SHORT-TIME EXISTENCE OF THE RICCI FLOW ON NONCOMPACT RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we give the first detailed proof of the short-time existence of Deane Yang’s local Ricci flow. Then, using the local Ricci flow, we prove the short-time existence of the Ricci flow on noncompact manifolds, whose Ricci curvature has global lower bound and sectional curvature has only local average integral bound. The short-time existence of the Ricci flow on noncompact manifolds with bounded curvature was studied by Wan-Xiong Shi in the 1990s. As a corollary of our main theorem, we obtain the short-time existence part of Shi’s theorem in this more general context.

1. INTRODUCTION

In his well-known paper [5], R. Hamilton introduced the Ricci flow on compact Riemannian manifolds which has proved to be very useful in the research of differential geometry and topology. Let us recall the definition of the Ricci flow: it is the solution of the evolution equation deforming the metric on any n -dimensional Riemannian manifold (M^n, g_{ij}) ,

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij}(x, t) &= -2R_{ij}(x, t), \\ g(x, 0) &= g_0(x), \end{cases}$$

where R_{ij} is the Ricci curvature of M^n at time t .

The first thing we mention in the study of the Ricci flow on Riemannian manifolds is the short-time existence of the solution. In the case where M^n is compact, it is well known that for any given initial metric g_{ij} on M^n , the evolution equation (1.1) always has a unique solution for a short time (see [4], [5]).

To prove Yau’s Uniformization Conjecture, W.-X. Shi initiated studying the Ricci flow on complete noncompact Riemannian manifolds. The short-time existence problem of the evolution equation (1.1) is more difficult than the compact case. In [12], Shi proved that if (M^n, g_{ij}) is complete noncompact with bounded curvature, then the Ricci flow (1.1) has a solution with bounded curvature on a short-time interval. Note that Shi’s theorem requires that the initial manifold (M^n, g_0) has the bounded geometry, i.e. the curvature of M^n must have a point-wise estimate.

In [15], Deane Yang introduced the local Ricci flow. We recall the definition of local Ricci flow first. Let M^n be a smooth n -dimensional manifold with Riemannian metric g_0 , Ω an open bounded domain of M^n and $n \geq 3$. Let χ be a nonnegative

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smooth compactly supported function on Ω , and $0 \leq \chi \leq 1$. The local Ricci flow is the solution of the following evolution equation:

$$(1.2) \quad \frac{\partial g}{\partial t} = -2\chi^2 Rc(g), \quad g(0) = g_0, \quad \chi \in C_0^\infty(\Omega).$$

In this paper, using the local Ricci flow, we prove

Theorem 1.1. *Assume (M^n, g_0) is an n -dimensional ($n \geq 3$) complete noncompact Riemannian manifold, satisfying the following conditions:*

$$(1.3) \quad \begin{cases} \left(\int_{B_x(4r_0)} h^{\frac{2n}{n-2}} dV_{g_0} \right)^{\frac{n-2}{n}} \leq A_0 r_0^2 \int_{B_x(4r_0)} |\nabla h|^2 dV_{g_0}, \\ \left(\int_{B_x(r_0)} |Rm(g_0)|^{p_0} dV_{g_0} \right)^{\frac{1}{p_0}} \leq K_1 \end{cases}$$

for any $x \in M$ and $h \in C_0^\infty(B_x(4r_0))$, where $A_0 \geq 1$, $p_0 > \frac{n}{2}$, K_1 and r_0 are positive constants.

Then the Ricci flow (1.1) has a smooth solution $g_{ij}(x, t)$ on $[0, \frac{T_2}{2}]$ (where T_2 is defined in (5.25)) and satisfies the following estimates. For any integer $m \geq 0$, there exists a positive constant $C(A_0, K_1, m, n, p_0, r_0)$ depending only on A_0 , K_1 , m , n , p_0 and r_0 , such that

$$(1.4) \quad \sup_{x \in M} |\nabla^m Rm(x, t)| \leq \frac{C(A_0, K_1, m, n, p_0, r_0)}{t^{\frac{m}{2}}} (t^{-\frac{n}{2p_0}} + t^{\frac{1}{p_0}}), \quad 0 < t \leq \frac{T_2}{2}.$$

In particular,

$$(1.5) \quad \sup_{x \in M} |Rm(x, t)| \leq C_2 (t^{-\frac{n}{2p_0}} + C_1^{\frac{n+2}{2p_0}} t^{\frac{1}{p_0}}), \quad 0 < t \leq \frac{T_2}{2},$$

where C_1 and C_2 are defined in (5.9) and (5.16).

Theorem 1.1 requires only the integral conditions on the curvature tensor of the initial manifold and a local Sobolev inequality. By [10], local Sobolev inequality is valid on Riemannian manifolds whose Ricci curvature has a lower bound. Hence we can generalize Shi's theorem in more general context in the following corollary.

Corollary 1.2. *Assume (M, g_0) is an n -dimensional ($n \geq 3$) complete noncompact Riemannian manifold, satisfying the following conditions:*

$$(1.6) \quad \begin{cases} Rc(g_0) \geq -K g_0, \\ \left(\int_{B_x(r_0)} |Rm(g_0)|^{p_0} dV_{g_0} \right)^{\frac{1}{p_0}} \leq K_1 \end{cases}$$

for any $x \in M$, where $p_0 > \frac{n}{2}$, r_0 , K and K_1 are positive constants.

Then the Ricci flow (1.1) has a smooth solution $g_{ij}(x, t)$ on $[0, T]$, and satisfies the following estimates. For any integer $m \geq 0$, there exists the positive constant $C(K, K_1, m, n, p_0, r_0)$ such that

$$\sup_{x \in M} |\nabla^m Rm(x, t)| \leq \frac{C(K, K_1, m, n, p_0, r_0)}{t^{\frac{m}{2}}} (t^{-\frac{n}{2p_0}} + t^{\frac{1}{p_0}}), \quad 0 < t \leq T,$$

where $T = T(K, K_1, n, p_0, r_0)$ is a positive constant depending only on K , K_1 , n , p_0 and r_0 .

In section 2, we give the detailed proof of short-time existence of the local Ricci flow following the brief sketch in [15]. In the sketch proof of Theorem 8.2 of [15],

Yang considered a similar but different differential system (see the top of page 91 of [15]) comparing (2.4), and the main difference is that our principal coefficients in (2.4) are $(\chi^2 + \epsilon)u^{\alpha\beta}$ instead of $(\chi^2 + \epsilon)v^{\alpha\beta}$. This is the key point as to why we can get the C^0 a-priori estimate of v_{ij} (see (2.11), etc.) and the higher order estimate further on in Proposition 2.5.

More concretely, for (2.4) we prove that $\Phi_\epsilon : U \rightarrow V$ is a contraction map on $[0, T]$, where T is independent of ϵ . Then we use the fixed point method to get a solution of (2.4) on $[0, T]$. Finally, letting $\epsilon \rightarrow 0$, we get the solution of (2.1) on $[0, T]$, and by regularity of parabolic system, the solution is smooth. All the estimates are about metric tensors, and the norms we considered are with respect to the initial metric g_0 of manifold M^n .

In spirit, our proof is close to the original proof of the Ricci flow's short-time existence on compact manifolds by R. Hamilton. In [5], R. Hamilton used the Nash-Moser inverse function theorem; in this paper we use the fixed-point argument in a Sobolev-type inequality.

In section 3, we discuss the evolution equations and inequalities of curvature tensors under local Ricci flow. In section 4, the extension of local Ricci flow is discussed. The main theorem (Theorem 4.8) of this section was proved in [15]; recently the complete proof was given by Wang, among other things (see [14]). The proof is nontrivial and different from the proof of the Ricci flow's extension under the bound of curvature. In the proof, the interplay among the estimates of $\nabla^i \chi$, $\nabla^j Rm$ and $\nabla^k (\chi^2 Rm)$ are interesting and subtle. Following the proof of Wang, we prove this extension theorem again to make this paper self-contained. Note all the estimates in sections 2, 3 and 4 are valid for any smooth initial manifold (M^n, g_0) .

After establishing the short-time existence of the local Ricci flow and its extension theorem, we try to get the existence time estimate of the local Ricci flow independent of the domain Ω . This is done in section 5 and section 6. Note that the time estimate which we get in the proof of the short-time existence of the local Ricci flow depends on Ω . To get the time estimate independent of Ω , we need to add some assumptions on the initial manifold (M^n, g_0) .

The key point of section 5 is: If we assume the initial local $L^p(p > \frac{n}{2})$ average integral bound of curvature tensor and local average Sobolev inequality, we can get the parabolic version of the "local local" energy estimate of curvature tensor, where the coefficient of $\int \xi_i^{2q-2} \chi^{2p'} |Rm|^p$ is independent of Ω . Here "local local" means that we use two cut-off functions with curvature tensor. One is ξ_i , which is from the partition of unity of the good cover mentioned before, and the other is χ , which is the cut-off function in the local Ricci flow.

Then using the modified Moser iteration, we can get the point-wise estimate of $\chi^2 Rm$ on $[0, T]$, where T is independent of Ω . By the local Ricci flow's extension theorem proved in section 4, we get our existence-time estimate independent of Ω . The precise-time estimate we get is expressed in the explicit form of curvature bound, diameter of balls in good cover and local Sobolev constant, and this explicit estimate has independent interest and application (see Remarks 5.15 and 7.1). Another reason we get a time estimate in such an explicit expression (with A_0, r_0, K_1 , see (5.21) and (5.24), etc.) is that we are interested in the $Rc \geq 0$ case, where we can allow $r_0 \rightarrow \infty$ if there is a suitable initial curvature integral bound on $B(r_0)$ as $r_0 \rightarrow \infty$.

In many situations, critical metrics are interesting, and we have only the $L^{\frac{n}{2}}$ bound of Rm . In section 6, using different energy estimates we get a similar

existence-time estimate of the local Ricci flow on manifolds with a small $L^{\frac{n}{2}}$ bound of Rm , an $L^p(p > \frac{n}{2})$ bound of Rc , local Sobolev inequality and volume doubling property.

In section 7, using a family of local Ricci flows with respect to a family of expanding domains $\{\Omega_i\}_{i=1}^\infty$ and the curvature estimates we get in sections 5 and 6, we can apply compactness results in Ricci flow to show that a subsequence of this family of local Ricci flows converges to the Ricci flow on the whole manifold when $i \rightarrow \infty$.

Finally, it is worth mentioning that our arguments in sections 5 and 6 also apply to compact manifolds. On compact manifolds, we can choose $\chi \equiv 1$ in (1.2); then local Ricci flow is just Ricci flow. Note the minimal existence time dependent on curvature bound is well known (see Corollary 7.7 in [1]), and the minimal existence time we get depends on the integral bound of curvature, which has its own interest.

2. SHORT-TIME EXISTENCE OF THE LOCAL RICCI FLOW

To prove the short-time existence of the local Ricci flow, we follow the brief sketch in the paper of Deane Yang [15]. By DeTurck's trick, we can equivalently consider the following local Ricci-DeTurck flow:

$$(2.1) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij} &= -2\chi^2 R_{ij} + \nabla_i(\chi^2 \mathscr{W})_j + \nabla_j(\chi^2 \mathscr{W})_i, \\ \frac{\partial}{\partial t} \phi_t(p) &= -\chi^2(p) \mathscr{W}(\phi_t(p), t), \quad p \in M, \\ \phi_0 &= Id_M, \quad g_{ij}(x, 0) = \tilde{g}_{ij}(x) \triangleq (g_0)_{ij}(x), \end{cases}$$

where $\mathscr{W}_j = g_{jk} g^{pq} (\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k)$. The second equation in (2.1) is a quasilinear ODE, and the solution on $M \setminus \Omega$ is $Id_{M \setminus \Omega}$, so one only need consider the first family of equations in (2.1).

Lemma 2.1. *The following modified evolution equations (2.2) is a degenerate parabolic system:*

$$(2.2) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij} &= -2\chi^2 R_{ij} + \nabla_i(\chi^2 \mathscr{W})_j + \nabla_j(\chi^2 \mathscr{W})_i, \\ g_{ij}(x, 0) &= \tilde{g}_{ij}(x). \end{cases}$$

Proof. On $M \setminus \Omega$, $g_{ij}(x, t) \equiv \tilde{g}_{ij}(x)$, one only needs to consider the evolution equations on Ω . Calculating (2.2) directly, using Lemma 2.1 in [12], we get

$$(2.3) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij} &= \chi^2 g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta g_{ij} - \chi^2 (g^{\alpha\beta} g_{ip} \tilde{g}^{pq} \tilde{R}_{j\alpha q\beta} + g^{\alpha\beta} g_{jp} \tilde{g}^{pq} \tilde{R}_{i\alpha q\beta}) \\ &\quad + \frac{1}{2} \chi^2 g^{\alpha\beta} g^{pq} (\tilde{\nabla}_i g_{p\alpha} \tilde{\nabla}_j g_{q\beta} + 2 \tilde{\nabla}_\alpha g_{jp} \tilde{\nabla}_q g_{i\beta} - 2 \tilde{\nabla}_\alpha g_{jp} \tilde{\nabla}_\beta g_{iq} \\ &\quad - 2 \tilde{\nabla}_j g_{p\alpha} \tilde{\nabla}_\beta g_{iq} - 2 \tilde{\nabla}_i g_{p\alpha} \tilde{\nabla}_\beta g_{jq}) \\ &\quad + \chi \chi_i g^{pq} (\tilde{\nabla}_p g_{jq} + \tilde{\nabla}_q g_{pj} - \tilde{\nabla}_j g_{pq}) + \chi \chi_j g^{pq} (\tilde{\nabla}_p g_{iq} + \tilde{\nabla}_q g_{pi} - \tilde{\nabla}_i g_{pq}), \\ g_{ij}(x, 0) &= \tilde{g}_{ij}(x), \quad x \in \Omega, \\ g_{ij}(x, t) &= \tilde{g}_{ij}(x), \quad x \in \partial\Omega, \quad t \in [0, T], \end{cases}$$

where $\chi_i = \tilde{\nabla}_i \chi$. Note that the principal coefficients of (2.3) are $(\chi^2 g^{\alpha\beta})$, and it is a degenerate parabolic system. $\square$

We define the following tensor-valued function spaces and norms:

Definition 2.2.

$$\begin{aligned} \mathfrak{A}_T &= \{U \mid \frac{1}{2}\tilde{g}_{ij} \leq u_{ij} \leq 2\tilde{g}_{ij}, \ U(x, 0) = \tilde{g}(x), x \in \Omega, \\ U(x, t) &= \tilde{g}, x \in \partial\Omega, \ U \in L^\infty(\bar{\Omega} \times [0, T], \otimes^2 T^*(\bar{\Omega}))\}, \\ |U(x, t)|_k &= \left(\sum_{|\alpha| \leq k} \int_{\Omega} |\tilde{\nabla}^\alpha U(x, t)|^2 d\mu_{\tilde{g}} \right)^{\frac{1}{2}}, \quad |U(x, t)|_{T, k} = \sup_{t \in [0, T]} (|U(x, t)|_k), \\ \|f\|_k &\triangleq \sum_{|\alpha| \leq k} |\tilde{\nabla}^\alpha f|_{L^\infty(\Omega)}, \quad k = 0, 1, 2, \dots \end{aligned}$$

where $U : (\bar{\Omega} \times [0, T] \rightarrow \otimes^2 T^*(\bar{\Omega}))$, f is a tensor field or function defined on Ω and $|\nabla^\alpha U(x, t)|^2 = \tilde{g}^{i_1 j_1} \dots \tilde{g}^{i_s j_s} \tilde{g}^{p_k} \tilde{g}^{q_l} (\tilde{\nabla}_{i_1} \dots \tilde{\nabla}_{i_s} u_{pq}) (\tilde{\nabla}_{j_1} \dots \tilde{\nabla}_{j_s} u_{kl})$, where $|\alpha| = s \geq 0$. Also,

$$\mathbb{B}_m(K)_T = \{U \mid |U|_{T, m} \leq K; U \in L^\infty(\bar{\Omega} \times [0, T], \otimes^2 T^*(\bar{\Omega}))\},$$

where $m \geq 0$, $K > 0$.

We consider the following strictly parabolic system by adding ϵ to χ^2 in the principal coefficients, and $\epsilon \leq 1$ is some positive constant:

$$(2.4) \quad \begin{cases} \frac{\partial}{\partial t} v_{ij} &= (\chi^2 + \epsilon) u^{\alpha\beta} (v_{ij})_{\alpha\beta} + D_{ij}(U, V) + B_{ij}(U, \tilde{\nabla}U), \\ v_{ij}(x, 0) &= \tilde{g}_{ij}(x), \quad x \in \Omega, \\ v_{ij}(x, t) &= \tilde{g}_{ij}(x), \quad x \in \partial\Omega, \ t \in [0, T], \end{cases}$$

where $U = (u_{ij})_{i,j=1}^n$, $V = (v_{ij})_{i,j=1}^n$, $(v_{ij})_{\alpha\beta} = \tilde{\nabla}_\alpha \tilde{\nabla}_\beta (v_{ij})$,

$$D_{ij}(U, V) = -\chi^2 [(u^{\alpha\beta} \tilde{g}^{pq} \tilde{R}_{j\alpha q\beta}) v_{ip} + (u^{\alpha\beta} \tilde{g}^{pq} \tilde{R}_{i\alpha q\beta}) v_{jp}],$$

$$B_{ij}(U, \tilde{\nabla}U) = E_{ij} + F_{ij},$$

$$\begin{aligned} E_{ij} &= \frac{1}{2} \chi^2 u^{\alpha\beta} u^{pq} (\tilde{\nabla}_i u_{p\alpha} \tilde{\nabla}_j u_{q\beta} + 2 \tilde{\nabla}_\alpha u_{jp} \tilde{\nabla}_q u_{i\beta} - 2 \tilde{\nabla}_\alpha u_{jp} \tilde{\nabla}_\beta u_{iq} \\ &\quad - 2 \tilde{\nabla}_j u_{p\alpha} \tilde{\nabla}_\beta u_{iq} - 2 \tilde{\nabla}_i u_{p\alpha} \tilde{\nabla}_\beta u_{jq}), \end{aligned}$$

$$F_{ij} = \chi \chi_i u^{pq} (\tilde{\nabla}_p u_{jq} + \tilde{\nabla}_q u_{pj} - \tilde{\nabla}_j u_{pq}) + \chi \chi_j u^{pq} (\tilde{\nabla}_p u_{iq} + \tilde{\nabla}_q u_{pi} - \tilde{\nabla}_i u_{pq}).$$

We define $\Phi_\epsilon(U) = V$ if V is the solution of (2.4). Also, $A_\epsilon^{\alpha\beta} \triangleq (\chi^2 + \epsilon) u^{\alpha\beta}$. Then

$$(2.5) \quad \frac{\partial}{\partial t} v_{ij} = A_\epsilon^{\alpha\beta} (v_{ij})_{\alpha\beta} + D_{ij}(U, V) + B_{ij}(U, \tilde{\nabla}U).$$

Remark 2.3. In (2.5), there is no ∇V term, and this is one of the reasons we can get a C^0 -estimate of V in Proposition 2.5.

We first prove a technical lemma about the norm of U^{-1} using interpolation inequalities. We define the Sobolev constant $C_s(\Omega, g(t))$ for $\Omega \subset (M, g(t))$ as the following:

$$(2.6) \quad |h|_{C^0(\Omega)} \leq C_s(\Omega, g(t)) \left(\int_{\Omega} |\nabla h|_{g(t)}^2 dV_{g(t)} \right)^{\frac{1}{2}}, \quad \forall h \in C_0^\infty(\Omega).$$

In this section we use the notation $C_s(\Omega, g_0) \triangleq C_s(\Omega, \tilde{g})$ because the covariant derivatives, tensor norm considered in this section are all with respect to $\tilde{g}$.

Choose $G = 2\sqrt{n}V_{\tilde{g}}(\Omega)^{\frac{1}{2}}$ and some fixed $\tilde{p} > \min\{\frac{n}{2}, 3\}$ in the rest of this section. Note $|\tilde{g}|_m = \sqrt{n}V_{\tilde{g}}(\Omega)^{\frac{1}{2}}$ for any $m \geq 0$.

Lemma 2.4. *Assume $U \in \mathfrak{A}_T$ and $|U^{-1}(x, t)|_{L^\infty} \leq C_0$ for some $t \in [0, T]$. Then*

$$|U^{-1}(x, t)|_{k+1} \leq C(k, C_0, C_s(\Omega, g_0))(|U(x, t)|_{\tilde{p}+1} + 1)^{k+1}(|U(x, t)|_{k+1} + 1),$$

where $k = 0, 1, 2, \dots$.

Proof.

$$\begin{aligned} |\tilde{\nabla}^{k+1}U^{-1}|_{L^2} &= |\tilde{\nabla}^k(U^{-1}U^{-1}\tilde{\nabla}U)|_{L^2} = |\tilde{\nabla}^k(U^{-2}\tilde{\nabla}U)|_{L^2} \\ &\leq C(k)((|U^{-2}|_{L^\infty} + 1)|\tilde{\nabla}U|_k + |U^{-2}|_k(|\tilde{\nabla}U|_{L^\infty} + 1)) \\ &\leq C(k, C_0)|\tilde{\nabla}U|_k + C(k, C_0, C_s(\Omega, g_0))|U^{-1}|_k(|U^{-1}|_{L^\infty} + 1)(|U|_{\tilde{p}+1} + 1) \\ &\leq C(k, C_0, C_s(\Omega, g_0))|U|_{k+1} + C(k, C_0, C_s(\Omega, g_0))|U^{-1}|_k(|U|_{\tilde{p}+1} + 1). \end{aligned}$$

Let $a_k = |U^{-1}|_k$, $c_k = |U|_k$, $b = (|U|_{\tilde{p}+1} + 1)$. Then

$$a_{k+1} \leq C(k, C_0, C_s(\Omega, g_0))c_{k+1} + C(k, C_0, C_s(\Omega, g_0))ba_k.$$

By induction, it is easy to get

$$a_{k+1} \leq C(k, C_0, C_s(\Omega, g_0))(b+1)^{k+1}(c_{k+1} + 1).$$

Hence

$$|U^{-1}|_{k+1} \leq C(k, C_0, C_s(\Omega, g_0))(|U|_{\tilde{p}+1} + 1)^{k+1}(|U|_{k+1} + 1). \quad \square$$

Proposition 2.5. *There exist some $t_3 = t_3(n, \tilde{p}, G, \|\chi\|_{\tilde{p}+2}, \|\widehat{Rm}\|_{\tilde{p}+2}, C_s(\Omega, g_0)) > 0$, such that $\Phi_\epsilon : \mathfrak{A}_{t_3} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_3} \rightarrow \mathfrak{A}_{t_3} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_3}$.*

Proof. Choose $U \in \mathfrak{A}_{t_1} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_1}$, $t_1 > 0$, to be determined later; we first show that $V \in \mathfrak{A}_{t_1}$ if $V = \Phi_\epsilon(U)$.

Choose a positive integer m such that $2^m > 2n$, and choose a normal coordinate system $\{x^i\}$, such that at one point

$$\begin{aligned} \{\tilde{g}_{ij}\} &= \begin{pmatrix} 1 & & \mathbf{0} \\ & 1 & \\ & & \ddots \\ \mathbf{0} & & & 1 \end{pmatrix}, \\ \{v_{ij}\} &= \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \lambda_2 & \\ & & \ddots \\ \mathbf{0} & & & \lambda_n \end{pmatrix}, \end{aligned}$$

then

$$\{v^{ij}\} = \begin{pmatrix} \lambda_1^{-1} & & \mathbf{0} \\ & \lambda_2^{-1} & \\ & & \ddots \\ \mathbf{0} & & & \lambda_n^{-1} \end{pmatrix}.$$

Now one can define

$$\phi_m \triangleq v^{\alpha_1\beta_1}\tilde{g}_{\beta_1\alpha_2}\cdots v^{\alpha_m\beta_m}\tilde{g}_{\beta_m\alpha_1} = \sum_{i=1}^n \lambda_i^{-m}.$$

By (2.5) and $\frac{\partial}{\partial t} v^{ij} = -v^{ik} v^{jl} \frac{\partial}{\partial t} v_{kl}$,

$$(2.7) \quad \frac{\partial}{\partial t} v^{ij} = -v^{ik} v^{jl} [A_\epsilon^{\alpha\beta}(v_{kl})_{\alpha\beta} + D_{kl}(U, V) + B_{kl}(U, \tilde{\nabla} U)].$$

By (61) on page 236 of [12],

$$(2.8) \quad (v^{ij})_{\alpha\beta} = \tilde{\nabla}_\alpha \tilde{\nabla}_\beta (v^{ij}) = -v^{ik} v^{jl} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta (v_{kl}) - \tilde{\nabla}_\alpha (v^{ik} v^{jl}) \tilde{\nabla}_\beta v_{kl}.$$

By (2.7) and (2.8),

$$(2.9) \quad \begin{aligned} \frac{\partial}{\partial t} v^{ij} &= A_\epsilon^{\alpha\beta} (v^{ij})_{\alpha\beta} + A_\epsilon^{\alpha\beta} (v^{ik} v^{jl})_\alpha (v_{kl})_\beta - v^{ik} v^{jl} [D_{kl}(U, V) + B_{kl}(U, \tilde{\nabla} U)] \\ (\phi_m)_t &= v^{\alpha_1 \beta_1} \dots \left(\frac{\partial}{\partial t} v^{\alpha_k \beta_k} \right) \dots \tilde{g}_{\beta_m \alpha_1} \\ &= v^{\alpha_1 \beta_1} \dots \left[A_\epsilon^{\alpha\beta} (v^{\alpha_k \beta_k})_{\alpha\beta} + A_\epsilon^{\alpha\beta} (v^{\alpha_k p} v^{\beta_k q})_\alpha (v_{pq})_\beta \right. \\ &\quad \left. - v^{\alpha_k p} v^{\beta_k q} [D_{pq}(U, V) + B_{pq}(U, \tilde{\nabla} U)] \right] \dots \tilde{g}_{\beta_m \alpha_1} \\ &= A_\epsilon^{\alpha\beta} (\phi_m)_{\alpha\beta} + 2\lambda_i^{-m} \chi^2 u^{\alpha\beta} R_{i\alpha i \beta} - \lambda_i^{-m-1} B_{ii} - 2A_\epsilon^{\alpha\beta} \lambda_p^{-m-1} \lambda_q^{-1} (v_{pq})_\alpha (v_{pq})_\beta \\ &\quad - 2 \sum_{k,l} \sum_{p < q} A_\epsilon^{\alpha\beta} (\lambda_k^{-1})^{m+p-q-3} (\lambda_l^{-1})^{q-p-3} (v_{kl})_\alpha (v_{kl})_\beta \end{aligned}$$

and

$$(2.10) \quad \begin{aligned} &-2 \sum_{k,l} \sum_{p < q} A_\epsilon^{\alpha\beta} (\lambda_k^{-1})^{m+p-q-3} (\lambda_l^{-1})^{q-p-3} (v_{kl})_\alpha (v_{kl})_\beta \\ &= -m \sum_{k,l} A_\epsilon^{\alpha\beta} \sum_{\sigma=-2}^{m-4} (\lambda_k^{-1})^{m-\sigma-6} (\lambda_l^{-1})^\sigma (v_{kl})_\alpha (v_{kl})_\beta. \end{aligned}$$

By (2.9) and (2.10),

$$\begin{aligned} (\phi_m)_t &= (\chi^2 + \epsilon) (u^{\alpha\beta}) (\phi_m)_{\alpha\beta} - 2(\lambda_p^{-1})^{m+1} \lambda_q^{-1} (\chi^2 + \epsilon) u^{\alpha\beta} (v_{pq})_\alpha (v_{pq})_\beta \\ &\quad - m(\chi^2 + \epsilon) \sum_{\sigma=-2}^{m-4} u^{\alpha\beta} (v_{kl})_\alpha (v_{kl})_\beta (\lambda_k^{-1})^{m-\sigma-6} (\lambda_l^{-1})^\sigma \\ &\quad + 2\lambda_i^{-m} \chi^2 u^{\alpha\beta} \tilde{R}_{i\alpha i \beta} - \lambda_i^{-m-1} B_{ii}; \end{aligned}$$

then

$$(2.11) \quad (\phi_m)_t \leq (\chi^2 + \epsilon) u^{\alpha\beta} (\phi_m)_{\alpha\beta} + C(n) \chi^2 |\widetilde{Rm}| \phi_m - \lambda_i^{-m-1} B_{ii}.$$

Because $U \in \mathfrak{A}_{t_1} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_1}$, by definition of B_{ij} and (2.6),

$$(2.12) \quad -\lambda_i^{-m-1} B_{ii} \leq C(n, \|\chi\|_1, C_s(\Omega, g_0)) \cdot (G+1)^2 \phi_m^{1+\frac{1}{m}}.$$

From (2.11), (2.12) and the Maximum Principle,

$$\left(\max_{x \in \Omega} \phi_m(x, t) \right)_t \leq C(n, \|\chi\|_1, \|\widetilde{Rm}\|_0, C_s(\Omega, g_0), G) \left(\max_{x \in \Omega} \phi_m(x, t) + 1 \right)^2,$$

$$\phi_m(x, t) \leq \frac{1}{C_0 - C t} - 1, \quad C_0 = \frac{1}{n+1}.$$

Note that $\phi_m(0) = 2n$. One can choose

$$t_{1,1} = t_{1,1}(n, \|\chi\|_1, \|\widetilde{Rm}\|_0, C_s(\Omega, g_0), G) > 0,$$

such that when $t \in [0, t_{1,1}]$, $\phi_m(x, t) \leq 2^m$, then

$$\lambda_i \geq \frac{1}{2}, \quad i = 1, 2, \dots, n.$$

Now define

$$(2.13) \quad \begin{aligned} \psi_m &= v_{\alpha_1\beta_1} \tilde{g}^{\beta_1\alpha_2} \dots v_{\alpha_m\beta_m} \tilde{g}^{\beta_m\alpha_1} = \sum_{i=1}^n \lambda_i^m, \\ (\psi_m)_t &= (\chi^2 + \epsilon) u^{\alpha\beta} (\psi_m)_{\alpha\beta} - 2\chi^2 \lambda_i^m u^{\alpha\beta} \tilde{R}_{i\alpha i\beta} + \lambda_i^{m-1} B_{ii} \\ &\quad - 2(\chi^2 + \epsilon) \sum_{j < k} \lambda_q^{m+j-k-1} \lambda_r^{k-j-1} u^{\alpha\beta} (v_{qr})_\alpha (v_{qr})_\beta \\ &\leq (\chi^2 + \epsilon) u^{\alpha\beta} (\psi_m)_{\alpha\beta} + C(\psi_m + 1)^2. \end{aligned}$$

Similarly, by (2.13) and the Maximum Principle,

$$\phi_m(x, t) \leq \frac{1}{C_0 - Ct} - 1, \quad C_0 = \frac{1}{n+1}.$$

Choose $t_{1,2} = t_{1,2}(n, \|\chi\|_1, \|\widetilde{Rm}\|_0, C_s(\Omega, g_0), G) > 0$, such that when $t \in [0, t_{1,2}]$, $\psi_m(x, t) \leq 2^m$, then

$$\lambda_i \leq 2 \quad i = 1, 2, \dots, n.$$

Set $t_1 = \min\{t_{1,1}, t_{1,2}\}$ (note that $t_1 = t_1(n, \|\chi\|_1, \|\widetilde{Rm}\|_0, C_s(\Omega, g_0), G)$). Then when $t \in [0, t_1]$, $\frac{1}{2} \leq \lambda_i \leq 2$; hence $V \in \mathfrak{A}_{t_1}$. In fact we proved that $V \in \mathfrak{A}_{t_1}$ if $U \in \mathfrak{A}_{t_1} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_1}$ and $V = \Phi_\epsilon(U)$.

By (2.5), for $m \geq \tilde{p} + 2$,

$$(2.14) \quad \frac{\partial}{\partial t} |V|_m^2 = 2 \sum_{|\alpha| \leq m} \int_\Omega (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha v_{kl} \tilde{\nabla}^\alpha \left(\frac{\partial}{\partial t} v_{ij} \right) = I_1 + I_2 + I_3 + I_4,$$

where

$$(2.15) \quad \begin{aligned} I_1 &= 2 \sum_{|\alpha| \leq m} \int_\Omega (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha v_{kl} \tilde{\nabla}^\alpha (v_{ij})_{pq} A_\epsilon^{pq}, \\ I_2 &= 2 \sum_{|\alpha| \leq m} \int_\Omega (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha v_{kl} \left[\tilde{\nabla}^\alpha (A_\epsilon^{pq} (v_{ij})_{pq}) - A_\epsilon^{pq} \tilde{\nabla}^\alpha (v_{ij})_{pq} \right], \\ I_3 &= 2 \sum_{|\alpha| \leq m} \int_\Omega (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha v_{kl} \tilde{\nabla}^\alpha D_{ij}, \quad I_4 = 2 \sum_{|\alpha| \leq m} \int_\Omega (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha v_{kl} \tilde{\nabla}^\alpha B_{ij}, \\ I_1 &\leq -2 \sum_{|\alpha| \leq m} \int (\tilde{g}^{-1})^{|\alpha|+2} \left((\tilde{\nabla}^\alpha v_{kl})_p \tilde{\nabla}^\alpha (v_{ij})_q A_\epsilon + (\tilde{\nabla}^\alpha v_{kl}) \tilde{\nabla}^\alpha (v_{ij})_q (A_\epsilon)_p \right) \\ &\leq -\frac{1}{2} \sum_{|\alpha|=m} \int (\chi^2 + \epsilon) |\tilde{\nabla}^{\alpha+1} V|^2 + C(n, m, G, C_s(\Omega, g_0), \|\chi\|_1) |V|_m^2, \end{aligned}$$

where we used the fact that $|\tilde{\nabla} U^{-1}| \leq |U^{-1}|^2 |\tilde{\nabla} U| \leq C(G, C_s(\Omega, g_0))$. Then

$$(2.16) \quad \begin{aligned} I_2 &\leq 2 \sum_{|\alpha| \leq m} \left| \tilde{\nabla}^\alpha V \right|_{L^2} \cdot \left[C(\|\chi\|_1) \left(\int (\chi^2 + \epsilon) |\tilde{\nabla}^{\alpha+1} V|^2 \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left| \tilde{\nabla}^{\alpha-2} [\tilde{\nabla}^2 A_\epsilon^{pq}(V)_{pq}] \right|_{L^2} \right]. \end{aligned}$$

By interpolation inequality, it is easy to get

$$\begin{aligned} \left| \tilde{\nabla}^{\alpha-2} [\tilde{\nabla}^2 A_\epsilon^{pq}(V)_{pq}] \right|_{L^2} &\leq C(\|\chi\|_2, m) |V|_m \\ &\quad + C(m, C_s(\Omega, g_0)) (|V|_{\tilde{p}+2} + 1) (|A_\epsilon^{pq}|_m + 1). \end{aligned}$$

From Lemma 2.4 and $|U^{-1}|_{L^\infty} \leq 2n$,

$$|U^{-1}|_{k+1} \leq C(k, C_s(\Omega, g_0)) (|U|_{\tilde{p}+1} + 1)^{k+1} (|U|_{k+1} + 1).$$

Hence

$$(2.17) \quad \begin{aligned} \left| \tilde{\nabla}^{\alpha-2} [\tilde{\nabla}^2 A^{pq}(V)_{pq}] \right|_{L^2} &\leq C(\|\chi\|_2, m) |V|_m \\ &\quad + C(m, C_s(\Omega, g_0)) (|V|_{\bar{p}+2} + 1) (|U|_m + 1). \end{aligned}$$

By (2.16) and (2.17),

$$(2.18) \quad \begin{aligned} I_2 &\leq \frac{1}{16} \sum_{|\alpha|=m} \int (\chi^2 + \epsilon) \left| \tilde{\nabla}^{\alpha+1} V \right|^2 + C(\|\chi\|_1) |V|_m^2 \\ &\quad + C(m, \|\chi\|_2, C_s(\Omega, g_0)) |V|_m (|V|_{\bar{p}+2} + 1) (|U|_m + 1). \end{aligned}$$

Similar to the estimate of I_2 ,

$$(2.19) \quad I_3 \leq C(m, \|\chi\|_m, \|\widetilde{Rm}\|_m, C_s(\Omega, g_0)) \left(|V|_m^2 + |V|_m (|V|_{\bar{p}} + 1) (|U|_m + 1) \right),$$

$$(2.20) \quad I_4 = I_5 + I_6,$$

where

$$\begin{aligned} I_5 &= 2 \sum_{|\alpha| \leq m} \int (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha v_{kl} \tilde{\nabla}^\alpha E_{ij}, \\ I_6 &= 2 \sum_{|\alpha| \leq m} \int (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha v_{kl} \tilde{\nabla}^\alpha F_{ij} \end{aligned}$$

for I_5 . We only estimate one term of it as in the following; the rest are similar:

$$\begin{aligned} &2 \sum_{|\alpha| \leq m} \int (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha v_{kl} \tilde{\nabla}^\alpha \left[\frac{1}{2} \chi^2 u^{rs} u^{pq} (u_{rp})_i (u_{sq})_j \right] \\ &\leq C(n) \sum_{\alpha} \left[|\tilde{\nabla}^\alpha V|_{L^2} \cdot |\tilde{\nabla}^{\alpha-1} [\tilde{\nabla}(\chi^2 U^{-2}) \cdot (\tilde{\nabla} U)^2]|_{L^2} \right] \\ &\quad + C(n) \sum_{\alpha} \int \tilde{\nabla}^\alpha V \cdot (\chi^2 U^{-2}) \tilde{\nabla}^\alpha (\tilde{\nabla} U \tilde{\nabla} U) \\ &\leq C(n, m, \|\chi\|_m, C_s(\Omega, g_0)) |V|_m \\ &\quad \cdot \left[|U^{-2}|_m (|\tilde{\nabla} U|^2|_{\bar{p}} + 1) + (|U^{-2}|_{\bar{p}+1} + 1) (\tilde{\nabla} U)^2|_{m-1} \right] \\ &\quad + C(n) \sum_{\alpha} \int |\tilde{\nabla}^{\alpha+1} V \cdot (\chi^2 U^{-2}) \tilde{\nabla}^{\alpha-1} (\tilde{\nabla} U \tilde{\nabla} U)| \\ &\quad + C(n) \sum_{\alpha} \int |\tilde{\nabla}^\alpha V \cdot \tilde{\nabla}(\chi^2 U^{-2}) \tilde{\nabla}^{\alpha-1} (\tilde{\nabla} U \tilde{\nabla} U)| \\ &\leq \frac{1}{64} \sum_{|\alpha|=m} \int \chi^2 |\tilde{\nabla}^{\alpha+1} V|^2 + C(n, m, \|\chi\|_m, C_s(\Omega, g_0), G) \\ &\quad \cdot \left(|V|_m (|U|_m + 1) + |U|_m^2 + |V|_m |U|_m \right). \end{aligned}$$

Hence

$$\begin{aligned} I_5 &\leq \frac{9}{64} \sum_{|\alpha|=m} \int \chi^2 |\tilde{\nabla}^{\alpha+1} V|^2 + C(n, m, \|\chi\|_m, C_s(\Omega, g_0), G) \\ &\quad \cdot \left(|V|_m (|U|_m + 1) + |U|_m^2 \right) \end{aligned}$$

for I_6 . We only estimate one term of it as in the following; the rest are similar:

$$\begin{aligned}
 & 2 \sum_{|\alpha| \leq m} \int (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha v_{kl} \tilde{\nabla}^\alpha [\chi \chi_i u^{pq} (u_{jq})_p] \\
 & \leq C(n) \sum_\alpha \int \tilde{\nabla}^\alpha V \cdot \tilde{\nabla}^{\alpha-1} [\tilde{\nabla}(\chi \tilde{\nabla} \chi U^{-1}) \cdot (\tilde{\nabla} U)] \\
 & \quad + 2 \sum_\alpha \int \tilde{\nabla}^\alpha V \cdot (\chi \tilde{\nabla} \chi U^{-1}) \tilde{\nabla}^{\alpha+1} U \\
 & \leq \frac{1}{64} \sum_{|\alpha|=m} \int \chi^2 |\tilde{\nabla}^{\alpha+1} V|^2 + C(n, m, G, \|\chi\|_m, C_s(\Omega, g_0)) \\
 & \quad \cdot \left(|V|_m (|U|_m + 1) + |U|_m^2 + |V|_m |U|_m \right).
 \end{aligned}$$

Hence

$$\begin{aligned}
 I_6 & \leq \frac{6}{64} \sum_{|\alpha|=m} \int \chi^2 |\tilde{\nabla}^{\alpha+1} V|^2 + C(n, m, G, \|\chi\|_m, C_s(\Omega, g_0)) \\
 & \quad \cdot \left(|U|_m^2 + |V|_m (|U|_m + 1) \right).
 \end{aligned}$$

Then we get

$$\begin{aligned}
 (2.21) \quad I_4 & \leq \frac{15}{64} \sum_{|\alpha|=m} \int \chi^2 |\tilde{\nabla}^{\alpha+1} V|^2 + C(n, m, G, \|\chi\|_m, C_s(\Omega, g_0)) \\
 & \quad \cdot \left(|U|_m^2 + |V|_m (|U|_m + 1) \right).
 \end{aligned}$$

By (2.14), (2.15), (2.18), (2.19) and (2.21), and noting that the sum of the coefficients of $\sum_{|\alpha| \leq m} \int \chi^2 |\tilde{\nabla}^{\alpha+1} V|^2$ is negative on the right side, then

$$(2.22) \quad \frac{\partial}{\partial t} |V|_m^2 \leq C \left(|V|_m^2 + |V|_m (|V|_{\tilde{p}+2} + 1) (|U|_m + 1) + |U|_m^2 \right),$$

where $C = C(n, m, G, C_s(\Omega, g_0), \|\chi\|_m, \|\widetilde{Rm}\|_m)$. For simplicity, we use C instead of $C(n, m, G, C_s(\Omega, g_0), \|\chi\|_m, \|\widetilde{Rm}\|_m)$ in the rest of the proof. Let $\phi(t) = |V|_m^2(t)$. Then

$$\phi_t \leq C\phi + C\phi^{\frac{1}{2}}Q_1 + Q_2,$$

where $Q_1 = (|V|_{\tilde{p}+2} + 1)(|U|_m + 1)$ and $Q_2 = C|U|_m^2$. We can get

$$(2.23) \quad \phi_t \leq C(\phi + 1)(Q_1^2 + Q_2 + 1) \leq CQ_4(\phi + 1),$$

where

$$(2.24) \quad Q_4 = (|V|_{\tilde{p}+2}^2 + 1)(|U|_m + 1)^2.$$

Choose $m = \tilde{p} + 2$, $\phi(t) = |V|_{\tilde{p}+2}^2(t)$, and define $\psi(t) = |U|_{\tilde{p}+2}(t)$. Then $Q_4 = (\phi + 1)(\psi + 1)^2$,

$$\phi_t \leq C(\phi + 1)^2(\psi + 1)^2 \leq C(G + 1)^2(\phi + 1)^2 = C(\phi + 1)^2,$$

where $\psi \leq G$ because of $U \in \mathbb{B}_{\tilde{p}+2}(G)_{t_1}$ and

$$C = C(n, \tilde{p}, G, \|\chi\|_{\tilde{p}+2}, \|\widetilde{Rm}\|_{\tilde{p}+2}, C_s(\Omega, g_0)),$$

$$\phi(t) \leq \frac{1}{C_0 - Ct} - 1, \quad C_0 = (\phi(0) + 1)^{-1}.$$

Note that $\phi(0) = (\frac{G}{2})^2$. Then there exists $t_2 > 0$, such that if $t \in [0, t_2]$, $\phi(t) \leq G^2$; that is

$$|V|_{\tilde{p}+2} \leq G.$$

Choose

$$(2.25) \quad t_3 = \min\{t_1, t_2\} = t_3(n, \tilde{p}, G, \|\chi\|_{\tilde{p}+2}, \|\widetilde{Rm}\|_{\tilde{p}+2}, C_s(\Omega, g_0)).$$

We get the a priori estimate that $V \in \mathfrak{A}_{t_3} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_3}$ if $U \in \mathfrak{A}_{t_3} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_3}$ and $V = \Phi_\epsilon(U)$. By Theorem 8.3 in [9] (which is for parabolic equations, but similar results apply for the parabolic system here), we in fact get the solution of (2.4) on $\Omega \times [0, t_3]$ and

$$\Phi_\epsilon : \mathfrak{A}_{t_3} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_3} \longrightarrow \mathfrak{A}_{t_3} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_3}. \quad \square$$

Remark 2.6. The proof of Proposition 2.5 (especially the energy estimate) has the same spirit as in [7], but the tensor version makes our estimate more subtle. Also note that t_3 is independent of ϵ .

Proposition 2.7. *For any $m \geq \tilde{p} + 2$, there exists*

$$t_5 = t_5(n, m, G, \|\chi\|_m, \|\widetilde{Rm}\|_m, C_s(\Omega, g_0)) > 0$$

such that

$$\Phi_\epsilon : \mathfrak{A}_{t_5} \cap \mathbb{B}_m(G)_{t_5} \longrightarrow \mathfrak{A}_{t_5} \cap \mathbb{B}_m(G)_{t_5}.$$

Proof. Choose $U \in \mathfrak{A}_T \cap \mathbb{B}_m(G)_T$, where $T \leq t_3$ is to be determined later. By Proposition 2.5, we get the solution $V = \Phi_\epsilon(U) \in \mathfrak{A}_T \cap \mathbb{B}_{\tilde{p}+2}(G)_T$, so $|V|_{\tilde{p}+2} \leq G$. We also have $|U|_m \leq G$. By (2.23) and (2.24) we get

$$\phi_t \leq C(G^2 + 1)(G + 1)^2(\phi + 1) \leq C(\phi + 1),$$

where $\phi(t) = |V|_m^2(t)$ and $C = C(n, m, G, \|\chi\|_m, \|\widetilde{Rm}\|_m, C_s(\Omega, g_0))$. Then

$$\phi(t) \leq e^{Ct}[\phi(0) + 1] - 1.$$

Note $\phi(0) = (\frac{G}{2})^2$. Choose $t_4 = t_4(n, m, G, \|\chi\|_m, \|\widetilde{Rm}\|_m, C_s(\Omega, g_0)) > 0$ such that $e^{Ct_4}[\phi(0) + 1] - 1 = G^2$, and let

$$t_5 = \min t_3, t_4 = t_5(n, m, G, \|\chi\|_m, \|\widetilde{Rm}\|_m, C_s(\Omega, g_0)).$$

If $t \in [0, t_5]$,

$$\phi(t) \leq e^{Ct_5}[(\frac{G}{2})^2 + 1] - 1 \leq G^2;$$

then $|V|_m \leq G$. Hence $V \in \mathfrak{A}_{t_5} \cap \mathbb{B}_m(G)_{t_5}$. $\square$

Next we show that Φ_ϵ is a contraction map.

Proposition 2.8. *For $0 < \delta < 1$, there exists*

$$t_7 = t_7(n, \tilde{p}, G, \|\chi\|_{\tilde{p}+2}, \|\widetilde{Rm}\|_{\tilde{p}+2}, C_s(\Omega, g_0), \delta) > 0$$

such that

$$|\Phi_\epsilon(U) - \Phi_\epsilon(\tilde{U})|_{t_7, \tilde{p}} \leq \delta |U - \tilde{U}|_{t_7, \tilde{p}},$$

where $U, \tilde{U} \in \mathfrak{A}_{t_7} \cap \mathbb{B}_{\tilde{p}+2}(G)_{t_7}$.

Proof. By (2.5) we have the following:

$$\begin{cases} V_t &= (\chi^2 + \epsilon)u^{\alpha\beta}(V)_{\alpha\beta} + P(U)V + B(U, \tilde{\nabla}U), \\ \tilde{V}_t &= (\chi^2 + \epsilon)\tilde{u}^{\alpha\beta}(\tilde{V})_{\alpha\beta} + P(\tilde{U})\tilde{V} + B(\tilde{U}, \tilde{\nabla}\tilde{U}), \end{cases}$$

where

$$\begin{aligned} V &= \Phi_\epsilon(U), \quad (P(U)V)_{ij} = D_{ij}(U, V), \\ (B(U, \tilde{\nabla}U))_{ij} &= B_{ij}(U, \tilde{\nabla}U), \quad (V)_{\alpha\beta} = \tilde{\nabla}_\alpha \tilde{\nabla}_\beta V. \end{aligned}$$

There is a similar notation for $\tilde{V}$, etc. By Proposition 2.5, if $T \leq t_3$, where t_3 is defined in (2.25), then $V, \tilde{V} \in \mathfrak{A}_T \cap \mathbb{B}_{\tilde{p}+2}(G)_T$.

Define $W \triangleq V - \tilde{V} = \Phi_\epsilon(U) - \Phi_\epsilon(\tilde{U})$. Then

$$W_t = (\chi^2 + \epsilon)u^{\alpha\beta}w_{\alpha\beta} + P(U)W + H,$$

where

$$H = (\chi^2 + \epsilon)(u^{\alpha\beta} - \tilde{u}^{\alpha\beta})(\tilde{V})_{\alpha\beta} + (P(U) - P(\tilde{U}))\tilde{V} + (B(U, \tilde{\nabla}U) - B(\tilde{U}, \tilde{\nabla}\tilde{U})).$$

We consider the evolution equation of $|W|_{\tilde{p}}^2$:

$$(2.26) \quad \frac{\partial}{\partial t}|W|_{\tilde{p}}^2 = J_1 + J_2 + J_3 + J_4,$$

where

$$\begin{aligned} J_1 &= 2 \sum_{|\alpha| \leq \tilde{p}} \int (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha w_{kl} \tilde{\nabla}^\alpha (w_{ij})_{pq} A_\epsilon^{pq}, \\ J_2 &= 2 \sum \int (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha w_{kl} [\tilde{\nabla}^\alpha (A_\epsilon^{pq}(w_{ij})_{pq}) - A_\epsilon^{pq} \tilde{\nabla}^\alpha (w_{ij})_{pq}], \\ J_3 &= 2 \sum_{|\alpha| \leq \tilde{p}} \int (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha w_{kl} \tilde{\nabla}^\alpha (P(U)W)_{ij}, \\ J_4 &= 2 \sum_{|\alpha| \leq \tilde{p}} \int (\tilde{g}^{-1})^{|\alpha|+2} \tilde{\nabla}^\alpha w_{kl} \tilde{\nabla}^\alpha H_{ij}. \end{aligned}$$

Recall $A_\epsilon^{pq} = (\chi^2 + \epsilon)u^{pq}$. Similar to (2.15), we have

$$(2.27) \quad J_1 \leq -\frac{1}{2} \sum_{|\alpha|=\tilde{p}} \int (\chi^2 + \epsilon) |\tilde{\nabla}^{\alpha+1} W|^2 + C(n, \tilde{p}, G, C_s(\Omega, g_0), \|\chi\|_1) |W|_{\tilde{p}}^2.$$

Similar to (2.16), we get

$$J_2 \leq 2 \sum_{|\alpha| \leq \tilde{p}} |\tilde{\nabla}^\alpha W|_{L^2} \cdot [C(\int (\chi^2 + \epsilon) |\tilde{\nabla}^{\alpha+1} W|^2)^{\frac{1}{2}} + |\tilde{\nabla}^{\alpha-2} [\nabla^2 A_\epsilon^{pq}(W)_{pq}]|_{L^2}],$$

and we have

$$\begin{aligned} |\tilde{\nabla}^{\alpha-2} [\tilde{\nabla}^2 A_\epsilon^{pq}(W)_{pq}]|_{L^2} &\leq C(\tilde{p}) \sum_{\beta+\gamma=\alpha} |\tilde{\nabla}^\beta (\tilde{\nabla}^2 A^{pq}) \tilde{\nabla}^\gamma W|_{L^2} \\ &\leq C(n, \tilde{p}, C_s(\Omega, g_0)) [(\|\tilde{\nabla}^2 A_\epsilon^{pq}\|_{L^\infty} + 1) |\tilde{\nabla}^\alpha W|_{L^2} + |W|_{L^\infty} (|\tilde{\nabla}^\alpha (\tilde{\nabla}^2 A_\epsilon^{pq})|_{L^2} + 1)] \\ &\leq C(n, \tilde{p}, C_s(\Omega, g_0), \|\chi\|_2) [(|U|_{\tilde{p}+2} + 1) |W|_{\tilde{p}} + |W|_{\tilde{p}} (|U|_{\tilde{p}+2} + 1)] \\ &\leq C(n, \tilde{p}, C_s(\Omega, g_0), \|\chi\|_2, G) |W|_{\tilde{p}}. \end{aligned}$$

Hence

$$(2.28) \quad J_2 \leq \frac{1}{16} \sum_{|\alpha|=\tilde{p}} \int (\chi^2 + \epsilon) |\tilde{\nabla}^{\alpha+1} W|^2 + C(n, \tilde{p}, C_s(\Omega, g_0), \|\chi\|_2, G) |W|_{\tilde{p}}^2.$$

Similar to (2.19), we get

$$(2.29) \quad J_3 \leq C(\tilde{p}, \|\chi\|_{\tilde{p}}, \|\widetilde{Rm}\|_{\tilde{p}}, C_s(\Omega, g_0), G) |W|_{\tilde{p}}^2.$$

In the rest of the proof, we use C instead of $C(n, \tilde{p}, \|\chi\|_{\tilde{p}}, \|\widetilde{Rm}\|_{\tilde{p}}, C_s(\Omega, g_0), G)$ for simplicity. We have

$$(2.30) \quad J_4 = J_5 + J_6 + J_7,$$

where

$$(2.31) \quad \begin{aligned} J_5 &= 2 \sum_{\alpha} \int (\tilde{g}^{-1})^{\alpha+2} \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [(\chi^2 + \epsilon)(U^{-1} - \tilde{U}^{-1}) \tilde{\nabla}^2 \tilde{V}], \\ J_6 &= 2 \sum_{\alpha} \int (\tilde{g}^{-1})^{\alpha+2} \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [(P(U) - P(\tilde{U})) \tilde{V}], \\ J_7 &= 2 \sum_{\alpha} \int (\tilde{g}^{-1})^{\alpha+2} \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [B(U, \tilde{\nabla} U) - B(\tilde{U} - \tilde{\nabla} \tilde{U})], \\ J_5 &\leq C|W|_{\tilde{p}} \left(\sum_{\alpha} |\tilde{\nabla}^{\alpha} [(\chi^2 + \epsilon) \tilde{\nabla}^2 \tilde{V} \cdot (U^{-1} - \tilde{U}^{-1})]|_{L^2} \right) \\ &\leq C|W|_{\tilde{p}} (|\tilde{V}|_{\tilde{p}+2} + 1) |U^{-1} - \tilde{U}^{-1}|_{\tilde{p}} \leq C|W|_{\tilde{p}} |\tilde{U} - U|_{\tilde{p}}. \end{aligned}$$

It is easy to get

$$(2.32) \quad J_6 \leq C|W|_{\tilde{p}} [|P(U) - P(\tilde{U})|_{\tilde{p}} (|\tilde{V}|_{\tilde{p}} + 1)] \leq C|W|_{\tilde{p}} |\tilde{U} - U|_{\tilde{p}}.$$

We only need to estimate two terms of J_7 such as in the following for J_8 and J_9 ; the others are similar:

$$\begin{aligned} J_8 &= \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [\chi^2 (U^{-2} (\tilde{\nabla} U)^2 - \tilde{U}^{-2} (\tilde{\nabla} \tilde{U})^2)], \\ J_9 &= \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [\chi \tilde{\nabla} \chi (U^{-1} \tilde{\nabla} U - \tilde{U}^{-1} \tilde{\nabla} \tilde{U})]. \end{aligned}$$

We first estimate J_8 :

$$(2.33) \quad J_8 \leq J_{10} + J_{11} + J_{12} + J_{13},$$

where

$$\begin{aligned} J_{10} &= C \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [(U^{-1} - \tilde{U}^{-1}) U^{-1} (\tilde{\nabla} U)^2 \chi^2], \\ J_{11} &= C \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [(U^{-1} - \tilde{U}^{-1}) \tilde{U}^{-1} (\tilde{\nabla} U)^2 \chi^2], \\ J_{12} &= C \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [(\tilde{U}^{-2} \tilde{\nabla} U \chi^2) \tilde{\nabla} (U - \tilde{U})], \\ J_{13} &= C \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [(\tilde{U}^{-2} \tilde{\nabla} \tilde{U} \chi^2) \tilde{\nabla} (U - \tilde{U})]. \end{aligned}$$

It is easy to get

$$(2.34) \quad J_{10} \leq C|W|_{\tilde{p}} \left[|U^{-1} - \tilde{U}^{-1}|_{\tilde{p}} (|U^{-1} (\nabla U)^2|_{\tilde{p}} + 1) \right] \leq C|W|_{\tilde{p}} |\tilde{U} - U|_{\tilde{p}}.$$

Similarly, we get

$$(2.35) \quad J_{11} \leq C|W|_{\tilde{p}}|\tilde{U} - U|_{\tilde{p}}.$$

We have $J_{12} \leq J_{12.1} + J_{12.2}$, where

$$(2.36) \quad \begin{aligned} J_{12.1} &= C \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha-1} [\tilde{\nabla}(\tilde{U}^{-2} \tilde{\nabla} U \chi^2) \tilde{\nabla}(U - \tilde{U})], \\ J_{12.2} &= C \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot (\tilde{U}^{-2} \tilde{\nabla} U \chi^2) \cdot \tilde{\nabla}^{\alpha+1}(U - \tilde{U}), \\ J_{12.1} &\leq C|W|_{\tilde{p}} \sum_{\alpha} |\tilde{\nabla}^{\alpha-1}[f \tilde{\nabla} g]|_{L^2}, \end{aligned}$$

where $f = \tilde{\nabla}(\tilde{U}^{-2} \chi^2 \tilde{\nabla} U)$, $g = U - \tilde{U}$. Note that $|f|_{\tilde{p}} \leq C(|u|_{\tilde{p}+2} + |\tilde{u}|_{\tilde{p}+2} + 1) \leq C$. Then

$$(2.37) \quad \begin{aligned} |\tilde{\nabla}^{\alpha-1}[f \tilde{\nabla} g]|_{L^2} &\leq C \sum_{\beta+\gamma=\alpha} |\tilde{\nabla}^{\beta} f \tilde{\nabla}^{\gamma} g|_{L^2} \\ &\leq C(|f|_{L^{\infty}} + 1)|\tilde{\nabla}^{\alpha} g|_{L^2} + |g|_{L^{\infty}}(|\tilde{\nabla}^{\alpha} f|_{L^2} + 1) \\ &\leq C|g|_{\tilde{p}} = C|U - \tilde{U}|_{\tilde{p}}. \end{aligned}$$

With (2.36) and (2.37), we get

$$(2.38) \quad J_{12.1} \leq C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}}.$$

It is easy to get

$$(2.39) \quad J_{12.2} \leq \frac{1}{1000} \sum_{|\alpha|=\tilde{p}} \int |\tilde{\nabla}^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}},$$

and by (2.38) and (2.39),

$$(2.40) \quad J_{12} \leq \frac{1}{1000} \sum_{|\alpha|=\tilde{p}} \int |\tilde{\nabla}^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}}.$$

Similarly,

$$(2.41) \quad J_{13} \leq \frac{1}{1000} \sum_{|\alpha|=\tilde{p}} \int |\tilde{\nabla}^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}}.$$

By (2.34), (2.35), (2.40), (2.41) and (2.33),

$$(2.42) \quad J_8 \leq \frac{1}{500} \sum_{|\alpha|=\tilde{p}} \int |\tilde{\nabla}^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}},$$

$$(2.43) \quad J_9 \leq J_{14} + J_{15},$$

where

$$\begin{aligned} J_{14} &= \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [(\chi \tilde{\nabla} \chi \tilde{\nabla} U)(U^{-1} - \tilde{U}^{-1})], \\ J_{15} &= \sum_{\alpha} \int (\tilde{g}^{-1})^{|\alpha|+2} \cdot \tilde{\nabla}^{\alpha} W \cdot \tilde{\nabla}^{\alpha} [(\chi \tilde{\nabla} \chi \tilde{U}^{-1}) \tilde{\nabla}(U - \tilde{U})]. \end{aligned}$$

It is easy to get

$$(2.44) \quad J_{14} \leq C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}}.$$

Similar to the argument for J_{12} , we can get

$$(2.45) \quad J_{15} \leq \frac{1}{1000} \sum_{|\alpha|=\tilde{p}} \int |\tilde{\nabla}^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}}.$$

By (2.44), (2.45) and (2.43), we get

$$(2.46) \quad J_9 \leq \frac{1}{1000} \sum_{|\alpha|=\tilde{p}} \int |\tilde{\nabla}^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}}.$$

By (2.42) and (2.46), we get

$$(2.47) \quad J_7 \leq \left(\frac{9}{500} + \frac{6}{1000} \right) \sum_{|\alpha|=\tilde{p}} \int |\tilde{\nabla}^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}}.$$

By (2.31), (2.32), (2.47) and (2.30), we get

$$(2.48) \quad J_4 \leq \frac{3}{100} \sum_{|\alpha|=\tilde{p}} \int |\tilde{\nabla}^{\alpha+1} W|^2 \chi^2 + C|U - \tilde{U}|_{\tilde{p}}^2 + C|W|_{\tilde{p}}|U - \tilde{U}|_{\tilde{p}}.$$

By (2.27), (2.28), (2.29), (2.48), and (2.26), we get

$$(2.49) \quad \frac{\partial}{\partial t} (|W|_{\tilde{p}}^2) \leq C_1 |W|_{\tilde{p}}^2 + C_2 |U - \tilde{U}|_{\tilde{p}}^2,$$

where $C_1 = C_1(n, \tilde{p}, G, \|\chi\|_{\tilde{p}}, \|\widetilde{Rm}\|_{\tilde{p}}, C_s(\Omega, g_0)) > 0$ and C_2 depends on the same parameter as C_1 .

By Gronwall's inequality and (2.49), noting $|W(x, 0)|_{\tilde{p}} = 0$, we get

$$(2.50) \quad |W|_{\tilde{p}}^2(t) \leq e^{C_1 t} \cdot \int_0^t C_2 |U - \tilde{U}|_{\tilde{p}}^2(s) ds \leq C_2 t e^{C_1 t} |U - \tilde{U}|_{t, \tilde{p}}^2.$$

We can choose $t_6 = t_6(\delta, C_1, C_2) = t_6(n, \tilde{p}, G, \|\chi\|_{\tilde{p}}, \|\widetilde{Rm}\|_{\tilde{p}}, C_s(\Omega, g_0), \delta) > 0$, such that $C_2 \cdot t_6 \cdot e^{C_1 t_6} = \delta^2 < 1$. Then we choose

$$(2.51) \quad t_7 = \min\{t_3, t_6\} = t_7(n, \tilde{p}, G, \|\chi\|_{\tilde{p}+2}, \|\widetilde{Rm}\|_{\tilde{p}+2}, C_s(\Omega, g_0), \delta).$$

If $t \in [0, t_7]$, we get $|W|_{\tilde{p}}^2(t) \leq \delta^2 |U - \tilde{U}|_{t_7, \tilde{p}}^2$. Hence

$$|W|_{t_7, \tilde{p}} \leq \delta |U - \tilde{U}|_{t_7, \tilde{p}}.$$

That is,

$$|\Phi_\epsilon(U) - \Phi_\epsilon(\tilde{U})|_{t_7, \tilde{p}} \leq \delta |U - \tilde{U}|_{t_7, \tilde{p}}. \quad \square$$

Theorem 2.9. *There exists $T = T(n, \tilde{p}, G, \|\chi\|_{2\tilde{p}+2}, \|\widetilde{Rm}\|_{2\tilde{p}+2}, C_s(\Omega, g_0)) > 0$, such that on $\Omega \times [0, T]$, the degenerate parabolic system (2.3) has a smooth solution.*

Proof. Choose $m = 2\tilde{p} + 2$ in Proposition 2.7 and $\delta = \frac{1}{2}$ in Proposition 2.8. Then we choose

$$T = \min\{t_5, t_7\} = T(n, \tilde{p}, G, \|\chi\|_{2\tilde{p}+2}, \|\widetilde{Rm}\|_{2\tilde{p}+2}, C_s(\Omega, g_0)).$$

Set $U^{(0)}(x, t) = \tilde{g}(x)$ and $U_\epsilon^{(i+1)} = \Phi_\epsilon(U^{(i)})$, where $U_\epsilon^{(i)}$ are $(2, 0)$ -tensors on $\Omega \times [0, T]$, $i = 0, 1, 2, \dots$. Note that $U^{(0)} \in \mathfrak{A}_T \cap \mathbb{B}_m(G)_T$ for any $m \geq \tilde{p} + 2$. By Proposition 2.7, we get $U_\epsilon^{(i)} \in \mathfrak{A}_T \cap \mathbb{B}_{2\tilde{p}+2}(G)_T$ for any i . Then we have the following:

$$(2.52) \quad \sup_{(x,t) \in \Omega \times [0,T]} \sup_{|\alpha| \leq \tilde{p}+2} |\tilde{\nabla}^\alpha U_\epsilon^{(i)}(x, t)| \leq C_1 |U_\epsilon^{(i)}|_{T, 2\tilde{p}+2} \leq C_1 G$$

for any $i \geq 0$, where $C_1 = C_1(C_s(\Omega, g_0), \tilde{p})$, and note that C_1 is independent of i .

By (2.4), we get

$$(2.53) \quad \sup_{|\beta| \leq \frac{\tilde{p}+2}{2}} \sup_{(x,t) \in \Omega \times [0,T]} |D_t^\beta U_\epsilon^{(i)}(x,t)| \leq C_2$$

for any $i \geq 0$, where $C_2 = C_2(C_s(\Omega, g_0), n, \tilde{p}, G, \|\chi\|_2, \|\widetilde{Rm}\|_0)$. Note that C_2 is independent of i .

Since $\{U_\epsilon^{(i)}\}_{i=1}^\infty$ are uniformly bounded in $\mathbb{B}_{2\tilde{p}+2}(G)_T$, we can choose the subsequence $U_\epsilon^{(k_i)} \rightarrow U_\epsilon^\infty$ in $C^{\tilde{p}+2, \frac{\tilde{p}+2}{2}}(\Omega \times [0, T])$ by the Rellich-Kondrachov Compactness theorem, and $U_\epsilon^\infty \in \mathbb{B}_{\tilde{p}+2}(G)_T$. That is why we chose $m = 2\tilde{p} + 2$ in Proposition 2.7 at the beginning of the proof. For $k_i \geq m > 0$, we get

$$\begin{aligned} |U_\epsilon^{(m)} - U_\epsilon^{(k_i)}|_{T, \tilde{p}} &= |\Phi_\epsilon^m(U^{(0)}) - \Phi_\epsilon^{k_i}(U^{(0)})|_{T, \tilde{p}} \leq \delta^m |U^{(0)} - \Phi_\epsilon^{k_i-m}(U^{(0)})|_{T, \tilde{p}} \\ &\leq 2\left(\frac{1}{2}\right)^m G. \end{aligned}$$

Hence

$$\lim_{m \rightarrow \infty} |U_\epsilon^{(m)} - U_\epsilon^\infty|_{T, \tilde{p}} \leq \lim_{m \rightarrow \infty} \left(|U_\epsilon^{(m)} - U_\epsilon^{(k_i)}|_{T, \tilde{p}} + |U_\epsilon^{(k_i)} - U_\epsilon^\infty|_{T, \tilde{p}} \right) = 0.$$

Now

$$\begin{aligned} |U_\epsilon^\infty - \Phi_\epsilon(U_\epsilon^\infty)|_{T, \tilde{p}} &= \lim_{m \rightarrow \infty} |U_\epsilon^{(m)} - \Phi_\epsilon(U_\epsilon^\infty)|_{T, \tilde{p}} \\ &= \lim_{m \rightarrow \infty} |\Phi_\epsilon(U_\epsilon^{(m-1)}) - \Phi_\epsilon(U_\epsilon^\infty)|_{T, \tilde{p}} \leq \frac{1}{2} \lim_{m \rightarrow \infty} |U_\epsilon^{(m-1)} - U_\epsilon^\infty|_{T, \tilde{p}} = 0. \end{aligned}$$

Then we have

$$|U_\epsilon^\infty - \Phi_\epsilon(U_\epsilon^\infty)|_{C^0} \leq C_s(\Omega, g_0) |U_\epsilon^\infty - \Phi_\epsilon(U_\epsilon^\infty)|_{T, \tilde{p}} = 0,$$

and hence

$$(2.54) \quad U_\epsilon^\infty = \Phi_\epsilon(U_\epsilon^\infty).$$

By (2.52) and (2.53), we get $\{U_\epsilon^\infty\}_{0 < \epsilon \leq 1}$ is uniformly bounded in $C^{\tilde{p}+2, \frac{\tilde{p}+2}{2}}(\Omega \times [0, T])$. So there exists a subsequence $\{U_{\epsilon_i}^\infty\}_{i=1}^\infty$, such that

$$\lim_{i \rightarrow \infty} U_{\epsilon_i}^\infty = \hat{U}^\infty \quad \text{in } C^{\tilde{p}, \frac{\tilde{p}}{2}}(\Omega \times [0, T]),$$

where $\lim_{i \rightarrow \infty} \epsilon_i = 0$.

On the other hand, by (2.54) and (2.4) we have

$$(2.55) \quad (U_{\epsilon_i}^\infty)_t = (\chi^2 + \epsilon_i)(U_{\epsilon_i}^\infty)^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta (U_{\epsilon_i}^\infty) + D(U_{\epsilon_i}^\infty) + B(U_{\epsilon_i}^\infty, \tilde{\nabla} U_{\epsilon_i}^\infty).$$

Let $i \rightarrow \infty$ in (2.55); we get

$$(\hat{U}^\infty)_t = \chi^2 (\hat{U}^\infty)^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta (\hat{U}^\infty) + D(\hat{U}^\infty) + B(\hat{U}^\infty, \tilde{\nabla} \hat{U}^\infty).$$

Then $\hat{U}^\infty$ is a solution of (2.3) on $\Omega \times [0, T]$, and $\hat{U}^\infty \in C^{\tilde{p}, \frac{\tilde{p}}{2}}(\Omega \times [0, T])$. By regularity theory of the parabolic system (note that we have local regularity property where $\chi(x) > 0$, and $\hat{U}^\infty(x, t) = \tilde{g}_{ij}(x)$ where $\chi(x) = 0$), we have

$$\hat{U}^\infty \in C^\infty(\Omega \times [0, T]).$$

Hence $\hat{U}^\infty$ is a smooth solution of (2.3) on $\Omega \times [0, T]$. □

3. EVOLUTION EQUATIONS OF CURVATURE UNDER THE LOCAL RICCI FLOW

Under local Ricci flow, the behavior of curvature tensors are affected by their evolution equations. These equations have simpler versions in Ricci flow context by just taking the cut-off function $\chi(x) \equiv 1$.

Now we calculate the evolution equation of the curvature tensor Rm . We use the following notation: $R_{ijk}^l = g^{lp} R_{ijkp}$, $R_i^r = g^{rp} R_{ip}$, and R is the scalar curvature.

Lemma 3.1. *Under the local Ricci flow (1.2),*

$$(3.1) \quad \frac{\partial}{\partial t} R_{ijkl} = \chi^2 \Delta R_{ijkl} + Q + I_1 + I_2 + I_3,$$

where Q , I_1 , I_2 , I_3 are defined in (3.8), (3.4), (3.5), (3.6).

Proof. First, if $\frac{\partial}{\partial t} g_{ij} = h_{ij}$, then (see [1])

$$\frac{\partial}{\partial t} R_{ijk}^l = \frac{1}{2} g^{lp} [\nabla_i \nabla_k h_{jp} + \nabla_j \nabla_p h_{ik} - \nabla_i \nabla_p h_{jk} - \nabla_j \nabla_k h_{ip} - R_{ijk}^q h_{qp} - R_{ijp}^q h_{kq}].$$

In the local Ricci flow case,

$$h_{ij} = -2\chi^2 R_{ij}.$$

Now

$$(3.2) \quad \frac{\partial}{\partial t} R_{ijkl} = (-2\chi^2 R_{lm}) R_{ijk}^m + g_{lm} \left(\frac{\partial}{\partial t} R_{ijk}^m \right).$$

We calculate $g_{lm} \frac{\partial}{\partial t} R_{ijk}^m$ first:

$$g_{lm} \frac{\partial}{\partial t} R_{ijk}^m = \frac{1}{2} [\nabla_i \nabla_k h_{jl} + \nabla_j \nabla_l h_{ik} - \nabla_i \nabla_l h_{jk} - \nabla_j \nabla_k h_{il} - R_{ijk}^q h_{ql} - R_{ijl}^q h_{kq}].$$

We calculate $\nabla_i \nabla_k h_{jl}$ as a sample, using the notation $\chi_i = \nabla_i \chi$ and $\chi_{ij} = \nabla_i \nabla_j \chi$:

$$\nabla_i \nabla_k h_{jl} = -2(\chi^2 \nabla_i \nabla_k R_{jl} + 2\chi \chi_i \nabla_k R_{jl} + 2\chi \chi_k \nabla_i R_{jl} + 2\chi_i \chi_k R_{jl} + 2\chi \chi_{ik} R_{jl}).$$

A straightforward computation gives

$$(3.3) \quad g_{lm} \frac{\partial}{\partial t} R_{ijk}^m = \chi^2 (\nabla_i \nabla_l R_{jk} + \nabla_j \nabla_k R_{il} - \nabla_i \nabla_k R_{jl} - \nabla_j \nabla_l R_{ik}) + \chi^2 (R_{ijk}^q R_{ql} + R_{ijl}^q R_{kq}) + I_1 + I_2 + I_3,$$

where

$$(3.4) \quad I_1 = 2\chi [(\chi_k \nabla_j R_{il} + \chi_j \nabla_k R_{il} + \chi_l \nabla_i R_{jk} + \chi_i \nabla_l R_{jk}) - (\chi_k \nabla_i R_{jl} + \chi_i \nabla_k R_{jl} + \chi_l \nabla_j R_{ik} + \chi_j \nabla_l R_{ik})],$$

$$(3.5) \quad I_2 = 2[(\chi_i \chi_l R_{jk} + \chi_j \chi_k R_{il}) - (\chi_i \chi_k R_{jl} + \chi_j \chi_l R_{ik})],$$

$$(3.6) \quad I_3 = 2\chi [(\chi_{il} R_{jk} + \chi_{jk} R_{il}) - (\chi_{ik} R_{jl} + \chi_{jl} R_{ik})].$$

Now we compute ΔR_{ijkl} . From the formula for ΔR_{ijk}^l in [1],

$$(3.7) \quad \begin{aligned} \Delta R_{ijkl} &= \Delta(g_{lm} R_{ijk}^m) = g_{lm} \Delta R_{ijk}^m \\ &= (\nabla_i \nabla_l R_{jk} + \nabla_j \nabla_k R_{il} - \nabla_i \nabla_k R_{jl} - \nabla_j \nabla_l R_{ik}) \\ &\quad + (R_{ijp}^r R_{irkl} - R_i^r R_{rkl}) - g^{pq} (R_{ijp}^r R_{rqkl} - 2R_{pi k}^r R_{jqrl} + 2R_{jqk}^r R_{pir l}). \end{aligned}$$

By (3.2), (3.3) and (3.7),

$$\frac{\partial}{\partial t} R_{ijkl} = \chi^2 \Delta R_{ijkl} + Q + I_1 + I_2 + I_3,$$

where

$$(3.8) \quad Q = Q_1 + Q_2,$$

$$(3.9) \quad Q_1 = \chi^2 g^{pq} (R_{ijp}^r R_{rqkl} - 2R_{pik}^r R_{jqrl} + 2R_{jqk}^r R_{pirl}),$$

$$(3.10) \quad Q_2 = -\chi^2 (R_{mjkl} R_i^m + R_{imkl} R_j^m + R_{ijml} R_k^m + R_{ijkm} R_l^m). \quad \square$$

For our applications, we need to estimate $\frac{\partial}{\partial t} |Rm|$.

Lemma 3.2. *Under the local Ricci flow (1.2), we have*

$$(3.11) \quad \begin{aligned} \frac{\partial}{\partial t} |Rm| &\leq \chi^2 \Delta |Rm| + \left(\frac{\chi^2 |\nabla |Rm||^2}{|Rm|} \right) - (1 - \epsilon) \left(\frac{\chi^2 |\nabla Rm|^2}{|Rm|} \right) \\ &\quad + 10\chi^2 |Rm|^2 + \left(\frac{10^5}{\epsilon} \right) n |\nabla \chi|^2 |Rm| + \left(\frac{8\chi g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} \chi_{il} R_{jk}}{|Rm|} \right), \end{aligned}$$

where ϵ is any positive constant satisfying $0 < \epsilon < 1$.

Proof.

$$\frac{\partial}{\partial t} |Rm|^2 = 2g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} \left(\frac{\partial}{\partial t} R_{ijkl} \right) + \frac{\partial}{\partial t} (g^{ri} g^{sj} g^{pk} g^{ql}) R_{rspq} R_{ijkl}.$$

Note that

$$\frac{\partial}{\partial t} (g^{ri}) g^{sj} g^{pk} g^{ql} R_{rspq} R_{ijkl} = 2\chi^2 g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} R_{mjkl} R_i^m.$$

Similarly,

$$\frac{\partial}{\partial t} (g^{ri} g^{sj} g^{pk} g^{ql}) R_{rspq} R_{ijkl} = -2g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} Q_2.$$

Hence

$$\frac{\partial}{\partial t} |Rm|^2 = \chi^2 [\Delta (|Rm|^2) - 2|\nabla Rm|^2] + 2g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} [Q_1 + I_1 + I_2 + I_3].$$

Letting $f = |Rm|$,

$$\frac{\partial}{\partial t} (f^2) = \chi^2 \Delta (f^2) - 2\chi^2 |\nabla Rm|^2 + 2g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} [Q_1 + I_1 + I_2 + I_3].$$

First we have the following estimate:

$$2g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} Q_1 \leq 10\chi^2 f^3.$$

It is easy to get

$$2g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} I_1 \leq 32\sqrt{n} \chi |\nabla \chi| |\nabla Rm| f \leq 2\epsilon (\chi |\nabla Rm|)^2 + \left(\frac{10^4}{\epsilon} \right) n |\nabla \chi|^2 f^2,$$

$$2g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} I_2 \leq 16\sqrt{n} |\nabla \chi|^2 f^2.$$

We can use the normal coordinate system to simplify:

$$2g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} I_3 = 16\chi \sum_{ijkl} (\chi_{il} R_{jk} R_{ijkl}) = 16\chi g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} \chi_{il} R_{jk}.$$

Using normal coordinates it is also easy to see

$$|\nabla |Rm|| \leq |\nabla Rm|.$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} (f^2) &\leq 2f\chi^2 \Delta f + 2\chi^2 |\nabla f|^2 - (2 - 2\epsilon)\chi^2 |\nabla Rm|^2 + 10\chi^2 f^3 \\ &\quad + \left(\frac{10^5}{\epsilon} \right) n |\nabla \chi|^2 f^2 + 16\chi g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} \chi_{il} R_{jk}. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{\partial}{\partial t} f &\leq \chi^2 \Delta f + \left(\frac{\chi^2 |\nabla f|^2}{f} \right) - (1 - \epsilon) \left(\frac{\chi^2 |\nabla Rm|^2}{f} \right) \\ &\quad + 10\chi^2 f^2 + \left(\frac{10^5}{\epsilon} \right) n |\nabla \chi|^2 f + \left(\frac{8\chi g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} \chi_{il} R_{jk}}{f} \right). \end{aligned}$$

Replacing f with $|Rm|$, we get our conclusion. $\square$

Using the formula in Chapter 3 of [1], we can get the following formula (because the calculation is very similar to the reduction of Lemma 3.1; we omit the calculation here, and just give the formula):

Lemma 3.3. *Under the local Ricci flow (1.2),*

$$\begin{aligned} (3.12) \quad \frac{\partial}{\partial t} R_{ij} &= \chi^2 \Delta(R_{ij}) + J_1 + J_2 + J_3 + J_4, \\ \text{where } J_1 &= 2\chi g^{pq} \chi_p \nabla_q R_{ij} + \chi(\chi_i \nabla_j R + \chi_j \nabla_i R) \\ &\quad - 2\chi \chi_p g^{pq} (\nabla_i R_{qj} + \nabla_j R_{qi}), \\ (3.13) \quad J_2 &= -2g^{pq} \chi(\chi_{ip} R_{qj} + \chi_{jp} R_{qi}) + 2\chi \chi_{ij} R, \\ J_3 &= -2g^{pq} (\chi_i \chi_p R_{qj} + \chi_j \chi_p R_{qi}) + 2\chi_i \chi_j R, \\ J_4 &= 2\chi^2 g^{pq} (R_{qij}^r R_{rp} - R_{ip} R_{qj}). \end{aligned}$$

Imitating the reduction of Lemma 3.2, we can also get the following theorem about $|Rc|$:

Lemma 3.4. *Under the local Ricci flow (1.2),*

$$\begin{aligned} (3.14) \quad \frac{\partial}{\partial t} |Rc| &\leq \chi^2 \Delta |Rc| + \left(\frac{\chi^2 |\nabla |Rc||^2}{|Rc|} \right) - (1 - \epsilon) \left(\frac{\chi^2 |\nabla Rc|^2}{|Rc|} \right) \\ &\quad + \left(\frac{10^5}{\epsilon} \right) n |\nabla \chi|^2 |Rc| + (2 + \sqrt{n}) \chi^2 |Rc| |Rm| + \left(\frac{g^{ik} g^{jl} R_{kl} J_2}{|Rc|} \right), \end{aligned}$$

where J_2 in the last line comes from (3.13), and ϵ is any positive constant satisfying $0 < \epsilon < 1$.

Proof. Similar to the proof of Lemma 3.2. $\square$

If T_1 and T_2 are two tensors, we use $T_1 * T_2$ to denote all possible tensor products between them.

Lemma 3.5.

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla^m Rm) &= \chi^2 \Delta (\nabla^m Rm) + \sum_{i+j+k=m+2, k \leq m+1} \nabla^i \chi * \nabla^j \chi * \nabla^k Rm \\ &\quad + \sum_{i+j+k=m} \nabla^i (\chi^2) * \nabla^j Rm * \nabla^k Rm. \end{aligned}$$

Proof. By induction, when $m = 0$, it is true from Lemma 3.1. Suppose the conclusion holds for $m - 1$. Then we have

$$\begin{aligned} \frac{\partial}{\partial t}(\nabla \nabla^{m-1} Rm) &= \nabla^{m-1} Rm * \nabla(\chi^2 Rc) + \nabla(\chi^2 \Delta \nabla^{m-1} Rm) \\ &\quad + \nabla \left[\sum_{i+j+k=m+1, k \leq m} \nabla^i \chi * \nabla^j \chi * \nabla^k Rm \right] \\ &\quad + \nabla \left[\sum_{i+j+k=m-1} \nabla^i(\chi^2) * \nabla^j Rm * \nabla^k Rm \right] \end{aligned}$$

and

$$\begin{aligned} \nabla \left[\chi^2 \Delta \nabla^{m-1} Rm \right] &= \chi^2 \Delta \nabla^m Rm + \chi^2 \nabla^{m-1} Rm * \nabla Rm \\ &\quad + \chi^2 \nabla^m Rm * Rm + \nabla(\chi^2) * \Delta \nabla^{m-1} Rm. \end{aligned}$$

We can consider $\nabla^{m-1} Rm * \nabla(\chi^2 Rc)$, $\chi^2 \nabla^{m-1} Rm * \nabla Rm$, $\chi^2 \nabla^m Rm * Rm$ as the terms in $\nabla \left[\sum_{i+j+k=m-1} \nabla^i(\chi^2) * \nabla^j Rm * \nabla^k Rm \right]$, and $\nabla(\chi^2) * \Delta \nabla^{m-1} Rm$ as one of the terms in $\nabla \left[\sum_{i+j+k=m+1, k \leq m} \nabla^i \chi * \nabla^j \chi * \nabla^k Rm \right]$. Then we get our equation. $\square$

Similarly, we have

Lemma 3.6.

$$\begin{aligned} \frac{\partial}{\partial t} \nabla^m(\chi^2 Rm) &= \chi^2 \Delta \nabla^m(\chi^2 Rm) \\ &\quad + \sum_{i+j+k=m+2, k \leq m+1} \nabla^i \chi * \nabla^j \chi * \nabla^k(\chi^2 Rm) \\ &\quad + \sum_{i+j+k=m, k \leq m-1} \nabla^i(\chi^2) * \nabla^j(\chi^2 Rm) * \nabla^k Rm + \chi^2 Rm * \nabla^m(\chi^2 Rm) \end{aligned}$$

and

$$\frac{\partial}{\partial t} |\nabla^m \chi| \leq C(n, k) \sum_{k \leq m-1} |\nabla^k(\chi^2 Rc)| \cdot |\nabla^{m-k} \chi|.$$

4. EXTENSION OF THE LOCAL RICCI FLOW

First we prove the following ODE comparison lemmas, which will be used in the later proofs of this section.

Lemma 4.1. Assume $f(t)$ and $g(t)$ are nonnegative functions, $t \in [0, T)$, b is a nonnegative constant, and

$$\frac{df}{dt} \leq g + bf.$$

Then on $[0, T)$:

$$f(t) \leq \left[f(0) + \int_0^t e^{-bs} g(s) ds \right] e^{bt}.$$

Proof. Let $h(t) = \left[f(0) + \int_0^t e^{-bs} g(s) ds \right] e^{bt}$. Then $h(0) = f(0)$, and $h'(t) \geq f'(t)$ by assumption. Hence $h(t) \geq f(t)$. $\square$

Lemma 4.2. Assume $f(t)$ is a nonnegative function, $t \in [0, T)$, a, b, c are non-negative constants, and

$$\frac{df}{dt} \leq a \int_0^t f(s) ds + bf(t) + c.$$

Then there exist positive constants w, k which depend on a, b, c and $f(0)$ such that

$$f(t) \leq we^{kt}, \quad t \in [0, T).$$

Proof. It suffices to choose w, k large enough to ensure

$$(we^{kt})' > a \left[\int_0^t (we^{ks}) ds \right] + b(we^{kt}) + c \quad \text{and} \quad w > f(0)$$

to get the comparison function. $\square$

We define $|\nabla \chi|_{(g(t), \infty)} = \max_{x \in \Omega} |\nabla \chi(x)|_{g(t)}$. Recall that we have defined $C_s(\Omega, g(t))$ in (2.6). For simplicity we use the notation $C_s(\Omega, t)$ to replace it.

Lemma 4.3. If $|\chi^2 Rm|_{(g(t), \infty)}$ is uniformly bounded along a given solution $g(t)$ of local Ricci flow, $t \in [0, T)$, $0 < T < +\infty$, then $|\nabla \chi|_{(g(t), \infty)}$ and the Sobolev constant $C_s(\Omega, t)$ are uniformly bounded on $[0, T)$.

Proof. Because Ω is a bounded domain in (M, g_0) , by the proof of Theorem 3.5 in [6] we get that $C_s(\Omega, 0)$ is finite. Then one only needs to notice that the metrics $\{g(t)\}_{t \in [0, T)}$ are uniformly equivalent to each other by assumption. $\square$

Lemma 4.4. Assume $g(t)$, $t \in [0, T)$, $0 < T < +\infty$, is a smooth solution to local Ricci flow on (M^n, g_0) , and $|\chi^2 Rm|_{(g(t), \infty)}$ is uniformly bounded on $[0, T)$. Then for any $p > 100$, $|Rm|_{(g(t), L^p(\Omega))} \triangleq \int_{\Omega} |Rm|_{g(t)}^p dV_{g(t)}$ is uniformly bounded on $[0, T)$.

Proof. Using Lemma 3.1,

$$\frac{\partial}{\partial t} \left(\int_{\Omega} |Rm|_{g(t)}^p dV_{g(t)} \right) = p(I_1 + I_2 + I_3 + I_4 + I_5),$$

where

$$\begin{aligned} I_1 &= - \int_{\Omega} \chi^2 |\nabla Rm|^2 |Rm|^{p-2}, \quad I_2 = \int_{\Omega} |Rm|^{p-2} \langle \chi \nabla \chi * \nabla Rm, Rm \rangle, \\ I_3 &= \int_{\Omega} |Rm|^{p-2} \langle \chi^2 Rm * Rm, Rm \rangle, \quad I_4 = \int_{\Omega} |Rm|^{p-2} \langle \nabla \chi * \nabla \chi * Rm, Rm \rangle, \\ I_5 &= \int_{\Omega} |Rm|^{p-2} \langle \chi \nabla^2 \chi * Rm, Rm \rangle \end{aligned}$$

and

$$\begin{aligned} I_2 &\leq C \int_{\Omega} |Rm|^p + \frac{1}{16} \int_{\Omega} \chi^2 |\nabla Rm|^2 |Rm|^{p-2}, \\ I_3 &\leq \max_{t \in [0, T)} |\chi^2 Rm|_{(g(t), \infty)} \cdot \int_{\Omega} |Rm|^p \leq C \int_{\Omega} |Rm|^p, \\ I_4 &\leq C \max_{t \in [0, T)} |\nabla \chi|_{(g(t), \infty)} \int_{\Omega} |Rm|^p \leq C \int_{\Omega} |Rm|^p. \end{aligned}$$

Applying integration by parts and Lemma 4.3 yields

$$I_5 = \int_{\Omega} |Rm|^{p-2} \langle \chi \nabla^2 \chi * Rm, Rm \rangle \leq C \int_{\Omega} |Rm|^p + \frac{1}{16} \int_{\Omega} \chi^2 |\nabla Rm|^2 |Rm|^{p-2}.$$

Then

$$\frac{\partial}{\partial t} \int_{\Omega} |Rm|^p \leq C \int_{\Omega} |Rm|^p.$$

We get

$$\int_{\Omega} |Rm|_{g(t)}^p dV_{g(t)} \leq e^{CT} \int_{\Omega} |Rm|_{g_0}^p dV_{g_0}.$$

Hence $|Rm|_{(g(t), L^p(\Omega))}$ are uniformly bounded on $[0, T)$. $\square$

Proposition 4.5. *If $g(t)$ is a smooth solution of the local Ricci flow on (M^n, g_0) , $t \in [0, T)$, $0 < T < +\infty$ and m is a nonnegative integer. Assume on $[0, T)$, for any $q > 100$, that $|\nabla^i \chi|_{(g(t), L^q(\Omega))}$ are uniformly bounded, where $0 \leq i \leq m+1$, and $|\chi^2 Rm|_{(g(t), \infty)}$ are uniformly bounded. Then for any $p > 100$, $|\nabla^m Rm|_{(g(t), L^p(\Omega))}$ is uniformly bounded on $[0, T)$.*

Proof. By induction. When $m = 0$, the conclusion follows from Lemma 4.4. For $m \geq 1$, assume that for any $q > 100$, $\int_{\Omega} |\nabla^k Rm|_{g(t)}^q dV_{g(t)}$ is uniformly bounded on $[0, T)$, where $0 \leq k \leq m-1$. For simplicity, we use $\int$ instead of $\int_{\Omega}$ in the rest of the proof.

Using Lemma 3.5,

$$(4.1) \quad \frac{\partial}{\partial t} \int |\nabla^m Rm|_{g(t)}^p dV_{g(t)} \leq p(I_1 + I_2 + I_3) + C \int |\nabla^m Rm|_{g(t)}^p dV_{g(t)},$$

where

$$\begin{aligned} I_1 &= \int |\nabla^m Rm|^{p-2} \langle \chi^2 \Delta \nabla^m Rm, \nabla^m Rm \rangle, \\ I_2 &= \int |\nabla^m Rm|^{p-2} \sum_{i+j+k=m+2, k \leq m+1} \langle \nabla^i \chi * \nabla^j \chi * \nabla^k Rm, \nabla^m Rm \rangle, \\ I_3 &= \int |\nabla^m Rm|^{p-2} \sum_{i+j+k=m} \langle \nabla^i (\chi^2) * \nabla^j (Rm) * \nabla^k Rm, \nabla^m Rm \rangle. \end{aligned}$$

Using integration by parts and the Hölder inequality,

$$\begin{aligned} I_2 &\leq C \int |\nabla^m Rm|^{p-1} \cdot |\nabla^{m+1} Rm| \cdot |\nabla \chi| \cdot |\chi| + C \int |\nabla^m Rm|^p \cdot |\nabla \chi|^2 \\ &\quad + C \sum_{i+j+k=m+2, 1 \leq k \leq m-1} \left[\left(\int |\nabla^m Rm|^p \right)^{\frac{p-1}{p}} \cdot \left(\int |\nabla^k Rm|^{3p} \right)^{\frac{1}{3p}} \right. \\ &\quad \cdot \left. \left(\int |\nabla^i \chi|^{3p} \right)^{\frac{1}{3p}} \cdot \left(\int |\nabla^j \chi|^{3p} \right)^{\frac{1}{3p}} \right] + I_4 \\ &\quad + C \sum_{i+j=m+2, i, j \leq m+1} \left[\left(\int |\nabla^m Rm|^p \right)^{\frac{p-1}{p}} \cdot \left(\int |Rm|^{3p} \right)^{\frac{1}{3p}} \right. \\ &\quad \cdot \left. \left(\int |\nabla^i \chi|^{3p} \right)^{\frac{1}{3p}} \cdot \left(\int |\nabla^j \chi|^{3p} \right)^{\frac{1}{3p}} \right]. \end{aligned}$$

Using the Schwartz inequality,

$$(4.2) \quad \begin{aligned} I_2 &\leq \frac{1}{16} \int \chi^2 |\nabla^{m+1} Rm|^2 |\nabla^m Rm|^{p-2} + C \int |\nabla^m Rm|^p \\ &\quad + C |\nabla^m Rm|_{(g(t), L^p(\Omega))}^{p-1} + I_4, \end{aligned}$$

where $I_4 \triangleq C \int |\nabla^m Rm|^{p-2} \cdot \langle \chi \nabla^{m+2} \chi * Rm, \nabla^m Rm \rangle$. Using integration by parts on I_4 ,

$$\begin{aligned} I_4 &\leq C \int \chi |\nabla^m Rm|^{p-2} \cdot |\nabla^{m+1} Rm| \cdot |\nabla^{m+1} \chi| \cdot |Rm| \\ &\quad + C \int |\nabla^m Rm|^{p-1} \cdot |\nabla^{m+1} \chi| \cdot |\nabla \chi| \cdot |Rm| + I_5 \\ &\leq C \left(\int \chi^2 |\nabla^m Rm|^{p-2} |\nabla^{m+1} Rm|^2 \right)^{\frac{1}{2}} \cdot \left(\int |\nabla^m Rm|^p \right)^{\frac{p-1}{2p}} \\ &\quad + C |\nabla^m Rm|_{(g(t), L^p(\Omega))}^{p-1} + I_5. \end{aligned}$$

In the last inequality, we used the assumption of the proposition and Lemma 4.4. Then

$$(4.3) \quad I_4 \leq \frac{1}{16} \int \chi^2 |\nabla^m Rm|^{p-2} |\nabla^{m+1} Rm|^2 + C |\nabla^m Rm|_{(g(t), L^p(\Omega))}^{p-1} + I_5,$$

where $I_5 \triangleq C \left| \int |\nabla^m Rm|^{p-2} \langle \chi \nabla^{m+1} \chi * \nabla Rm, \nabla^m Rm \rangle \right|$.

To estimate I_5 , there are two cases:

If $m = 1$, then $\nabla Rm = \nabla^m Rm$, and by definition of I_5 ,

$$\begin{aligned} I_5 &\leq C \int |\nabla^m Rm|^{p-1} |\nabla^{m+1} Rm| \cdot |\nabla^m \chi| \cdot |\chi| + C \int |\nabla^m Rm|^p \cdot |\nabla \chi|^2 \\ &\leq \frac{1}{16} \int \chi^2 |\nabla^m Rm|^{p-2} |\nabla^{m+1} Rm|^2 + C \int |\nabla^m Rm|^p \left[|\nabla^m \chi|^2 + |\nabla \chi|^2 \right] \\ (4.4) \quad &\leq \frac{1}{16} \int \chi^2 |\nabla^m Rm|^{p-2} |\nabla^{m+1} Rm|^2 + C \int |\nabla^m Rm|^p. \end{aligned}$$

The last inequality above uses the fact that $m = 1$ and $|\nabla \chi|_{(g(t), \infty)}$ are uniformly bounded on $[0, T]$:

If $m \geq 2$, by induction assumption, for any $q > 100$, $\int |\nabla Rm|^q$ is uniformly bounded on $[0, T]$:

$$\begin{aligned} I_5 &\leq C \int |\nabla^m Rm|^{p-1} \cdot |\nabla^{m+1} \chi| \cdot |\nabla Rm| \\ &\leq C \left(\int |\nabla^m Rm|^p \right)^{\frac{p-1}{p}} \cdot \left(\int |\nabla^{m+1} \chi|^{2p} \right)^{\frac{1}{2p}} \cdot \left(\int |\nabla Rm|^{2p} \right)^{\frac{1}{2p}}, \\ (4.5) \quad &I_5 \leq C |\nabla^m Rm|_{(g(t), L^p(\Omega))}^{p-1}. \end{aligned}$$

By (4.4) and (4.5),

$$(4.6) \quad I_5 \leq \frac{1}{16} \int \chi^2 |\nabla^m Rm|^{p-2} |\nabla^{m+1} Rm|^2 + C \int |\nabla^m Rm|^p + C |\nabla^m Rm|_{(g(t), L^p(\Omega))}^{p-1}.$$

By (4.3) and (4.6),

$$(4.7) \quad I_4 \leq \frac{1}{8} \int \chi^2 |\nabla^m Rm|^{p-2} |\nabla^{m+1} Rm|^2 + C \int |\nabla^m Rm|^p + C |\nabla^m Rm|_{(g(t), L^p(\Omega))}^{p-1}.$$

By (4.2) and (4.7),

$$(4.8) \quad I_2 \leq \frac{3}{16} \int \chi^2 |\nabla^{m+1} Rm|^2 \cdot |\nabla^m Rm|^{p-2} + C \int |\nabla^m Rm|^p + C |\nabla^m Rm|_{(g(t), L^p(\Omega))}^{p-1}.$$

To estimate I_3 ,

$$\begin{aligned}
 I_3 &\leq C \int |\nabla^m Rm|^p |\chi^2 Rm| \\
 &\quad + C \sum_{i+j+k=m, j, k \leq m-1} \int |\nabla^m Rm|^{p-1} |\nabla^i (\chi^2)| |\nabla^j Rm| |\nabla^k Rm| \\
 &\leq C \int |\nabla^m Rm|^p + C \sum_{i_1+i_2+j+k=m, j, k \leq m-1} \\
 &\quad \cdot \left[\left(\int |\nabla^m Rm|^{p-1} \right)^{\frac{p-1}{p}} \cdot \left(\int |\nabla^{i_1} \chi|^{4p} \right)^{\frac{1}{4p}} \right. \\
 &\quad \cdot \left(\int |\nabla^{i_2} \chi|^{4p} \right)^{\frac{1}{4p}} \cdot \left(\int |\nabla^j Rm|^{4p} \right)^{\frac{1}{4p}} \left(\int |\nabla^k Rm|^{4p} \right)^{\frac{1}{4p}} \left. \right].
 \end{aligned}$$

By induction assumption and the assumption of the proposition,

$$(4.9) \quad I_3 \leq C \int |\nabla^m Rm|^p + C |\nabla^m Rm|_{(g(t), L^p(\Omega))}^{p-1}.$$

For I_1 , integration by parts yields

$$I_1 \leq 2 \int |\nabla^m Rm|^{p-1} \chi |\nabla \chi| |\nabla^{m+1} Rm| - \int \chi^2 |\nabla^m Rm|^{p-2} |\nabla^{m+1} Rm|^2.$$

Using the Hölder and Schwartz inequalities,

$$(4.10) \quad I_1 \leq -\frac{15}{16} \int \chi^2 |\nabla^m Rm|^{p-2} |\nabla^{m+1} Rm|^2 + C \int |\nabla^m Rm|^p.$$

By (4.8), (4.9), (4.10) and (4.1),

$$\begin{aligned}
 \frac{\partial}{\partial t} \int |\nabla^m Rm|^p &\leq -\frac{3}{4} \int \chi^2 |\nabla^m Rm|^{p-2} |\nabla^{m+1} Rm|^2 \\
 &\quad + C \int |\nabla^m Rm|^p + C |\nabla^m Rm|_{(g(t), L^p(\Omega))}^{p-1}.
 \end{aligned}$$

By Young's inequality,

$$\frac{\partial}{\partial t} \int |\nabla^m Rm|^p \leq C \left(\int |\nabla^m Rm|^p + 1 \right).$$

Then $\int |\nabla^m Rm|_{g(t)}^p \leq \left[\int |\nabla^m Rm|_{g_0}^p + 1 \right] e^{CT} - 1$, $t \in [0, T]$. Hence $|\nabla^m Rm|_{(g(t), L^p(\Omega))}$ is uniformly bounded on $[0, T]$. $\square$

Theorem 4.6. *If $g(t)$, $t \in [0, T]$, $0 < T < +\infty$ is a smooth solution of the local Ricci flow on (M^n, g_0) , assume $|\chi^2 Rm|_{(g(t), \infty)}$ is uniformly bounded on $[0, T]$. Then for any $m \geq 0$, $|\nabla^m (\chi^2 Rm)|_{(g(t), \infty)}$ is uniformly bounded on $[0, T]$.*

Proof. By Lemma 4.3, if $|\nabla \chi|_{(g(t), \infty)}$ and $C_s(\Omega, t)$ are uniformly bounded on $[0, T]$, and $g(t)$ are uniformly equivalent to each other, then it suffices to prove

(C1) “For any $p > 100$, $m \geq 0$, $|\nabla^m (\chi^2 Rm)|_{(g(t), L^p(\Omega))}$ is uniformly bounded on $[0, T]$.”

We prove (C1) by induction on m . When $m = 0$, the conclusion follows from the assumption of the theorem. Assume

(A1) “For any $p > 100$, $|\nabla^l (\chi^2 Rm)|_{(g(t), L^p(\Omega))}$ is uniformly bounded on $[0, T]$, where $0 \leq l \leq m-1$ and $m \geq 1$.”

We will prove

(C2) “For any $p > 100$, $|\nabla^m(\chi^2 Rm)|_{(g(t), L^p(\Omega))}$ is uniformly bounded on $[0, T)$.”

First we claim:

(C3) “For any $p > 100$, $\int |\nabla^k \chi|^p$ is uniformly bounded on $[0, T)$, where $0 \leq k \leq m$.”

We prove (C3) by induction on k . When $k = 0$, the conclusion follows from the fact that $g(t)$ are uniformly equivalent to each other under the assumption in the theorem. Assume

(A2) “For any $q > 100$, $\int |\nabla^i \chi|^q$ is uniformly bounded on $[0, T)$, where $0 \leq i \leq k - 1$.”

Note $k \leq m$ in our claim. Using Lemma 3.6,

$$\begin{aligned} \frac{\partial}{\partial t} \int |\nabla^k \chi|^p &\leq C \sum_{1 \leq i \leq k-1} \left(\int |\nabla^i \chi|^p \right)^{\frac{p-1}{p}} \cdot \left(\int |\nabla^i(\chi^2 Rm)|^{2p} \right)^{\frac{1}{2p}} \cdot \left(\int |\nabla^{k-i} \chi|^{2p} \right)^{\frac{1}{2p}} \\ &\quad + C \int |\nabla^k \chi|^p \cdot |\chi^2 Rm| \\ &\leq C |\nabla^k \chi|_{(g(t), L^p(\Omega))}^{p-1} + C \int |\nabla^k \chi|^p \leq C \left(\int |\nabla^k \chi|^p + 1 \right). \end{aligned}$$

For the last two inequalities, we used the assumption of the theorem, (A1), (A2) and Young's inequality. Hence $\int |\nabla^k \chi|_{g(t)}^p dV_{g(t)}$ is uniformly bounded on $[0, T)$. From the induction method, claim (C3) is proved.

From the assumption of the theorem, (C3) and Proposition 4.5, we get

(C4) “For any $p > 100$, $|\nabla^i Rm|_{g(t), L^p(\Omega)}$ is uniformly bounded on $[0, T)$, where $0 \leq i \leq m - 1$.”

By Lemma 3.6 and integration by parts,

$$(4.11) \quad \frac{\partial}{\partial t} \int |\nabla^m(\chi^2 Rm)|^p \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= - \int \chi^2 |\nabla^{m+1}(\chi^2 Rm)|^2 |\nabla^m(\chi^2 Rm)|^{p-2}, \\ I_2 &= \sum_{i+j+k=m+2, k \leq m+1} \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), \nabla^i \chi * \nabla^j \chi * \nabla^k(\chi^2 Rm) \rangle, \\ I_3 &= \sum_{i+j+k=m, k \leq m-1} \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), \nabla^i(\chi^2) * \nabla^j(\chi^2 Rm) * \nabla^k(Rm) \rangle, \\ I_4 &= \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), \chi^2 Rm * \nabla^m(\chi^2 Rm) \rangle. \end{aligned}$$

To estimate I_2 ,

$$\begin{aligned} I_2 &= \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), \chi \nabla^{m+2} \chi * (\chi^2 Rm) \rangle \\ &\quad + \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), \nabla \chi * \nabla^{m+1} \chi * (\chi^2 Rm) \rangle \\ &\quad + \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), \chi \nabla^{m+1} \chi * \nabla(\chi^2 Rm) \rangle \\ &\quad + \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), \chi \nabla \chi * \nabla^{m+1}(\chi^2 Rm) \rangle \end{aligned}$$

$$\begin{aligned}
& + \sum_{i+j+k=m+2, i, j \leq m, k \leq m-1} \int |\nabla^m(\chi^2 Rm)|^{p-2} \\
& \quad \cdot \langle \nabla^m(\chi^2 Rm), \nabla^i \chi * \nabla^j \chi * \nabla^k(\chi^2 Rm) \rangle \\
& + \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), (\chi \nabla^2 \chi + \nabla \chi * \nabla \chi) * \nabla^m(\chi^2 Rm) \rangle.
\end{aligned}$$

Using integration by parts,

$$(4.12) \quad I_2 \leq \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5,$$

where

$$\begin{aligned}
\Gamma_1 &= C \int |\nabla^m(\chi^2 Rm)|^{p-2} \cdot |\nabla^{m+1}(\chi^2 Rm)| \cdot \chi |\nabla^{m+1} \chi| \cdot |\chi^2 Rm|, \\
\Gamma_2 &= C \int |\nabla^m(\chi^2 Rm)|^{p-1} \cdot |\nabla \chi| \cdot |\nabla^{m+1} \chi| \cdot |\chi^2 Rm|, \\
\Gamma_3 &= \left| \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), \chi \nabla^{m+1} \chi * \nabla(\chi^2 Rm) \rangle \right|, \\
\Gamma_4 &= C \int \chi |\nabla^{m+1}(\chi^2 Rm)| \cdot |\nabla^m(\chi^2 Rm)|^{p-1} + C \int |\nabla^m(\chi^2 Rm)|^p, \\
\Gamma_5 &= C \sum_{i+j+k=m+2, i, j \leq m, k \leq m-1} \int |\nabla^m(\chi^2 Rm)|^{p-2} \cdot |\nabla^i \chi| \cdot |\nabla^j \chi| \cdot |\nabla^k(\chi^2 Rm)|.
\end{aligned}$$

Using the Hölder and Young inequalities,

$$\begin{aligned}
(4.13) \quad \Gamma_1 &\leq \frac{1}{16} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p \\
&\quad + C \int |\nabla^{m+1} \chi|^p.
\end{aligned}$$

We can also estimate Γ_2 , Γ_4 and Γ_5 in a similar way:

$$(4.14) \quad \Gamma_2 \leq C \int |\nabla^m(\chi^2 Rm)|^p + C \int |\nabla^{m+1} \chi|^p,$$

$$(4.15) \quad \Gamma_4 \leq \frac{1}{16} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p,$$

$$\begin{aligned}
(4.16) \quad \Gamma_5 &\leq C \int |\nabla^m(\chi^2 Rm)|^p + C \sum_{i+j+k=m+2, i, j \leq m, k \leq m-1} \left[\int |\nabla^i \chi|^{3p} \right. \\
&\quad \left. + \int |\nabla^j \chi|^{3p} + \int |\nabla^k(\chi^2 Rm)|^{3p} \right] \\
&\leq C \int |\nabla^m(\chi^2 Rm)|^p + C.
\end{aligned}$$

The last inequality in estimating Γ_5 follows from (A1) and (C3).

To estimate Γ_3 , there are two cases: Case (1) and Case (2).

Case (1): If $m = 1$, using integration by parts,

$$\Gamma_3 \leq C \int |\nabla^m(\chi^2 Rm)|^{p-1} |\nabla^{m+1}(\chi^2 Rm)| (\chi |\nabla \chi|) + C \int |\nabla^m(\chi^2 Rm)|^p |\nabla \chi|^2.$$

Using the fact that $|\nabla \chi|_{(g(t), \infty)}$ is uniformly bounded, we get that if $m = 1$,

$$(4.17) \quad \Gamma_3 \leq \frac{1}{16} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p.$$

Case (2): If $m \geq 2$, using integration by parts,

$$\begin{aligned}
\Gamma_3 &\leq C \int |\nabla^m(\chi^2 Rm)|^{p-2} \cdot |\nabla^{m+1}(\chi^2 Rm)| \cdot |\nabla(\chi^2 Rm)| \cdot \chi |\nabla^m \chi| \\
&\quad + C \int |\nabla^m(\chi^2 Rm)|^{p-1} |\nabla \chi| |\nabla^m \chi| |\nabla(\chi^2 Rm)| + \Gamma_{3,1},
\end{aligned}$$

where

$$\Gamma_{3,1} = \left| \int |\nabla^m(\chi^2 Rm)|^{p-2} \langle \nabla^m(\chi^2 Rm), \chi \nabla^m \chi * \nabla^2(\chi^2 Rm) \rangle \right|.$$

Using the Hölder and Young inequalities,

$$\begin{aligned} \Gamma_3 &\leq \frac{1}{16} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p \\ &\quad + C \int |\nabla(\chi^2 Rm)|^{2p} + C \int |\nabla^m \chi|^{2p} + \Gamma_{3,1}. \end{aligned}$$

Note that $m \geq 2$; using (A1) and (C3),
(4.18)

$$\Gamma_3 \leq \frac{1}{16} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p + C + \Gamma_{3,1}.$$

To estimate $\Gamma_{3,1}$, note that $m \geq 2$. There are still two cases: Case (2.1) and Case (2.2).

Case (2.1): If $m = 2$, using integration by parts and the fact that $|\nabla \chi|_{(g(t), \infty)}$ is uniformly bounded,

$$(4.19) \quad \Gamma_{3,1} \leq \frac{1}{16} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p.$$

Case (2.2): If $m \geq 3$, using the Hölder and Young inequalities,

$$(4.20) \quad \begin{aligned} \Gamma_{3,1} &\leq C \int |\nabla^m(\chi^2 Rm)|^p + C \int |\nabla^m \chi|^{2p} + C \int |\nabla^2(\chi^2 Rm)|^{2p} \\ &\leq C \int |\nabla^m(\chi^2 Rm)|^p + C. \end{aligned}$$

The last inequality follows from (A1) and (C3).

By (4.19) and (4.20),

$$(4.21) \quad \Gamma_{3,1} \leq \frac{1}{16} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p + C.$$

From (4.18) and (4.21), we get that if $m \geq 2$,

$$(4.22) \quad \Gamma_3 \leq \frac{1}{8} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p + C.$$

From (4.17) and (4.22), when $m \geq 1$, (4.22) holds.

By (4.12), (4.13), (4.14), (4.15), (4.16) and (4.22),

$$(4.23) \quad \begin{aligned} I_2 &\leq \frac{1}{4} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p \\ &\quad + C \int |\nabla^{m+1} \chi|^p + C. \end{aligned}$$

To estimate I_3 ,

$$\begin{aligned} I_3 &\leq C \left[\sum_{i_1+i_2+j+k=m, \ k \leq m-1, \ j \leq m-1} \left(\int |\nabla^m(\chi^2 Rm)|^p \right)^{\frac{1}{p}} \cdot \left(\int |\nabla^{i_1} \chi|^{4p} \right)^{\frac{1}{4p}} \right. \\ &\quad \cdot \left(\int |\nabla^{i_2} \chi|^{4p} \right)^{\frac{1}{4p}} \cdot \left(\int |\nabla^j(\chi^2 Rm)|^{4p} \right)^{\frac{1}{4p}} \cdot \left. \left(\int |\nabla^k Rm|^{4p} \right)^{\frac{1}{4p}} \right] \\ &\quad + C \int |\nabla^m(\chi^2 Rm)|^p. \end{aligned}$$

Hence from (A1), (C3), (C4) and Young's inequality,

$$(4.24) \quad I_3 \leq C \int |\nabla^m(\chi^2 Rm)|^p + C.$$

From the assumption of the theorem,

$$(4.25) \quad I_4 \leq C \int |\nabla^m(\chi^2 Rm)|^p.$$

By (4.23), (4.24), (4.25) and (4.11),

$$(4.26) \quad \frac{\partial}{\partial t} \int |\nabla^m(\chi^2 Rm)|^p \leq -\frac{3}{4} \int \chi^2 |\nabla^m(\chi^2 Rm)|^p |\nabla^{m+1}(\chi^2 Rm)|^2 + C \int |\nabla^m(\chi^2 Rm)|^p + C \int |\nabla^{m+1}\chi|^p + C.$$

By Lemma 3.6,

$$\begin{aligned} & \frac{\partial}{\partial t} \int |\nabla^{m+1}\chi|^p \\ & \leq C \int |\nabla^{m+1}\chi|^p \cdot |\chi^2 Rc| + C \int |\nabla^{m+1}\chi|^{p-1} \cdot |\nabla^m(\chi^2 Rm)| \cdot |\nabla\chi| \\ & \quad + C \sum_{1 \leq k \leq m-1} \int |\nabla^{m+1}\chi|^{p-1} \cdot |\nabla^k(\chi^2 Rm)| \cdot |\nabla^{m+1-k}\chi| \\ & \leq C \int |\nabla^{m+1}\chi|^p + C \left(\int |\nabla^{m+1}\chi|^p \right)^{\frac{1}{p}} \cdot \left(\int |\nabla^m(\chi^2 Rm)|^p \right)^{\frac{1}{p}} \\ & \quad + C \sum_{1 \leq k \leq m-1} \left[\left(\int |\nabla^{m+1}\chi|^p \right)^{\frac{1}{p}} \cdot \left(\int |\nabla^k\chi|^{2p} \right)^{\frac{1}{2p}} \cdot \left(\int |\nabla^{m+1-k}\chi|^{2p} \right)^{\frac{1}{2p}} \right]. \end{aligned}$$

The last inequality follows from the Hölder inequality and the assumption of the theorem.

Hence by Young's inequality, (A1) and (C3),

$$(4.27) \quad \frac{\partial}{\partial t} \int |\nabla^{m+1}\chi|^p \leq C_1 \int |\nabla^{m+1}\chi|^p + C_2 \int |\nabla^m(\chi^2 Rm)|^p + C_3.$$

By (4.27) and Lemma 4.1, on $[0, T)$,

$$\begin{aligned} \int |\nabla^{m+1}\chi|_{g(t)}^p dV_{g(t)} & \leq e^{C_1 t} \left[\int |\nabla^{m+1}\chi|_{g_0}^p dV_{g_0} \right. \\ & \quad \left. + \int_0^t e^{-C_1 s} \left(C_3 + C_2 |\nabla^m(\chi^2 Rm)|_{(g(s), L^p(\Omega))}^p \right) ds \right]. \end{aligned}$$

Hence on $[0, T)$,

$$(4.28) \quad \int |\nabla^{m+1}\chi|^p \leq C + C \int_0^t |\nabla^m(\chi^2 Rm)|_{(g(s), L^p(\Omega))}^p ds.$$

By (4.26) and (4.28),

$$(4.29) \quad \begin{aligned} \frac{\partial}{\partial t} |\nabla^m(\chi^2 Rm)|_{(g(t), L^p(\Omega))}^p & \leq C_4 \int_0^t |\nabla^m(\chi^2 Rm)|_{(g(s), L^p(\Omega))}^p ds \\ & \quad + C_5 |\nabla^m(\chi^2 Rm)|_{(g(t), L^p(\Omega))}^p + C_6. \end{aligned}$$

By (4.29) and Lemma 4.2, $|\nabla^m(\chi^2 Rm)|_{(g(t), L^p(\Omega))}^p$ is uniformly bounded on $[0, T)$. Hence (C2) is proved. By the induction method, (C1) is proved. $\square$

Proposition 4.7. *If $g(t)$, $t \in [0, T)$ is a solution to the local Ricci flow on (M^n, g_0) , and for any $m \geq 0$, $|\nabla^m(\chi^2 Rm)|_{(g(t), \infty)}$ is uniformly bounded on $[0, T)$. Then the local Ricci flow can be extended smoothly through time T .*

Proof. Using a similar argument to one in section 7 of chapter 6 of [1], the difference is that $\nabla^m(\chi^2 Rm)$ plays the key role in our local Ricci flow case instead of $\nabla^m Rm$ in Ricci flow case. $\square$

Theorem 4.8. *If $g(t)$, $t \in [0, T)$, $0 < T < +\infty$ is a smooth solution of the local Ricci flow on (M^n, g_0) . Assume $|\chi^2 Rm|_{(g(t), \infty)}$ is uniformly bounded on $[0, T)$, then the local Ricci flow can be extended smoothly through time T .*

Proof. By Theorem 4.6 and Proposition 4.7, we get our conclusion. $\square$

5. EXISTENCE TIME ESTIMATES (THE FIRST CASE: $p_0 > \frac{n}{2}$)

In this section, we use the product of two different cut-off functions to get curvature estimates, so we call them “local local” curvature estimates. Also in this section, we assume that (M^n, g_0) is an n -dimensional ($n \geq 3$) complete noncompact Riemannian manifold satisfying the following assumptions:

$$(5.1) \quad \begin{cases} \left(\int_{B_x(4r_0)} f_{B_x(4r_0)} h^{\frac{2n}{n-2}} dV_{g_0} \right)^{\frac{n-2}{n}} \leq A_0 r_0^2 \int_{B_x(4r_0)} |\nabla h|^2 dV_{g_0}, \\ \left(\int_{B_x(r_0)} |Rm(g_0)|^{p_0} dV_{g_0} \right)^{\frac{1}{p_0}} \leq K_1 \end{cases}$$

for any $x \in M$ and $h \in C_0^\infty(B_x(4r_0))$, where $A_0 \geq 1$, $p_0 > \frac{n}{2}$, K_1 , r_0 are some positive constants, and $B_x(r)$ is the open geodesic ball of radius r centered at x on (M, g_0) . Also, $\bar{f}$ means the usual average of the integrand.

We always assume $\|\nabla \chi\|_\infty \triangleq \|\nabla \chi\|_{(g_0, \infty)} = \sup_{x \in \Omega} (g_0^{ij} \nabla_i \chi \nabla_j \chi)^{\frac{1}{2}} \leq 1$. Now we state some lemmas about local Sobolev constants under the scaling of metrics. There is the well-known local Sobolev inequality

$$(5.2) \quad \left(\int_{B_x(r)} f^{\frac{2n}{n-2}} dV_{g(t)} \right)^{\frac{n-2}{n}} \leq A(B_x(r), t) r^2 \int_{B_x(r)} |\nabla f|^2 dV_{g(t)}, \quad \forall f \in C_0^\infty(B_x(r))$$

with respect to each metric $g(t)$, and $A(B_x(r), t)$ depends on both t and $B_x(r)$. Note that in (5.2), although $B_x(r)$ is the geodesic ball with respect to g_0 , $\bar{f}_{B_x(r)} f^{\frac{2n}{n-2}} dV_{g(t)} \triangleq \frac{1}{V_{g(t)}(B_x(r))} \int_{B_x(r)} f^{\frac{2n}{n-2}} dV_{g(t)}$. When r is fixed, $A(t) \triangleq \sup_{x \in M^n} A(B_x(r), t)$.

Lemma 5.1. *If $a^{-1}g_0 \leq g(t) \leq ag_0$, then*

$$a^{-(n+1)} A_0 \leq A(t) \leq a^{n+1} A_0,$$

where $A_0 = A(0)$ is with respect to g_0 , and $a \geq 1$ is some positive constant.

Proof. It is straightforward from the scaling argument and (5.2). $\square$

Remark 5.2. We have defined two Sobolev constants. The Sobolev constant for $(\Omega, g(t))$ is $C_s(\Omega, g(t))$ defined in (2.6), and local Sobolev constant $A(B_x(r), t)$ is defined in (5.2). They are different from each other, although they satisfy a similar property as in Lemma 5.1.

Lemma 5.3. *Assume the inequality*

$$\left(\int_{B_x(4r_0)} h^{\frac{2n}{n-2}} dV_{g_0} \right)^{\frac{n-2}{n}} \leq A_0 r_0^2 \int_{B_x(4r_0)} |\nabla h|^2 dV_{g_0},$$

where $h \in C_0^\infty(B_x(4r_0))$, $A_0 \geq 1$ and r_0 are fixed positive constants. Then

$$V_{g_0}(B_x(\rho)) \geq N_1 \left(\frac{\rho}{4r_0} \right)^n V_{g_0}(B_x(4r_0)), \quad 0 < \rho \leq 4r_0,$$

where $N_1 = 2^{2n - \frac{n}{2}} A_0^{-\frac{n}{2}}$ and $N_1 \leq 1$.

Proof. For simplicity, we use $V(4r_0)$ instead of $V_{g_0}(B_x(4r_0))$ in the proof. By assumption,

$$\left(\int_{B(4r_0)} |h|^{\frac{2n}{n-1}} \right)^{\frac{n-2}{2n}} \leq A_0^{\frac{1}{2}} r_0 V(4r_0)^{-\frac{1}{n}} \left(\int_{B(4r_0)} |\nabla h|^2 \right)^{\frac{1}{2}}.$$

We recall Theorem 3.1.5 in [11].

Theorem. Assume that the inequality

$$\forall f \in C_0^\infty(M), \quad \|f\|_r \leq (C \|\nabla f\|_p)^\theta \|f\|_s^{1-\theta}$$

is satisfied for some r, s, θ with $0 < s \leq r \leq \infty$ and $0 < \theta \leq 1$. Assume also that

$$\frac{\theta}{p} + \frac{1-\theta}{s} - \frac{1}{r} > 0.$$

Then $V(x, \rho) \geq c\rho^\nu$ with ν defined by

$$\frac{\theta}{\nu} = \frac{\theta}{p} + \frac{1-\theta}{s} - \frac{1}{r}$$

and the constant c given by $c = 2^{-\frac{\nu^2}{\theta r} - \frac{\nu}{\theta}} C^{-\nu}$.

Let $\theta = 1$, $s = r = \frac{2n}{n-2}$, $p = 2$ in the above theorem. Also, $C = \frac{A_0^{\frac{1}{2}} r_0}{V^{\frac{1}{n}}}$, where $V = V(4r_0)$, and $\nu = n$. Then $c = 2^{-\frac{n^2}{2}} (A_0^{\frac{1}{2}} r_0)^{-n} V$, and by the above theorem,

$$V(\rho) \geq c\rho^\nu = 2^{2n-\frac{n^2}{2}} A_0^{-\frac{n}{2}} \left(\frac{\rho}{4r_0} \right)^n V \geq N_1 \left(\frac{\rho}{4r_0} \right)^n V(4r_0). \quad \square$$

Lemma 5.4. (M^n, g) is a smooth, complete Riemannian manifold with the property:

$$V_g(B_x(\rho)) \geq N_1 \left(\frac{\rho}{4r_0} \right)^n \text{Vol}(B_x(4r_0)), \quad 0 < \rho \leq 4r_0, \quad \forall x \in M,$$

where r_0 and N_1 are fixed positive constants. Then there exists a sequence (x_i) of points of M such that

(I) $M = \bigcup_{i=1}^\infty B_{x_i}(r_0)$.

(II) For any $i \neq j$, $B_{x_i}(\frac{r_0}{2}) \cap B_{x_j}(\frac{r_0}{2}) = \emptyset$.

(III) For any j , $B_{x_j}(r_0)$ intersects at most N balls $B_{x_i}(r_0)$, $i \neq j$, where $N = 2^{4n} N_1^{-2} = 2^{n^2} A_0^n$.

Proof. Let $X_{r_0} = \{(x_i)_I, x_i \in M, \text{ s.t. } I \text{ is countable and } \forall i \neq j, d_g(x_i, x_j) \geq r_0\}$. As one can easily check, X_{r_0} is partially ordered by inclusion and every chain in X_{r_0} has an upper bound. Hence, by Zorn's lemma, X_{r_0} contains a maximal element (x_i) and (x_i) satisfies (I) and (II). From now on, let (x_i) be such that (I) and (II) are satisfied. For $x \in M$, we define

$$I_{2r_0}(x) = \{i : x \in B_{x_i}(2r_0)\}.$$

Then

$$(5.3) \quad V(B_x(2r_0)) \geq \frac{N_1}{2^n} V(B_x(4r_0)) \geq \frac{N_1}{2^n} \sum_{i \in I_{2r_0}(x)} V(B_{x_i}(\frac{r_0}{2})).$$

The last inequality follows from $\bigcup_{i \in I_{2r_0}(x)} B_{x_i}(\frac{r_0}{2}) \subseteq B_x(4r_0)$ and $B_{x_i}(\frac{r_0}{2}) \cap B_{x_j}(\frac{r_0}{2}) = \emptyset$ if $i \neq j$. By the assumption, $V(B_{x_i}(\frac{r_0}{2})) \geq \frac{N_1}{8^n} V(B_{x_i}(4r_0))$. Also, for $i \in I_{2r_0}(x)$, $B_x(2r_0) \subseteq B_{x_i}(4r_0)$,

$$(5.4) \quad V(B_{x_i}(\frac{r_0}{2})) \geq \frac{N_1}{8^n} V(B_x(2r_0)).$$

By (5.3) and (5.4),

$$V(B_x(2r_0)) \geq N_1^2 2^{-4n} \text{Card}(I_{2r_0}(x)) V(B_x(2r_0)),$$

where Card stands for the cardinality. For any $x \in M$, $\text{Card}(I_{2r_0}(x)) \leq 2^{4n} N_1^{-2}$. For fixed x_j , if $B_{x_j}(r_0) \cap B_{x_i}(r_0) \neq \emptyset$, then $x_i \in B_{x_j}(2r_0)$, $i \in I_{2r_0}(x_j)$ and we get that $N \leq \text{Card}(I_{2r_0}(x_j)) \leq 2^{4n} N_1^{-2}$ $\square$

Lemma 5.5. *For the covering $(B_{x_i}(r_0))$ of (M, g) as in Lemma 5.4, we can find $\xi_i \in C_0^{0,1}(B_{x_i}(r_0))$, $0 \leq \xi_i \leq 1$, $\sum_i \xi_i^2 = 1$, $|\nabla \xi_i|^2 \leq C(r_0, N)$, where $C(r_0, N) = \frac{8}{r_0^2}(N+1)$ and N is as in Lemma 5.4.*

Proof. Let $\rho : [0, \infty) \rightarrow [0, 1]$ be defined by

$$(5.5) \quad \rho(t) = \begin{cases} 1, & 0 \leq t \leq \frac{r_0}{2}, \\ (2 - \frac{2}{r_0}t)^2, & \frac{r_0}{2} \leq t \leq r_0, \\ 0, & r_0 \leq t. \end{cases}$$

Let $\alpha_i(x) = \rho(d_g(x, x_i))$, where d_g denotes the distance associated to g and $x \in M$. Clearly, α_i is Lipschitz with compact support. Note that for any m , $|\nabla \alpha_m| \leq 2(2 - \frac{2}{r_0}t) \frac{2}{r_0} = \frac{4}{r_0} \alpha_m^{\frac{1}{2}}$; that is, $\frac{|\nabla \alpha_m|^2}{|\alpha_m|} \leq \left(\frac{4}{r_0}\right)^2$. Let $\eta_i = \frac{\alpha_i}{\sum_m \alpha_m}$. Then η_i is a partition of unity subordinate to the covering $(B_{x_i}(r_0))$:

$$\begin{aligned} |\nabla \eta_i| &\leq \left| \frac{\nabla \alpha_i}{(\sum_m \alpha_m)} \right| + \frac{|\alpha_i| \cdot |\sum_m \nabla \alpha_m|}{(\sum_m \alpha_m)^2} \leq |\nabla \alpha_i| + \frac{4\eta_i}{r_0} \cdot \frac{\sum_m \alpha_m^{\frac{1}{2}}}{\sum_m \alpha_m} \\ &\leq |\nabla \alpha_i| + \frac{4\eta_i}{r_0} \sqrt{N}. \end{aligned}$$

In the last equality, we use the Cauchy-Schwartz inequality and property (III) in Lemma 5.4.

Let $\xi_i = \eta_i^{\frac{1}{2}}$; then $\xi_i \in C_0^{0,1}(B_{x_i}(r_0))$, $0 \leq \xi_i \leq 1$, $\sum_i \xi_i^2 = 1$. Hence

$$|\nabla \xi_i|^2 = \frac{1}{4} \cdot \frac{|\nabla \eta_i|^2}{|\eta_i|} \leq \frac{1}{4|\alpha_i|} \cdot 2 \left(|\nabla \alpha_i|^2 + \left(\frac{4}{r_0} \eta_i \sqrt{N}\right)^2 \right) \leq \frac{8}{r_0^2} (N+1). \quad \square$$

For simplicity, we use the notation B_i to replace $B_{x_i}(r_0)$ in the rest of this section. Note that (x_i) are chosen as in Lemma 5.4 and are fixed in the rest of this section and in section 6. We also use ξ to replace ξ_i in some proofs of this section; it will be clear from the context when we do such a replacement. We use the following notation in the rest of this section:

$$\oint h dV_{g(t)} \triangleq \frac{1}{V_{g(t)}(B_i)} \int_{B_i \cap \Omega} h dV_{g(t)}$$

for any $h \in C_0^\infty(B_i)$, and it will be clear from the context which B_i we choose.

From Theorem 2.9 and DeTurck's trick, local Ricci flow (1.2) has a smooth solution on a sufficiently small time interval starting at $t = 0$. By (5.1), we can assume that $[0, T_{\max})$ is the maximal time interval, on which local Ricci flow (1.2) has a smooth solution and such that the following hold for each metric $g(t)$ (where $g(0) = g_0$):

$$(5.6) \quad \left(\int_{B_x(4r_0)} h^{\frac{2n}{n-2}} dV_{g(t)} \right)^{\frac{n-2}{n}} \leq 4A_0 r_0^2 \int_{B_x(4r_0)} |\nabla h|^2 dV_{g(t)},$$

$$(5.7) \quad \frac{1}{2}g_0 \leq g(t) \leq 2g_0,$$

$$(5.8) \quad \left(\frac{1}{V_{g(t)}(B_i)} \int_{B_i \cap \Omega} |Rm(g(t))|^{p_0} dV_{g(t)} \right)^{\frac{1}{p_0}} \leq (2N)^{\frac{1}{p_0}} K_1, \quad i = 1, 2, 3, \dots,$$

for any $x \in M$, $h \in C_0^\infty(B_x(4r_0))$ and B_i chosen as in Lemma 5.4.

In the following lemma, we get the parabolic version of energy estimates on $[0, T_{\max})$ under the local Ricci flow.

Lemma 5.6. *For $0 \leq t < T_{\max}$, any i , $q \geq 1$, $p \geq \frac{n}{2}$, $p' \geq 0$ and $p \geq p'$, there exists some positive constant*

$$(5.9) \quad C_1(A_0, K_1, n, p_0, r_0) = C(n, p_0) A_0^{\frac{(2p_0+3)n}{2p_0-n}} K_1^{\frac{2p_0}{2p_0-n}} \left[r_0^{-2} + r_0^{\frac{2n}{2p_0-n}} \right]$$

such that

$$(5.10) \quad \begin{aligned} & \frac{\partial}{\partial t} \left(\int \xi_i^{2q} \chi^{2p'} |Rm|^p dV_{g(t)} \right) + \frac{1}{10} \left(\int |\nabla(\xi_i^q \chi^{p'+1} |Rm|^{\frac{p}{2}})|^2 dV_{g(t)} \right) \\ & \leq C_1(A_0, K_1, n, p_0, r_0) p^{\frac{n}{2p_0-n}+3} q^2 \left(\int \xi_i^{2q-2} \chi^{2p'} |Rm|^p dV_{g(t)} \right). \end{aligned}$$

Proof. Set $f(x, t) = |Rm(x)|_{g(t)}$. We get the following:

$$(5.11) \quad \frac{\partial}{\partial t} \left(\int \xi^{2q} \chi^{2p'} f^p \right) \leq \sum_{k=1}^7 I_k,$$

where

$$I_1 = p \int \xi^{2q} \chi^{2p'+2} f^{p-1} \Delta f, \quad I_2 = p \int \xi^{2q} \chi^{2p'+2} f^{p-2} |\nabla f|^2,$$

$$I_3 = -(1-\epsilon) p \int \xi^{2q} \chi^{2p'+2} f^{p-2} |\nabla Rm|^2,$$

$$I_4 = 10p \int \xi^{2q} \chi^{2p'+2} f^{p+1} - \int \xi^{2q} \chi^{2p'+2} f^p R,$$

$$I_5 = \left(\frac{10^5}{\epsilon} \right) np \int \xi^{2q} \chi^{2p'} |\nabla \chi|^2 f^p,$$

$$I_6 = 8p \int \xi^{2q} \chi^{2p'+1} f^{p-2} g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} R_{jkl} \chi_{il},$$

$$I_7 = n(n-1) \left(\int \xi^{2q} \chi^{2p'} f^p \right) \cdot \left(\frac{1}{V_{g(t)}(B_i)} \int_{B_i} \chi^2 f \right).$$

For simplification, we set the following notation:

$$\begin{aligned} \tilde{a} &= \xi^q \chi^{p'} f^{\frac{p}{2}} |\nabla \chi|, \quad \tilde{b} = \xi^q \chi^{p'+1} f^{\frac{p}{2}-1} |\nabla f|, \\ \tilde{c} &= \xi^q \chi^{p'+1} f^{\frac{p}{2}-1} |\nabla Rm|, \quad \tilde{d} = f |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2, \quad \tilde{e} = \xi^{q-1} \chi^{p'+1} f^{\frac{p}{2}} |\nabla \xi|. \end{aligned}$$

Then

$$I_1 \leq -p(p-1)f\tilde{b}^2 + 2\epsilon pf\tilde{b}^2 + \frac{pq^2}{\epsilon}f\tilde{e}^2 + \frac{p(p'+1)^2}{\epsilon}f\tilde{a}^2,$$

$$I_2 \leq p\int \tilde{b}^2, \quad I_3 = -(1-\epsilon)p\int \tilde{c}^2, \quad I_5 \leq \left(\frac{10^5}{\epsilon}\right)n p\int \tilde{a}^2,$$

where $0 < \epsilon < \frac{1}{8}$ is some constant to be determined later.

Using integration by parts and Young's inequality,

$$I_6 \leq \epsilon p\int \tilde{c}^2 + \frac{1}{\epsilon}10^6 n^3 p^3 q^2 \left(\int \tilde{a}^2 + \int \tilde{e}^2\right).$$

Also,

$$I_7 \leq n(n-1)\int \xi^{2q}\chi^{2p'}f^p \cdot \left(\frac{1}{V_{g(t)}(B_i)} \int_{B_i \cap \Omega} f^{p_0}\right)^{\frac{1}{p_0}}$$

$$\leq C(n, p_0)A_0^{\frac{n}{p_0}}K_1 \left(\int \xi_i^{2q-2}\chi^{2p'}f^p dV_{g(t)}\right).$$

In the last inequality above, we use the fact that $N = 2^{n^2}A_0^n$ by Lemma 5.3 and Lemma 5.4.

Note that $|R| \leq n|Rm| \leq 2pf$, $p'_0 < \frac{n}{n-2}$, where $\frac{1}{p_0} + \frac{1}{p'_0} = 1$, and

$$I_4 \leq 12pf\xi^{2q}\chi^{2p'+2}f^{p+1} \leq 12p\|f\|_{p_0, B_i \cap \Omega} \cdot \left[f(\xi^{2q}\chi^{2p'+2}f^p)^{p'_0}\right]^{\frac{1}{p'_0}}$$

$$\leq 12p(2N)^{\frac{1}{p_0}}K_1 \cdot \left[C(\epsilon, p_0, n)(2NpK_1A_0r_0^{\frac{n}{2p_0-n}}\int \xi^{2q}\chi^{2p'+2}f^p\right.$$

$$\left. + \frac{\epsilon}{2NpK_1A_0r_0^{\frac{n}{2p_0-n}}}\left(f(\xi^{2q}\chi^{2p'+2}f^p)^{\frac{n-2}{n-2}}\right)^{\frac{n-2}{n}}\right]$$

$$\leq 100\epsilon\tilde{d} + C(\epsilon, n, p_0)p^{\frac{n}{2p_0-n}+1}q^2A_0^{\frac{n^2+3n}{2p_0-n}}K_1^{\frac{2p_0}{2p_0-n}}r_0^{\frac{2n}{2p_0-n}}f_{B_i \cap \Omega}\xi^{2q-2}\chi^{2p'+2}f^p,$$

where $\|f\|_{p_0, B_i \cap \Omega} = \left(\frac{1}{V_{g(t)}(B_i)} \int_{B_i \cap \Omega} f^{p_0}\right)^{\frac{1}{p_0}}$.

Combining the above estimates, note that $\tilde{b} \leq \tilde{c}$. Then

$$(5.12) \quad \begin{aligned} \frac{\partial}{\partial t} \left(\int \xi^{2q}\chi^{2p'}f^p \right) &\leq p(1+4\epsilon-p)f\tilde{b}^2 + 100\epsilon\tilde{d} \\ &\quad + \frac{2}{\epsilon}10^6 n^3 p^3 q^2 \left(\int \tilde{a}^2 + \int \tilde{e}^2 \right) \\ &\quad + C(\epsilon, n, p_0)p^{\frac{n}{2p_0-n}+1}q^2A_0^{\frac{n^2+3n}{2p_0-n}}K_1^{\frac{2p_0}{2p_0-n}}(r_0^{\frac{2n}{2p_0-n}}+1) \\ &\quad \cdot \left(\int \xi^{2q-2}\chi^{2p'+2}f^p \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \left| \nabla(\xi^q\chi^{p'+1}f^{\frac{p}{2}}) \right|^2 &\leq (q\tilde{e} + (p'+1)\tilde{a} + \frac{p}{2}\tilde{b})^2 \\ &\leq \left(2 + \frac{1}{2\epsilon}\right)q^2\tilde{e}^2 + \left(2 + \frac{1}{2\epsilon}\right)(p'+1)^2\tilde{a}^2 + \left(\frac{1}{4} + \epsilon\right)p^2\tilde{b}^2. \end{aligned}$$

Then

$$(5.13) \quad \int \tilde{b}^2 \geq \left[\left(\frac{1}{4} + \epsilon\right)p^2\right]^{-1}\tilde{d} - \left[\left(\frac{1}{4} + \epsilon\right)p^2\right]^{-1}\left(2 + \frac{1}{2\epsilon}\right)(q+p'+1)^2\int(\tilde{a}^2 + \tilde{e}^2).$$

By (5.12), (5.13), and the estimate $f\tilde{\epsilon}^2$, using $|\tilde{\nabla}\xi_i|^2 \leq C(r_0, N) = \frac{8}{r_0^2}(N+1)$ as in Lemma 5.5, $0 \leq \xi_i \leq 1$, $0 \leq \chi \leq 1$, also noting that $1 + 4\epsilon - p < 0$,

$$(5.14) \quad \begin{aligned} & \frac{\partial}{\partial t} \left(\int \xi^{2q} \chi^{2p'} f^p \right) + \left(\frac{4(p-1-4\epsilon)}{(1+4\epsilon)p} - 100\epsilon \right) \tilde{d} \\ & \leq C(\epsilon, n, p_0) p^{\frac{n}{2p_0-n}+3} q^2 A_0^{\frac{(2p_0+3)n}{2p_0-n}} K_1^{\frac{2p_0}{2p_0-n}} [r_0^2 + r_0^{\frac{2n}{2p_0-n}}] \int \xi^{2q-2} \chi^{2p'} f^p. \end{aligned}$$

Choose $\epsilon = \frac{1}{1000}$ in (5.14). Then

$$(5.15) \quad \begin{aligned} & \frac{\partial}{\partial t} \left(\int \xi^{2q} \chi^{2p'} f^p \right) + \frac{1}{10} \left(\int |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2 \right) \\ & \leq C(n, p_0) p^{\frac{n}{2p_0-n}+3} q^2 A_0^{\frac{(2p_0+3)n}{2p_0-n}} K_1^{\frac{2p_0}{2p_0-n}} \left[r_0^2 + r_0^{\frac{2n}{2p_0-n}} \right] \int \xi^{2q-2} \chi^{2p'} f^p. \end{aligned}$$

Replacing f with $|Rm|$, the lemma is proved. $\square$

For simplicity, we use C_1 instead of $C_1(A_0, K_1, n, p_0, r_0)$ in the rest of this section; similarly for C_2 , etc. In the next lemma, we use the energy estimate and modified Moser iteration to get the local C^0 -estimate of $|\chi^2 Rm(g(t))|$.

Lemma 5.7. *If $0 < t < T_{\max}$, then there exists some positive constant*

$$(5.16) \quad C_2(A_0, K_1, n, p_0, r_0) = C(n, p_0) A_0^{\left(\frac{3}{2p_0}+1\right)n} \left(r_0^{\frac{n}{p_0}} K_1 \right)$$

such that

$$(5.17) \quad |\chi^2 Rm(g(t))| \leq C_2 \left[C_1^{\frac{n+2}{2p_0}} t^{\frac{1}{p_0}} + t^{-\frac{n}{2p_0}} \right].$$

Proof. Let $f(x, t) = |Rm(x)|_{g(t)}$. Given any t_1 and t_2 such that $0 < t_1 < t_2 < T_{\max}$, set

$$\psi(t) = \begin{cases} 0, & 0 \leq t \leq t_1, \\ \frac{t-t_1}{t_2-t_1}, & t_1 \leq t \leq t_2, \\ 1, & t_2 \leq t < T_{\max}. \end{cases}$$

By Lemma 5.6,

$$(5.18) \quad \begin{aligned} & \frac{\partial}{\partial t} \left(\int \xi_i^{2q} \chi^{2p'} f^p dV_{g(t)} \right) + \frac{1}{10} \left(\int |\nabla(\xi_i^q \chi^{p'+1} f^{\frac{p}{2}})|^2 dV_{g(t)} \right) \\ & \leq C_1 p^{\frac{n}{2p_0-n}+3} q^2 \left(\int \xi_i^{2q-2} \chi^{2p'} f^p dV_{g(t)} \right). \end{aligned}$$

Multiply (5.18) by $\psi(t)$, and integrate it from 0 to t_0 with respect to t , where $t_2 \leq t_0 < T_{\max}$. Then

$$(5.19) \quad \begin{aligned} & \int (\xi^{2q} \chi^{2p'} f^p) dV_{g(t_0)} + \frac{1}{10} \int_{t_2}^{t_0} \int |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2 \\ & \leq C A_0^{\frac{(2p_0+3)n}{2p_0-n}} K_1^{\frac{2p_0}{2p_0-n}} \left[r_0^2 + r_0^{\frac{2n}{2p_0-n}} \right] p^{\frac{n}{2p_0-n}+3} q^2 \left(1 + \frac{1}{t_2-t_1} \right) \int_{t_1}^{t_0} \\ & \quad \cdot \left(\int \xi^{2(q-1)} \chi^{2p'} f^p \right). \end{aligned}$$

Note (5.19) is in fact valid for any $0 < t_1 < t_2 \leq t_0 < T_{\max}$.

For $0 < t \leq t_0$, denote

$$H(p, p', q, t) = \int_t^{t_0} \int \xi^{2q} \chi^{2p'} f^p.$$

Then for any $0 < \iota < \iota' \leq t_0 < T_{\max}$, using (5.6), (5.19) and the Hölder inequality,

$$\begin{aligned} H(p(1 + \frac{2}{n}), p'(1 + \frac{2}{n}) + 1, q(1 + \frac{2}{n}), \iota') &= \int_{\iota'}^{t_0} f \left(\xi^{2q} \chi^{2p'} f^p \right)^{\frac{2}{n}} \left(\xi^q \chi^{p'+1} f^{\frac{p}{2}} \right)^2 \\ &\leq \int_{\iota'}^{t_0} \left[f \left(\xi^{2q} \chi^{2p'} f^p \right) \right]^{\frac{2}{n}} \cdot 4A_0 r^2 \left[f \left| \nabla \left(\xi^q \chi^{p'+1} f^{\frac{p}{2}} \right) \right|^2 \right] dt \\ &\leq 40A_0 r_0^2 \left[C_1 p^{\frac{n}{2p_0-n}+3} q^2 + \frac{1}{\iota'-\iota} \right]^{1+\frac{2}{n}} H(p, p', q-1, \iota)^{1+\frac{2}{n}}. \end{aligned}$$

Denote $\alpha = \frac{n}{2p_0-n} + 3$, $\nu = 1 + \frac{2}{n}$, $\eta = \nu^{n+2}$, fix $t \in (0, t_0)$, and set

$$\begin{aligned} p_k &= p_0 \nu^k, \quad p'_k = p_k - \frac{n}{2}, \quad p'_0 = p_0 - \frac{n}{2}, \\ q_k &= n \nu^k - \left(\frac{n}{2} + 1 \right), \quad q_0 = \frac{n}{2} - 1, \quad \iota_k = t(1 - \eta^{-k}), \\ H_k &= H(p_k, p'_k, q_k, \iota_k), \quad \Phi_k = H_k^{\frac{1}{p_0}}, \end{aligned}$$

$$\Phi_0 = \left(\int_0^{t_0} \int \xi^{n-2} \chi^{2p_0-n} f^{p_0} \right)^{\frac{1}{p_0}}.$$

Then for any $k \geq 0$,

$$\begin{aligned} H_{k+1} &= H(p_{k+1}, p'_{k+1}, q_{k+1}, \iota_{k+1}) \\ &= H\left(p_k(1 + \frac{2}{n}), p'_k(1 + \frac{2}{n}) + 1, (q_k + 1)(1 + \frac{2}{n}), \iota_{k+1}\right) \\ &\leq 40A_0 r_0^2 \left[C_1 p_k^\alpha (q_k + 1)^2 + \frac{1}{\iota_{k+1}-\iota_k} \right]^\nu \cdot H(p_k, p'_k, q_k, \iota_k)^\nu \\ &\leq 40A_0 r_0^2 \left[C_1 p_0^\alpha n^2 \nu^{(\alpha+2)k} + t^{-1} \frac{n}{\eta-1} \eta^k \right]^\nu H_k^\nu \\ &\leq C(n, p_0) A_0 r_0^2 [C_1 + t^{-1}]^\nu \nu^{(\alpha+n+2)k\nu} H_k^\nu. \end{aligned}$$

Taking the p_{k+1} root on both sides of the above inequality,

$$\Phi_{k+1} \leq \left[C(n, p_0) A_0 r_0^2 \right]^{p_0^{-1} \nu^{-(k+1)}} \nu^{(\alpha+n+2) \frac{k\nu-k}{p_0}} \left[C_1 + t^{-1} \right]^{p_0^{-1} \nu^{-k}} \Phi_k.$$

By induction,

$$\Phi_{k+1} \leq \left[C(n, p_0) A_0 r_0^2 \right]^{p_0^{-1} (\sigma_{k+1}-1)} \nu^{(\alpha+n+2) \frac{\sigma'_k}{p_0}} \left[C_1 + t^{-1} \right]^{p_0^{-1} \sigma_k} \Phi_0,$$

where $\sigma_k = \sum_{i=0}^k \nu^{-i}$ and $\sigma'_k = \sum_{i=0}^k i \nu^{-i}$; note that $\frac{\sigma_{k+1}-1}{p_0} \leq \frac{n}{2p_0}$. Then

$$\begin{aligned} |\xi_i^{\frac{n}{p_0}} \chi^2 f(x, t)| &\leq \lim_{k \rightarrow \infty} \Phi_{k+1} \\ &\leq C(n, p_0) A_0^{\frac{n}{2p_0}} \left[C_1 + t^{-1} \right]^{\frac{n+2}{2p_0}} r_0^{\frac{n}{p_0}} \cdot \left(\int_0^{t_0} \int_{B_i \cap \Omega} \xi_i^{n-2} \chi^{2p_0-n} f^{p_0} \right)^{\frac{1}{p_0}}. \end{aligned}$$

Letting $t_0 \rightarrow t$ in the above and using (5.8),

$$\begin{aligned} |\xi_i^{\frac{n}{p_0}} \chi^2 f(x, t)| &\leq C(n, p_0) A_0^{\frac{n}{2p_0}} \left[C_1 + t^{-1} \right]^{\frac{n+2}{2p_0}} r_0^{\frac{n}{p_0}} \left[(2N)^{\frac{1}{p_0}} K_1 \right] t^{\frac{1}{p_0}} \\ &\leq C(n, p_0) A_0^{\frac{3n}{2p_0}} \left(r_0^{\frac{n}{p_0}} K_1 \right) \left[C_1^{\frac{n+2}{2p_0}} t^{\frac{1}{p_0}} + t^{-\frac{n}{2p_0}} \right]. \end{aligned}$$

Now for any $x \in \Omega$, we assume $x \in B_{i_0} \cap \Omega$. Note that $|\xi_i^2 \chi^2 f| \leq |\xi_{i_0}^{\frac{n}{p_0}} \chi^2 f|$; then

$$|\chi^2 f|(x, t) = \left| \sum_{j=1}^N \xi_j^2 \chi^2 f(x, t) \right| \leq C(n, p_0) A_0^{\left(\frac{3}{2p_0} + 1 \right)n} \left(r_0^{\frac{n}{p_0}} K_1 \right) \left[C_1^{\frac{n+2}{2p_0}} t^{\frac{1}{p_0}} + t^{-\frac{n}{2p_0}} \right].$$

By replacing f with $|Rm|$, we obtain our conclusion. $\square$

Straightforward from Lemma 5.7, we have the following two corollaries.

Corollary 5.8. *If $0 < t < T_{\max}$, there exists some positive constant*

$$(5.20) \quad C_3(A_0, K_1, n, p_0, r_0) = C(n, p_0) A_0^{\left(\frac{3}{2p_0} + 1\right)n} \left(r_0^{\frac{n}{p_0}} K_1\right)$$

such that

$$(5.21) \quad |\chi^2 Rc(g(t))| \leq C_3 \left(t^{-\frac{n}{2p_0}} + C_1^{\frac{n+2}{2p_0}} t^{\frac{1}{p_0}} \right).$$

Corollary 5.9. *If $0 \leq t < T_{\max}$, there exists some positive constant C , which is independent of t , such that*

$$|\chi^2 Rm(g(t))| \leq C.$$

Choose $T_1 > 0$ such that

$$\exp \left(C_1 p_0^{\frac{n}{2p_0-n}+3} N T_1 \right) = \frac{3}{2},$$

where C_1 is from Lemma 5.6 and N is from Lemma 5.4. Then

$$(5.22) \quad T_1 = C(n, p_0) A_0^{-\frac{5p_0 n}{2p_0-n}} K_1^{-\frac{2p_0}{2p_0-n}} \left[r^{-2} + r_0^{\frac{2n}{2p_0-n}} \right]^{-1}.$$

Proposition 5.10. *If $0 \leq t < T_{\max} < T_1$, then*

$$\left(\frac{1}{V_{g(t)}(B_i)} \int_{B_i \cap \Omega} |Rm(g(t))|^{p_0} dV_{g(t)} \right)^{\frac{1}{p_0}} \leq \left(\frac{3}{2} N \right)^{\frac{1}{p_0}} K_1, \quad i = 1, 2, 3, \dots$$

Proof. Set $f(x, t) \doteq |Rm(x)|_{g(t)}$. Let $q = 1$, $p' = 0$, $p = p_0$ in Lemma 5.6; then

$$\frac{\partial}{\partial t} \left(\int_{\Omega} \xi_i^2 f^{p_0} dV_{g(t)} \right) \leq C_1 p_0^{\frac{n}{2p_0-n}+3} \int_{B_i \cap \Omega} f^{p_0} dV_{g(t)}.$$

For Ω , define $Cov(\Omega) = \{B_i | B_i \cap \bar{\Omega} \neq \emptyset\}$ and note that $Cov(\Omega)$ has only a finite number of elements. We define

$$\phi(t) = \sup_{B_i \in Cov(\Omega)} \left[\int_{B_i \cap \Omega} \xi_i^2 |Rm|^{p_0} dV_{g(t)} \right].$$

Assume $\phi(t_0) = \int_{B_{i_0} \cap \Omega} \xi_{i_0}^2 |Rm|^{p_0} dV_{g(t_0)}$; then

$$(5.23) \quad \begin{aligned} \frac{\partial}{\partial t} \phi(t_0) &\leq \frac{\partial}{\partial t} \left(\int_{\Omega} \xi_{i_0}^2 f^{p_0} \right)(t_0) \leq C_1 p_0^{\frac{n}{2p_0-n}+3} \int_{B_{i_0} \cap \Omega} f^{p_0} dV_{g(t_0)} \\ &\leq C_1 p_0^{\frac{n}{2p_0-n}+3} \sum_{j=1}^N \left(\int_{B_{i_0} \cap \Omega} \xi_j^2 f^{p_0} dV_{g(t_0)} \right) \leq C_1 p_0^{\frac{n}{2p_0-n}+3} N \phi(t_0). \end{aligned}$$

Because t_0 is chosen freely,

$$(5.24) \quad \frac{\partial}{\partial t} \phi(t) \leq \left[C_1 p_0^{\frac{n}{2p_0-n}+3} N \right] \phi(t).$$

By (5.1), $\phi(0) \leq \sup_{B_i} \int_{B_i \cap \Omega} |Rm|^{p_0} dV_{g_0} \leq K_1^{p_0}$. Then by the definition of T_1 and (5.24),

$$\phi(t) \leq \frac{3}{2} K_1^{p_0}.$$

Now

$$\int_{B_i \cap \Omega} |Rm|^{p_0} dV_{g(t)} = \sum_{j=1}^N \int_{B_i \cap \Omega} \xi_j^2 f^{p_0} \leq N \phi(t) \leq \frac{3}{2} N K_1^{p_0}. \quad \square$$

Define

$$T_{1,1} \doteq \left[\frac{(2p_0 - n) \ln 2}{8p_0(n+1)} C_3^{-1} \right]^{\frac{2p_0}{2p_0-n}}, \quad T_{1,2} \doteq \left[\frac{p_0 \ln 2}{4(p_0+1)(n+1)} C_1^{-1} C_3^{-\left(\frac{n+2}{2p_0}\right)} \right]^{\frac{p_0}{p_0+1}}$$

and recall T_1 is defined in (5.22). Define

$$(5.25) \quad T_2 \doteq \min\{T_1, T_{1,1}, T_{1,2}\}.$$

Proposition 5.11. *If $0 \leq t < T_{\max} < T_2$, then*

$$(5.26) \quad 2^{-\frac{1}{n+1}} g_0 \leq g(t) \leq 2^{\frac{1}{n+1}} g_0.$$

Proof. Choose $0 \leq t_0 < T_{\max} < T_2$. Then by Corollary 5.8, we have

$$\int_0^{t_0} \left| \frac{\partial}{\partial t} g \right| dt = \int_0^{t_0} |2\chi^2 \text{Rc}(g(t))| dt \leq \int_0^{t_0} 2C_3 \left(t^{-\frac{n}{2p_0}} + C_1^{\frac{n+2}{2p_0}} t^{\frac{1}{p_0}} \right) dt \leq \frac{\ln 2}{n+1}.$$

By Lemma 6.49 in [1], we get our conclusion. $\square$

Proposition 5.12. *If $0 \leq t < T_{\max} < T_2$, then*

$$(5.27) \quad \left(\int_{B_x(4r_0)} h^{\frac{2n}{n-2}} dV_{g(t)} \right)^{\frac{n-2}{n}} \leq 2A_0 r^2 \int_{B_x(4r_0)} |\nabla h|^2 dV_{g(t)}, \quad \forall h \in C_0^\infty(B_x(4r_0)).$$

Proof. By Lemma 5.1 and Proposition 5.11, we get our conclusion. $\square$

Proposition 5.13. *If $T_{\max} < T_2$, then on $[0, T_{\max})$, we have*

$$(5.28) \quad \left\{ \begin{array}{l} \left(\int_{B_x(4r_0)} h^{\frac{2n}{n-2}} dV_{g(t)} \right)^{\frac{n-2}{n}} \leq 2A_0 r^2 \int_{B_x(4r_0)} |\nabla h|^2 dV_{g(t)}, \\ 2^{-\frac{1}{4}} g_0 \leq g(t) \leq 2^{\frac{1}{4}} g_0, \\ \left(\int_{B_i \cap \Omega} |Rm(g(t))|^{p_0} dV_{g(t)} \right)^{\frac{1}{p_0}} \leq \left(\frac{3}{2} N \right)^{\frac{1}{p_0}} K_1, \\ |\chi^2 Rm(g(t))| \leq C \end{array} \right.$$

for any $x \in M$ and $h \in C_0^\infty(B_x(4r_0))$, where C is independent of t .

Proof. Combining Propositions 5.10, 5.11, 5.12 and Corollary 5.9, we get our conclusion. $\square$

Theorem 5.14. *Assume (M^n, g_0) is an n -dimensional ($n \geq 3$) complete noncompact Riemannian manifold, which satisfies (5.1). Then the local Ricci flow (1.2) has a smooth solution on $[0, \frac{T_2}{2}]$, where T_2 is defined in (5.25). Moreover, for $t \in [0, \frac{T_2}{2}]$, the metric satisfies (5.7) and the curvature tensors satisfy the bounds (5.17) and (5.21).*

Proof. First, by Theorem 2.9 and DeTurck's trick, equation (1.2) has a smooth solution on a sufficiently small time interval starting at $t = 0$. Let $[0, T_{\max})$ be a maximal time interval on which (1.2) has a smooth solution and such that (5.6), (5.7), (5.8) hold for each metric $g(t)$. We claim that $T_{\max} > \frac{T_2}{2}$; we prove it by contradiction.

If $T_{\max} \leq \frac{T_2}{2}$, then $T_{\max} < T_2$. By Proposition 5.13, $|\chi^2 Rm|_{(g(t), \infty)}$ is uniformly bounded on $[0, T_{\max})$. Then by Theorem 4.8, we can extend the solution of local Ricci flow to $[0, T_{\max} + \delta)$, where $\delta > 0$ is some constant. By Proposition 5.13, we can furthermore assume that (5.6), (5.7), (5.8) hold on $[0, T_{\max} + \delta)$, so it is in contradiction with the definition of $T_{\max}$. Then we have $T_{\max} > \frac{T_2}{2}$, and (1.2) has a smooth solution on $[0, \frac{T_2}{2}]$. We prove the first conclusion and $g(t)$ satisfies (5.7).

Note that $\frac{T_2}{2} < T_{\max}$. Then by Lemma 5.7 and Corollary 5.8, we get (5.17) and (5.21). $\square$

Remark 5.15. In fact we have the following conclusion by the observation that the power ratio between K_1 and r_0 is exactly $\frac{p_0}{n}$ in the local Ricci flows' existence time $\frac{T_2}{2}$ (in fact in T_1 , $T_{1,1}$ and $T_{1,2}$).

Assume (M^n, g_0) is an n -dimensional ($n \geq 3$) complete noncompact Riemannian manifold, which satisfies

$$(5.29) \quad \begin{cases} Rc \geq 0, \\ \lim_{r \rightarrow \infty} \frac{V(B_{x_0}(r))}{r^{\frac{n}{p_0}}} = \delta > 0, \\ \left(\int_{M^n} |Rm(g_0)|^{p_0} dV_{g_0} \right)^{\frac{1}{p_0}} \leq C_0 < \infty, \end{cases}$$

where $x_0 \in M^n$ is a fixed point, $p_0 > \frac{n}{2}$ and C_0 are positive constants.

Let $\Omega_r \doteq B_{x_0}(r)$, $r \geq 1$, be a positive constant. Then the local Ricci flow (1.2) with respect to Ω_r has a smooth solution on $[0, T]$, where $T > 0$ is independent of r .

The proof of the above conclusion is similar to Theorem 5.14, but not using $\{\xi_i\}_{i=1}^\infty$ in the whole procedure.

6. EXISTENCE TIME ESTIMATES (THE SECOND CASE: SCALE-INVARIANT EXPONENT)

In this section, we assume that (M^n, g_0) is an n -dimensional ($n \geq 3$) complete noncompact Riemannian manifold satisfying the following assumptions:

$$(6.1) \quad \begin{cases} V_{g_0}(B_x(\rho)) \geq \hat{N}_1 \left(\frac{\rho}{4r_0} \right)^n V_{g_0}(B_x(4r_0)), & 0 < \rho \leq 4r_0, \\ \left(\int_{B_x(r_0)} h^{\frac{2n}{n-2}} dV_{g_0} \right)^{\frac{n-2}{n}} \leq \hat{A}_0 \int_{B_x(r_0)} |\nabla h|^2 dV_{g_0}, \\ \left(\int_{B_x(r_0)} |Rm(g_0)|^{\frac{n}{2}} dV_{g_0} \right)^{\frac{2}{n}} \leq \left(\tau n \hat{A}_0 \right)^{-1}, \\ \left(\int_{B_x(r_0)} |Rc(g_0)|^{p_0} dV_{g_0} \right)^{\frac{1}{p_0}} \leq \hat{K}_1 \end{cases}$$

for any $x \in M^n$, $h(x) \in C_0^\infty(B_x(r_0))$, where $p_0 > \frac{n}{2}$, $\hat{A}_0 \geq 1$, $\hat{N}_1 \leq 1$, $\hat{K}_1$ and r_0 are positive constants, $\tau = 300\hat{N}p_0$ and $\hat{N} = 2^{4n}\hat{N}_1^{-2} \geq 1$ as in Lemma 5.4, but using $\hat{N}_1$, not N_1 .

By the doubling property assumption in (6.1), we can choose B_i as in Lemma 5.4. By an argument similar to the one in section 5, we can assume that $[0, T_{\max})$ is the maximal time interval on which the local Ricci flow (1.2) has a smooth solution and the following holds for each metric $g(t)$ (where $g(0) = g_0$):

$$(6.2) \quad \left(\int_{B_x(r_0)} h^{\frac{2n}{n-2}} dV_{g(t)} \right)^{\frac{n-2}{n}} \leq 4\hat{A}_0 \int_{B_x(r_0)} |\nabla h|^2 dV_{g(t)},$$

$$(6.3) \quad \frac{1}{2}g_0 \leq g(t) \leq 2g_0,$$

$$(6.4) \quad \left(\int_{B_i \cap \Omega} |Rm(g(t))|^{\frac{n}{2}} dV_{g(t)} \right)^{\frac{2}{n}} \leq (2\hat{N})^{\frac{2}{n}} \left(\tau n \hat{A}_0 \right)^{-1}, \quad i = 1, 2, 3, \dots,$$

for any $x \in M^n$, $h \in C_0^\infty(B_x(r_0))$.

Lemma 6.1. *When $0 \leq t < T_{\max}$, for any i , $q \geq 1$, $p \geq \frac{n}{2}$, $p' \geq 0$ and $p \geq p'$, there exists the constant $\widehat{C}_1(n, \widehat{N}_1, r_0) = C(n, \widehat{N}_1)(r_0^{-2} + 1) > 0$ such that*

$$(6.5) \quad \begin{aligned} & \frac{\partial}{\partial t} \left(\int_{\Omega} \xi_i^{2q} \chi^{2p'} |Rm|^p dV_{g(t)} \right) + \theta \left(\int_{\Omega} |\nabla(\xi_i^q \chi^{p'+1} |Rm|^{\frac{p}{2}})|^2 dV_{g(t)} \right) \\ & \leq \frac{1}{\epsilon} \widehat{C}_1(n, \widehat{N}_1, r_0) p^3 q^2 \left(\int_{B_i \cap \Omega} \xi_i^{2q-2} \chi^{2p'} |Rm|^p dV_{g(t)} \right), \end{aligned}$$

where

$$(6.6) \quad \theta = \left(\frac{4(p-1-4\epsilon)}{(1+4\epsilon)p} \right) - 48p\widehat{A}_0 |Rm|_{(g(t), L^{\frac{n}{2}}(B_i \cap \Omega))}$$

and $|Rm|_{(g(t), L^{\frac{n}{2}}(B_i \cap \Omega))} \doteq \left(\int_{B_i \cap \Omega} |Rm(x, t)|^{\frac{n}{2}} dV_{g(t)} \right)^{\frac{2}{n}}$, $\epsilon > 0$, is any positive constant.

Proof. Let $f(x, t) = |Rm(x, t)|_{g(t)}$ and

$$(6.7) \quad \frac{\partial}{\partial t} \left(\int_{\Omega} \xi^{2q} \chi^{2p'} f^p \right) \leq \sum_{k=1}^6 I_k,$$

where

$$\begin{aligned} I_1 &= p \int \xi^{2q} \chi^{2p'+2} f^{p-1} \Delta f, \quad I_2 = p \int \xi^{2q} \chi^{2p'+2} f^{p-2} |\nabla f|^2, \\ I_3 &= -(1-\epsilon)p \int \xi^{2q} \chi^{2p'+2} f^{p-2} |\nabla Rm|^2, \\ I_4 &= 10p \int \xi^{2q} \chi^{2p'+2} f^{p+1} - \int \xi^{2q} \chi^{2p'+2} f^p R, \\ I_5 &= \left(\frac{10^5}{\epsilon} \right) np \int \xi^{2q} \chi^{2p'} |\nabla \chi|^2 f^p, \\ I_6 &= 8p \int \xi^{2q} \chi^{2p'+1} f^{p-2} g^{ri} g^{sj} g^{pk} g^{ql} R_{rspq} \chi_{il} R_{jk}. \end{aligned}$$

For simplification, we set the following notation:

$$\begin{aligned} \tilde{a} &= \xi^q \chi^{p'} f^{\frac{p}{2}} |\nabla \chi|, \quad \tilde{b} = \xi^q \chi^{p'+1} f^{\frac{p}{2}-1} |\nabla f|, \\ \tilde{c} &= \xi^q \chi^{p'+1} f^{\frac{p}{2}-1} |\nabla Rm|, \quad \tilde{d} = \int |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2, \quad \tilde{e} = \xi^{q-1} \chi^{p'+1} f^{\frac{p}{2}} |\nabla \xi|. \end{aligned}$$

Note that $|R| \leq n|Rm| = 2pf$ and

$$\begin{aligned} I_4 &\leq 12p \int \xi^{2q} \chi^{2p'+2} f^{p+1} \leq 12p|f|_{(g(t), L^{\frac{n}{2}}(B_i \cap \Omega))} \cdot 4\widehat{A}_0 \cdot \int |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2 \\ &\leq 48p\widehat{A}_0 |f|_{(g(t), L^{\frac{n}{2}}(B_i \cap \Omega))} \cdot \tilde{d}. \end{aligned}$$

Similar to the proof of Lemma 5.1,

$$(6.8) \quad \begin{aligned} \frac{\partial}{\partial t} \left(\int_{\Omega} \xi^{2q} \chi^{2p'} f^p \right) &\leq p(1+4\epsilon-p) \int \tilde{b}^2 + 48p\widehat{A}_0 \|f\|_{\frac{n}{2}, B_i \cap \Omega} \cdot \tilde{d} \\ &\quad + \frac{2}{\epsilon} 10^6 n^3 p^3 q^2 \left(\int \tilde{a}^2 + \int \tilde{e}^2 \right). \end{aligned}$$

Then by (5.13),

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int \xi^{2q} \chi^{2p'} f^p \right) + \theta \tilde{d} &\leq \frac{4}{\epsilon} 10^6 n^3 p^3 q^2 \int (\tilde{a}^2 + \tilde{e}^2) \\ &\leq \frac{8}{\epsilon} 10^6 n^3 p^3 q^2 \left(\int_{\Omega} \xi^{2q} \chi^{2p'} |Rm|^p dV_{g(t)} + \int_{\Omega} \xi^{2q-2} \chi^{2p'+2} |Rm|^p |\nabla \xi|^2 dV_{g(t)} \right), \end{aligned}$$

where θ is defined in (6.6). Replace f with $|Rm|$; by Lemma 5.5, $|\tilde{\nabla} \xi_i| \leq \frac{8}{r_0^2} (\hat{N} + 1)$, and the lemma is proved. $\square$

Lemma 6.2. *When $0 < t < T_{\max}$, for $i = 1, 2, 3, \dots$, there exists a constant $C(n, p_0, \hat{N}_1)$ such that*

$$(6.9) \quad \int_{\Omega} \xi_i^4 \chi^2 |Rm(x)|^{1+\frac{n}{2}} dV_{g(t)} \leq C(n, p_0, \hat{N}_1) (r_0^{-2} + 1) (\hat{A}_0)^{-\frac{n}{2}} (t+1)^2 t^{-1}.$$

Proof. Let $f(x, t) = |Rm(x)|_{g(t)}$. Using Lemma 6.1, letting $q = 1$, $p' = 0$, $p = \frac{n}{2}$, $\epsilon = \frac{1}{16}$, and plugging the above values into (6.6), we get $\theta \geq \frac{1}{6}$. Then

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_{\Omega} \xi^2 f^{\frac{n}{2}} dV_{g(t)} \right) + \frac{1}{6} \int_{\Omega} |\nabla(\xi \chi f^{\frac{n}{4}})|^2 dV_{g(t)} \\ \leq \frac{1}{\epsilon_0} C(n, \hat{N}_1) (r_0^{-2} + 1) \left(\int_{B_i \cap \Omega} f^{\frac{n}{2}} dV_{g(t)} \right). \end{aligned}$$

Multiply the above by $\psi_{t_1, t_2}(t)$, which is defined in the proof of Lemma 5.7. Integrate from 0 to t_0 with respect to t , where t_0 satisfies $t_2 \leq t_0 < T_{\max}$. Then let $t_1 = \frac{t_0}{4}$, $t_2 = \frac{t_0}{2}$, and by (6.4),

$$\begin{aligned} \int_{\frac{t_0}{2}}^{t_0} \int_{\Omega} |\nabla(\xi \chi f^{\frac{n}{4}})|^2 \leq C(n, \hat{N}_1) (r_0^{-2} + 1) \left(1 + \frac{1}{t_0}\right) \int_{\frac{t_0}{4}}^{t_0} \int_{B_i \cap \Omega} f^{\frac{n}{2}} \\ (6.10) \quad \leq C(n, p_0, \hat{N}_1) (r_0^{-2} + 1) (\hat{A}_0)^{-\frac{n}{2}} \left(1 + \frac{1}{t_0}\right) t_0. \end{aligned}$$

Again in Lemma 6.1, let $q = 2$, $p' = 1$, $p = 1 + \frac{n}{2}$, and choose $\epsilon = \frac{1}{16}$. Then $\theta \geq \frac{1}{3} > 0$ and

$$\frac{\partial}{\partial t} \left(\int_{\Omega} \xi^4 \chi^2 f^{1+\frac{n}{2}} dV_{g(t)} \right) \leq C(n, \hat{N}_1) (r_0^{-2} + 1) \left(\int_{\Omega} \xi^2 \chi^2 f^{1+\frac{n}{2}} dV_{g(t)} \right).$$

Multiply the above with ψ , and integrate from 0 to t_0 , where $t_2 \leq t_0 < T_{\max}$. Then let $t_1 = \frac{t_0}{2}$, $t_2 = t_0$, and from (6.2), (6.4) and (6.10),

$$\begin{aligned} \int_{\Omega} \xi^4 \chi^2 f^{1+\frac{n}{2}} dV_{g(t_0)} \leq C(n, \hat{N}_1) (r_0^{-2} + 1) \left(1 + \frac{1}{t_0}\right) \left(\int_{\frac{t_0}{2}}^{t_0} \int_{\Omega} \xi^2 \chi^2 f^{1+\frac{n}{2}} \right) \\ \leq C(n, p_0, \hat{N}_1) (r_0^{-2} + 1) (\hat{A}_0)^{-\frac{n}{2}} \left(1 + \frac{1}{t_0}\right)^2 t_0. \end{aligned}$$

Because we choose t_0 arbitrarily, for any $0 < t < T_{\max}$,

$$\int_{\Omega} \xi^4 \chi^2 f^{1+\frac{n}{2}} dV_{g(t)} \leq C(n, p_0, \hat{N}_1) (r_0^{-2} + 1) (\hat{A}_0)^{-\frac{n}{2}} (t+1)^2 t^{-1}.$$

Replacing f with $|Rm|$, we get our conclusion. $\square$

Lemma 6.3. *When $0 < t < T_{\max}$, $q \geq 2$, $p \geq \frac{n}{2}$, $p' \geq 0$ and $p \geq p'$, there exists a constant $C(n, p_0, \widehat{N}_1)$ such that*

$$(6.11) \quad \begin{aligned} & \frac{\partial}{\partial t} \left(\int_{\Omega} \xi_i^{2q} \chi^{2p'} |Rc|^p \right) + \frac{1}{9} \left(\int_{\Omega} |\nabla(\xi_i^q \chi^{p'+1} |Rc|^{\frac{p}{2}})|^2 \right) \\ & \leq C(n, p_0, \widehat{N}_1) p^{2+\frac{n}{2}} q^2 (r_0^{-2} + 1) (t+1)^2 t^{-1} \left(\int_{B_i \cap \Omega} \xi_i^{2(q-2)} \chi^{2p'} |Rc|^p \right). \end{aligned}$$

There also exists $\widehat{C}_2(n, \widehat{N}_1) > 0$ such that

$$(6.12) \quad \begin{aligned} & \frac{\partial}{\partial t} \left(\int_{\Omega} \xi_i^2 \chi^{2p'} |Rc|^p dV_{g(t)} \right) + \theta \left(\int_{\Omega} \left| \nabla(\xi_i \chi^{p'+1} |Rc|^{\frac{p}{2}}) \right|^2 dV_{g(t)} \right) \\ & \leq \frac{1}{\epsilon} \widehat{C}_2(n, \widehat{N}_2) p^3 (r_0^{-2} + 1) \left(\int_{B_i \cap \Omega} \chi^{2p'} |Rc|^p \right), \end{aligned}$$

where $\theta = \left[\frac{4(p-1-4\epsilon)}{(1+4\epsilon)p} - \frac{32p\widehat{N}_2^{\frac{2}{n}}}{\tau\sqrt{n}} \right]$ and $\epsilon > 0$ is any positive constant.

Proof. Let $f(x, t) = |Rc(x)|_{g(t)}$. By Lemma 3.4

$$(6.13) \quad \frac{\partial}{\partial t} \left(\int_{\Omega} \xi^{2q} \chi^{2p'} f^p \right) \leq \sum_{k=1}^6 I_k,$$

where

$$\begin{aligned} I_1 &= p \int \xi^{2q} \chi^{2p'+2} f^{p-1} \Delta f, \quad I_2 = p \int \xi^{2q} \chi^{2p'+2} f^{p-2} |\nabla f|^2, \\ I_3 &= -(1-\epsilon)p \int \xi^{2q} \chi^{2p'+2} f^{p-2} |\nabla Rc|^2, \quad I_4 = 2(1+\sqrt{n})p \int \xi^{2q} \chi^{2p'+2} f^p |Rm|, \\ I_5 &= \left(\frac{10^5}{\epsilon} \right) np \int \xi^{2q} \chi^{2p'} |\nabla \chi|^2 f^p, \quad I_6 = p \int \xi^{2q} \chi^{2p'} g^{ik} g^{jl} R_{kl} J_2, \end{aligned}$$

where J_2 in I_6 is from (3.13).

Similar to the proof of Lemma 1.1, we set

$$\begin{aligned} \tilde{a} &= \xi^q \chi^{p'} f^{\frac{p}{2}} |\nabla \chi|, \quad \tilde{b} = \xi^q \chi^{p'+1} f^{\frac{p}{2}-1} |\nabla f|, \\ \tilde{c} &= \xi^q \chi^{p'+1} f^{\frac{p}{2}-1} |\nabla Rc|, \quad \tilde{d} = \int |\nabla(\xi^q \chi^{p'+1} f^{\frac{p}{2}})|^2, \quad \tilde{e} = \xi^{q-1} \chi^{p'+1} f^{\frac{p}{2}} |\nabla \xi|. \end{aligned}$$

Then

$$(6.14) \quad I_1 + I_2 + I_3 + I_5 + I_6 \leq \frac{4(1+4\epsilon-p)}{(1+4\epsilon)p} \tilde{d} + \frac{4}{\epsilon} 10^6 n^3 p^3 q^2 \left(\int \tilde{a}^2 + \int \tilde{e}^2 \right).$$

Finally for I_4 , when $q \geq 2$,

$$\begin{aligned} I_4 &\leq 2(1+\sqrt{n})p \left(\int_{\Omega} \xi^4 \chi^2 |Rm|^{1+\frac{n}{2}} \right)^{\frac{2}{n+2}} \cdot \left(\int_{B_i \cap \Omega} \xi^{2(q-2)} \chi^{2p'} f^p \right)^{\frac{2}{n+2}} \\ &\quad \cdot \left[\int_{\Omega} \left(\xi^q \chi^{p'+1} f^{\frac{p}{2}} \right)^{\frac{2n}{n-2}} \right]^{\frac{n-2}{n+2}} \\ &\leq \hat{a} \cdot \hat{b}. \end{aligned}$$

In the last inequality we used Lemma 6.2 and the following notation:

$$\begin{aligned}\hat{a} &= C(n, p_0, \hat{N}_1) p \left[(r_0^{-2} + 1)(t + 1)^2 t^{-1} \right]^{\frac{2}{n+2}} \left(\int_{B_i \cap \Omega} \xi^{2(q-2)} \chi^{2p'} f^p \right)^{\frac{2}{n+2}}, \\ \hat{b} &= \left(\int_{\Omega} \left| \nabla (\xi^q \chi^{p'+1} f^{\frac{p}{2}}) \right|^2 \right)^{\frac{n}{n+2}}, \\ s &= \frac{n+2}{2}, \quad s' = \frac{n+2}{n}, \quad \frac{1}{s} + \frac{1}{s'} = 1.\end{aligned}$$

For any $\delta > 0$,

$$\hat{a}\hat{b} \leq \frac{1}{s} \left(\hat{a} \delta^{-\frac{1}{ss'}} \right)^s + \frac{1}{s'} \left(\hat{b} \delta^{\frac{1}{ss'}} \right)^{s'} \leq \hat{a}^s \delta^{-\frac{1}{s'}} + \hat{b}^{s'} \delta^{\frac{1}{s}}.$$

Let $\delta = k_0^{\frac{n+2}{2}}$, where k_0 is some positive constant which will be determined later. Then

(6.15)

$$I_4 \leq k_0^{-\frac{n}{2}} C(n, p_0, \hat{N}_1) p^{\frac{n+2}{2}} \left[(r_0^{-2} + 1)(t + 1)^2 t^{-1} \right] \left(\int_{B_i \cap \Omega} \xi^{2(q-2)} \chi^{2p'} f^p \right) + k_0 \tilde{d}.$$

Now we choose $\epsilon = \frac{1}{16}$, $k_0 = \frac{1}{9}$ in (6.14) and (6.15), and from (6.13),

$$\begin{aligned}& \frac{\partial}{\partial t} \left(\int_{\Omega} \xi^{2q} \chi^{2p'} f^p \right) + \frac{1}{3} \int_{\Omega} \left| \nabla (\xi^q \chi^{p'+1} f^{\frac{p}{2}}) \right|^2 \\ & \leq C(n, p_0, \hat{N}_1) p^{2+\frac{n}{2}} q^2 (r_0^{-2} + 1)(t + 1)^2 t^{-1} \left(\int_{B_i \cap \Omega} \xi^{2(q-2)} \chi^{2p'} f^p \right).\end{aligned}$$

Hence (6.11) is proved.

When $q = 1$, from (6.2), (6.4) and the Hölder inequality,

$$(6.16) \quad I_4 \leq 8(1 + \sqrt{n}) p \hat{A}_0 |Rm|_{(g(t), L^{\frac{n}{2}}(B_i \cap \Omega))} \cdot \tilde{d} \leq \frac{32p \hat{N}^{\frac{2}{n}}}{\tau \sqrt{n}} \tilde{d}.$$

From (6.13), (6.14) and (6.16),

$$\begin{aligned}& \frac{\partial}{\partial t} \left(\int_{\Omega} \xi_i^2 \chi^{2p'} f^p dV_{g(t)} \right) + \theta \left(\int_{\Omega} \left| \nabla (\xi_i \chi^{p'+1} f^{\frac{p}{2}}) \right|^2 dV_{g(t)} \right) \\ & \leq \frac{4}{\epsilon} 10^6 n^3 p^3 \left(|\nabla \chi|_{\infty}^2 \int_{\Omega} \xi_i^2 \chi^{2p'} f^p + \int_{B_i \cap \Omega} \chi^{2p'+2} f^p |\nabla \xi_i|^2 \right).\end{aligned}$$

Replacing f by $|Rc|$ and simplifying it, we get (6.12). □

Choose $\hat{T}_1 > 0$ such that

$$(6.17) \quad \exp \left(16 \hat{C}_2 p_0^3 \hat{N} (r_0^{-2} + 1) \hat{T}_1 \right) = \frac{3}{2}.$$

Corollary 6.4. When $0 \leq t < T_{\max} \leq \hat{T}_1$,

$$(6.18) \quad \left(\int_{B_i \cap \Omega} |Rc|^{p_0} dV_{g(t)} \right)^{\frac{1}{p_0}} \leq \left(\frac{3}{2} \hat{N} \right)^{\frac{1}{p_0}} \hat{K}_1, \quad i = 1, 2, 3, \dots$$

Proof. Set $f = |Rc|$ and choose $p' = 0$, $p = p_0$ and $\epsilon = \frac{1}{16}$ in (6.12). Then it is easy to check that $\theta \geq \frac{1}{15} \geq 0$ from the definition of θ in Lemma 6.3. Then

$$\frac{\partial}{\partial t} \int_{\Omega} \xi_i^2 |Rc|^{p_0} \leq 16 \hat{C}_2 p_0^3 (r_0^{-2} + 1) \left(\int_{B_i \cap \Omega} |Rc|^{p_0} \right).$$

Define $\phi(t) = \sup_{B_i \in \text{Cov}(\Omega)} \int_{B_i \cap \Omega} \xi_i^2 |Rc|^{p_0} dV_{g(t)}$, and

$$\frac{\partial}{\partial t} \phi(t) \leq 16 \widehat{C}_2 p_0^3 (r_0^{-2} + 1) \widehat{N} \phi(t).$$

From the last inequality in (6.1),

$$\phi(t) \leq \exp \left(16 \widehat{C}_2 p_0^3 \widehat{N} (r_0^{-2} + 1) t \right) \phi(0) \leq \frac{3}{2} \widehat{K}_1^{p_0}.$$

Now we have

$$\int_{B_i \cap \Omega} |Rc|^{p_0} dV_{g(t)} \leq \widehat{N} \phi(t) \leq \left(\frac{3}{2} \widehat{N} \right) \widehat{K}_1^{p_0}.$$

The conclusion is proved. $\square$

We still use Moser iteration to get the local C^0 -estimate $|\chi^2 Rc(g(t))|$.

Proposition 6.5. *If $0 < t < T_{\max} \leq \widehat{T}_1$, there exists some constant $\widehat{C}_3(n, \widehat{N}_1, p_0) > 0$ such that*

$$(6.19) \quad |\chi^2 Rc(g(t))| \leq \widehat{C}_3 \left[(r_0^{-2} + 1) \widehat{A}_0 \right]^{\frac{n}{2p_0}} \widehat{K}_1 \left(t^{-\frac{n}{2p_0}} + t^{\frac{n}{2p_0}} \right).$$

Proof. Let $f(x, t) = |Rc(x)|_{g(t)}$. From (6.11), use a similar argument as in Lemma 5.7, except define H as the following:

$$H(p, p', q, t) = \int_t^{t_0} \int_{B_i \cap \Omega} \xi^{2q} \chi^{2p'} f^p.$$

Denote $\nu = 1 + \frac{2}{n}$, $\eta = \nu^{n+2}$ and fix $t \in (0, t_0)$; set

$$\begin{aligned} p_k &= p_0 \nu^k, & p'_k &= p_k - \frac{n}{2}, \\ q_k &= 2n \nu^k - (n+2), & q_0 &= n-2, & \tau_k &= t(1 - \eta^{-k}), \\ H_k &= H(p_k, p'_k, q_k, \tau_k), & \Phi_k &= H_k^{\frac{1}{p_k}}. \end{aligned}$$

Then

$$\Phi_0 = \left(\int_0^{t_0} \int_{\Omega} \xi^{2(n-2)} \chi^{2p_0-n} f^{p_0} \right)^{\frac{1}{p_0}}.$$

We can get

$$\begin{aligned} \left| \xi_i^{\frac{2n}{p_0}} \chi^2 f(x, t) \right| &\leq \lim_{k \rightarrow \infty} \Phi_{k+1} \\ &\leq C(n, \widehat{N}_1, p_0) (r_0^{-2} + 1)^{\frac{n}{2p_0}} (t_0 + 1)^{\frac{n}{p_0}} \widehat{A}_0^{\frac{n}{2p_0}} t^{-\frac{n+2}{2p_0}} \left(\int_0^{t_0} \int_{B_i \cap \Omega} \xi_i^{2(n-2)} \chi^{2p_0-n} f^{p_0} \right)^{\frac{1}{p_0}}. \end{aligned}$$

Letting $t_0 \rightarrow t$ in the above and using Corollary 6.4,

$$\left| \xi_i^{\frac{2n}{p_0}} \chi^2 f(x, t) \right| \leq C(n, \widehat{N}_1, p_0) \left[(r_0^{-2} + 1) \widehat{A}_0 \right]^{\frac{n}{2p_0}} \widehat{K}_1 \left(t^{-\frac{n}{2p_0}} + t^{\frac{n}{2p_0}} \right).$$

For any $x \in \Omega$, one can assume $x \in B_{i_0} \cap \Omega$, and note that $|\xi_i^4 \chi^2 f| \leq \left| \xi_i^{\frac{2n}{p_0}} \chi^2 f \right|$. Then

$$\begin{aligned} |\chi^2 f|(x, t) &= \left| \sum_{j=1}^{\widehat{N}} \xi_j^2 \chi^2 f(x, t) \right| = \left| \left(\sum_{j=1}^{\widehat{N}} \xi_j^2 \right) \chi^2 f \right| \\ &\leq \widehat{C}(n, \widehat{N}_1, p_0) \left[(r_0^{-2} + 1) \widehat{A}_0 \right]^{\frac{n}{2p_0}} \widehat{K}_1 \left(t^{-\frac{n}{2p_0}} + t^{\frac{n}{2p_0}} \right). \end{aligned}$$

Replacing f with $|Rc|$, we obtain our conclusion. $\square$

Lemma 6.6. *When $0 < t < T_{\max}$, for $q > 2$, $p \geq \frac{n}{2}$, $p' \geq 0$ and $p \geq p'$, there exists a constant $C(n, p_0, \widehat{N}_1)$ such that*

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\int \xi_i^{2q} \chi^{2p'} |Rm|^p \right) + \frac{1}{9} \left(\int \left| \nabla (\xi_i^q \chi^{p'+1} |Rm|^{\frac{p}{2}}) \right|^2 \right) \\ &\leq C(n, p_0, \widehat{N}_1) p^{2+\frac{n}{2}} q^2 (r_0^{-2} + 1) (t+1)^2 t^{-1} \left(\int_{B_i \cap \Omega} \xi_i^{2(q-2)} \chi^{2p'} |Rm|^p \right). \end{aligned}$$

Proof. Using Lemma 6.2, the proof is similar to the proof of (6.11) in Lemma 6.3. $\square$

Lemma 6.7. *If $0 < t < T_{\max}$, there exists some positive constant $\widehat{C}_4(n, \widehat{N}_1, p_0)$ such that*

$$(6.20) \quad |\chi^2 Rm(g(t))| \leq \widehat{C}_4(r_0^{-2} + 1)(t^{-1} + t).$$

Proof. Similar to the proof of Proposition 6.5, but replacing p_0 with $\frac{n}{2}$ and using Lemma 6.6 and (6.4) instead of Lemma 6.3 and Corollary 6.4. $\square$

Corollary 6.8. *If $0 < t < T_{\max}$, then there exists some positive constant C independent of t such that*

$$|\chi^2 Rm(g(t))| \leq C.$$

Proof. Straightforward from Lemma 6.7.

Choose $\widehat{T}_2 > 0$ such that

$$(6.21) \quad \exp \left(2n^3 \widehat{C}_1 \widehat{N} \widehat{T}_2 \right) = \frac{3}{2}$$

and choose $q \geq 1$, $p' \geq 0$, $p \geq \frac{n}{2}$, $\epsilon \geq \frac{1}{16}$ in Lemma 6.1, and apply a similar argument as shown in Corollary 6.4, if $0 \leq t \leq T_{\max} \leq \widehat{T}_2$,

$$\left(\int_{B_i \cap \Omega} |Rm(g(t))|^{\frac{n}{2}} dV_{g(t)} \right)^{\frac{2}{n}} \leq \left(\frac{3}{2} \widehat{N} \right)^{\frac{2}{n}} \left(\tau n \widehat{A}_0 \right)^{-1}, \quad i = 1, 2, 3, \dots.$$

Define

$$\begin{aligned} \widehat{T}_{3,1} &= \left[\frac{(2p_0 - n) \ln 2}{32p_0} \widehat{C}_3^{-1} \widehat{K}_1^{-1} [(r_0^{-2} + 1) \widehat{A}_0]^{-\frac{n}{2p_0}} \right]^{\frac{2p_0}{2p_0 - n}}, \\ \widehat{T}_{3,2} &= \left[\frac{(2p_0 + n) \ln 2}{32p_0} \widehat{C}_3^{-1} \widehat{K}_1^{-1} [(r_0^{-2} + 1) \widehat{A}_0]^{-\frac{n}{2p_0}} \right]^{\frac{2p_0}{2p_0 + n}}, \\ (6.22) \quad \widehat{T}_3 &= \min\{\widehat{T}_{3,1}, \widehat{T}_{3,2}\}. \end{aligned}$$

By Proposition 6.5 and a similar argument as in Proposition 5.11, if $0 \leq t < T_{\max} \leq \min\{\widehat{T}_1, \widehat{T}_3\}$, then $2^{-\frac{1}{4}} g_0 \leq g(t) \leq 2^{\frac{1}{4}} g_0$.

We define

$$(6.23) \quad \widehat{T}_4 = \min\{\widehat{T}_1, \widehat{T}_2, \widehat{T}_3\}.$$

By the above argument, similar to Proposition 5.13, we have

Proposition 6.9. *If $0 < T_{\max} \leq \widehat{T}_4$, then on $[0, T_{\max})$, we have*

$$(6.24) \quad \left\{ \begin{array}{l} \left(\int_{B_x(r_0)} h^{\frac{2n}{n-2}} dV_{g_0} \right)^{\frac{n-2}{n}} \leq 2\widehat{A}_0 \int_{B_x(r_0)} |\nabla h|^2 dV_{g_0}, \\ 2^{-\frac{1}{4}} g_0 \leq g(t) \leq 2^{\frac{1}{4}} g_0, \\ \left(\int_{B_i \cap \Omega} |Rm(g(t))|^{\frac{n}{2}} dV_{g(t)} \right)^{\frac{2}{n}} \leq \left(\frac{3}{2} \widehat{N} \right)^{\frac{2}{n}} \left(\tau n \widehat{A}_0 \right)^{-1}, \\ |\chi^2 Rm(g(t))| \leq C \end{array} \right.$$

for any $x \in M^n$ and $h \in C_0^\infty(B_x(r_0))$, where C is independent of t .

By a similar argument to the proof of Theorem 5.14, we have the following theorem:

Theorem 6.10. *Assume (M^n, g_0) is an n -dimensional ($n \geq 3$) complete noncompact Riemannian manifold, which satisfies (6.1). Then the local Ricci flow (1.2) has a smooth solution on $[0, \frac{\widehat{T}_4}{2}]$. Moreover, for $t \in (0, \frac{\widehat{T}_4}{2}]$, the metric satisfies (6.3) and the curvature tensors satisfy the estimates (6.19), (6.20).*

7. SHORT-TIME EXISTENCE OF THE RICCI FLOW

In this section we consider the family of local Ricci flows converging to the Ricci flow on a noncompact manifold M^n . We can get the short-time existence of the Ricci flow on M^n when we have the uniform lower bound of existence-time for the family of the local Ricci flows. At many points in this section, we will take a subsequence; to simplify notation, at each stage a sequence such as $\{\chi_k\}$ will be re-indexed to continue to be $\{\chi_k\}$.

Proof of Theorem 1.1. Choose

$$\Omega_i = \{x \mid d_{g_0}(x, \mathbf{O}) \leq 4^i, x \in M\}, \quad i = 1, 2, \dots,$$

where $\mathbf{O}$ is some fixed point in M . Construct χ_i , which is a smooth cut-off function on M , such that

$$\chi_{i+1}(x) = \begin{cases} 1, & x \in \Omega_i, \\ 0, & x \in M \setminus \Omega_{i+1} \end{cases}$$

and

$$(7.1) \quad 0 \leq \chi_i \leq 1, \quad \|\nabla \chi_i\|_\infty \leq 1, \quad \lim_{i \rightarrow \infty} \chi_i(x) = 1.$$

Note that T_2 is independent of χ_i . By Theorem 5.14, for each k , there exists $g_k(t)$ which is the solution of the following local Ricci flow on $[0, \frac{T_2}{2}]$:

$$(7.2) \quad \begin{cases} \frac{\partial}{\partial t} g_k &= -2\chi_k^2 Rc, & x \in M, \\ g_k(x, 0) &= g_0(x), & x \in M, \end{cases}$$

Then fix i . For any $0 < t_1 < \frac{T_2}{2}$, we have

$$(7.3) \quad \|\nabla_k^m Rm_k(x, t_1)\|_{(g_k(t_1), \infty)} \leq C(m, \Omega_i, t_1), \quad x \in \Omega_i, \quad k \leq i,$$

where ∇_k and Rm_k are with respect to $g_k(t)$ and $C_{(m)}^i$ is a sequence of constants independent of k . For simplicity, we use the notation $|\nabla_k^m Rm_k(x, t_1)|_\infty$ instead of

$\|\nabla_k^m Rm_k(x, t_1)\|_{(g_k(t_1), \infty)}$. If $k > i$ on Ω_i , g_k satisfies (1.1), which is the Ricci flow. From (5.17)

$$(7.4) \quad |\chi_k^2 Rm_k(x, t_1)|_\infty \leq C_2 \left(t_1^{-\frac{n}{2p_0}} + C_1^{\frac{n+2}{2p_0}} t^{\frac{1}{p_0}} \right), \quad x \in M, \quad \forall k.$$

Note that C_1, C_2 are independent of k . Hence by the definition of χ_i ,

$$(7.5) \quad |Rm_k(x, t_1)|_\infty \leq C_2 \left(t_1^{-\frac{n}{2p_0}} + C_1^{\frac{n+2}{2p_0}} t^{\frac{1}{p_0}} \right), \quad x \in \Omega_i, \quad k > i.$$

Then by Theorem 14.14 in [3],

$$(7.6) \quad \begin{aligned} |\nabla_k^m Rm_k(x, t_1)|_\infty &\leq C(A_0, K_1, n, m, p_0, r_0, t_1, \Omega_i) t_1^{-\frac{m}{2}} \\ &= C(A_0, K_1, n, m, p_0, r_0, t_1, \Omega_i) \end{aligned}$$

for any $x \in \Omega_i$ and $k > i$.

By (7.3) and (7.6),

$$(7.7) \quad |\nabla_k^m Rm_k(x, t_1)|_\infty \leq C_{(m)}^i \quad x \in \Omega_i, \quad \forall k \geq 0,$$

where $C_{(m)}^i$ is a sequence of constants independent of k .

When $t \in [0, \frac{T_2}{2}]$, by Theorem 5.14

$$(7.8) \quad \frac{1}{2}g_0 \leq g_k(t) \leq 2g_0;$$

then

$$C_1 \sqrt{\det g_0} \leq \sqrt{\det g_k(t_1)} \leq C_2 \sqrt{\det g_0}.$$

By Lemma 5.2 of [15], there exist some $\tau_0 > 0$ such that

$$(7.9) \quad \text{inj}_{g_k(t_1)}(\mathbf{O}) \geq \tau_0.$$

By (7.7) and (7.9), Theorem 3.9 in [2] applies to $(\Omega_i, g_k(t_1), \mathbf{O})_{k \in \mathbb{N}}$. Then

$$\lim_{k \rightarrow \infty} g_k(x, t_1) = g_\infty(x, t_1),$$

where the limit is in C^∞ sense on Ω_i . Hence

$$(7.10) \quad |\hat{\nabla}^m g_k(x, t_1)|_\infty \triangleq |\nabla_{g_0}^m g_k(x, t_1)|_\infty \leq C_{(m)}^i, \quad x \in \Omega_i,$$

where $C_{(m)}^i$ is a sequence of constants independent of k .

From Theorem 14.14 in [3] and the argument for obtaining (7.6), when $k > i$, we have

$$(7.11) \quad |\nabla_k^m Rm_k(x, t)|_\infty \leq C_{(m)}^i, \quad (x, t) \in \Omega_i \times [t_1, \frac{T_2}{2}].$$

If $k \leq i$, it is easy to see that (7.11) is also true. So for any k , (7.11) is true, where $C_{(m)}^i$ is independent of k .

Then by (7.7), (7.10), (7.11) and Lemma 3.11 in [2],

$$\left| \frac{\partial^q}{\partial t^q} \hat{\nabla}^m g_k(x, t) \right|_\infty \leq C_{(m,q)}^i, \quad (x, t) \in \Omega_i \times [t_1, \frac{T_2}{2}], \quad \forall k \geq 0,$$

where $C_{(m,q)}^i$ is a sequence of constants independent of k .

Now by the Arzela-Ascoli theorem, we get

$$\lim_{k \rightarrow \infty} g_k(x, t) = g_\infty(x, t), \quad (x, t) \in \Omega_i \times [t_1, \frac{T_2}{2}],$$

where the limit is in the C^∞ sense. Because i and t_1 are arbitrary, by the diagonal method,

$$(7.12) \quad g(x, t) \triangleq g_\infty(x, t) = \lim_{k \rightarrow \infty} g_k(x, t), \quad (x, t) \in M \times (0, \frac{T_2}{2}].$$

Then

$$\begin{aligned} |\lim_{t \rightarrow 0} (g(x, t) - g_0(x))| &= |\lim_{t \rightarrow 0} \lim_{k \rightarrow \infty} (g_k(x, t) - g_k(x, 0))| \\ &= |\lim_{t \rightarrow 0} \lim_{k \rightarrow \infty} \int_0^t \frac{\partial}{\partial s} g_k(x, s) ds| \\ &\leq \lim_{t \rightarrow 0} \lim_{k \rightarrow \infty} \int_0^t |2\chi_k^2 Ric_k|_\infty \\ &\leq C \lim_{t \rightarrow 0} (t^{1-\frac{n}{2p_0}} + t^{1+\frac{1}{p_0}}) = 0. \end{aligned}$$

In the last inequality above, we used (5.21). Hence we can define

$$(7.13) \quad g(x, 0) = g_0(x), \quad x \in M^n.$$

Then by (7.2), (7.12) and (7.13), we get that $g(t)$ is the solution of the Ricci flow (1.1) on $M^n \times [0, T]$. This is the first conclusion of Theorem 1.1; there remains to prove the estimates (1.4).

Now by (7.5) and (7.12),

$$|Rm_g(x, t_1)|_\infty \leq \lim_{k \rightarrow \infty} |Rm_k(x, t_1)|_\infty \leq C_2 \left(t_1^{-\frac{n}{2p_0}} + C_1^{\frac{n+2}{2p_0}} t_1^{\frac{1}{p_0}} \right)$$

for any $x \in M^n$ and $t_1 \in (0, \frac{T_2}{2}]$.

We take $(M^n, g(\frac{t_1}{2}))$ as the initial conditions of the Ricci flow. Then the Ricci flow has a solution on $M^n \times [\frac{t_1}{2}, \frac{T_2}{2}]$ and

$$(7.14) \quad \left| Rm_{g(\frac{t_1}{2})} \right|_\infty \leq C(A_0, K_1, n, p_0, r_0) (t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}}).$$

Now define

$$(7.15) \quad \tilde{g}(x, t) = \left[C(t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}}) \right] \cdot g\left(x, \frac{t_1}{2} + t \left[C(t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}}) \right]^{-1} \right),$$

where $C = C(A_0, K_1, n, p_0, r_0)$ in (7.15) is the same as in (7.14). Then from the scaling argument,

$$\begin{cases} \frac{\partial}{\partial t} \tilde{g}_{ij}(x, t) &= -2\tilde{R}_{ij}(x, t), \\ \tilde{g}(x, 0) &= \left[C(A_0, K_1, n, p_0, r_0) (t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}}) \right] g(x, \frac{t_1}{2}). \end{cases}$$

We already know that g_{ij} and its Ricci flow exist on $[\frac{t_1}{2}, \frac{T_2}{2}]$ and that $\tilde{g}_{ij}$ and its Ricci flow exist on $\left[0, (\frac{T_2}{2} - \frac{t_1}{2}) C(A_0, K_1, n, p_0, r_0) (t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}}) \right]$. We use $\{\tilde{R}_{ijkl}(x)\}$ and $\tilde{\nabla}$ to denote, respectively, the Riemannian curvature tensor and the covariant derivative with respect to $\tilde{g}_{ij}(x, 0)$.

Note that $|\widetilde{Rm}| \leq 1$ on M , and by Lemma 7.1 of [12], there exists a positive constant $C(n, m)$, such that for any $m \geq 0$, if

$$t \in \left(0, \left(\frac{T_2}{2} - \frac{t_1}{2}\right) C(A_0, K_1, n, p_0, r_0) \left(t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}}\right) \right],$$

we have

$$(7.16) \quad |\tilde{\nabla}^m \widetilde{Rm}(x, t)|^2 \leq \frac{C(n, m)}{t^m}.$$

By (7.15) and (7.16), for $\frac{t_1}{2} < t \leq \frac{T_2}{2}$,

$$\begin{aligned} |\nabla^m Rm(x, t)|^2 &\leq \frac{C(n, m) \left[C(A_0, K_1, n, p_0, r_0) (t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}}) \right]^{m+2}}{\left[C(A_0, K_1, n, p_0, r_0) (t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}}) (t - \frac{t_1}{2}) \right]^m} \\ &\leq \frac{C(A_0, K_1, m, n, p_0, r_0)}{(t - t_1)^m} (t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}})^2. \end{aligned}$$

Then

$$|\nabla^m Rm(x, t_1)|^2 \leq \frac{C(A_0, K_1, m, n, p_0, r_0)}{t_1^m} (t_1^{-\frac{n}{2p_0}} + t_1^{\frac{1}{p_0}})^2, \quad m \geq 0.$$

Because t_1 was chosen arbitrarily,

$$|\nabla^m Rm(x, t)| \leq \frac{C(A_0, K_1, m, n, p_0, r_0)}{t^{\frac{m}{2}}} (t^{-\frac{n}{2p_0}} + t^{\frac{1}{p_0}}), \quad \text{for } 0 < t \leq \frac{T_2}{2}, \quad m \geq 0. \quad \square$$

Remark 7.1. In fact, the assumption about the integral of the curvature tensor in the above theorem can be weakened as

$$\left(\int_{B_{x_i}(r_0)} |Rm|^{p_0} \right)^{\frac{1}{p_0}} \leq K_1,$$

where $B_{x_i}(r_0)$ is chosen as in Lemma 5.4.

Remark 7.2. By Remark 5.15 and the argument in Theorem 1.1, if (M^n, g_0) satisfies (5.29), the Ricci flow (1.1) has a smooth solution on $[0, T]$, where $T > 0$ is some positive constant.

Proof of Corollary 1.2. Because $Rc(g_0) \geq -Kg_0$, by Theorem 3.1 in [10] and Corollary 1.1 in [8], $A_0 \leq C(n, K, r_0)$. Then applying Theorem 1.1, we get our conclusion. $\square$

We also have the following corollary which is part of Shi's Theorem.

Corollary 7.3. *Let (M, g_0) be an n -dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfying $|Rm| \leq k_0$ on M , where $0 < k_0 < +\infty$ is a constant. Then there is a constant $T(n, k_0) > 0$ depending only on n and k_0 such that the evolution equation (1.1) has a smooth solution $g_{ij}(x, t)$ on $[0, T(n, k_0)]$.*

Proof. By $|Rm| \leq k_0$, there exists $K = nk_0$ such that

$$Rc(g_0) \geq -Kg_0,$$

and it is obvious that

$$\left(\int_{\bar{B}_x(1)} |Rm(g_0)|^n dV_{g_0} \right)^{\frac{1}{n}} \leq k_0.$$

Now we choose

$$K = nk_0, \quad K_1 = k_0, \quad p_0 = n, \quad r = 1.$$

Then by all the above choices of K , K_1 , p_0 and r , (1.6) in Corollary 1.2 is satisfied. We get our conclusion. $\square$

By Theorem 6.10, we can use a similar argument in the proof of Theorem 1.1 to get the following theorem.

Theorem 7.4. *Assume (M^n, g_0) is an n -dimensional ($n \geq 3$) complete noncompact Riemannian manifold satisfying (6.1). Then the Ricci flow (1.1) has a smooth solution $g_{ij}(x, t)$ on $[0, \hat{T}_4]$ (where $\hat{T}_4$ is defined in (6.23)) and satisfies the following estimates. For any integer $m \geq 0$, there exists a positive constant $C(\hat{A}_0, \hat{K}_1, \hat{N}_1, m, n, r_0)$ such that*

$$\sup_{x \in M^n} |\nabla^m Rm(x, t)| \leq \frac{C(\hat{A}_0, \hat{K}_1, \hat{N}_1, m, n, r_0)}{t^{\frac{m}{2}}}(t^{-1} + t), \quad 0 < t \leq \frac{\hat{T}_4}{2}.$$

Especially,

$$\sup_{x \in M^n} |Rm(x, t)| \leq C(n, \hat{N}_1, r_0)(t^{-1} + t), \quad 0 < t \leq \frac{\hat{T}_4}{2},$$

where $C(n, \hat{N}_1, r_0)$ depends only on n , $\hat{N}_1$ and r_0 .

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