TRANSLATING SOLUTIONS TO LAGRANGIAN MEAN CURVATURE FLOW

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ABSTRACT. We prove some non-existence theorems for translating solutions to Lagrangian mean curvature flow. More precisely, we show that translating solutions with an L^2 bound on the mean curvature are planes and that almost-calibrated translating solutions which are static are also planes. Recent work of D. Joyce, Y.-I. Lee, and M.-P. Tsui shows that these conditions are optimal.

1. INTRODUCTION

It was shown in [5] that finite-time singularities are, in some sense, unavoidable. More precisely, the first author gave examples of Lagrangians in $\mathbb{C}^2$ having the Lagrangian angle as small as we want and for which the Lagrangian mean curvature flow develops a finite-time singularity. Thus, if one aims to use Lagrangian mean curvature flow in order to understand the existence problem for Special Lagrangians (i.e. Lagrangians which are minimal surfaces), it is crucial to understand how finite-time singularities form. The next example shows that this is a rather non-trivial problem.

Example 1.1. Let γ_0 be the curve in $\mathbb{C}$ given in Figure 1.

The curve can be made so that, under curve-shortening flow $(\gamma_t)_{t \geq 0}$, the small loop collapses at time T and γ_T becomes a curve with a cusp point. Moreover, γ_0 can be chosen so that the angle the tangent vector makes with the x -axis has an oscillation not much larger than π . Let L_t be the Lagrangian surface in $\mathbb{C}^2$ given by

$$L_t = \gamma_t \times \mathbb{R} \subset \mathbb{C} \times \mathbb{C}.$$

Then, L_0 is a zero-Maslov class Lagrangian with oscillation of the Lagrangian angle as close to π as we want, which develops a singularity at time T . The singular set is a line of cusp-points and hence has Hausdorff dimension one.

This example shows that in order to develop a regularity theory for the flow, i.e., show that singularities for Lagrangian mean curvature flow are isolated, we need to require the oscillation of the Lagrangian angle to be strictly smaller than π (*almost-calibrated*).

So far, the only known evidence that such regularity theory is possible was given in [5]. There, assuming the initial condition is a rational and almost-calibrated Lagrangian, the first author showed that if one rescales the flow around a fixed point in space-time, connected components of this rescaled flow converge to an area-minimizing union of planes. The fact that the rescaled flow converges to a

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FIGURE 1. Curves γ_0 and γ_T .

union of planes is an almost trivial consequence of Huisken's monotonicity formula, and so the interesting part is that the configuration of planes needs to be area-minimizing. Without this property it would be hopeless to expect any regularity theory.

Nonetheless, we should point out that the property mentioned above is not sufficient to develop a regularity theory. One needs to understand dilations of the flow where the point at which we center the dilation changes with the scale (called *Type II dilations*). From general theory, it follows that Type II dilations converge to an eternal solution with second fundamental form uniformly bounded. If singularities are indeed isolated, the expectation is that this eternal solution has vanishing mean curvature. We now describe heuristically what could happen regarding Example 1.1. If we rescale around the fixed point in space-time at which the singularity is developing, the rescaled flow converges to a plane with multiplicity two. On the other hand, if we rescale the flow around the point of highest second fundamental form at some time t_1 close to T and choose the scale so that the second fundamental form for the new solution becomes bounded by one, the rescaled flow converges to the eternal solution given by

$$(1) \quad L_t = \{(-\log \cos y_1 + t, y_1, x_2, 0) \mid -\pi/2 < y_1 < \pi/2, x_2 \in \mathbb{R}\}.$$

This solution is called the *grim-reaper* and is an example of translating solutions to mean curvature flow.

Definition 1.2. A Lagrangian L is a *translating solution* to Lagrangian mean curvature flow if we can find an ambient vector e_1 so that

$$L_t = L + te_1$$

is a solution to mean curvature flow.

Remark 1.3. Without loss of generality, we assume that $e_1 = (1, 0, 0, 0)$. We can always achieve this by scaling L and then choosing a suitable coordinate system.

In a surprising new result, Dominic Joyce, Yng-Ing Lee, and Mao-Pei Tsui [4] found translating solutions to Lagrangian mean curvature flow with oscillation of

the Lagrangian angle arbitrarily small. They are described as follows. Let w be a curve in $\mathbb{C}$ such that

$$w_t = \sqrt{2t}w \quad \text{for } t > 0$$

is a solution to curve shortening flow in $\mathbb{C}$. This curve can be chosen so that the angle θ that the tangent vector makes with the x -axis has arbitrarily small oscillation. Set

$$(2) \quad L = \left\{ \left(\frac{|w|^2(y) - x^2}{2} - i\theta(y), xw(y) \right) \mid x, y \in \mathbb{R} \right\} \subset \mathbb{C} \times \mathbb{C}.$$

Using the fact that the curvature of w satisfies

$$\vec{k} = w^\perp,$$

it is a straightforward computation to check that L is Lagrangian and that

$$L_t = L + t(1, 0, 0, 0)$$

is a solution to Lagrangian mean curvature flow. Moreover, the Lagrangian angle of L coincides with θ , and hence its oscillation can be made arbitrarily small.

The main purpose of this paper is to give conditions that exclude the existence of non-trivial translating solutions to Lagrangian mean curvature flow. In order to do so, we need one more definition.

Let $(L_t)_{-\infty < t < \infty}$ be an eternal solution to Lagrangian mean curvature flow in $\mathbb{C}^2$ which is almost-calibrated. Given a sequence $(\lambda_i)_{i \in \mathbb{N}}$ converging to zero, we can consider the sequence of *blow-downs*

$$L_s^i = \lambda_i L_{s/\lambda_i^2}, \quad \text{where } -\infty < s < \infty.$$

It follows from Theorem 3.1 that we can extract a subsequence L_s^i converging weakly to the same union of planes for every $s \leq 0$, which we denote by L^∞ (see Theorem 3.1 for the notion of weak convergence we are considering).

Definition 1.4. An eternal solution to Lagrangian mean curvature flow is called *static* if we can find a convergent sequence of blow-downs L_s^i that converges in the Radon measure sense to L^∞ for every $s > 0$.

In Theorem 3.1 we characterize the blow-downs of eternal solutions. In particular, it will be seen in Theorem 3.1 that assuming L is exact (see the next section for a definition) and $\inf_L \cos \theta > 0$, where θ denotes the Lagrangian angle, then if $L_{s_0}^i$ converges to a union of planes for some $s_0 > 0$, L_s^i must converge to L^∞ for every $s > 0$. The reason is that in that case the limit of $L_{s_0}^i$ will be stationary, and thus the last statement of Theorem 3.1 applies.

Theorem A. Let L be a translating solution for Lagrangian mean curvature flow in $\mathbb{C}^2$ for which we can find a constant C_1 such that

$$(H) \quad \begin{cases} \text{The first Betti number of } L \text{ is finite,} \\ \sup_L |\theta| \leq C_1, \\ \mathcal{H}^2(L \cap B_R(0)) \leq C_1 R^2 \quad \text{for all } R > 0, \\ \sup_L |A|^2 \leq C_1. \end{cases}$$

If

$$\int_L |H|^2 d\mu \quad \text{is finite}$$

or if

$$L \text{ is static and } \inf_L \cos \theta \geq \varepsilon_1$$

for some $\varepsilon_1 > 0$, then L is a plane.

Remark 1.5. (1) $n = 2$ is necessary for the following reason. Special Lagrangians which are invariant under translation by e_1 are translating solutions. For instance, if L is any Special Lagrangian in $\mathbb{C}^2$, then $\tilde{L} = \mathbb{R} \times L \subset \mathbb{C}^3$ is also a translating solution, and so the theorem is false when $n = 3$. It is a simple fact that smooth Special Lagrangians in $\mathbb{C}^2$ which are invariant under e_1 must be a plane, and this fact will be used in the proof of Theorem 4.3.

- (2) The static condition is necessary for the following reason. Consider the translating solution $(L_t)_{-\infty < t < \infty}$ described in (2) and denote by $\tilde{w}$ the curve in $\mathbb{C}$ given by the union of w with $-w$. If $(\lambda_i)_{i \in \mathbb{N}}$ is a sequence converging to zero, the curves $\sqrt{\lambda_i} \tilde{w}$ converge to a union of two lines crossing at the origin which we denote by $\tilde{w}_0$. A simple computation shows that the sequence of blow-downs L_s^i converges to the union of two planes given by

$$\mathbb{R} \times \tilde{w}_0 \subseteq \mathbb{C} \times \mathbb{C}$$

for every $s \leq 0$ and to a self-expander given by

$$\sqrt{2s}(\mathbb{R} \times \tilde{w}) = \mathbb{R} \times \sqrt{2s} \tilde{w} \subseteq \mathbb{C} \times \mathbb{C}$$

for every $s > 0$.

- (3) The condition

$$\inf_L \cos \theta \geq \varepsilon_1$$

is necessary because otherwise the grim-reaper described in (1) would be a counterexample.

The paper is organized as follows. In Section 2 we introduce some basic notation and derive some simple identities for translating solutions, and in Section 3 we prove a compactness theorem for the blow-down sequence of an eternal solution.

In Section 4 we assume that, outside a compact set, L can be decomposed into N components $L_1, \dots, L_N$, where each L_j is the graph of a multi-valued function defined over a plane P_j minus a disc, and show that in this case L needs to be a plane (Theorem 4.3). The argument consists of using barriers to show that on each component L_j the Lagrangian angle converges to a constant sufficiently fast, and this will imply that, by choosing R sufficiently large, $L_j \setminus B_R(0)$ can be made as close to a plane as we want. In particular,

$$\lim_{R \rightarrow \infty} \oint_{\partial(L \cap B_R(0))} \langle \nu, e_1 \rangle d\sigma = 0,$$

where ν denotes the exterior unit normal to $\partial(L \cap B_R(0))$ in L . The result will follow because Proposition 2.2 (ii) implies that

$$\oint_{\partial(L \cap B_R(0))} \langle \nu, e_1 \rangle d\sigma = \int_{L \cap B_R(0)} \Delta x_1 d\mu = \int_{L \cap B_R(0)} |H|^2 d\mu,$$

and so L will have zero mean curvature. In that section we also prove a lemma (Lemma 4.6) that gives us conditions under which a translating solution to Lagrangian mean curvature flow admits a graphical decomposition.

In Section 5 we show that if L has the L^2 -norm of $|H|$ bounded or if L is static and almost-calibrated, then L satisfies the conditions specified in Lemma 4.6 and hence admits a graphical decomposition.

1.1. Open questions. We propose two questions whose answers could provide some valuable insight as to whether it is reasonable to expect any good behavior for singularities of Lagrangian mean curvature flow.

The first question is whether the translating solutions found by D. Joyce, Y.-I. Lee, and M.-P. Tsui can arise as a blow-up of a finite time singularity for Lagrangian mean curvature flow.

If one aims to develop a regularity theory for the flow, it is absolutely necessary to answer this question. This relation has been observed before in codimension one mean curvature flow and Ricci flow. In the first case, Brian White used his work on mean convex solutions to mean curvature flow [9] to show that no grim-reaper appears as the limit of a sequence of rescaled flows [10, Corollary 4]. In the second case, one of the first breakthroughs of Perelman [6, Section 4] was to show that the cigar soliton does not arise as a finite-time singularity model.

If one could show that, assuming the initial condition for the flow is an exact and almost-calibrated Lagrangian, no blow-down of a Type II rescale can give rise to a non-stationary self-expander, then, in view of Theorem 3.1, the question above would be solved for a large class of initial conditions.

The second question addresses the issue of uniqueness of translating solutions. Suppose that $(L_t)_{t \in \mathbb{R}}$ and $(L'_t)_{t \in \mathbb{R}}$ are two translating solutions which are almost calibrated and exact. If after blow-down they produce the same self-expander, do they have to differ only by a rigid motion?

2. DEFINITIONS AND BASIC IDENTITIES

Let J and ω denote, respectively, the standard complex structure on $\mathbb{C}^2$ and the standard symplectic form on $\mathbb{C}^2$. We also consider the closed complex-valued 2-form given by

$$\Omega \equiv dz_1 \wedge dz_2,$$

where $z_j = x_j + iy_j$ are complex coordinates of $\mathbb{C}^2$, and the Liouville form

$$\lambda = \sum_{j=1}^2 x_j dy_j - y_j dx_j.$$

We denote by $\mathcal{H}^1$ and $\mathcal{H}^2$ the one dimensional and two dimensional Hausdorff measures in $\mathbb{C}^2$, respectively.

A smooth two-dimensional submanifold L in $\mathbb{C}^2$ is said to be *Lagrangian* if $\omega_L = 0$, and this implies that

$$\Omega_L = e^{i\theta} \text{vol}_L,$$

where vol_L denotes the volume form of L and θ is a multi-valued function called the *Lagrangian angle*. When the Lagrangian angle is a single-valued function the Lagrangian is called *zero-Maslov class*, and if

$$\cos \theta \geq \varepsilon_0$$

for some positive ε_0 , then L is said to be *almost-calibrated*. Furthermore, if $\theta \equiv \theta_0$, then L is calibrated by

$$\operatorname{Re}(e^{-i\theta_0}\Omega)$$

and hence area-minimizing. In this case, L is referred to as being *Special Lagrangian*. The Lagrangian L is said to be *exact* if the Liouville form is an exact form on L .

Finally, the relation between the Lagrangian angle and the mean curvature is given by

$$H = J\nabla\theta.$$

Given a point x_0 in $\mathbb{C}^2$ and a time T , the backwards heat kernel is defined as

$$\Phi_{x_0,T}(x,t) = \frac{\exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)}{4\pi(T-t)}.$$

When x_0 is the origin and $T = 0$, we denote it by Φ . When it is clear from the context at which instant t we are evaluating $\Phi_{x_0,T}(x,t)$, we denote it simply by $\Phi_{x_0,T}$. Moreover, $\mathbf{x}$ and $\mathbf{x}^\perp$ stand for the position vector associated with the point x in $\mathbb{C}^2$ and its projection on the normal space of $T_x L$, respectively.

Lemma 2.1 (Huisken's monotonicity formula). *Let f_t be a smooth family of functions in L_t . Then*

$$\frac{d}{dt} \int_{L_t} f_t \Phi_{x_0,T} d\mu = \int_{L_t} (\partial_t f_t - \Delta f_t) \Phi_{x_0,T} d\mu - \int_{L_t} \left| H + \frac{(\mathbf{x} - x_0)^\perp}{2(T-t_0)} \right|^2 \Phi_{x_0,T} d\mu.$$

Throughout this paper, $(L_t)_{-\infty < t < \infty}$ will be a translating solution to Lagrangian mean curvature flow in $\mathbb{C}^2$, where L satisfies hypothesis (H). We denote by $|e_1^\perp|$ the projection of $e_1 = (1, 0, 0, 0)$ on the normal space of L . The intrinsic ball of radius r around a point x in L is defined by $\widehat{B}_r(x)$, and we fix $r_0 < 1$ to be such that $\widehat{B}_{r_0}(x)$ is simply connected for every x in L .

Proposition 2.2. *The following equations hold on L .*

(i) *There is a constant θ_0 such that*

$$\theta(x) = -\langle Je_1, \mathbf{x} \rangle + \theta_0 \quad \text{and} \quad |H| = |e_1^\perp|;$$

(ii)

$$\Delta x_1 = |H|^2;$$

(iii)

$$\Delta\theta + \langle \nabla\theta, e_1 \rangle = 0.$$

Proof. Because $(L_t)_{-\infty < t < \infty}$ is a translating solution to mean curvature flow we have that

$$H = e_1^\perp,$$

and thus

$$\nabla\theta = -Je_1^\perp = -(Je_1)^\top.$$

This implies the first property. The second one follows from

$$\Delta x_1 = \langle H, e_1 \rangle = |H|^2,$$

and the third property is a consequence of

$$\Delta\theta = -(Je_1)^\top = (Je_1)^\perp = -\langle H, Je_1 \rangle = -\langle \nabla\theta, e_1 \rangle.$$

□

Given a sequence $(\lambda_i)_{i \in \mathbb{N}}$ converging to zero, we define the sequence of blow-downs

$$L_s^i = \lambda_i L_{s/\lambda_i^2}, \quad \text{where} \quad -\infty < s < \infty.$$

Finally, we use the notation

$$\{\theta = \alpha\} = \{x \in L \mid \theta(x) = \alpha\}$$

and

$$\{|\theta - \alpha| \leq \delta\} = \{x \in L \mid |\theta(x) - \alpha| \leq \delta\}.$$

3. BLOW-DOWN THEOREM

Let $(\Sigma_t)_{-\infty < t < \infty}$ be an eternal solution to Lagrangian mean curvature flow for which we can find a constant D such that, for every t ,

$$\begin{cases} \mathcal{H}^2(\Sigma_t \cap B_R(0)) \leq DR^2 & \text{for all } R > 0, \\ \cos(\theta_t) \geq D^{-1}. \end{cases}$$

Given a sequence $(\lambda_i)_{i \in \mathbb{N}}$ converging to zero, consider the blow-downs

$$\Sigma_s^i = \lambda_i \Sigma_{s/\lambda_i^2} \quad \text{for} \quad -\infty < s < \infty.$$

Theorem 3.1. *There exist a finite set*

$$\{\bar{\theta}_1, \dots, \bar{\theta}_N\}$$

and Lagrangian planes

$$P_1, \dots, P_N$$

such that, after passing to a subsequence, we have for every smooth function ϕ compactly supported, every f in $C^2(\mathbb{R})$, and every $s \leq 0$

$$(3) \quad \lim_{i \rightarrow \infty} \int_{\Sigma_s^i} f(\theta_{i,s}) \phi d\mu = \sum_{j=1}^N m_j f(\bar{\theta}_j) \int_{P_j} \phi d\mu,$$

where m_j denotes the multiplicity of P_j . The set

$$\{\bar{\theta}_1, \dots, \bar{\theta}_N\}$$

does not depend on the sequence of rescales chosen.

If Σ_0 is exact, there exists an integer rectifiable 2-varifold Σ_1^∞ satisfying

$$H = \frac{\mathbf{x}^\perp}{2}$$

such that, after passing to a subsequence, Σ_s^i converges as Radon measures to $\sqrt{s}\Sigma_1^\infty$ for every $s > 0$.

If Σ_1^∞ is stationary, then identity (3) holds for every s , and hence $(\Sigma_t)_{-\infty < t < \infty}$ is static.

Remark 3.2. (i) Whenever identity (3) holds we say that Σ_s^i converges weakly to $\Sigma^\infty = m_1 P_1 + \dots + m_N P_N$.

(ii) From (3) it follows that the integers m_j do not depend on the subsequence chosen. The planes P_j could, in principle, depend on the sequence chosen.

- (iii) If one blows-down the grim-reaper defined in Section 1, we see that the limit flow $(L_s^\infty)_{s \in \mathbb{R}}$ is a line with multiplicity two for $s < 0$, half-line with multiplicity two if $s = 0$, and the empty set if $s > 0$. Hence, equation (3) holds only for $s < 0$ and does not extend to $s = 0$. This implies that being almost-calibrated is essential to showing that (3) holds when $s = 0$.

Proof. From the compactness for integral Brakke motions [2, Section 7.1] we know that, after passing to a subsequence, $(\Sigma_s^i)_{-\infty < s < \infty}$ converges to an integral Brakke motion $(\Sigma_s^\infty)_{-\infty < s < \infty}$. Arguing as in the proof of [5, Theorem A], it is straightforward to see the existence of Lagrangian planes

$$P_1, \dots, P_N,$$

each having multiplicity m_i and Lagrangian angle $\bar{\theta}_i$, such that (3) holds for every $s < 0$. In particular, $\Sigma_{-1}^\infty = m_1 P_1 + \dots + m_N P_N$.

We now argue that (3) also holds when $s = 0$. The main technical issue is trying to overcome the fact that Σ_0^i could converge to Σ_0^∞ only in the Radon measure sense and not in the varifold sense.

Federer and Fleming compactness for integral currents implies that, after passing to a subsequence, Σ_0^i converges to an integral current Σ . We claim that, after rearranging the labeling of the Lagrangian planes $\{P_1, \dots, P_N\}$, there is $Q \leq N$ and $n_i \leq m_i$, $i = 1, \dots, Q$, such that the varifold associated to the integral current Σ (which we still denote by Σ) coincides with $n_1 P_1 + \dots + n_Q P_Q$.

We start by remarking that for every $\phi \geq 0$,

$$\int_{\Sigma_0^\infty} \phi d\mu \leq D \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \phi \cos \theta d\mu = D \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \phi \operatorname{Re} \Omega = D \int_{\Sigma} \phi \operatorname{Re} \Omega$$

and

$$D^{-1} \int_{\Sigma} \phi d\mu \leq \int_{\Sigma} \phi \operatorname{Re} \Omega = \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \phi \operatorname{Re} \Omega \leq \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \phi d\mu = \int_{\Sigma_0^\infty} \phi d\mu,$$

and so the support of Σ coincides with the support of Σ_0^∞ . From Huisken's monotonicity formula, Lemma 2.1, we have that, for every x_0 in $\mathbb{C}^2$, every positive T , and every $s < 0$,

$$\int_{\Sigma_0^i} \Phi_{x_0, T}(\cdot, 0) d\mu \leq \int_{\Sigma_s^i} \Phi_{x_0, T}(\cdot, s) d\mu,$$

and thus

$$\int_{\Sigma_0^\infty} \Phi_{x_0, T}(\cdot, 0) d\mu \leq \sum_{j=1}^N m_j \int_{P_j} \Phi_{x_0, T}(\cdot, s) d\mu.$$

Denote the density function of Σ_0^∞ and Σ_{-1}^∞ by $\Theta_0(x)$ and $\Theta(x)$, respectively. Make T go to zero so that the left-hand side converges to $\Theta_0(x_0)$ for almost all x_0 . After that, make s converge to zero to obtain $\Theta(x_0) \geq \Theta_0(x_0)$, and so the support of Σ is contained in the support of Σ_{-1}^∞ . Because $\partial \Sigma = 0$, the Constancy Theorem [7, Theorem 26.27] implies, after rearranging the labeling of the Lagrangian planes, the existence of $Q \leq N$ such that the support of Σ consists of a union of planes P_1 through P_Q . Thus, one can find integers $r_i \leq m_i$ and n_i , $i = 1, \dots, Q$, such that

$$\Sigma_0^\infty = r_1 P_1 + \dots + r_Q P_Q \quad \text{and} \quad \Sigma = n_1 P_1 + \dots + n_Q P_Q.$$

If we show that $n_i \leq r_i$, the claim follows. It suffices to show that $n_1 \leq r_1$. Without loss of generality we assume that the Lagrangian angle of P_1 equals zero. We have

$$\begin{aligned} n_1 \int_{P_1} \Phi_{x_0, T}(\cdot, 0) d\mu &\leq \int_{\Sigma} \Phi_{x_0, T}(\cdot, 0) \operatorname{Re} \Omega \\ &= \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \Phi_{x_0, T}(\cdot, 0) \operatorname{Re} \Omega = \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \Phi_{x_0, T}(\cdot, 0) \cos \theta d\mu \\ &\leq \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \Phi_{x_0, T}(\cdot, 0) d\mu = \int_{\Sigma_0^\infty} \Phi_{x_0, T}(\cdot, 0) d\mu. \end{aligned}$$

Choosing x_0 in P_1 but not in any other P_j and making T tend to zero, we conclude that $n_1 \leq \Theta_0(x_0) = r_1$.

Next, we show that Σ_0^∞ coincides with Σ_{-1}^∞ . We know that

$$(4) \quad \frac{d \cos \theta_t}{dt} = \Delta \cos \theta_t + \cos \theta_t |H|^2,$$

and thus, for every $T > 0$ and $s < 0$, we have from the monotonicity formula of Lemma 2.1 that

$$\begin{aligned} \left| \frac{d}{ds} \int_{\Sigma_s^i} \cos \theta_{i,s} \Phi_{0,T} d\mu \right| &= \left| \int_{\Sigma_s^i} \cos \theta_{i,s} \left(|H|^2 - \left| H + \frac{\mathbf{x}^\perp}{2(T-s)} \right|^2 \right) \Phi_{0,T} d\mu \right| \\ &\leq \int_{\Sigma_s^i} \cos \theta_{i,s} \left(\delta |H|^2 + C \left| \frac{\mathbf{x}^\perp}{2(T-s)} \right|^2 \right) \Phi_{0,T} d\mu, \end{aligned}$$

where $C = C(\delta)$. Combining the evolution equation

$$\frac{d\theta_t^2}{dt} = \Delta \theta_t^2 - 2|H|^2$$

with Huisken's monotonicity formula, Lemma 2.1, we have that for all $s < 0$,

$$\begin{aligned} (5) \quad \lim_{i \rightarrow \infty} \int_s^0 \int_{\Sigma_t^i} 2|H|^2 \Phi_{0,T} d\mu dt &\leq \lim_{i \rightarrow \infty} \int_{\Sigma_s^i} \theta_{i,s}^2 \Phi_{0,T} d\mu \\ &= \sum_{j=1}^N m_j \bar{\theta}_j^2 \int_{P_j} \Phi_{0,T}(\cdot, s) d\mu = \sum_{j=1}^N m_j \bar{\theta}_j^2 = B. \end{aligned}$$

Moreover,

$$(6) \quad \lim_{i \rightarrow \infty} \int_s^0 \int_{\Sigma_t^i} \left| \frac{\mathbf{x}^\perp}{2(T-t)} \right|^2 \Phi_{0,T} d\mu dt = 0,$$

and so

$$(7) \quad \lim_{i \rightarrow \infty} \left| \int_s^0 \frac{d}{dt} \int_{\Sigma_t^i} \cos \theta_{i,t} \Phi_{0,T} d\mu dt \right| \leq \delta \frac{B}{2}$$

for every $\delta > 0$. Thus, making δ tend to zero, we have

$$\begin{aligned} \sum_{j=1}^Q n_j \cos \bar{\theta}_j &= \int_{\Sigma} \Phi_{0,T}(\cdot, 0) \operatorname{Re} \Omega \\ &= \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \cos \theta_{i,0} \Phi_{0,T} d\mu = \lim_{i \rightarrow \infty} \int_{\Sigma_s^i} \cos \theta_{i,s} \Phi_{0,T} d\mu \\ &= \sum_{j=1}^N m_j \cos \bar{\theta}_j \int_{P_j} \Phi_{0,T}(\cdot, s) d\mu = \sum_{j=1}^N m_j \cos \bar{\theta}_j. \end{aligned}$$

The fact that $n_j \leq m_j$ implies that $n_j = m_j$ and $Q = N$. Because $n_j \leq r_j \leq m_j$, the claim follows.

As a result, we have that for every T positive

$$\lim_{i \rightarrow \infty} \int_{\Sigma_{-1}^i} \Phi_{0,T}(\cdot, -1) d\mu = \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \Phi_{0,T}(\cdot, 0) d\mu,$$

and so Huisken's monotonicity formula implies that

$$(8) \quad \lim_{i \rightarrow \infty} \int_{-1}^0 \int_{\Sigma_i^s} (|H|^2 + |\mathbf{x}^\perp|^2) \exp(-|x|^2) d\mu ds = 0$$

because, due to (6), we can find a constant C for which

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{-1}^0 \int_{\Sigma_i^s} (|H|^2 + |\mathbf{x}^\perp|^2) \exp(-|x|^2) d\mu ds &= \lim_{i \rightarrow \infty} \int_{-1}^0 \int_{\Sigma_i^s} |H|^2 \exp(-|x|^2) d\mu ds \\ &\leq \lim_{i \rightarrow \infty} C \int_{-1}^0 \int_{\Sigma_i^s} \left| H + \frac{\mathbf{x}^\perp}{2(1-s)} \right|^2 \Phi_{0,1} d\mu ds \\ &= \lim_{i \rightarrow \infty} C \left(\int_{\Sigma_{-1}^i} \Phi_{0,T}(\cdot, -1) d\mu - \int_{\Sigma_0^i} \Phi_{0,T}(\cdot, 0) d\mu \right) = 0. \end{aligned}$$

We can then argue in the same way as was done at the end of page 471 and the beginning of page 472 in [5] (where $a = -1$ and $b = 0$) to conclude that identity (3) holds for $s = 0$.

Recall that Σ_0^i converges as a Radon measure and as a current to a union of planes with possible multiplicities. Consider the complex-valued 2-form

$$S = \lambda \wedge (\Omega \lrcorner J\mathbf{x}).$$

Because

$$S_{\Sigma_0^i} = (|\mathbf{x}^\perp|^2 + i\langle J\mathbf{x}, \mathbf{x}^\top \rangle) e^{i\theta} \operatorname{vol}_{\Sigma_0^i},$$

we have

$$\lim_{i \rightarrow \infty} \int_{\Sigma_0^i \cap B_R(0)} |\mathbf{x}^\perp|^2 d\mu = 0$$

for every positive R . Moreover, the fact that Σ_0 is exact implies the existence of β_s^i defined on Σ_s^i for which

$$d\beta_s^i = \lambda \quad \text{and} \quad |\nabla \beta_s^i| = |\mathbf{x}^\perp|.$$

Hence, because Σ_0^i is connected, we can apply Proposition B.1 and conclude the existence of $\bar{\beta}$ such that, after passing to a subsequence,

$$\lim_{i \rightarrow \infty} \int_{\Sigma_0^i \cap B_R(0)} (\beta_0^i - \bar{\beta})^2 d\mu = 0.$$

According to [5, Section 6], we have that

$$\frac{d}{ds} (f_s^i)^2 = \Delta (f_s^i)^2 - 2|\mathbf{x}^\perp - 2sH|^2, \quad \text{where} \quad f_s^i = \beta_s^i + 2s\theta_s^i - \bar{\beta},$$

and so we can apply Huisken's monotonicity formula, Lemma 2.1, to the functions f_s^i to conclude that, for every positive T ,

$$(9) \quad \lim_{i \rightarrow \infty} \int_0^T \int_{\Sigma_s^i} 2|\mathbf{x}^\perp - 2sH|^2 \Phi_{0,T} d\mu ds \leq \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} (\beta_0^i - \bar{\beta})^2 \Phi_{0,T} d\mu = 0.$$

We want to show that, for every compactly supported function ϕ in $\mathbb{C}^2$,

$$\frac{1}{s} \int_{\Sigma_s^\infty} \phi(x/\sqrt{s}) d\mu$$

is constant as a function of s for every $s > 0$.

A standard computation shows that

$$\begin{aligned} \frac{d}{ds} \left(\frac{1}{s} \int_{\Sigma_s^i} \phi(x/\sqrt{s}) d\mu \right) &= -\frac{1}{s^2} \int_{\Sigma_s^i} \phi d\mu - \frac{1}{2s^{5/2}} \int_{\Sigma_s^i} \langle D\phi, \mathbf{x} \rangle d\mu \\ &\quad + \frac{1}{s^{3/2}} \int_{\Sigma_s^i} \langle D\phi, H \rangle d\mu - \frac{1}{s} \int_{\Sigma_s^i} \phi |H|^2 d\mu. \end{aligned}$$

Due to

$$\Delta|x|^2 = 4 + 2\langle \mathbf{x}, H \rangle$$

we obtain that

$$2 \int_{\Sigma_s^i} \phi(x/\sqrt{s}) d\mu = - \int_{\Sigma_s^i} \langle \mathbf{x}, H \rangle \phi d\mu - \frac{1}{\sqrt{s}} \int_{\Sigma_s^i} \langle \mathbf{x}^\top, D\phi \rangle d\mu.$$

Hence,

$$\begin{aligned} \frac{d}{ds} \left(\frac{1}{s} \int_{\Sigma_s^i} \phi(x/\sqrt{s}) d\mu \right) &= \frac{1}{s^{3/2}} \int_{\Sigma_s^i} \left\langle D\phi, H - \frac{x^\perp}{2s} \right\rangle d\mu \\ &\quad + \frac{1}{s} \int_{\Sigma_s^i} \phi \left\langle H, \frac{x^\perp}{2s} - H \right\rangle d\mu. \end{aligned}$$

For every $0 < a < b$ and $R > 0$, we have that

$$\int_a^b \int_{\Sigma_s^i} |D\phi|^2 d\mu ds \quad \text{and} \quad \int_a^b \int_{\Sigma_s^i \cap B_R(0)} |H|^2 d\mu ds$$

are uniformly bounded. Therefore, (9) implies that

$$\frac{1}{s} \int_{\Sigma_s^\infty} \phi(x/\sqrt{s}) d\mu$$

is indeed independent of s for all $s > 0$, and this is equivalent to $\Sigma_s^\infty = \sqrt{s}\Sigma_1^\infty$ for all $s > 0$.

We now show the last property. Because Σ_1^∞ has $\mathbf{x}^\perp = 0$ and $\Sigma_s^\infty = \sqrt{s}\Sigma_1^\infty$ for all $s > 0$, we obtain that $\mathbf{x}^\perp = 0$ on Σ_s^∞ for all s , and thus, from varifold convergence, we have for every $0 < a < b$ and every $R > 0$,

$$(10) \quad \lim_{i \rightarrow \infty} \int_a^b \int_{\Sigma_s^i \cap B_R(0)} |\mathbf{x}^\perp|^2 d\mu ds = 0.$$

Hence, identity (9) combined with (10) implies that, for some universal constant C ,

$$\begin{aligned} (11) \quad & \lim_{i \rightarrow \infty} \int_a^b \int_{\Sigma_s^i} (|H|^2 + |\mathbf{x}^\perp|^2) \exp(-|x|^2) d\mu ds \\ &= \lim_{i \rightarrow \infty} \int_a^b \int_{\Sigma_s^i} |H|^2 \exp(-|x|^2) d\mu ds \\ &\leq \lim_{i \rightarrow \infty} C \int_a^b \int_{\Sigma_s^i} 2|\mathbf{x}^\perp - 2sH|^2 \exp(-|x|^2) d\mu ds \\ &\leq \lim_{i \rightarrow \infty} C \int_a^b \int_{\Sigma_s^i} 2|\mathbf{x}^\perp - 2sH|^2 \Phi_{0,2b} d\mu ds = 0. \end{aligned}$$

Thus, assuming without loss of generality that

$$\int_{\Sigma_1^i} (|H|^2 + |\mathbf{x}^\perp|^2) \exp(-|x|^2) d\mu ds = 0,$$

we obtain from [5, Proposition 5.1] (which applies with no modifications) that Σ_1^∞ is a union of Lagrangian planes with multiplicities. In order to prove the result, it suffices to show that Σ_1^∞ equals Σ_0^∞ .

Huisken's monotonicity formula, Lemma 2.1, applied to (4) implies that for every $T > 0$ and $0 < s < T$,

$$\int_{\Sigma_0^\infty} \Phi_{x_0, T} d\mu \leq \int_{\Sigma_s^\infty} \Phi_{x_0, T}(\cdot, s) d\mu = \int_{\Sigma_1^\infty} \Phi_{x_0, T}(\cdot, s) d\mu.$$

Thus making s converge to T and then T converge to zero, we obtain that

$$\Theta_0(x_0) \geq \Theta_1(x_0),$$

where Θ_1 denotes the density function of Σ_1^∞ . Hence the support of Σ_1^∞ is contained in the support of Σ_0^∞ .

Lemma 3.3. *For every x_0 and $s < T$ we have*

$$\lim_{i \rightarrow \infty} \int_{\Sigma_s^i} \cos(\theta_{i,s}) \Phi_{x_0, T}(\cdot, s) d\mu = \lim_{i \rightarrow \infty} \int_{\Sigma_s^i} \cos(\theta_{i,s}) \Phi_{x_0, T}(\cdot, s) d\mu.$$

Proof. The same reasoning that was used to show (5) shows the existence of B so that

$$(12) \quad \lim_{i \rightarrow \infty} \int_0^s \int_{\Sigma_t^i} 2|H|^2 \Phi_{x_0, T} d\mu dt \leq B.$$

Moreover, identity (10) readily implies the following analogue of (6):

$$(13) \quad \lim_{i \rightarrow \infty} \int_0^s \int_{\Sigma_t^i} \left| \frac{\mathbf{x}^\perp}{2(T-t)} \right|^2 \Phi_{x_0, T} d\mu dt = 0.$$

With the help of (12) and (10), we can now apply the same reasoning used to show (7) in order to conclude that

$$\lim_{i \rightarrow \infty} \left| \int_0^s \frac{d}{dt} \int_{\Sigma_t^i} \cos \theta_{i,t} \Phi_{x_0,T} d\mu dt \right| \leq \delta \frac{B}{2}.$$

The result follows from making δ tend to zero. $\square$

Choosing x_0 to be in exactly one P_{j_0} we use the above lemma to conclude that for any $0 < s < T$,

$$\begin{aligned} \int_{\Sigma_1^\infty} \cos(\bar{\theta}_j) \Phi_{x_0,T}(\cdot, s) d\mu &= \lim_{i \rightarrow \infty} \int_{\Sigma_s^i} \cos(\theta_{i,s}) \Phi_{x_0,T}(\cdot, s) d\mu \\ &= \lim_{i \rightarrow \infty} \int_{\Sigma_0^i} \cos(\theta_{i,s}) \Phi_{x_0,T}(\cdot, 0) d\mu = \sum_{j=1}^N m_j \cos(\bar{\theta}_j) \int_{P_j} \Phi_{x_0,T}(\cdot, 0) d\mu \\ &\geq \cos(\bar{\theta}_{j_0}) m_{j_0}, \end{aligned}$$

where the last inequality follows from the almost-calibrated condition. Making s converge to T , the almost-calibrated condition implies that

$$\Theta_0(x_0) \leq \Theta_1(x_0).$$

$\square$

4. GRAPHICAL IMPLIES FLATNESS

We start by defining what it means for a Lagrangian L to have a graphical decomposition. Recall that r_0 was chosen (see Section 2) so that $\widehat{B}_{r_0}(x)$ is simply connected for all x in L .

Definition 4.1. A Lagrangian L is said to admit a *graphical decomposition* if, outside a compact set, L can be decomposed into N connected components L_j , $j = 1, \dots, N$, having the following property.

For each $j = 1, \dots, N$ there are constants $\bar{\theta}_j$, R_j , S_j , n_j a positive integer, and a Lagrangian plane P_j ,

$$P_j = \{(u, 0, v \cos \bar{\theta}_j, v \sin \bar{\theta}_j) \mid (u, v) \in \mathbb{R}^2\},$$

such that

- (i) for every x in L_j , $\widehat{B}_{r_0}(x)$ can be written as the graph of a function defined over P_j with its derivatives bounded by S_j ;
- (ii) the projection of L_j on P_j ,

$$\text{Proj}_{P_j}: L_j \longrightarrow P_j \setminus B_{R_j}(0),$$

is an n_j -fold covering map;

(iii)

$$\lim_{R \rightarrow \infty} \sup_{L_j \setminus B_R(0)} |\theta - \bar{\theta}_j| = 0.$$

In particular, properties (i) and (iii) imply that

$$\lim_{R \rightarrow \infty} \sup_{L \setminus B_R(0)} |H| = 0.$$

Remark 4.2. In case L is not embedded, property (ii) in Definition 4.1 should be interpreted as follows. If F denotes the immersion of the surface L in $\mathbb{C}^2$, then $\text{Proj}_{P_j} \circ F$ is an n_j -fold covering map.

In this section we show that any translating solution L to Lagrangian mean curvature flow which admits a graphical decomposition is a plane.

Theorem 4.3. *If L is a translating solution to Lagrangian mean curvature flow with uniformly bounded second fundamental form and admitting a graphical decomposition, then L is a plane.*

Proof. It suffices to show that the mean curvature of L is zero because, in that case, we have from Proposition 2.2 that L is a Special Lagrangian contained in some hyperplane and therefore a plane. The idea consists in finding a sequence of compact sets K_n exhausting L for which

$$\lim_{n \rightarrow \infty} \oint_{\partial K_n} \langle \nu, e_1 \rangle d\sigma = 0,$$

where ν denotes the exterior unit normal to ∂K_n in L . The theorem follows because, due to Proposition 2.2 (ii),

$$\oint_{\partial K_n} \langle \nu, e_1 \rangle d\sigma = \int_{K_n} |H|^2 d\mu.$$

The following barrier will be needed to prove the main theorem.

Lemma 4.4. *For every $\alpha < 1/3$, there is a constant $R_0 = R_0(\alpha)$ such that, for every constant δ and B , the function*

$$V_{\delta,B} = B|x|^{-\alpha} \exp(-|x|/2) + \delta \exp(x_1/2)$$

satisfies

$$\Delta V_{\delta,B} \leq V_{\delta,B} \frac{|H|^2 + 1}{4} \quad \text{for all } |x| \geq R_0.$$

Proof. Denote by ∂_r , $\partial_r^\top$, and $\partial_r^\perp$ the radial vector, its tangential projection on L , and its projection on the normal bundle of L , respectively. Set

$$f(|x|) = |x|^{-\alpha} \exp(-|x|/2).$$

Away from the origin, we have

$$\Delta|x| = \langle \partial_r, H \rangle + \frac{2}{|x|} - \frac{|\partial_r^\top|^2}{|x|},$$

and so, because

$$f' = -\alpha \frac{f}{|x|} - \frac{f}{2} \quad \text{and} \quad f'' = \alpha(\alpha+1) \frac{f}{|x|^2} + \alpha \frac{f}{|x|} + \frac{f}{4},$$

we obtain that

$$\begin{aligned} \Delta f = f \left(\frac{|\partial_r^\top|^2}{4} - \frac{\langle \partial_r, H \rangle}{2} \right) + \frac{f}{|x|} \left(\alpha |\partial_r^\top|^2 - 1 - \alpha \langle \partial_r, H \rangle + \frac{|\partial_r^\top|^2}{2} \right) \\ + \frac{f}{|x|^2} (\alpha(\alpha+2) |\partial_r^\top|^2 - 2\alpha). \end{aligned}$$

Hence

$$\begin{aligned}\Delta f &\leq f \frac{|H|^2 + 1}{4} + \frac{f}{|x|} \left(\frac{\alpha |H|^2}{2} + \alpha - \frac{1}{2} \right) + \frac{f}{|x|^2} \alpha^2 \\ &\leq f \frac{|H|^2 + 1}{4} + \frac{f}{|x|} \frac{3\alpha - 1}{2} + \frac{f}{|x|^2} \alpha^2.\end{aligned}$$

Therefore, we can choose $R_0 = R_0(\alpha)$ so that for all $|x| \geq R_0$ we have

$$\Delta f \leq f \frac{|H|^2 + 1}{4}.$$

Proposition 2.2 (ii) implies that

$$(14) \quad \Delta \exp(x_1/2) = \exp(x_1/2) \frac{2|H|^2 + |e_1^\top|^2}{4} = \exp(x_1/2) \frac{|H|^2 + 1}{4},$$

and so the lemma follows. $\square$

This lemma implies the following decay for $|\theta - \bar{\theta}_j|$ on each component L_j , $j = 1, \dots, N$, given by the graphical decomposition.

Lemma 4.5. *For each $\alpha < 1/3$, there are constants B and R_0 such that, on each component L_j , $j = 1, \dots, N$, we have*

$$|\theta(x) - \bar{\theta}_j| + |\nabla \theta|(x) \leq B|x|^{-\alpha} \exp(-|x|/2 - x_1/2) \quad \text{for all } |x| \geq R_0.$$

Proof. For each $j = 1, \dots, N$, set

$$u_j = (\theta - \bar{\theta}_j) \exp(x_1/2).$$

From Proposition 2.2 and (14) we know that

$$\Delta u_j = u_j \frac{|H|^2 + 1}{4} + \exp(x_1/2) (\Delta \theta + \langle e_1^\top, \nabla \theta \rangle) = u_j \frac{|H|^2 + 1}{4},$$

and thus, we obtain from Lemma 4.4 that

$$\Delta(V_{\delta,B} - u_j) \leq (V_{\delta,B} - u_j) \frac{|H|^2 + 1}{4} \quad \text{for all } |x| \geq R_0,$$

where R_0 is chosen large enough so that $\partial L_j \subseteq B_{R_0}(0)$ and the constant B is chosen so that, for every $j = 1, \dots, N$, we have

$$x \in L_j \cap \partial B_{R_0}(0) \implies B|x|^{-\alpha} \exp(-|x|/2) > |u_j(x)|.$$

Note that, for all R sufficiently large, we have from Definition 4.1 (iii) that

$$\sup_{L_j \cap \partial B_R(0)} |u_j \exp(-x_1/2)| \leq \delta,$$

and thus

$$\sup_{L_j \cap \partial B_R(0)} (V_{\delta,B} - u_j) > 0 \quad \text{for every } j = 1, \dots, N.$$

Applying the maximum principle to $L_j \cap (B_R(0) \setminus B_{R_0}(0))$ for all R sufficiently large, we have that

$$u_j(x) \leq B|x|^{-\alpha} \exp(-|x|/2) + \delta \exp(x_1/2) \quad \text{for all } |x| \geq R_0.$$

As a result, after making δ tend to zero, we obtain

$$\theta(x) - \bar{\theta}_j \leq B|x|^{-\alpha} \exp(-|x|/2 - x_1/2) \quad \text{for all } |x| \geq R_0.$$

The correspondent estimate for $|\theta(x) - \bar{\theta}_j|$ follows in the same way by considering the function $V_{\delta,B} + u_j$. The estimate for $|\nabla\theta|$ is a consequence of Proposition 2.2 (i) and interior Schauder estimates. $\square$

The graphical decomposition implies the existence of r_1 so that, for every p in L_j , $j = 1, \dots, N$, the projection of $\hat{B}_{r_0}(p)$ on P_j contains $B_{r_1}(\bar{p}) \cap P_j$, where $\bar{p}$ stands for the projection of p on P_j . Thus, after an appropriate change of coordinates, a neighborhood of p can be described as

$$(u, v, \partial_u f, \partial_v f) \quad \text{with} \quad (u, v) \in B_{r_1}(\bar{p})$$

for some function f with $|\text{Hess } f|$ uniformly bounded, where the coordinate x_1 equals u and the coordinate y_1 equals $\partial_u f$. Furthermore, a direct computation shows that we can find a constant D depending only on the constants S_j (see Definition 4.1 (i)) for which one of the eigenvalues of $\text{Hess } f$ (called $\bar{\lambda}$) satisfies

$$|\bar{\lambda}| \leq D|e_1^\perp| = D|\nabla\theta| \quad \text{on} \quad B_{r_1}(\bar{p}).$$

Thus, we obtain from Lemma 4.5 that, provided $|p|$ is large enough, one of the eigenvalues $\bar{\lambda}$ has the decay

$$(15) \quad |\bar{\lambda}| \leq B|x|^{-\alpha} \exp(-|x|/2 - u/2)$$

for some constant B .

An explicit computation shows that the function f satisfies the equation

$$\arctan \lambda_1 + \arctan \lambda_2 = \theta(x) - \bar{\theta}_j,$$

where λ_1 and λ_2 are the eigenvalues of $\text{Hess } f$. As a result, Lemma 4.5 and estimate (15) imply that, provided $|p|$ is large enough,

$$(16) \quad |\text{Hess } f| \leq B|x|^{-\alpha} \exp(-|x|/2 - u/2),$$

where $\alpha < 1/3$ and B depends on α .

On each of the connected components L_j , denote by $\gamma_{j,r}$ the lift to L_j of the path on P_j given by

$$c_j(t) = (r \cos t, r \sin t) \quad 0 \leq t \leq 2n_j\pi,$$

where the variable r will be made as large as we want.

There is $t_0 = t_0(r, r_1)$ such that, for every $t_1 \leq 2n_j\pi$, we can find a function f for which

$$\gamma_{j,r}(t) = (r \cos t, r \sin t, \partial_u f, \partial_v f) \quad \text{for all } t_1 - t_0 \leq t \leq t_1 + t_0,$$

where, due to (16), the function f restricted to this portion of $\gamma_{j,r}$ satisfies

$$|\text{Hess } f|(t) \leq Br^{-\alpha} \exp(-r/2 - r \cos t/2).$$

Using an obvious abuse of notation, the tangent vector $\gamma'_{j,r}(t)$ is given by

$$\gamma'_{j,r}(t) = r(\partial_t, (\text{Hess } f)(\partial_t)), \quad \text{where} \quad \partial_t = (-\sin t, \cos t),$$

and, denoting by $\bar{\nu}$ the vector in $|x|^2$ for which

$$\nu = (\bar{\nu}, (\text{Hess } f)(\bar{\nu})),$$

then

$$\langle \bar{\nu}, \partial_t + (\text{Hess } f)^*(\text{Hess } f)(\partial_t) \rangle = 0 \quad \text{and} \quad 1 = |\bar{\nu}|^2 + |(\text{Hess } f)(\bar{\nu})|^2,$$

where $(\text{Hess } f)^*$ denotes the transpose of $\text{Hess } f$. Thus, provided we choose r sufficiently large, we can find a constant C such that

$$|\bar{\nu} - \partial_r| \leq Cr^{-2\alpha} \exp(-r - r \cos t)$$

and

$$|\gamma'_{j,r}(t)| \leq r(1 + Cr^{-2\alpha} \exp(-r - r \cos t)).$$

For this reason,

$$\int_{t_1-t_0}^{t_1+t_0} \langle \nu, e_1 \rangle |\gamma'_{j,r}(t)| dt = \int_{t_1-t_0}^{t_1+t_0} r \cos t dt + Q,$$

where we can find a constant C such that

$$|Q| \leq \int_{t_1-t_0}^{t_1+t_0} Cr^{1-2\alpha} \exp(-r - r \cos t) dt.$$

Therefore, we obtain from unique continuation that

$$\begin{aligned} \left| \oint_{\gamma_{j,r}} \langle \nu, e_1 \rangle d\sigma \right| &\leq \int_0^{2n_j\pi} Cr^{1-2\alpha} \exp(-r - r \cos t) dt \\ &= n_j Cr^{1-2\alpha} \int_0^{2\pi} \exp(-r - r \cos t) dt. \end{aligned}$$

Choose δ so that

$$|\cos(y) + 1| \leq (y - \pi)^2 \quad \text{for all } |y - \pi| \leq \delta.$$

Then, we can find a positive constant $D = D(\delta)$ so that, for all r sufficiently large,

$$\begin{aligned} \int_0^{2\pi} \exp(-r - r \cos t) dt &\leq \int_{\pi-\delta}^{\pi+\delta} \exp(-r(t - \pi)^2) dt + 2\pi \exp(-Dr) \\ &\leq r^{-1/2} \int_{-\sqrt{r}\delta}^{\sqrt{r}\delta} \exp(-s^2) ds + 2\pi \exp(-Dr) \\ &\leq Dr^{-1/2}. \end{aligned}$$

Hence, provided we choose $\alpha > 1/4$ in Lemma 4.5, we have that

$$\lim_{r \rightarrow \infty} \left| \oint_{\gamma_{j,r}} \langle \nu, e_1 \rangle d\sigma \right| = \lim_{r \rightarrow \infty} r^{1/2-2\alpha} = 0.$$

Properties (i) and (ii) of Definition 4.1 imply that we can find a sequence of compact sets K_n exhausting L and such that

$$\partial K_n = \gamma_{1,r_n} \cup \cdots \cup \gamma_{N,r_n},$$

where $(r_n)_{n \in \mathbb{N}}$ is a sequence converging to infinity. This finishes the proof. $\square$

The next lemma gives conditions under which a translating solution L satisfying (H) admits a graphical decomposition.

Lemma 4.6. *Assume that for almost all $-C_1 < a < C_1$ we have*

$$\{\theta = a\} \subset B_{R(a)}$$

for some positive $R(a) > 0$. Then L admits a graphical decomposition.

Proof. For every α , there is only one Lagrangian plane P_α with Lagrangian angle α and $|e_1^\perp| = 0$ which is given by

$$(17) \quad P_\alpha = \{(u, 0, v \cos \alpha, v \sin \alpha) \mid (u, v) \in |x|^2\}.$$

Set ω_α to be the volume form of that plane extended by parallel translation to $\mathbb{C}^2$ and denote its Hodge-dual on L by $*\omega_\alpha$.

We claim that for every ε , there is δ_1 such that, for every x in L and every α , we have

$$(18) \quad \sup_{\widehat{B}_{r_0}(x)} |\theta - \alpha| \leq \delta_1 \implies \inf_{\widehat{B}_{r_0}(x)} *\omega_\alpha \geq \varepsilon.$$

A simple continuity argument shows the existence of δ_2 such that if Q is a Lagrangian plane with Lagrangian angle $\theta(Q)$, then for every α

$$|e_1^\perp| \leq \delta_2 \quad \text{and} \quad |\theta(Q) - \alpha| \leq \delta_2 \implies \omega_\alpha(Q) \geq \varepsilon,$$

where $\omega_\alpha(Q)$ denotes the evaluation of ω_α on Q . Moreover, as we argue next, we can find $\delta_1 \leq \delta_2$ such that for all x in L ,

$$\sup_{\widehat{B}_{r_0}(x)} |\theta - \theta(x)| \leq \delta_1/2 \implies \sup_{\widehat{B}_{r_0}(x)} |H| \leq \delta_2.$$

If not, we could find a sequence of translating solutions $(T_j)_{j \in \mathbb{N}}$ converging smoothly on compact sets to another translating solution T_∞ with Lagrangian angle constant on $\widehat{B}_{r_0}(0)$ and $|H|$ not identically zero on $\widehat{B}_{r_0}(0)$. This proves the desired claim.

Let $(x_i)_{i \in \mathbb{N}}$ be a sequence in L with $|x_i|$ going to infinity and $\theta(x_i)$ converging to some α . Set $\varepsilon = 1/2$ in identity (18) and choose $\delta < \delta_1/2$ so that

$$\{\theta = \alpha \pm \delta\} \subset B_{R(\alpha \pm \delta)}.$$

Set

$$R = \max\{R(\alpha + \delta), R(\alpha - \delta)\} + 2 \quad \text{and} \quad \Sigma = \{|\theta - \alpha| \leq \delta\}.$$

We have

$$\partial\Sigma \subseteq B_{R-2}(0),$$

and if x belongs to $\Sigma \setminus B_R(0)$, identity (18) implies that

$$\inf_{\widehat{B}_{r_0}(x)} *\omega_\alpha \geq 1/2.$$

Moreover, from [5, page 476], there exists a constant D such that

$$(19) \quad \inf_{x \in L, r \leq r_0} r^{-2} \mathcal{H}^2(\widehat{B}_r(x)) \geq D^{-1},$$

and so we can apply Lemma A.1 in order to conclude that, outside a compact set K ,

$$\Sigma \setminus K = L_1 \cup \cdots \cup L_{N_1},$$

where, for each $j = 1, \dots, N_1$,

$$\text{Proj}_{P_\alpha} : L_j \longrightarrow P_\alpha \setminus B_R(0)$$

is an n_j -fold covering map and, for every x in $\Sigma \setminus B_R(0)$, $\widehat{B}_{r_0}(x)$ can be written as the graph of a function defined over P_α with its derivatives uniformly bounded.

Take an unbounded sequence (y_i) in L_j such that $\theta(y_i)$ converges to some $\bar{\theta}_j$ and suppose there is another unbounded sequence p_i in L_j such that $\theta(p_i)$ converges to some β distinct from $\bar{\theta}_j$. Using the map Proj_{P_α} and the local graphical property, we can find, for every a between β and $\bar{\theta}_j$, an unbounded sequence w_i in L_j such

that $\theta(w_i) = a$. This contradicts our hypothesis and proves the third property of Definition 4.1. The first and second properties of Definition 4.1 follow because, by choosing $R_j > R$, we can replace α by $\bar{\theta}_j$ on each L_j .

We can repeat the whole process, but this time applied to $L \setminus \Sigma$. We only need to do this finitely many times because

$$\lim_{R \rightarrow \infty} \mathcal{H}^2(L_j \cap B_R(0))R^{-2} \geq C$$

for some universal constant C . $\square$

5. PROOF OF THE MAIN THEOREM

We now prove Theorem A. We start by showing

Theorem 5.1. *If L is a translating solution that satisfies (H), is almost-calibrated, and is static, then L is a plane.*

Proof. In view of Lemma 4.6 and Theorem 4.3, it suffices to show

Proposition 5.2. *Let L be a translating solution satisfying hypothesis (H), almost-calibrated, and static. Then for almost all $-\pi/2 < a < \pi/2$ we have*

$$\{\theta = a\} \subset B_{R(a)}$$

for some positive $R(a) > 0$.

Proof. The static condition implies the existence of a sequence of blow-downs

$$L_s^i = \lambda_i L_s / \lambda_i^2, \quad \text{where} \quad \lim_{i \rightarrow \infty} \lambda_i = 0,$$

which converges weakly to

$$(20) \quad L^\infty = m_1 P_1 + \cdots + m_N P_N$$

for every $s \in \mathbb{R}$, where $P_1, \dots, P_N$ are Lagrangian planes with multiplicities $m_1, \dots, m_N$ and Lagrangian angles $\bar{\theta}_1, \dots, \bar{\theta}_N$.

We claim that

$$P_j = \{(u, 0, v \cos \bar{\theta}_j, v \sin \bar{\theta}_j) \mid (u, v) \in |x|^2\}.$$

The reason is that the coordinate y_1 equals, up to a constant, $-\theta$ (Proposition 2.2 (i)) and so is bounded for every L_t . Thus each plane P_j must have $e_1^\perp = 0$ because, for every $R > 0$ and $s \leq 0$,

$$\lim_{i \rightarrow \infty} \sup\{y_1 \mid x \in L_s^i \cap B_R(0)\} = 0.$$

Hence the almost-calibrated condition implies that P_j is uniquely determined by its Lagrangian angle.

From Theorem 3.1 we know that the set of limiting Lagrangian angles does not depend on the sequence of rescalings chosen, and so any other convergent sequence of blow-downs $(\bar{L}_s^i)$ must converge to L^∞ for every s . This observation will be used later.

Sard's Theorem implies that, for almost all $-\pi/2 < a < \pi/2$, the set $\{\theta = a\}$ is a smooth submanifold of L (possibly empty). We argue that only finitely many curves contained in $\{\theta = a\}$ have finite length. Suppose that $(C_j)_{j \in \mathbb{N}}$ is a sequence of distinct curves contained in $\{\theta = a\}$ having finite length. Hypothesis (H) implies that, for some j_0 , we can find integers $b_1, \dots, b_{j_0}$ such that

$$b_1[C_1] + \cdots + b_{j_0}[C_{j_0}] = 0,$$

where $[C_j]$ denotes the homology class of C_j in $H_1(L, \mathbb{Z})$. Thus, there is a compact set $K \subseteq L$ such that

$$\partial K = b_1 C_1 + \cdots + b_{j_0} C_{j_0}.$$

From Proposition 2.2 (i) we know that θ cannot have any interior maximum or minimum, and therefore, because θ equals a on ∂K , θ must be constant on K . Analytic continuation implies that θ is constant on L , and this gives us a contradiction.

From (8) and (11) we can assume, without loss of generality, that after passing to a subsequence,

$$\lim_{i \rightarrow \infty} \int_{L_{-1}^i} |H|^2 \exp(-|x|^2) d\mu + \lim_{i \rightarrow \infty} \int_{L_1^i} |H|^2 \exp(-|x|^2) d\mu = 0.$$

Hence, from the coarea formula

$$\begin{aligned} & \int_{-\pi/2}^{\pi/2} \lambda_i \mathcal{H}^1(\{\theta = a\} \cap B_{\lambda_i^{-1}}(\lambda_i^{-2} e_1)) da \\ &= \int_{-\pi/2}^{\pi/2} \mathcal{H}^1\{x \in L_{-1}^i \cap B_1(0) \mid \theta_{-1}^i = a\} da = \int_{L_{-1}^i \cap B_1(0)} |H| d\mu \\ &\leq \left(\int_{L_{-1}^i \cap B_1(0)} |H|^2 d\mu \right)^{1/2} (\mathcal{H}^2(L_{-1}^i \cap B_1(0)))^{1/2}, \end{aligned}$$

and this implies that, for almost all $-\pi/2 < a < \pi/2$,

$$\lim_{i \rightarrow \infty} \lambda_i \mathcal{H}^1(\{\theta = a\} \cap B_{\lambda_i^{-1}}(\lambda_i^{-2} e_1)) = 0.$$

Likewise, we also obtain

$$\lim_{i \rightarrow \infty} \lambda_i \mathcal{H}^1(\{\theta = a\} \cap B_{\lambda_i^{-1}}(-\lambda_i^{-2} e_1)) = 0.$$

Choose a distinct from $\bar{\theta}_1, \dots, \bar{\theta}_N$ such that

$$\begin{aligned} (21) \quad & \lim_{i \rightarrow \infty} \lambda_i \mathcal{H}^1\left(\{\theta = a\} \cap B_{\lambda_i^{-1}}(\lambda_i^{-2} e_1)\right) \\ & + \lim_{i \rightarrow \infty} \lambda_i \mathcal{H}^1\left(\{\theta = a\} \cap B_{\lambda_i^{-1}}(-\lambda_i^{-2} e_1)\right) = 0 \end{aligned}$$

and such that $\{\theta = a\}$ is a smooth submanifold of L .

It suffices to show that there is no connected curve C with infinite length contained in $\{\theta = a\}$. If not, we could find an unbounded sequence (x_i) in C :

$$x_i = t_i e_1 + u_i, \quad \text{where } |x_i| = \lambda_i^{-2} \quad \text{and} \quad \langle u_i, e_1 \rangle = 0.$$

Lemma 5.3.

$$\liminf_{i \rightarrow \infty} |u_i|^2 |t_i|^{-1} > 0.$$

Proof. Suppose for some subsequence

$$\lim_{i \rightarrow \infty} |u_i|^2 t_i^{-1} = 0.$$

Then

$$\lim_{i \rightarrow \infty} |\lambda_i^{-2} - |t_i|| = \lim_{i \rightarrow \infty} |t_i| \left| \sqrt{1 + |u_i|^2 t_i^{-2}} - 1 \right| \leq \lim_{i \rightarrow \infty} |t_i|^{-1} |u_i|^2 = 0$$

and

$$\lim_{i \rightarrow \infty} \lambda_i^2 |x_i - t_i e_1|^2 = \lim_{i \rightarrow \infty} \lambda_i^2 |t_i| |u_i|^2 |t_i|^{-1} = \lim_{i \rightarrow \infty} \frac{|t_i|}{\sqrt{t_i^2 + |u_i|^2}} |u_i|^2 |t_i|^{-1} = 0.$$

Thus, for every i sufficiently large, x_i belongs to either

$$B_{\lambda_i^{-1}/2}(\lambda_i^{-2} e_1) \text{ or } B_{\lambda_i^{-1}/2}(-\lambda_i^{-2} e_1),$$

and so

$$\mathcal{H}^1(C \cap B_{\lambda_i^{-1}}(\pm \lambda_i^{-2} e_1)) \geq \pi \lambda_i^{-1}.$$

This contradicts identity (21). $\square$

After passing to a subsequence we can assume that

$$\lim_{i \rightarrow \infty} t_i |u_i|^{-2} = s_1.$$

Moreover, we also assume without loss of generality that

$$\bar{L}_s^i = |u_i|^{-1} L_{s|u_i|^2}, \quad \text{where } -\infty < s < \infty$$

converges for all s , the sequence of manifolds $L_i = L - x_i$ converges to a smooth translating solution L_∞ with $\theta_\infty(0) = a$, and $v_i = u_i |u_i|^{-1}$ converges to a vector v perpendicular to e_1 . The comments made at the beginning of this proof imply $\bar{L}_s^i$ converges to L^∞ (given in (20)) for every s .

For every β , $r > 0$ and $s < 0$,

$$\begin{aligned} \int_{L_\infty} (\theta_\infty - \beta)^2 \Phi_{0,r}(\cdot, 0) d\mu &= \lim_{i \rightarrow \infty} \int_L (\theta - \beta)^2 \Phi_{x_i, r}(\cdot, 0) d\mu \\ &= \lim_{i \rightarrow \infty} \int_{L - t_i} (\theta - \beta)^2 \Phi_{u_i, r - t_i}(\cdot, -t_i) \\ &\leq \lim_{i \rightarrow \infty} \int_{L - t_i + s|u_i|^2} (\theta - \beta)^2 \Phi_{u_i, r - t_i}(\cdot, -t_i + s|u_i|^2) \\ &= \lim_{i \rightarrow \infty} \int_{\bar{L}_s^i} (\theta - \beta)^2 \frac{\exp\left(-\frac{|x - v_i|^2}{4(r|u_i|^{-2} - s)}\right)}{4\pi(r|u_i|^{-2} - s)} d\mu \\ &= \int_{L_s^\infty} (\theta_{s-t_0}^\infty - \beta)^2 \frac{\exp\left(\frac{|x - v|^2}{4s}\right)}{-4\pi s} d\mu \\ &= \sum_{j=1}^N m_j (\bar{\theta}_j - \beta)^2 \int_{P_j} \frac{\exp\left(\frac{|x - v|^2}{4s}\right)}{-4\pi s} d\mu. \end{aligned}$$

If v did not belong to any P_j , we could make s go to zero to conclude that the leftmost-hand side of the inequalities above is zero for every β , which is impossible. The fact that v is perpendicular to e_1 implies that it can belong at most to one

P_{j_0} , and thus, making r go to zero and then s go to zero, we see that

$$(a - \beta)^2 \leq m_{j_0}(\bar{\theta}_{j_0} - \beta)^2$$

for every β . Therefore, $a = \bar{\theta}_{j_0}$, and this is a contradiction. $\square$

$\square$

The proof of Theorem A will be completed after we show

Theorem 5.4. *If L is a translating solution that satisfies hypothesis (H) and has*

$$\int_L |H|^2 d\mu \leq C_2$$

for some C_2 , then L is a plane.

Proof. In light of Lemma 4.6 and Theorem 4.3, it suffices to show that for almost all $-C_1 < a < C_1$ we have

$$\{\theta = a\} \subset B_{R(a)}$$

for some positive $R(a) > 0$. Moreover, we can argue as in Proposition 5.2 and see the fact that L has a finite first Betti number implies that it is sufficient to show that for almost all $-C_2 \leq s \leq C_2$ there is no curve contained in $\{\theta = s\}$ having infinite length.

Suppose that such a smooth curve exists and denote it by C . Then, for all r sufficiently large, $C \cap \{|x| = r\}$ is not empty, and so

$$(22) \quad \mathcal{H}^1(C \cap \{r \leq |x| \leq 2r\}) \geq r.$$

On the other hand, using the coarea formula, we have for all r ,

$$\begin{aligned} \int_{-C_1}^{C_1} \mathcal{H}^1(\{\theta = s\} \cap \{r \leq |x| \leq 2r\}) ds &= \int_{L \cap \{r \leq |x| \leq 2r\}} |H| d\mu \\ &\leq \left(\int_{L \cap \{r \leq |x| \leq 2r\}} |H|^2 d\mu \right)^{1/2} (\mathcal{H}^2(L \cap \{r \leq |x| \leq 2r\}))^{1/2} \\ &\leq \sqrt{C_1} r \left(\int_{L \cap \{r \leq |x| \leq 2r\}} |H|^2 d\mu \right)^{1/2}. \end{aligned}$$

Note that

$$\lim_{r \rightarrow \infty} \int_{L \cap \{r \leq |x| \leq 2r\}} |H|^2 d\mu = 0,$$

and thus, for almost all $-C_2 \leq s \leq C_2$, we can find a sequence r_i going to infinity such that

$$\lim_{i \rightarrow \infty} r_i^{-1} \mathcal{H}^1(\{\theta = s\} \cap \{r_i \leq |x| \leq 2r_i\}) = 0.$$

This contradicts (22). $\square$

APPENDIX A

Let τ be the volume form of a plane P extended by parallel translation to all of $\mathbb{C}^2$. We denote by Proj_P the projection onto the plane P and assume that P contains the line spanned by e_1 . Given any surface Σ in $\mathbb{C}^2$, we denote by $*\tau$ the Hodge-dual of τ , and if F denotes an immersion of Σ on $\mathbb{C}^2$, we also use Proj_P to represent $\text{Proj}_P \circ F$.

In what follows, Σ will be a complete non-compact surface with smooth boundary such that $\widehat{B}_{r_0}(x)$ is simply connected for all x in L ,

$$\sup_{R>0} R^{-2} \mathcal{H}^2(\Sigma \cap B_R(0)) \leq D \quad \text{and} \quad \inf_{x \in \Sigma, r \leq r_0} r^{-2} \mathcal{H}^2(\widehat{B}_r(x)) \geq D^{-1}$$

for some constant D .

Lemma A.1. *Assume that we can find $R > 0$ and $\varepsilon > 0$ such that*

$$\partial\Sigma \subset B_R(0)$$

and

$$\inf_{\widehat{B}_{r_0}(x)} *\tau \geq \varepsilon \quad \text{for all } x \in \Sigma \setminus B_R(0).$$

Outside a compact set, Σ can be decomposed into N connected components Σ_j , $j = 1, \dots, N$, having the following property.

- (i) For every x in $\Sigma \setminus B_R(0)$, $\widehat{B}_{r_0}(x)$ can be written as the graph of a function defined over P with its derivatives bounded by a constant depending only on ε .
- (ii)

$$\text{Proj}_P: \Sigma_j \longrightarrow P \setminus B_R(0)$$

is an n_j -fold covering map.

Proof. The first property is an immediate consequence of the fact that $\widehat{B}_{r_0}(x)$ is simply connected. Consider the map

$$\text{Proj}_P: \Sigma \setminus B_R(0) \longrightarrow P.$$

The local graphical property combined with the uniform lower bounds on area densities implies that $\text{Proj}_P^{-1}(B_R(0))$ is a compact subset of $\Sigma \setminus B_R(0)$.

Decompose $\text{Proj}_P^{-1}(P \setminus B_R(0))$ into connected components $(\Sigma_j)_{j \in \mathbb{N}}$. The local graphical property implies the existence of an integer n_j so that

$$\text{Proj}_P: \Sigma_j \longrightarrow P \setminus B_R(0)$$

is an n_j -fold covering map with

$$\text{Proj}_P(\partial\Sigma_j) \subseteq \partial(P \setminus B_R(0)).$$

Moreover, there is a constant $C = C(\varepsilon)$ such that

$$\lim_{R \rightarrow \infty} R^{-2} \mathcal{H}^2(\Sigma_j \cap B_R(0)) \geq C,$$

and so there can only exist finitely many connected components Σ_j . □

APPENDIX B

The next proposition was proven in [5, Appendix A] with a slightly different hypothesis. For that reason we will only indicate the modifications in the proof.

Proposition B.1. *Let (N^i) and (α_i) be a sequence of smooth Lagrangian surfaces in $|x|^4$ and smooth functions on N^i , respectively, such that (N^i) converges as Radon measure and as currents to a union of planes with positive integer multiplicities N . We assume that, for some $R > 0$, the following properties hold:*

a) *There exists a constant D_0 such that*

$$\mathcal{H}^2(N^i \cap B_{3R}) \leq D_0 R^2$$

and

$$\cos \theta^i \geq D_0^{-1}$$

for all $i \in \mathbb{N}$.

b)

$$\lim_{i \rightarrow \infty} \int_{N^i \cap B_{3R}(0)} |\nabla \alpha_i|^2 d\mu = 0.$$

c) *There exists a constant D_1 for which*

$$\sup_{N^i \cap B_{3R}(0)} |\nabla \alpha_i| + R^{-1} \sup_{N^i \cap B_{3R}(0)} |\alpha_i| \leq D_1$$

for all $i \in \mathbb{N}$.

d) *For all $i \in \mathbb{N}$,*

$$N^i \cap B_{2R}(0) \quad \text{is connected}$$

and

$$\partial(N^i \cap B_{3R}(0)) \subset \partial B_{3R}(0).$$

Then, there is a real number α such that, after passing to a subsequence, we have for all ϕ with compact support in $B_R(0)$ and all f in $C(\mathbb{R})$,

$$\lim_{i \rightarrow \infty} \int_{N^i} f(\alpha_i) \phi d\mu = f(\alpha) \mu_N(\phi),$$

where μ_N denotes the Radon measure associated to N .

Proof. It suffices to find $\alpha \in \mathbb{R}$ and a sequence (ε_j) converging to zero such that, for some appropriate subsequence, we have for all $j \in \mathbb{N}$

$$\lim_{i \rightarrow \infty} \mathcal{H}^2(\{|\alpha_i - \alpha| \leq \varepsilon_j\} \cap B_R(0)) = \mathcal{H}^2(N \cap B_R(0)).$$

For the rest of this proof, $K = K(D_0, D_1, k)$ will denote a generic constant depending only on the mentioned quantities. Choose any sequence (x_i) in $N^i \cap B_R(0)$. After passing to a subsequence, we have that

$$\lim_{i \rightarrow \infty} x_i = x_0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \alpha_i(x_i) = \alpha$$

for some $x_0 \in B_R(0)$ and $\alpha \in \mathbb{R}$. Furthermore, also consider a sequence (ε_j) converging to zero such that, for all $j \in \mathbb{N}$,

$$\lim_{i \rightarrow \infty} \mathcal{H}^1(\{\alpha_i = \alpha \pm \varepsilon_j\} \cap B_{3R}) = 0.$$

Such a subsequence exists because, by the coarea formula, we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{-\infty}^{\infty} \mathcal{H}^1(\{\alpha_i = s\} \cap B_{3R}) ds &= \lim_{i \rightarrow \infty} \int_{N^i \cap B_{3R}} |\nabla \alpha_i| d\mu \\ &\leq \lim_{i \rightarrow \infty} KR \left(\int_{N^i \cap B_{3R}} |\nabla \alpha_i|^2 d\mu \right)^{1/2} = 0. \end{aligned}$$

Define

$$N^{i,\alpha,j} = \{|\alpha_i - \alpha| \leq \varepsilon_j\} \cap B_{3R}.$$

The Federer and Fleming Compactness Theorem implies that, after passing to a subsequence, we have convergence to an integral current $\bar{N}^{\alpha,j}$ with no boundary on B_{3R} . Its support is contained in a union of planes, and so the Constancy Theorem [7, Theorem 26.27] implies that $N^{\alpha,j}$ is a union of planes (with multiplicities) intersected with B_{3R} .

The almost-calibrated condition implies that for every open set B ,

$$D_0^{-1} \mathcal{H}^2(N^{i,\alpha,j} \cap B) \leq \int_{N^{i,\alpha,j} \cap B} \text{Re } \Omega \leq \mathcal{H}^2(N^{i,\alpha,j} \cap B).$$

Moreover, N^i being almost-calibrated implies the existence (see [5, Lemma 7.1] of some constant D such that

$$(\mathcal{H}^2(A))^{1/2} \leq D \mathcal{H}^1(\partial A),$$

where A is any open subset of N with rectifiable boundary. Hence, [5, Lemma A.2] applies with no modification, and we conclude the existence of K such that for every $j \in \mathbb{N}$,

$$\mathcal{H}^2(N^{\alpha,j} \cap B_R(x_0)) \geq KR^2.$$

Suppose that for some positive integer j we have

$$\mathcal{H}^k(N^{\alpha,j} \cap B_R(0)) < \mathcal{H}^k(N \cap B_R(0)).$$

Repeating the same type of argument, we can find y_0 in $B_R(0)$ and a closed interval I disjoint from $[\alpha - \varepsilon_j, \alpha + \varepsilon_j]$ so that, after passing to a subsequence,

$$\lim_{i \rightarrow \infty} \mathcal{H}^2(\alpha_i^{-1}(I) \cap B_R(y_0)) \geq KR^2.$$

Given any positive integer p , pick disjoint closed intervals

$$I_1, \dots, I_p$$

lying between I and $[\alpha - \varepsilon_j, \alpha + \varepsilon_j]$. The connectedness of $N^i \cap B_{2R}(0)$ implies that all $\alpha_i^{-1}(I_l) \cap B_{2R}(0)$ are non-empty for i sufficiently large. Hence, arguing as before, we find $y_1, \dots, y_p$ in $B_{2R}(0)$ such that, after passing to a subsequence,

$$\lim_{i \rightarrow \infty} \mathcal{H}^k(\alpha_i^{-1}(I_l) \cap B_R(y_l)) \geq KR^k,$$

for all l in $\{1, \dots, p\}$. This implies that

$$\begin{aligned} \lim_{i \rightarrow \infty} \mathcal{H}^2(N^i \cap B_{2R}(0)) &\geq \lim_{i \rightarrow \infty} \sum_{l=1}^p \mathcal{H}^2(\alpha_i^{-1}(I_l) \cap B_R(y_l)) \\ &\geq pKR^2. \end{aligned}$$

Choosing p sufficiently large, we get a contradiction. $\square$

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