WEIGHTED LOCAL ORLICZ-HARDY SPACES ON DOMAINS AND THEIR APPLICATIONS IN INHOMOGENEOUS DIRICHLET AND NEUMANN PROBLEMS

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ABSTRACT. Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$, and $\omega \in A_\infty(\mathbb{R}^n)$ (the class of Muckenhoupt weights). Let L be a second-order divergence form elliptic operator on $L^2(\Omega)$ with the Dirichlet or Neumann boundary condition, and assume that the heat semigroup generated by L has the Gaussian property (G_1) with the regularity of their kernels measured by $\mu \in (0, 1]$. Let Φ be a continuous, strictly increasing, subadditive, positive and concave function on $(0, \infty)$ of critical lower type index $p_\Phi^- \in (0, 1]$. In this paper, the authors first introduce the “geometrical” weighted local Orlicz-Hardy spaces $h_{\omega, r}^\Phi(\Omega)$ and $h_{\omega, z}^\Phi(\Omega)$ via the weighted local Orlicz-Hardy spaces $h_\omega^\Phi(\mathbb{R}^n)$, and obtain their two equivalent characterizations in terms of the nontangential maximal function and the Lusin area function associated with the heat semigroup generated by L when $p_\Phi^- \in (n/(n + \mu), 1]$. Second, the authors furthermore establish three equivalent characterizations of $h_{\omega, r}^\Phi(\Omega)$ in terms of the grand maximal function, the radial maximal function and the atomic decomposition when the complement of Ω is unbounded and $p_\Phi^- \in (0, 1]$. Third, as applications, the authors prove that the operators $\nabla^2 \mathbb{G}_D$ are bounded from $h_{\omega, r}^\Phi(\Omega)$ to the weighted Orlicz space $L_\omega^\Phi(\Omega)$, and from $h_{\omega, r}^\Phi(\Omega)$ to itself when Ω is a bounded semiconvex domain in $\mathbb{R}^n$ and $p_\Phi^- \in (\frac{n}{n+1}, 1]$, and the operators $\nabla^2 \mathbb{G}_N$ are bounded from $h_{\omega, z}^\Phi(\Omega)$ to $L_\omega^\Phi(\Omega)$, and from $h_{\omega, z}^\Phi(\Omega)$ to $h_{\omega, r}^\Phi(\Omega)$ when Ω is a bounded convex domain in $\mathbb{R}^n$ and $p_\Phi^- \in (\frac{n}{n+1}, 1]$, where $\mathbb{G}_D$ and $\mathbb{G}_N$ denote, respectively, the Dirichlet Green operator and the Neumann Green operator.

CONTENTS

1. Introduction	4730
2. Preliminaries	4741
2.1. Some properties of the weight class $A_\infty(\mathbb{R}^n)$	4741
2.2. The divergence form elliptic operator L	4743
2.3. Orlicz functions	4745

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3. Proof of Theorem 1.4	4746
4. Proof of Theorem 1.7	4781
5. Proof of Theorem 1.8	4789
6. Proof of Theorem 1.9	4798
Acknowledgements	4805
References	4805

1. INTRODUCTION

The theory of Hardy spaces on the n -dimensional Euclidean space $\mathbb{R}^n$ was originally initiated by Stein and Weiss in [77]. Later, Fefferman and Stein [30] systematically developed a real-variable theory for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$, which are designed to behave well under the Calderón-Zygmund operators; see, for example, [19, 67, 75, 76]. In particular, they respect translations, rotations, and dilations. However, there are two shortcomings of the spaces $H^p(\mathbb{R}^n)$. It is known that the spaces $H^p(\mathbb{R}^n)$ are not closed under compositions with diffeomorphisms nor under multiplication by smooth functions with compact support. In order to overcome these issues, Goldberg [37] developed the theory of the local Hardy spaces $h^p(\mathbb{R}^n)$ with $p \in (0, 1]$, which plays an important role in partial differential equations and harmonic analysis; see, for example, [12, 37, 74, 78] and their references. In particular, one may prove that pseudo-differential operators of order zero are bounded on the spaces $h^p(\mathbb{R}^n)$ with $p \in (0, 1]$; see [37] (also [79, 80]). In [12], Bui studied the weighted version $h_\omega^p(\mathbb{R}^n)$ of the local Hardy space $h^p(\mathbb{R}^n)$ with $\omega \in A_\infty(\mathbb{R}^n)$, where and in what follows, $A_q(\mathbb{R}^n)$ for $q \in [1, \infty]$ denotes the class of Muckenhoupt weights; see, for example, [33] for their definitions and properties.

Rychkov [74] introduced a class of local weights, denoted by $A_\infty^{\text{loc}}(\mathbb{R}^n)$, and studied the weighted Besov-Lipschitz spaces and Triebel-Lizorkin spaces with weights belonging to $A_\infty^{\text{loc}}(\mathbb{R}^n)$, which contains $A_\infty(\mathbb{R}^n)$ weights as special cases. In particular, Rychkov [74] generalized some of the results of Bui [12] on weighted local Hardy spaces $h_\omega^p(\mathbb{R}^n)$ to $A_\infty^{\text{loc}}(\mathbb{R}^n)$ weights. Very recently, Tang [78] established the weighted atomic decomposition characterization of the weighted local Hardy space $h_\omega^p(\mathbb{R}^n)$ with $\omega \in A_\infty^{\text{loc}}(\mathbb{R}^n)$ via the local grand maximal function.

On the other hand, as a generalization of $L^p(\mathbb{R}^n)$, the Orlicz space was introduced by Birnbaum-Orlicz in [11] and Orlicz in [68]. Since then, the theory of the Orlicz spaces themselves has been well developed and these spaces have been widely used in probability, statistics, potential theory, and partial differential equations, as well as harmonic analysis and some other fields of mathematics; see, for example, [13, 45, 59, 70, 71]. Moreover, Orlicz-Hardy spaces are also suitable substitutions of the Orlicz spaces in the study of boundedness of operators; see, for example, [46, 48, 49, 51, 82]. Recall that Orlicz-Hardy spaces and their dual spaces were studied by Janson [46] on $\mathbb{R}^n$ and Viviani [82] on spaces of homogeneous type in the sense of Coifman and Weiss [21] (see also [22]). Let Φ be a continuous, strictly increasing, subadditive, positive and concave function on $(0, \infty)$ of strictly critical lower type index $p_\Phi \in (0, 1]$ (see (2.13) below for the definition of p_Φ). Based on the works of Rychkov [74] and Tang [78], the weighted local Orlicz-Hardy spaces $h_\omega^\Phi(\mathbb{R}^n)$ with $\omega \in A_\infty^{\text{loc}}(\mathbb{R}^n)$ were introduced and studied in [86]. We point out that the

assumptions on p_Φ in [86] can be relaxed into the same assumptions on p_Φ^- ; see (2.12) below for the definition of p_Φ^- and also Remark 2.7 below.

As we mentioned at the beginning, Hardy spaces $H^p(\mathbb{R}^n)$ are essentially related to the second-order elliptic operator with constant coefficients,

$$L := \sum_{j,k=1}^n a_{jk} \frac{\partial^2}{\partial x_i^2},$$

where $\{a_{jk}\}_{j,k=1}^n$ are constants and $(a_{jk})_{n \times n} > 0$. In recent years, the research of the real-variable theory of various function spaces associated with different differential operators has inspired great interests; see, for example, [6, 7, 28, 29, 38, 39, 40, 41, 50, 85]. Moreover, Orlicz-Hardy spaces associated with some differential operators and their dual spaces were introduced and studied in [48, 49, 51]. In particular, the local Hardy space $h_L^1(\mathbb{R}^n)$, associated with a linear operator L in $L^2(\mathbb{R}^n)$ which generates an analytic semigroup with kernels satisfying an upper bound of Poisson type, was also studied in [52].

Next, it is natural to develop a theory of Hardy spaces on domains of $\mathbb{R}^n$; see, for example, [8, 16, 17, 18, 27, 43, 44, 66, 81]. As we may expect, there are several ways to define Hardy spaces on domains. In particular, the second author of this paper, Krantz and Stein [18] introduced the Hardy spaces $H_r^p(\Omega)$ and $H_z^p(\Omega)$ on domains Ω of $\mathbb{R}^n$, respectively, by restricting arbitrary elements of $H^p(\mathbb{R}^n)$ to Ω , and restricting elements of $H^p(\mathbb{R}^n)$ which are zero outside $\overline{\Omega}$ to Ω , where and in what follows, $\overline{\Omega}$ denotes the *closure* of Ω in $\mathbb{R}^n$. For these Hardy spaces, atomic decompositions have been obtained in [18] when Ω is a special Lipschitz domain or a bounded Lipschitz domain of $\mathbb{R}^n$. The second author of this paper, Krantz and Stein [18] also introduced the local Hardy spaces $h_r^p(\Omega)$ and $h_z^p(\Omega)$ in a similar way and obtained atomic decompositions for these local Hardy spaces when Ω is a special Lipschitz domain or a bounded Lipschitz domain of $\mathbb{R}^n$. Furthermore, the dual spaces of these Hardy spaces and local Hardy spaces were studied in [15]. Let Ω be a strongly Lipschitz domain and let $H_r^1(\Omega)$ and $H_z^1(\Omega)$ be defined as in [18]. Auscher and Russ [8] proved that $H_r^1(\Omega)$ and $H_z^1(\Omega)$ are characterized by the nontangential maximal function and the Lusin area function associated with $\{e^{-t\sqrt{L}}\}_{t \geq 0}$, respectively, under the so-called Dirichlet and Neumann boundary conditions, where L is a second-order divergence form elliptic operator such that for all $t \in (0, \infty)$, the kernel of e^{-tL} has the Gaussian property (G_∞) (see, for example, [8, Definition 3] or Definition 2.4 below). Let Φ be a continuous, strictly increasing, subadditive, positive function on $(0, \infty)$ of strictly critical lower type index

$$p_\Phi \in \left(\frac{n}{n+\mu}, 1 \right].$$

The Orlicz-Hardy spaces $H_r^\Phi(\Omega)$ and $H_z^\Phi(\Omega)$ were first introduced, respectively, in [87] and [88]. Similar to [86], we point out that the assumptions on p_Φ in [87, 88] can also be relaxed into the same assumptions on p_Φ^- ; see Remark 2.7 below.

Let $h_r^1(\Omega)$ and $h_z^1(\Omega)$ be defined as in [18]. Auscher and Russ [8] also showed that $h_r^1(\Omega)$ and $h_z^1(\Omega)$ can be characterized by the local nontangential maximal function and the local Lusin area function associated with $\{e^{-t\sqrt{L}}\}_{t \geq 0}$, respectively, under the Dirichlet and the Neumann boundary conditions. Here L is a second-order divergence form elliptic operator satisfying the Gaussian property (G_1) .

Let $p \in (0, 1]$ and Ω be a proper open subset of $\mathbb{R}^n$. Let $H_r^p(\Omega)$ be defined as in [18]. Miyachi [66] obtained three equivalent characterizations of $H_r^p(\Omega)$ in terms of the grand maximal function, the radial maximal function and the atomic decomposition. The dual theory of $H_r^p(\Omega)$ was also obtained in [66]. Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$. Wang and Yang [84] introduced the weighted local Hardy spaces on Ω by restricting arbitrary elements of $h_\omega^p(\mathbb{R}^n)$ to Ω with $\omega \in A_\infty(\mathbb{R}^n)$. They characterized the space $h_{\omega, r}^p(\Omega)$ in terms of the grand maximal function, the radial maximal function and the atomic decomposition. They also applied these characterizations to harmonic functions defined on bounded Lipschitz domains.

Let Ω be a domain of $\mathbb{R}^n$. In what follows, we denote by $W^{1,2}(\Omega)$ the usual Sobolev space on Ω equipped with the norm

$$\left\{ \|f\|_{L^2(\Omega)}^2 + \|\nabla f\|_{L^2(\Omega)}^2 \right\}^{1/2},$$

where ∇f denotes the *distributional gradient* of f . In what follows, $W_0^{1,2}(\Omega)$ stands for the *closure of $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega)$* , where $C_c^\infty(\Omega)$ denotes the *set of all C^∞ functions on Ω with compact support*.

Now we recall the inhomogeneous Dirichlet problem and Neumann problem on bounded domains of $\mathbb{R}^n$. Given an open, bounded subset Ω of $\mathbb{R}^n$ and $f \in C^\infty(\overline{\Omega})$, we denote by $\mathbb{G}_D(f)$ the *unique solution in $W_0^{1,2}(\Omega)$ of the inhomogeneous Dirichlet problem*

$$(1.1) \quad \begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and refer to $\mathbb{G}_D$ as the *Dirichlet Green operator*. Given an open, bounded subset Ω of $\mathbb{R}^n$ and $f \in C^\infty(\overline{\Omega})$, we denote by $\mathbb{G}_N(f)$ the *unique solution in $W^{1,2}(\Omega)$ of the inhomogeneous Neumann problem*

$$(1.2) \quad \begin{cases} \Delta u = f, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \end{cases}$$

where it is assumed that $\int_\Omega f(x) dx = 0$ and the solution is normalized by requiring that $\int_\Omega u(x) dx = 0$, $\nu(x)$ denotes the *unit outward normal* to $\partial\Omega$ at $x \in \partial\Omega$, and $\partial_\nu := \nabla \cdot \nu$ stands for the *normal derivative*.

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. The regularity of the operators $\mathbb{G}_D$ and $\mathbb{G}_N$ on $L^p(\Omega)$ spaces, for $p \in (1, \infty)$, is well known; see, for example, [4, 57]. More results, including extensions to local Hardy spaces $h_r^p(\Omega)$ or $h_z^p(\Omega)$ for $p \in (0, 1]$, were obtained in [16, 17, 18]. A natural question is to study the regularity of these Green operators on L^p spaces for $p \in (1, \infty)$ on a bounded domain Ω in $\mathbb{R}^n$ under weaker smoothness hypotheses on the boundary $\partial\Omega$ of Ω . Similarly, one may ask whether the $L^p(\Omega)$ estimates can be replaced by local Hardy spaces, $h_r^p(\Omega)$ or $h_z^p(\Omega)$, for $p \in (0, 1]$. We give a brief survey of the progress in this direction via (i) through (vi) as follows (see also [26]):

(i) One early result in this line of work obtained by Kadlec [54] is that the mappings

$$(1.3) \quad f \mapsto \frac{\partial^2 \mathbb{G}_D(f)}{\partial x_i \partial x_j}, \quad i, j \in \{1, \dots, n\},$$

are well defined and bounded on $L^2(\Omega)$ whenever Ω is a bounded convex domain in $\mathbb{R}^n$; see also [1, 35].

(ii) Assume that Ω is a bounded convex domain in $\mathbb{R}^n$. The mappings in (1.3) were shown to be of weak type $(1, 1)$, independently, by Dahlberg et al. [25] and Fromm [31], and to be bounded on a suitable Hardy space by Adolfsen [2]. By interpolation, these mappings are bounded on $L^p(\Omega)$ for $p \in (1, 2)$.

(iii) The L^2 -boundedness of the mappings,

$$(1.4) \quad f \mapsto \frac{\partial^2 \mathbb{G}_N(f)}{\partial x_i \partial x_j}, \quad i, j \in \{1, \dots, n\},$$

has been known since the mid 1970s (see, for example, [35]) when Ω is a bounded convex domain in $\mathbb{R}^n$, but the optimal $L^p(\Omega)$ estimates, valid in the range $p \in (1, 2]$, have only been proved in 1994 by Adolfsen and Jerison [3]. Their method was to obtain an endpoint estimate for atoms in a suitable Hardy space $H^1(\Omega)$, and then to use interpolation with the $L^2(\Omega)$ results.

(iv) For $p \in (0, 1]$, the regularity of the Green operators $\mathbb{G}_D$ and $\mathbb{G}_N$ on scales of local Hardy spaces, $h_r^p(\Omega)$ or $h_z^p(\Omega)$, have been studied by Mayboroda and Mitrea [60, 61] when Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and the results were formulated in terms of a pair of Hardy spaces, $h_r^p(\Omega)$ and $h_z^p(\Omega)$, for the range $p \in (\frac{n}{n+\epsilon}, 1)$ for some $\epsilon \in (0, 1]$. Recently, Mitrea et al. [63] studied the mapping properties of the mappings in (1.3) on Besov and Triebel-Lizorkin spaces in a bounded Lipschitz domain satisfying a uniform exterior ball condition. In particular, the mappings in (1.3) are bounded from $h_r^p(\Omega)$ into itself when $p \in (\frac{n}{n+1}, 1]$.

(v) Very recently, Mitrea et al. [64] proved that these mappings in (1.3) and (1.4) are bounded on $L^2(\Omega)$ when Ω is a bounded semiconvex domain in $\mathbb{R}^n$ (which contains bounded convex domains). Moreover, Duong et al. [26] obtained the boundedness of these mappings in (1.3) from the local Hardy spaces $h_{\Delta_D}^p(\Omega)$ to $L^p(\Omega)$ for the range $p \in (0, 1]$ and proved that these mappings are also of weak type $(1, 1)$, when Ω is a bounded simply connected semiconvex domain in $\mathbb{R}^n$. By interpolation, these mappings are bounded on $L^p(\Omega)$ for all $p \in (1, 2)$. In [26], Duong et al. also showed that these mappings in (1.4) are bounded from the local Hardy spaces $h_{\Delta_N}^p(\Omega)$ to $L^p(\Omega)$ for the range $p \in (0, 1]$ and are of weak type $(1, 1)$, when Ω is a bounded simply connected semiconvex domain in $\mathbb{R}^n$. Hence, by interpolation again, these mappings are bounded on $L^p(\Omega)$ for $p \in (1, 2)$. Based on these results, Duong et al. [26] proved that the mappings in (1.3) are bounded on the local Hardy space $h_r^p(\Omega)$, for the range $p \in (\frac{n}{n+1}, 1]$, when Ω is a bounded simply connected semiconvex domain in $\mathbb{R}^n$; and the mappings in (1.4) are bounded from the space $h_z^p(\Omega)$ to the space $h_r^p(\Omega)$ when $p \in (\frac{n}{n+1}, 1]$ and Ω is a bounded simply connected convex domain in $\mathbb{R}^n$.

(vi) In relation to (ii) and (iii) above, it should be mentioned that the aforementioned $L^p(\Omega)$ boundedness of the mappings in (1.3) and (1.4) may fail in the class of Lipschitz domains Ω for any $p \in (1, \infty)$ and in the class of convex domains Ω for any $p \in (2, \infty)$ (see [2, 3, 24, 47] for counterexamples; recall that every convex domain is a Lipschitz domain).

Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$, and $\omega \in A_\infty(\mathbb{R}^n)$. Let L be a second-order divergence form elliptic operator on $L^2(\Omega)$ with the Dirichlet or Neumann boundary condition, and assume that the heat semigroup generated by L has the Gaussian property (G_1) with the regularity of their kernels measured

by $\mu \in (0, 1]$. Let Φ be a continuous, strictly increasing, subadditive, positive and concave function on $(0, \infty)$ of critical lower type index $p_{\Phi}^{-} \in (0, 1]$. A typical example of such functions is

$$\Phi(t) := t^p \quad \text{for all } t \in (0, \infty) \quad \text{and } p \in (0, 1].$$

More examples are given in Section 2.3 below. First, motivated by [8, 16, 18, 49, 51, 86], we introduce the weighted local Orlicz-Hardy spaces $h_{\omega, r}^{\Phi}(\Omega)$ and $h_{\omega, z}^{\Phi}(\Omega)$, respectively, by restricting arbitrary elements of $h_{\omega}^{\Phi}(\mathbb{R}^n)$ to Ω or by restricting elements of $h_{\omega}^{\Phi}(\mathbb{R}^n)$, which are zero outside $\overline{\Omega}$, to Ω , where $h_{\omega}^{\Phi}(\mathbb{R}^n)$ denotes the weighted local Orlicz-Hardy space introduced in [86]. Then we establish the atomic decompositions of these spaces by means of the Lusin area function associated with $\{e^{-tL}\}_{t \geq 0}$. Applying this, we obtain two equivalent characterizations of $h_{\omega, r}^{\Phi}(\Omega)$ and $h_{\omega, z}^{\Phi}(\Omega)$, respectively, in terms of the nontangential maximal function and the Lusin area function associated with $\{e^{-tL}\}_{t \geq 0}$. Second, motivated by [66], we furthermore establish three equivalent characterizations of the spaces $h_{\omega, r}^{\Phi}(\Omega)$, respectively, in terms of the grand maximal function, the radial maximal function and the atomic decomposition, when the complement of Ω is unbounded and $p_{\Phi}^{-} \in (0, 1]$. Third, as applications, we prove that the operators $\nabla^2 \mathbb{G}_D$ are bounded from $h_{\omega, r}^{\Phi}(\Omega)$ to the weighted Orlicz space $L_{\omega}^{\Phi}(\Omega)$ and from $h_{\omega, z}^{\Phi}(\Omega)$ to itself, when Ω is a bounded semiconvex domain in $\mathbb{R}^n$ and $p_{\Phi}^{-} \in (\frac{n}{n+1}, 1]$, and the operators $\nabla^2 \mathbb{G}_N$ are bounded from $h_{\omega, z}^{\Phi}(\Omega)$ to $L_{\omega}^{\Phi}(\Omega)$ and from $h_{\omega, z}^{\Phi}(\Omega)$ to $h_{\omega, r}^{\Phi}(\Omega)$, when Ω is a bounded convex domain in $\mathbb{R}^n$ and $p_{\Phi}^{-} \in (\frac{n}{n+1}, 1]$, where $\mathbb{G}_D$ and $\mathbb{G}_N$ denote, respectively, the Dirichlet Green operator and the Neumann Green operator.

To state the main results of this paper, we first recall some necessary notions. Throughout the whole paper, we always assume that Ω is a *strongly Lipschitz domain* of $\mathbb{R}^n$; namely, Ω is a proper open connected set in $\mathbb{R}^n$ whose boundary is a finite union of parts of rotated graphs of Lipschitz maps, at most one of these parts possibly unbounded. It is well known that strongly Lipschitz domains include *special Lipschitz domains*, *bounded Lipschitz domains* and *exterior domains*; see, for example, [8, 10] for their definitions and properties.

Also, throughout the entire paper, for the sake of convenience, we choose the norm on $\mathbb{R}^n$ to be the *supremum norm*; namely, for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$,

$$|x| := \max\{|x_1|, \dots, |x_n|\},$$

for which balls determined by this norm are cubes associated with the usual Euclidean norm with sides parallel to the axes.

Remark 1.1. Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. Then Ω is a space of homogeneous type in the sense of Coifman and Weiss [21]. Furthermore, as a space of homogeneous type, the collection of all balls of Ω is given by the set

$$\{Q \cap \Omega : \text{cube } Q \subset \mathbb{R}^n \text{ satisfying } x_Q \in \Omega \text{ and } l(Q) \leq 2 \text{diam}(\Omega)\},$$

where x_Q denotes the *center* of Q , $l(Q)$ the *sidelength* of Q and $\text{diam}(\Omega)$ the *diameter* of Ω , namely,

$$\text{diam}(\Omega) := \sup\{|x - y| : x, y \in \Omega\};$$

see, for example, [8].

To introduce the spaces $h_{\omega, r}^{\Phi}(\Omega)$ and $h_{\omega, z}^{\Phi}(\Omega)$, we first recall the definition of the weighted local Orlicz-Hardy space $h_{\omega}^{\Phi}(\mathbb{R}^n)$ introduced in [86]. Let $\mathcal{D}(\mathbb{R}^n)$ denote

the space of all infinitely differentiable functions with compact support in $\mathbb{R}^n$ endowed with the inductive topology, and $\mathcal{D}'(\mathbb{R}^n)$ its topological dual with the weak-* topology which is called the space of distributions on $\mathbb{R}^n$. For all $f \in \mathcal{D}'(\mathbb{R}^n)$, let $\mathcal{G}^{\text{loc}}(f)$ denote its local grand maximal function; see [86, Definition 3.1].

Definition 1.2. Let Φ satisfy Assumption (A) (see Section 2.2 for the definition of Assumption (A)) and $\omega \in A_\infty(\mathbb{R}^n)$ (see Definition 2.1 below for the definition of $A_\infty(\mathbb{R}^n)$). Define

$$h_\omega^\Phi(\mathbb{R}^n) := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\mathcal{G}^{\text{loc}}(f)(x)) \omega(x) dx < \infty \right\}$$

and

$$\|f\|_{h_\omega^\Phi(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi\left(\frac{\mathcal{G}^{\text{loc}}(f)(x)}{\lambda}\right) \omega(x) dx \leq 1 \right\}.$$

In what follows, let $\mathcal{D}(\Omega)$ denote the space of all infinitely differentiable functions with compact support in Ω endowed with the inductive topology, and $\mathcal{D}'(\Omega)$ its topological dual with the weak-* topology which is called the space of distributions on Ω .

Definition 1.3. Let Φ and ω be as in Definition 1.2, and Ω a subdomain of $\mathbb{R}^n$. A distribution f on Ω is said to be in the weighted local Orlicz-Hardy space $h_{\omega,r}^\Phi(\Omega)$ if f is the restriction to Ω of a distribution F in $h_\omega^\Phi(\mathbb{R}^n)$; namely,

$$\begin{aligned} h_{\omega,r}^\Phi(\Omega) &:= \{f \in \mathcal{D}'(\Omega) : \text{there exists an } F \in h_\omega^\Phi(\mathbb{R}^n) \text{ such that } F|_\Omega = f\} \\ &= h_\omega^\Phi(\mathbb{R}^n) / \{F \in h_\omega^\Phi(\mathbb{R}^n) : F = 0 \text{ on } \Omega\}. \end{aligned}$$

Moreover, for all $f \in h_{\omega,r}^\Phi(\Omega)$, the norm of f in $h_{\omega,r}^\Phi(\Omega)$ is defined by

$$\|f\|_{h_{\omega,r}^\Phi(\Omega)} := \inf \left\{ \|F\|_{h_\omega^\Phi(\mathbb{R}^n)} : F \in h_\omega^\Phi(\mathbb{R}^n) \text{ and } F|_\Omega = f \right\},$$

where the infimum is taken over all $F \in h_\omega^\Phi(\mathbb{R}^n)$ satisfying $F = f$ on Ω .

The weighted local Orlicz-Hardy space $h_{\omega,z}^\Phi(\Omega)$ is defined by

$$h_{\omega,z}^\Phi(\Omega) := \left\{ f \in h_\omega^\Phi(\mathbb{R}^n) : f = 0 \text{ on } (\overline{\Omega})^c \right\} / \{f \in h_\omega^\Phi(\mathbb{R}^n) : f = 0 \text{ on } \Omega\},$$

where $(\overline{\Omega})^c$ denotes the set $\mathbb{R}^n \setminus \overline{\Omega}$. Moreover, for any $f \in h_{\omega,z}^\Phi(\Omega)$, its norm in $h_{\omega,z}^\Phi(\Omega)$ is defined by

$$\|f\|_{h_{\omega,z}^\Phi(\Omega)} := \inf \left\{ \|F\|_{h_\omega^\Phi(\mathbb{R}^n)} : F \in h_\omega^\Phi(\mathbb{R}^n), F = 0 \text{ on } (\overline{\Omega})^c \text{ and } F|_\Omega = f \right\}.$$

Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$. Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$ and L be a divergence form elliptic operator on $L^2(\Omega)$ with the Dirichlet boundary condition (for simplicity, DBC) or the Neumann boundary condition (for simplicity, NBC) (see (2.7) below for the definition of L and Definition 2.3 below for DBC and NBC). Let the spaces $h_{\mathcal{N}_h,\omega}^\Phi(\Omega)$, $h_{\mathcal{S}_h,\omega}^\Phi(\Omega)$ and $h_{\mathcal{S}_h,\omega}^\Phi(\Omega)$ be respectively as in Definitions 3.1 and 3.3 below. Then one of the main results of this paper is as follows.

Theorem 1.4. Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$ and L be as in (2.7). Let q_ω , r_ω , μ , p_Φ^+ and p_Φ^- be respectively as in (2.5), (2.6), (2.9), (2.11) and (2.12).

Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$. Assume that q_ω , r_ω , μ , p_Φ^+ and p_Φ^- satisfy the inequalities $\frac{q_\omega}{p_\Phi} < \frac{n+\mu}{n}$,

$$\frac{2q_\omega}{p_\Phi^-} < \frac{n+1}{n} + \frac{r_\omega - 1}{p_\Phi^+ r_\omega}$$

and $r_\omega > \frac{2}{2-q_\omega}$, and the semigroup generated by L has the Gaussian property (G_1) .

(i) If $\Omega := \mathbb{R}^n$, then the spaces

$$h_\omega^\Phi(\mathbb{R}^n), h_{\mathcal{N}_h, \omega}^\Phi(\mathbb{R}^n), h_{\tilde{S}_h, \omega}^\Phi(\mathbb{R}^n) \text{ and } h_{S_h, \omega}^\Phi(\mathbb{R}^n)$$

coincide with equivalent quasi-norms.

(ii) Under DBC, if $\Omega^\mathbb{G}$ is unbounded, then the spaces

$$h_{\omega, r}^\Phi(\Omega), h_{\mathcal{N}_h, \omega}^\Phi(\Omega), h_{\tilde{S}_h, \omega}^\Phi(\Omega) \text{ and } h_{S_h, \omega}^\Phi(\Omega)$$

coincide with equivalent quasi-norms.

(iii) Under NBC, the spaces

$$h_{\omega, z}^\Phi(\Omega), h_{\mathcal{N}_h, \omega}^\Phi(\Omega), h_{\tilde{S}_h, \omega}^\Phi(\Omega) \text{ and } h_{S_h, \omega}^\Phi(\Omega)$$

coincide with equivalent quasi-norms.

We first point out that the coincidence between $h_\omega^\Phi(\mathbb{R}^n)$ and $h_{\mathcal{N}_h, \omega}^\Phi(\mathbb{R}^n)$, or between $h_{\omega, r}^\Phi(\Omega)$ and $h_{\mathcal{N}_h, \omega}^\Phi(\Omega)$, or between $h_{\omega, z}^\Phi(\Omega)$ and $h_{\mathcal{N}_h, \omega}^\Phi(\Omega)$ of Theorem 1.4 when $\Phi(t) := t$ for all $t \in (0, \infty)$ and $\omega \equiv 1$ was already obtained by Auscher and Russ in [8, Theorems 2 and 20].

The proofs of (i) through (iii) of Theorem 1.4 are similar and can be given by following the main approach used by Auscher and Russ in [8] with some subtle modifications. Here we only give the framework of the proof of Theorem 1.4(ii). The following chains of inequalities give the strategy of the proof of Theorem 1.4(ii). For all $f \in h_{\omega, r}^\Phi(\Omega) \cap L^2(\Omega)$ and any given $R_0 \in [\frac{1}{2}, \infty)$, we have

$$\begin{aligned} (1.5) \quad \|f\|_{h_{\omega, r}^\Phi(\Omega)} &\gtrsim \left\| \mathcal{N}_h^{\text{loc}, 2R_0}(f) \right\|_{L_\omega^\Phi(\Omega)} \\ &\gtrsim \left\| \tilde{S}_{h, R_0}^{\text{loc}}(f) \right\|_{L_\omega^\Phi(\Omega)} + \mathbf{I}(f) \\ &\gtrsim \left\| S_{h, R_0}^{\text{loc}}(f) \right\|_{L_\omega^\Phi(\Omega)} + \mathbf{I}(f) \gtrsim \|f\|_{h_{\omega, r}^\Phi(\Omega)}, \end{aligned}$$

where the implicit constants are independent of f , and $\mathcal{N}_h^{\text{loc}, 2R_0}(f)$, $\tilde{S}_{h, R_0}^{\text{loc}}(f)$ and $S_{h, R_0}^{\text{loc}}(f)$ are respectively as in Definitions 3.1 and 3.3 below, and

$$\mathbf{I}(f) := \inf \left\{ \lambda \in (0, \infty) : \sum_{\tilde{Q}_k \in \mathcal{Q}_\Omega} \omega(\tilde{Q}_k \cap \Omega) \Phi \left(\frac{m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\}$$

with $m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)$ being the average of $|e^{-R_0^2 L}(f)|$ on $\tilde{Q}_k \cap \Omega$ (see (3.1) below). Moreover, see Section 3 for the definitions of $\mathcal{Q}_\Omega$ and $\tilde{Q}_k$. Theorem 1.4(ii) is deduced from (1.5) and the arbitrariness of $R_0 \in [\frac{1}{2}, \infty)$. The proof of the first inequality in (1.5) is standard by applying the atomic decomposition of $h_\omega^\Phi(\mathbb{R}^n)$ obtained in [86] and the relation between $h_{\omega, r}^\Phi(\Omega)$ and $h_\omega^\Phi(\mathbb{R}^n)$; see Proposition 3.4 below. We prove the second and the third inequalities, respectively, in Propositions 3.8 and 3.12 below. We point out that Proposition 3.8 plays an important role in the proof of Theorem 1.4(ii), and the key step in the proof of Proposition 3.8

is to establish a “good- λ inequality” concerning $\mathcal{N}_h^{\text{loc}, 2R_0}(f)$ and $\tilde{S}_{h, R_0}^{\text{loc}}(f)$ (see Lemma 3.10 below), which is a subtle weighted variant on the local nontangential maximal function and the local Lusin-area function of [8, Lemma 9] and even if its special case that $\omega \equiv 1$ also improves [8, Lemma 9] in the sense that [8, Lemma 9] presents a “good- λ inequality” on the nontangential maximal function and the average of the Lusin-area function, not the Lusin-area function itself (see also [88]). To show the last inequality of (1.5) in Proposition 3.13(i) below, for all $f \in L^2(\Omega)$ satisfying $\|S_{h, R_0}^{\text{loc}}(f)\|_{L_\omega^\Phi(\Omega)} < \infty$, we establish its atomic decomposition by using a local Calderón reproducing formula (see (3.39) below) on $L^2(\Omega)$ associated with L , the atomic decomposition of functions in the tent space on Ω , and the reflection technology related to Lipschitz domains on $\mathbb{R}^n$ which was proved by Auscher and Russ in [8, p. 183] and plays a key role in the proof of Theorem 1.4 (see also Lemma 3.15 below). But, this reflection technology was not necessary in the proof of Theorem 1.4(iii) (see also [8]). We point out that the method used in the proof of the last inequality of (1.5) is also quite different from that used in the proof of [8, Proposition 19] for a similar inequality on $h_r^1(\Omega)$. Since $h_{\omega, r}^\Phi(\Omega)$ when $p_\Phi^- < 1$ is only a quasi-Banach space, not a Banach space, and the norms of elements in quasi-Banach spaces cannot be achieved by duality, the method used in the proof of [8, Proposition 19] cannot be applied to the current situation; see also [88].

Let $p \in (\frac{n}{n+1}, 1]$. Recall that *bounded convex domains and semiconvex domains in $\mathbb{R}^n$ are strongly Lipschitz domains of $\mathbb{R}^n$* . We remark that when $L = -\Delta$ with the Dirichlet boundary condition, Ω is a bounded semiconvex domain in $\mathbb{R}^n$, $\omega \equiv 1$ and $\Phi(t) := t^p$ for all $t \in (0, \infty)$, Theorem 1.4(ii) coincides with [26, Proposition 5.3(ii)]; when $L = -\Delta$ with the Neumann boundary condition, Ω is a bounded convex domain in $\mathbb{R}^n$, $\omega \equiv 1$ and $\Phi(t) := t^p$ for all $t \in (0, \infty)$, Theorem 1.4(iii) was also obtained in [26, Proposition 5.3(i)]. We remark that the approach used in the proofs of [26, Theorem 3.5 and Proposition 5.3] is quite different from that used in the proof of Theorem 1.4. A key tool used in the proofs of [26, Theorem 3.5 and Proposition 5.3] is the atomic characterization closely associated with $-\Delta$ on domains, while in the proof of Theorem 1.4, we fully use the atomic characterization of $h_\omega^\Phi(\mathbb{R}^n)$ in [86].

To state the second main theorem of this paper, we first recall the following notions of the radial and the grand maximal functions on Ω .

Definition 1.5. Let Ω be an open subset of $\mathbb{R}^n$. Denote the boundary of Ω by $\partial\Omega$. For all $x \in \Omega$, let $\delta(x) := \text{dist}(x, \partial\Omega)$. Let

$$(1.6) \quad \varphi \in \mathcal{D}(\mathbb{R}^n) \text{ with } \text{supp}(\varphi) \subset B(0, 1) \text{ and } \int_{\mathbb{R}^n} \varphi(x) dx = 1.$$

Let $c_0 \in (1, \infty)$. For any $f \in \mathcal{D}'(\Omega)$, the *radial maximal function* $f_{\varphi, \Omega}^+$ of f associated to φ on Ω is defined by setting, for all $x \in \Omega$,

$$(1.7) \quad f_{\varphi, \Omega}^+(x) := \sup_{0 < t < \delta(x)/c_0} |\varphi_t * f(x)|,$$

where and in what follows, for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$\varphi_t(x) := \frac{1}{t^n} \varphi\left(\frac{x}{t}\right).$$

The *grand maximal function* f_Ω^* of f on Ω is defined by setting, for all $x \in \Omega$,

$$(1.8) \quad f_\Omega^*(x) := \sup_{0 < t < \delta(x)/c_0} \sup_{\psi \in \mathcal{F}_t(x)} |\langle f, \psi \rangle|,$$

where for all $x \in \Omega$,

$$\mathcal{F}_t(x) := \left\{ \psi \in \mathcal{D}(\mathbb{R}^n) : \text{supp}(\psi) \subset B(x, t), \right. \\ \left. \sup_{y \in \mathbb{R}^n} |\partial^\alpha \psi(y)| \leq t^{-|\alpha|-n} \text{ for every } \alpha \in \mathbb{Z}_+^n \right\}.$$

We also need the following notions of local $(\rho, q, s)_\omega$ -atoms on Ω . Recall that the space $L_\omega^q(\Omega)$, for $q \in [1, \infty]$ and $\omega \in A_\infty(\mathbb{R}^n)$, denotes the weighted Lebesgue space endowed with the norm that, for any $f \in L_\omega^q(\Omega)$, when $q \in [1, \infty)$,

$$\|f\|_{L_\omega^q(\Omega)} := \left\{ \int_\Omega |f(x)|^q \omega(x) dx \right\}^{1/q},$$

and

$$\|f\|_{L_\omega^\infty(\Omega)} := \|f\|_{L^\infty(\Omega)}.$$

Definition 1.6. Let Ω be an open subset of $\mathbb{R}^n$ and $\omega \in A_\infty(\mathbb{R}^n)$, q_ω, p_Φ^- and ρ be respectively as in (2.5), (2.12) and (2.14) below. A triplet $(\rho, q, s)_\omega$ is called *admissible* if $q \in (q_\omega, \infty]$ and $s \in \mathbb{Z}_+$ with $s \geq \lfloor n(\frac{q_\omega}{p_\Phi} - 1) \rfloor$. A function a supported on a cube $Q \subset \Omega$ is called a *type (a) local $(\rho, q, s)_\omega$ -atom* if $4Q \cap \partial\Omega = \emptyset$ with $l(Q) < 1$,

$$\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(Q)]^{\frac{1}{q}-1} [\rho(\omega(Q))]^{-1},$$

and

$$\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$$

for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$.

A function a supported on a cube $Q \subset \Omega$ is called a *type (b) local $(\rho, q, s)_\omega$ -atom* if either $l(Q) \geq 1$ or $2Q \cap \partial\Omega = \emptyset$ and $4Q \cap \partial\Omega \neq \emptyset$, and

$$\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(Q)]^{\frac{1}{q}-1} [\rho(\omega(Q))]^{-1}.$$

Theorem 1.7. Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$, Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ satisfying that Ω^c is unbounded, and φ be as in (1.6). Then the following are equivalent:

- (i) $f \in h_{\omega, r}^\Phi(\Omega)$;
- (ii) $f \in \mathcal{D}'(\Omega)$ and $f_{\varphi, \Omega}^+ \in L_\omega^\Phi(\Omega)$, where $f_{\varphi, \Omega}^+$ is as in (1.7);
- (iii) $f \in \mathcal{D}'(\Omega)$ and $f_\Omega^* \in L_\omega^\Phi(\Omega)$, where f_Ω^* is as in (1.8);
- (iv) $f \in \mathcal{D}'(\Omega)$ and

$$f = \sum_{\text{type (a) atoms}} \lambda_Q a_Q + \sum_{\text{type (b) atoms}} \mu_Q b_Q$$

in $\mathcal{D}'(\Omega)$, where $(\rho, q, s)_\omega$ is an admissible triplet, $\{a_Q\}_Q$ is a sequence of type (a) local $(\rho, q, s)_\omega$ -atoms, $\{b_Q\}_Q$ is a sequence of type (b) local $(\rho, q, s)_\omega$ -atoms and $\{\lambda_Q\}_Q \cup \{\mu_Q\}_Q \subset \mathbb{C}$ satisfying

$$\sum_{\text{type (a) atoms}} \omega(Q) \Phi \left(\frac{|\lambda_Q|}{\omega(Q) \rho(\omega(Q))} \right)$$

$$+ \sum_{\text{type } (b) \text{ atoms}} \omega(Q) \Phi \left(\frac{|\lambda_Q|}{\omega(Q) \rho(\omega(Q))} \right) < \infty.$$

Moreover, define

$$\begin{aligned} & \Lambda(\{\lambda_Q a_Q\}_Q \cup \{\mu_Q b_Q\}_Q) \\ &:= \inf \left\{ \lambda \in (0, \infty) : \sum_{\text{type } (a) \text{ atoms}} \omega(Q) \Phi \left(\frac{|\lambda_Q|}{\lambda \omega(Q) \rho(\omega(Q))} \right) \right. \\ & \quad \left. + \sum_{\text{type } (b) \text{ atoms}} \omega(Q) \Phi \left(\frac{|\lambda_Q|}{\lambda \omega(Q) \rho(\omega(Q))} \right) \leq 1 \right\} \end{aligned}$$

and

$$\|f\|_{h_{\omega}^{\rho, q, s}(\Omega)} := \inf \{ \Lambda(\{\lambda_Q a_Q\}_Q \cup \{\mu_Q b_Q\}_Q) \},$$

where the infimum is taken over all the decompositions of f as above.

Furthermore, for all $f \in h_{\omega, r}^{\Phi}(\Omega)$,

$$\|f\|_{h_{\omega, r}^{\Phi}(\Omega)} \sim \|f_{\varphi, \Omega}^{+}\|_{L_{\omega}^{\Phi}(\Omega)} \sim \|f_{\Omega}^{*}\|_{L_{\omega}^{\Phi}(\Omega)} \sim \|f\|_{h_{\omega}^{\rho, q, s}(\Omega)},$$

where the implicit constants are independent of f .

The following outline gives the strategy of the proof of Theorem 1.7: (i) $\iff$ (iv) $\implies$ (iii) $\iff$ (ii) $\implies$ (i). We divide the proof of Theorem 1.7 into the following four steps. Step I: (i) $\iff$ (iv); Step II: (iii) $\iff$ (ii); Step III: (iv) $\implies$ (iii); Step IV: (ii) $\implies$ (i). In Step I, we prove that (i) implies (iv) by using the atomic characterization of $h_{\omega}^{\Phi}(\mathbb{R}^n)$ and the definition of the space $h_{\omega, r}^{\Phi}(\Omega)$; moreover, for any given $f \in \mathcal{D}'(\Omega)$ satisfying (iv) of Theorem 1.7, we construct an $F \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ such that $F|_{\Omega} = f$ and

$$\|F\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|f\|_{h_{\omega}^{\rho, q, s}(\Omega)}$$

by using the reflection technology related to Lipschitz domains on $\mathbb{R}^n$ from [8] (see also Lemma 3.15 below), which completes the proof that (iv) implies (i). In Step II, it is obvious that (iii) implies (ii); by using a useful estimate concerning $f_{\Omega, \varphi}^{+}$ and f_{Ω}^{*} , which was established by Miyachi [65] (see also (4.5) below), we prove that (ii) implies (iii). The proof of Step III is standard, which is similar to that of Proposition 3.4(i) below. In Step IV, for any given $f \in \mathcal{D}'(\Omega)$ satisfying $f_{\varphi, \Omega}^{+} \in L_{\omega}^{\Phi}(\Omega)$, we construct an $F \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ such that $F|_{\Omega} = f$ by using two useful estimates established by Miyachi [66] (see also (4.7) and (4.8) below), which shows that (ii) implies (i). We remark that except for its own interest, Theorem 1.7 is also needed to obtain Theorems 1.8 and 1.9 below.

Theorem 1.8. *Let Φ , ω , q_{ω} , r_{ω} and p_{Φ}^{-} be as in Theorem 1.4. Let Ω be a bounded, simply connected, semiconvex domain in $\mathbb{R}^n$, and $\mathbb{G}_D$ the Dirichlet Green operator for the problem (1.1). Then*

(i) *the operators in (1.3), originally defined on $C^{\infty}(\overline{\Omega})$, can be extended to bounded operators from $h_{\omega, r}^{\Phi}(\Omega)$ to $L_{\omega}^{\Phi}(\Omega)$;*

(ii) *the operators in (1.3), originally defined on $C^{\infty}(\overline{\Omega})$, can be extended to bounded operators from $h_{\omega, r}^{\Phi}(\Omega)$ to $h_{\omega, r}^{\Phi}(\Omega)$.*

The proof of Theorem 1.8 below follows the approach used in the proofs of Duong et al. [26, Theorems 4.1 and 5.4]. We prove Theorem 1.8(i) by using the atomic characterization of $h_{\omega,r}^{\Phi}(\Omega)$ obtained in Theorem 1.7, some useful estimates about the integral kernel of the Dirichlet Green operator $\mathbb{G}_D$ from Fromm [31] and Grüter-Widman [36] (see also Lemma 5.2 below), and the $L^p(\Omega)$ -boundedness of the mappings in (1.3) from Duong et al. [26, Theorem 4.1] (see also Lemma 5.3 below). Furthermore, we prove Theorem 1.8(ii) by using Theorem 1.8(i), the radial maximal function characterization of $h_{\omega,r}^{\Phi}(\Omega)$ obtained in Theorem 1.7 and the reflection technology related to Lipschitz domains on $\mathbb{R}^n$ from [8].

Let $p \in (n/(n+1), 1]$. We point out that Theorem 1.8(i) when $\Phi(t) := t^p$ for all $t \in (0, \infty)$ and $\omega \equiv 1$ was obtained in [26, Theorem 4.1]; and Theorem 1.8(ii) completely covers [26, Theorem 5.4] by taking $\Phi(t) := t^p$ for all $t \in (0, \infty)$ and $\omega \equiv 1$.

Theorem 1.9. *Let Φ , ω , q_{ω} , r_{ω} and p_{Φ}^{-} be as in Theorem 1.4. Let Ω be a bounded, simply connected, convex domain in $\mathbb{R}^n$. Let $\mathbb{G}_N$ be the Neumann Green operator for the problem (1.2). Then*

(i) *the operators in (1.4), originally defined on*

$$\left\{ f \in C^{\infty}(\overline{\Omega}) : \int_{\Omega} f(x) dx = 0 \right\},$$

can be extended to bounded operators from $h_{\omega,z}^{\Phi}(\Omega)$ to $L_{\omega}^{\Phi}(\Omega)$;

(ii) *the operators in (1.4), originally defined on*

$$\left\{ f \in C^{\infty}(\overline{\Omega}) : \int_{\Omega} f(x) dx = 0 \right\},$$

can be extended to bounded operators from $h_{\omega,z}^{\Phi}(\Omega)$ to $h_{\omega,r}^{\Phi}(\Omega)$.

Because the integral kernel of the Neumann Green operator $\mathbb{G}_N$ and its gradients do not have good enough size estimates which are similar to those in Lemma 5.3, the method used in the proof of Theorem 1.8 is not valid in the proof of Theorem 1.9. Motivated by Duong et al. [26, Theorem 3.5 and Proposition 4.11], we establish a new atomic decomposition characterization of $h_{\omega,z}^{\Phi}(\Omega)$ via a class of atoms associated with the operator L , where L denotes the Laplace operator on $L^2(\Omega)$ with the Neumann boundary condition; see Lemma 6.3 below. By using this auxiliary result and some integral estimates of the kernel $\{K_t\}_{t \geq 0}$ of the semigroup $\{e^{-tL}\}_{t \geq 0}$ (see Lemma 6.4 below), and the $L^p(\Omega)$ -boundedness of the mappings in (1.4) from Duong et al. [26, Theorem 4.2] (see also Lemma 6.5 below), we prove Theorem 1.9(i). Furthermore, from the new atomic decomposition characterization of $h_{\omega,z}^{\Phi}(\Omega)$, Theorem 1.9(i) and the radial maximal function characterization of $h_{\omega,r}^{\Phi}(\Omega)$ obtained in Theorem 1.7, we deduce Theorem 1.9(ii).

Let $p \in (\frac{n}{n+1}, 1]$. Theorem 1.9(i) when $\Phi(t) := t^p$ for all $t \in (0, \infty)$ and $\omega \equiv 1$ was obtained in [26, Theorem 4.2 and Proposition 5.3(i)]; also Theorem 1.9(ii) completely covers [26, Theorem 5.7] by taking $\Phi(t) := t^p$ for all $t \in (0, \infty)$ and $\omega \equiv 1$.

By [74], we know that the local weight class $A_{\infty}^{\text{loc}}(\mathbb{R}^n)$ satisfies the exponential growth in the whole space $\mathbb{R}^n$, which implies that when $\omega \in A_{\infty}^{\text{loc}}(\mathbb{R}^n)$ but $\omega \notin A_{\infty}(\mathbb{R}^n)$, (3.9) and (3.10) in the proof of Proposition 3.4, and (3.20) in the proof of Proposition 3.8 may not be true, respectively, due to the lack of the compact

support of the kernel of $e^{-t^2 L}$ or the exponential growth of ω . Thus, throughout the whole paper, we *always assume* that $\omega \in A_\infty(\mathbb{R}^n)$.

The layout of this paper is as follows. In Section 2, we first recall the definition of the weight class $A_\infty(\mathbb{R}^n)$ and some of its properties; then we recall some properties of the divergence form elliptic operator L on $\mathbb{R}^n$ or a strongly Lipschitz domain Ω , and then describe some basic assumptions on L ; finally we describe some basic assumptions on Orlicz functions and present some properties of these functions. In Section 3, we give the proof of Theorem 1.4. Section 4 is devoted to the proof of Theorem 1.7. In Sections 5 and 6, we respectively give the proofs of Theorems 1.8 and 1.9.

Finally we make some conventions on notation. Throughout the entire paper, L always denotes the *second-order divergence form elliptic operator* as in (2.7). We denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. We also use $C(\gamma, \beta, \dots)$ to denote a *positive constant* depending on the indicated parameters $\gamma, \beta, \dots$. The *symbol* $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. The *symbol* $\lfloor s \rfloor$ for $s \in \mathbb{R}$ denotes the maximal integer not more than s ; $Q(x, t)$ denotes a *closed cube* in $\mathbb{R}^n$ with *center* $x \in \mathbb{R}^n$ and *sidelength* $l(Q) := t$ and $CQ(x, t) := Q(x, Ct)$. For any given normed spaces $\mathcal{A}$ and $\mathcal{B}$ with the corresponding norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, $\mathcal{A} \subset \mathcal{B}$ means that for all $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. For any subset G of $\mathbb{R}^n$, we denote by G^c the *set* $\mathbb{R}^n \setminus G$, and by χ_G its *characteristic function*. We also set $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any $\theta := (\theta_1, \dots, \theta_n) \in \mathbb{Z}_+^n$, let

$$|\theta| := \theta_1 + \dots + \theta_n \quad \text{and} \quad \partial_x^\theta := \frac{\partial^{|\theta|}}{\partial x_1^{\theta_1} \dots \partial x_n^{\theta_n}}.$$

For any sets $E, F \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, let

$$\text{dist}(E, F) := \inf_{x \in E, y \in F} |x - y| \quad \text{and} \quad \text{dist}(z, E) := \inf_{x \in E} |x - z|.$$

2. PRELIMINARIES

In Subsection 2.1, we first recall some properties of the class of Muckenhoupt weights; in Subsection 2.2, we then recall some properties of the divergence form elliptic operator L on $\mathbb{R}^n$ or a strongly Lipschitz domain Ω , and describe some basic assumptions on L ; in Subsection 2.3, we describe some basic assumptions of Orlicz functions and then present some properties of these functions.

2.1. Some properties of the weight class $A_\infty(\mathbb{R}^n)$. In this subsection, we first recall the definitions and some properties of the class of Muckenhoupt weights and the reverse Hölder class; see, for example, [32, 33, 34].

Definition 2.1. (i) Let $p \in (1, \infty)$. The *weight class* $A_p(\mathbb{R}^n)$ is defined to be the set of all nonnegative locally integrable functions ω on $\mathbb{R}^n$ such that

$$(2.1) \quad A_p(\omega) := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|^p} \int_Q \omega(x) dx \left(\int_Q [\omega(x)]^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

When $p = 1$, the *weight class* $A_1(\mathbb{R}^n)$ is defined to be the set of all nonnegative locally integrable functions ω on $\mathbb{R}^n$ such that

$$(2.2) \quad A_1(\omega) := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q \omega(x) dx \left(\operatorname{ess\,sup}_{y \in Q} [\omega(y)]^{-1} \right) < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

(ii) Let $r \in (1, \infty)$. The *reverse Hölder class* $RH_r(\mathbb{R}^n)$ is defined to be the set of all nonnegative locally integrable functions ω on $\mathbb{R}^n$ such that

$$(2.3) \quad RH_r(\omega) := \sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q [\omega(x)]^r dx \right)^{1/r} \left[\frac{1}{|Q|} \int_Q \omega(x) dx \right]^{-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

When $r = \infty$, the *reverse Hölder class* $RH_\infty(\mathbb{R}^n)$ is defined to be the set of all nonnegative locally integrable functions ω on $\mathbb{R}^n$ such that

$$(2.4) \quad RH_\infty(\omega) := \sup_{Q \subset \mathbb{R}^n} \left[\operatorname{ess\,sup}_{y \in Q} \omega(y) \right] \left[\frac{1}{|Q|} \int_Q \omega(x) dx \right]^{-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

In what follows, for a Lebesgue measurable set $E \subset \mathbb{R}^n$ and a weight $\omega \in A_\infty(\mathbb{R}^n)$, let

$$\omega(E) := \int_E \omega(x) dx.$$

Then we recall some properties of the Muckenhoupt weights as follows.

Lemma 2.2. (i) If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}(\mathbb{R}^n) \subset A_{p_2}(\mathbb{R}^n)$.

(ii) If $\omega \in A_p(\mathbb{R}^n)$ with $p \in (1, \infty)$, then there exists $q \in (1, p)$ such that $\omega \in A_q(\mathbb{R}^n)$.

(iii) If $\omega \in A_p(\mathbb{R}^n)$ with $p \in [1, \infty)$, then there exists a positive constant C such that for any cube $Q \subset \mathbb{R}^n$ and any measurable subset E of Q ,

$$\frac{\omega(Q)}{\omega(E)} \leq C \left[\frac{|Q|}{|E|} \right]^p.$$

(iv) If $1 < p_1 \leq p_2 < \infty$, then

$$RH_\infty(\mathbb{R}^n) \subset RH_{p_2}(\mathbb{R}^n) \subset RH_{p_1}(\mathbb{R}^n).$$

(v) If $\omega \in RH_r(\mathbb{R}^n)$ with $r \in (1, \infty]$, then there exists a positive constant C such that for any cube $Q \subset \mathbb{R}^n$ and any measurable subset E of Q ,

$$\frac{\omega(E)}{\omega(Q)} \leq C \left[\frac{|E|}{|Q|} \right]^{\frac{r-1}{r}}.$$

(vi)

$$\bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n) = \bigcup_{1 < q \leq \infty} RH_q(\mathbb{R}^n).$$

(vii) Let $\omega \in A_p(\mathbb{R}^n)$ with $p \in (1, \infty)$, $R_0 \in (0, \infty)$ and $\mathcal{N}_h^{\text{loc}, R_0}$ be as in Definition 3.1 below. Then $\mathcal{N}_h^{\text{loc}, R_0}$ is bounded on $L_\omega^p(\mathbb{R}^n)$.

Proof. The statements (i) through (vi) of this lemma and their proofs can be found in [33, 34]. We omit the details. Now we prove Lemma 2.2(vii). Denote by M the *Hardy-Littlewood maximal operator* on $\mathbb{R}^n$. Let $p \in (1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$. From (2.8) below, we deduce that

$$\mathcal{N}_h^{\text{loc}, R_0}(f) \lesssim M(f)$$

for all $f \in L_\omega^p(\mathbb{R}^n)$. By this and the well-known fact that M is bounded on $L_\omega^p(\mathbb{R}^n)$, we know that $\mathcal{N}_h^{\text{loc}, R_0}$ is bounded on $L_\omega^p(\mathbb{R}^n)$, which completes the proof of Lemma 2.2(vii) and hence Lemma 2.2. $\square$

Let

$$A_\infty(\mathbb{R}^n) := \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n) = \bigcup_{1 < q \leq \infty} RH_q(\mathbb{R}^n).$$

For any given $\omega \in A_\infty(\mathbb{R}^n)$, define the *critical indexes* q_ω and r_ω of ω , respectively, by

$$(2.5) \quad q_\omega := \inf \{p \in [1, \infty) : \omega \in A_p(\mathbb{R}^n)\}$$

and

$$(2.6) \quad r_\omega := \sup \{r \in (1, \infty] : \omega \in RH_r(\mathbb{R}^n)\}.$$

Recall that if $q_\omega \in (1, \infty)$, then $\omega \notin A_{q_\omega}(\mathbb{R}^n)$, and there exists $\omega \notin A_1(\mathbb{R}^n)$ such that $q_\omega = 1$ (see [53]). Similarly, if $r_\omega \in (1, \infty)$, then $\omega \notin RH_{r_\omega}(\mathbb{R}^n)$, and there exists $\omega \notin RH_\infty(\mathbb{R}^n)$ such that $r_\omega = \infty$ (see [23]).

2.2. The divergence form elliptic operator L . In this subsection, we describe the divergence form elliptic operators considered in this paper and the most typical example is the Laplace operator on the Lipschitz domain of $\mathbb{R}^n$ with the Dirichlet boundary condition or the Neumann boundary condition.

For $A : \mathbb{R}^n \rightarrow M_n(\mathbb{C})$ a measurable function, where $M_n(\mathbb{C})$ denotes the *set of all $n \times n$ complex-valued matrixes*, let

$$\|A\|_\infty := \operatorname{ess\,sup}_{\substack{x \in \mathbb{R}^n, \xi, \eta \in \mathbb{C}^n, \\ |\xi| = |\eta| = 1}} |A(x)\xi \cdot \bar{\eta}|,$$

where $\bar{\eta}$ denotes the *conjugate vector* of η . For all $\delta \in (0, 1]$, denote by $\mathcal{A}(\delta)$ the *class of all measurable functions* $A : \mathbb{R}^n \rightarrow M_n(\mathbb{C})$ *satisfying the ellipticity condition*; namely, for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{C}^n$,

$$\|A\|_\infty \leq \delta^{-1} \text{ and } \Re(A(x)\xi \cdot \bar{\xi}) \geq \delta|\xi|^2,$$

where and in what follows, $\Re(A(x)\xi \cdot \bar{\xi})$ denotes the *real part* of $A(x)\xi \cdot \bar{\xi}$. Denote by $\mathcal{A}$ the *union of all $\mathcal{A}(\delta)$ for $\delta \in (0, 1]$* .

When $A \in \mathcal{A}$ and V is a *closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$* , denote by L the *maximal-accretive operator* (see [69, p.23, Definition 1.46] for the definition) on $L^2(\Omega)$ with largest domain $D(L) \subset V$ such that for all $f \in D(L)$ and $g \in V$,

$$\langle Lf, g \rangle = \int_\Omega A(x) \nabla f(x) \cdot \overline{\nabla g(x)} dx,$$

where $\langle \cdot, \cdot \rangle$ denotes the *interior product* in $L^2(\Omega)$. In this sense, for all $f \in D(L)$, we write

$$(2.7) \quad Lf := -\operatorname{div}(A \nabla f).$$

We recall the following Dirichlet and Neumann boundary conditions of L from [8, p. 152].

Definition 2.3. Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$ and L be as in (2.7). The operator L is said to satisfy the *Dirichlet boundary condition* (for simplicity, DBC) if $V := W_0^{1,2}(\Omega)$, and the *Neumann boundary condition* (for simplicity, NBC) if $V := W^{1,2}(\Omega)$.

Recall that if $\Omega := \mathbb{R}^n$, then

$$W_0^{1,2}(\Omega) = W^{1,2}(\Omega).$$

Thus, in this case, DBC and NBC are identical.

Let L be as in (2.7). Then L generates a semigroup $\{e^{-tL}\}_{t \geq 0}$ of operators that is *analytic* (namely, it has a holomorphic extension to a complex half cone $z \neq 0$ and $|\arg z| < \mu$ for some $\mu \in (0, \pi/2)$) and *contracting* on $L^2(\Omega)$ (namely, for all $f \in L^2(\Omega)$ and $t \in (0, \infty)$, $\|e^{-tL}f\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$); see, for example, [69, Proposition 1.51 and Theorem 1.52] for the details. Also, L has a unique *maximal accretive square root* $\sqrt{L}$ such that $-\sqrt{L}$ generates an analytic and $L^2(\Omega)$ -contracting semigroup $\{P_t\}_{t \geq 0}$ with $P_t := e^{-t\sqrt{L}}$, the *Poisson semigroup* for L ; see, for example, [55] for the details.

Now we recall the Gaussian property of $\{e^{-tL}\}_{t \geq 0}$ on a strongly Lipschitz domain; see, for example, [8, Definition 3] and also [9, 10].

Definition 2.4. Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$, and let L be as in (2.7). Let $\beta \in (0, \infty]$. The semigroup generated by L is said to have the *Gaussian property* (G_β) if the following (i) and (ii) hold:

(i) The kernel of e^{-tL} , denoted by K_t , is a measurable function on $\Omega \times \Omega$ and there exist positive constants C and α such that for all $t \in (0, \beta)$ and all $x, y \in \Omega$,

$$(2.8) \quad |K_t(x, y)| \leq \frac{C}{t^{n/2}} e^{-\alpha \frac{|x-y|^2}{t}}.$$

(ii) For all $x \in \Omega$ and $t \in (0, \beta)$, the functions $y \mapsto K_t(x, y)$ and $y \mapsto K_t(y, x)$ are Hölder continuous in Ω and there exist positive constants C and $\mu \in (0, 1]$ such that for all $t \in (0, \beta)$ and $x, y_1, y_2 \in \Omega$,

$$(2.9) \quad |K_t(x, y_1) - K_t(x, y_2)| + |K_t(y_1, x) - K_t(y_2, x)| \leq \frac{C}{t^{n/2}} \frac{|y_1 - y_2|^\mu}{t^{\mu/2}}.$$

Remark 2.5. (i) The assumption (G_∞) is always satisfied if L is the Laplacian or a real operator (under DBC or NBC) on $\mathbb{R}^n$ or on Lipschitz domains except under NBC with Ω bounded; see, for example, [10].

(ii) The assumption (G_∞) implies that for all $\beta \in (0, \infty)$, (G_β) holds. If β is finite, by [8, p. 178, Lemma A.1] and the property of semigroups, we know that (G_β) and (G_1) are equivalent.

The following well-known fact is a simple corollary of the analyticity of the semigroup $\{e^{-tL}\}_{t \geq 0}$. We omit the details.

Lemma 2.6. Let $\beta \in (0, \infty]$. Assume that L has the Gaussian property (G_β) . Then the estimate (2.8) also holds for $t\partial_t K_t$.

2.3. Orlicz functions. Let Φ be a positive function on $\mathbb{R}_+ := (0, \infty)$. The function Φ is said to be of *upper type* p (resp. *lower type* p) for some $p \in [0, \infty)$ if there exists a positive constant C such that for all $t \in [1, \infty)$ (resp. $t \in (0, 1]$) and $s \in (0, \infty)$,

$$(2.10) \quad \Phi(st) \leq Ct^p \Phi(s).$$

Obviously, if Φ is of lower type p for some $p \in (0, \infty)$, then $\lim_{t \rightarrow 0^+} \Phi(t) = 0$. So, for the sake of convenience, if it is necessary, we may assume that $\Phi(0) = 0$. If Φ is of both upper type p_1 and lower type p_0 , then Φ is said to be of *type* (p_0, p_1) . Let

$$(2.11) \quad p_\Phi^+ := \inf\{p \in (0, \infty) : \text{there exists } C \in (0, \infty) \\ \text{such that (2.10) holds for all } t \in [1, \infty) \text{ and } s \in (0, \infty)\}$$

and

$$(2.12) \quad p_\Phi^- := \sup\{p \in (0, \infty) : \text{there exists } C \in (0, \infty) \\ \text{such that (2.10) holds for all } t \in (0, 1] \text{ and } s \in (0, \infty)\}.$$

The function Φ is said to be of *strictly lower type* p if for all $t \in (0, 1)$ and $s \in (0, \infty)$, $\Phi(st) \leq t^p \Phi(s)$, and we define

$$(2.13) \quad p_\Phi := \sup\{p \in (0, \infty) : \Phi(st) \leq t^p \Phi(s) \text{ holds} \\ \text{for all } t \in (0, 1) \text{ and } s \in (0, \infty)\}.$$

It is easy to see that

$$p_\Phi \leq p_\Phi^- \leq p_\Phi^+$$

for all Φ . In what follows, p_Φ , p_Φ^- and p_Φ^+ are respectively called the *strictly critical lower type index*, the *critical lower type index* and the *critical upper type index* of Φ . Moreover, it was proved in [49, Remark 2.1] that Φ is also of strictly lower type p_Φ . In other words, p_Φ is *attainable*.

Throughout the entire paper, we always assume that Φ satisfies the following assumptions.

Assumption (A). Let Φ be a positive function defined on $\mathbb{R}_+$ which is of critical lower type $p_\Phi^- \in (0, 1]$. Also assume that Φ is continuous, strictly increasing, subadditive and concave.

Remark 2.7. We observe that, via the Aoki-Rolewicz theorem in [5, 72], all results in [86, 87, 88] are still true if the assumptions on p_Φ therein are relaxed into the same assumptions on p_Φ^- .

It is worth noticing that p_Φ^- and p_Φ^+ may not be attainable. For example, for $p \in (0, 1]$, if $\Phi(t) := t^p$ for all $t \in (0, \infty)$, then Φ satisfies Assumption (A) and $p_\Phi = p_\Phi^- = p_\Phi^+ = p$; for $p \in [\frac{1}{2}, 1]$, if

$$\Phi(t) := t^p / \ln(e + t)$$

for all $t \in (0, \infty)$, then Φ satisfies Assumption (A) and $p_\Phi^- = p = p_\Phi^+$, p_Φ^- is not attainable, but p_Φ^+ is attainable; for $p \in (0, \frac{1}{2}]$, if

$$\Phi(t) := t^p \ln(e + t)$$

for all $t \in (0, \infty)$, then Φ satisfies Assumption (A) and $p_\Phi^- = p = p_\Phi^+$, p_Φ^- is attainable, but p_Φ^+ is not attainable.

Notice that if Φ satisfies Assumption (A), then $\Phi(0) = 0$ and Φ is obviously of upper type 1. For any concave and positive function $\tilde{\Phi}$ of lower type p , if we set

$$\Phi(t) := \int_0^t \frac{\tilde{\Phi}(s)}{s} ds$$

for $t \in [0, \infty)$, then by [82, Proposition 3.1], Φ is equivalent to $\tilde{\Phi}$; namely, there exists a positive constant C such that

$$C^{-1}\tilde{\Phi}(t) \leq \Phi(t) \leq C\tilde{\Phi}(t)$$

for all $t \in [0, \infty)$. Moreover, Φ is a strictly increasing, concave, subadditive and continuous function of lower type p . Notice that all our results are invariant on equivalent functions satisfying Assumption (A). From this, we deduce that all results in this paper with Φ as in Assumption (A) also hold for all concave and positive functions $\tilde{\Phi}$ of the same critical lower type $p_{\tilde{\Phi}}^-$ as Φ .

Since Φ is strictly increasing, we can define the function $\rho(t)$ on $\mathbb{R}_+$ by setting, for all $t \in (0, \infty)$,

$$(2.14) \quad \rho(t) := \frac{t^{-1}}{\Phi^{-1}(t^{-1})},$$

where Φ^{-1} is the inverse function of Φ . Then the types of Φ and ρ have the following relation: *If $0 < p_0 \leq p_1 \leq 1$ and Φ is an increasing function, then Φ is of type (p_0, p_1) if and only if ρ is of type $(p_1^{-1} - 1, p_0^{-1} - 1)$; see [82] for its proof.*

3. PROOF OF THEOREM 1.4

In this section, we first introduce the Orlicz-Hardy spaces

$$h_{\mathcal{N}_h, \omega}^{\Phi}(\Omega), h_{S_h, \omega}^{\Phi}(\Omega) \text{ and } h_{\tilde{S}_h, \omega}^{\Phi}(\Omega),$$

and then give the proof of Theorem 1.4.

Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$ and $\omega \in A_{\infty}(\mathbb{R}^n)$. Recall that for an Orlicz function Φ on $(0, \infty)$, a measurable function f on Ω is said to be in the space $L_{\omega}^{\Phi}(\Omega)$ if

$$\int_{\Omega} \Phi(|f(x)|) \omega(x) dx < \infty.$$

Moreover, for any $f \in L_{\omega}^{\Phi}(\Omega)$, define its *quasi-norm* by

$$\|f\|_{L_{\omega}^{\Phi}(\Omega)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\Omega} \Phi \left(\frac{|f(x)|}{\lambda} \right) \omega(x) dx \leq 1 \right\}.$$

If $p \in (0, 1]$ and $\Phi(t) := t^p$ for all $t \in (0, \infty)$, we then denote $L_{\omega}^{\Phi}(\Omega)$ simply by $L_{\omega}^p(\Omega)$.

Definition 3.1. Let Φ satisfy Assumption (A), $\omega \in A_{\infty}(\mathbb{R}^n)$, Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$ and L be as in (2.7). Let $R_0 \in (0, \infty)$. For all $f \in L^2(\Omega)$ and $x \in \Omega$, let

$$\mathcal{N}_h^{\text{loc}, R_0}(f)(x) := \sup_{y \in \Omega, t \in (0, R_0], |y-x| < t} \left| e^{-t^2 L}(f)(y) \right|.$$

When $R_0 := 1$, denote $\mathcal{N}_h^{\text{loc}, R_0}(f)$ simply by $\mathcal{N}_h^{\text{loc}}(f)$. A function $f \in L^2(\Omega)$ is said to be in $\tilde{h}_{\mathcal{N}_h, \omega}^\Phi(\Omega)$ if $\mathcal{N}_h^{\text{loc}}(f) \in L_\omega^\Phi(\Omega)$; moreover, define

$$\begin{aligned} \|f\|_{h_{\mathcal{N}_h, \omega}^\Phi(\Omega)} &:= \|\mathcal{N}_h^{\text{loc}}(f)\|_{L_\omega^\Phi(\Omega)} \\ &:= \inf \left\{ \lambda \in (0, \infty) : \int_\Omega \Phi \left(\frac{\mathcal{N}_h^{\text{loc}}(f)(x)}{\lambda} \right) \omega(x) dx \leq 1 \right\}. \end{aligned}$$

The *weighted local Orlicz-Hardy space* $h_{\mathcal{N}_h, \omega}^\Phi(\Omega)$ is defined to be the completion of $\tilde{h}_{\mathcal{N}_h, \omega}^\Phi(\Omega)$ in the quasi-norm $\|\cdot\|_{h_{\mathcal{N}_h, \omega}^\Phi(\Omega)}$.

Remark 3.2. (i) It is easy to see that $\|\cdot\|_{h_{\mathcal{N}_h, \omega}^\Phi(\Omega)}$ is a quasi-norm.

(ii) From the Aoki-Rolewicz theorem in [5, 72], it follows that there exist a *quasi-norm* $\|\cdot\|$ on $\tilde{h}_{\mathcal{N}_h, \omega}^\Phi(\Omega)$ and $\gamma \in (0, 1]$ such that for all $f \in \tilde{h}_{\mathcal{N}_h, \omega}^\Phi(\Omega)$,

$$\|f\| \sim \|f\|_{h_{\mathcal{N}_h, \omega}^\Phi(\Omega)},$$

and that for any sequence $\{f_j\}_j \subset \tilde{h}_{\mathcal{N}_h, \omega}^\Phi(\Omega)$,

$$\left\| \sum_j f_j \right\|^\gamma \leq \sum_j \|f_j\|^\gamma.$$

By the theorem of completion of Yosida [89, p. 56], it follows that $(\tilde{h}_{\mathcal{N}_h, \omega}^\Phi(\Omega), \|\cdot\|)$ has a completion space $(h_{\mathcal{N}_h, \omega}^\Phi(\Omega), \|\cdot\|)$; namely, for any $f \in (h_{\mathcal{N}_h, \omega}^\Phi(\Omega), \|\cdot\|)$, there exists a Cauchy sequence $\{f_k\}_{k=1}^\infty \subset (\tilde{h}_{\mathcal{N}_h, \omega}^\Phi(\Omega), \|\cdot\|)$ such that

$$\lim_{k \rightarrow \infty} \|f_k - f\| = 0.$$

Moreover, if $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in $(\tilde{h}_{\mathcal{N}_h, \omega}^\Phi(\Omega), \|\cdot\|)$, then there exists a unique $f \in (h_{\mathcal{N}_h, \omega}^\Phi(\Omega), \|\cdot\|)$ such that

$$\lim_{k \rightarrow \infty} \|f_k - f\| = 0.$$

Furthermore, by $\|f\| \sim \|f\|_{h_{\mathcal{N}_h, \omega}^\Phi(\Omega)}$ for all $f \in \tilde{h}_{\mathcal{N}_h, \omega}^\Phi(\Omega)$, we know that the *spaces* $(h_{\mathcal{N}_h, \omega}^\Phi(\Omega), \|\cdot\|_{h_{\mathcal{N}_h, \omega}^\Phi(\Omega)})$ and $(h_{\mathcal{N}_h, \omega}^\Phi(\Omega), \|\cdot\|)$ coincide with equivalent quasi-norms.

In what follows, $Q(x, t)$ denotes the *closed cube of $\mathbb{R}^n$ centered at x and of sidelength t with sides parallel to the axes*. Similarly, given $Q := Q(x, t)$ and $\lambda \in (0, \infty)$, we write λQ for the λ -*dilated cube*, which is the cube with the same center x and with sidelength λt . Let $R_0 \in (0, \infty)$. For any $f \in L^2(\Omega)$ and $x \in \Omega$, the *Lusin area functions* $S_{h, R_0}^{\text{loc}}(f)(x)$ and $\tilde{S}_{h, R_0}^{\text{loc}}(f)(x)$ associated with $\{e^{-t^2 L}\}_{t \geq 0}$ are respectively defined by

$$S_{h, R_0}^{\text{loc}}(f)(x) := \left[\int_{\Gamma^{R_0}(x)} \left| t^2 L e^{-t^2 L}(f)(y) \right|^2 \frac{dy dt}{t |Q(x, t) \cap \Omega|} \right]^{\frac{1}{2}}$$

and

$$\tilde{S}_{h, R_0}^{\text{loc}}(f)(x) := \left[\int_{\Gamma^{R_0}(x)} \left| t \nabla e^{-t^2 L}(f)(y) \right|^2 \frac{dy dt}{t |Q(x, t) \cap \Omega|} \right]^{\frac{1}{2}},$$

where $\Gamma^{R_0}(x)$ is the *cone* defined by

$$\Gamma^{R_0}(x) := \{(y, t) \in \Omega \times (0, R_0] : |y - x| < t\}.$$

When $R_0 := 1$, we denote S_{h, R_0}^{loc} and $\tilde{S}_{h, R_0}^{\text{loc}}$ simply, respectively, by S_h^{loc} and $\tilde{S}_h^{\text{loc}}$.

To give the definitions of the weighted local Orlicz-Hardy spaces associated with the Lusin area functions S_h^{loc} and $\tilde{S}_h^{\text{loc}}$, we need the following notation.

In what follows, denote by $\mathcal{Q}$ the set of all unit cubes of $\mathbb{R}^n$ whose interiors are disjoint. Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$. For any given $Q \in \mathcal{Q}$ satisfying $Q \cap \Omega \neq \emptyset$, if the center of Q , $x_Q \in Q \cap \Omega$, let $\tilde{Q} := Q$; if the center of Q , $x_Q \in Q \cap \Omega^c$, take the cube $\tilde{Q} \subset \mathbb{R}^n$ satisfying $l(\tilde{Q}) = 2$, $Q \subset \tilde{Q}$ and the center of $\tilde{Q}$, $x_{\tilde{Q}} \in \tilde{Q} \cap \Omega$, where and in what follows, Ω^c denotes the complement of Ω in $\mathbb{R}^n$. Denote by $\mathcal{Q}_\Omega$ the set of all cubes $\tilde{Q}$ as above. For any given measurable subset E of $\mathbb{R}^n$ satisfying $|E| < \infty$ and $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, let

$$(3.1) \quad m_E(f) := \frac{1}{|E|} \int_E f(y) dy,$$

where and in what follows, the space $L_{\text{loc}}^1(\mathbb{R}^n)$ denotes the set of all locally integrable functions on $\mathbb{R}^n$.

Definition 3.3. Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$, Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$ and L be as in (2.7). A function $f \in L^2(\Omega)$ is said to be in $\tilde{h}_{\omega, S_h}^\Phi(\Omega)$ if $S_h^{\text{loc}}(f) \in L_\omega^\Phi(\Omega)$ and

$$\sum_{Q \in \mathcal{Q}_\Omega} \omega(Q \cap \Omega) \Phi(m_{Q \cap \Omega}(|e^{-L}(f)|)) < \infty.$$

Furthermore, define

$$\begin{aligned} \|f\|_{h_{\omega, S_h}^\Phi(\Omega)} &:= \inf \left\{ \lambda \in (0, \infty) : \int_\Omega \Phi \left(\frac{S_h^{\text{loc}}(f)(x)}{\lambda} \right) \omega(x) dx \leq 1 \right\} \\ &+ \inf \left\{ \lambda \in (0, \infty) : \sum_{Q \in \mathcal{Q}_\Omega} \omega(Q \cap \Omega) \Phi \left(\frac{m_{Q \cap \Omega}(|e^{-L}(f)|)}{\lambda} \right) \leq 1 \right\}. \end{aligned}$$

The *weighted local Orlicz-Hardy space* $h_{S_h, \omega}^\Phi(\Omega)$ is defined to be the completion of $\tilde{h}_{S_h, \omega}^\Phi(\Omega)$ in the norm $\|\cdot\|_{h_{S_h, \omega}^\Phi(\Omega)}$.

The *weighted local Orlicz-Hardy space* $h_{S_h, \omega}^\Phi(\Omega)$ is defined via replacing S_h^{loc} in the definition of $h_{S_h, \omega}^\Phi(\Omega)$ by $\tilde{S}_h^{\text{loc}}$.

In this section, we present the proof of Theorem 1.4. To this end, we need some auxiliary area functions as follows. Let $R_0 \in (0, \infty)$, $\alpha \in (0, \infty)$, $\epsilon, R \in (0, R_0)$ and $\epsilon < R$. For all given $f \in L^2(\Omega)$ and $x \in \Omega$, let

$$\tilde{S}_{h, R_0}^{\text{loc}, \alpha}(f)(x) := \left[\int_{\Gamma_\alpha^{R_0}(x)} \left| t \nabla e^{-t^2 L}(f)(y) \right|^2 \frac{dy dt}{t |Q(x, t) \cap \Omega|} \right]^{\frac{1}{2}}$$

and

$$\tilde{S}_{h, R_0}^{\epsilon, R, \alpha}(f)(x) := \left[\int_{\Gamma_\alpha^{\epsilon, R}(x)} \left| t \nabla e^{-t^2 L}(f)(y) \right|^2 \frac{dy dt}{t |Q(x, t) \cap \Omega|} \right]^{\frac{1}{2}},$$

where and in what follows, for all $x \in \Omega$, $\Gamma_\alpha^{R_0}(x)$ and $\Gamma_\alpha^{\epsilon, R}(x)$ are the *local cone* and the *truncated cone* defined, respectively, by

$$\Gamma_\alpha^{R_0}(x) := \{(y, t) \in \Omega \times (0, R_0] : |y - x| < \alpha t\}$$

and

$$\Gamma_\alpha^{\epsilon, R}(x) := \{(y, t) \in \Omega \times (\epsilon, R] : |y - x| < \alpha t\}$$

for $\alpha \in (0, \infty)$ and $0 < \epsilon < R < R_0$. When $\alpha = 1$, we denote $\tilde{S}_{h, R_0}^{\text{loc}, \alpha}(f)$, $\tilde{S}_{h, R_0}^{\epsilon, R, \alpha}(f)$ and $\Gamma_\alpha^{R_0}(x)$ simply, respectively, by $\tilde{S}_{h, R_0}^{\text{loc}}(f)$, $\tilde{S}_{h, R_0}^{\epsilon, R}(f)$ and $\Gamma^{R_0}(x)$. To show Theorem 1.4, we first establish the following Proposition 3.4.

Proposition 3.4. *Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$, $R_0 \in (0, \infty)$ and L be as in (2.7). Let q_ω , μ and p_Φ^- be respectively as in (2.5), (2.9) and (2.12). Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$. Assume that q_ω , μ and p_Φ^- satisfy the inequality $q_\omega/p_\Phi^- < (n + \mu)/n$, and that the semigroup generated by L has the Gaussian property (G_1) .*

(i) *If $\Omega := \mathbb{R}^n$, then there exists a positive constant C such that for all $f \in h_\omega^\Phi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,*

$$\left\| \mathcal{N}_h^{\text{loc}, R_0}(f) \right\|_{L_\omega^\Phi(\mathbb{R}^n)} \leq C \|f\|_{h_\omega^\Phi(\mathbb{R}^n)}.$$

(ii) *Under DBC, there exists a positive constant C such that for all $f \in h_{\omega, r}^\Phi(\Omega) \cap L^2(\Omega)$,*

$$\left\| \mathcal{N}_h^{\text{loc}, R_0}(f) \right\|_{L_\omega^\Phi(\Omega)} \leq C \|f\|_{h_{\omega, r}^\Phi(\Omega)}.$$

(iii) *Under NBC, there exists a positive constant C such that for all $f \in h_{\omega, z}^\Phi(\Omega) \cap L^2(\Omega)$,*

$$\left\| \mathcal{N}_h^{\text{loc}, R_0}(f) \right\|_{L_\omega^\Phi(\Omega)} \leq C \|f\|_{h_{\omega, z}^\Phi(\Omega)}.$$

To show Proposition 3.4, we need the atomic decomposition characterization of the weighted local Orlicz-Hardy space $h_\omega^\Phi(\mathbb{R}^n)$ established in [86]. To state this, we begin with the notions of $(\rho, q, s)_\omega$ -atoms and the weighted atomic local Orlicz-Hardy space $h_{\omega, q, s}^\rho(\mathbb{R}^n)$.

Definition 3.5. Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$, p_Φ^- , q_ω and ρ be respectively as in (2.12), (2.5) and (2.14). A triplet $(\rho, q, s)_\omega$ is called *admissible* if $q \in (q_\omega, \infty]$ and $s \in \mathbb{Z}_+$ with

$$s \geq \left\lfloor n \left(\frac{q_\omega}{p_\Phi^-} - 1 \right) \right\rfloor.$$

A function a on $\mathbb{R}^n$ is called a $(\rho, q, s)_\omega$ -atom if there exists a cube $Q \subset \mathbb{R}^n$ such that

- (i) $\text{supp}(a) \subset Q$;
- (ii)

$$\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(Q)]^{\frac{1}{q}-1} [\rho(\omega(Q))]^{-1};$$

- (iii) for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$, when $l(Q) < 1$,

$$\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0.$$

Definition 3.6. Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$, q_ω, ρ be respectively as in (2.5) and (2.14), and $(\rho, q, s)_\omega$ admissible. The *weighted atomic local Orlicz-Hardy space* $h_\omega^{\rho, q, s}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfying that

$$f = \sum_{i=1}^{\infty} \lambda_i a_i$$

in $\mathcal{D}'(\mathbb{R}^n)$, where $\{a_i\}_{i \in \mathbb{N}}$ are $(\rho, q, s)_\omega$ -atoms with $\text{supp}(a_i) \subset Q_i$, $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$, and

$$\sum_{i=1}^{\infty} \omega(Q_i) \Phi\left(\frac{|\lambda_i|}{\omega(Q_i) \rho(\omega(Q_i))}\right) < \infty.$$

Moreover, letting

$$\Lambda(\{\lambda_i a_i\}_{i \in \mathbb{N}}) := \inf \left\{ \lambda \in (0, \infty) : \sum_{i=1}^{\infty} \omega(Q_i) \Phi\left(\frac{|\lambda_i|}{\lambda \omega(Q_i) \rho(\omega(Q_i))}\right) \leq 1 \right\},$$

the *quasi-norm* of $f \in h_\omega^{\rho, q, s}(\mathbb{R}^n)$ is defined by

$$\|f\|_{h_\omega^{\rho, q, s}(\mathbb{R}^n)} := \inf \{ \Lambda(\{\lambda_i a_i\}_{i \in \mathbb{N}}) \},$$

where the infimum is taken over all the decompositions of f as above.

The $(\rho, q, s)_\omega$ -atom and the weighted atomic local Orlicz-Hardy space $h_\omega^{\rho, q, s}(\mathbb{R}^n)$ were introduced in [86], in which the following Lemma 3.7 was also obtained (see [86, Theorem 5.6]).

Lemma 3.7. *Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$, q_ω, p_Φ^- and ρ be respectively as in (2.5), (2.12) and (2.14). If $q \in (q_\omega, \infty]$ and integer s satisfies $s \geq \lfloor n(\frac{q_\omega}{p_\Phi} - 1) \rfloor$, then*

$$h_\omega^{\rho, q, s}(\mathbb{R}^n) = h_\omega^\Phi(\mathbb{R}^n)$$

with equivalent quasi-norms.

We point out that in [86, Theorem 5.6], the assumption $p_\Phi^- \in (0, 1]$ was replaced by the assumption $p_\Phi \in (0, 1]$, which, however, can be relaxed; see Remark 2.7.

Now we prove Proposition 3.4 by applying Lemma 3.7.

Proof of Proposition 3.4. We first prove Proposition 3.4(i). To this end, it suffices to prove that for any $(\rho, q, 0)$ -atom a supported in $Q_0 := Q(x_0, r_0)$ with some $q \in (q_\omega, \infty]$ and any $\lambda \in \mathbb{C}$,

$$(3.2) \quad \int_{\mathbb{R}^n} \Phi\left(\mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x)\right) \omega(x) dx \lesssim \omega(Q_0) \Phi\left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))}\right).$$

Indeed, if (3.2) holds, by Lemma 3.7 and an argument in [86, p.43], we know that for any $f \in h_\omega^\Phi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, there exist $\{\lambda_i\}_i \subset \mathbb{C}$ and a sequence $\{a_i\}_i$ of $(\rho, q, 0)$ -atoms such that

$$(3.3) \quad f = \sum_i \lambda_i a_i \text{ in } L^2(\mathbb{R}^n)$$

and

$$(3.4) \quad \Lambda(\{\lambda_i a_i\}_i) := \inf \left\{ \lambda \in (0, \infty) : \sum_i \omega(Q_i) \Phi \left(\frac{|\lambda_i|}{\lambda \omega(Q_i) \rho(\omega(Q_i))} \right) \leq 1 \right\} \\ \lesssim \|f\|_{h_\omega^\Phi(\mathbb{R}^n)},$$

where for each i , $\text{supp}(a_i) \subset Q_i$. By (3.3), we conclude that for all $x \in \mathbb{R}^n$,

$$\mathcal{N}_h^{\text{loc}, R_0}(f)(x) \leq \sum_i \mathcal{N}_h^{\text{loc}, R_0}(\lambda_i a_i)(x),$$

which, together with the assumptions that Φ is strictly increasing, subadditive and continuous, and that for any $\lambda \in (0, \infty)$ and each i ,

$$\mathcal{N}_h^{\text{loc}, R_0}(f/\lambda) = \mathcal{N}_h^{\text{loc}, R_0}(f)/\lambda, \quad \mathcal{N}_h^{\text{loc}, R_0}(a_i/\lambda) = \mathcal{N}_h^{\text{loc}, R_0}(a_i)/\lambda$$

and (3.2), implies that for any $\lambda \in (0, \infty)$,

$$\begin{aligned} & \int_{\mathbb{R}^n} \Phi \left(\frac{\mathcal{N}_h^{\text{loc}, R_0}(f)(x)}{\lambda} \right) \omega(x) dx \\ &= \int_{\mathbb{R}^n} \Phi \left(\mathcal{N}_h^{\text{loc}, R_0} \left(\frac{f}{\lambda} \right) (x) \right) \omega(x) dx \\ &\leq \sum_i \int_{\mathbb{R}^n} \Phi \left(\mathcal{N}_h^{\text{loc}, R_0} \left(\frac{\lambda_i a_i}{\lambda} \right) (x) \right) \omega(x) dx \\ &\lesssim \sum_i \omega(Q_i) \Phi \left(\frac{|\lambda_i|}{\lambda \omega(Q_i) \rho(\omega(Q_i))} \right). \end{aligned}$$

From this and (3.4), we deduce that

$$\left\| \mathcal{N}_h^{\text{loc}, R_0}(f) \right\|_{L_\omega^\Phi(\mathbb{R}^n)} \lesssim \|f\|_{h_\omega^\Phi(\mathbb{R}^n)},$$

which is as desired.

Now we prove (3.2). Let $\tilde{Q}_0 := 4(R_0 + 1)Q_0$. Then we write

$$(3.5) \quad \begin{aligned} & \int_{\mathbb{R}^n} \Phi \left(\mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x) \right) \omega(x) dx \\ &= \int_{\tilde{Q}_0} \Phi \left(\mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x) \right) \omega(x) dx + \int_{\tilde{Q}_0^c} \cdots =: I_1 + I_2. \end{aligned}$$

We first estimate I_1 . From the assumption that $\frac{q_\omega}{p_\Phi} < \frac{n+\mu}{n}$, we deduce that there exist $q \in (q_\omega, \infty)$, $p_0 \in (0, p_\Phi^-)$ and $\tilde{\mu} \in (0, \mu)$ such that Φ is of lower type p_0 and

$$(3.6) \quad \frac{q}{p_0} < \frac{n + \tilde{\mu}}{n}.$$

Since $q \in (q_\omega, \infty)$, by the definition of q_ω , we know that $\omega \in A_q(\mathbb{R}^n)$. Since Φ is concave, by Jensen's inequality, Hölder's inequality, (vii) and (iii) of Lemma 2.2, we have

$$(3.7) \quad \begin{aligned} I_1 &\leq \omega(\tilde{Q}_0) \Phi \left(\frac{1}{\omega(\tilde{Q}_0)} \int_{\tilde{Q}_0} \mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x) \omega(x) dx \right) \\ &\leq \omega(\tilde{Q}_0) \Phi \left(\frac{1}{[\omega(\tilde{Q}_0)]^{\frac{1}{q}}} \left\{ \int_{\tilde{Q}_0} [\mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x)]^q \omega(x) dx \right\}^{\frac{1}{q}} \right) \end{aligned}$$

$$\begin{aligned}
&\lesssim \omega(\tilde{Q}_0) \Phi \left(\frac{1}{[\omega(\tilde{Q}_0)]^{\frac{1}{q}}} |\lambda| \|a\|_{L_\omega^q(\mathbb{R}^n)} \right) \\
&\lesssim \omega(\tilde{Q}_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right) \lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right),
\end{aligned}$$

which is as desired.

Now we deal with I_2 by considering the following two cases for Q_0 .

Case 1) $l(Q_0) < 1$. In this case, let $x \in (\tilde{Q}_0)^c$, $t \in (0, R_0]$ and $y \in \mathbb{R}^n$ satisfy $|x - y| < t$. Let $\tilde{\mu}$ be as in (3.6). By an argument similar to that in [87, p. 18], we know that for all $z \in Q_0$,

$$|K_{t^2}(y, z) - K_{t^2}(y, x_0)| \lesssim \frac{|z - x_0|^{\tilde{\mu}}}{|x - x_0|^{n+\tilde{\mu}}},$$

which, together with the moment condition of a , Hölder's inequality and (2.1), implies that

$$\begin{aligned}
(3.8) \quad |e^{-t^2 L}(\lambda a)(y)| &= \left| \int_{\mathbb{R}^n} \lambda [K_{t^2}(y, z) - K_{t^2}(y, x_0)] a(z) dz \right| \\
&\lesssim \int_{\mathbb{R}^n} \frac{|\lambda| |z - x_0|^{\tilde{\mu}}}{|x - x_0|^{n+\tilde{\mu}}} |a(z)| dz \\
&\lesssim |\lambda| |Q_0|^{\tilde{\mu}/n} \|a\|_{L_\omega^q(\mathbb{R}^n)} \left\{ \int_{Q_0} [\omega(x)]^{-q'/q} dx \right\}^{\frac{1}{q'}} |x - x_0|^{-(n+\tilde{\mu})} \\
&\lesssim |\lambda| |Q_0|^{\frac{\tilde{\mu}}{n}} \|a\|_{L_\omega^q(\mathbb{R}^n)} \frac{|Q_0|}{[\omega(Q_0)]^{\frac{1}{q}}} |x - x_0|^{-(n+\tilde{\mu})} \\
&\lesssim \frac{|\lambda| |Q_0|^{\frac{n+\tilde{\mu}}{n}}}{\omega(Q_0) \rho(\omega(Q_0))} |x - x_0|^{-(n+\tilde{\mu})},
\end{aligned}$$

where and in what follows, $\frac{1}{q} + \frac{1}{q'} = 1$. From this, it further follows that for all $x \in (\tilde{Q}_0)^c$,

$$\mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x) \lesssim \frac{|\lambda| |Q_0|^{\frac{n+\tilde{\mu}}{n}}}{\omega(Q_0) \rho(\omega(Q_0))} |x - x_0|^{-(n+\tilde{\mu})},$$

which, together with the lower type p_0 and upper type 1 properties of Φ , Lemma 2.2(iii) and (3.6), implies that

$$\begin{aligned}
(3.9) \quad I_2 &\lesssim \int_{(\tilde{Q}_0)^c} \Phi \left(\frac{|\lambda| |Q_0|^{(n+\tilde{\mu})/n}}{\omega(Q_0) \rho(\omega(Q_0))} |x - x_0|^{-(n+\tilde{\mu})} \right) \omega(x) dx \\
&\lesssim \sum_{k=2}^{\infty} \int_{2^{k+1} Q_0 \setminus 2^k Q_0} \Phi \left(2^{-k(n+\tilde{\mu})} |\lambda| [\omega(Q_0) \rho(\omega(Q_0))]^{-1} \right) \omega(x) dx \\
&\lesssim \sum_{k=2}^{\infty} 2^{-k[(n+\tilde{\mu})p_0 - nq]} \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right) \\
&\lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right).
\end{aligned}$$

Case 2) $l(Q_0) \geq 1$. In this case, let $x \in (\tilde{Q}_0)^c$, $t \in (0, R_0]$ and $y \in \mathbb{R}^n$ satisfy $|x - y| < t$. Then for all $z \in Q_0$,

$$\begin{aligned} |y - z| &\geq |x - z| - |x - y| \geq |x - x_0| - |x_0 - z| - |x - y| \\ &\geq |x - x_0| - \frac{1}{4(R_0 + 1)}|x - x_0| - \frac{R_0}{2(R_0 + 1)}|x - x_0| > \frac{1}{2}|x - x_0|, \end{aligned}$$

which, together with (2.8), Hölder's inequality and (2.1), implies that

$$\begin{aligned} \left| e^{-t^2 L}(\lambda a)(y) \right| &= \left| \int_{Q_0} \lambda K_{t^2}(y, z) a(z) dz \right| \lesssim \int_{Q_0} \frac{|\lambda| t}{|y - z|^{n+1}} |a(z)| dz \\ &\lesssim \frac{|\lambda|}{|x - x_0|^{n+1}} \int_{Q_0} |a(z)| dz \lesssim |\lambda| \|a\|_{L_\omega^q(\mathbb{R}^n)} \frac{|Q_0|}{[\omega(Q_0)]^{\frac{1}{q}}} \frac{1}{|x - x_0|^{n+1}} \\ &\lesssim |\lambda| |Q_0| [\omega(Q_0) \rho(\omega(Q_0))]^{-1} |x - x_0|^{-(n+1)}, \end{aligned}$$

where we omitted some computations similar to (3.8). From this, we infer that for all $x \in (\tilde{Q}_0)^c$,

$$\mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x) \lesssim |\lambda| |Q_0| [\omega(Q_0) \rho(\omega(Q_0))]^{-1} |x - x_0|^{-(n+1)},$$

which, together with the lower type p_0 and upper type 1 properties of Φ , Lemma 2.2(iii), $l(Q_0) \geq 1$ and (3.6), implies that

$$\begin{aligned} (3.10) \quad I_2 &\lesssim \int_{(\tilde{Q}_0)^c} \Phi \left(|\lambda| |Q_0| [\omega(Q_0) \rho(\omega(Q_0))]^{-1} |x - x_0|^{-(n+1)} \right) \omega(x) dx \\ &\lesssim \sum_{k=2}^{\infty} \int_{2^{k+1}Q_0 \setminus 2^kQ_0} \Phi \left(2^{-k(n+1)} |\lambda| [\omega(Q_0) \rho(\omega(Q_0))]^{-1} \right) \omega(x) dx \\ &\lesssim \sum_{k=2}^{\infty} 2^{-k[(n+1)p_0 - nq]} \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right) \\ &\lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right). \end{aligned}$$

Thus, by (3.5), (3.7), (3.9) and (3.10), we conclude that

$$\int_{\mathbb{R}^n} \Phi \left(\mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x) \right) \omega(x) dx \lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right),$$

which implies that (3.2) holds and hence completes the proof of Proposition 3.4(i).

Now we prove Proposition 3.4(ii). Let $f \in h_{\omega, r}^\Phi(\Omega) \cap L^2(\Omega)$. By the definition of $h_{\omega, r}^\Phi(\Omega)$, we know that there exists $\tilde{f} \in h_\omega^\Phi(\mathbb{R}^n)$ such that $\tilde{f}|_\Omega = f$ and

$$(3.11) \quad \left\| \tilde{f} \right\|_{h_\omega^\Phi(\mathbb{R}^n)} \lesssim \|f\|_{h_{\omega, r}^\Phi(\Omega)}.$$

Let q be as in (3.6). To show Proposition 3.4(ii), we only need to prove that for any $(\rho, q, 0)_\omega$ -atom a supported in $Q_0 := Q(x_0, r_0)$ and $\lambda \in \mathbb{C}$,

$$(3.12) \quad \int_{\Omega} \Phi \left(\mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x) \right) \omega(x) dx \lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right).$$

Indeed, since $\tilde{f} \in h_\omega^\Phi(\mathbb{R}^n)$, by Lemma 3.7, there exist $\{\lambda_i\}_i \subset \mathbb{C}$ and a sequence $\{a_i\}_i$ of $(\rho, q, 0)_\omega$ -atoms such that $\tilde{f} = \sum_i \lambda_i a_i$ in $\mathcal{D}'(\mathbb{R}^n)$ and

$$\Lambda(\{\lambda_i a_i\}_i) \sim \left\| \tilde{f} \right\|_{h_\omega^\Phi(\mathbb{R}^n)}.$$

Moreover, by the proof of [86, Theorem 5.1], we know that the supports of $\{a_i\}_i$ are of *finite intersection property*. By this, $f \in L^2(\Omega)$,

$$\tilde{f} = \sum_i \lambda_i a_i$$

in $\mathcal{D}'(\mathbb{R}^n)$ and $\tilde{f}|_{\Omega} = f$, we see that $f = \sum_i \lambda_i a_i$ almost everywhere on Ω , which further implies that

$$\int_{\Omega} K_{t^2}(x, y) f(y) dy = \sum_i \lambda_i \int_{\Omega} K_{t^2}(x, y) a_i(y) dy.$$

From this, we deduce that for all $x \in \Omega$,

$$\mathcal{N}_h^{\text{loc}, R_0}(f)(x) \leq \sum_i \mathcal{N}_h^{\text{loc}, R_0}(\lambda_i a_i)(x),$$

which, together with the assumptions that Φ is strictly increasing, subadditive and continuous, and that for any $\lambda \in (0, \infty)$ and each i ,

$$\mathcal{N}_h^{\text{loc}, R_0}(f/\lambda) = \mathcal{N}_h^{\text{loc}, R_0}(f)/\lambda, \quad \mathcal{N}_h^{\text{loc}, R_0}(a_i/\lambda) = \mathcal{N}_h^{\text{loc}, R_0}(a_i)/\lambda$$

and (3.12), implies that for any $\lambda \in (0, \infty)$,

$$\begin{aligned} & \int_{\Omega} \Phi \left(\frac{\mathcal{N}_h^{\text{loc}, R_0}(f)(x)}{\lambda} \right) \omega(x) dx \\ &= \int_{\Omega} \Phi \left(\mathcal{N}_h^{\text{loc}, R_0} \left(\frac{f}{\lambda} \right) (x) \right) \omega(x) dx \\ &\leq \sum_i \int_{\Omega} \Phi \left(\mathcal{N}_h^{\text{loc}, R_0} \left(\frac{\lambda_i a_i}{\lambda} \right) (x) \right) \omega(x) dx \\ &\lesssim \sum_i \omega(Q_i) \Phi \left(\frac{|\lambda_i|}{\lambda \omega(Q_i) \rho(\omega(Q_i))} \right), \end{aligned}$$

where for each i , $\text{supp}(a_i) \subset Q_i$. From this, Lemma 3.7 and (3.11), we deduce that

$$\left\| \mathcal{N}_h^{\text{loc}, R_0}(f) \right\|_{L_{\omega}^{\Phi}(\Omega)} \lesssim \Lambda(\{\lambda_i a_i\}_i) \sim \left\| \tilde{f} \right\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|f\|_{h_{\omega, r}^{\Phi}(\Omega)},$$

which, together with the arbitrariness of $f \in h_{\omega, r}^{\Phi}(\Omega) \cap L^2(\Omega)$, implies the conclusions of Proposition 3.4(ii).

It is easy to see that for all $x \in \Omega$,

$$(3.13) \quad e^{-t^2 L}(\lambda a)(x) = \int_{Q_0 \cap \Omega} \lambda K_{t^2}(x, y) a(y) dy.$$

Now we show (3.12) by considering the following three cases for Q_0 .

Case i) $Q_0 \cap \Omega = \emptyset$. In this case, by (3.13), we know that $\mathcal{N}_h^{\text{loc}, R_0}(\lambda a)(x) = 0$ for all $x \in \Omega$. From this, it follows that (3.12) holds.

Case ii) $Q_0 \subset \Omega$. In this case, the proof of (3.12) is similar to that of (3.2). We omit the details.

Case iii) $Q_0 \cap \partial\Omega \neq \emptyset$. In this case, recall that for any $x \in \Omega$, $t \in (0, \infty)$ and $y \in \partial\Omega$, $K_t(x, y) = 0$ (see, for example, [8, p. 156]). Take $y_0 \in Q_0 \cap \partial\Omega$. Then we

see that for any $x \in \Omega$ and $t \in (0, R_0]$, $K_{t^2}(x, y_0) = 0$, which further implies that for any $x \in \Omega$,

$$e^{-t^2 L}(\lambda a)(x) = \int_{Q_0 \cap \Omega} \lambda [K_{t^2}(x, y) - K_{t^2}(x, y_0)] a(y) dy,$$

provided $l(Q_0) < 1$. The remaining estimates are similar to those of Proposition 3.4(i). We omit the details, which completes the proof of Proposition 3.4(ii).

Finally we give the proof of Proposition 3.4(iii). Let $f \in h_{\omega, z}^\Phi(\Omega) \cap L^2(\Omega)$ and $\tilde{f}$ be the zero extension out of Ω of f . Then $\tilde{f} \in h_{\omega, z}^\Phi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. For all $t \in (0, R_0]$, $x \in \Omega$ and $y \in \Omega$, define

$$F_{x, t}(y) := t^n \left(1 + \frac{|x - y|}{t} \right)^{n+1} K_{t^2}(x, y).$$

It was proved in [8, p. 156] that $F_{x, t}$ can be extended to a bounded Hölder continuous function on $\mathbb{R}^n$ with the Hölder index $\tilde{\mu} \in (0, \mu)$. We denote this extension by $\tilde{F}_{x, t}$. For all $t \in (0, R_0]$, $x \in \Omega$ and $y \in \mathbb{R}^n$, define

$$\tilde{K}_{t^2}(x, y) := t^n \left(1 + \frac{|x - y|}{t} \right)^{-(n+1)} \tilde{F}_{x, t}(y).$$

It was also proved in [8, p. 157] that $\tilde{K}_{t^2}$ satisfies (2.8) and (2.9) with μ replaced by any $\tilde{\mu} \in (0, \mu)$. Obviously, for all $t \in (0, \infty)$ and all $x, y \in \Omega$,

$$K_{t^2}(x, y) = \tilde{K}_{t^2}(x, y).$$

By Lemma 3.7 and an argument in [86, p. 43], we know that there exist $\{\lambda_i\}_i \subset \mathbb{C}$ and a sequence $\{a_i\}_i$ of $(\rho, q, 0)_\omega$ -atoms, with $q \in (q_\omega, \infty]$, such that

$$\tilde{f} = \sum_i \lambda_i a_i$$

in $L^2(\mathbb{R}^n)$. Thus, for all $t \in (0, R_0]$ and $x \in \Omega$,

$$\begin{aligned} e^{-t^2 L}(f)(x) &= \int_{\Omega} K_{t^2}(x, y) f(y) dy \\ &= \int_{\mathbb{R}^n} \tilde{K}_{t^2}(x, y) \tilde{f}(y) dy = \sum_i \int_{\mathbb{R}^n} \lambda_i \tilde{K}_{t^2}(x, y) a_i(y) dy, \end{aligned}$$

which implies that

$$\mathcal{N}_h^{\text{loc}, R_0}(f)(x) \leq \sum_i \mathcal{N}_h^{\text{loc}, R_0}(\lambda_i a_i)(x).$$

The remainder of the proof is similar to that of Proposition 3.4(i). We omit the details. This finishes the proof of Proposition 3.4. $\square$

To show Theorem 1.4, we need the following key proposition.

Proposition 3.8. *Let Φ , Ω , ω and L be as in Proposition 3.4, and $R_0 \in (0, \infty)$. Then there exists a positive constant C such that for all $f \in L^2(\Omega)$ satisfying*

$$\begin{aligned} \left\| \mathcal{N}_h^{\text{loc}, 2R_0}(f) \right\|_{L_\omega^\Phi(\Omega)} &< \infty, \\ \left\| \tilde{S}_{h, R_0}^{\text{loc}}(f) \right\|_{L_\omega^\Phi(\Omega)} &\leq C \left\| \mathcal{N}_h^{\text{loc}, 2R_0}(f) \right\|_{L_\omega^\Phi(\Omega)}. \end{aligned}$$

To show Proposition 3.8, we need the following Lemmas 3.9 through 3.11.

In [8, p.183], Auscher and Russ proved the following geometric property of strongly Lipschitz domains, which plays an important role in this paper.

Lemma 3.9. *Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. Then there exists a constant $C \in (0, 1]$ such that for all cubes Q centered in Ω with*

$$l(Q) \in (0, \infty) \cap (0, 2 \operatorname{diam}(\Omega)],$$

$$|Q \cap \Omega| \geq C|Q|.$$

In what follows, we denote by $B((z, \tau), r)$ the ball in $\mathbb{R}^n \times (0, \infty)$ with center (z, τ) and radius r ; namely,

$$B((z, \tau), r) := \{(x, t) \in \mathbb{R}^n \times (0, \infty) : \max(|x - z|, |t - \tau|) < r\}.$$

Lemma 3.10. *Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$, $R_0 \in (0, \infty)$ and L be as in (2.7). Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$. Assume that the semigroup generated by L has the Gaussian property (G_1). Then there exist $\epsilon_0 \in (0, \infty)$ and a positive constant C such that for all $\gamma \in (0, 1]$, $\lambda \in (0, \infty)$, $\epsilon, R \in (0, R_0)$ with $\epsilon < R$ and $f \in L^2(\Omega)$ satisfying $\|\mathcal{N}_h^{\operatorname{loc}, 2R_0}(f)\|_{L^\Phi_\omega(\Omega)} < \infty$,*

$$(3.14) \quad \omega \left(\left\{ x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x) > 2\lambda, \mathcal{N}_h^{\operatorname{loc}, 2R_0}(f)(x) \leq \gamma\lambda \right\} \right) \\ \leq C\gamma^{\epsilon_0} \omega \left(\left\{ x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{2}}(f)(x) > \lambda \right\} \right).$$

We point out that in the proof of Proposition 3.8, Lemma 3.10 plays a key role. The inequality (3.14) is usually called the “good- λ inequality” concerning the maximal function $\mathcal{N}_h^{\operatorname{loc}, 2R_0}(f)$ and the truncated area functions $\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)$ and $\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{2}}(f)$.

Proof of Lemma 3.10. To prove this lemma, we borrow some ideas from [7] and [8]. Fix $0 < \epsilon < R < R_0$, $\gamma \in (0, 1]$ and $\lambda \in (0, \infty)$. Let $f \in L^2(\Omega)$ satisfy

$$\left\| \mathcal{N}_h^{\operatorname{loc}, 2R_0}(f) \right\|_{L^\Phi_\omega(\Omega)} < \infty$$

and

$$O := \left\{ x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{2}}(f)(x) > \lambda \right\}.$$

It is easy to show that O is an open subset of Ω .

Now we show (3.14) by considering the following two cases for O .

Case 1) $O \neq \Omega$. In this case, let

$$(3.15) \quad O = \bigcup_k (Q_k \cap \Omega)$$

be the Whitney decomposition of O , where $\{Q_k\}_k$ are dyadic cubes of $\mathbb{R}^n$ with disjoint interiors and $2Q_k \cap \Omega \subset O \subset \Omega$, but

$$4Q_k \cap \Omega \cap (\Omega \setminus O) \neq \emptyset.$$

To show (3.14), by (3.15) and the disjoint property of $\{Q_k\}_k$, it suffices to show that for all k ,

$$(3.16) \quad \omega \left(\left\{ x \in Q_k \cap \Omega : \tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x) > 2\lambda, \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda \right\} \right) \lesssim \gamma^{\epsilon_0} \omega(Q_k \cap \Omega).$$

From now on, we fix k and denote by l_k the *sidelength* of Q_k .

If $x \in Q_k \cap \Omega$, then

$$(3.17) \quad \tilde{S}_{h, R_0}^{\max\{10l_k, \epsilon\}, R, \frac{1}{20}}(f)(x) \leq \lambda.$$

Indeed, pick $x_k \in 4Q_k \cap \Omega$ with $x_k \notin O$. For any $(y, t) \in \Omega \times (0, R_0]$, if $|x - y| < \frac{t}{20}$ and $t \geq \max\{10l_k, \epsilon\}$, then

$$|x_k - y| \leq |x_k - x| + |x - y| < 4l_k + \frac{t}{20} < \frac{t}{2},$$

which implies that

$$\Gamma_{\frac{1}{20}}^{\max\{10l_k, \epsilon\}, R}(x) \subset \Gamma_{\frac{1}{2}}^{\max\{10l_k, \epsilon\}, R}(x_k).$$

By this, we obtain that

$$\tilde{S}_{h, R_0}^{\max\{10l_k, \epsilon\}, R, \frac{1}{20}}(f)(x) \leq \tilde{S}_{h, R_0}^{\max\{10l_k, \epsilon\}, R, \frac{1}{2}}(f)(x_k) \leq \lambda.$$

Thus, the claim (3.17) holds.

If $\epsilon \geq 10l_k$, by (3.17), we see that (3.16) trivially holds. If $\epsilon < 10l_k$, to show (3.16), by the fact that

$$\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f) \leq \tilde{S}_{h, R_0}^{\epsilon, 10l_k, \frac{1}{20}}(f) + \tilde{S}_{h, R_0}^{10l_k, R, \frac{1}{20}}(f)$$

and (3.17), it remains to show that

$$(3.18) \quad \omega(\{x \in Q_k \cap F : g(x) > \lambda\}) \lesssim \gamma^{\epsilon_0} \omega(Q_k \cap \Omega),$$

where $g := \tilde{S}_{h, R_0}^{\epsilon, 10l_k, \frac{1}{20}}(f)$ and

$$F := \{x \in \Omega : \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda\}.$$

Now we prove (3.18). Similar to the proof of [87, (3.18)], we have

$$(3.19) \quad |\{x \in \Omega \cap Q_k : g(x) > \lambda\}| \lesssim \gamma^2 |Q_k \cap \Omega|.$$

By $\omega \in A_\infty(\mathbb{R}^n)$, Lemma 2.2(vi), we know that there exists $r \in (1, \infty]$ such that $\omega \in RH_r(\mathbb{R}^n)$, which, together with Lemma 2.2(v), Lemma 3.9 and (3.19), implies that

$$(3.20) \quad \frac{\omega(\{x \in Q_k \cap F : g(x) > \lambda\})}{\omega(Q_k)} \lesssim \left(\frac{|\{x \in Q_k \cap F : g(x) > \lambda\}|}{|Q_k|} \right)^{\frac{r-1}{r}} \lesssim \left(\frac{\gamma^2 |Q_k \cap \Omega|}{|Q_k|} \right)^{\frac{r-1}{r}} \sim \gamma^{\frac{2(r-1)}{r}}.$$

Take $\epsilon_0 := 2(r-1)/r$. Then from (3.20) and Lemmas 3.9 and 2.2(iii), we deduce that

$$\omega(\{x \in Q_k \cap F : g(x) > \lambda\}) \lesssim \gamma^{\epsilon_0} \omega(Q_k) \sim \gamma^{\epsilon_0} \omega(Q_k \cap \Omega),$$

which implies that (3.18) holds. Thus, (3.14) holds in the case that $O \neq \Omega$.

Case 2) $O = \Omega$. In this case, similar to the proof of [87, Lemma 3.5], we know that Ω is bounded.

Also, similar to the proof of [87, Lemma 3.5], we know that there exists a positive constant C_1 such that for all $R \in (R_0/2, R_0)$ and $x \in \Omega$,

$$(3.21) \quad \tilde{S}_{h, R_0}^{R_0/2, R, \frac{1}{20}}(f)(x) \leq C_1 \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x).$$

Now we continue the proof of Lemma 3.10 by using (3.21). Without loss of generality, we may assume that $R \geq R_0/2$. Otherwise, we replace R just by $R_0/2$ in (3.15). If $\gamma \geq \frac{1}{C_1}$, then

$$\begin{aligned} & \omega \left(\left\{ x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x) > 2\lambda, \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda \right\} \right) \\ & \leq \omega(\Omega) \leq C_1^{\epsilon_0} \gamma^{\epsilon_0} \omega(O) \lesssim \gamma^{\epsilon_0} \omega(O), \end{aligned}$$

which shows (3.14) in the case that $O = \Omega$ and $\gamma \geq \frac{1}{C_1}$.

If $\gamma < \frac{1}{C_1}$, by the fact that $\mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda$ for all $x \in F$ and (3.21), we conclude that for any $R \geq R_0/2$ and $x \in F$,

$$\tilde{S}_{h, R_0}^{R_0/2, R, \frac{1}{20}}(f)(x) \leq C_1 \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) < \frac{1}{\gamma} \gamma\lambda = \lambda,$$

which implies that

$$(3.22) \quad \begin{aligned} & \left\{ x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x) > 2\lambda, \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda \right\} \\ & \subset \left\{ x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R_0/2, \frac{1}{20}}(f)(x) > \lambda, \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda \right\}. \end{aligned}$$

Thus, to finish the proof of (3.14) in this case, it suffices to show that

$$(3.23) \quad \left| \left\{ x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R_0/2, \frac{1}{20}}(f)(x) > \lambda, \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda \right\} \right| \lesssim \gamma^2 |O|.$$

Indeed, if (3.23) holds, by Lemma 3.9 and the assumption that Ω is bounded, we know that there exists a cube $Q_0 \subset \mathbb{R}^n$ such that $\Omega \subset Q_0$ and $|Q_0| \sim |\Omega|$. From this, $O = \Omega$, (3.22), (3.23), and (iii) and (v) of Lemma 2.2, we infer that

$$\begin{aligned} & \frac{\omega(\{x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x) > 2\lambda, \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda\})}{\omega(Q_0)} \\ & \leq \frac{\omega(\{x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R_0/2, \frac{1}{20}}(f)(x) > \lambda, \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda\})}{\omega(Q_0)} \\ & \lesssim \left\{ \frac{|\{x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R_0/2, \frac{1}{20}}(f)(x) > \lambda, \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda\}|}{|Q_0|} \right\}^{\frac{r-1}{r}} \\ & \lesssim \left\{ \frac{\gamma^2 |\Omega|}{|Q_0|} \right\}^{\frac{r-1}{r}} \sim \gamma^{\epsilon_0}. \end{aligned}$$

By this, the fact that $|Q_0| \sim |\Omega|$, (iii) and (v) of Lemma 2.2 and $O = \Omega$, we conclude that

$$\omega \left(\left\{ x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x) > 2\lambda, \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \leq \gamma\lambda \right\} \right) \lesssim \gamma^{\epsilon_0} \omega(Q_0) \sim \gamma^{\epsilon_0} \omega(O),$$

which shows (3.14) in the case that $O = \Omega$ and $\gamma < \frac{1}{C_1}$.

Finally, we point out that the proof of (3.23) is similar to that of (3.19) with $10l_k$ and $Q_k \cap F$, respectively, replaced by $R_0/2$ and Ω . We omit the details, which completes the proof of Lemma 3.10. $\square$

Lemma 3.11. *Let Φ , Ω and L be as in Proposition 3.4. Let $\omega \in A_\infty(\mathbb{R}^n)$ and $R_0 \in (0, \infty)$. For all $\alpha, \beta \in (0, \infty)$, $0 < \epsilon < R < R_0$ and all $f \in L^2(\Omega)$,*

$$\int_{\Omega} \Phi \left(\tilde{S}_{h, R_0}^{\epsilon, R, \alpha}(f)(x) \right) \omega(x) dx \sim \int_{\Omega} \Phi \left(\tilde{S}_{h, R_0}^{\epsilon, R, \beta}(f)(x) \right) \omega(x) dx,$$

where the implicit constants are independent of ϵ, R and f .

The proof of Lemma 3.11 is similar to that of [20, Proposition 4]. We omit the details.

Now we show Proposition 3.8 by virtue of Lemmas 3.10 and 3.11.

Proof of Proposition 3.8. Let $f \in L^2(\Omega)$ satisfy $\|\mathcal{N}_h^{\text{loc}, 2R_0}(f)\|_{L_{\omega}^{\Phi}(\Omega)} < \infty$. By the upper type 1 and the lower type p_0 properties of Φ , where $p_0 \in (0, p_{\Phi}^-)$, we know that for all $t \in (0, \infty)$,

$$\Phi(t) \sim \int_0^t \frac{\Phi(s)}{s} ds.$$

From this, Fubini's theorem and Lemma 3.10, it follows that for all $\epsilon, R \in (0, R_0]$ with $\epsilon < R$ and $\gamma \in (0, 1]$,

$$\begin{aligned} (3.24) \quad & \int_{\Omega} \Phi \left(\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x) \right) \omega(x) dx \\ & \sim \int_{\Omega} \left\{ \int_0^{\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x)} \frac{\Phi(t)}{t} dt \right\} \omega(x) dx \sim \int_0^{\infty} \frac{\Phi(t)}{t} \sigma_{\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)}(t) dt \\ & \lesssim \int_0^{\infty} \frac{\Phi(t)}{t} \sigma_{\mathcal{N}_h^{\text{loc}, 2R_0}(f)}(\gamma t) dt + \gamma^{\epsilon_0} \int_0^{\infty} \frac{\Phi(t)}{t} \sigma_{\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{2}}(f)}(t/2) dt \\ & \lesssim \frac{1}{\gamma} \int_0^{\infty} \frac{\Phi(t)}{t} \sigma_{\mathcal{N}_h^{\text{loc}, 2R_0}(f)}(t) dt + \gamma^{\epsilon_0} \int_0^{\infty} \frac{\Phi(t)}{t} \sigma_{\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{2}}(f)}(t) dt \\ & \sim \frac{1}{\gamma} \int_{\Omega} \Phi \left(\mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \right) \omega(x) dx + \gamma^{\epsilon_0} \int_{\Omega} \Phi \left(\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{2}}(f)(x) \right) \omega(x) dx, \end{aligned}$$

where

$$\sigma_{\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)}(t) := \omega \left(\left\{ x \in \Omega : \tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x) > t \right\} \right)$$

and ϵ_0 is as in Lemma 3.10.

Furthermore, by Lemma 3.11, (3.24) and $\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{2}}(f) \leq \tilde{S}_{h, R_0}^{\epsilon, R}(f)$, we know that for all $\epsilon, R \in (0, R_0]$ with $\epsilon < R$ and $\gamma \in (0, 1]$,

$$\begin{aligned} \int_{\Omega} \Phi \left(\tilde{S}_{h, R_0}^{\epsilon, R}(f)(x) \right) \omega(x) dx & \sim \int_{\Omega} \Phi \left(\tilde{S}_{h, R_0}^{\epsilon, R, \frac{1}{20}}(f)(x) \right) \omega(x) dx \\ & \lesssim \frac{1}{\gamma} \int_{\Omega} \Phi \left(\mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \right) \omega(x) dx \\ & \quad + \gamma^{\epsilon_0} \int_{\Omega} \Phi \left(\tilde{S}_{h, R_0}^{\epsilon, R}(f)(x) \right) \omega(x) dx, \end{aligned}$$

which, together with the facts that for all $\lambda \in (0, \infty)$,

$$\tilde{S}_{h,R_0}^{\epsilon,R}(f/\lambda) = \tilde{S}_{h,R_0}^{\epsilon,R}(f)/\lambda \quad \text{and} \quad \mathcal{N}_h^{\text{loc},2R_0}(f/\lambda) = \mathcal{N}_h^{\text{loc},2R_0}(f)/\lambda,$$

implies that there exists a positive constant C_2 such that

$$(3.25) \quad \begin{aligned} & \int_{\Omega} \Phi \left(\frac{\tilde{S}_{h,R_0}^{\epsilon,R}(f)(x)}{\lambda} \right) \omega(x) dx \\ & \leq C_2 \left\{ \frac{1}{\gamma} \int_{\Omega} \Phi \left(\frac{\mathcal{N}_h^{\text{loc},2R_0}(f)(x)}{\lambda} \right) \omega(x) dx \right. \\ & \quad \left. + \gamma^{\epsilon_0} \int_{\Omega} \Phi \left(\frac{\tilde{S}_{h,R_0}^{\epsilon,R}(f)(x)}{\lambda} \right) \omega(x) dx \right\}. \end{aligned}$$

Take $\gamma \in (0, 1]$ such that $C_2 \gamma^{\epsilon_0} = \frac{1}{2}$. Then from (3.25), it follows that for all $\lambda \in (0, \infty)$,

$$\int_{\Omega} \Phi \left(\frac{\tilde{S}_{h,R_0}^{\epsilon,R}(f)(x)}{\lambda} \right) \omega(x) dx \lesssim \int_{\Omega} \Phi \left(\frac{\mathcal{N}_h^{\text{loc},2R_0}(f)(x)}{\lambda} \right) \omega(x) dx.$$

By the Fatou lemma and letting $\epsilon \rightarrow 0$ and $R \rightarrow R_0$, we see that for any $\lambda \in (0, \infty)$,

$$\int_{\Omega} \Phi \left(\frac{\tilde{S}_{h,R_0}^{\text{loc}}(f)(x)}{\lambda} \right) \omega(x) dx \lesssim \int_{\Omega} \Phi \left(\frac{\mathcal{N}_h^{\text{loc},2R_0}(f)(x)}{\lambda} \right) \omega(x) dx,$$

which implies that

$$\left\| \tilde{S}_{h,R_0}^{\text{loc}}(f) \right\|_{L_{\omega}^{\Phi}(\Omega)} \lesssim \left\| \mathcal{N}_h^{\text{loc},2R_0}(f) \right\|_{L_{\omega}^{\Phi}(\Omega)}.$$

This finishes the proof of Proposition 3.8. $\square$

Proposition 3.12. *Let Φ , Ω , ω and L be as in Proposition 3.4 and $R_0 \in (0, \infty)$. Then there exists a positive constant C such that for all $f \in L^2(\Omega)$,*

$$\left\| S_{h,R_0}^{\text{loc}}(f) \right\|_{L_{\omega}^{\Phi}(\Omega)} \leq C \left\| \tilde{S}_{h,R_0}^{\text{loc}}(f) \right\|_{L_{\omega}^{\Phi}(\Omega)}.$$

Proof. Let $f \in L^2(\Omega)$, $x \in \Omega$ and $p_0 \in (0, p_{\Phi}^-)$. Then by the definition of p_{Φ}^- , we know that Φ is of lower type p_0 . Similar to the proof of [87, (3.38)], we see that there exists a positive constant C_3 , independent of f and x , such that for all $\varepsilon \in (0, 1)$,

$$(3.26) \quad S_{h,R_0}^{\text{loc}}(f)(x) \leq \frac{C_3}{\varepsilon} \tilde{S}_{h,R_0}^{\text{loc},\frac{3}{2}}(f)(x) + \varepsilon S_{h,R_0}^{\text{loc},2}(f)(x).$$

Also, similar to the proof of Lemma 3.11, we conclude that there exists a positive constant C_4 such that for all $g \in L^2(\Omega)$,

$$\int_{\Omega} \Phi \left(S_{h,R_0}^{\text{loc},2}(g)(y) \right) \omega(y) dy \leq C_4 \int_{\Omega} \Phi \left(S_{h,R_0}^{\text{loc}}(g)(y) \right) \omega(y) dy.$$

From this, (3.26), the lower type p_0 and the upper type 1 properties of Φ , it follows that there exists a positive constant $\tilde{C}$ such that

$$(3.27) \quad \int_{\Omega} \Phi \left(S_{h,R_0}^{\text{loc}}(f)(x) \right) \omega(x) dx$$

$$\begin{aligned}
&\leq \int_{\Omega} \Phi \left(\frac{C_3}{\varepsilon} \tilde{S}_{h,R_0}^{\text{loc}, \frac{3}{2}}(f)(x) \right) \omega(x) dx + \int_{\Omega} \Phi \left(\varepsilon S_{h,R_0}^{\text{loc}, 2}(f)(x) \right) \omega(x) dx \\
&\leq \tilde{C} \left\{ \frac{C_3}{\varepsilon} \int_{\Omega} \Phi \left(\tilde{S}_{h,R_0}^{\frac{3}{2}}(f)(x) \right) \omega(x) dx \right. \\
&\quad \left. + C_4 \varepsilon^{p_0} \int_{\Omega} \Phi \left(S_{h,R_0}^{\text{loc}}(f)(x) \right) \omega(x) dx \right\}.
\end{aligned}$$

Take $\varepsilon \in (0, 1)$ small enough such that $\tilde{C} C_4 \varepsilon^{p_0} \leq \frac{1}{2}$. By this, (3.27) and Lemma 3.11, we see that

$$\int_{\Omega} \Phi \left(S_{h,R_0}^{\text{loc}}(f)(x) \right) \omega(x) dx \lesssim \int_{\Omega} \Phi \left(\tilde{S}_{h,R_0}^{\text{loc}}(f)(x) \right) \omega(x) dx,$$

which, together with the facts that for all $\lambda \in (0, \infty)$,

$$S_{h,R_0}^{\text{loc}}(f/\lambda) = S_{h,R_0}^{\text{loc}}(f)/\lambda \quad \text{and} \quad \tilde{S}_{h,R_0}^{\text{loc}}(f/\lambda) = \tilde{S}_{h,R_0}^{\text{loc}}(f)/\lambda,$$

implies that

$$\int_{\Omega} \Phi \left(\frac{S_{h,R_0}^{\text{loc}}(f)(x)}{\lambda} \right) \omega(x) dx \lesssim \int_{\Omega} \Phi \left(\frac{\tilde{S}_{h,R_0}^{\text{loc}}(f)(x)}{\lambda} \right) \omega(x) dx.$$

From this, the desired conclusion follows, which completes the proof of Proposition 3.12. $\square$

To complete the proof of Theorem 1.4, we need the following key proposition.

Proposition 3.13. *Let Φ , Ω , L , ω , p_{Φ}^{-} , p_{Φ}^{+} and r_{ω} be as in Theorem 1.4, and $R_0 \in (0, \infty)$. Assume that the semigroup generated by L has the Gaussian property (G_1) .*

(i) *If $\Omega := \mathbb{R}^n$, then there exists a positive constant C such that for all $f \in L^2(\mathbb{R}^n)$ satisfying $\|S_{h,R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\mathbb{R}^n)} < \infty$,*

$$\begin{aligned}
\|f\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} &\leq C \left[\|S_{h,R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\mathbb{R}^n)} \right. \\
&\quad \left. + \inf \left\{ \lambda \in (0, \infty) : \sum_{Q_k \in \mathcal{Q}} \omega(Q_k) \Phi \left(\frac{m_{Q_k}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\} \right].
\end{aligned}$$

(ii) *Under DBC, if $\Omega^{\complement}$ is unbounded, then there exists a positive constant C such that for all $f \in L^2(\Omega)$ satisfying $\|S_{h,R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\Omega)} < \infty$,*

$$\begin{aligned}
\|f\|_{h_{\omega}^{\Phi}, r(\Omega)} &\leq C \left[\|S_{h,R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\Omega)} + \inf \left\{ \lambda \in (0, \infty) : \right. \right. \\
&\quad \left. \left. \sum_{\tilde{Q}_k \in \mathcal{Q}_{\Omega}} \omega(\tilde{Q}_k \cap \Omega) \Phi \left(\frac{m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\} \right].
\end{aligned}$$

(iii) Under NBC, there exists a positive constant C such that for all $f \in L^2(\Omega)$ satisfying $\|S_{h, R_0}^{\text{loc}}(f)\|_{L_\omega^\Phi(\Omega)} < \infty$,

$$\|f\|_{h_{\omega, r}^\Phi(\Omega)} \leq C \left[\|S_{h, R_0}^{\text{loc}}(f)\|_{L_\omega^\Phi(\Omega)} + \inf \left\{ \lambda \in (0, \infty) : \sum_{\tilde{Q}_k \in \mathcal{Q}_\Omega} \omega(\tilde{Q}_k \cap \Omega) \Phi \left(\frac{m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\} \right].$$

Let Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$. To show Proposition 3.13, we need the atomic decomposition of functions in the local tent space on Ω . Now we recall some definitions and notation about the tent space, which was initially introduced by Coifman, Meyer and Stein [20] on $\mathbb{R}^n$, and then generalized by Russ [73] to spaces of homogeneous type in the sense of Coifman and Weiss [21, 22]. Recently, Carbonaro, McIntosh and Morris [14] developed the theory of the local tent space on locally doubling metric measure spaces. Recall that it is well known that the strongly Lipschitz domain Ω is a space of homogeneous type.

Let $R_0 \in (0, \infty)$ and $\omega \in A_\infty(\mathbb{R}^n)$. For all measurable functions g on $\Omega \times (0, R_0]$ and $x \in \Omega$, define

$$\mathcal{A}^{\text{loc}, R_0}(g)(x) := \left[\int_{\Gamma^{R_0}(x)} |g(x, t)|^2 \frac{dy}{|Q(x, t) \cap \Omega|} \frac{dt}{t} \right]^{\frac{1}{2}},$$

where

$$\Gamma^{R_0}(x) := \{(y, t) \in \Omega \times (0, R_0] : |y - x| < t\}.$$

In what follows, we denote by $T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)$ the space of all measurable functions g on $\Omega \times (0, R_0]$ such that $\mathcal{A}^{\text{loc}, R_0}(g) \in L_\omega^\Phi(\Omega)$ and for any $g \in T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)$, define its norm by

$$\begin{aligned} \|g\|_{T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)} &:= \|\mathcal{A}^{\text{loc}, R_0}(g)\|_{L_\omega^\Phi(\Omega)} \\ &:= \inf \left\{ \lambda \in (0, \infty) : \int_\Omega \Phi \left(\frac{\mathcal{A}^{\text{loc}, R_0}(g)(x)}{\lambda} \right) \omega(x) dx \leq 1 \right\}. \end{aligned}$$

A function a on $\Omega \times (0, R_0]$ is called a $T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)$ -atom if

(i) there exists a cube $Q := Q(x_Q, l(Q)) \subset \mathbb{R}^n$ with $x_Q \in \Omega$ and

$$l(Q) \in (0, \infty) \cap (0, \text{diam}(\Omega))$$

such that $\text{supp}(a) \subset \widehat{Q \cap \Omega}$, where and in what follows,

$$\widehat{Q \cap \Omega} := \left\{ (y, t) \in \Omega \times (0, R_0] : |y - x_Q| < \frac{l(Q)}{2} - t \right\};$$

(ii)

$$\|a\|_{T_2^2(\Omega \times (0, R_0])}^2 := \int_{\widehat{Q \cap \Omega}} |a(y, t)|^2 \frac{dy dt}{t} \leq |Q \cap \Omega| [\omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))]^{-2}.$$

Since Φ is concave, by Jensen's inequality, it is easy to see that for all $T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)$ -atoms a , we have

$$\|a\|_{T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)} \leq 1;$$

see [49]. By a slight modification on the proof of [48, Theorem 3.1], we have the following atomic decomposition for functions in $T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)$. We omit the details.

Lemma 3.14. *Let Φ satisfy Assumption (A), Ω be either $\mathbb{R}^n$ or a strongly Lipschitz domain of $\mathbb{R}^n$ and $\omega \in A_\infty(\mathbb{R}^n)$. Then for all $f \in T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)$, there exist a sequence $\{a_j\}_j$ of $T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)$ -atoms and a sequence $\{\lambda_j\}_j \subset \mathbb{C}$ such that for almost every $(x, t) \in \Omega \times (0, R_0]$,*

$$f(x, t) = \sum_j \lambda_j a_j(x, t).$$

Moreover, there exists a positive constant C such that for all $f \in T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)$,

$$\begin{aligned} & \Lambda(\{\lambda_j a_j\}_j) \\ &:= \inf \left\{ \lambda \in (0, \infty) : \sum_j \omega(Q_j \cap \Omega) \Phi \left(\frac{|\lambda_j|}{\lambda \omega(Q_j \cap \Omega) \rho(\omega(Q_j \cap \Omega))} \right) \leq 1 \right\} \\ &\leq C \|f\|_{T_{\Phi, \omega}^{\text{loc}, R_0}(\Omega)}, \end{aligned}$$

where for each j , $Q_j \cap \Omega$ appears in the support of a_j .

In [8, p. 183], Auscher and Russ showed the following property of strongly Lipschitz domains, which plays an important role in the proof of Proposition 3.13.

Lemma 3.15. *Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. Then there exists $\gamma_\Omega \in (0, \infty)$ such that for any cube Q satisfying $l(Q) < \gamma_\Omega$ and $2Q \subset \Omega$ but $4Q \cap \partial\Omega \neq \emptyset$, where $\partial\Omega$ denotes the boundary of Ω , there exists a cube $\tilde{Q} \subset \Omega^c$ such that $l(\tilde{Q}) = l(Q)$ and the distance from $\tilde{Q}$ to Q is comparable to $l(Q)$. Furthermore, $\gamma_\Omega = \infty$ if Ω^c is unbounded.*

Now we show Proposition 3.13 by applying Lemmas 3.7, 3.14 and 3.15.

Proof of Proposition 3.13. We first prove Proposition 3.13(i) by borrowing some ideas from the proof of [22, p. 594, Theorem C] (see also [42] and [51]). Let a be a $T_{\Phi, \omega}^{\text{loc}, R_0}(\mathbb{R}^n)$ -atom, $\text{supp}(a) \subset \tilde{Q}$ with $Q := Q(x_0, r_0)$, and

$$(3.28) \quad \alpha := 8 \int_0^{R_0} t^2 L e^{-t^2 L}(a) \frac{dt}{t}.$$

Set $R_k(Q) := 2^{k+1}Q \setminus 2^k Q$ when $k \in \mathbb{N}$ and $R_0(Q) := 2Q$. For $k \in \mathbb{Z}_+$, let $\chi_k := \chi_{R_k(Q)}$, $\tilde{\chi}_k := |R_k(Q)|^{-1} \chi_k$,

$$m_k := \int_{R_k(Q)} \alpha(x) dx$$

and $M_k := \alpha \chi_k - m_k \tilde{\chi}_k$. Then we have

$$(3.29) \quad \alpha = \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} m_k \tilde{\chi}_k.$$

For $j \in \mathbb{Z}_+$, let

$$N_j := \sum_{k=j}^{\infty} m_k.$$

By [8, Lemma A.5(a)], we see that

$$\int_{\mathbb{R}^n} \alpha(x) dx = 0,$$

which, together with (3.29), yields that

$$(3.30) \quad \alpha = \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} N_{k+1} (\tilde{\chi}_{k+1} - \tilde{\chi}_k).$$

Obviously, for all $k \in \mathbb{Z}_+$,

$$(3.31) \quad \int_{\mathbb{R}^n} M_k(x) dx = 0.$$

In what follows, if (2.10) holds with p_{Φ}^+ for all $t \in [1, \infty)$ and $s \in (0, \infty)$, then we choose $\tilde{p}_{\Phi} := p_{\Phi}^+$; otherwise, since Φ is concave, we know $p_{\Phi}^+ < 1$ and we choose $\tilde{p}_{\Phi} \in (p_{\Phi}^+, 1)$ to be close enough to p_{Φ}^+ . Then we know that Φ has the upper type $\tilde{p}_{\Phi}$ property. From the hypotheses

$$\frac{q_{\omega}}{p_{\Phi}} < \frac{n+\mu}{n}, \frac{2q_{\omega}}{p_{\Phi}} < \frac{n+1}{n} + \frac{r_{\omega}-1}{p_{\Phi}^+ r_{\omega}} \text{ and } r_{\omega} > \frac{2}{2-q_{\omega}},$$

we deduce that there exist $p_0 \in (0, p_{\Phi}^-)$, $r_0 \in (1, r_{\omega})$, and $q_1, q_2, q_3 \in (q_{\omega}, 2)$ such that

$$\frac{q_1}{p_0} < \frac{n+\mu}{n}, \frac{2q_2}{p_0} < \frac{n+1}{n} + \frac{r_0-1}{\tilde{p}_{\Phi} r_0} \text{ and } \frac{2}{2-q_3} < r_0.$$

Take $q := \min\{q_1, q_2, q_3\}$. Then $q \in (q_{\omega}, 2)$,

$$(3.32) \quad \frac{q}{p_0} < \frac{n+\mu}{n}, \frac{2q}{p_0} < \frac{n+1}{n} + \frac{r_0-1}{\tilde{p}_{\Phi} r_0} \text{ and } \frac{2}{2-q} < r_0.$$

By the third inequality in (3.32), we know that $\omega \in RH_{2/(2-q)}(\mathbb{R}^n)$. Moreover, similar to the proof of [87, (3.51)], we know that

$$\|\alpha\|_{L^2(\mathbb{R}^n)} \lesssim \|a\|_{T_2^2(\mathbb{R}^n \times (0, R_0])}.$$

Then from this, $\omega \in RH_{2/(2-q)}(\mathbb{R}^n)$, Hölder's inequality, (2.1), (2.3) and Lemma 2.2(iii), we infer that when $k = 0$, there exists a positive constant C_5 such that

$$\begin{aligned} (3.33) \quad \|M_0\|_{L_{\omega}^q(\mathbb{R}^n)} &\leq \|\alpha\|_{L_{\omega}^q(2Q)} + |m_0| \frac{[\omega(2Q)]^{\frac{1}{q}}}{|2Q|} \\ &\leq \|\alpha\|_{L_{\omega}^q(2Q)} + \left\{ \int_{2Q} |\alpha(x)|^q \omega(x) dx \right\}^{\frac{1}{q}} \\ &\quad \times \left\{ \int_{2Q} [\omega(x)]^{-\frac{q'}{q}} dx \right\}^{\frac{1}{q'}} \frac{[\omega(2Q)]^{\frac{1}{q}}}{|2Q|} \\ &\lesssim \|\alpha\|_{L_{\omega}^q(2Q)} \lesssim \|\alpha\|_{L^2(2Q)} \left\{ \int_{2Q} [\omega(x)]^{\frac{2}{2-q}} dx \right\}^{\frac{1}{q} - \frac{1}{2}} \\ &\lesssim \|a\|_{T_2^2(\mathbb{R}^n \times (0, R_0])} \frac{[\omega(2Q)]^{\frac{1}{q}}}{|2Q|^{\frac{1}{2}}} \leq C_5 [\omega(2Q)]^{\frac{1}{q}-1} [\rho(\omega(2Q))]^{-1}, \end{aligned}$$

which, together with (3.31) and $\text{supp}(M_0) \subset 2Q$, implies that M_0/C_5 is a $(\rho, q, 0)_\omega$ -atom. When $k \geq 1$, similar to the proof of [87, (3.53)], we know that for all $x \in R_k(Q)$,

$$(3.34) \quad |\alpha(x)| \lesssim 2^{-k(n+1)} |Q|^{-\frac{1}{2}} \|a\|_{T_2^2(\mathbb{R}^n \times (0, R_0])}.$$

Thus, for all $k \in \mathbb{N}$, from (3.34), Hölder's inequality, (2.1), Lemma 2.2(iii), the upper type $\frac{1}{p_0} - 1$ property of ρ and the first inequality of (3.32), it follows that there exists a positive constant C_6 such that

$$(3.35) \quad \begin{aligned} \|M_k\|_{L_\omega^q(\mathbb{R}^n)} &\leq \|\alpha\|_{L_\omega^q(R_k(Q))} + |m_k| \frac{[\omega(R_k(Q))]^{\frac{1}{q}}}{|R_k(Q)|} \\ &\leq \|\alpha\|_{L_\omega^q(R_k(Q))} \\ &\quad + \|\alpha\|_{L_\omega^q(R_k(Q))} \left\{ \int_{R_k(Q)} [\omega(x)]^{-q'/q} dx \right\}^{\frac{1}{q'}} \frac{[\omega(R_k(Q))]^{\frac{1}{q}}}{|R_k(Q)|} \\ &\lesssim \|\alpha\|_{L_\omega^q(R_k(Q))} \lesssim 2^{-k(n+1)} [\omega(Q)\rho(\omega(Q))]^{-1} [\omega(2^{k+1}Q)]^{\frac{1}{q}} \\ &\lesssim 2^{-k(n+1)} 2^{knq} [\omega(2^{k+1}Q)]^{\frac{1}{q}-1} [\rho(2^{-knq}\omega(2^{k+1}Q))]^{-1} \\ &\lesssim C_6 2^{-kn(\frac{n+1}{n}-\frac{q}{p_0})} [\omega(2^{k+1}Q)]^{\frac{1}{q}-1} [\rho(\omega(2^{k+1}Q))]^{-1}, \end{aligned}$$

which, together with (3.31) and the fact that for each $k \in \mathbb{N}$, $\text{supp}(M_k) \subset 2^{k+1}Q$, implies that for each $k \in \mathbb{N}$, $2^{kn(\frac{n+1}{n}-\frac{q}{p_0})} M_k/C_6$ is a $(\rho, q, 0)_\omega$ -atom. Moreover, by an argument similar to that used in [87, p. 44], we know that $\sum_{k=0}^\infty M_k$ converges in $L^2(\mathbb{R}^n)$. Let

$$\lambda_{1,k} := C_6 2^{-kn(\frac{n+1}{n}-\frac{q}{p_0})} \text{ and } a_{1,k} := 2^{kn(\frac{n+1}{n}-\frac{q}{p_0})} M_k/C_6$$

when $k \in \mathbb{N}$, $\lambda_{1,0} := C_5$ and $a_{1,0} := M_0/C_5$. Then $\{a_{1,k}\}_{k=0}^\infty$ is a sequence of $(\rho, q, 0)_\omega$ -atoms. Furthermore, by the definitions of $\{\lambda_{1,k}\}_{k=0}^\infty$,

$$\omega \in A_q(\mathbb{R}^n) \cap RH_{r_0}(\mathbb{R}^n),$$

(iii) and (v) of Lemma 2.2, the lower type $\frac{1}{p_\Phi} - 1$ property of ρ and the second inequality in (3.32), we know that for all $\lambda \in (0, \infty)$,

$$(3.36) \quad \begin{aligned} &\sum_{k=0}^\infty \omega(2^{k+1}Q) \Phi \left(\frac{|\lambda_{1,k}|}{\lambda \omega(2^{k+1}Q) \rho(\omega(2^{k+1}Q))} \right) \\ &\lesssim \sum_{k=0}^\infty 2^{(k+1)nq} \omega(Q) \Phi \left(\frac{2^{-kn(\frac{n+1}{n}-\frac{q}{p_0})}}{\lambda \omega(2^{k+1}Q) \rho(\omega(2^{k+1}Q))} \right) \\ &\lesssim \sum_{k=0}^\infty 2^{(k+1)nq} \omega(Q) \Phi \left(\frac{2^{-kn(\frac{n+1}{n}-\frac{q}{p_0})} 2^{-\frac{kn(r_0-1)}{p_\Phi r_0}}}{\lambda \omega(Q) \rho(\omega(Q))} \right) \\ &\lesssim \sum_{k=0}^\infty 2^{-kn[\frac{(n+1)p_0}{n} + \frac{p_0(r_0-1)}{p_\Phi r_0} - 2q]} \omega(Q) \Phi \left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))} \right) \\ &\lesssim \omega(Q) \Phi \left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))} \right), \end{aligned}$$

which, together with Lemma 3.7, implies that

$$\sum_{k=0}^{\infty} M_k \in h_{\omega}^{\Phi}(\mathbb{R}^n).$$

To deal with the second sum in (3.32), by Hölder's inequality,

$$\omega \in A_q(\mathbb{R}^n) \cap RH_{\frac{2}{2-q}}(\mathbb{R}^n),$$

Lemma 2.2(v), (3.34), $|\tilde{\chi}_{k+1} - \tilde{\chi}_k| \lesssim |2^k Q|^{-1}$, Lemma 2.2(iii) and the upper type $\frac{1}{p_0} - 1$ property of ρ , we know that there exists a positive constant C_7 such that for all $k \in \mathbb{Z}_+$,

$$\begin{aligned} (3.37) \quad & \|N_{k+1}(\tilde{\chi}_{k+1} - \tilde{\chi}_k)\|_{L_{\omega}^q(\mathbb{R}^n)} \\ & \leq \|N_{k+1}(\tilde{\chi}_{k+1} - \tilde{\chi}_k)\|_{L^2(2^{k+1}Q)} \left\{ \int_{2^{k+1}Q} [\omega(x)]^{\frac{2}{2-q}} dx \right\}^{\frac{1}{q} - \frac{1}{2}} \\ & \leq |2^{k+1}Q|^{-\frac{1}{2}} |N_{k+1}| \frac{[\omega(2^{k+1}Q)]^{\frac{1}{q}}}{|2^{k+1}Q|^{\frac{1}{2}}} \\ & \lesssim |2^{k+1}Q|^{-\frac{1}{2}} \left(\sum_{j=k+1}^{\infty} 2^{-j} \right) |Q|^{\frac{1}{2}} \|a\|_{T_2^2(\mathbb{R}^n \times (0, R_0])} \frac{[\omega(2^{k+1}Q)]^{\frac{1}{q}}}{|2^{k+1}Q|^{\frac{1}{2}}} \\ & \lesssim 2^{-k(n+1)} \frac{[\omega(2^{k+1}Q)]^{\frac{1}{q}}}{\omega(Q)\rho(\omega(Q))} \lesssim 2^{-k(n+1)} 2^{knq} \frac{[\omega(2^{k+1}Q)]^{\frac{1}{q}}}{\omega(2^{k+1}Q)\rho(2^{-knq}\omega(2^{k+1}Q))} \\ & \lesssim C_7 2^{-kn(\frac{n+1}{n} - \frac{q}{p_0})} [\omega(2^{k+1}Q)]^{\frac{1}{q}-1} [\rho(\omega(2^{k+1}Q))]^{-1}. \end{aligned}$$

This, combined with

$$\int_{\mathbb{R}^n} [\tilde{\chi}_{k+1}(x) - \tilde{\chi}_k(x)] dx = 0$$

and $\text{supp}(\tilde{\chi}_{k+1} - \tilde{\chi}_k) \subset 2^{k+1}Q$, yields that for each $k \in \mathbb{Z}_+$,

$$2^{-kn(\frac{n+1}{n} - \frac{q}{p_0})} N_{k+1}(\tilde{\chi}_{k+1} - \tilde{\chi}_k) / C_7$$

is a $(\rho, q, 0)_{\omega}$ -atom. Moreover, by an argument similar to that used in [87, p. 44], we know that

$$\sum_{k=0}^{\infty} N_{k+1}(\tilde{\chi}_{k+1} - \tilde{\chi}_k)$$

converges in $L^2(\mathbb{R}^n)$. For all $k \in \mathbb{Z}_+$, let

$$\lambda_{2,k} := C_7 2^{-kn(\frac{n+1}{n} - \frac{q}{p_0})} \text{ and } a_{2,k} := 2^{kn(\frac{n+1}{n} - \frac{q}{p_0})} N_{k+1}(\tilde{\chi}_{k+1} - \tilde{\chi}_k) / C_7.$$

Then $\{a_{2,k}\}_{k=0}^{\infty}$ is a sequence of $(\rho, q, 0)_{\omega}$ -atoms. Similar to the proof of (3.36), we also see that for all $\lambda \in (0, \infty)$,

$$(3.38) \quad \sum_{k=0}^{\infty} \omega(2^{k+1}Q) \Phi \left(\frac{|\lambda_{2,k}|}{\lambda \omega(2^{k+1}Q) \rho(\omega(2^{k+1}Q))} \right) \lesssim \omega(Q) \Phi \left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))} \right),$$

which, together with Lemma 3.7, implies that

$$\sum_{k=0}^{\infty} N_{k+1}(\tilde{\chi}_{k+1} - \tilde{\chi}_k) \in h_{\omega}^{\Phi}(\mathbb{R}^n).$$

Let $f \in L^2(\mathbb{R}^n)$ satisfy $\|S_{h,R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\mathbb{R}^n)} < \infty$. It is easy to see that for all $z \in \mathbb{C}$ satisfying $z \neq 0$ and $|\arg z| \in (0, \pi/2)$,

$$8 \int_0^{R_0} (t^2 z e^{-t^2 z}) (t^2 z e^{-t^2 z}) \frac{dt}{t} + (2R_0^2 z + 1) e^{-2R_0^2 z} = 1,$$

which, together with the H^∞ -functional calculus for L (see, for example, [62]), implies that

$$(3.39) \quad f = 8 \int_0^{R_0} (t^2 L e^{-t^2 L}) (t^2 L e^{-t^2 L})(f) \frac{dt}{t} + \left[2R_0^2 L e^{-2R_0^2 L}(f) + e^{-2R_0^2 L}(f) \right] \\ =: f_1 + f_2.$$

From the assumption $\|S_{h,R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\mathbb{R}^n)} < \infty$, we deduce that

$$t^2 L e^{-t^2 L}(f) \in T_{\Phi, \omega}^{\text{loc}, R_0}(\mathbb{R}^n)$$

and

$$\|S_{h,R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\mathbb{R}^n)} = \|t^2 L e^{-t^2 L}(f)\|_{T_{\Phi, \omega}^{\text{loc}, R_0}(\mathbb{R}^n)}.$$

Then by Lemma 3.14, we know that there exist $\{\mu_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $T_{\Phi, \omega}^{\text{loc}, R_0}(\mathbb{R}^n)$ -atoms such that for almost every $(x, t) \in \mathbb{R}^n \times (0, R_0]$,

$$(3.40) \quad t^2 L e^{-t^2 L}(f)(x) = \sum_j \mu_j a_j(x, t).$$

For each j , let

$$\alpha_j := 8 \int_0^{R_0} t^2 L e^{-t^2 L}(a_j) \frac{dt}{t}.$$

Then by (3.39) and (3.40), similar to the proof of [49, Proposition 4.2], we see that

$$(3.41) \quad f_1 = \sum_j \mu_j \alpha_j$$

in $L^2(\mathbb{R}^n)$. Replacing α in (3.28) by α_j , consequently, we then denote M_k , N_k and $\tilde{\chi}_k$ in (3.30), $\lambda_{1,k}$, $\lambda_{2,k}$, $a_{1,k}$ and $a_{2,k}$, respectively, by $M_{j,k}$, $N_{j,k}$, $\tilde{\chi}_{j,k}$, $\lambda_{1,j,k}$, $\lambda_{2,j,k}$, $a_{1,j,k}$ and $a_{2,j,k}$. Repeating the above procedure, we obtain

$$f_1 = \sum_j \sum_{k=0}^{\infty} \mu_j M_{j,k} + \sum_j \sum_{k=0}^{\infty} \mu_j N_{j,k+1} (\tilde{\chi}_{j,k+1} - \tilde{\chi}_{j,k}) \\ =: \sum_j \sum_{k=0}^{\infty} \mu_j \lambda_{1,j,k} a_{1,j,k} + \sum_j \sum_{k=0}^{\infty} \mu_j \lambda_{2,j,k} a_{2,j,k},$$

where for each j ,

$$\{\lambda_{1,j,k}\}_{k \in \mathbb{Z}_+} \cup \{\lambda_{2,j,k}\}_{k \in \mathbb{Z}_+} \subset \mathbb{C}$$

and, $\{a_{1,j,k}\}_{k \in \mathbb{Z}_+}$ and $\{a_{2,j,k}\}_{k \in \mathbb{Z}_+}$ are sequences of $(\rho, q, 0)_\omega$ -atoms and both summations hold in $L^2(\mathbb{R}^n)$, and hence in $\mathcal{D}'(\mathbb{R}^n)$. Moreover, from (3.36) with Q and $\lambda_{1,k}$ replaced by Q_j and $\lambda_{1,j,k}$, (3.38) with Q and $\lambda_{2,k}$ replaced by Q_j and $\lambda_{2,j,k}$, and Lemma 3.14, we deduce that

$$\Lambda(\{\mu_j \lambda_{1,j,k} a_{1,j,k}\}_{j,k}) + \Lambda(\{\mu_j \lambda_{2,j,k} a_{2,j,k}\}_{j,k}) \lesssim \Lambda(\{\mu_j a_j\}_j) \\ \lesssim \|S_{h,R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\mathbb{R}^n)}.$$

This, combined with Lemma 3.7, implies that $f_1 \in h_\omega^\Phi(\mathbb{R}^n)$ and

$$(3.42) \quad \|f_1\|_{h_\omega^\Phi(\mathbb{R}^n)} \lesssim \|S_{h, R_0}^{\text{loc}}(f)\|_{L_\omega^\Phi(\mathbb{R}^n)}.$$

Now we deal with f_2 . Denote by $\tilde{K}_{R_0}$ the kernel of $(2R_0^2L + 1)e^{-R_0^2L}$. Then by the mean value theorem for integrals, we know that for all $x \in \mathbb{R}^n$,

$$\begin{aligned} f_2(x) &= \int_{\mathbb{R}^n} \tilde{K}_{R_0}(x, y) e^{-R_0^2L}(f)(y) dy = \sum_{Q_k \in \mathcal{Q}} \int_{Q_k} \tilde{K}_{R_0}(x, y) e^{-R_0^2L}(f)(y) dy \\ &= \sum_{Q_k \in \mathcal{Q}} |Q_k| m_{Q_k} \left(e^{-R_0^2L}(f) \right) \tilde{K}_{R_0}(x, y_k), \end{aligned}$$

where $\mathcal{Q}$ denotes the set of all unit cubes of $\mathbb{R}^n$ whose interiors are disjoint, and for each $k \in \mathbb{N}$, $y_k \in Q_k$ may depend on x . For each k , we have

$$\tilde{K}_{R_0}(x, y_k) = \sum_{i=0}^{\infty} \tilde{K}_{R_0}(x, y_k) \chi_{S_i(Q_k)} =: \sum_{i=0}^{\infty} H_{k, i},$$

where $S_0(Q_k) := 2Q_k$ and for each $i \in \mathbb{N}$, $S_i(Q_k) := 2^{i+1}Q_k \setminus 2^iQ_k$. For each k , from (2.8), it follows that

$$(3.43) \quad \left| \tilde{K}_{R_0}(x, y_k) \right| \lesssim \frac{1}{(1 + |x - y_k|)^{n+1}}.$$

By this, we conclude that there exists a positive constant C_8 such that

$$(3.44) \quad \|H_{k, 0}\|_{L_\omega^q(\mathbb{R}^n)} \lesssim \{C_8\omega(2Q_k)\rho(\omega(2Q_k))\}[\omega(2Q_k)]^{\frac{1}{q}-1}[\rho(\omega(2Q_k))]^{-1}.$$

Thus, $\{C_8\omega(2Q_k)\rho(\omega(2Q_k))\}^{-1}H_{k, 0}$ is a $(\rho, q, 0)_\omega$ -atom. For all $i \in \mathbb{N}$, from (3.43), it follows that there exists a positive constant C_9 such that

$$\begin{aligned} (3.45) \quad \|H_{k, i}\|_{L_\omega^q(\mathbb{R}^n)} &\lesssim \left\{ \int_{S_i(Q_k)} \frac{\omega(x)}{[2^i l(Q_k)]^{(n+1)q}} dx \right\}^{\frac{1}{q}} \\ &\lesssim \left\{ C_9 |2^i Q_k|^{-\frac{n+1}{n}} \omega(2^{i+1}Q_k) \rho(\omega(2^{i+1}Q_k)) \right\} \\ &\quad \times [\omega(2^{i+1}Q_k)]^{\frac{1}{q}-1} [\rho(\omega(2^{i+1}Q_k))]^{-1}, \end{aligned}$$

which implies that $C_9^{-1}|2^i Q_k|^{\frac{n+1}{n}}[\omega(2^{i+1}Q_k)\rho(\omega(2^{i+1}Q_k))]^{-1}H_{k, i}$ is a $(\rho, q, 0)_\omega$ -atom. Let

$$\lambda_{3, k, i} := C_9 |Q_k| m_{Q_k} \left(e^{-R_0^2L}(f) \right) |2^i Q_k|^{-\frac{n+1}{n}} \omega(2^{i+1}Q_k) \rho(\omega(2^{i+1}Q_k))$$

and

$$a_{3, k, i} := C_9^{-1} |2^i Q_k|^{\frac{n+1}{n}} [\omega(2^{i+1}Q_k) \rho(\omega(2^{i+1}Q_k))]^{-1} H_{k, i}$$

for $i \in \mathbb{N}$,

$$\lambda_{3, k, 0} := C_8 |Q_k| m_{Q_k} \left(e^{-R_0^2L}(f) \right) \omega(2Q_k) \rho(\omega(2Q_k))$$

and

$$a_{3, k, 0} := \{C_8\omega(2Q_k)\rho(\omega(2Q_k))\}^{-1}H_{k, 0}.$$

Then

$$f_2 = \sum_k \sum_{i=0}^{\infty} \lambda_{3, k, i} a_{3, k, i}$$

and $\{a_{3,k,i}\}_{k,i \in \mathbb{Z}_+}$ is a sequence of $(\rho, q, 0)_\omega$ -atoms. From this, (3.44), (3.45), $l(Q_k) = 1$, the lower type p_0 property of Φ and the first inequality in (3.32), we deduce that for all $\lambda \in (0, \infty)$,

$$\begin{aligned} & \sum_{Q_k \in \mathcal{Q}} \sum_{i=0}^{\infty} \omega(2^{i+1}Q_k) \Phi \left(\frac{|\lambda_{3,k,i}|}{\lambda \omega(2^{i+1}Q_k) \rho(\omega(2^{i+1}Q_k))} \right) \\ & \lesssim \sum_{Q_k \in \mathcal{Q}} \sum_{i=0}^{\infty} \omega(2^{i+1}Q_k) \Phi \left(\frac{2^{-i(n+1)} m_{Q_k}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \\ & \lesssim \sum_{Q_k \in \mathcal{Q}} \sum_{i=0}^{\infty} 2^{-inq} 2^{-(n+1)p_0} \omega(Q_k) \Phi \left(\frac{m_{Q_k}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \\ & \lesssim \sum_{Q_k \in \mathcal{Q}} \omega(Q_k) \Phi \left(\frac{m_{Q_k}(|e^{-R_0^2 L}(f)|)}{\lambda} \right), \end{aligned}$$

which, together with Lemma 3.7, implies that $f_2 \in h_\omega^\Phi(\mathbb{R}^n)$ and

$$(3.46) \quad \|f_2\|_{h_\omega^\Phi(\mathbb{R}^n)} \lesssim \inf \left\{ \lambda \in (0, \infty) : \sum_{Q_k \in \mathcal{Q}} \omega(Q_k) \Phi \left(\frac{m_{Q_k}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\}.$$

From this, (3.42) and (3.46), we infer that $f \in h_\omega^\Phi(\mathbb{R}^n)$ and

$$\begin{aligned} \|f\|_{h_\omega^\Phi(\mathbb{R}^n)} & \lesssim \|S_{h,R_0}^{\text{loc}}(f)\|_{L_\omega^\Phi(\mathbb{R}^n)} \\ & + \inf \left\{ \lambda \in (0, \infty) : \sum_{Q_k \in \mathcal{Q}} \omega(Q_k) \Phi \left(\frac{m_{Q_k}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\}, \end{aligned}$$

which completes the proof of Proposition 3.13(i).

Now we prove Proposition 3.13(ii). Let $f \in L^2(\Omega)$ satisfy $\|S_{h,R_0}^{\text{loc}}(f)\|_{L_\omega^\Phi(\Omega)} < \infty$. Similar to the proof of (3.39), we know that (3.39) also holds in this case. Let f_1 and f_2 be as in (3.39).

We first deal with f_1 . Similar to the proof of (3.41), we know that

$$f_1 = \sum_j \mu_j \alpha_j$$

in $L^2(\Omega)$, where $\{\mu_j\}_j \subset \mathbb{C}$ and for each j ,

$$\alpha_j := 8 \int_0^{R_0} t^2 L e^{-t^2 L}(a_j) \frac{dt}{t}$$

and a_j is a $T_{\Phi,\omega}^{\text{loc},R_0}(\Omega)$ -atom. For any $T_{\Phi,\omega}^{\text{loc},R_0}(\Omega)$ -atom a supported in $\widehat{Q \cap \Omega}$, let

$$\alpha := 8 \int_0^{R_0} t^2 L e^{-t^2 L}(a) \frac{dt}{t}.$$

To show $f_1 \in h_{\omega,r}^\Phi(\Omega)$, it suffices to show that there exist a function $\tilde{\alpha}$ on $\mathbb{R}^n$, a sequence $\{\lambda_i\}_i$ of numbers and a sequence $\{b_i\}_i$ of $(\rho, q, 0)_\omega$ -atoms such that

$$\begin{aligned} (3.47) \quad & \tilde{\alpha}|_\Omega = \alpha, \\ & \tilde{\alpha} = \sum_i \lambda_i b_i \text{ in } L^2(\mathbb{R}^n) \end{aligned}$$

and for all $\lambda \in (0, \infty)$,

$$(3.48) \quad \sum_i \omega(Q_i) \Phi \left(\frac{|\lambda_i|}{\lambda \omega(Q_i) \rho(\omega(Q_i))} \right) \lesssim \omega(Q \cap \Omega) \Phi \left(\frac{1}{\lambda \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))} \right),$$

where for each i , $\text{supp}(b_i) \subset Q_i$ and $Q \cap \Omega$ appears in the support of a . Indeed, if (3.47) and (3.48) hold, then by (3.47), we know that for each j , there exists a function $\tilde{\alpha}_j$ on $\mathbb{R}^n$ such that $\tilde{\alpha}_j|_\Omega = \alpha_j$. Let

$$\tilde{f}_1 := \sum_j \mu_j \tilde{\alpha}_j.$$

Then $\tilde{f}_1|_\Omega = f_1$. Furthermore, from (3.48), we deduce that there exist $\{\lambda_{j,i}\}_{j,i} \subset \mathbb{C}$ and a sequence $\{b_{j,i}\}_{j,i}$ of $(\rho, q, 0)_\omega$ -atoms such that

$$\tilde{f}_1 = \sum_j \sum_i \mu_j \lambda_{j,i} b_{j,i}$$

and for all $\lambda \in (0, \infty)$,

$$\begin{aligned} & \sum_{j,i} \omega(Q_{j,i}) \Phi \left(\frac{|\mu_j \lambda_{j,i}|}{\lambda \omega(Q_{j,i}) \rho(\omega(Q_{j,i}))} \right) \\ & \lesssim \sum_j \omega(Q_j \cap \Omega) \Phi \left(\frac{|\mu_j|}{\lambda \omega(Q_j \cap \Omega) \rho(\omega(Q_j \cap \Omega))} \right), \end{aligned}$$

which, together with Lemmas 3.7 and 3.14, implies that

$$\left\| \tilde{f}_1 \right\|_{h_\omega^\Phi(\mathbb{R}^n)} \sim \left\| \tilde{f}_1 \right\|_{h_{\omega, q, 0}^{\rho, \omega}(\mathbb{R}^n)} \lesssim \|S_{h, R_0}^{\text{loc}}(f)\|_{L_\omega^\Phi(\Omega)}.$$

From this and Definition 1.3, we deduce that $\tilde{f}_1 \in h_{\omega, r}^\Phi(\Omega)$ and

$$(3.49) \quad \|f_1\|_{h_{\omega, r}^\Phi(\Omega)} \lesssim \|S_{h, R_0}^{\text{loc}}(f)\|_{L_\omega^\Phi(\Omega)}.$$

Let $Q := Q(x_0, r_0)$. Now we show (3.47) and (3.48) by considering the following two cases for Q which appears in the support of a .

Case 1) $8Q \cap \Omega^\complement \neq \emptyset$. In this case, let

$$R_k(Q) := (2^{k+1}Q \setminus 2^k Q) \cap \Omega$$

if $k \geq 3$ and $R_0(Q) := 8Q \cap \Omega$. Let

$$J_\Omega := \{k \in \mathbb{N} : k \geq 3, |R_k(Q)| > 0\}.$$

For $k \in J_\Omega \cup \{0\}$, let $\chi_k := \chi_{R_k(Q)}$, $\tilde{\chi}_k := |R_k(Q)|^{-1} \chi_k$ and

$$m_k := \int_{R_k(Q)} \alpha(x) dx.$$

Then we have

$$\alpha = \alpha \chi_0 + \sum_{k \in J_\Omega} \alpha \chi_k$$

almost everywhere and also in $L^2(\Omega)$. Take the cube $\tilde{Q} \subset \mathbb{R}^n$ such that the center of $\tilde{Q}$, $x_{\tilde{Q}} \in \Omega^\complement$, $l(\tilde{Q}) = l(Q)$ and $\text{dist}(Q, \tilde{Q}) \sim l(Q)$. Then there exists a cube Q_0^* such that $(8Q \cup \tilde{Q}) \subset Q_0^*$ and

$$(3.50) \quad l(Q_0^*) \sim l(Q).$$

Let

$$H_0 := \alpha \chi_0 - \frac{1}{|\tilde{Q} \cap \Omega^c|} \left\{ \int_{R_0(Q)} \alpha(x) dx \right\} \chi_{\tilde{Q} \cap \Omega^c}.$$

Then

$$\int_{\mathbb{R}^n} H_0(x) dx = 0$$

and $\text{supp}(H_0) \subset Q_0^*$. Similar to the proof of [87, (3.51)], we conclude that

$$(3.51) \quad \|\alpha\|_{L^2(\Omega)} \lesssim \|a\|_{T_2^2(\Omega \times (0, R_0])}.$$

By the assumption that Ω^c is an unbounded strongly Lipschitz domain and Lemma 3.10, we know that $|\tilde{Q} \cap \Omega^c| \sim |\tilde{Q}|$. From this, Hölder's inequality,

$$\omega \in A_q(\mathbb{R}^n) \cap RH_{2/(2-q)}(\mathbb{R}^n),$$

(2.3), (3.50), (3.51) and, (iii) and (v) of Lemma 2.2, it follows that there exists a positive constant C_{10} such that

$$\begin{aligned} \|H_0\|_{L_\omega^q(\mathbb{R}^n)} &\leq \|H_0\|_{L^2(Q_0^*)} \left\{ \int_{Q_0^*} [\omega(x)]^{2/(2-q)} dx \right\}^{\frac{1}{q} - \frac{1}{2}} \\ &\lesssim \|H_0\|_{L^2(Q_0^*)} \frac{[\omega(Q_0^*)]^{\frac{1}{q}}}{|Q_0^*|^{\frac{1}{2}}} \\ &\lesssim \left\{ \|\alpha\|_{L^2(\Omega)} + \frac{1}{|\tilde{Q} \cap \Omega^c|^{\frac{1}{2}}} \left(\int_{R_0(Q)} |\alpha(x)|^2 dx \right)^{\frac{1}{2}} |Q \cap \Omega|^{\frac{1}{2}} \right\} \frac{[\omega(Q_0^*)]^{\frac{1}{q}}}{|Q_0^*|^{\frac{1}{2}}} \\ &\lesssim \frac{[\omega(Q_0^*)]^{\frac{1}{q}}}{|Q_0^*|^{\frac{1}{2}}} \|\alpha\|_{L^2(\Omega)} \lesssim \frac{[\omega(Q_0^*)]^{\frac{1}{q}}}{|Q_0^*|^{\frac{1}{2}}} \|a\|_{T_2^2(\Omega \times (0, R_0])} \\ &\lesssim \frac{[\omega(Q_0^*)]^{\frac{1}{q}} |Q \cap \Omega|^{\frac{1}{2}}}{|Q_0^*|^{\frac{1}{2}} \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))} \lesssim C_{10} [\omega(Q_0^*)]^{\frac{1}{q}-1} [\rho(\omega(Q_0^*))]^{-1}. \end{aligned}$$

Let $\lambda_0 := C_{10}$ and $b_0 := H_0/C_{10}$. Then $H_0 = \lambda_0 b_0$ and b_0 is a $(\rho, q, 0)_\omega$ -atom.

To finish the proof in this case, we need the following Fact 1, whose proof is similar to the usual Whitney decomposition of an open set in $\mathbb{R}^n$; see, for example, [76]. We omit the details.

Fact 1. For all $k \in J_\Omega$, there exists the Whitney decomposition $\{Q_{k,i}\}_i$ of $R_k(Q)$ about $\partial\Omega$, where $\{Q_{k,i}\}_i$ are dyadic cubes of $\mathbb{R}^n$ with disjoint interiors and for each i , $2Q_{k,i} \subset \Omega$ but $4Q_{k,i} \cap \partial\Omega \neq \emptyset$.

Let $\{Q_{k,i}\}_{k \in J_\Omega, i}$ be as in Fact 1. Then for each $k \in J_\Omega$,

$$\alpha \chi_{R_k(Q)} = \sum_i \alpha \chi_{Q_{k,i}}$$

almost everywhere. Similar to the proof of [87, (3.53)], we know that for all $x \in R_k(Q)$,

$$(3.52) \quad |\alpha(x)| \lesssim 2^{-k(n+1)} |Q \cap \Omega|^{-\frac{1}{2}} \|a\|_{T_2^2(\Omega \times (0, R_0])}.$$

Moreover, by Lemma 3.15, we see that for each k and i , there exists a cube $\tilde{Q}_{k,i} \subset \Omega^c$ such that $l(\tilde{Q}_{k,i}) = l(Q_{k,i})$ and

$$\text{dist}(\tilde{Q}_{k,i}, Q_{k,i}) \sim l(Q_{k,i}).$$

Then for each k and i , there exists a cube $Q_{k,i}^*$ such that $(Q_{k,i} \cup \tilde{Q}_{k,i}) \subset Q_{k,i}^*$ and $l(Q_{k,i}^*) \sim l(Q_{k,i})$. For each k and i , let

$$H_{k,i} := \alpha \chi_{Q_{k,i}} - \frac{1}{|\tilde{Q}_{k,i}|} \left\{ \int_{Q_{k,i}} \alpha(x) dx \right\} \chi_{\tilde{Q}_{k,i}}.$$

Then

$$\int_{\mathbb{R}^n} H_{k,i}(x) dx = 0 \quad \text{and} \quad \text{supp}(H_{k,i}) \subset Q_{k,i}^*.$$

Furthermore, from Hölder's inequality, $\omega \in RH_{2/(2-q)}(\mathbb{R}^n)$, (2.3), (3.52) and

$$l(Q_{k,i}^*) \sim l(Q_{k,i}),$$

we infer that there exists a positive constant C_{11} such that

$$\begin{aligned} (3.53) \quad \|H_{k,i}\|_{L_\omega^q(\mathbb{R}^n)} &\leq \|\alpha\|_{L^2(Q_{k,i}^*)} \left\{ \int_{Q_{k,i}^*} [\omega(x)]^{2/(2-q)} dx \right\}^{\frac{1}{q}-\frac{1}{2}} \\ &\lesssim 2^{-k(n+1)} |Q \cap \Omega|^{-\frac{1}{2}} |Q_{k,i}^*|^{\frac{1}{2}} \|a\|_{T_2^2(\Omega \times (0, R_0])} \frac{[\omega(Q_{k,i}^*)]^{\frac{1}{q}}}{|Q_{k,i}^*|^{\frac{1}{2}}} \\ &\lesssim C_{11} 2^{-k(n+1)} [\omega(Q_{k,i}^*)]^{\frac{1}{q}} [\omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))]^{-1}. \end{aligned}$$

For each k and j , let

$$\lambda_{k,j} := \frac{C_{11} 2^{-k(n+1)} \omega(Q_{k,i}^*) \rho(\omega(Q_{k,i}^*))}{\omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))}$$

and

$$b_{k,j} := \frac{2^{k(n+1)} H_{k,j} \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))}{C_{11} \omega(Q_{k,i}^*) \rho(\omega(Q_{k,i}^*))}.$$

Then for each k and j , $b_{k,j}$ is a $(\rho, q, 0)_\omega$ -atom and $H_{k,j} := \lambda_{k,j} b_{k,j}$. Let

$$\tilde{\alpha} := H_0 + \sum_{k \in J_\Omega} \sum_i H_{k,i} = \lambda_0 b_0 + \sum_{k \in J_\Omega} \sum_i \lambda_{k,i} b_{k,j}.$$

Then by the constructions of H_0 and $\{H_{k,i}\}_{k \in J_\Omega, i}$, we know that $\tilde{\alpha}|_\Omega = \alpha$. Similar to [87, (3.55)], we know that

$$\tilde{\alpha} = \lambda_0 b_0 + \sum_{k \in J_\Omega} \sum_i \lambda_{k,i} b_{k,j}$$

in $L^2(\mathbb{R}^n)$. Moreover, by the definitions of λ_0 and $\lambda_{k,i}$, the lower type p_0 property of Φ , Lemma 2.2(iii) and the first inequality in (3.32), we know that for all $\lambda \in (0, \infty)$,

$$\begin{aligned} (3.54) \quad &\omega(Q_0^*) \Phi\left(\frac{\lambda_0}{\lambda \omega(Q_0^*) \rho(\omega(Q_0^*))}\right) + \sum_{k \in J_\Omega} \sum_i \omega(Q_{k,i}^*) \Phi\left(\frac{\lambda_{k,i}}{\lambda \omega(Q_{k,i}^*) \rho(\omega(Q_{k,i}^*))}\right) \\ &\lesssim \omega(Q \cap \Omega) \Phi\left(\frac{1}{\lambda \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))}\right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{k \in J_\Omega} \sum_i \omega(Q_{k,i}) \Phi \left(\frac{2^{-k(n+1)}}{\lambda \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))} \right) \\
& \lesssim \omega(Q \cap \Omega) \Phi \left(\frac{1}{\lambda \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))} \right) \\
& + \sum_{k=3}^{\infty} \omega(2^{k+1}Q \cap \Omega) \Phi \left(\frac{2^{-k(n+1)}}{\lambda \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))} \right) \\
& \lesssim \omega(Q \cap \Omega) \Phi \left(\frac{1}{\lambda \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))} \right) \left\{ 1 + \sum_{k=3}^{\infty} 2^{-[k(n+1)p_0 - kqn]} \right\} \\
& \lesssim \omega(Q \cap \Omega) \Phi \left(\frac{1}{\omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))} \right),
\end{aligned}$$

which implies that (3.48) holds in Case 1).

Case 2) $8Q \subset \Omega$. In this case, let $k_0 \in \mathbb{N}$ be such that $2^{k_0}Q \subset \Omega$ but

$$2^{k_0+1}Q \cap \partial\Omega \neq \emptyset.$$

Then $k_0 \geq 3$. Let

$$R_k(Q) := (2^{k+1}Q \setminus 2^kQ) \cap \Omega$$

for $k \in \mathbb{N}$, and $R_0(Q) := 2Q$. Let

$$J_{\Omega, k_0} := \{k \in \mathbb{N} : k \geq k_0 + 1, |R_k(Q)| > 0\}.$$

For $k \in \mathbb{Z}_+$, let $\chi_k := \chi_{R_k(Q)}$, $\tilde{\chi}_k := |R_k(Q)|^{-1} \chi_k$,

$$m_k := \int_{R_k(Q)} \alpha(x) dx, \quad M_k := \alpha \chi_k - m_k \tilde{\chi}_k$$

and $\tilde{M}_k := \alpha \tilde{\chi}_k$. Then

$$\alpha = \sum_{k=0}^{k_0} M_k + \sum_{k \in J_{\Omega, k_0}} \tilde{M}_k + \sum_{k=0}^{k_0} m_k \tilde{\chi}_k.$$

For $k \in \{0, 1, \dots, k_0\}$, by the definition of M_k , we know that

$$\int_{\mathbb{R}^n} M_k(x) dx = 0$$

and $\text{supp}(M_k) \subset 2^{k+1}Q$. Moreover, similar to the estimates of (3.33) and (3.35), we see that there exists a positive constant C_{12} such that for all $k \in \{0, 1, \dots, k_0\}$,

$$\|M_k\|_{L^q_\omega(\mathbb{R}^n)} \leq C_{12} 2^{-kn(\frac{n+1}{n} - \frac{q}{p_0})} [\omega(2^{k+1}Q)]^{\frac{1}{q}-1} [\rho(\omega(2^{k+1}Q))]^{-1}.$$

For each $k \in \{0, \dots, k_0\}$, let

$$\lambda_{1,k} := C_{12} 2^{-kn(\frac{n+1}{n} - \frac{q}{p_0})} \text{ and } b_{1,k} := C_{12}^{-1} 2^{kn(\frac{n+1}{n} - \frac{q}{p_0})} M_k.$$

Thus, for each $k \in \{0, \dots, k_0\}$, $b_{1,k}$ is a $(\rho, q, 0)_\omega$ -atom and $M_k = \lambda_{1,k} b_{1,k}$. Moreover, similar to the estimate of (3.36), we know that for all $\lambda \in (0, \infty)$,

$$\begin{aligned}
(3.55) \quad & \sum_{k=0}^{k_0} \omega(2^{k+1}Q) \Phi \left(\frac{\lambda_{1,k}}{\lambda \omega(2^{k+1}Q) \rho(\omega(2^{k+1}Q))} \right) \\
& \lesssim \omega(Q) \Phi \left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))} \right).
\end{aligned}$$

For each $k \in J_{\Omega, k_0}$, by Fact 1, there exists the Whitney decomposition $\{Q_{k,i}\}_i$ of $R_k(Q)$ about $\partial\Omega$ such that $\bigcup_i Q_{k,i} = R_k(Q)$ and for each i , $Q_{k,i}$ satisfies $2Q_{k,i} \subset \Omega$ and $4Q_{k,i} \cap \partial\Omega \neq \emptyset$. Then

$$\widetilde{M}_k = \sum_i \alpha \chi_{Q_{k,i}}$$

almost everywhere. Moreover, by Lemma 3.15, for each k and i , there exists a cube $\widetilde{Q}_{k,i} \subset \Omega^c$ such that $l(\widetilde{Q}_{k,i}) = l(Q_{k,i})$ and $\text{dist}(\widetilde{Q}_{k,i}, Q_{k,i}) \sim l(Q_{k,i})$. Then for each k and i , there exists a cube $Q_{k,i}^*$ such that $Q_{k,i} \cup \widetilde{Q}_{k,i} \subset Q_{k,i}^*$ and $l(Q_{k,i}^*) \sim l(Q_{k,i})$. For each k and i , let

$$H_{k,i} := \alpha \chi_{Q_{k,i}} - \frac{1}{|\widetilde{Q}_{k,i}|} \left\{ \int_{Q_{k,i}} \alpha(x) dx \right\} \chi_{\widetilde{Q}_{k,i}}.$$

Then

$$\int_{\mathbb{R}^n} H_{k,i}(x) dx = 0$$

and $\text{supp}(H_{k,i}) \subset Q_{k,i}^*$. Furthermore, similar to the proof of (3.53), we conclude that there exists a positive constant C_{13} such that for each $k \in J_{\Omega, k_0}$ and i ,

$$\|H_{k,i}\|_{L_{\omega}^q(\mathbb{R}^n)} \leq C_{13} 2^{-k(n+1)} [\omega(Q_{k,i}^*)]^{\frac{1}{q}} [\omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))]^{-1}.$$

For each k and j , let

$$\lambda_{k,j} := \frac{C_{13} 2^{-k(n+1)} \omega(Q_{k,i}^*) \rho(\omega(Q_{k,i}^*))}{\omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))}$$

and

$$b_{k,j} := \frac{2^{k(n+1)} H_{k,j} \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))}{C_{13} \omega(Q_{k,i}^*) \rho(\omega(Q_{k,i}^*))}.$$

Then for each k and j , $b_{k,j}$ is a $(\rho, q, 0)_{\omega}$ -atom and $H_{k,j} := \lambda_{k,j} b_{k,j}$. Furthermore, similar to the proof of (3.54), we see that for all $\lambda \in (0, \infty)$,

$$\begin{aligned} (3.56) \quad & \sum_{k \in J_{\Omega, k_0}} \sum_i \omega(Q_{k,i}^*) \Phi \left(\frac{\lambda_{k,i}}{\lambda \omega(Q_{k,i}^*) \rho(\omega(Q_{k,i}^*))} \right) \\ & \lesssim \omega(Q \cap \Omega) \Phi \left(\frac{1}{\lambda \omega(Q \cap \Omega) \rho(\omega(Q \cap \Omega))} \right). \end{aligned}$$

For $j \in \{0, 1, \dots, k_0\}$, let

$$N_j := \sum_{k=j}^{k_0} m_k.$$

It is easy to see that

$$\sum_{k=0}^{k_0} m_k \widetilde{\chi}_k = \sum_{k=1}^{k_0} (\widetilde{\chi}_k - \widetilde{\chi}_{k-1}) N_k + N_0 \widetilde{\chi}_0.$$

Similar to the proof of (3.37), we know that there exists a positive constant C_{14} such that for each $k \in \{1, 2, \dots, k_0\}$,

$$\begin{aligned} (3.57) \quad & \|N_k(\widetilde{\chi}_k - \widetilde{\chi}_{k-1})\|_{L_{\omega}^q(\mathbb{R}^n)} \\ & \leq C_{14} 2^{-kn(\frac{n+1}{n} - \frac{q}{p_0})} [\omega(2^k Q)]^{\frac{1}{q}-1} [\rho(\omega(2^k Q))]^{-1}. \end{aligned}$$

For all $k \in \{1, 2, \dots, k_0\}$, let

$$\lambda_{2,k} := C_{14} 2^{-kn(\frac{n+1}{n} - \frac{q}{p_0})} \text{ and } b_{2,k} := C_{14}^{-1} 2^{kn(\frac{n+1}{n} - \frac{q}{p_0})} N_k(\tilde{\chi}_k - \tilde{\chi}_{k-1}).$$

Then from (3.57),

$$\int_{\mathbb{R}^n} [\tilde{\chi}_k(x) - \tilde{\chi}_{k-1}(x)] dx = 0$$

and $\text{supp}(\tilde{\chi}_k - \tilde{\chi}_{k-1}) \subset 2^k Q$, we deduce that for each $k \in \{1, 2, \dots, k_0\}$, $b_{2,k}$ is a $(\rho, q, 0)_\omega$ -atom. Similar to the proof of (3.36), we know that for all $\lambda \in (0, \infty)$,

$$(3.58) \quad \sum_{k=1}^{k_0} \omega(2^k Q) \Phi \left(\frac{\lambda_{2,k}}{\lambda \omega(2^k Q) \rho(\omega(2^k Q))} \right) \lesssim \omega(Q) \Phi \left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))} \right).$$

Finally we deal with $N_0 \tilde{\chi}_0$. By

$$2^{k_0-1} r_0 < \text{dist}(x_0, \partial\Omega) \leq 2^{k_0} r_0,$$

we conclude that there exist a positive integer M and a sequence $\{Q_{0,i}\}_{i=1}^M$ of cubes such that

- (i) $M \sim 2^{k_0}$;
 - (ii) for all $i \in \{1, 2, \dots, M\}$, $l(Q_{0,i}) = 2r_0$ and $Q_{0,i} \subset \Omega$;
 - (iii) for all $i \in \{1, 2, \dots, M-1\}$, $Q_{0,i} \cap Q_{0,i+1} \neq \emptyset$ and
- $$\text{dist}(Q_{0,i}, \partial\Omega) \geq \text{dist}(Q_{0,i+1}, \partial\Omega);$$

- (iv) $2Q_{0,M} \cap \partial\Omega \neq \emptyset$.

Then by Lemma 3.15, there exists a cube $Q_{0,M+1} \subset \Omega^c$ such that $l(Q_{0,M+1}) = r_0$ and

$$\text{dist}(Q_{0,M}, Q_{0,M+1}) \sim r_0.$$

Let

$$H_{0,1} := N_0 \tilde{\chi}_0 - \frac{N_0}{|Q_{0,1}|} \chi_{Q_{0,1}}$$

and

$$H_{0,i} := \frac{N_0}{|Q_{0,i-1}|} \chi_{Q_{0,i-1}} - \frac{N_0}{|Q_{0,i}|} \chi_{Q_{0,i}}$$

with $i \in \{2, \dots, M+1\}$. Obviously, for all $i \in \{1, 2, \dots, M+1\}$, by the definition of $H_{0,i}$, we see that

$$\int_{\mathbb{R}^n} H_{0,i}(x) dx = 0$$

and there exists a cube $Q_{0,i}^* \subset \mathbb{R}^n$ such that $\text{supp}(H_{0,i}) \subset Q_{0,i}^*$ and

$$(3.59) \quad l(Q_{0,i}^*) \sim l(Q).$$

Similar to the proof of [87, (3.66)], we know that

$$(3.60) \quad |N_0| \lesssim 2^{-k_0(n+1)/n} |Q|^{\frac{1}{2}} \|a\|_{T_2^2(\Omega \times (0, R_0])}.$$

For each $i \in \{1, 2, \dots, M+1\}$, from Hölder's inequality, $\omega \in RH_{\frac{2}{2-q}}(\mathbb{R}^n)$, (2.3), the definition of $H_{0,i}$, (3.59) and (3.60), it follows that there exists a positive constant C_{15} such that

$$(3.61) \quad \|H_{0,i}\|_{L_\omega^q(\mathbb{R}^n)} \leq \|H_{0,i}\|_{L^2(\mathbb{R}^n)} \left\{ \int_{Q_{0,i}^*} [\omega(x)]^{\frac{2}{2-q}} dx \right\}^{\frac{1}{q} - \frac{1}{2}}$$

$$\begin{aligned}
&\lesssim |N_0| |Q|^{-\frac{1}{2}} \frac{[\omega(Q_{0,i}^*)]^{\frac{1}{q}}}{|Q_{0,i}^*|^{\frac{1}{2}}} \\
&\lesssim 2^{-k_0(n+1)/n} \frac{[\omega(Q_{0,i}^*)]^{\frac{1}{q}}}{|Q_{0,i}^*|^{\frac{1}{2}}} \|a\|_{T_2^2(\Omega \times (0, R_0])} \\
&\lesssim C_{15} 2^{-k_0(n+1)/n} [\omega(Q_{0,i}^*)]^{\frac{1}{q}} [\omega(Q) \rho(\omega(Q))]^{-1}.
\end{aligned}$$

For each $i \in \{1, \dots, M+1\}$, let

$$\lambda_{3,i} := C_{15} 2^{-k_0(n+1)/n} \frac{\omega(Q_{0,i}^*) \rho(\omega(Q_{0,i}^*))}{\omega(Q) \rho(\omega(Q))}$$

and

$$b_{3,i} := \frac{2^{k_0(n+1)/n} \omega(Q) \rho(\omega(Q)) H_{0,i}}{C_{15} \omega(Q_{0,i}^*) \rho(\omega(Q_{0,i}^*))}.$$

Then for any $i \in \{1, \dots, M+1\}$, from (3.61),

$$\int_{\mathbb{R}^n} H_{0,i}(x) dx = 0$$

and $\text{supp}(H_{0,i}) \subset Q_{0,i}^*$, we infer that $b_{3,i}$ is a $(\rho, q, 0)_\omega$ -atom. By the construction of $\{Q_{0,i}\}_{i=1}^{M+1}$, we know that there exists a positive constant C_{16} such that

$$\bigcup_{i=1}^{M+1} Q_{0,i} \subset C_{16} 2^{k_0} Q.$$

By this, the lower p_0 property of Φ , Lemma 2.2(iii) and the first inequality in (3.32), we conclude that for all $\lambda \in (0, \infty)$,

$$\begin{aligned}
(3.62) \quad &\sum_{i=1}^{M+1} \omega(Q_{0,i}^*) \Phi \left(\frac{\lambda_{3,i}}{\lambda \omega(Q_{0,i}^*) \rho(\omega(Q_{0,i}^*))} \right) \\
&\lesssim \sum_{i=1}^{M+1} \omega(Q_{0,i}) \Phi \left(\frac{2^{-k_0(n+1)/n}}{\lambda \omega(Q) \rho(\omega(Q))} \right) \\
&\lesssim \omega(C_{16} 2^{k_0} Q) 2^{-\frac{k_0(n+1)p_0}{n}} \Phi \left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))} \right) \\
&\lesssim 2^{k_0[nq-(n+1)p_0/n]} \omega(Q) \Phi \left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))} \right) \\
&\lesssim \omega(Q) \Phi \left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))} \right).
\end{aligned}$$

Let

$$\tilde{\alpha} := \sum_{i=1}^{k_0} M_k + \sum_{k \in J_{\Omega, k_0}} \sum_i b_{k,i} + \sum_{k=1}^{k_0} (\tilde{\chi}_k - \tilde{\chi}_{k-1}) N_k + \sum_{i=1}^{M+1} H_{0,i}.$$

Similar to the proof of [87, (3.55)], we see that the above equality holds in $L^2(\mathbb{R}^n)$. By the definition of $\tilde{\alpha}$, we know that $\tilde{\alpha}|_\Omega = \alpha$. Furthermore, from (3.55), (3.56), (3.58) and (3.62), it follows that $\tilde{\alpha} \in h_\omega^\Phi(\mathbb{R}^n)$ and (3.48) holds in Case 2).

Finally, we deal with f_2 . Denote by $\tilde{K}_{R_0}$ the kernel of $(2R_0^2 + 1)L e^{-R_0^2 L}$. Then by the mean value theorem for integrals, we know that

$$\begin{aligned} f_2 &= \int_{\Omega} \tilde{K}_{R_0}(x, y) e^{-R_0^2 L}(f)(y) dy \\ &= \sum_{Q_k \in \mathcal{Q}, Q_k \cap \Omega \neq \emptyset} \int_{Q_k \cap \Omega} \tilde{K}_{R_0}(x, y) e^{-R_0^2 L}(f)(y) dy \\ &= \sum_{Q_k \in \mathcal{Q}, Q_k \cap \Omega \neq \emptyset} |Q_k \cap \Omega| m_{Q_k \cap \Omega} \left(e^{-R_0^2 L}(f) \right) \tilde{K}_{R_0}(x, y_k), \end{aligned}$$

where for each $k \in \mathbb{N}$, $y_k \in Q_k \cap \Omega$ may depend on x . For each k , we have

$$\tilde{K}_{R_0}(x, y_k) = \sum_{i=0}^{\infty} \tilde{K}_{R_0}(x, y_k) \chi_{S_i(Q_k)} := \sum_{i=0}^{\infty} H_{k,i},$$

where $S_0(Q_k) := 2Q_k \cap \Omega$ and for each $i \in \mathbb{N}$,

$$S_i(Q_k) := (2^{i+1}Q_k \setminus 2^i Q_k) \cap \Omega.$$

For each k , by (2.8), we see that for all $x \in \Omega$,

$$(3.63) \quad \left| \tilde{K}_{R_0}(x, y_k) \right| \lesssim \frac{1}{(1 + |x - y_k|)^{n+1}}.$$

From this, we infer that there exists a positive constant C_{17} such that

$$\|H_{k,0}\|_{L_{\omega}^q(\mathbb{R}^n)} \lesssim [\omega(2Q_k)]^{\frac{1}{q}} \lesssim C_{17} \omega(2Q_k) \rho(\omega(2Q_k)) [\omega(2Q_k)]^{\frac{1}{q}-1} [\rho(\omega(2Q_k))]^{-1}.$$

Thus, $\{C_{17} \omega(2Q_k) \rho(\omega(2Q_k))\}^{-1} b_{k,0}$ is a $(\rho, q, 0)_{\omega}$ -atom. For all $i \in \mathbb{N}$, by (3.63), we conclude that there exists a positive constant C_{18} such that

$$\begin{aligned} (3.64) \quad \|H_{k,i}\|_{L_{\omega}^q(\mathbb{R}^n)} &\lesssim \left\{ \int_{S_i(Q_k)} \frac{\omega(x)}{[2^i l(Q_k)]^{(n+1)q}} dx \right\}^{\frac{1}{q}} \\ &\lesssim |2^i Q_k|^{-\frac{n+1}{n}} [\omega(2^{i+1}Q_k)]^{1/q} \\ &\lesssim \left\{ C_{18} |2^i Q_k|^{-\frac{n+1}{n}} \omega(2^{i+1}Q_k) \rho(\omega(2^{i+1}Q_k)) \right\} \\ &\quad \times [\omega(2^{i+1}Q_k)]^{\frac{1}{q}-1} [\rho(\omega(2^{i+1}Q_k))]^{-1}, \end{aligned}$$

which implies that

$$|2^i Q_k|^{(n+1)/n} [\omega(2^{i+1}Q_k) \rho(\omega(2^{i+1}Q_k))]^{-1} H_{k,i} / C_{18}$$

is a $(\rho, q, 0)_{\omega}$ -atom. Let

$$\lambda_{3,k,i} := C_{18} |Q_k \cap \Omega| m_{Q_k \cap \Omega} \left(e^{-R_0^2 L}(f) \right) |2^i Q_k|^{-(n+1)/n} \omega(2^{i+1}Q_k) \rho(\omega(2^{i+1}Q_k))$$

and

$$a_{3,k,i} := C_{18}^{-1} |2^i Q_k|^{\frac{n+1}{n}} [\omega(2^{i+1}Q_k) \rho(\omega(2^{i+1}Q_k))]^{-1} H_{k,i}$$

when $i \in \mathbb{N}$,

$$\lambda_{3,k,0} := C_{17} |Q_k \cap \Omega| m_{Q_k \cap \Omega} \left(e^{-R_0^2 L}(f) \right) \omega(2Q_k) \rho(\omega(2Q_k))$$

and

$$a_{3,k,0} := \{C_{17} \omega(2Q_k) \rho(\omega(2Q_k))\}^{-1} H_{k,0}.$$

Then

$$f_2 = \sum_k \sum_{i=0}^{\infty} \lambda_{3,k,i} a_{3,k,i}$$

and $\{a_{3,k,i}\}_{k,i \in \mathbb{Z}_+}$ is a sequence of $(\rho, q, 0)_\omega$ -atoms. From this, (3.44), (3.45) and $l(Q_k) = 1$, Lemma 2.2(iii) and the first inequality in (3.32), we deduce that for all $\lambda \in (0, \infty)$,

$$\begin{aligned} & \sum_{Q_k \in \mathcal{Q}, Q_k \cap \Omega \neq \emptyset} \sum_{i=0}^{\infty} \omega(2^{i+1}Q_k) \Phi \left(\frac{|\lambda_{3,k,i}|}{\lambda \omega(2^{i+1}Q_k) \rho(\omega(2^{i+1}Q_k))} \right) \\ & \lesssim \sum_{Q_k \in \mathcal{Q}, Q_k \cap \Omega \neq \emptyset} \sum_{i=0}^{\infty} \omega(2^{i+1}Q_k) \Phi \left(\frac{2^{-i(n+1)} m_{Q_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \\ & \lesssim \sum_{\tilde{Q}_k \in \mathcal{Q}_\Omega} \sum_{i=0}^{\infty} 2^{-inq} 2^{-(n+1)p_0} \omega(\tilde{Q}_k) \Phi \left(\frac{m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \\ & \lesssim \sum_{\tilde{Q}_k \in \mathcal{Q}_\Omega} \omega(\tilde{Q}_k \cap \Omega) \Phi \left(\frac{m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right), \end{aligned}$$

where for each k , $\tilde{Q}_k$ is as in Definition 3.3, which, together with Lemma 3.7, implies that $f_2 \in h_{\omega,z}^\Phi(\Omega)$ and hence $f_2 \in h_{\omega,r}^\Phi(\Omega)$, and

$$\begin{aligned} (3.65) \quad & \|f_2\|_{h_{\omega,r}^\Phi(\Omega)} \leq \|f_2\|_{h_{\omega,z}^\Phi(\Omega)} \\ & \lesssim \inf \left\{ \lambda \in (0, \infty) : \sum_{\tilde{Q}_k \in \mathcal{Q}_\Omega} \omega(\tilde{Q}_k \cap \Omega) \right. \\ & \quad \left. \times \Phi \left(\frac{m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\}. \end{aligned}$$

From (3.39), (3.49) and (3.65), we infer that $f \in h_{\omega,r}^\Phi(\Omega)$ and

$$\begin{aligned} \|f\|_{h_{\omega,r}^\Phi(\Omega)} & \lesssim \|S_{h,R_0}^{\text{loc}}(f)\|_{L_\omega^\Phi(\Omega)} + \inf \left\{ \lambda \in (0, \infty) : \sum_{\tilde{Q}_k \in \mathcal{Q}_\Omega} \omega(\tilde{Q}_k \cap \Omega) \right. \\ & \quad \left. \times \Phi \left(\frac{m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\}, \end{aligned}$$

which completes the proof of Proposition 3.13(ii).

Now we prove Proposition 3.13(iii). Let $f \in L^2(\Omega)$ satisfy

$$\|S_{h,R_0}(f)\|_{L_\omega^\Phi(\Omega)} < \infty.$$

By the proof of (3.39), we know that (3.39) also holds in this case. Let f_1 and f_2 be as in (3.39). Denote the *zero extensions* out of Ω of f_1 and f_2 respectively by $\tilde{f}_1$ and $\tilde{f}_2$. Similar to the proof of $f_1 \in h_\omega^\Phi(\mathbb{R}^n)$ in Proposition 3.13(i), we conclude

that $\tilde{f}_1 \in h_{\omega}^{\Phi}(\mathbb{R}^n)$, and hence $f_1 \in h_{\omega, z}^{\Phi}(\Omega)$ and

$$\|f_1\|_{h_{\omega, z}^{\Phi}(\Omega)} = \|\tilde{f}_1\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|S_{h, R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\Omega)}.$$

Similar to the proof of $f_2 \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ in Proposition 3.13(ii), we know that $\tilde{f}_2 \in h_{\omega}^{\Phi}(\mathbb{R}^n)$, and hence $f_2 \in h_{\omega, z}^{\Phi}(\Omega)$ and

$$\begin{aligned} \|f_2\|_{h_{\omega, z}^{\Phi}(\Omega)} &= \|\tilde{f}_2\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \\ &\lesssim \inf \left\{ \lambda \in (0, \infty) : \sum_{\tilde{Q}_k \in \mathcal{Q}_{\Omega}} \omega(\tilde{Q}_k \cap \Omega) \right. \\ &\quad \left. \times \Phi \left(\frac{m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\}. \end{aligned}$$

Let $\tilde{f} := \tilde{f}_1 + \tilde{f}_2$. Then $\tilde{f}$ is the zero extension out of Ω of f . By the above argument, we know that $\tilde{f} \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ and hence $f \in h_{\omega, z}^{\Phi}(\Omega)$. Furthermore,

$$\begin{aligned} \|f\|_{h_{\omega, z}^{\Phi}(\Omega)} &\lesssim \|S_{h, R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\Omega)} + \inf \left\{ \lambda \in (0, \infty) : \sum_{\tilde{Q}_k \in \mathcal{Q}_{\Omega}} \omega(\tilde{Q}_k \cap \Omega) \right. \\ &\quad \left. \times \Phi \left(\frac{m_{\tilde{Q}_k \cap \Omega}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\}, \end{aligned}$$

which completes the proof of Proposition 3.13(iii) and hence Proposition 3.13. $\square$

Now we prove Theorem 1.4 by using Propositions 3.4, 3.8, 3.12 and 3.13.

Proof of Theorem 1.4. We first show Theorem 1.4(i). Let $f \in h_{\omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $R_0 \in [\frac{1}{2}, \infty)$ and $Q \in \mathcal{Q}$. Then

$$m_Q(|e^{-R_0^2 L}(f)|) \leq \inf_{x \in Q} \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x),$$

which implies that for all $\lambda \in (0, \infty)$,

$$\begin{aligned} \sum_{Q_k \in \mathcal{Q}} \omega(Q_k) \Phi \left(\frac{m_{Q_k}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) &\leq \sum_{Q_k \in \mathcal{Q}} \int_{Q_k} \Phi \left(\frac{\mathcal{N}_h^{\text{loc}, 2R_0}(f)(x)}{\lambda} \right) \omega(x) dx \\ &\lesssim \int_{\mathbb{R}^n} \Phi \left(\frac{\mathcal{N}_h^{\text{loc}, 2R_0}(f)(x)}{\lambda} \right) \omega(x) dx. \end{aligned}$$

From this, together with Propositions 3.4(i), 3.8, 3.12 and 3.13(i), we deduce that

$$\begin{aligned} \|f\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} &\sim \left\| \mathcal{N}_h^{\text{loc}, 2R_0}(f)(x) \right\|_{L_{\omega}^{\Phi}(\mathbb{R}^n)} \\ &\sim \left\| \tilde{S}_{h, R_0}^{\text{loc}}(f) \right\|_{L_{\omega}^{\Phi}(\mathbb{R}^n)} \\ &\quad + \inf \left\{ \lambda \in (0, \infty) : \sum_{Q_k \in \mathcal{Q}} \omega(Q_k) \Phi \left(\frac{m_{Q_k}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\} \end{aligned}$$

$$\sim \|S_{h,R_0}^{\text{loc}}(f)\|_{L_{\omega}^{\Phi}(\mathbb{R}^n)} + \inf \left\{ \lambda \in (0, \infty) : \sum_{Q_k \in \mathcal{Q}} \omega(Q_k) \Phi \left(\frac{m_{Q_k}(|e^{-R_0^2 L}(f)|)}{\lambda} \right) \leq 1 \right\},$$

which, together with the arbitrariness of $R_0 \in [\frac{1}{2}, \infty)$ and $f \in h_{\omega}^{\Phi}(\mathbb{R}^n)$, implies that

$$\begin{aligned} h_{\omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) &= h_{\mathcal{N}_h, \omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \\ &= h_{\bar{S}_h, \omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) = h_{S_h, \omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \end{aligned}$$

with equivalent quasi-norms.

To finish the proof of Theorem 1.4(i), we claim that $h_{\omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $h_{\omega}^{\Phi}(\mathbb{R}^n)$. We now prove the claim. For any $q \in (2q_{\omega}, \infty]$ and $s \in \mathbb{Z}_+$ satisfying $s \geq \lfloor n(q_{\omega}/p_{\Phi}^- - 1) \rfloor$, denote the *vector space of all finite linear combinations of* $(\rho, q, s)_{\omega}$ -atoms by $h_{\omega, \text{fin}}^{\rho, q, s}(\mathbb{R}^n)$. By Lemma 3.7 and the definition of $h_{\omega, \text{fin}}^{\rho, q, s}(\mathbb{R}^n)$, we know that $h_{\omega, \text{fin}}^{\rho, q, s}(\mathbb{R}^n)$ is a dense subspace of $h_{\omega}^{\Phi}(\mathbb{R}^n)$. For any $f \in h_{\omega, \text{fin}}^{\rho, q, s}(\mathbb{R}^n)$, let

$$g := \sum_{i=1}^N \lambda_i a_i,$$

where $N \in \mathbb{N}$, and for each $i \in \{1, \dots, N\}$, $\lambda_i \in \mathbb{C}$ and a_i is a $(\rho, q, s)_{\omega}$ -atom. For each $i \in \{1, \dots, N\}$, let $\text{supp}(a_i) \subset Q_i$. By $q \in (2q_{\omega}, \infty]$ and the definition of q_{ω} , we see that $\omega \in A_{q/2}(\mathbb{R}^n)$. From this and Hölder's inequality, we deduce that for each $i \in \{1, \dots, N\}$,

$$\begin{aligned} \|a_i\|_{L^2(\mathbb{R}^n)} &\leq \|a_i\|_{L_{\omega}^q(\mathbb{R}^n)} \left\{ \int_{Q_i} [\omega(x)]^{-\frac{2}{q-2}} \right\}^{\frac{1}{2} - \frac{1}{q}} \\ &\lesssim [\omega(Q_i)]^{\frac{1}{q} - 1} [\rho(\omega(Q_i))]^{-1} \frac{|Q_i|^{\frac{1}{2}}}{[\omega(Q_i)]^{\frac{1}{q}}} \lesssim \frac{|Q_i|^{\frac{1}{2}}}{\omega(Q_i) \rho(\omega(Q_i))}, \end{aligned}$$

which implies that $a_i \in L^2(\mathbb{R}^n)$. By this and the definition of f , we conclude that $f \in h_{\omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, which, together with the fact that $h_{\omega, \text{fin}}^{\rho, q, s}(\mathbb{R}^n)$ is dense in $h_{\omega}^{\Phi}(\mathbb{R}^n)$, implies that $h_{\omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $h_{\omega}^{\Phi}(\mathbb{R}^n)$. Thus, the claim holds. From this, (3.64), the fact that

$$h_{\mathcal{N}_h, \omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), h_{\bar{S}_h, \omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \text{ and } h_{S_h, \omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

are, respectively, dense in $h_{\mathcal{N}_h, \omega}^{\Phi}(\mathbb{R}^n)$, $h_{\bar{S}_h, \omega}^{\Phi}(\mathbb{R}^n)$ and $h_{S_h, \omega}^{\Phi}(\mathbb{R}^n)$, and a density argument, we deduce that the spaces $h_{\omega}^{\Phi}(\mathbb{R}^n)$, $h_{\mathcal{N}_h, \omega}^{\Phi}(\mathbb{R}^n)$, $h_{\bar{S}_h, \omega}^{\Phi}(\mathbb{R}^n)$ and $h_{S_h, \omega}^{\Phi}(\mathbb{R}^n)$ coincide with equivalent quasi-norms. This finishes the proof of Theorem 1.4(i).

From the definitions of the spaces $h_{\omega, r}^{\Phi}(\Omega)$ and $h_{\omega, z}^{\Phi}(\Omega)$ and the fact that $h_{\omega}^{\Phi}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $h_{\omega}^{\Phi}(\mathbb{R}^n)$, we deduce that $h_{\omega, r}^{\Phi}(\Omega) \cap L^2(\Omega)$ and $h_{\omega, z}^{\Phi}(\Omega) \cap L^2(\Omega)$ are, respectively, dense in $h_{\omega, r}^{\Phi}(\Omega)$ and $h_{\omega, z}^{\Phi}(\Omega)$. The remainder of the proofs of (ii) and (iii) of Theorem 1.4 is similar to that of Theorem 1.4(i). We omit the details. This finishes the proof of Theorem 1.4. $\square$

4. PROOF OF THEOREM 1.7

In this section, we give the proof of Theorem 1.7.

Proof of Theorem 1.7. We borrow some ideas from [66, 65] and [84]. We prove Theorem 1.7 by using the following strategy: first, we show that (i) and (iv) are equivalent; then we prove the equivalence between (ii) and (iii); finally, we show that (ii) implies (i), which, together with the standard proof of the implication (iv) $\implies$ (iii), completes the proof of Theorem 1.7. Thus, we divide the whole proof into the following four steps.

Step I. (i) $\iff$ (iv). First we prove that (i) implies (iv). Let $f \in h_{\omega, r}^{\Phi}(\Omega)$. Then there exists $F \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ such that $F|_{\Omega} = f$ and

$$(4.1) \quad \|F\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \sim \|f\|_{h_{\omega, r}^{\Phi}(\Omega)}.$$

From Lemma 3.7, it follows that there exist a sequence $\{a_i\}_i$ of $(\rho, \infty, s)_{\omega}$ -atoms and $\{\lambda_i\}_i \subset \mathbb{C}$ such that $F = \sum_i \lambda_i a_i$ in $\mathcal{D}'(\mathbb{R}^n)$ and

$$(4.2) \quad \|F\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_i a_i\}_i),$$

where $\Lambda(\{\lambda_i a_i\}_i)$ is as in Definition 3.6.

For any $(\rho, \infty, s)_{\omega}$ -atom a , let $\text{supp}(a) \subset Q$. Since $f = F|_{\Omega}$, we only need to consider the case that $Q \cap \Omega \neq \emptyset$. If $2Q \subset \Omega$ and $4Q \cap \partial\Omega = \emptyset$, then a is a type (a) local $(\rho, \infty, s)_{\omega}$ -atom. If $2Q \subset \Omega$ and $4Q \cap \partial\Omega \neq \emptyset$, then a is a type (b) local $(\rho, \infty, s)_{\omega}$ -atom. If $2Q \cap \partial\Omega \neq \emptyset$, by the Whitney decomposition over $Q \cap \Omega$ with $\partial\Omega$, we know that there exists a family of cubes, $\{Q_j\}_j$, with disjoint interiors such that $2Q_j \subset \Omega$, $4Q_j \cap \partial\Omega \neq \emptyset$ and

$$Q \cap \Omega = \bigcup_j Q_j.$$

Thus, $a|_{\Omega} = \sum_j a \chi_{Q_j}$. For each j , let

$$b_{Q_j} := \frac{\omega(Q)\rho(\omega(Q))}{\omega(Q_j)\rho(\omega(Q_j))} a \chi_{Q_j}$$

and

$$\mu_j := \frac{\omega(Q_j)\rho(\omega(Q_j))}{\omega(Q)\rho(\omega(Q))}.$$

Then b_{Q_j} is a type (b) local $(\rho, \infty, s)_{\omega}$ -atom, $a|_{\Omega} = \sum_j \mu_j b_{Q_j}$ and for all $\lambda \in (0, \infty)$,

$$(4.3) \quad \begin{aligned} \sum_j \omega(Q_j) \Phi\left(\frac{\mu_j}{\lambda \omega(Q_j) \rho(\omega(Q_j))}\right) &\leq \sum_j \omega(Q_j) \Phi\left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))}\right) \\ &\leq \omega(Q) \Phi\left(\frac{1}{\lambda \omega(Q) \rho(\omega(Q))}\right). \end{aligned}$$

Thus, from the above observation, we deduce that

$$f = \sum_k \mu_{1,k} b_{1,k} + \sum_j \mu_{2,j} b_{2,j}$$

in $\mathcal{D}'(\mathbb{R}^n)$, where $\{b_{1,k}\}_k$ is a sequence of type (a) local $(\rho, \infty, s)_{\omega}$ -atoms, $\{b_{2,j}\}_j$ a sequence of type (b) local $(\rho, \infty, s)_{\omega}$ -atoms, and $\{\mu_{1,k}\}_k \cup \{\mu_{2,j}\}_j \subset \mathbb{C}$. Moreover,

by (4.3), we know that for all $\lambda \in (0, \infty)$,

$$\begin{aligned} & \sum_k \omega(Q_{1,k}) \Phi \left(\frac{|\mu_{1,k}|}{\lambda \omega(Q_{1,k}) \rho(\omega(Q_{1,k}))} \right) + \sum_j \omega(Q_{2,j}) \Phi \left(\frac{|\mu_{2,j}|}{\lambda \omega(Q_{2,j}) \rho(\omega(Q_{2,j}))} \right) \\ & \leq \sum_i \omega(Q_i) \Phi \left(\frac{|\lambda_i|}{\lambda \omega(Q_i) \rho(\omega(Q_i))} \right), \end{aligned}$$

where for each k, j and i , $\text{supp}(b_{1,k}) \subset Q_{1,k}$, $\text{supp}(b_{2,j}) \subset Q_{2,j}$ and $\text{supp}(a_i) \subset Q_i$. This, combined with (4.1) and (4.2), implies that

$$\|f\|_{h_{\omega}^{\rho, \infty, s}(\Omega)} \lesssim \|F\|_{h_{\omega}^{\rho, \infty, s}(\mathbb{R}^n)} \sim \|F\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \sim \|f\|_{h_{\omega, r}^{\Phi}(\Omega)}.$$

Thus, we prove that (i) implies (iv).

Now we prove that (iv) implies (i). Let $(\rho, q, s)_{\omega}$ be an admissible triplet and $f \in \mathcal{D}'(\Omega)$ such that

$$f = \sum_i \lambda_{Q_{1,i}} a_{Q_{1,i}} + \sum_j \lambda_{Q_{2,j}} a_{Q_{2,j}},$$

where for each i , $a_{Q_{1,i}}$ is a type (a) local $(\rho, q, s)_{\omega}$ -atom supported in the cube $Q_{1,i}$, $\lambda_{Q_{1,i}} \in \mathbb{C}$, and for each j , $a_{Q_{2,j}}$ is a type (b) local $(\rho, q, s)_{\omega}$ -atom supported in the cube $Q_{2,j}$ and $\lambda_{Q_{2,j}} \in \mathbb{C}$.

To finish the proof of this case, we first construct an $F \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ such that $F|_{\Omega} = f$. For all i and j , let $A_{Q_{1,i}} := a_{Q_{1,i}}$, and if $l(Q_{2,j}) \geq 1$, let $A_{Q_{2,j}} := a_{Q_{2,j}}$. Then both $A_{Q_{1,i}}$ and $A_{Q_{2,j}}$ are $(\rho, q, s)_{\omega}$ -atoms.

Now we consider the case when $l(Q_{2,j}) < 1$. It is known that for all $N \in \mathbb{N} \cup \{0\}$, there exist $\{\varphi_{\alpha}\}_{\alpha} \subset C_c^{\infty}(B(0, 1))$ such that

$$\int_{\mathbb{R}^n} x^{\beta} \varphi_{\alpha}(x) dx = \delta_{\alpha, \beta},$$

where $\alpha, \beta \in \mathbb{Z}_+^n$ such that $|\alpha|, |\beta| \leq N$, and $\delta_{\alpha, \beta} = 1$ when $\alpha = \beta$, and $\delta_{\alpha, \beta} = 0$ when $\alpha \neq \beta$; see, for example, [76]. Since $\Omega^{\mathbb{G}}$ is unbounded, from Lemma 3.15, it follows that for any cube $Q \subset \Omega$ such that $2Q \subset \Omega$ and $4Q \cap \partial\Omega \neq \emptyset$, there exist $\tilde{Q}$ and $Q^* \subset \mathbb{R}^n$ such that $Q \cup Q^* \subset \tilde{Q}$, $Q^* \subset (\tilde{\Omega})^{\mathbb{G}}$ and

$$(4.4) \quad l(Q) = l(Q^*) \sim l(\tilde{Q}).$$

Let $N := s \geq \lfloor n(\frac{q_{\omega}}{p_{\Phi}} - 1) \rfloor$. For all $x \in \mathbb{R}^n$ and $Q_{2,j}$ satisfying $l(Q_{2,j}) < 1$, let

$$A_{Q_{2,j}}(x) := a_{Q_{2,j}}(x) - \sum_{|\alpha| \leq s} b_{\alpha} \varphi_{\alpha} \left(\frac{x - x_{Q_{2,j}^*}}{l(Q_{2,j}^*)} \right),$$

where for each $\alpha \in \mathbb{Z}_+^n$, b_{α} is a constant which will be determined later and, for each j , $x_{Q_{2,j}^*}$ denotes the center of $Q_{2,j}^*$. For all $\alpha \in \mathbb{Z}_+^n$ with $0 \leq |\alpha| \leq s$, to show that

$$\int_{\mathbb{R}^n} A_{Q_{2,j}}(x) x^{\alpha} dx = 0,$$

we set

$$b_{\alpha} := \frac{1}{[l(Q_{2,j}^*)]^{|\alpha|+n}} \int_{\mathbb{R}^n} a_{Q_{2,j}}(x) \left(x - x_{Q_{2,j}^*} \right)^{\alpha} dx.$$

Then, for all $\alpha \in \mathbb{Z}_+^n$ with $0 \leq |\alpha| \leq s$, we see that

$$\begin{aligned} \int_{\mathbb{R}^n} A_{Q_{2,j}}(x) x^\alpha dx &= \int_{\mathbb{R}^n} a_{Q_{2,j}}(x) x^\alpha dx - \sum_{|\beta| \leq s} b_\beta \int_{\mathbb{R}^n} \varphi_\beta \left(\frac{x - x_{Q_{2,j}^*}}{l(Q_{2,j}^*)} \right) x^\alpha dx \\ &= \int_{\mathbb{R}^n} a_{Q_{2,j}}(x) x^\alpha dx \\ &\quad - \sum_{|\beta| \leq s} b_\beta [l(Q_{2,j}^*)]^n \int_{\mathbb{R}^n} \varphi_\beta(x) \left(l(Q_{2,j}^*)x + x_{Q_{2,j}^*} \right)^\alpha dx = 0. \end{aligned}$$

Moreover, by (4.4) and Lemma 2.2(iii), we conclude that

$$\begin{aligned} \|A_{Q_{2,j}}\|_{L_\omega^q(\mathbb{R}^n)} &\leq \|a_{Q_{2,j}}\|_{L_\omega^q(\mathbb{R}^n)} + \sum_{|\alpha| \leq s} |b_\alpha| \left\| \varphi_\alpha \left(\frac{\cdot - x_{Q_{2,j}^*}}{l(Q_{2,j}^*)} \right) \right\|_{L_\omega^q(\mathbb{R}^n)} \\ &\lesssim \|a_{Q_{2,j}}\|_{L_\omega^q(\mathbb{R}^n)} \\ &\quad + \sum_{|\alpha| \leq s} |Q_{2,j}|^{-\frac{n+|\alpha|}{n}} \left| \int_{\mathbb{R}^n} a_{Q_{2,j}}(x) (x - x_{Q_{2,j}^*})^\alpha dx \right| [\omega(Q_{2,j}^*)]^{\frac{1}{q}} \\ &\lesssim [\omega(Q_{2,j})]^{\frac{1}{q}-1} [\rho(\omega(Q_{2,j}))]^{-1} \\ &\quad + |Q_{2,j}|^{-1} \|a_{Q_{2,j}}\|_{L_\omega^q(\mathbb{R}^n)} \frac{|Q_{2,j}|}{[\omega(Q_{2,j})]^{\frac{1}{q}}} [\omega(Q_{2,j}^*)]^{\frac{1}{q}} \\ &\lesssim [\omega(Q_{2,j})]^{\frac{1}{q}-1} [\rho(\omega(Q_{2,j}))]^{-1} \sim [\omega(\tilde{Q}_{2,j})]^{\frac{1}{q}-1} [\rho(\omega(\tilde{Q}_{2,j}))]^{-1}. \end{aligned}$$

Let

$$F := \sum_i \lambda_{Q_{1,i}} A_{Q_{1,i}} + \sum_j \lambda_{Q_{2,j}} A_{Q_{2,j}}.$$

We then see that $F \in h_\omega^\Phi(\mathbb{R}^n)$, $F|_\Omega = f$ and

$$\begin{aligned} \|F\|_{h_\omega^\Phi(\mathbb{R}^n)} &\lesssim \inf \left\{ \lambda \in (0, \infty) : \sum_i \omega(\tilde{Q}_{1,i}) \Phi \left(\frac{|\lambda_{Q_{1,i}}|}{\lambda \omega(\tilde{Q}_{1,i}) \rho(\omega(\tilde{Q}_{1,i}))} \right) \right. \\ &\quad \left. + \sum_j \omega(\tilde{Q}_{2,j}) \Phi \left(\frac{|\lambda_{Q_{2,j}}|}{\lambda \omega(\tilde{Q}_{2,j}) \rho(\omega(\tilde{Q}_{2,j}))} \right) \leq 1 \right\} \lesssim \|f\|_{h_\omega^{p,q,s}(\Omega)}. \end{aligned}$$

From this and the definition of $h_{\omega,r}^\Phi(\Omega)$, we deduce that $f \in h_{\omega,r}^\Phi(\Omega)$ and

$$\|f\|_{h_{\omega,r}^\Phi(\Omega)} \lesssim \|f\|_{h_\omega^{p,q,s}(\Omega)}.$$

Thus, (vi) implies (i). This finishes the proof of Step I.

Step II. (ii) $\iff$ (iii). Obviously, (ii) implies (iii). We now prove that (iii) implies (ii). Let $f \in \mathcal{D}'(\Omega)$ such that $f_{\Omega,\varphi}^+ \in L_\omega^\Phi(\Omega)$. Let

$$\left[f_{\Omega,\varphi}^+ \right]^e(x) := \begin{cases} f_{\Omega,\varphi}^+(x), & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

In what follows, for all $g \in L_{\text{loc}}^q(\mathbb{R}^n)$ with $q \in (0, \infty)$ and $x \in \mathbb{R}^n$, let

$$\mathcal{M}_q(g)(x) := \sup_{t>0} \left[\frac{1}{|B(x,t)|} \int_{B(x,t)} |g(y)|^q dy \right]^{\frac{1}{q}},$$

and denote $\mathcal{M}_1(g)$ simply by $\mathcal{M}(g)$, which is just the Hardy-Littlewood maximal function. Miyachi [65] proved that for all $x \in \Omega$,

$$(4.5) \quad f_{\Omega}^*(x) \lesssim \mathcal{M}_{\gamma} \left([f_{\Omega, \varphi}^+]^e \right) (x),$$

where $\gamma \in (0, 1]$ such that $\gamma < \frac{p_{\Phi}^-}{q_{\omega}}$. Similar to the proof of [86, (3.15)], we know that for any given $q \in (q_{\omega}, \infty)$, and all $g \in L_{\text{loc}}^1(\mathbb{R}^n)$ and $\alpha \in (0, \infty)$,

$$\omega(\{x \in \mathbb{R}^n : \mathcal{M}(g)(x) > 2\alpha\}) \lesssim \frac{1}{\alpha^q} \int_{\{x \in \mathbb{R}^n : |g(x)| > \alpha\}} |g(x)|^q \omega(x) dx.$$

From this, it follows that for all $\alpha \in (0, \infty)$,

$$(4.6) \quad \begin{aligned} & \omega \left(\left\{ x \in \mathbb{R}^n : \mathcal{M}_{\gamma} \left([f_{\Omega, \varphi}^+]^e \right) (x) > 2\alpha \right\} \right) \\ & \lesssim \frac{1}{\alpha^{\gamma q}} \int_{\{x \in \mathbb{R}^n : ([f_{\Omega, \varphi}^+]^e(x))^{\gamma} > \frac{\alpha^{\gamma}}{2}\}} \left([f_{\Omega, \varphi}^+]^e(x) \right)^{\gamma q} \omega(x) dx \\ & \sim \sigma_{[f_{\Omega, \varphi}^+]^e} \left(\frac{\alpha}{2^{1/\gamma}} \right) + \frac{1}{\alpha^{\gamma q}} \int_{\frac{\alpha}{2^{1/\gamma}}}^{\infty} \gamma q s^{\gamma q - 1} \sigma_{[f_{\Omega, \varphi}^+]^e}(s) ds, \end{aligned}$$

where and in what follows,

$$\sigma_{[f_{\Omega, \varphi}^+]^e}(t) := \omega(\{x \in \mathbb{R}^n : [f_{\Omega, \varphi}^+]^e(x) > t\}).$$

Choose $q \in (q_{\omega}, \infty)$ and $p_0 \in (0, p_{\Phi}^-)$ such that $\gamma q < p_0$. Then Φ is of lower type p_0 and $\omega \in A_q(\mathbb{R}^n)$. From the assumption that Φ is of upper type 1 and of lower type p_0 , we infer that

$$\Phi(t) \sim \int_0^t \frac{\Phi(s)}{s} ds$$

for all $t \in (0, \infty)$. By this, (4.5), (4.6), and the upper type 1 and the lower type p_0 properties of Φ , we conclude that

$$\begin{aligned} & \int_{\Omega} \Phi(f_{\Omega}^*(x)) \omega(x) dx \\ & \sim \int_{\Omega} \left\{ \int_0^{f_{\Omega}^*(x)} \frac{\Phi(t)}{t} dt \right\} \omega(x) dx \\ & \sim \int_0^{\infty} \frac{\Phi(t)}{t} \tilde{\sigma}_{f_{\Omega}^*}(t) dt \lesssim \int_0^{\infty} \frac{\Phi(t)}{t} \sigma_{\mathcal{M}_{\gamma}([f_{\Omega, \varphi}^+]^e)}(t) dt \\ & \lesssim \int_0^{\infty} \frac{\Phi(t)}{t} \left\{ \sigma_{[f_{\Omega, \varphi}^+]^e} \left(\frac{t}{2^{1/\gamma}} \right) dt + \frac{1}{t^{\gamma q}} \int_{\frac{t}{2^{1/\gamma}}}^{\infty} \gamma q s^{\gamma q - 1} \sigma_{[f_{\Omega, \varphi}^+]^e}(s) ds \right\} dt \\ & \lesssim \int_0^{\infty} \frac{\Phi(t)}{t} \sigma_{[f_{\Omega, \varphi}^+]^e}(t) dt \\ & \quad + \int_0^{\infty} \gamma q s^{\gamma q - 1} \sigma_{[f_{\Omega, \varphi}^+]^e}(s) \Phi(2^{1/\gamma} s) \left\{ \int_0^{2^{1/\gamma} s} \left(\frac{t}{2^{1/\gamma} s} \right)^{p_0} \frac{1}{t^{\gamma q + 1}} dt \right\} ds \\ & \sim \int_0^{\infty} \frac{\Phi(t)}{t} \sigma_{[f_{\Omega, \varphi}^+]^e}(t) dt \sim \int_0^{\infty} \frac{\Phi(t)}{t} \omega \left(\left\{ x \in \Omega : |f_{\Omega, \varphi}^+(x)| > t \right\} \right) dt \\ & \sim \int_{\Omega} \Phi(f_{\Omega, \varphi}^+(x)) \omega(x) dx, \end{aligned}$$

where

$$\tilde{\sigma}_{f_{\Omega}^*}(t) := \omega(\{x \in \Omega : f_{\Omega}^*(x) > t\}).$$

From this and the facts that for all $\lambda \in (0, \infty)$,

$$(f/\lambda)_{\Omega}^* = f_{\Omega}^*/\lambda \text{ and } (f/\lambda)_{\Omega, \varphi}^+ = f_{\Omega, \varphi}^+/\lambda,$$

we deduce that for all $\lambda \in (0, \infty)$,

$$\int_{\Omega} \Phi(f_{\Omega}^*(x)/\lambda) \omega(x) dx \lesssim \int_{\Omega} \Phi(f_{\Omega, \varphi}^+(x)/\lambda) \omega(x) dx,$$

which implies that

$$\|f_{\Omega}^*\|_{L_{\omega}^{\Phi}(\Omega)} \lesssim \|f_{\Omega, \varphi}^+\|_{L_{\omega}^{\Phi}(\Omega)}.$$

This finishes the proof of (ii) $\implies$ (iii) and hence Step II.

Step III. (vi) $\implies$ (iii). The proof of Step III is similar to that of Proposition 3.4(i). We omit the details.

Step IV. (ii) $\implies$ (i). Let $f \in \mathcal{D}'(\Omega)$ such that $f_{\Omega, \varphi}^+ \in L_{\omega}^{\Phi}(\Omega)$. First, we show that there exists $F \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ such that $F|_{\Omega} = f$ and

$$\|F\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|f_{\Omega, \varphi}^+\|_{L_{\omega}^{\Phi}(\Omega)}.$$

For the strongly Lipschitz domain Ω , let the family of cubes, $\{Q_k\}_k$, be the Whitney decomposition over Ω ; namely, there exist a positive constant $c_1 \in (1, 5/4)$ and a family of cubes, $\{Q_k\}_k$, such that the $\{Q_k\}_k$ have disjoint interiors, $\bigcup_k Q_k = \Omega$,

$$\text{diam}(Q_k) \leq \text{dist}(Q_k, \Omega^c) \leq 4 \text{diam}(Q_k)$$

and $\{c_1 Q_k\}_k$ have the bounded intersection property. In the remainder of the proof of this step, for each k , let $Q_k^* := \frac{1+c_1}{2} Q_k$, and $\{\varphi_k\}_k$ be a *partition of unity* associated to $\{Q_k^*\}_k$, that is, for all $k \in \mathbb{N}$, $\varphi_k \in C_c^{\infty}(Q_k^*)$, $0 \leq \varphi_k \leq 1$ and $\varphi_k \equiv 1$ on Q_k . For each k , let $P_{Q_k} \in \mathcal{P}_s(\mathbb{R}^n)$ such that for all $P \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\langle f\varphi_k - P_{Q_k}\chi_{Q_k^*}, P \rangle = 0,$$

where $\mathcal{P}_s(\mathbb{R}^n)$ denotes the *linear space of polynomials in n variables of degrees no more than s* . Let

$$g(x) := \begin{cases} f(x) - \sum_k \chi_{Q_k^*}(x) P_{Q_k}(x), & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

Now, we prove that $g \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ and

$$\|g\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|f_{\Omega, \varphi}^+\|_{L_{\omega}^{\Phi}(\Omega)}.$$

Recall that Miyachi [66, Lemma 1 and (3.25)] proved that for all $f \in \mathcal{D}'(\Omega)$, $k \in \mathbb{N}$ and $\psi \in C_c^{\infty}(Q_k^*)$,

$$(4.7) \quad |\langle f, \psi \rangle| \lesssim |Q_k| \sum_{|\alpha| \leq s+1} \sup_{y \in \Omega} |\partial_y^{\alpha} (\psi(x_{Q_k} + l(Q_k)y))| \inf_{x \in Q_k^*} f_{\Omega}^*(x)$$

and

$$(4.8) \quad \|P_{Q_k}\chi_{Q_k^*}\|_{L^{\infty}(Q_k^*)} \lesssim \inf_{x \in Q_k^*} f_{\Omega}^*(x).$$

Let $\psi \in C_c^\infty(B(0, 1))$ and

$$\int_{\mathbb{R}^n} \psi(y) dy = 1.$$

To prove $g \in h_\omega^\Phi(\mathbb{R}^n)$, we consider two cases as follows.

Case (i) $x \in \Omega$. In this case, from the definition of g , we deduce that

$$\begin{aligned} (4.9) \quad & |\psi_t * g(x)| \\ &= \left| \sum_k \int_{B(x, t)} [f(y)\varphi_k(y) - P_{Q_k}(y)\chi_{Q_k^*}(y)] \psi_t(x-y) dy \right| \\ &\leq \sum_{\{k: |x-x_{Q_k}| < \frac{c_1}{2}l(Q_k)\}} \int_{B(x, t)} |[f(y)\varphi_k(y) - P_{Q_k}(y)\chi_{Q_k^*}(y)] \psi_t(x-y)| dy \\ &\quad + \sum_{\{k: |x-x_{Q_k}| \geq \frac{c_1}{2}l(Q_k)\}} \left| \int_{Q_k^*} [f(y)\varphi_k(y) - P_{Q_k}(y)\chi_{Q_k^*}(y)] \right. \\ &\quad \times \left. \left[\psi_t(x-y) - \sum_{|\alpha| \leq s} \partial^\alpha(\psi_t)(x-x_{Q_k})(y-x_{Q_k})^\alpha \right] dy \right| =: I_1 + I_2. \end{aligned}$$

For I_1 , by (4.8), we know that

$$\begin{aligned} (4.10) \quad I_1 &\lesssim \sum_{\{k: |x-x_{Q_k}| < \frac{c_1}{2}l(Q_k)\}} f_\Omega^*(x)\chi_{\{z: |z-x_{Q_k}| < \frac{c_1}{2}l(Q_k)\}}(x) \\ &\quad + \left\{ \inf_{y \in Q_k^*} f_\Omega^*(y) \right\} \chi_{\{z: |z-x_{Q_k}| < \frac{c_1}{2}l(Q_k)\}}(x). \end{aligned}$$

Now we estimate I_2 . By Taylor's remainder theorem, we conclude that

$$\begin{aligned} (4.11) \quad I_2 &\leq \sum_{\{k: |x-x_{Q_k}| \geq \frac{c_1}{2}l(Q_k)\}} \left| \int_{Q_k^*} f(y)\varphi_k(y) \sum_{|\alpha|=s+1} \partial^\alpha(\psi_t)(x-\xi) \right. \\ &\quad \times \left. (y-x_{Q_k})^\alpha dy \right| + \sum_{\{k: |x-x_{Q_k}| \geq \frac{c_1}{2}l(Q_k)\}} \left| \int_{Q_k^*} \chi_{Q_k^*}(y)P_{Q_k}(y) \right. \\ &\quad \times \left. \sum_{|\alpha|=s+1} \partial^\alpha(\psi_t)(x-\xi)(y-x_{Q_k})^\alpha dy \right| =: A_t(x) + B_t(x), \end{aligned}$$

where $\xi := \theta y + (1-\theta)x_{Q_k}$ for some $\theta \in (0, 1)$. We now estimate $A_t(x)$ and $B_t(x)$. From $|x-x_{Q_k}| \geq \frac{c_1}{2}l(Q_k)$ and $t > |x-\xi|$, we infer that

$$t > |x-\xi| \geq |x-x_{Q_k}| - \theta|y-x_{Q_k}| > |x-x_{Q_k}| - \frac{1+c_1}{4}l(Q_k).$$

Thus,

$$(4.12) \quad t \gtrsim |x-x_{Q_k}| \gtrsim l(Q_k).$$

Therefore, from (4.7) and (4.12), we deduce that

$$(4.13) \quad A_t(x) \lesssim \sum_{\{k: |x-x_{Q_k}| \geq \frac{c_1}{2}l(Q_k)\}} \left\{ \inf_{y \in Q_k^*} f_\Omega^*(y) \right\} |Q_k| \frac{[l(Q_k)]^{s+1}}{t^{n+s+1}}$$

$$\lesssim \sum_{\{k: |x-x_{Q_k}| \geq \frac{c_1}{2}l(Q_k)\}} \left\{ \inf_{y \in Q_k^*} f_{\Omega}^*(y) \right\} \left[1 + \frac{|x-x_{Q_k}|}{l(Q_k)} \right]^{-(n+s+1)}.$$

For $B_t(x)$, by (4.8) and (4.12), and an estimate similar to $A_t(x)$, we know that

$$(4.14) \quad B_t(x) \lesssim \sum_{\{k: |x-x_{Q_k}| \geq \frac{c_1}{2}l(Q_k)\}} \left\{ \inf_{y \in Q_k^*} f_{\Omega}^*(y) \right\} \left[1 + \frac{|x-x_{Q_k}|}{l(Q_k)} \right]^{-(n+s+1)}.$$

Thus, combining (4.9), (4.10), (4.11), (4.13) and (4.14), we conclude that

$$(4.15) \quad \begin{aligned} \psi^+(g)(x) &:= \sup_{0 < t \leq 1} |\psi_t * g(x)| \\ &\lesssim f_{\Omega}^*(x) + \sum_{\{k: |x-x_{Q_k}| \geq \frac{c_1}{2}l(Q_k)\}} \left\{ \inf_{y \in Q_k^*} f_{\Omega}^*(y) \right\} \\ &\quad \times \left[1 + \frac{|x-x_{Q_k}|}{l(Q_k)} \right]^{-(n+s+1)}. \end{aligned}$$

Case (ii) $x \notin \Omega$. In this case, we have

$$\begin{aligned} \psi_t * g(x) &= \int_{B(x,t) \cap \Omega} \sum_k [f(y)\varphi_k(y) - \chi_{Q_k^*}(y)P_{Q_k}(y)] \psi_t(x-y) dy \\ &= \int_{B(x,t) \cap \Omega} \sum_{\{k: Q_k^* \cap B(x,t) \neq \emptyset\}} [f(y)\varphi_k(y) - \chi_{Q_k^*}(y)P_{Q_k}(y)] \psi_t(x-y) dy. \end{aligned}$$

By an argument similar to Case (i), we see that

$$(4.16) \quad \psi^+(g)(x) \lesssim \sum_{\{k: Q_k^* \cap B(x,t) \neq \emptyset\}} \inf_{y \in Q_k^*} f_{\Omega}^*(y) \left[1 + \frac{|x-x_{Q_k}|}{l(Q_k)} \right]^{-(n+s+1)}.$$

Furthermore, by $s \geq \lfloor n(q_{\omega}/p_{\Phi}^- - 1) \rfloor$, we know that $(n+s+1)p_{\Phi}^- > nq_{\omega}$, which, together with the definitions of q_{ω} and p_{Φ}^- , implies that there exist $p_0 \in (0, p_{\Phi}^-)$ and $q \in (q_{\omega}, \infty)$ such that $\omega \in A_q(\mathbb{R}^n)$, Φ is of lower type p_0 and $(n+s+1)p_0 > nq$. From this and Lemma 2.2(iii), we deduce that

$$\int_{\mathbb{R}^n} \left[1 + \frac{|x-x_{Q_k}|}{l(Q_k)} \right]^{-(n+s+1)p_0} \omega(x) dx \lesssim \omega(Q_k).$$

By this, (4.15), (4.16), the lower type p_0 property of Φ and the equivalence between (ii) and (iii) established in Step II, we conclude that

$$\begin{aligned} &\int_{\mathbb{R}^n} \Phi(\psi^+(g)(x)) \omega(x) dx \\ &\lesssim \int_{\Omega} \Phi(f_{\Omega}^*(x)) \omega(x) dx \\ &\quad + \int_{\mathbb{R}^n} \sum_k \Phi \left(\left\{ \inf_{y \in Q_k^*} f_{\Omega}^*(y) \right\} \left[1 + \frac{|x-x_{Q_k}|}{l(Q_k)} \right]^{-(n+s+1)} \right) \omega(x) dx \\ &\lesssim \int_{\Omega} \Phi(f_{\Omega}^*(x)) \omega(x) dx + \sum_k \Phi \left(\inf_{y \in Q_k^*} f_{\Omega}^*(y) \right) \end{aligned}$$

$$\begin{aligned}
& \times \int_{\mathbb{R}^n} \left[1 + \frac{|x - x_{Q_k}|}{l(Q_k)} \right]^{-(n+s+1)p_0} \omega(x) dx \\
& \lesssim \int_{\Omega} \Phi(f_{\Omega}^*(x)) \omega(x) dx + \sum_k \Phi \left(\inf_{y \in Q_k^*} f_{\Omega}^*(y) \right) \omega(Q_k) \\
& \lesssim \int_{\Omega} \Phi(f_{\Omega}^*(x)) \omega(x) dx \lesssim \int_{\Omega} \Phi(f_{\Omega, \varphi}^+(x)) \omega(x) dx,
\end{aligned}$$

which implies that $g \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ and

$$(4.17) \quad \|g\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|f_{\Omega, \varphi}^+\|_{L_{\omega}^{\Phi}(\Omega)}.$$

For all $x \in \mathbb{R}^n$, let

$$R(x) := \sum_k P_{Q_k}(x) \chi_{Q_k^*}(x).$$

Let $q \in (q_{\omega}, \infty]$, $s \in \mathbb{Z}_+$ and $s \geq \lfloor n(\frac{q_{\omega}}{p_{\Phi}} - 1) \rfloor$. For all $k \in \mathbb{N}$, let

$$\lambda_k := \frac{\omega(Q_k^*) \rho(\omega(Q_k^*))}{[\omega(Q_k^*)]^{\frac{1}{q}}} \|P_{Q_k}\|_{L_{\omega}^q(Q_k^*)}$$

and

$$a_k := \frac{[\omega(Q_k^*)]^{\frac{1}{q}}}{\omega(Q_k^*) \rho(\omega(Q_k^*))} \frac{P_{Q_k} \chi_{Q_k^*}}{\|P_{Q_k}\|_{L_{\omega}^q(Q_k^*)}}.$$

Then, obviously,

$$R = \sum_k \lambda_k a_k.$$

Moreover, a_k is a type (b) local $(\rho, q, s)_{\omega}$ -atom and, by (4.8) and Lemma 2.2(iii), we further conclude that

$$\begin{aligned}
& \sum_k \omega(Q_k^*) \Phi \left(\frac{|\lambda_k|}{\omega(Q_k^*) \rho(\omega(Q_k^*))} \right) \\
& \leq \sum_k \omega(Q_k^*) \Phi \left(\frac{\|P_{Q_k}\|_{L_{\omega}^q(Q_k^*)}}{[\omega(Q_k^*)]^{\frac{1}{q}}} \right) \leq \sum_k \omega(Q_k^*) \Phi \left(\|P_{Q_k}\|_{L^{\infty}(Q_k^*)} \right) \\
& \lesssim \sum_k \omega(Q_k^*) \Phi \left(\inf_{y \in Q_k^*} f_{\Omega}^*(y) \right) \lesssim \sum_k \omega(Q_k) \Phi \left(\inf_{y \in Q_k^*} f_{\Omega}^*(y) \right) \\
& \lesssim \int_{\Omega} \Phi(f_{\Omega}^*(y)) \omega(y) dy.
\end{aligned}$$

From this, it follows that

$$\|R\|_{h_{\omega}^{\rho, q, s}(\Omega)} \lesssim \|f_{\Omega}^*\|_{L_{\omega}^{\Phi}(\Omega)}.$$

By this and the equivalence between (i) and (iv) established in Step I, we see that there exists $\tilde{R} \in h_{\omega}^{\Phi}(\mathbb{R}^n)$ such that $\tilde{R}|_{\Omega} = R$ and

$$(4.18) \quad \left\| \tilde{R} \right\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|R\|_{h_{\omega}^{\rho, q, s}(\Omega)} \sim \|R\|_{h_{\omega}^{\rho, q, s}(\Omega)} \lesssim \|f_{\Omega}^*\|_{L_{\omega}^{\Phi}(\Omega)} \lesssim \|f_{\Omega, \varphi}^+\|_{L_{\omega}^{\Phi}(\Omega)}.$$

Let $F := g + \tilde{R}$. By (4.17) and (4.18), we have $F \in h_{\omega}^{\Phi}(\mathbb{R}^n)$, $F|_{\Omega} = f$ and

$$\|F\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|g\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} + \left\| \tilde{R} \right\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|f_{\Omega, \varphi}^+\|_{L_{\omega}^{\Phi}(\Omega)},$$

which completes the proof of Step VI and hence Theorem 1.7. $\square$

5. PROOF OF THEOREM 1.8

In this section, we give the proof of Theorem 1.8. In what follows, we always assume that Ω is a bounded, simply connected, semiconvex domain in $\mathbb{R}^n$, and $\mathbb{G}_D$ the Dirichlet Green operator for the problem (1.1). Denote the integral kernel of $\mathbb{G}_D$ by G_D . We first recall the notion of semiconvex domains in $\mathbb{R}^n$ and some useful estimates for G_D ; see [26].

Definition 5.1. (i) Let O be an open set in $\mathbb{R}^n$. The collection of *semiconvex functions* on O consists of continuous functions $u : O \mapsto \mathbb{R}$ with the property that there exists a positive constant C such that for all $x, h \in \mathbb{R}^n$ with the ball $B(x, |h|) \subset O$,

$$2u(x) - u(x+h) - u(x-h) \leq C|h|^2.$$

The best constant C above is referred to as the *semiconvexity constant* of u .

(ii) A nonempty, proper open subset Ω of $\mathbb{R}^n$ is called *semiconvex* provided there exist $b, c \in (0, \infty)$ with the property that for every $x_0 \in \partial\Omega$, there exist an $(n-1)$ -dimensional affine variety $H \subset \mathbb{R}^n$ passing through x_0 , a choice N of the unit normal to H , and an open set

$$\mathcal{C} := \{\tilde{x} + tN : \tilde{x} \in H, |\tilde{x} - x_0| < b, |t| < c\},$$

called a *coordinate cylinder* near x_0 (with axis along N), such that for some semiconvex function $\varphi : H \rightarrow \mathbb{R}$ satisfying

$$\mathcal{C} \cap \Omega = \mathcal{C} \cap \{\tilde{x} + tN : \tilde{x} \in H, t > \varphi(\tilde{x})\},$$

$$\mathcal{C} \cap \partial\Omega = \mathcal{C} \cap \{\tilde{x} + tN : \tilde{x} \in H, t = \varphi(\tilde{x})\},$$

$$\mathcal{C} \cap \overline{\Omega}^c = \mathcal{C} \cap \{\tilde{x} + tN : \tilde{x} \in H, t < \varphi(\tilde{x})\},$$

we have

$$\varphi(x_0) = 0 \text{ and } |\varphi(\tilde{x})| < c/2 \text{ if } |\tilde{x} - x_0| \leq b.$$

The following Lemma 5.2 was established in [36, 31]. Recall that for all $y \in \Omega$,

$$\delta(y) := \text{dist}(y, \partial\Omega).$$

Lemma 5.2. Let Ω and $\mathbb{G}_D$ be as in Theorem 1.8. Denote the integral kernel of $\mathbb{G}_D$ by G_D . Then there exists a positive constant C such that for all $x, y \in \Omega$ with $x \neq y$,

- (i) $0 \leq G_D(x, y) \leq C|x - y|^{2-n}$;
- (ii) $|\nabla_x G_D(x, y)| \leq C|x - y|^{1-n}$;
- (iii) $|\nabla_y G_D(x, y)| \leq C\delta(y)|x - y|^{-n}$;
- (iv) $|\nabla_y G_D(x, y)| \leq C|x - y|^{1-n}$;
- (v) $|\nabla_x \nabla_y G_D(x, y)| \leq C|x - y|^{-n}$.

The following Lemma 5.3 is just [26, Theorem 4.1].

Lemma 5.3. Let Ω and $\mathbb{G}_D$ be as in Theorem 1.8. Then the operators in (1.3), originally defined on $C^\infty(\overline{\Omega})$, can be extended to bounded operators on $L^p(\Omega)$ for $p \in (1, 2]$.

Now we prove Theorem 1.8 by using Lemmas 5.2 and 5.3 and Theorem 1.7.

Proof of Theorem 1.8. We first prove Theorem 1.8(i). Fix $i, j \in \{1, \dots, n\}$ and denote by T the operator $\frac{\partial^2 \mathbb{G}_D}{\partial x_i \partial x_j}$. First let $f \in h_{\omega, r}^\Phi(\Omega) \cap L^2(\Omega)$. By the assumption that $r_\omega > \frac{2}{2-q_\omega}$, we know that $\frac{r_\omega q_\omega}{r_\omega - 1} < 2$. Take $q \in (\frac{r_\omega q_\omega}{r_\omega - 1}, 2]$ and $q_1 \in (\frac{r_\omega}{r_\omega - 1}, \frac{2}{q_\omega}]$ such that $\frac{q}{q_1} > q_\omega$. Then

$$\frac{2}{2 - q_\omega} \leq q'_1 < r_\omega,$$

where and in what follows, $\frac{1}{q_1} + \frac{1}{q'_1} = 1$. Thus, $\omega \in RH_{q'_1}(\mathbb{R}^n)$ and $\omega \in A_{q/q_1}(\mathbb{R}^n)$. By Theorem 1.7, we conclude that

$$(5.1) \quad f = \sum_{\text{type } (a)\text{-atoms}} \lambda_{1,k} a_{1,k} + \sum_{\text{type } (b)\text{-atoms}} \lambda_{2,m} a_{2,m},$$

where $\{a_{1,k}\}_k$ and $\{a_{2,m}\}_m$ are respectively sequences of type (a) local $(\rho, q, 0)_\omega$ -atoms and type (b) local $(\rho, q, 0)_\omega$ -atoms, and $(\{\lambda_{1,k}\}_k \cup \{\lambda_{2,m}\}_m) \subset \mathbb{C}$. Moreover,

$$(5.2) \quad \Lambda(\{\lambda_{1,k} a_{1,k}\}_k \cup \{\lambda_{2,m} a_{2,m}\}_m) \sim \|f\|_{h_{\omega, r}^\Phi(\Omega)}.$$

To finish the proof of Theorem 1.8(i), we only need to show that for any type (a) local $(\rho, q, 0)_\omega$ -atom or any type (b) local $(\rho, q, 0)_\omega$ -atom a supported in the cube Q_0 and any $\lambda \in \mathbb{C}$,

$$(5.3) \quad \int_\Omega \Phi(T(\lambda a)(x)) \omega(x) dx \lesssim \omega(Q_0) \Phi\left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))}\right).$$

Indeed, if (5.3) holds, then by (5.1) and the assumption that Φ is subadditive, we know that for all $\lambda \in (0, \infty)$,

$$\begin{aligned} & \int_\Omega \Phi\left(T\left(\frac{f}{\lambda}\right)(x)\right) \omega(x) dx \\ & \leq \sum_k \int_\Omega \Phi\left(T\left(\frac{\lambda_{1,k} a_{1,k}}{\lambda}\right)(x)\right) \omega(x) dx \\ & \quad + \sum_m \int_\Omega \Phi\left(T\left(\frac{\lambda_{2,m} a_{2,m}}{\lambda}\right)(x)\right) \omega(x) dx \\ & \lesssim \sum_k \omega(Q_{1,k}) \Phi\left(\frac{|\lambda_{1,k}|}{\lambda \omega(Q_{1,k}) \rho(\omega(Q_{1,k}))}\right) \\ & \quad + \sum_m \omega(Q_{2,m}) \Phi\left(\frac{|\lambda_{2,m}|}{\lambda \omega(Q_{2,m}) \rho(\omega(Q_{2,m}))}\right), \end{aligned}$$

where for each k and m , $\text{supp}(a_{1,k}) \subset Q_{1,k}$ and $\text{supp}(a_{2,m}) \subset Q_{2,m}$, which, together with Theorem 1.7 and (5.2), implies that

$$\|T(f)\|_{L^\Phi_\omega(\Omega)} \lesssim \|f\|_{h_{\omega, r}^\Phi(\Omega)}.$$

From this, the fact that $h_{\omega, r}^\Phi(\Omega) \cap L^2(\Omega)$ is dense in $h_{\omega, r}^\Phi(\Omega)$ and a density argument, we deduce that Theorem 1.8(i) holds.

Now we prove (5.3) by considering the following three cases for Q_0 .

Case i) $l(Q_0) \geq 1$. In this case, by

$$\omega \in RH_{q'_1}(\mathbb{R}^n) \cap A_{q/q_1}(\mathbb{R}^n),$$

Jensen's inequality, Hölder's inequality, Lemma 5.3, (iii) and (v) of Lemma 2.2, the assumptions that Ω is bounded and $l(Q_0) \geq 1$, we conclude that

$$\begin{aligned}
 (5.4) \quad & \int_{\Omega} \Phi(T(\lambda a)(x)) \omega(x) dx \\
 & \leq \omega(\Omega) \Phi \left(\frac{1}{\omega(\Omega)} \left\{ \int_{\Omega} |T(\lambda a)(x)|^{q_1} dx \right\}^{\frac{1}{q_1}} \left\{ \int_{\Omega} [\omega(x)]^{q'_1} dx \right\}^{\frac{1}{q'_1}} \right) \\
 & \lesssim \omega(\Omega) \Phi \left(\frac{1}{|\Omega|^{\frac{1}{q_1}}} \|\lambda a\|_{L^{q_1}(Q_0)} \right) \lesssim \omega(\Omega) \Phi \left(\frac{|\lambda| |Q_0|^{\frac{1}{q_1}}}{|\Omega|^{\frac{1}{q_1}} [\omega(Q_0)]^{\frac{1}{q}}} \|a\|_{L^q_{\omega}(Q_0)} \right) \\
 & \lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right).
 \end{aligned}$$

Case ii) $l(Q_0) < 1$, $2Q_0 \subset \Omega$ and $4Q_0 \cap \partial\Omega \neq \emptyset$. In this case, we have

$$\begin{aligned}
 (5.5) \quad & \int_{\Omega} \Phi(T(\lambda a)(x)) \omega(x) dx = \int_{4Q_0 \cap \Omega} \Phi(T(\lambda a)(x)) \omega(x) dx \\
 & + \sum_{j=2}^{\infty} \int_{(2^{j+1}Q_0 \setminus 2^jQ_0) \cap \Omega} \cdots =: I_1 + I_2.
 \end{aligned}$$

Similar to the proof of (5.4), we have

$$(5.6) \quad I_1 \lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right).$$

Now we deal with I_2 . For all $j \in \mathbb{N}$ with $j \geq 2$, let

$$R_j(Q_0) := (2^{j+1}Q_0 \setminus 2^jQ_0) \cap \Omega.$$

Similar to the proof of [26, (5.34)] (or [83, p. 346, (8)]), we know that

$$\begin{aligned}
 (5.7) \quad & \int_{R_j(Q_0)} |T(\lambda a)(x)| \omega(x) dx \lesssim \frac{|\lambda| l(Q_0)}{[2^j l(Q_0)]^{n+1}} \|a\|_{L^1(Q_0)} \omega(R_j(Q_0)) \\
 & \lesssim 2^{-j(n+1)} |\lambda| \frac{\omega(R_j(Q_0))}{[\omega(Q_0)]^{\frac{1}{q}}} \|a\|_{L^q_{\omega}(Q_0)} \\
 & \lesssim 2^{-j(n+1)} |\lambda| \frac{\omega(R_j(Q_0))}{\omega(Q_0) \rho(\omega(Q_0))}.
 \end{aligned}$$

By the assumption that $nq_{\omega} < (n+1)p_{\Phi}^-$, we know that there exist $q_0 \in (q_{\omega}, \infty)$ and $p_0 \in (0, p_{\Phi}^-)$ such that Φ is of lower type p_0 and $nq_0 < (n+1)p_0$. Moreover, by the definition of q_{ω} , we see that $\omega \in A_{q_0}(\mathbb{R}^n)$. Then, from Jensen's inequality, Hölder's inequality, (iii) and (v) of Lemma 2.2, the lower type p_0 property of Φ , (5.7) and $nq_0 < (n+1)p_0$, it follows that

$$\begin{aligned}
 (5.8) \quad & I_2 \leq \sum_{j=2}^{\infty} \omega(R_j(Q_0)) \Phi \left(\frac{1}{\omega(R_j(Q_0))} \int_{R_j(Q_0)} |T(\lambda a)(x)| \omega(x) dx \right) \\
 & \lesssim \sum_{j=2}^{\infty} \omega(R_j(Q_0)) \Phi \left(2^{-k(n+1)} \frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right) \\
 & \lesssim \sum_{j=2}^{\infty} 2^{-j[(n+1)p_0 - nq_0]} \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right)
 \end{aligned}$$

$$\lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right).$$

Thus, by (5.5), (5.6) and (5.8), we conclude that (5.3) holds in this case.

Case iii) $l(Q_0) < 1$ and $4Q_0 \subset \Omega$. In this case, we have

$$(5.9) \quad \int_{\Omega} \Phi(T(\lambda a)(x)) \omega(x) dx \\ = \int_{2Q_0} \Phi(T(\lambda a)(x)) \omega(x) dx + \sum_{j=1}^{\infty} \int_{(2^{j+1}Q_0 \setminus 2^j Q_0) \cap \Omega} \cdots =: J_0 + \sum_{j=1}^{\infty} J_j.$$

For $j \in \{0, 1, 2\}$, similar to the estimate of (5.6), we know that

$$(5.10) \quad J_j \lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right).$$

Now we deal with J_j for $j \in \mathbb{N}$ with $j \geq 3$. Pick $\varphi \in C_c^\infty(\mathbb{R}_+)$ satisfying $\varphi(t) \equiv 0$ if $t \leq \frac{1}{2}$ or $t \geq 4$, and $\varphi(t) \equiv 1$ if $1 \leq t \leq 2$. Furthermore, for each $j \geq 3$, let

$$\varphi_j(x) := \varphi(|x - x_{Q_0}| / (2^j l(Q_0)))$$

for all $x \in \mathbb{R}^n$, where x_{Q_0} denotes the center of Q_0 . Similar to the proof of [26, p. 49, (5.29)], we have

$$(5.11) \quad \int_{R_j(Q_0)} |T(\lambda a)(x)| \omega(x) dx \lesssim \int_{R_j(Q_0)} |\mathbb{G}_D(\lambda a)(x) \Delta \varphi_j(x)| \omega(x) dx \\ + \int_{R_j(Q_0)} |\nabla_x \mathbb{G}_D(\lambda a)(x) \cdot \nabla \varphi_j(x)| \omega(x) dx.$$

In this case, we know that a is a type local (a) $(\rho, q, 0)_\omega$ -atom, and hence

$$\int_{\Omega} a(x) dx = 0.$$

From this, Lemma 5.2(iv) and Hölder's inequality, it follows that for all $x \in R_j(Q_0)$,

$$\begin{aligned} |\mathbb{G}_D(\lambda a)(x)| &= \left| \lambda \int_{Q_0} \mathbb{G}_D(x, y) a(y) dy \right| \\ &= \left| \lambda \int_{Q_0} [\mathbb{G}_D(x, y) - \mathbb{G}_D(x, x_{Q_0})] a(y) dy \right| \\ &\leq |\lambda| \int_{Q_0} |\nabla_y \mathbb{G}_D(x, y_1)| |y - x_{Q_0}| |a(y)| dy \\ &\lesssim |\lambda| \int_{Q_0} \frac{|y - x_{Q_0}|}{|x - y_1|^{n-1}} |a(y)| dy \lesssim \frac{|\lambda| l(Q_0)}{|x - x_{Q_0}|^{n-1}} \|a\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \frac{|\lambda| l(Q_0)}{|x - x_{Q_0}|^{n-1}} \frac{|Q_0|}{[\omega(Q_0)]^{\frac{1}{q}}} \|a\|_{L^q_\omega(Q_0)}, \end{aligned}$$

where $y_1 \in Q_0$ depends on y and x_{Q_0} , and in the penultimate inequality we used the fact that $|x - y_1| \sim |x - x_{Q_0}|$, which, together with the fact that for all $x \in \mathbb{R}^n$, $|\Delta \varphi_j(x)| \lesssim [2^j l(Q_0)]^{-2}$, implies that

$$(5.12) \quad \int_{\Omega} |\mathbb{G}_D(\lambda a)(x) \Delta \varphi_j(x)| \omega(x) dx \lesssim 2^{-j(n+1)} \frac{|\lambda| \omega(R_j(Q_0))}{\omega(Q_0) \rho(\omega(Q_0))}.$$

Moreover, by the fact that $\int_{Q_0} a(x) dx = 0$, Lemma 5.2(v) and Hölder's inequality, we know that for all $x \in R_j(Q_0)$,

$$\begin{aligned} |\nabla_x \mathbb{G}_D(\lambda a)(x)| &= \left| \lambda \int_{Q_0} \nabla_x G_D(x, y) a(y) dy \right| \\ &= \left| \lambda \int_{Q_0} [\nabla_x G_D(x, y) - \nabla_x G_D(x, x_{Q_0})] a(y) dy \right| \\ &\leq |\lambda| \int_{Q_0} |y - x_{Q_0}| \sup_{y_1 \in Q_0} |\nabla_x \nabla_y G_D(x, y_1)| |a(y)| dy \\ &\lesssim \frac{|\lambda| l(Q_0)}{|x - x_{Q_0}|^n} \|a\|_{L^1(Q_0)} \lesssim \frac{|\lambda| [l(Q_0)]^{n+1} \|a\|_{L^\omega_\omega(Q_0)}}{|x - x_{Q_0}|^n [\omega(Q_0)]^{\frac{1}{q}}}, \end{aligned}$$

which, together with the fact that for all $x \in \mathbb{R}^n$, $|\nabla \varphi_j(x)| \lesssim [2^j l(Q_0)]^{-1}$, implies that

$$(5.13) \quad \int_{\Omega} |\nabla \mathbb{G}_D(\lambda a)(x) \cdot \nabla \varphi_j(x)| \omega(x) dx \lesssim 2^{-j(n+1)} \frac{|\lambda| \omega(R_j(Q_0))}{\omega(Q_0) \rho(\omega(Q_0))}.$$

Thus, by (5.11), (5.12) and (5.13), we conclude that for each $j \in \mathbb{N}$ with $j \geq 3$,

$$\int_{R_j(Q_0)} |T(\lambda a)(x)| \omega(x) dx \lesssim 2^{-j(n+1)} \frac{|\lambda| \omega(R_j(Q_0))}{\omega(Q_0) \rho(\omega(Q_0))},$$

which, together with Jensen's inequality, (iii) and (v) of Lemma 2.2, the lower type p_0 property of Φ and $nq_0 < (n+1)p_0$, implies that

$$\begin{aligned} \sum_{j=3}^{\infty} J_j &\leq \sum_{j=3}^{\infty} \omega(R_j(Q_0)) \Phi \left(\frac{1}{\omega(R_j(Q_0))} \int_{R_j(Q_0)} |T(\lambda a)(x)| \omega(x) dx \right) \\ &\lesssim \sum_{j=3}^{\infty} \omega(R_j(Q_0)) \Phi \left(2^{-j(n+1)} \frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right) \\ &\lesssim \sum_{j=3}^{\infty} 2^{-j[(n+1)p_0 - nq_0]} \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right) \\ &\lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right). \end{aligned}$$

This, combined with (5.9) and (5.10), implies that (5.3) holds in this case and hence completes the proof of Theorem 1.8(i).

Now we prove Theorem 1.8(ii). We borrow some ideas from [26] and [83]. First let

$$f \in h_{\omega, r}^{\Phi}(\Omega) \cap L^2(\Omega).$$

By the assumption that $r_\omega > \frac{2}{2-q_\omega}$ and the definition of q_ω , we know that there exists $q_2 \in (q_\omega, 2)$ such that

$$\frac{2}{2-q_2} < r_\omega.$$

Then $(\frac{2}{q_2})' < r_\omega$ and hence $\omega \in A_{(2/q_2)'}(\mathbb{R}^n)$. Take $q_3 \in (2q_\omega, \infty)$. Then $\omega \in A_{q_3/2}(\mathbb{R}^n)$. Then by Theorem 1.7, we know that there exist a sequence $\{a_{1,k}\}_k$ of

type (a) local $(\rho, q_2, 0)_\omega$ -atoms, a sequence $\{a_{2,m}\}_m$ of type (b) local $(\rho, q_3, 0)_\omega$ -atoms and $(\{\lambda_{1,k}\}_k \cup \{\lambda_{2,m}\}_m) \subset \mathbb{C}$ such that (5.1) and (5.2) hold. Moreover, by Lemma 5.3, we see that

$$T(f) = \sum_{\text{type (a) atoms}} \lambda_{1,k} T(a_{1,k}) + \sum_{\text{type (b) atoms}} \lambda_{2,m} T(a_{2,m}) \text{ in } L^2(\Omega).$$

To finish the proof of Theorem 1.8(ii), we only need to show that for all type (a) local $(\rho, q_2, 0)_\omega$ -atoms or type (b) local $(\rho, q_2, 0)_\omega$ -atoms a supported in the cube Q_0 and all $\lambda \in \mathbb{C} \setminus \{0\}$, there exist a sequence $\{b_s\}_s$ of type (a) local $(\rho, q_1, 0)$ -atoms and type (b) local $(\rho, q_1, 0)$ -atoms, and $\{\mu_s\}_s \subset \mathbb{C}$ such that

$$(5.14) \quad T(\lambda a) = \sum_s \mu_s b_s$$

and for all $\mu \in (0, \infty)$,

$$(5.15) \quad \sum_s \omega(Q_s) \Phi \left(\frac{|\mu_s|}{\mu \omega(Q_s) \rho(\omega(Q_s))} \right) \lesssim \omega(Q_0) \Phi \left(\frac{|\lambda|}{\mu \omega(Q_0) \rho(\omega(Q_0))} \right),$$

where for each s , $\text{supp}(b_s) \subset Q_s$.

Indeed, if (5.14) and (5.15) hold, then for each k and m , there exist sequences $\{b_{1,k,s}\}_{k,s}$ and $\{b_{2,m,s}\}_{m,s}$ of type (a) local $(\rho, q_1, 0)$ -atoms and type (b) local $(\rho, q_1, 0)$ -atoms, and $\{\mu_{1,k,s}\}_{k,s} \cup \{\mu_{2,m,s}\}_{m,s} \subset \mathbb{C}$ such that

$$T(f) = \sum_{k,s} \lambda_{1,k} \mu_{1,k,s} b_{1,k,s} + \sum_{m,s} \lambda_{2,m} \mu_{2,m,s} b_{2,m,s},$$

and for all $\lambda \in (0, \infty)$,

$$\begin{aligned} & \sum_{k,s} \omega(Q_{1,k,s}) \Phi \left(\frac{|\lambda_{1,k} \mu_{1,k,s}|}{\lambda \omega(Q_{1,k,s}) \rho(\omega(Q_{1,k,s}))} \right) \\ & + \sum_{m,s} \omega(Q_{2,m,s}) \Phi \left(\frac{|\lambda_{2,m} \mu_{2,m,s}|}{\lambda \omega(Q_{2,m,s}) \rho(\omega(Q_{2,m,s}))} \right) \\ & \lesssim \sum_k \omega(Q_{1,k}) \Phi \left(\frac{|\lambda_{1,k}|}{\lambda \omega(Q_{1,k}) \rho(\omega(Q_{1,k}))} \right) \\ & + \sum_m \omega(Q_{2,m}) \Phi \left(\frac{|\lambda_{2,m}|}{\lambda \omega(Q_{2,m}) \rho(\omega(Q_{2,m}))} \right), \end{aligned}$$

where for each k, m, s , $\text{supp}(a_{1,k}) \subset Q_{1,k}$, $\text{supp}(a_{2,m}) \subset Q_{2,m}$, $\text{supp}(b_{1,k,s}) \subset Q_{1,k,s}$ and $\text{supp}(b_{2,m,s}) \subset Q_{2,m,s}$, which, together with Theorem 1.7 and (5.2), implies that $T(f) \in h_{\omega,r}^\Phi(\Omega)$ and

$$\|T(f)\|_{h_{\omega,r}^\Phi(\Omega)} \lesssim \|f\|_{h_{\omega,r}^\Phi(\Omega)}.$$

From this, the fact that $h_{\omega,r}^\Phi(\Omega) \cap L^2(\Omega)$ is dense in $h_{\omega,r}^\Phi(\Omega)$ and a density argument, we deduce Theorem 1.8(ii).

Now we prove (5.14) and (5.15) by considering the following two cases for a .

Case i) a is a type (a) local $(\rho, q_3, 0)_\omega$ -atom. Recall that the standard fundamental solution of the Laplace operator

$$\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

on $\mathbb{R}^n$ ($n \geq 2$) is given by

$$(5.16) \quad \Gamma(x) := \begin{cases} \frac{1}{2\pi} \ln |x|, & \text{when } n = 2, \\ \frac{c_n}{|x|^{n-2}}, & \text{when } n \geq 3, \end{cases}$$

where $c_n := [(n-2)\omega_n]^{-1}$ and ω_n denotes the *area of the unit sphere in $\mathbb{R}^n$* . This allows us to solve the Poisson problem for the Laplacian in the whole space via integral operators. Indeed, as is well known, the Newtonian potential

$$E(f)(x) := \int_{\Omega} \Gamma(x-y)f(y) dy, \quad x \in \Omega,$$

satisfies $\Delta E(f) = f$ in Ω when $f \in C^\infty(\Omega)$.

For each $y \in \Omega$, let $U(\cdot, y)$ be the solution of the Dirichlet problem

$$(5.17) \quad \begin{cases} \Delta U(\cdot, y) = 0, & \text{in } \Omega, \\ U(x, y) = \Gamma(x-y), & \text{for } x \in \partial\Omega. \end{cases}$$

Then the Green function G_D for Δ_D on Ω (which is the integral kernel of the Dirichlet Green potential $\mathbb{G}_D$) can be expressed as

$$G_D(x, y) = \Gamma(x-y) - U(x, y), \quad x, y \in \Omega, \quad x \neq y.$$

As a consequence, the solution of the inhomogeneous Dirichlet problem (1.1) is given by

$$(5.18) \quad \begin{aligned} \mathbb{G}_D(f)(x) &= \int_{\Omega} G_D(x, y)f(y) dy \\ &= \int_{\Omega} \Gamma(x-y)f(y) dy - \int_{\Omega} U(x, y)f(y) dy \\ &=: E(f)(x) - U(f)(x), \end{aligned}$$

where $f \in C^\infty(\Omega)$ and $x \in \Omega$.

By abuse of notation, we denote by a the *zero extension* of a out of Ω . Then similar to the proof of [86, Theorem 8.2], we know that

$$\frac{\partial^2 E(\lambda a)}{\partial x_i \partial x_j} \in h_{\omega}^{\Phi}(\mathbb{R}^n) \quad \text{and} \quad \left\| \frac{\partial^2 E(\lambda a)}{\partial x_i \partial x_j} \right\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|\lambda a\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)},$$

which, together with the definition of $h_{\omega, r}^{\Phi}(\Omega)$, implies that

$$(5.19) \quad \left\| \frac{\partial^2 E(\lambda a)}{\partial x_i \partial x_j} \right\|_{h_{\omega, r}^{\Phi}(\Omega)} \leq \left\| \frac{\partial^2 E(\lambda a)}{\partial x_i \partial x_j} \right\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)} \lesssim \|\lambda a\|_{h_{\omega}^{\Phi}(\mathbb{R}^n)}.$$

For $i, j \in \{1, \dots, n\}$, let

$$H_{i,j}(\lambda a) := \frac{\partial^2 U(\lambda a)}{\partial x_i \partial x_j}.$$

From (5.17) and (5.18), we infer that $H_{i,j}(\lambda a)$ is a harmonic function in Ω . Thus, an application of Theorem 1.7, in which we take the function φ to be radial, yields, on account of the mean value property for harmonic functions, that for all $x \in \Omega$,

$$(5.20) \quad \begin{aligned} (H_{i,j}(\lambda a))_{\Omega, \varphi}^+(x) &= \sup_{0 < t < \delta(x)/c_0} \left| \int_{\Omega} H_{i,j}(\lambda a)(y) \varphi_t(x-y) dy \right| \\ &= |H_{i,j}(\lambda a)(x)|. \end{aligned}$$

Similar to the proof of Theorem 1.8(i), we know that

$$\left\| \frac{\partial^2 E(\lambda a)}{\partial x_i \partial x_j} \right\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim \|\lambda a\|_{h^\Phi_\omega(\mathbb{R}^n)}.$$

By this, (5.20), Theorem 1.7 and Theorem 1.8(i), we conclude that

$$\begin{aligned} \|H_{i,j}(\lambda a)\|_{h^\Phi_{\omega,r}(\Omega)} &= \left\| (H_{i,j}(\lambda a))_{\Omega,\varphi}^+ \right\|_{L^\Phi_\omega(\Omega)} = \|H_{i,j}(\lambda a)\|_{L^\Phi_\omega(\Omega)} \\ &\lesssim \left\| \frac{\partial^2 E(\lambda a)}{\partial x_i \partial x_j} \right\|_{L^\Phi_\omega(\Omega)} + \left\| \frac{\partial^2 \mathbb{G}_D(\lambda a)}{\partial x_i \partial x_j} \right\|_{L^\Phi_\omega(\Omega)} \lesssim \|\lambda a\|_{h^\Phi_\omega(\mathbb{R}^n)}, \end{aligned}$$

which, together with (5.19), implies that $T(\lambda a) \in h^\Phi_{\omega,r}(\Omega)$ and

$$\|T(\lambda a)\|_{h^\Phi_{\omega,r}(\Omega)} \lesssim \|\lambda a\|_{h^\Phi_\omega(\mathbb{R}^n)}.$$

From this and Theorem 1.7, we infer that (5.14) and (5.15) hold in this case.

Case ii) a is a type (b) local $(\rho, q_3, 0)_\omega$ -atom. Let

$$R_k(Q_0) := (2^{k+1}Q_0 \setminus 2^kQ_0) \cap \Omega$$

when $k \geq 3$, $R_0(Q_0) := 8Q_0 \cap \Omega$ and

$$J_\Omega := \{k \in \mathbb{N} : k \geq 3, |R_k(Q_0)| > 0\}.$$

For all $k \in \mathbb{N} \cup \{0\}$, let $\chi_k := \chi_{R_k(Q_0)}$, $\tilde{\chi}_k := \frac{\chi_k}{|R_k(Q_0)|}$ and

$$m_k := \int_{R_k(Q_0)} T(\lambda a) dx.$$

Then, we have

$$T(\lambda a) = T(\lambda a)\chi_0 + \sum_{k \in J_\Omega} T(\lambda a)\chi_k.$$

Let $\tilde{Q}_0 \subset \mathbb{R}^n$ satisfy $x_{\tilde{Q}_0} \in \Omega^\complement$, $l(\tilde{Q}_0) = l(Q_0)$ and $\text{dist}(Q_0, \tilde{Q}_0) \sim l(Q_0)$, which implies that there exists $Q_0^* \subset \mathbb{R}^n$ such that $(8Q_0 \cup \tilde{Q}_0) \subset Q_0^*$ and $l(Q_0^*) \sim l(Q_0)$. Moreover, let

$$H_0 := T(\lambda a)\chi_0 - \frac{1}{|\tilde{Q}_0 \cap \Omega|} \left\{ \int_{R_0(Q_0)} T(\lambda a)(y) dy \right\} \chi_{\tilde{Q}_0 \cap \Omega^\complement}.$$

We have

$$\int_{\mathbb{R}^n} H_0(x) dx = 0 \quad \text{and} \quad \text{supp}(H_0) \subset Q_0^*.$$

From $\omega \in RH_{(2/q_2)'}(\mathbb{R}^n) \cap A_{q_3/2}(\mathbb{R}^n)$, Hölder's inequality and Lemma 5.3, it follows that

$$\begin{aligned} \|H_0\|_{L^{q_2}_\omega(\mathbb{R}^n)} &\lesssim \|T(\lambda a)\|_{L^{q_2}_\omega(R_0(Q_0))} \\ &\leq \left\{ \int_{R_0(Q_0)} |T(\lambda a)(x)|^2 dx \right\}^{\frac{1}{2}} \left\{ \left(\int_{R_0(Q_0)} [\omega(x)]^{(\frac{2}{q_2})'} dx \right)^{\frac{1}{(\frac{2}{q_2})'}} \right\}^{\frac{1}{q_2}} \\ &\leq \|\lambda a\|_{L^2(\Omega)} \left\{ \frac{\omega(R_0(Q_0))}{|Q_0|^{q_2/2}} \right\}^{\frac{1}{q_2}} \leq \|\lambda a\|_{L^{q_3}_\omega(\Omega)} \frac{|Q_0|^{\frac{1}{2}}}{[\omega(Q_0)]^{\frac{1}{q_3}}} \frac{[\omega(R_0(Q_0))]^{\frac{1}{q_2}}}{|Q_0|^{\frac{1}{2}}} \\ &\lesssim |\lambda|[\omega(Q_0)]^{\frac{1}{q_2}-1} [\rho(\omega(Q_0))]^{-1} \sim |\lambda|[\omega(Q_0^*)]^{\frac{1}{q_2}-1} [\rho(\omega(Q_0^*))]^{-1}. \end{aligned}$$

Thus, there exists a positive constant $\tilde{C}$ such that $\frac{H_0}{C|\lambda|}$ is a $(\rho, q_1, 0)_\omega$ -atom.

If $k \in J_\Omega$, by Fact 1 in the proof of Proposition 3.13, we know that there exist $\{Q_{k,j}\}_j \subset \mathbb{R}^n$ such that for all j , $2Q_{k,j} \subset \Omega$, $4Q_{k,j} \cap \partial\Omega \neq \emptyset$ and

$$R_k(Q_0) = \bigcup_j Q_{k,j}.$$

From Lemma 3.15, it follows that for each k and j , there exists $\tilde{Q}_{k,j} \subset \Omega^\complement$ such that $l(\tilde{Q}_{k,j}) = l(Q_{k,j})$ and $\text{dist}(\tilde{Q}_{k,j}, Q_{k,j}) \sim l(Q_{k,j})$. For each k and j , take cube $Q_{k,j}^* \subset \mathbb{R}^n$ such that

$$Q_{k,j} \cup \tilde{Q}_{k,j} \subset Q_{k,j}^* \text{ and } l(Q_{k,j}^*) \sim l(Q_{k,j}).$$

By this, Hölder's inequality, Lemma 5.3, (iii) and (v) of Lemma 2.2, we see that

$$\begin{aligned} \|H_{k,i}\|_{L_\omega^{q_2}(\mathbb{R}^n)} &\lesssim \|T(\lambda a)\|_{L_\omega^{q_2}(Q_{k,j})} \\ &\quad + \frac{[\omega(\tilde{Q}_{k,j})]^{1/q_2}}{|\tilde{Q}_{k,j}|} \|T(\lambda a)\|_{L_\omega^{q_2}(Q_{k,j})} \left(\int_{\tilde{Q}_{k,j}} [\omega(x)]^{-\frac{q_2'}{q_2}} dx \right)^{1/q_2'} \\ &\leq \tilde{C} \|T(\lambda a)\|_{L_\omega^{q_2}(Q_{k,j})}, \end{aligned}$$

where $\tilde{C}$ is a positive constant. Let

$$\lambda_{k,i} := \tilde{C} \|T(\lambda a)\|_{L_\omega^{q_2}(Q_{k,j})} [\omega(Q_{k,j}^*)]^{1-1/q_2} \rho(\omega(Q_{k,j}^*))$$

and $b_{k,i} := H_{k,i}/\lambda_{k,i}$. We then have $H_{k,i} = \lambda_{k,i} b_{k,i}$ and $b_{k,i}$ is a $(\rho, q_2, 0)$ -atom. Moreover, by the assumption that $nq_\omega < (n+1)p_\Phi^-$ and the definitions of q_ω and p_Φ^- , we know that there exist $\tilde{q} \in (q_\omega, \infty)$ and $p_0 \in (0, p_\Phi^-)$ such that Φ is of lower type p_0 and $n\tilde{q} < (n+1)p_0$. Similar to the proof of (5.7), we know that

$$\int_{R_k(Q_0)} |T(\lambda a)(x)|^{q_2} \omega(x) dx \lesssim 2^{-k(n+1)q_2} \frac{\|\lambda a\|_{L_\omega^{q_2}(Q_0)}^{q_2} \omega(R_k(Q_0))}{\omega(Q_0)}.$$

This, combined with Hölder's inequality, the lower type p_0 and the upper type 1 properties of Φ , Lemma 2.2(iii) and $n\tilde{q} < (n+1)p_0$, implies that

$$\begin{aligned} &\sum_k \sum_i \omega(Q_{k,i}^*) \Phi \left(\frac{|\lambda| \|T(a)\|_{L_\omega^{q_2}(Q_{k,i})} [\omega(Q_{k,i}^*)]^{1-1/q_2} \rho(\omega(Q_{k,i}^*))}{[\omega(Q_{k,i}^*)] \rho(\omega(Q_{k,i}^*))} \right) \\ &\lesssim \sum_k \sum_i \omega(Q_{k,i}^*) \frac{[\omega(R_k(Q_0))]^{1/q_2}}{[\omega(Q_{k,i})]^{1/q_2}} \left[\frac{\|T(a)\|_{L_\omega^{q_2}(Q_{k,i})}}{\|T(a)\|_{L_\omega^{q_2}(R_k(Q_0))}} \right]^{p_0} \\ &\quad \times \Phi \left(\frac{|\lambda| \|T(a)\|_{L_\omega^{q_2}(R_k(Q_0))}}{[\omega(R_k(Q_0))]^{1/q_2}} \right) \\ &\lesssim \sum_k \left[\sum_i \omega(Q_{k,i}^*) \right]^{1/q_2'} \frac{[\omega(R_k(Q_0))]^{1/q_2}}{\|T(a)\|_{L_\omega^{q_2}(R_k(Q_0))}^{p_0}} \left[\sum_i \int_{Q_{k,i}} |T(a)(x)|^{q_2} \omega(x) dx \right]^{p_0/q_2} \\ &\quad \times \Phi \left(\frac{2^{-k(n+1)} |\lambda|}{\omega(Q_0) \rho(\omega(Q_0))} \right) \\ &\lesssim \sum_k \left[\int_{R_k(Q_0)} \omega(x) dx \right]^{1/q_2'} \frac{[\omega(R_k(Q_0))]^{1/q_2}}{\|T(a)\|_{L_\omega^{q_2}(R_k(Q_0))}^{p_0}} \end{aligned}$$

$$\begin{aligned}
& \times \|T(a)\|_{L_{\omega}^{q_2(R_k(Q_0))}}^{p_0} \Phi\left(\frac{2^{-k(n+1)}|\lambda|}{\omega(Q_0)\rho(\omega(Q_0))}\right) \\
& \lesssim \sum_k \omega(R_k(Q_0)) \Phi\left(\frac{2^{-k(n+1)}|\lambda|}{\omega(Q_0)\rho(\omega(Q_0))}\right) \\
& \lesssim \sum_k 2^{-k[(n+1)p_0-n\bar{q}]} \omega(Q_0) \Phi\left(\frac{|\lambda|}{\omega(Q_0)\rho(\omega(Q_0))}\right) \\
& \lesssim \omega(Q_0) \Phi\left(\frac{|\lambda|}{\omega(Q_0)\rho(\omega(Q_0))}\right).
\end{aligned}$$

Therefore, we obtain $T(\lambda a) \in h_{\omega, r}^{\Phi}(\Omega)$ and

$$\begin{aligned}
\|T(\lambda a)\|_{h_{\omega, r}^{\Phi}(\Omega)} & \lesssim \inf \left\{ s \in (0, \infty) : \omega(Q_0^*) \Phi\left(\frac{|\lambda|}{s\omega(Q_0^*)\rho(\omega(Q_0^*))}\right) \right. \\
& \quad \left. + \sum_{k,i} \omega(Q_{k,i}^*) \Phi\left(\frac{\lambda_{k,i}}{s\omega(Q_{k,i}^*)\rho(\omega(Q_{k,i}^*))}\right) \leq 1 \right\} \\
& \lesssim \inf \left\{ s \in (0, \infty) : \omega(Q_0) \Phi\left(\frac{|\lambda|}{s\omega(Q_0)\rho(\omega(Q_0))}\right) \leq 1 \right\}.
\end{aligned}$$

From this and Theorem 1.7, we deduce that (5.14) and (5.15) hold in this case. This finishes the proof of Theorem 1.8(ii) and hence Theorem 1.8. $\square$

6. PROOF OF THEOREM 1.9

In this section, we give the proof of Theorem 1.9. In what follows, we always assume that Ω is a bounded convex domain in $\mathbb{R}^n$. We first establish some auxiliary lemmas.

Let $L := -\Delta$ with the Neumann boundary condition on Ω . Denote by $\mathcal{D}(L)$ the domain of the operator L , and by L^k the k -fold composition of L with itself, in the sense of unbounded operators. In what follows, to simplify the notation, we just use B^Ω for $B(x_B, r_B) \cap \Omega$. For given $\lambda \in (0, \infty)$, we denote by λB^Ω the set $B(x_B, \lambda r_B) \cap \Omega$. Let

$$U_0^\Omega(B) := B^\Omega \quad \text{and} \quad U_j^\Omega(B) := (2^j B^\Omega \setminus 2^{j-1} B^\Omega) \text{ for } j \in \mathbb{N}.$$

Definition 6.1. Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$ and Ω be an open subset of $\mathbb{R}^n$. Let p_Φ^- , ρ and q_ω be respectively as in (2.12), (2.14) and (2.5). Let

$$(6.1) \quad M \in \mathbb{N} \text{ and } M > \left\lfloor \frac{n}{2} \left(\frac{q_\omega}{p_\Phi^-} - 1 \right) \right\rfloor.$$

A measurable function a on Ω is called a $(\rho, 2, M)_\omega$ -atom if there exists a ball B of $\mathbb{R}^n$ centered in Ω (but not necessarily included in Ω) with radius $r_B \leq 2 \operatorname{diam}(\Omega)$ such that

$$\|a\|_{L^2(\Omega)} \leq |B^\Omega|^{\frac{1}{2}} [\omega(B^\Omega) \rho(\omega(B^\Omega))]^{-1}$$

and either

- (i) $r_B > 1$, or

(ii) $r_B \leq 1$ and there exists a function $b \in \mathcal{D}(L^M)$ such that $a = L^M b$, $\text{supp}(L^k b) \subset B \cap \overline{\Omega}$ for all $k \in \{0, 1, \dots, M\}$, and

$$\left\| (r_B^2 L)^k b \right\|_{L^2(\Omega)} \leq r_B^{2M} |B^\Omega|^{\frac{1}{2}} [\omega(B^\Omega) \rho(\omega(B^\Omega))]^{-1}$$

for all $k \in \{0, 1, \dots, M\}$.

Definition 6.2. Let Φ satisfy Assumption (A), $\omega \in A_\infty(\mathbb{R}^n)$, Ω be a bounded, simply connected convex domain of $\mathbb{R}^n$ and $L = -\Delta$ with the Neumann boundary condition. Let ρ and M be respectively as in (2.14) and (6.1). A function $f \in L^2(\Omega)$ is said to be in $\tilde{h}_{L,at,M,\omega}^\Phi(\Omega)$ if there exist a sequence $\{a_i\}_i$ of $(\rho, 2, M)_\omega$ -atoms and $\{\lambda_i\}_i \subset \mathbb{C}$ such that $f = \sum_i \lambda_i a_i$ in $L^2(\Omega)$ and

$$\sum_i \omega(B_i^\Omega) \Phi\left(\frac{|\lambda_i|}{\omega(B_i^\Omega) \rho(\omega(B_i^\Omega))}\right) < \infty,$$

where for each i , $\text{supp } a_i \subset B_i^\Omega \cap \overline{\Omega}$. Moreover, letting

$$\Lambda(\{\lambda_i a_i\}_i) := \inf \left\{ \lambda \in (0, \infty) : \sum_i \omega(B_i^\Omega) \Phi\left(\frac{|\lambda_i|}{\lambda \omega(B_i^\Omega) \rho(\omega(B_i^\Omega))}\right) \leq 1 \right\},$$

the *quasi-norm* of $f \in \tilde{h}_{L,at,M,\omega}^\Phi(\Omega)$ is defined by

$$\|f\|_{h_{L,at,M,\omega}^\Phi(\Omega)} := \inf \{ \Lambda(\{\lambda_i a_i\}_i) \},$$

where the infimum is taken over all the decompositions of f as above. The *weighted local atomic Orlicz-Hardy space* $h_{L,at,M,\omega}^\Phi(\Omega)$ is defined to be the completion of $\tilde{h}_{L,at,M,\omega}^\Phi(\Omega)$ in the quasi-norm $\|\cdot\|_{h_{L,at,M,\omega}^\Phi(\Omega)}$.

Lemma 6.3. Let Φ , L , Ω and M be as in Definition 6.2 and ω as in Theorem 1.4. Then the spaces $h_{\omega,z}^\Phi(\Omega)$ and $h_{L,at,M,\omega}^\Phi(\Omega)$ coincide with equivalent quasi-norms.

Proof. Similar to the proof of [26, Theorem 3.5], we have

$$h_{\mathcal{N}_h,\omega}^\Phi(\Omega) \cap L^2(\Omega) = h_{L,at,M,\omega}^\Phi(\Omega) \cap L^2(\Omega)$$

with equivalent quasi-norms. From this, the following two facts that

$$h_{\mathcal{N}_h,\omega}^\Phi(\Omega) \cap L^2(\Omega) \text{ and } h_{L,at,M,\omega}^\Phi(\Omega) \cap L^2(\Omega)$$

are, respectively, dense in $h_{\mathcal{N}_h,\omega}^\Phi(\Omega)$ and $h_{L,at,M,\omega}^\Phi(\Omega)$, and a density argument, we deduce that the spaces $h_{\mathcal{N}_h,\omega}^\Phi(\Omega)$ and $h_{L,at,M,\omega}^\Phi(\Omega)$ coincide with equivalent quasi-norms. By the assumption that Ω is a bounded convex domain of $\mathbb{R}^n$ and [26, Lemma 2.8], we know that (G_1) holds with $\mu = 1$ for L . From this and Theorem 1.4(iii), we infer that the spaces $h_\omega^\Phi(\Omega)$ and $h_{L,at,M,\omega}^\Phi(\Omega)$ coincide with equivalent quasi-norms. This finishes the proof of Lemma 6.3. $\square$

To show Theorem 1.9, we need the following useful estimates.

Lemma 6.4. Let Ω and L be as in Definition 6.2. Denote by $\{K_t\}_{t \geq 0}$ the kernels of the semigroup $\{e^{-tL}\}_{t \geq 0}$. Let $q \in [1, 2)$. Then there exist positive constants γ and C such that for all $y \in \Omega$ and $s, t \in (0, \infty)$,

$$(6.2) \quad \left[\int_{\{x \in \Omega: |x-y| \geq \sqrt{s}\}} |\nabla_x^2 K_t(x, y)|^q dx \right]^{\frac{1}{q}} \leq C t^{-1} |B^\Omega(y, \sqrt{t})|^{\frac{1}{q}-1} e^{-\gamma \frac{s}{t}}.$$

Furthermore, for each $k \in \mathbb{N}$, there exist positive constants γ_k and $C(k)$, depending on k , such that the k -th order time derivative $\frac{d^k}{dt^k} K_t$ of the kernel K_t satisfies that for all $y \in \Omega$ and $s, t \in (0, \infty)$,

$$(6.3) \quad \left[\int_{\{x \in \Omega: |x-y| \geq \sqrt{s}\}} \left| \nabla_x^2 \left(\frac{d^k}{dt^k} K_t(x, y) \right) \right|^q dx \right]^{\frac{1}{q}} \leq C(k) t^{-(k+1)} |B^\Omega(y, \sqrt{t})|^{\frac{1}{q}-1} e^{-\gamma_k \frac{s}{t}}.$$

Proof. We first prove (6.2). It was shown in [26, Proposition 4.15] that for some $\gamma_1 \in (0, \infty)$, there exists a positive constant $C(\gamma_1)$ such that for all $y \in \Omega$ and $t \in (0, \infty)$,

$$(6.4) \quad \int_{\Omega} |\nabla_x^2 K_t(x, y)|^2 e^{\frac{\gamma_1 |x-y|^2}{t}} dx \leq C(\gamma_1) t^{-2} |B(y, \sqrt{t})|^{-1}.$$

Moreover, it was obtained in [26, Lemma 4.13] that for any $\gamma_2 \in (0, \infty)$, there exists a positive constant $C(\gamma_2)$ such that for all $s \in [0, \infty)$, $t \in (0, \infty)$ and $y \in \Omega$,

$$\int_{\{x \in \Omega: |x-y| \geq \sqrt{s}\}} e^{-2\gamma_2 \frac{|x-y|^2}{t}} dx \leq C(\gamma_2) |B^\Omega(y, \sqrt{t})| e^{-\gamma_2 \frac{s}{t}}.$$

This, together with Hölder's inequality and (6.4), implies that

$$\begin{aligned} & \left[\int_{\{x \in \Omega: |x-y| \geq \sqrt{s}\}} |\nabla_x^2 K_t(x, y)|^q dx \right]^{\frac{1}{q}} \\ & \leq \left[\int_{\Omega} |\nabla_x^2 K_t(x, y)|^2 e^{\frac{\gamma_1 |x-y|^2}{t}} dx \right]^{\frac{1}{2}} \left[\int_{\{x \in \Omega: |x-y| \geq \sqrt{s}\}} e^{-\frac{\gamma_1 q}{2-q} \frac{|x-y|^2}{t}} dx \right]^{\frac{1}{q}-\frac{1}{2}} \\ & \lesssim t^{-1} |B^\Omega(y, \sqrt{t})|^{-\frac{1}{2}} e^{-\gamma s/t} |B^\Omega(y, \sqrt{t})|^{\frac{1}{q}-\frac{1}{2}} \lesssim t^{-1} e^{-\gamma s/t} |B^\Omega(y, \sqrt{t})|^{\frac{1}{q}-1}, \end{aligned}$$

where $\gamma := \gamma_1/4$, which implies that (6.2) holds.

The proof of (6.3) is similar to that of [26, Lemma 4.12]. We omit the details. This finishes the proof of Lemma 6.4. $\square$

The following Lemma 6.5 is just [26, Theorem 4.2].

Lemma 6.5. *Let Ω be a bounded, simply connected semiconvex domain in $\mathbb{R}^n$ and $\mathbb{G}_N$ the Neumann Green operator for the problem (1.2). Then the operators in (1.4), originally defined on $C^\infty(\overline{\Omega})$, can be extended to bounded operators on $L^p(\Omega)$ for $p \in (1, 2]$.*

Now we prove Theorem 1.9 by using Lemmas 6.3 and 6.4.

Proof of Theorem 1.9. We borrow some ideas from [26]. Fix $m, s \in \{1, \dots, n\}$ and denote by T the operator $\frac{\partial^2 \mathbb{G}_N}{\partial x_m \partial x_s}$. We first prove Theorem 1.9(i). Let $f \in h_{\omega, z}^\Phi(\Omega) \cap L^2(\Omega)$ and M be as in (6.1) satisfying $M > \frac{nq_\omega}{2p_\Phi}$. Then by Lemma 6.3, we know that there exist a sequence $\{a_k\}_k$ of $(\rho, 2, M)_\omega$ -atoms and a sequence $\{\lambda_k\}_k$ of numbers such that

$$(6.5) \quad f = \sum_k \lambda_k a_k$$

and

$$(6.6) \quad \|f\|_{h_{\omega, z}^{\Phi}(\Omega)} \sim \Lambda(\{\lambda_k a_k\}_k),$$

where $\Lambda(\{\lambda_k a_k\}_k)$ is as in Definition 6.2.

To finish the proof of Theorem 1.9(i), we need to show that for any $(\rho, 2, M)_{\omega}$ -atom a supported in the ball $B(x_0, r_0) \cap \overline{\Omega}$ and any $\lambda \in \mathbb{C}$,

$$(6.7) \quad \int_{\Omega} \Phi(T(\lambda a)(x)) \omega(x) dx \lesssim \omega(B_0^{\Omega}) \Phi\left(\frac{|\lambda|}{\omega(B_0^{\Omega}) \rho(\omega(B_0^{\Omega}))}\right),$$

where and in what follows, $B_0^{\Omega} := B(x_0, r_0) \cap \Omega$.

Indeed, if (6.7) holds, then by (6.5) and the assumption that Φ is subadditive, we know that for all $\lambda \in (0, \infty)$,

$$\begin{aligned} \int_{\Omega} \Phi\left(T\left(\frac{f}{\lambda}\right)(x)\right) \omega(x) dx &\leq \sum_k \int_{\Omega} \Phi\left(T\left(\frac{\lambda_k a_k}{\lambda}\right)(x)\right) \omega(x) dx \\ &\lesssim \sum_k \omega(B_k^{\Omega}) \Phi\left(\frac{|\lambda_k|}{\lambda \omega(B_k^{\Omega}) \rho(\omega(B_k^{\Omega}))}\right), \end{aligned}$$

where for each k , $B_k^{\Omega} := B(x_k, r_k) \cap \Omega$ and $\text{supp } a_k \subset B(x_k, r_k) \cap \overline{\Omega}$, which, together with (6.6), implies that

$$\|T(f)\|_{L_{\omega}^{\Phi}(\Omega)} \lesssim \|f\|_{h_{\omega, z}^{\Phi}(\Omega)}.$$

From this, the fact that $h_{\omega, z}^{\Phi}(\Omega) \cap L^2(\Omega)$ is dense in $h_{\omega, z}^{\Phi}(\Omega)$ and a density argument, we deduce that Theorem 1.9(i) holds.

Now we prove (6.7) by considering the following two cases for r_0 .

Case i) $r_0 \geq 1$. In this case, the proof of (6.7) is similar to the proof of (5.4). We omit the details.

Case ii) $r_0 < 1$. In this case, we have

$$\int_{\Omega} \Phi(T(\lambda a)(x)) \omega(x) dx = \sum_{j=0}^{\infty} \int_{U_j^{\Omega}(B_0)} \Phi(T(\lambda a)(x)) \omega(x) dx =: \sum_{j=0}^{\infty} I_j.$$

For $j \in \{0, 1, 2\}$, similar to the estimate of (5.6), we know that

$$(6.8) \quad I_j \lesssim \omega(B_0^{\Omega}) \Phi\left(\frac{|\lambda|}{\omega(B_0^{\Omega}) \rho(\omega(B_0^{\Omega}))}\right).$$

Now we deal with I_j for $j \in \mathbb{N}$ with $j \geq 3$. Take $q \in (1, 2)$ such that $\omega \in RH_{q'}(\mathbb{R}^n)$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Then from Jensen's inequality, Hölder's inequality, and Lemma 2.2(v), we deduce that

$$\begin{aligned} (6.9) \quad I_j &\leq \omega(U_j^{\Omega}(B_0)) \Phi\left(\frac{1}{\omega(U_j^{\Omega}(B_0))} \left\{ \int_{U_j^{\Omega}(B_0)} |T(\lambda a)(x)|^q dx \right\}^{\frac{1}{q}} \right. \\ &\quad \left. \times \left\{ \int_{U_j^{\Omega}(B_0)} [\omega(x)]^{q'} dx \right\}^{\frac{1}{q'}} \right) \\ &\lesssim \omega(U_j^{\Omega}(B_0)) \Phi\left(\frac{1}{|U_j^{\Omega}(B_0)|^{\frac{1}{q}}} \left\{ \int_{U_j^{\Omega}(B_0)} |T(\lambda a)(x)|^q dx \right\}^{\frac{1}{q}} \right). \end{aligned}$$

Moreover, since

$$T(\lambda a) = 2 \int_0^\infty \frac{\partial^2 e^{-2tL}(\lambda a)}{\partial x_m \partial x_s} dt,$$

we conclude that for each $j \in \mathbb{N}$ with $j \geq 3$,

$$\begin{aligned} (6.10) \quad & \|T(\lambda a)\|_{L^q(U_j^\Omega(B_0))} \\ & \leq 2 \left\| \int_0^{r_0^2} \frac{\partial^2 e^{-2tL}(\lambda a)}{\partial x_m \partial x_s} dt \right\|_{L^q(U_j^\Omega(B_0))} + 2 \left\| \int_{r_0^2}^{[\text{diam}(\Omega)]^2} \cdots dt \right\|_{L^q(U_j^\Omega(B_0))} \\ & + 2 \left\| \int_{[\text{diam}(\Omega)]^2}^\infty \cdots dt \right\|_{L^q(U_j^\Omega(B_0))} =: \text{I} + \text{II} + \text{III}. \end{aligned}$$

We first estimate I. By $j \geq 3$, we know that $\text{dist}(U_j^\Omega(B_0), B_0^\Omega) \geq 2^{j-2}r_0$. From this and Minkowski's inequality, Lemma 3.9 and (6.2), we infer that

$$\begin{aligned} \left\| \frac{\partial^2 e^{-2tL}(\lambda a)}{\partial x_m \partial x_s} \right\|_{L^q(U_j^\Omega(B_0))} & \leq \int_{B_0^\Omega} \left[\int_{U_j^\Omega(B_0)} |\nabla_x^2 K_t(x, y)|^q dx \right]^{\frac{1}{q}} |\lambda a(y)| dy \\ & \lesssim |\lambda| \|a\|_{L^1(\Omega)} t^{\frac{n}{2}(\frac{1}{q}-1)-1} e^{-\gamma \frac{[2^j r_0]^2}{t}}, \end{aligned}$$

which, together with Minkowski's inequality, implies that, for any $M_0 \in (\frac{n}{2}(1 - \frac{1}{q}), \infty)$,

$$\begin{aligned} (6.11) \quad \text{I} & \lesssim |\lambda| \|a\|_{L^1(\Omega)} \int_0^{r_0^2} e^{-\gamma \frac{[2^j r_0]^2}{t}} t^{\frac{n}{2}(\frac{1}{q}-1)-1} dt \\ & \lesssim |\lambda| \|a\|_{L^2(\Omega)} |B_0^\Omega|^{\frac{1}{2}} \int_0^{r_0^2} \left(\frac{t}{[2^j r_0]^2} \right)^{M_0} t^{\frac{n}{2}(\frac{1}{q}-1)-1} dt \\ & \lesssim 2^{-2jM_0} \frac{|\lambda| |B_0|^{\frac{1}{q}}}{\omega(B_0^\Omega) \rho(\omega(B_0^\Omega))}. \end{aligned}$$

Now we deal with II. Pick $M_1 \in (\frac{nq\omega}{2p_\Phi}, M)$. By $a = L^M b$, the fact that for each $k \in \mathbb{N}$,

$$(-1)^k L^k e^{-tL} = \frac{d^k}{dt^k} e^{-tL},$$

Lemma 3.9 and (6.3), we conclude that

$$\begin{aligned} (6.12) \quad \text{II} & \lesssim |\lambda| \int_{r_0^2}^{[\text{diam}(\Omega)]^2} \left\| \frac{\partial^2}{\partial x_m \partial x_s} \left(\frac{d^M}{dt^M} e^{-2tL} b \right) \right\|_{L^q(U_j^\Omega(B_0))} dt \\ & \lesssim |\lambda| \int_{r_0^2}^{[\text{diam}(\Omega)]^2} \int_{B_0^\Omega} \left[\int_{U_j^\Omega(B_0)} \left| \nabla_x^2 \left(\frac{d^M}{dt^M} K_t \right) (x, y) \right|^q dx \right]^{\frac{1}{q}} |b(y)| dy dt \\ & \lesssim |\lambda| \|b\|_{L^1(\Omega)} \int_{r_0^2}^{[\text{diam}(\Omega)]^2} e^{-\gamma \frac{[2^j r_0]^2}{t}} t^{\frac{n}{2}(\frac{1}{q}-1)-(M+1)} dt \\ & \lesssim |\lambda| \|b\|_{L^2(\Omega)} |B_0^\Omega|^{\frac{1}{2}} \int_{r_0^2}^{[\text{diam}(\Omega)]^2} \left(\frac{t}{[2^j r_0]^2} \right)^{M_1} t^{\frac{n}{2}(\frac{1}{q}-1)-(M+1)} dt \end{aligned}$$

$$\lesssim 2^{-2jM_1} \frac{|\lambda| |B_0|^{\frac{1}{q}}}{\omega(B_0^\Omega) \rho(\omega(B_0^\Omega))}.$$

For III, similar to (6.12), we have

$$\begin{aligned} (6.13) \quad \text{III} &\lesssim |\lambda| \int_{[\text{diam}(\Omega)]^2}^\infty \left\| \frac{\partial^2}{\partial x_m \partial x_s} \left(\frac{d^M}{dt^M} e^{-2tL} b \right) \right\|_{L^q(U_j^\Omega(B_0))} dt \\ &\lesssim |\lambda| \int_{[\text{diam}(\Omega)]^2}^\infty \int_{B_0^\Omega} \left[\int_{U_j^\Omega(B_0)} \left| \nabla_x^2 \left(\frac{d^M}{dt^M} K_t \right) (x, y) \right|^q dx \right]^{\frac{1}{q}} |b(y)| dy dt \\ &\lesssim |\lambda| \|b\|_{L^1(\Omega)} \int_{[\text{diam}(\Omega)]^2}^\infty e^{-\gamma \frac{[2^j r_0]^2}{t}} t^{-(M+1)} dt \\ &\lesssim |\lambda| \|b\|_{L^2(\Omega)} |B_0^\Omega|^{\frac{1}{2}} \int_{[\text{diam}(\Omega)]^2}^\infty \left(\frac{t}{[2^j r_0]^2} \right)^{M_1} t^{-(M+1)} dt \\ &\lesssim 2^{-2jM_1} \frac{|\lambda| |B_0| r_0^{2(M-M_1)}}{\omega(B_0^\Omega) \rho(\omega(B_0^\Omega))} \lesssim 2^{-2jM_1} \frac{|\lambda| |B_0|^{\frac{1}{q}}}{\omega(B_0^\Omega) \rho(\omega(B_0^\Omega))}, \end{aligned}$$

where we used the assumption that Ω is bounded and $B(y, \sqrt{t}) \cap \Omega = \Omega$ for all $t \in ([\text{diam}(\Omega)]^2, \infty)$ in the third inequality, and the assumption that $r_B < 1$ in the last inequality.

By $M_0 > \frac{nq_\omega}{2p_\Phi}$, we know that there exist $p_0 \in (0, p_\Phi^-)$ and $q_0 \in (q_\omega, \infty)$ such that Φ is of lower type p_0 and $M_0 > \frac{nq_0}{2p_0}$. Thus, from (6.9), (6.10), (6.11), (6.12), (6.13), Lemma 2.2(iii), the lower type p_0 property of Φ and $M_0 > \frac{nq_0}{2p_0}$, we deduce that

$$\begin{aligned} \sum_{j=3}^\infty \text{I}_j &\lesssim \sum_{j=3}^\infty \omega(U_j^\Omega(B_0)) \Phi \left(\frac{2^{-2jM_1} |\lambda|}{|U_j^\Omega(B_0)|^{\frac{1}{q}}} \frac{|B_0|^{\frac{1}{q}}}{\omega(B_0^\Omega) \rho(\omega(B_0^\Omega))} \right) \\ &\lesssim \sum_{j=3}^\infty 2^{jnq_0} 2^{-(2M_1+n/q)jp_0} \omega(B_0^\Omega) \Phi \left(\frac{|\lambda|}{\omega(B_0^\Omega) \rho(\omega(B_0^\Omega))} \right) \\ &\lesssim \omega(B_0^\Omega) \Phi \left(\frac{|\lambda|}{\omega(B_0^\Omega) \rho(\omega(B_0^\Omega))} \right), \end{aligned}$$

which, together with (6.8), implies that (6.7) holds in this case. This finishes the proof of Theorem 1.9(i).

Now we prove Theorem 1.9(ii). For each $y \in \Omega$, let $V(\cdot, y)$ be the solution of the Neumann problem

$$(6.14) \quad \begin{cases} \Delta V(\cdot, y) = |\Omega|^{-1}, & \text{in } \Omega, \\ \partial_{\nu(x)}[V(x, y)] = \partial_{\nu(x)}[\Gamma(x - y)], & \text{for } x \in \partial\Omega, \end{cases}$$

where $\nu(x)$ denotes the *outward unit normal* to $\partial\Omega$ at $x \in \partial\Omega$. Then a convenient way of expressing the Green function G_N for $L = -\Delta$ with the Neumann boundary condition on Ω (namely, the integral kernel of the Neumann Green operator $\mathbb{G}_N$) is

$$G_N(x, y) = \Gamma(x - y) - V(x, y), \quad x, y \in \Omega, \quad x \neq y.$$

The Neumann problem (1.3) has a unique solution, up to an additive constant, given by

$$\begin{aligned}
 (6.15) \quad \mathbb{G}_N(f)(x) &= \int_{\Omega} G_N(x, y) f(y) dy \\
 &= \int_{\Omega} \Gamma(x - y) f(y) dy - \int_{\Omega} V(x, y) f(y) dy \\
 &=: E(f)(x) - V(f)(x),
 \end{aligned}$$

where $f \in C^\infty(\Omega)$ satisfies that

$$\int_{\Omega} f(y) dy = 0.$$

Let $f \in h_{\omega, z}^\Phi(\Omega) \cap L^2(\Omega)$. Then by Lemma 6.3, we know that there exist a sequence $\{a_k\}_k$ of $(\rho, 2, M)_\omega$ -atoms and a sequence $\{\lambda_k\}_k$ of numbers such that (6.5) and (6.6) hold.

To finish the proof of Theorem 1.9(ii), we only need to show that for any $(\rho, 2, M)_\omega$ -atom a supported in the ball $B(x_0, r_0) \cap \overline{\Omega}$ and any $\lambda \in \mathbb{C}$,

$$(6.16) \quad \int_{\Omega} \Phi \left([T(\lambda a)]_{\Omega, \varphi}^+(x) \right) \omega(x) dx \lesssim \omega(B_0^\Omega) \Phi \left(\frac{|\lambda|}{\omega(B_0^\Omega) \rho(\omega(B_0^\Omega))} \right),$$

where and in what follows, $B_0^\Omega := B(x_0, r_0) \cap \Omega$, and $[T(\lambda a)]_{\Omega, \varphi}^+$ is as in (1.7).

Indeed, if (6.16) holds, then by (6.5) and the assumption that Φ is subadditive, we conclude that for all $\lambda \in (0, \infty)$,

$$\begin{aligned}
 \int_{\Omega} \Phi \left(\left[T \left(\frac{f}{\lambda} \right) \right]_{\Omega, \varphi}^+(x) \right) \omega(x) dx &\leq \sum_k \int_{\Omega} \Phi \left(\left[T \left(\frac{\lambda_k a_k}{\lambda} \right) \right]_{\Omega, \varphi}^+(x) \right) \omega(x) dx \\
 &\lesssim \sum_k \omega(B_k^\Omega) \Phi \left(\frac{|\lambda_k|}{\lambda \omega(B_k^\Omega) \rho(\omega(B_k^\Omega))} \right),
 \end{aligned}$$

where for each k , $B_k^\Omega := B(x_k, r_k) \cap \Omega$ and $\text{supp } a_k \subset B(x_k, r_k) \cap \overline{\Omega}$, which, together with Theorem 1.7 and (6.6), implies that

$$\|T(f)\|_{h_{\omega, r}^\Phi(\Omega)} \lesssim \|f\|_{h_{\omega, z}^\Phi(\Omega)}.$$

From this, the fact that $h_{\omega, z}^\Phi(\Omega) \cap L^2(\Omega)$ is dense in $h_{\omega, z}^\Phi(\Omega)$ and a density argument, we deduce that Theorem 1.9(ii) holds.

Now we prove (6.16). By (6.15), it suffices to show

$$\begin{aligned}
 &\int_{\Omega} \Phi \left(\left[\frac{\partial^2 E(\lambda a)}{\partial x_m \partial x_s} \right]_{\Omega, \varphi}^+(x) \right) \omega(x) dx + \int_{\Omega} \Phi \left(\left[\frac{\partial^2 V(\lambda a)}{\partial x_m \partial x_s} \right]_{\Omega, \varphi}^+(x) \right) \omega(x) dx \\
 &\lesssim \omega(B_0^\Omega) \Phi \left(\frac{|\lambda|}{\omega(B_0^\Omega) \rho(\omega(B_0^\Omega))} \right).
 \end{aligned}$$

By the fact that L conserves probability, namely, $e^{-tL}1 = 1$ for all $t \in (0, \infty)$, we know that

$$\int_{\Omega} a(x) dx = 0$$

(see also [26, (5.4)]). Then, similar to the proof of [86, Theorem 8.2], we have

$$(6.17) \quad \int_{\Omega} \Phi \left(\left[\frac{\partial^2 E(\lambda a)}{\partial x_m \partial x_s} \right]_{\Omega, \varphi}^+ (x) \right) \omega(x) dx \sim \left\| \frac{\partial^2 E(\lambda a)}{\partial x_m \partial x_s} \right\|_{h_{\omega, r}^{\Phi}(\Omega)} \lesssim \|\lambda a\|_{h_{\omega, r}^{\Phi}(\Omega)}.$$

Moreover, by (6.14) and (6.15), we see that $\frac{\partial^2 V(\lambda a)}{\partial x_m \partial x_s}$ is harmonic in Ω . Hence, an application of Theorem 1.7, in which we take the function φ to be radial, yields, on account of the mean value property for harmonic functions, that for all $x \in \Omega$,

$$\left(\frac{\partial^2 V(\lambda a)}{\partial x_m \partial x_s} \right)_{\Omega, \varphi}^+ (x) = \left| \frac{\partial^2 V(\lambda a)}{\partial x_m \partial x_s} (x) \right|,$$

which, together with Theorems 1.7 and 1.9(i), and Lemma 6.3, implies that

$$\begin{aligned} \left\| \frac{\partial^2 V(\lambda a)}{\partial x_m \partial x_s} \right\|_{h_{\omega, r}^{\Phi}(\Omega)} &\sim \left\| \left(\frac{\partial^2 V(\lambda a)}{\partial x_m \partial x_s} \right)_{\Omega, \varphi}^+ \right\|_{L_{\omega}^{\Phi}(\Omega)} \sim \left\| \frac{\partial^2 V(\lambda a)}{\partial x_m \partial x_s} \right\|_{L_{\omega}^{\Phi}(\Omega)} \\ &\lesssim \left\| \frac{\partial^2 E(\lambda a)}{\partial x_m \partial x_s} \right\|_{L_{\omega}^{\Phi}(\Omega)} + \left\| \frac{\partial^2 G_N(\lambda a)}{\partial x_m \partial x_s} \right\|_{L_{\omega}^{\Phi}(\Omega)} \lesssim \|\lambda a\|_{h_{\omega, r}^{\Phi}(\Omega)}. \end{aligned}$$

This, combined with (6.17) and Lemma 6.3, implies that (6.16) holds, which completes the proof of Theorem 1.9. $\square$

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