda_{22})^{abM_2(M_2-1)}(\lambda_{11})^{b^2M_1M_2}(\lambda_{22})^{a^2M_1M_2} = 1.$$

Since the rank of $\langle \lambda_{11}, \lambda_{22} \rangle$ is 2, we have

$$abM_1(M_1 - 1) + b^2M_1M_2 = abM_2(M_2 - 1) + a^2M_1M_2 = 0$$

or

$$a(M_1 - 1) + bM_2 = b(M_2 - 1) + aM_1 = 0.$$

Since a, b are nonzero, this means that $(M_1 - 1)(M_2 - 1) - M_1M_2 = 0$. This is impossible when $M_1 + M_2 \geq 2$, which yields a contradiction. $\square$

The rest of this section is devoted to the proof of Theorem 0.4. The next definition was given in the introduction, but perhaps it should be reviewed here. Let y be a $(1, g)$ -primitive element in a Hopf algebra H . Let $T_{g^{-1}}$ denote the inverse conjugation by g , namely, $T_{g^{-1}} : a \rightarrow g^{-1}ag$.

Definition 3.5. Let y be a $(1, g)$ -primitive element of H not in C_0 . A nonzero scalar λ is called the commutator of y of level n if $(T_{g^{-1}} - \lambda Id_H)^n(y) \in C_0$ and $(T_{g^{-1}} - \lambda Id_H)^{n-1}(y) \notin C_0$. In this case we write $\gamma(y) = \lambda$. When $n = 1$, $\gamma(y)$ is the commutator of y defined as in (I0.3.2) or equivalently in (I0.3.3).

In general, the commutator of y may not exist. We also need a generalization of Definition 3.5. Recall that $W_\times$ is the subset of W consisting of weights $\mu(y)$ such that the commutator of y is either 1 or not a root of unity. Throughout the rest of the section let G_0 be the subgroup $\langle W_\times \rangle$ and suppose that G_0 is abelian. A 1-dimensional representation of G_0 is equivalent to a multiplicative map $\lambda : G_0 \rightarrow k^\times$. Let G_0^* denote the set of 1-dimensional representations of G_0 , which is also called the character group of G_0 .

Definition 3.6. Let y be a skew primitive element in $H \setminus C_0$ and let $\lambda \in G_0^*$. We say λ is the *generalized commutator* of y of level n if there is an n such that $(T_{g^{-1}} - \lambda(g)Id_H)^n(y) \in C_0$ for all $g \in G_0$ and $(T_{g^{-1}} - \lambda(g)Id_H)^{n-1}(y) \notin C_0$ for some $g \in G_0$. If $n = 1$, λ is called the *generalized commutator* of y .

Lemma 3.7. In parts (a) and (c) suppose that $\text{GKdim } H < \infty$.

- (a) Every skew primitive element $y \in H$ is a linear combination of skew primitives with commutator of finite level and weight $\mu(y)$.
- (b) Suppose V is a conjugation G_0 -stable finite dimensional subspace spanned by skew primitives with their weights in G_0 . Then every element $y \in V$ is a linear combination of skew primitives with generalized commutator of finite level.

- (c) Every skew primitive $y \in H$ with commutator of level 1 such that $\gamma(y) \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}$ is a linear combination of skew primitives with generalized commutator of finite level. Further, each nonzero summand of the linear combination has weight commutator $\omega(y)$.

Proof. (a) If $y \in C_0$, then $y = \alpha(\mu(y) - 1)$ for some $\alpha \in k$ [Lemma 1.6] and the commutator of y has level 0 by definition.

If $y \notin C_0$, then, by Lemma 2.5, $V := \sum_{n \in \mathbb{Z}} k(g^{-n}yg^n) + k(g - 1)$ is finite dimensional over k , where $g = \mu(y)$. Then $T_{g^{-1}}$ acts on V as an invertible linear map. Pick a basis of V so that the presentation of $T_{g^{-1}}$ with respect to the basis is in the Jordan canonical form. Then each basis element is a skew primitive element with commutator of finite level. The assertion follows.

(b) Write $V = \bigoplus_{j=1}^m V_j$, where each V_j is spanned by skew primitives with weight g_j for distinct group-like elements $g_1, \dots, g_m \in G_0$. Since G_0 is abelian, each V_j is conjugation G_0 -stable. Passing from V to V_j we may assume that each element in V is a skew primitive of weight g .

For every $h \in G_0$, $T_{h^{-1}}$ acts on V as an invertible linear map. It is clear that V is a finite dimensional kG_0 -module. Since kG_0 is commutative, every finite dimensional simple G_0 -module is 1-dimensional and $\text{Ext}_{kG_0}^1(S, S') = 0$ if S and S' are distinct simple modules over kG_0 . Then V is a finite direct sum of submodules V_i so that the support of each V_i is a single closed point of $\text{Spec } kG_0$. This closed point corresponds to a 1-dimensional G_0 -representation λ_i . Fix any i , every element in V_i has a generalized commutator λ_i with level no more than $\dim V_i$.

(c) Let V be the vector space spanned by all skew primitive elements z with commutator of level 1 such that $\omega(z) = \omega(y)$. Pick any finite set $\{z_1, \dots, z_w\}$ which is linearly independent in $V/(V \cap k(\mu(y) - 1))$. Then (I1.2.1)-(I1.2.3) hold for $D = C_0$ and $A = k[\mu(y)^{\pm 1}]$. By Theorem 1.5(b), $w \leq \text{GKdim } H$. Thus V is finite dimensional. Clearly, $T_{g^{-1}}$ stabilizes V for all $g \in G_0$. The first assertion follows from part (b). The final assertion is clear since every element in V has weight commutator equal $\omega(y)$. $\square$

We need to introduce some conventions. Let $\mu \in W$. Define P_μ to be the k -linear space spanned by all skew primitives $y \in H$ with $\mu(y) = \mu$. Let γ be a nonzero scalar and let $P_{\mu, \gamma, n}$ be the k -linear space spanned by all skew primitives $y \in P_\mu$ with commutator γ of level no more than n . Let $P_{\mu, \gamma, *} = \sum_{n \geq 0} P_{\mu, \gamma, n}$, $P_{*, *, n} = \sum_{\mu, \gamma} P_{\mu, \gamma, n}$ and $P_{*, *, *} = \sum_{\mu, \gamma, n} P_{\mu, \gamma, n}$.

Given any $\mu \in W$ and $\lambda \in G_0^*$, let $P_{\mu, \lambda, n}$ be the k -linear space spanned by all skew primitives $y \in H$ with generalized commutator λ of level no more than n and $\mu(y) = \mu$. Let $P_{\mu, \lambda, *} = \sum_{n \geq 0} P_{\mu, \lambda, n}$, $P_{*, G^*, n} = \sum_{\mu, \lambda} P_{\mu, \lambda, n}$ and $P_{*, G^*, *} = \sum_{\mu, \lambda, n} P_{\mu, \lambda, n}$.

Lemma 3.7(a) says that $P_{*, *, *}$ contains all skew primitive elements, and Lemma 3.7(c) says that $P_{\mu, \gamma, 1}$ is a subspace of $P_{*, G^*, *}$ when $\gamma \in k^\times$ and γ is 1 or not a root of unity.

Lemma 3.8. *Retain the above notation. Suppose that H does not contain a free subalgebra of rank 2.*

- (a) If $P_{\mu, \gamma, 1} \subset k(\mu - 1)$, then $P_{\mu, \gamma, *} \subset k(\mu - 1)$.
 (b) If $P_{\mu, \gamma, 1} \not\subset k(\mu - 1)$ and γ is not a root of unity, then $P_{\mu, \gamma_0, *} \subset k(\mu - 1)$ for every root of unity γ_0 .
 (c) If γ is not a root of unity, then $P_{\mu, \gamma, 1}/k(\mu - 1)$ has dimension at most 1.

- (d) Suppose γ_1 and γ_2 are two distinct scalars, neither of which is a root of unity. If $P_{\mu,\gamma_i,1} \not\subset k(\mu-1)$ for $i = 1, 2$, then $\gamma_1^N \gamma_2^M = 1$ for some positive integers N, M . Further, there is no $\gamma_3 \in k^\times \setminus \{\gamma_1, \gamma_2\}$ such that $P_{\mu,\gamma_3,1} \not\subset k(\mu-1)$.
- (e) Suppose that $\text{GKdim } H < \infty$. If γ is not a root of unity, then $P_{\mu,\gamma,*} = P_{\mu,\gamma,1} \subset P_{*,G*,*}$.

Proof. (a) This is clear by induction.

(b) First assume that γ_0 is not 1. Pick $y_2 \in P_{\mu,\gamma,1} \setminus C_0$. If $P_{\mu,\gamma_0,*} \not\subset k(\mu-1)$ for a root of unity γ_0 , by part (a), there is a $y_1 \in P_{\mu,\gamma_0,1} \setminus C_0$. Since γ_0 and γ are not 1, after adding a suitable term $\alpha(\mu(y_i) - 1)$ to y_i , the hypothesis in Proposition 3.3(ii) holds. Then we are in the situation of Proposition 3.3(b)(2), (b)(3) (where $d_1 = d_2 = 1$). By Proposition 3.3, H contains a free subalgebra of rank 2, a contradiction.

Second, assume that $\gamma_0 = 1$. For the statement we only need to consider the Hopf subalgebra generated by group-like and skew primitive elements. So we may assume that H is pointed. Passing to the associated graded Hopf algebra of H with respect to its coradical filtration, one can assume the hypothesis in Proposition 3.3(ii) holds. So the above proof works.

(c) This follows from Proposition 3.3(b)(1).

(d) By the proof of Proposition 3.3, see (I3.3.1),

$$q_1^{d_1(M_1(M_1-1))+d_2(M_1M_2)} q_2^{d_2(M_2(M_2-1))+d_1(M_1M_2)} = 1,$$

where M_1 and M_2 are nonnegative integers with $M_1 + M_2 \geq 2$. Note that $\gamma_i = q_i$ and $d_1 = d_2 = 1$ in this case. Let $N = M_1(M_1 - 1) + M_1M_2$ and $M = M_2(M_2 - 1) + M_1M_2$. Then N and M are nonnegative and $N + M \geq 2$. Since γ_1 and γ_2 are not roots of unity, both N and M must be positive.

If such a γ_3 exists, by part (b) it is not a root of unity. By the above assertion we have positive integers A, B, C, D such that $\gamma_1^A \gamma_3^B = 1$ and $\gamma_2^C \gamma_3^D = 1$. Then

$$1 = (\gamma_1^N \gamma_2^M)^{AC} = \gamma_1^{ACN} \gamma_2^{CAM} = \gamma_3^{-BCN-DM},$$

which contradicts the fact that γ_3 is not a root of unity.

(e) If $P_{\mu,\gamma,*} \neq P_{\mu,\gamma,1}$, pick a skew primitive $y_2 \in P_{\mu,\gamma,2}$. This means that $\mu^{-1}y_2\mu - \gamma y_2 = y_1 \in P_{\mu,\gamma,1} \setminus k(\mu-1)$. Consider the Hopf subalgebra K generated as an algebra by y_2, y_1, μ, μ^{-1} . By possibly adding a term $\beta(\mu-1)$ to y_1 for some $\beta \in k$ and adding a term $\alpha(\mu-1)$ to y_2 for some $\alpha \in k$, we can assume that $y_1\mu = \gamma\mu y_1$ and $y_2\mu = \gamma\mu y_2 + \mu y_1$.

It remains to show that K contains a free subalgebra of rank 2. Define a filtration F_i of K inductively as follows:

$$F_0 = C_0(K) = k[\mu^{\pm 1}],$$

$$F_1 = F_0 + F_0 y_1 = F_0 y_1 + F_0,$$

$$F_2 = F_0 + F_0 y_1 + F_0 y_1^2 + F_0 y_2 = F_0 + y_1 F_0 + y_1^2 F_0 + y_2 F_0,$$

$$F_n = \sum_{i=1}^{n-1} F_i F_{n-i} \text{ for all } n \geq 3.$$

It is easy to check that F is a Hopf algebra filtration and $\text{gr}_F K$ is a Hopf algebra. In $\text{gr}_F K$, y_1, y_2 are linearly independent in $\text{gr}_F K/k(\mu-1)$. Since $y_1, y_2 \in P_{\mu,\gamma,1}(K)$, by part (c), $\text{gr}_F K$ contains a free subalgebra of rank 2. Therefore K contains a free subalgebra of rank 2. This yields a contradiction. Therefore the first assertion (namely, the first equation) follows. Finally, by Lemma 3.7(c), $P_{\mu,\gamma,1} \subset P_{*,G*,*}$. $\square$

Remark 3.9. Suppose $\text{GKdim } H < \infty$. By Lemmas 2.1(b) and 3.7(a), $W \setminus W_{\sqrt{\cdot}} \subseteq W_{\times}$. In practice it often happens that $W \setminus W_{\sqrt{\cdot}} = W_{\times}$.

Now we are ready to prove the Third Lower Bound Theorem. Let Y_* be the k -linear vector space spanned by all skew primitive elements y with commutator of finite level such that $\gamma(y) \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}$.

Theorem 3.10. *Suppose G_0 is abelian. Then*

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_*/(Y_* \cap C_0).$$

Proof. Nothing needs to be proved if $\text{GKdim } H = \infty$, so we assume $\text{GKdim } H < \infty$.

Let

$$Y_n = \sum_{\mu \in W_{\times}, \gamma \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}} P_{\mu, \gamma, n} \quad \text{and} \quad Y_{Gn} = \sum_{\mu \in W_{\times}, \lambda \in G_0^*, \gamma := \lambda(\mu) \in \Gamma \setminus \Gamma_{\sqrt{\cdot}}} P_{\mu, \lambda, n}$$

for all $n \geq 1$. Then $Y_* = \sum_n Y_n$. Let $Y_{G*} = \sum_n Y_{Gn}$. We prove the following claim by induction:

Claim A.

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_{Gn}/(Y_{Gn} \cap C_0)$$

for all $n \geq 1$. When $n = 1$, let $\{y_i\}$ be a basis of $Y_{G1}/(Y_{G1} \cap C_0)$ such that each y_i is in $P_{\mu_i, \lambda_i, 1}$ for some μ_i, λ_i . Then we have

$$\mu_j^{-1} y_i \mu_j = \lambda_i(\mu_j) y_i + \tau_{ij}(\mu_i - 1),$$

where $\lambda_i(\mu_i) = \gamma(y_i)$ is either 1 or not a root of unity and where $\tau_{ij} \in k$. By Theorem 1.5 (with $D = C_0$ and $A = kG_0$),

$$\text{GKdim } H \geq \text{GKdim } C_0 + \#\{y_i\} = \text{GKdim } C_0 + \dim Y_{G1}/(Y_{G1} \cap C_0),$$

which proves Claim A for $n = 1$.

Now assume that Claim A holds for n . Without loss of generality we may assume that H is generated as an algebra by C_0 and all skew primitive elements of H . Define a filtration F_i of H as follows:

$$F_0 = C_0,$$

$$F_1 = F_0 + F_0 Y_{G1} = F_0 + Y_{G1} F_0,$$

$F_2 = F_1^2 + F_0 Z$, where Z is the k -linear span of all skew primitive elements of H , and

$$F_m = \sum_{i=1}^{m-1} F_i F_{m-i} \text{ for all } m \geq 3.$$

Then $\{F_m\}$ is a Hopf algebra filtration of H . Let K be the associated graded Hopf algebra $\text{gr}_F H$. Note that if $y \in P_{\mu, \lambda, 2} \subset Y_{G2}(H)$, then $y \in F_2$ and the associated element in K is $\text{gr } y \in F_2/F_1$, and $g^{-1}(\text{gr } y)g = \lambda(g)(\text{gr } y)$ for all $g \in G_0$ (or, when trivially $y \in F_1$, we have $\text{gr } y \in F_1/F_0$ or $\text{gr } y \in F_0$). Thus $\text{gr } y \in Y_{G1}(K)$ (which is easy to see when $y \in F_1$). By induction one sees that

$$Y_{Gn}(K) \supseteq \{\text{gr } y \mid y \in Y_{G(n+1)}(H)\} =: \text{gr } Y_{G(n+1)}(H)$$

for all n . Applying the induction hypothesis to K ,

$$\text{GKdim } K \geq \text{GKdim } C_0 + \dim Y_{Gn}(K)/(Y_{Gn}(K) \cap C_0).$$

By [KL, Lemma 6.5], $\text{GKdim } H \geq \text{GKdim } K$. It is clear that

$$\dim Y_{G(n+1)}(H)/(Y_{G(n+1)}(H) \cap C_0) = \dim \text{gr } Y_{G(n+1)}(H)/(\text{gr } Y_{G(n+1)}(H) \cap C_0).$$

Therefore,

$$\begin{aligned} \text{GKdim } H &\geq \text{GKdim } K \geq \text{GKdim } C_0 + \dim Y_{G^n}(K)/(Y_{G^n}(K) \cap C_0) \\ &\geq \text{GKdim } C_0 + \dim \text{gr } Y_{G(n+1)}(H)/(\text{gr } Y_{G(n+1)}(H) \cap C_0) \\ &= \text{GKdim } C_0 + \dim Y_{G(n+1)}(H)/(Y_{G(n+1)}(H) \cap C_0), \end{aligned}$$

which finishes the induction step. Therefore we proved Claim A.

When n goes to infinity, we have

$$(I3.10.1) \quad \text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_{G^*}(H)/(Y_{G^*}(H) \cap C_0).$$

Next we prove the following claim by induction:

Claim B.

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_n/(Y_n \cap C_0).$$

When $n = 1$, this follows from (I3.10.1) since $Y_1(H) \subset Y_{G^*}(H)$ by Lemma 3.7(c). Note that Y_1 is G_0 -stable. Using an argument similar to the proof of Claim A by passing to the associated graded Hopf algebra $\text{gr}_F H$ (and replacing Y_{G^n} by Y_n), one sees that Claim B holds. When n goes to infinity, we have

$$\text{GKdim } H \geq \text{GKdim } C_0 + \dim Y_*(H)/(Y_*(H) \cap C_0).$$

□

The proof of Theorem 3.10 also shows the following.

Corollary 3.11. *Suppose G_0 is abelian and $\text{GKdim } H < \infty$. Let $\mu \in W$ and suppose that $\gamma \in k^\times$ is not a root of unity. Then*

- (a) $P_{\mu,1,*}$ is finite dimensional.
- (b) $P_{\mu,1,*}$ and $P_{\mu,\gamma,*}$ are subspaces of $P_{*,G^*,*}$.
- (c) $Y_* = Y_{G^*}$.

By Lemma 3.8(c), (e), $P_{\mu,\gamma,*}$ is finite dimensional.

Proof of Corollary 3.11. (a) By Theorem 3.10, $Y_*/(Y_* \cap C_0)$ is finite dimensional. Since $P_{\mu,1,*} \subset Y_*$ and $P_{\mu,1,*} \cap C_0 = k(\mu - 1)$, we have

$$\dim P_{\mu,1,*}/k(\mu - 1) \leq \dim Y_*/(Y_* \cap C_0) < \infty,$$

which implies that $P_{\mu,1,*}$ is finite dimensional.

(b) By Lemma 3.8(e), $P_{\mu,\gamma,*}$ is a subspace of $P_{*,G^*,*}$.

By part (a) $P_{\mu,1,*}$ is finite dimensional. It is clear that $P_{\mu,1,*}$ is G_0 -stable. By Lemma 3.7(b), (c) $P_{\mu,1,*}$ is a subspace of $P_{*,G^*,*}$.

(c) As a consequence of part (b), $P_{\mu,\lambda,*} = P_{\mu,\lambda(\mu),*}$ when $\lambda(\mu)$ is either 1 or not a root of unity. The assertion follows. □

Theorem 0.4 is an immediate consequence of Theorem 3.10.

Proof of Theorem 0.4. Without loss of generality we assume that $\text{GKdim } H < \infty$. First we claim that $Y_* \cap (Y_{\sqrt{}} + C_0) \subset C_0$. Suppose $y = z + c$ is in $Y_* \cap (Y_{\sqrt{}} + C_0)$, where $y \in Y_* \setminus C_0$ and $z \in Y_{\sqrt{}}$ and $c \in C_0$. It is easily reduced to the case when $\mu(y) = \mu(z) = g$ and $c = \alpha(g - 1)$ for some $\alpha \in k$. Then $y \in Y_{\sqrt{}}$, a contradiction. Therefore the claim holds.

By Lemma 3.7(a), every skew primitive element is a linear combination of skew primitives with commutator of finite level. This says that $Z = Y_* + Y_{\sqrt{}} + C_0$. Since $Y_* \cap (Y_{\sqrt{}} + C_0) \subset C_0$, we have

$$Z/(C_0 + Y_{\sqrt{}}) \cong Y_*/Y_* \cap (C_0 + Y_{\sqrt{}}) = Y_*/(Y_* \cap C_0).$$

The assertion follows by Theorem 3.10. $\square$

Another way of proving Theorem 0.3 (with a slightly stronger hypothesis that $\langle W_{\times} \rangle$ is abelian) is by using Theorem 0.4 and the following lemma, which is due to an anonymous referee.

Lemma 3.12. *Suppose $\text{GKdim } H < \infty$. Then*

$$\dim Z/(C_0 + Y_{\sqrt{}}) \geq \#(W \setminus W_{\sqrt{}}).$$

Proof. For any $w \leq \#(W \setminus W_{\sqrt{}})$, take distinct elements $x_1, \dots, x_w \in W \setminus W_{\sqrt{}}$; we need to prove that $\dim Z/(C_0 + Y_{\sqrt{}}) \geq w$.

For each i , pick a skew primitive element $z_i \in H \setminus C_0$ such that $x_i = \mu(z_i)$. Lemma 2.5 shows how to get a suitable z_i such that $\gamma(z_i)$ is defined. Since $\mu(z_i) = x_i \notin W_{\sqrt{}}$, Lemma 2.1(b) says that $\gamma(z_i)$ is either 1 or not a root of unity.

It suffices to show that $z_1, \dots, z_w$ are linearly independent in $Z/(C_0 + Y_{\sqrt{}})$, so it is enough to show that $z_1, \dots, z_w, y$ are linearly independent in Z/C_0 for any $y \in Y_{\sqrt{}} \setminus C_0$. We can arrange $y = y_1 + \dots + y_v$ for some skew primitive elements $y_j \notin C_0$, where y_j has a commutator $\gamma(y_j)$ of finite level which is a nontrivial root of unity and where the pairs $(\mu(y_j), \gamma(y_j))$ are distinct. The pairs $(\mu(z_i), \gamma(z_i))$ are already distinct, and $(\mu(z_i), \gamma(z_i)) \neq (\mu(y_j), \gamma(y_j))$ for all i, j because $\gamma(z_i)$ is either 1 or not a root of unity.

An improved version of Lemma 2.3 for skew primitive elements with commutators of finite level says that $z_1, \dots, z_w, y_1, \dots, y_v$ are linearly independent in Z/C_0 . Consequently, $z_1, \dots, z_w, y$ are linearly independent in Z/C_0 , as desired. $\square$

Finally, we end with a simple example in which $Z = Y_{\sqrt{}} + C_0$.

Example 3.13. Let H be generated as an algebra by x and $\{y_i\}_{i=1}^{\infty}$ subject to the relations

$$\begin{aligned} x^2 &= 1, \\ y_i^2 &= 0, \\ y_i y_j + y_j y_i &= 0, \\ x y_i + y_i x &= 0 \end{aligned}$$

for all $i, j \in \mathbb{N}$. The coalgebra structure and the antipode of H are determined by

$$\begin{aligned} \Delta(x) &= x \otimes x, & \Delta(y_i) &= y_i \otimes 1 + x \otimes y_i, \\ \epsilon(x) &= 1, & \epsilon(y_i) &= 0, \\ S(x) &= x, & S(y_i) &= -x y_i = y_i x \end{aligned}$$

for all i . It is easy to check the following:

- (a) $\text{GKdim } H = 0$,
- (b) $\Omega = \{(x, -1)\} = \Omega_{\sqrt{}}$,
- (c) $Z = Y_{\sqrt{}} + C_0$ and $\dim Y_{\sqrt{}} = \infty$,
- (d) $P_{x, -1, *} = P_{*, -1, 1} = Y_{\sqrt{}}$.

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