THE NONLINEAR HEAT EQUATION WITH HIGH ORDER MIXED DERIVATIVES OF THE DIRAC DELTA AS INITIAL VALUES

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ABSTRACT. In this paper we prove local existence of solutions of the nonlinear heat equation $u_t = \Delta u + |u|^\alpha u$, $t \in (0, T)$, $x \in \mathbb{R}^N$, with initial value $u(0) = K\partial_1\partial_2 \cdots \partial_m \delta$, $K \neq 0$, $m \in \{1, 2, \dots, N\}$, $0 < \alpha < 2/(N + m)$ and δ is the Dirac distribution. In particular, this gives a local existence result with an initial value in a high order negative Sobolev space $H^{s,q}(\mathbb{R}^N)$ with $s \leq -2$.

As an application, we prove the existence of initial values $u_0 = \lambda f$ for which the resulting solution blows up in finite time if $\lambda > 0$ is sufficiently small. Here, f satisfies in particular $f \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and is anti-symmetric with respect to $x_1, x_2, \dots, x_m$. Moreover, we require $\int_{\mathbb{R}^N} x_1 \cdots x_m f(x) dx \neq 0$. This extends the known “small lambda” blow up results which require either that $\int_{\mathbb{R}^N} f(x) dx \neq 0$ (Dickstein (2006)) or $\int_{\mathbb{R}^N} x_1 f(x) dx \neq 0$ (Ghoul (2011), (2012)).

1. INTRODUCTION

In this paper we study local existence and uniqueness of solutions to the semi-linear heat equation in integral form,

$$(1.1) \quad u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\sigma)\Delta} (|u(\sigma)|^\alpha u(\sigma)) d\sigma,$$

where $e^{t\Delta}$ is the heat semigroup on $\mathbb{R}^N$, $\alpha > 0$ and $u_0 \in \mathcal{S}'(\mathbb{R}^N)$ is a multiple of

$$(1.2) \quad (-1)^m \partial_1 \partial_2 \cdots \partial_m \delta,$$

$1 \leq m \leq N$ and δ is the Dirac point mass at the origin. Recall that $e^{t\Delta} f = G_t \star f$, where

$$(1.3) \quad G_t(x) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^N.$$

For $x = (x_1, x_2, \dots, x_N)$, we denote ∂_{x_i} by ∂_i , $1 \leq i \leq N$. We seek a solution $u \in C((0, T]; C_0(\mathbb{R}^N))$ such that $u(t) \rightarrow u_0$ in $\mathcal{S}'(\mathbb{R}^N)$ as $t \rightarrow 0$. $C_0(\mathbb{R}^N)$ denotes the set of continuous functions on $\mathbb{R}^N$ which tend to zero at infinity and $\mathcal{S}'(\mathbb{R}^N)$ denotes the space of tempered distributions on $\mathbb{R}^N$.

It is well known that (1.1) is locally well posed in $C_0(\mathbb{R}^N)$. In particular, given any $u_0 \in C_0(\mathbb{R}^N)$, there exist $T_{\max}(u_0) > 0$ and a unique continuous solution

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$u : [0, T_{\max}(u_0)) \rightarrow C_0(\mathbb{R}^N)$ of (1.1) with $u(0) = u_0$. This solution is a classical solution of

$$(1.4) \quad u_t = \Delta u + |u|^\alpha u$$

for $0 < T < T_{\max}(u_0)$. Furthermore, if $T_{\max}(u_0) < \infty$, then $\|u(t)\|_\infty \rightarrow \infty$ as $t \rightarrow T_{\max}(u_0)$.

The equation (1.1) is also locally well posed in other function spaces. For example, if $\alpha < 2q/N$ and $q \geq 1$, then (1.1) is locally well posed in $L^q(\mathbb{R}^N)$. See [20, 21]. If $u_0 \in L^q(\mathbb{R}^N)$, then the resulting solution is continuous $u : (0, T_{\max}(u_0)) \rightarrow C_0(\mathbb{R}^N)$ and a classical solution of (1.4) for $0 < T < T_{\max}(u_0)$. (Here we are not concerned with the limiting case $\alpha = 2q/N$ or the precise formulation of uniqueness in $L^q(\mathbb{R}^N)$.) In the case $q = 1$ and $\alpha < 2/N$, (1.1) is also locally well posed in $\mathcal{M}$, the space of finite Radon measures. In particular, this includes the Dirac mass, i.e. example (1.2) in the case $m = 0$.

In addition, (1.1) is locally well posed in certain negative order Sobolev spaces $H^{s,q}(\mathbb{R}^N)$, i.e. with $s < 0$. See Ribaud [18], Wu [25], and Ghoul [7]. Without going into the details of these results, it suffices to remark that if $\alpha < 1/(N+1)$, then (1.1) is locally well posed in a negative Sobolev space which contains the distribution $\partial_i \delta$. See the proof of Theorem 18 in [7]. This is the case $m = 1$ in (1.2).

The novelty of our methods is that we can treat the cases $2 \leq m \leq N$. Also, we improve the result in the case $m = 1$ treated in [7]. We emphasize that our methods do not enable us to prove that (1.1) is locally well posed in high negative order Sobolev spaces. Indeed, it seems there is an obstacle to such a result. If $s \leq -2$, then the semigroup part of the integral term is too singular to be handled in standard function spaces. The known results, for example Corollary 11 in [7], explicitly require $s > -2$. We can only handle very specific initial values in these spaces. For example, if $N = 2$, we can prove the existence of solutions to (1.1) with initial value $u_0 = \partial_{xy} \delta$ but not $u_0 = \partial_{xx} \delta$ or $u_0 = \partial_{yy} \delta$.

Our first main result is the following.

Theorem 1.1. *Let the positive integer m and the real number α be such that*

$$1 \leq m \leq N, \quad 0 < \alpha < \frac{2}{N+m}.$$

Let $K \in \mathbb{R}$, $K \neq 0$, and set

$$(1.5) \quad u_0 = K(-1)^m \partial_1 \partial_2 \cdots \partial_m \delta.$$

There exist $T > 0$ and a continuous solution $u : (0, T] \rightarrow C_0(\mathbb{R}^N)$ of the integral equation (1.1) with initial value u_0 and such that $u(t) \rightarrow u_0$ in $\mathcal{S}'(\mathbb{R}^N)$ as $t \rightarrow 0$.

The uniqueness part of Theorem 1.1 is rather technical to state here and is better understood in the following sections. See Theorem 2.3 and Proposition 3.3.

It turns out that the values of $\alpha > 0$, for which a solution exists with initial value given by (1.5), are precisely those predicted by a formal scaling argument. The initial value $\partial_1 \partial_2 \cdots \partial_m \delta$ is in the Sobolev space $H^{s,q}(\mathbb{R}^N)$ when $s < \frac{N}{q} - N - m$. On the other hand, the “scaling critical exponent s_c ” is given by $s_c = \frac{N}{q} - \frac{2}{\alpha}$. To see this, we observe that the transformation $\lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x)$, which leaves invariant the set of solutions to $u_t = \Delta u + |u|^\alpha u$, acts on initial values as $\lambda^{\frac{2}{\alpha}} u_0(\lambda \cdot)$. Furthermore,

$$\|\lambda^{\frac{2}{\alpha}} u_0(\lambda \cdot)\|_{\dot{H}^{s,q}(\mathbb{R}^N)} = \lambda^{\frac{2}{\alpha} - \frac{N}{q} + s} \|u_0(\cdot)\|_{\dot{H}^{s,q}(\mathbb{R}^N)}, \quad \forall \lambda > 0.$$

In other words, the homogeneous Sobolev norm of $\dot{H}^{s,q}(\mathbb{R}^N)$ is invariant under the action of $\lambda^{\frac{2}{\alpha}}u_0(\lambda \cdot)$ precisely if $s = s_c$. We would “hope” that solutions exist if the initial value is in some $H^{s,q}(\mathbb{R}^N)$ with $s > s_c$. With the initial value given by (1.5), this happens when $\frac{N}{q} - N - m > \frac{N}{q} - \frac{2}{\alpha}$, i.e. $\alpha < 2/(N + m)$. We refer the reader to [2] for the definitions and basic properties of the Sobolev spaces $H^{s,q}(\mathbb{R}^N)$, the homogeneous Sobolev spaces $\dot{H}^{s,q}(\mathbb{R}^N)$, and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^N)$.

Our method to prove Theorem 1.1, going back to Fujita [6] and an earlier work of the second author [22], is based on the following idea: one looks for solutions u such that $u(t) \leq Ce^{t\Delta}u_0$. Fujita [6] used this idea to show, in the case $\alpha > 2/N$, that if u_0 is bounded by a small enough Gaussian, then the resulting solution of (1.1) is global in time. In [22], this idea was used to find new sufficient conditions on u_0 which imply that the resulting solution of (1.1) is global. In this paper, we give a more general version of the arguments in [22] and apply them to the heat semigroup on domains of the form

$$(1.6) \quad \Omega = \{x \in \mathbb{R}^N : x_1 > 0, x_2 > 0, \dots, x_m > 0\}.$$

Functions in $C_0(\mathbb{R}^N)$, which are anti-symmetric in $x_1, x_2, \dots, x_m$, restrict to functions in $C_0(\Omega)$, and functions in $C_0(\Omega)$ extend uniquely to functions in $C_0(\mathbb{R}^N)$ which are anti-symmetric in $x_1, x_2, \dots, x_m$. This enables us to prove the existence of solutions to (1.1) in $C_0(\Omega)$, with the appropriate heat semigroup, and then consider them as solutions in $C_0(\mathbb{R}^N)$ which are anti-symmetric in $x_1, x_2, \dots, x_m$. Solutions with initial values u_0 of the form (1.5) fit precisely into this situation, as can be seen by looking at $e^{t\Delta}u_0$. The solutions we construct will be positive (or negative, depending on the sign of K) on Ω .

Furthermore, the solution $u : (0, T] \rightarrow C_0(\mathbb{R}^N)$ constructed as above can be extended to a maximal solution of (1.1), $u : (0, T_{\max}) \rightarrow C_0(\mathbb{R}^N)$ with u_0 given by (1.5). Since these solutions have a definite sign on Ω it follows from the well-known Fujita-type results on sectors (see [16, 13, 1, 15]) that $T_{\max} < \infty$.

In principle, our method should apply more generally, for other highly singular initial values such as

$$u_0 = K(-1)^m \partial_1 \partial_2 \cdots \partial_m f$$

for certain $f \in L^1(\mathbb{R}^N) + L^q(\mathbb{R}^N)$. But for the moment, such a result eludes us.

Aside from being of interest in its own right, the existence of solutions to (1.1) with initial values of the form (1.2) has important implications for the finite time blow up properties of solutions with regular initial values. The relationship we have in mind was discovered by Dickstein [4]. Suppose $f \in L^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} f \neq 0$ and that $\alpha < 2/N$ (the Fujita exponent). It follows that $T_{\max}(\lambda f) < \infty$ for all sufficiently small $\lambda > 0$. This extends Fujita's result [6], for the same values of α , that $T_{\max}(u_0) < \infty$ for all positive $u_0 \in C_0(\mathbb{R}^N)$. Dickstein's proof is based on a rescaling of λf as $u_0^{\lambda,\mu} = \lambda \mu^{2/\alpha} f(\mu \cdot)$, with $\mu > 0$ chosen so that $u_0^{\lambda,\mu} \rightarrow c\delta$ as $\lambda \rightarrow 0$, where $c = \int_{\mathbb{R}^N} f \neq 0$. Since the solution with initial value $c\delta$ blows up in finite time (by Fujita's result), Dickstein is able to show the same for $u_0^{\lambda,\mu}$, and thus λf , for sufficiently small $\lambda > 0$.

Recently, Ghoul [7] has shown that if $\alpha < 1/(N + 1)$, the condition $\int_{\mathbb{R}^N} f \neq 0$ can be replaced by the condition $\int_{\mathbb{R}^N} f = 0$, but $f = \partial_i g$, where $g \in L^1(\mathbb{R}^N)$ and

$\int_{\mathbb{R}^N} g \neq 0$. Ghoul uses a different rescaling of the initial value so that the rescaled initial value converges to a nonzero multiple of $\partial_i \delta$. Ghoul's result mentioned above that (1.1) is well posed in certain negative order Sobolev spaces which contain $\partial_i \delta$ leads to the same conclusion that $T_{\max}(\lambda f) < \infty$ for all sufficiently small $\lambda > 0$. For this, Ghoul needs to use the Fujita-type result on a half space due to Meier [16], Bandle and Levine [1], Levine and Meier [15], and Kavian [13].

Our results on existence of solutions to (1.1) with initial values of the form (1.2) suggest that a further extension of Dickstein's result is possible. Indeed we establish such a result under the condition $\alpha < 2/(N+m)$. More precisely, we prove the following.

Theorem 1.2. *Let the positive integer m and the real number α be such that*

$$1 \leq m \leq N, \quad 0 < \alpha < \frac{2}{N+m}.$$

Let $f \in C_0(\mathbb{R}^N)$ be anti-symmetric with respect to $x_1, x_2, \dots, x_m$, and suppose that there exist $t_0 > 0$ and $K > 0$ such that

$$(1.7) \quad |f(x)| \leq K |\partial_1 \partial_2 \cdots \partial_m G_{t_0}(x)|, \quad \forall x \in \mathbb{R}^N.$$

Moreover, assume that

$$(1.8) \quad K_0 \equiv \int_{\mathbb{R}^N} x_1 x_2 \cdots x_m f(x) dx_1 dx_2 \cdots dx_N \neq 0.$$

It follows that the maximal solution $u : [0, T_{\max}) \rightarrow C_0(\mathbb{R}^N)$ of the integral equation (1.1) with initial value $u_0 = \lambda f$ blows up in finite time if $\lambda > 0$ is sufficiently small.

See Proposition 4.1 for additional properties of a function f satisfying the above conditions, and Remark 4.2 for the existence of such functions.

It is the assumption $K_0 \neq 0$ which replaces Dickstein's assumption that the initial value has nonzero mean value. Here, we clearly have $\int_{\mathbb{R}^N} u_0 = 0$ since u_0 is anti-symmetric with respect to at least one variable. Also, the case $m = 1$ was treated by Ghoul [7]. The hypothesis on u_0 needed in [7] is formally equivalent (and in many cases genuinely equivalent) to the condition $K_0 \neq 0$ with $m = 1$. See Corollary 1 in [8] and its proof. Thus, our result is a genuine extension of those in [4, 7].

Also, while $(-1)^m \partial_1 \partial_2 \cdots \partial_m G_{t_0}(x) > 0$ for all $x \in \Omega$, we most definitely do not make that assumption for f . This is the whole point of Theorem 1.2. If $f > 0$ on $x \in \Omega$, then the solution with initial value λf would blow up in finite time for all $\lambda > 0$ because of the Fujita-type results on Ω in [16, 13, 1, 15]. In addition, we note that the condition $\alpha < \frac{2}{N+m}$ in Theorem 1.2 is sharp. If $\alpha > \frac{2}{N+m}$, then all initial data antisymmetric in $x_1, x_2, \dots, x_m$ and satisfying (1.7) for some $t_0 > 0$ and sufficiently small K produce solutions which are global in time. See Remark 4.3 below.

To prove Theorem 1.2, we use a dilation for which the rescaled initial value converges to a nonzero multiple of (1.2), i.e. (1.5). As already noted, the maximal solution with initial value (1.5) blows up in finite time. Since we do not prove well posedness of (1.1) in any standard function space, we need to provide an ad hoc proof that the solutions with the rescaled initial values converge in $C_0(\mathbb{R}^N)$ at some positive time to the solution with initial value of the form (1.5).

If f is as in the statement of Theorem 1.2, and also in $H^1(\mathbb{R}^N)$, then by standard energy arguments [14], the maximal solution $u : [0, T_{\max}) \rightarrow C_0(\mathbb{R}^N)$ of the integral equation (1.1) with initial value $u_0 = \lambda f$ blows up in finite time if $\lambda > 0$ is sufficiently *large* as well. Thus, it would be interesting to find such a function so that the solution with initial value $u_0 = f$, i.e. with $\lambda = 1$, is in fact global. For such a function f , the solution with initial value $u_0 = \lambda f$ would blow up in finite time for large and small $\lambda > 0$, but it would be global for $\lambda = 1$. Such a result was proved in the context of Dickstein's small lambda theorem [4], i.e. in the case $m = 0$ and with the requirement that $\int_{\mathbb{R}^N} f dx \neq 0$, in the paper [3]. A similar result was proved in the case $m = 1$ in [8]. This result allows $\int_{\mathbb{R}^N} f dx = 0$, but requires $\int_{\mathbb{R}^N} x_j f dx \neq 0$ for some $1 \leq j \leq N$. Here, we obtain such a result in the case $m = 2$. We allow both $\int_{\mathbb{R}^N} f dx = 0$ and $\int_{\mathbb{R}^N} x_j f dx = 0$, but require $\int_{\mathbb{R}^N} x_j x_k f dx \neq 0$ for some $j \neq k$. Unfortunately, it is not clear how our proof could extend to general m , with $2 \leq m \leq N$, since that would require higher order regularity of the solution, which is unlikely for $\alpha < 1$.

Theorem 1.3. *Let the real number α be such that*

$$0 < \alpha < \frac{2}{N+2}.$$

There exists a function $f \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, verifying the hypotheses of Theorem 1.2 in the case $m = 2$, such that the maximal solution $u : [0, T_{\max}) \rightarrow C_0(\mathbb{R}^N)$ of the integral equation (1.1) with initial value $u_0 = f$ is global in time.

The rest of this paper is organized as follows. In Section 2, we establish the local existence of solutions to an integral equation in an abstract setting. This result does not give information about the initial value. In Section 3, we first use the result in Section 2 to prove the existence of solutions to equation (1.1) with u_0 given by (1.5). We also provide a proof that $u(t) \rightarrow u_0$ in $\mathcal{S}'(\mathbb{R}^N)$ as $t \rightarrow 0$. In Section 4, we establish the blow up result, Theorem 1.2, and in Section 5, we establish the global existence result, Theorem 1.3.

2. LOCAL EXISTENCE IN A GENERAL CONTEXT

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a regular domain not necessarily bounded. The Banach space $C_0(\Omega)$ is the set of all continuous functions $f : \overline{\Omega} \rightarrow \mathbb{R}$ such that $f = 0$ on the boundary $\partial\Omega$ and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$ in Ω . The *sup* norm on $C_0(\Omega)$ is designated by $\|f\|_\infty$. We assume that Ω is regular enough so that the heat semigroup $e^{t\Delta}$ is a C_0 contraction, positivity preserving semigroup on $C_0(\Omega)$. Since $e^{t\Delta}$ is positivity preserving, it satisfies the inequality

$$(2.1) \quad |e^{t\Delta}(fg)| \leq (e^{t\Delta}|f|)\|g\|_\infty,$$

for all $f, g \in C_0(\Omega)$, for example.

Definition 2.1. A continuous function $\varphi : (0, \infty) \rightarrow C_0(\Omega)$ is called a solution of the linear heat equation if

$$e^{t\Delta}\varphi(s) = \varphi(t+s)$$

for all $t > 0$ and $s > 0$.

It is important to note that no initial value $\varphi(0)$ is required or specified. Also, note that if $\varphi : (0, \infty) \rightarrow C_0(\Omega)$ is a solution of the linear heat equation, then $\|\varphi(t)\|_\infty$ is a nonincreasing function of $t > 0$.

Definition 2.2. Let $\Phi : (0, \infty) \rightarrow C_0(\Omega)$ be a positive solution of the linear heat equation, i.e. $\Phi(t, x) > 0$ for all $t > 0$, $x \in \Omega$. For a given $T > 0$, let $X = X_T = X_{T, \Phi}$ be the set of all continuous functions $u : (0, T] \rightarrow C_0(\Omega)$ which are bounded by a multiple of Φ in that there exists $M > 0$ such that

$$|u(t, x)| \leq M\Phi(t, x), \quad \forall t \in (0, T], \quad \forall x \in \Omega.$$

We define on X the norm

$$|||u||| = \sup_{0 < t \leq T, x \in \Omega} \frac{|u(t, x)|}{\Phi(t, x)}.$$

The set X equipped with this norm is a Banach space. Note that

$$(2.2) \quad |u(t)| \leq |||u||| \Phi(t)$$

for all $0 < t \leq T$. If needed for clarity, we may denote the norm $|||u|||$ by $|||u|||_{X_T}$ or even $|||u|||_{X_{T, \Phi}}$.

We have the following result.

Theorem 2.3. Fix a positive solution of the linear heat equation $\Phi : (0, \infty) \rightarrow C_0(\Omega)$. Let $\alpha > 0$, and suppose that Φ verifies

$$(2.3) \quad \int_0^A \|\Phi(\sigma)\|_\infty^\alpha d\sigma < \infty$$

for some (hence all) $A > 0$. Let $K > 0$, $M > 0$ and $T > 0$ be such that

$$(2.4) \quad K + M^{\alpha+1} \int_0^T \|\Phi(t)\|_\infty^\alpha dt \leq M$$

and

$$(2.5) \quad 2(\alpha + 1)M^\alpha \int_0^T \|\Phi(t)\|_\infty^\alpha dt < 1.$$

Given $\varphi \in X_T$ with $|||\varphi||| \leq K$, there is a unique solution $u \in X_T$ with $|||u||| \leq M$ of the integral equation

$$(2.6) \quad u(t) = \varphi(t) + \int_0^t e^{(t-\sigma)\Delta} (|u(\sigma)|^\alpha u(\sigma)) d\sigma.$$

Moreover, the integral in (2.6) is absolutely convergent in $C_0(\Omega)$, and

$$(2.7) \quad \lim_{t \rightarrow 0} \left\| \frac{u(t) - \varphi(t)}{\Phi(t)} \right\|_\infty = 0.$$

Furthermore, there exists $\mathcal{C} > 0$ such that if $\varphi \in X_T$ and $\psi \in X_T$, with $|||\varphi||| \leq K$ and $|||\psi||| \leq K$, and if u and v are respectively the solutions of the corresponding integral equations, then

$$(2.8) \quad |||u - v||| \leq \mathcal{C} |||\varphi - \psi|||.$$

Finally, if $\varphi \in X_T$ with $|||\varphi||| \leq K$ is positive, $\varphi(t) \geq 0$, for all $0 < t \leq T$, then the same is true of the resulting solution $u \in X_T$ of (2.6) with $|||u||| \leq M$, i.e. $u(t) \geq 0$ for all $0 < t \leq T$.

Remark 2.4. The previous theorem gives a local existence result of solutions for (1.1) with $u_0 \in L^q(\mathbb{R}^N)$, $q \geq 1$, $q > \frac{N\alpha}{2}$. In fact, by the smoothing properties of the semigroup of the linear heat equation, we have that $\|e^{t\Delta}|u_0|\|_{L^\infty(\Omega)} \leq t^{-\frac{N}{2q}} \|u_0\|_{L^q(\Omega)}$. Then we apply Theorem 2.3 with $\varphi(t) = e^{t\Delta}u_0$, $\Phi(t) = e^{t\Delta}|u_0|$ and $K = 1$. Clearly, by the conditions on q , (2.3) is satisfied. This result, of course, is well known [20, 21, 17].

Remark 2.5. We emphasize that Theorem 2.3 neither requires nor mentions an initial value to the integral equation (2.6). Moreover, the function φ need not itself be a solution of the linear heat equation.

Proof of Theorem 2.3. The proof is based on a contraction mapping argument, and is inspired by the construction in [22]. We thank Thierry Cazenave for pointing out how the construction in [22] could be reformulated as a contraction mapping argument. Moreover, a similar argument was used by Fujita [6] to prove global existence of solutions for initial values bounded by a small Gaussian function.

Let $K > 0$, $M > 0$ and $T > 0$ satisfying (2.4) and (2.5). Let

$$(2.9) \quad B(0, M) := \{u \in X \mid \|u\| \leq M\}.$$

$B(0, M)$ is a complete metric space. Fix $\varphi \in X$ such that $\|\varphi\| \leq K$ and consider, for $u \in B(0, M)$,

$$\mathcal{F}u(t) = \varphi(t) + \int_0^t e^{(t-\sigma)\Delta} (|u(\sigma)|^\alpha u(\sigma)) d\sigma.$$

We will show that $\mathcal{F}$ is a strict contraction on $B(0, M)$.

First, using the inequalities (2.1) and (2.2), we have that

$$\begin{aligned} |\mathcal{F}u(t)| &\leq |\varphi(t)| + \int_0^t (e^{(t-\sigma)\Delta} |u(\sigma)|) \|u(\sigma)\|_\infty^\alpha d\sigma \\ &\leq K\Phi(t) + \|u\|^{\alpha+1} \int_0^t (e^{(t-\sigma)\Delta} \Phi(\sigma)) \|\Phi(\sigma)\|_\infty^\alpha d\sigma \\ &\leq K\Phi(t) + M^{\alpha+1} \int_0^t \Phi(t) \|\Phi(\sigma)\|_\infty^\alpha d\sigma \\ &\leq \Phi(t) [K + M^{\alpha+1} \int_0^T \|\Phi(\sigma)\|_\infty^\alpha d\sigma]. \end{aligned}$$

Hence,

$$\frac{|\mathcal{F}u(t)|}{\Phi(t)} \leq K + M^{\alpha+1} \int_0^T \|\Phi(\sigma)\|_\infty^\alpha d\sigma, \quad \forall t \in (0, T].$$

Thus by (2.4), we have that $\|\mathcal{F}u\| \leq M$, and so $\mathcal{F}$ maps $B(0, M)$ into itself.

Next, to see that $\mathcal{F}$ is a contraction, we let $u_1, u_2 \in B(0, M)$ so that

$$\mathcal{F}(u_1)(t) - \mathcal{F}(u_2)(t) = \int_0^t e^{(t-\sigma)\Delta} (|u_1|^\alpha u_1(\sigma) - |u_2|^\alpha u_2(\sigma)) d\sigma.$$

It follows that

$$\begin{aligned}
& |\mathcal{F}(u_1)(t) - \mathcal{F}(u_2)(t)| \\
& \leq (\alpha + 1) \int_0^t e^{(t-\sigma)\Delta} \left(|u_1(\sigma) - u_2(\sigma)| (|u_1(\sigma)|^\alpha + |u_2(\sigma)|^\alpha) \right) d\sigma \\
& \leq (\alpha + 1) \int_0^t \left(e^{(t-\sigma)\Delta} \Phi(\sigma) \frac{|u_1(\sigma) - u_2(\sigma)|}{\Phi(\sigma)} \right) \\
& \quad \left(\|u_1(\sigma)\|_\infty^\alpha + \|u_2(\sigma)\|_\infty^\alpha \right) d\sigma \\
& \leq (\alpha + 1) \int_0^t \left(e^{(t-\sigma)\Delta} \Phi(\sigma) \right) \|u_1 - u_2\| \\
& \quad \left(\|u_1(\sigma)\|_\infty^\alpha + \|u_2(\sigma)\|_\infty^\alpha \right) d\sigma \\
& \leq 2M^\alpha (\alpha + 1) \int_0^t \Phi(t) \|\Phi(\sigma)\|_\infty^\alpha d\sigma \|u_1 - u_2\|.
\end{aligned}$$

Therefore,

$$\frac{|\mathcal{F}(u_1)(t) - \mathcal{F}(u_2)(t)|}{\Phi(t)} \leq \left(2(\alpha + 1)M^\alpha \int_0^T \|\Phi(\sigma)\|_\infty^\alpha d\sigma \right) \|u_1 - u_2\|,$$

and so by (2.5), there exists $\nu < 1$ such that

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\| \leq \nu \|u_1 - u_2\|.$$

Hence $\mathcal{F}$ is a strict contraction from $B(0, M)$ into itself. By the Banach fixed point theorem there exists a unique solution $u \in X_T$ of the integral equation (2.6) such that $\|u\| \leq M$.

Let u be the solution of the integral equation constructed above. Since

$$\begin{aligned}
|u(t) - \varphi(t)| &= \left| \int_0^t e^{(t-\sigma)\Delta} |u(\sigma)|^\alpha u(\sigma) d\sigma \right| \\
&\leq \int_0^t (e^{(t-\sigma)\Delta} |u(\sigma)|) \|u(\sigma)\|_\infty^\alpha d\sigma \\
&\leq M^{\alpha+1} \Phi(t) \int_0^t \|\Phi(\sigma)\|_\infty^\alpha d\sigma,
\end{aligned}$$

assertion (2.7) follows. In the same way,

$$(2.10) \quad \int_0^t \|e^{(t-\sigma)\Delta} |u(\sigma)|^\alpha u(\sigma)\|_\infty d\sigma \leq M^{\alpha+1} \|\Phi(t)\|_\infty \int_0^t \|\Phi(\sigma)\|_\infty^\alpha d\sigma,$$

which shows that the integral in (2.6) is absolutely convergent in $C_0(\Omega)$.

Let $\varphi \in X_T$ and $\psi \in X_T$ with $\|\phi\| \leq K$ and $\|\psi\| \leq K$. Let u and v be the solutions of the corresponding integral equations, i.e.

$$u(t) = \varphi(t) + \int_0^t e^{(t-\sigma)\Delta} |u(\sigma)|^\alpha u(\sigma) d\sigma$$

and

$$v(t) = \psi(t) + \int_0^t e^{(t-\sigma)\Delta} |v(\sigma)|^\alpha v(\sigma) d\sigma.$$

Then, by a calculation similar to the proof above that $\mathcal{F}$ is a contraction, we have that

$$|||u - v||| \leq |||\varphi - \psi||| + \left(2(\alpha + 1)M^\alpha \int_0^T \|\Phi(\sigma)\|_\infty^\alpha d\sigma \right) |||u - v|||.$$

Thus (2.8) follows with

$$\mathcal{C} = \frac{1}{1 - \left(2(\alpha + 1)M^\alpha \int_0^T \|\Phi(\sigma)\|_\infty^\alpha d\sigma \right)}.$$

Finally, if $\varphi(t) \geq 0$ for all $0 < t \leq T$, then one can carry out the contraction mapping argument on the subset of positive elements of $B(0, M)$. $\square$

Remark 2.6. If in the hypothesis of Theorem 2.3 we replace the condition (2.3) by

$$\int_0^\infty \|\Phi(\sigma)\|_\infty^\alpha d\sigma < \infty$$

and choose $K > 0$, $M > 0$ such that

$$K + M^{\alpha+1} \int_0^\infty \|\Phi(t)\|_\infty^\alpha dt \leq M, \quad 2(\alpha + 1)M^\alpha \int_0^\infty \|\Phi(t)\|_\infty^\alpha dt < 1,$$

then, if $\varphi \in X_\infty$ satisfies $|||\varphi|||_{X_\infty} \leq K$, there exists a unique global solution to the integral equation (2.6), $u \in X_\infty$ with $|||u|||_{X_\infty} \leq M$.

Remark 2.7. In the proof of Theorem 2.3 we only need that $\Phi : (0, T) \rightarrow C_0(\Omega)$ is positive and satisfies

$$e^{t\Delta}\Phi(s) \leq \Phi(t+s), \quad \forall t > 0, s > 0, s+t < T.$$

In other words, Φ could be a “super solution” of the linear heat equation. In this case, we define $|||u||| = \sup_{0 < t < T, x \in \Omega} \frac{|u(t,x)|}{\Phi(t,x)}$. This allows, in principle, that $\|\Phi(t)\|_\infty \rightarrow \infty$ as $t \rightarrow T$. Of course the resulting solution is only defined for $t \in (0, T)$.

3. HIGHLY SINGULAR INITIAL VALUES

The goal of this section is to use the abstract framework of the previous section as a tool to construct a solution of the nonlinear heat equation (1.4) on $\mathbb{R}^N$ with initial value u_0 , which is an arbitrary given multiple of

$$(3.1) \quad (-1)^m \partial_1 \partial_2 \cdots \partial_m \delta,$$

where $\delta \in \mathcal{S}'(\mathbb{R}^N)$ is the Dirac point mass at the origin and $1 \leq m \leq N$. The solution of the linear heat equation on $\mathbb{R}^N$ with initial value (3.1) is well known. Recall that G_t , given by (1.3), is the Gauss (heat semigroup) kernel, and that G_t is the solution of the linear heat equation on $C_0(\mathbb{R}^N)$ with initial value δ . It follows that

$$(3.2) \quad \Phi_0(t) = (-1)^m \partial_1 \partial_2 \cdots \partial_m G_t, \quad t > 0, x \in \mathbb{R}^N,$$

is the solution of the linear heat equation on $C_0(\mathbb{R}^N)$ with initial value (3.1). However, it is clearly not positive everywhere.

The key (albeit elementary) observation is that $\Phi_0(t)$ is a positive solution of the linear heat equation, as in Definition 2.1, in $C_0(\Omega)$, where

$$(3.3) \quad \Omega = \mathbb{R}_{m+}^N = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N \mid x_1 > 0, x_2 > 0, \dots, x_m > 0\}.$$

Moreover, viewed as an element of $C_0(\mathbb{R}^N)$, $\Phi_0(t)$ is anti-symmetric with respect to the variables $x_1, x_2, \dots, x_m$. Indeed, by a simple calculation, we have that

$$(3.4) \quad \Phi_0(t, x) = (4\pi t)^{-\frac{N}{2}} (2t)^{-m} e^{-\frac{|x|^2}{4t}} \prod_{i=1}^m x_i.$$

It is clear that $\Phi_0(t, x) > 0$ for all $x \in \Omega$, and that $\Phi_0(t) \in C_0(\Omega)$ for all $t > 0$.

In order to put this observation in an appropriate context, we need to be more precise about the relationship between solutions of the linear and nonlinear equations in $C_0(\mathbb{R}^N)$ and in $C_0(\Omega)$. Throughout this section, Ω will be as in (3.3).

The first step is to identify the heat semigroup on $C_0(\Omega)$. As noted in the introduction, $e^{t\Delta}$ is the heat semigroup on $\mathbb{R}^N$, i.e. convolution with the Gauss kernel (1.3), i.e.

$$e^{t\Delta} f = G_t \star f$$

for all (for example) $f \in C_0(\mathbb{R}^N)$. We denote by $e^{t\Delta_\Omega}$ the heat semigroup on $C_0(\Omega)$.

For $1 \leq i \leq N$, let $T_i : C_0(\mathbb{R}^N) \rightarrow C_0(\mathbb{R}^N)$ be the operator defined by

$$[T_i f](x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N) = f(x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_N).$$

If $f \in C_0(\mathbb{R}^N)$ satisfies

$$(3.5) \quad T_1 f = T_2 f = \dots = T_m f = -f,$$

then f restricted to $\overline{\Omega}$ is clearly in $C_0(\Omega)$, $f|_{\overline{\Omega}} \in C_0(\Omega)$. On the other hand, if $g \in C_0(\Omega)$, then clearly g can be extended uniquely to $\tilde{g} \in C_0(\mathbb{R}^N)$ so that $g = \tilde{g}|_{\overline{\Omega}}$ and

$$T_1 \tilde{g} = T_2 \tilde{g} = \dots = T_m \tilde{g} = -\tilde{g}.$$

It is immediate that $e^{t\Delta} T_i = T_i e^{t\Delta}$ for all $1 \leq i \leq m$. Consequently, if $g \in C_0(\Omega)$ and $\tilde{g} \in C_0(\mathbb{R}^N)$ is the extension of g just defined, then

$$T_i e^{t\Delta} \tilde{g} = e^{t\Delta} T_i \tilde{g} = -e^{t\Delta} \tilde{g}$$

for all $1 \leq i \leq m$. In other words, $(e^{t\Delta} \tilde{g})|_{\overline{\Omega}} \in C_0(\Omega)$. We claim that the heat semigroup on $C_0(\Omega)$, denoted $e^{t\Delta_\Omega}$, is given by

$$(3.6) \quad e^{t\Delta_\Omega} g = (e^{t\Delta} \tilde{g})|_{\overline{\Omega}},$$

where $e^{t\Delta}$ is the heat semigroup on $C_0(\mathbb{R}^N)$ and $\tilde{g}$ and g are related as above. In fact, $e^{t\Delta_\Omega} g$ is the solution of the linear heat equation on $C_0(\Omega)$ with initial value g . On the other hand, $e^{t\Delta} \tilde{g}$ is the solution of the linear heat equation on $C_0(\mathbb{R}^N)$ with initial value $\tilde{g}$. Since $g = \tilde{g}|_{\overline{\Omega}}$, (3.6) follows by uniqueness of solutions for the linear heat equations on $C_0(\Omega)$. In fact, the explicit expression for $e^{t\Delta_\Omega} g$ is well known.

Proposition 3.1. *The heat semigroup $e^{t\Delta_\Omega}$ on $C_0(\Omega)$ is given by*

$$e^{t\Delta_\Omega} g(x) = (4\pi t)^{-\frac{N}{2}} \int_{\Omega} \prod_{j=m+1}^N e^{-\frac{|x_j - y_j|^2}{4t}} \prod_{i=1}^m \left[e^{-\frac{|x_i - y_i|^2}{4t}} - e^{-\frac{|x_i + y_i|^2}{4t}} \right] g(y) dy,$$

for all $g \in C_0(\Omega)$.

Proof. First, let $N = 1 = m$. Hence $\Omega =]0, \infty[$. Assume that $g \in C_0(]0, \infty[)$. Then $\tilde{g} \in C_0(\mathbb{R})$ and

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x > 0, \\ -g(-x) & \text{if } x < 0. \end{cases}$$

This gives

$$\begin{aligned}
e^{t\Delta}\tilde{g}(x) &= \int_{\mathbb{R}} G_t(x-y)\tilde{g}(y)dy \\
&= (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y)dy \\
&= (4\pi t)^{-\frac{1}{2}} \int_0^\infty e^{-\frac{|x-y|^2}{4t}} g(y)dy + (4\pi t)^{-\frac{1}{2}} \int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}} (-g(-y))dy \\
&= (4\pi t)^{-\frac{1}{2}} \int_0^\infty e^{-\frac{|x-y|^2}{4t}} g(y)dy - (4\pi t)^{-\frac{1}{2}} \int_0^\infty e^{-\frac{|x+y|^2}{4t}} g(y)dy \\
&= \int_0^\infty (4\pi t)^{-\frac{1}{2}} [e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x+y|^2}{4t}}] g(y)dy.
\end{aligned}$$

Hence, for $x \in \Omega$,

$$e^{t\Delta_\Omega}g(x) = \int_0^\infty (4\pi t)^{-\frac{1}{2}} [e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x+y|^2}{4t}}] g(y)dy.$$

Now, for $N \geq 1$, since the function $G_t(x)$ satisfies

$$G_t(x) = (4\pi t)^{-\frac{N}{2}} \prod_{j=1}^N e^{-\frac{|x_j - y_j|^2}{4t}}, \quad x = (x_1, x_2, \dots, x_N),$$

the above calculation can be carried out successively for the variables x_i , $i = 1, \dots, m$, yielding the desired formula. $\square$

This enables us to prove the following relationship between solutions of the linear heat equation in $C_0(\mathbb{R}^N)$ and $C_0(\Omega)$, in the sense of Definition 2.1.

Proposition 3.2. *Let $\Psi : (0, \infty) \rightarrow C_0(\mathbb{R}^N)$ be a solution of the linear heat equation such that $T_i\Psi(t) = -\Psi(t)$ for all $1 \leq i \leq m$. It follows that $\Phi : (0, \infty) \rightarrow C_0(\Omega)$ defined by $\Phi(t) = \Psi(t)|_{\overline{\Omega}}$ is a solution to the linear heat equation.*

Let $\Phi : (0, \infty) \rightarrow C_0(\Omega)$ be a solution of the linear heat equation. It follows that $\Psi : (0, \infty) \rightarrow C_0(\mathbb{R}^N)$ is a solution of the linear heat equation, where $\Psi(t) = \tilde{\Phi}(t)$ is the unique extension of $\Phi(t)$ to $\mathbb{R}^N$ such that $T_i\Psi(t) = -\Psi(t)$ for all $1 \leq i \leq m$.

Proof. The first statement follows from (3.6), since

$$e^{t\Delta_\Omega}\Phi(s) = e^{t\Delta_\Omega}(\Psi(s)|_{\overline{\Omega}}) = (e^{t\Delta}\Psi(s))|_{\overline{\Omega}} = \Psi(t+s)|_{\overline{\Omega}} = \Phi(t+s).$$

The other statement is proved similarly, using the fact that $\widetilde{e^{t\Delta_\Omega}f} = e^{t\Delta}\tilde{f}$, $f \in C_0(\Omega)$, which follows from Proposition 3.1. In fact, since $\Phi(t+s) = e^{t\Delta_\Omega}\Phi(s)$, then

$$\Psi(t+s) = \widetilde{e^{t\Delta_\Omega}\Phi(s)} = e^{t\Delta}\tilde{\Phi}(s) = e^{t\Delta}\Psi(s).$$

$\square$

Let $\Psi : (0, \infty) \rightarrow C_0(\mathbb{R}^N)$ be a solution to the linear heat equation such that $T_i\Psi(t) = -\Psi(t)$ for all $1 \leq i \leq m$ and let $\Phi : (0, \infty) \rightarrow C_0(\Omega)$ be the solution of the linear heat equation on Ω defined by $\Phi(t) = \Psi(t)|_{\overline{\Omega}}$. Note that $\|\Psi(t)\|_{L^\infty(\mathbb{R}^N)} = \|\Phi(t)\|_{L^\infty(\Omega)}$ for all $t > 0$. The proof of the following proposition is now obvious.

Proposition 3.3. *Let $\Psi : (0, \infty) \rightarrow C_0(\mathbb{R}^N)$ be a solution to the linear heat equation such that $T_i\Psi(t) = -\Psi(t)$ for all $1 \leq i \leq m$ and let $\Phi : (0, \infty) \rightarrow C_0(\Omega)$ be the solution of the linear heat equation on Ω defined by $\Phi(t) = \Psi(t)|_{\overline{\Omega}}$, where Ω is the*

domain given by (3.3). Suppose $\Phi(t, x) > 0$ for all $t > 0$ and all $x \in \Omega$. For each $T > 0$, let $X = X_{T, \Phi}$ be the space of functions given in Definition 2.2. Suppose $\alpha > 0$ is such that condition (2.3) of Theorem 2.3 holds, and let $K > 0$, $M > 0$ and $T > 0$ satisfy (2.4) and (2.5). Let $\varphi \in X_T$ such that $\|\varphi\| \leq K$ and let $u \in X_T$ be the unique solution with $\|u\| \leq M$ of the integral equation

$$(3.7) \quad u(t) = \varphi(t) + \int_0^t e^{(t-\sigma)\Delta_\Omega} (|u(\sigma)|^\alpha u(\sigma)) d\sigma,$$

given by Theorem 2.3. Let $\tilde{u} : (0, T] \rightarrow C_0(\mathbb{R}^N)$ and $\tilde{\varphi} : (0, T] \rightarrow C_0(\mathbb{R}^N)$ be the extensions to $\mathbb{R}^N$ of u and φ such that $\tilde{u}(t)$ and $\tilde{\varphi}(t)$ all satisfy (3.5). It follows that

$$(3.8) \quad \tilde{u}(t) = \tilde{\varphi}(t) + \int_0^t e^{(t-\sigma)\Delta} (|\tilde{u}(\sigma)|^\alpha \tilde{u}(\sigma)) d\sigma,$$

for all $0 < t \leq T$. The integral in (3.8) is absolutely convergent in $C_0(\mathbb{R}^N)$ and

$$\int_0^t \|e^{(t-\sigma)\Delta} (|\tilde{u}(\sigma)|^\alpha \tilde{u}(\sigma))\|_\infty d\sigma \leq M^{\alpha+1} \|\Phi(t)\|_\infty \int_0^t \|\Phi(\sigma)\|_\infty^\alpha d\sigma.$$

This last estimate follows from (2.10).

Proposition 3.4. *Under the hypotheses of Proposition 3.3, if, in addition, $\Phi(t) = \Phi_0(t)$, given by (3.2), and $\alpha < 2/(N + m)$ so that (2.3) holds, it follows that*

$$(3.9) \quad \tilde{u}(t) - \tilde{\varphi}(t) = \int_0^t e^{(t-\sigma)\Delta} (|\tilde{u}(\sigma)|^\alpha \tilde{u}(\sigma)) d\sigma \rightarrow 0$$

in $\mathcal{S}'(\mathbb{R}^N)$ as $t \rightarrow 0$.

For the proof of the last proposition, we need the following two lemmas.

Lemma 3.5. *For $t > 0$, we have*

$$\int_{\mathbb{R}^N} \Phi_0(t, x) x_1 \cdots x_m dx = 1.$$

Proof. This is a straightforward calculation since

$$\Phi_0(t, x) x_1 \cdots x_m = H_t(x),$$

where

$$H_t(x) = t^{-N/2} H_1\left(\frac{x}{\sqrt{t}}\right), \quad H_1(x) = (4\pi)^{-\frac{N}{2}} (2)^{-m} e^{-\frac{|x|^2}{4}} \left(\prod_{i=1}^m x_i\right)^2, \quad t > 0, \quad x \in \mathbb{R}^N,$$

$$\text{and } \int_{\mathbb{R}^N} H_1(x) dx = 1. \quad \square$$

Lemma 3.6. *Assume that $f \in C^m(\mathbb{R}^N)$ and f is anti-symmetric with respect to the variables $x_1, \dots, x_m$; that is, f verifies (3.5). It follows that*

$$(3.10) \quad f(x) = \int_0^{x_m} \cdots \int_0^{x_1} \partial_1 \cdots \partial_m f(t_1, \dots, t_m, x_{m+1}, \dots, x_N) dt_1 \cdots dt_m,$$

for all $x \in \mathbb{R}^N$. In particular,

$$(3.11) \quad |f(x)| \leq \left(\sup_{\mathbb{R}^N} |\partial_1 \cdots \partial_m f| \right) |x_1 x_2 \cdots x_m|, \quad \forall x \in \mathbb{R}^N.$$

Proof. To prove equality (3.10), we argue by induction on m . For $m = 1$, the formula (3.10) is obvious. Assume that (3.10) is satisfied for $m - 1$. Since f is anti-symmetric with respect to $x_1, \dots, x_{m-1}$, then $\partial_m f$ is anti-symmetric with respect to $x_1, \dots, x_{m-1}$, and then by the induction hypothesis,

$$\begin{aligned} & \partial_m f(x_1, \dots, x_{m-1}, t_m, x_{m+1}, \dots, x_N) \\ &= \int_0^{x_{m-1}} \cdots \int_0^{x_1} \partial_1 \cdots \partial_{m-1} \partial_m f(t_1, t_2, \dots, t_{m-1}, t_m, x_{m+1}, \dots, x_N) dt_1 dt_2 \cdots dt_{m-1}. \end{aligned}$$

Integrating the last equality with respect to t_m and using the anti-symmetry of f with respect to x_m , we get (3.10) for m . (3.11) follows by (3.10) since f is C^m . $\square$

Proof of Proposition 3.4. We begin by noting that if $f, g \in C_0(\Omega)$, with $\tilde{f}, \tilde{g}$ their extensions to $C_0(\mathbb{R}^N)$ verifying (3.5), and if $\tilde{g} \in L^1(\mathbb{R}^N)$, then

$$\langle \tilde{f}, \tilde{g} \rangle_{\mathbb{R}^N} = 2^m \langle f, g \rangle_{\Omega},$$

where $\langle \tilde{f}, \tilde{g} \rangle_{\mathbb{R}^N}$ denotes $\int_{\mathbb{R}^N} \tilde{f}(x) \tilde{g}(x) dx$ and $\langle f, g \rangle_{\Omega}$ denotes $\int_{\Omega} f(x) g(x) dx$.

Let $\eta \in \mathcal{S}(\mathbb{R}^N)$. We need to show that

$$\begin{aligned} (3.12) \quad & \lim_{t \rightarrow 0} \left\langle \int_0^t e^{(t-\sigma)\Delta} (|\tilde{u}(\sigma)|^\alpha \tilde{u}(\sigma)) d\sigma, \eta \right\rangle_{\mathbb{R}^N} \\ &= \lim_{t \rightarrow 0} \int_0^t \langle e^{(t-\sigma)\Delta} (|\tilde{u}(\sigma)|^\alpha \tilde{u}(\sigma)), \eta \rangle_{\mathbb{R}^N} d\sigma = 0. \end{aligned}$$

Since $e^{(t-\sigma)\Delta} (|\tilde{u}(\sigma)|^\alpha \tilde{u}(\sigma))$ satisfies (3.5) for every $0 < \sigma < t$, it follows that if η is symmetric in any one of the variables $x_1, \dots, x_m$, i.e. if $T_i \eta = \eta$ for any one $i \in \{1, 2, \dots, m\}$, then

$$\langle e^{(t-\sigma)\Delta} (|\tilde{u}(\sigma)|^\alpha \tilde{u}(\sigma)), \eta \rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} [e^{(t-\sigma)\Delta} (|\tilde{u}(\sigma)|^\alpha \tilde{u}(\sigma))](x) \eta(x) dx = 0.$$

On the other hand, as is shown in the appendix, any function on $\mathbb{R}^N$ can be written explicitly as the sum of functions, one of which verifies (3.5), i.e. is anti-symmetric in all the variables $x_1, \dots, x_m$, plus other functions, each of which is symmetric in at least one of the variables $x_1, \dots, x_m$. This decomposition, which generalizes the fact that a function of one variable is the sum of an even and an odd function, preserves all the differentiability and decay properties of the original function; see (6.1). (Formula (6.1) is for the case $m = N$. The general case $1 \leq m \leq N$ is similar.) Thus, replacing η by this decomposition, using the fact just noted that the pairing is zero for functions which are symmetric with respect to any one of the variables, we may suppose from now on that $\eta \in \mathcal{S}(\mathbb{R}^N)$ satisfies condition (3.5), i.e. is anti-symmetric with respect to all the variables $x_1, \dots, x_m$. As such, it restricts to an element of $C_0(\Omega)$ still denoted by η . By the remark made at the start of the proof, it suffices to show that

$$\begin{aligned} (3.13) \quad & \lim_{t \rightarrow 0} \int_0^t \langle e^{(t-\sigma)\Delta} (|u(\sigma)|^\alpha u(\sigma)), \eta \rangle_{\Omega} d\sigma \\ &= \lim_{t \rightarrow 0} \int_0^t \int_{\Omega} [e^{(t-\sigma)\Delta} (|u(\sigma)|^\alpha u(\sigma))](x) \eta(x) dx d\sigma = 0. \end{aligned}$$

As in the proof of Theorem 2.3, we see that, on Ω , using $|u(\sigma)| \leq M\Phi_0(\sigma)$,

$$\begin{aligned} |e^{(t-\sigma)\Delta_\Omega}(|u(\sigma)|^\alpha u(\sigma))| &\leq e^{(t-\sigma)\Delta_\Omega}(|u(\sigma)|)\|u(\sigma)\|_\infty^\alpha \\ &\leq e^{(t-\sigma)\Delta_\Omega}(M\Phi_0(\sigma))M^\alpha\|\Phi_0(\sigma)\|_\infty^\alpha \\ &= M^{\alpha+1}\Phi_0(t)\|\Phi_0(\sigma)\|_\infty^\alpha, \end{aligned}$$

and so

$$(3.14) \quad \left| \int_0^t \int_\Omega [e^{(t-\sigma)\Delta_\Omega}(|u(\sigma)|^\alpha u(\sigma))](x) \eta(x) dx d\sigma \right| \leq M^{\alpha+1} \langle \Phi_0(t), |\eta| \rangle_\Omega \int_0^t \|\Phi_0(\sigma)\|_\infty^\alpha d\sigma.$$

Thus, to establish (3.13), it suffices by (3.14) to show that $\langle \Phi_0(t), |\eta| \rangle_\Omega$ is bounded as $t \rightarrow 0$.

Since $\eta \in \mathcal{S}(\mathbb{R}^N)$ and verifies (3.5), it follows from Lemma 3.6, specifically formula (3.11), that there exists $R > 0$ such that

$$|\eta(x)| \leq R|x_1x_2 \cdots x_m|, \quad \forall x \in \mathbb{R}^N.$$

Thus,

$$\begin{aligned} |\langle \Phi_0(t), |\eta| \rangle_\Omega| &\leq R \int_\Omega \Phi_0(t, x) x_1 x_2 \cdots x_m dx \\ &= 2^{-m} R \int_{\mathbb{R}^N} \Phi_0(t, x) x_1 x_2 \cdots x_m dx \\ &= 2^{-m} R, \end{aligned}$$

where we have used Lemma 3.5. This finishes the proof of the proposition. $\square$

Proof of Theorem 1.1. Consider $\Phi_0(t)$, given by (3.2) and (3.4). Since

$$\Phi_0(t, x) = t^{-(N+m)/2} \Phi_0(1, x/\sqrt{t}),$$

it is clear that

$$\|\Phi_0(t)\|_\infty = \|\Phi_0(1)\|_\infty t^{-(N+m)/2}.$$

Thus, if we consider $\Phi_0(t)$ as a solution of the linear heat equation in $C_0(\Omega)$, we may apply Theorem 2.3 for $\alpha < 2/(N+m)$. Let $K > 0$. Choose $M > 0$ and $T > 0$ so that (2.4) and (2.5) are satisfied. (If $K < 0$, we replace K by $|K|$ in (2.4) and (2.5).) We set $\varphi(t) = K\Phi_0(t)$. Theorem 2.3 gives a local solution to the integral equation (3.7) in $C_0(\Omega)$ and therefore by Proposition 3.3 a solution of (3.8) in $C_0(\mathbb{R}^N)$. Since $\varphi(t) > 0$ on Ω , it is clear that $u(t) > 0$ on Ω . (If $K < 0$, then $u(t) < 0$ on Ω .) By abuse of notation, we write u and φ instead of $\tilde{u}$ and $\tilde{\varphi}$, and we note that in $\mathbb{R}^N$, $\varphi(t) = e^{t\Delta}u_0$, where $u_0 = K(-1)^m \partial_1 \partial_2 \cdots \partial_m \delta$. We thus obtain a solution to the integral equation (1.1) in $\mathbb{R}^N$ with $u_0 = K(-1)^m \partial_1 \partial_2 \cdots \partial_m \delta$. The last statement follows by Proposition 3.4. $\square$

4. AN EXTENSION OF DICKSTEIN'S "SMALL LAMBDA" BLOW UP RESULT

We continue with the notation of Section 3, and assume that $\alpha < 2/(N+m)$. In particular, $\Phi_0(t)$ is given by (3.2) and (3.4), and Ω is given by (3.3). Before proving Theorem 1.2, we need the following preliminary result concerning the hypotheses of that theorem.

Proposition 4.1. *Let $f \in C_0(\mathbb{R}^N)$ be anti-symmetric with respect to the variables $x_1, \dots, x_m$, i.e. such that f satisfies condition (3.5), and suppose there exist $t_0 > 0$ and $K > 0$ such that*

$$(4.1) \quad |f(x)| \leq K|\Phi_0(t_0, x)|$$

for all $x \in \mathbb{R}^N$. Let $g : \mathbb{R}^N \rightarrow \mathbb{R}$ be given by

$$(4.2) \quad \begin{aligned} g(x_1, x_2, \dots, x_N) \\ = (-1)^m \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} f(y_1, y_2, \dots, y_m, x_{m+1}, \dots, x_N) dy_1 dy_2 \cdots dy_m. \end{aligned}$$

It follows that $g \in C_0(\mathbb{R}^N)$ and

$$(4.3) \quad f = (-1)^m \partial_1 \partial_2 \cdots \partial_m g.$$

Furthermore, g decays exponentially as $|x| \rightarrow \infty$, so (in particular) $g \in L^1(\mathbb{R}^N)$ and

$$(4.4) \quad \int_{\mathbb{R}^N} g = \int_{\mathbb{R}^N} x_1 x_2 \cdots x_m f(x) dx_1 dx_2 \cdots dx_N.$$

If we set

$$(4.5) \quad f_\tau(x) = \tau^{-(N+m)/2} f\left(\frac{x}{\sqrt{\tau}}\right),$$

then, for any given $t > 0$,

$$(4.6) \quad e^{t\Delta} f_\tau \rightarrow e^{t\Delta} K_0 (-1)^m \partial_1 \partial_2 \cdots \partial_m \delta = K_0 \Phi_0(t)$$

in $\mathcal{S}'(\mathbb{R}^N)$ as $\tau \rightarrow 0$, where

$$(4.7) \quad K_0 = \int_{\mathbb{R}^N} g.$$

Finally, it must be that $|K_0| \leq K$.

Proof. The fact that $g \in C_0(\mathbb{R}^N)$ is an immediate consequence of (4.1) and the anti-symmetry of f in the first m variables. The decay estimate follows easily since $|\Phi_0(t_0, x)| \leq e^{-(|x_1| + \cdots + |x_N|)}$ for sufficiently large $|x|$ and the anti-symmetry of f . To show (4.4), we use a calculation inspired by the proof of Corollary 1 in [8]. We note that g is symmetric with respect to $x_1, \dots, x_m$ so that $\int_{\mathbb{R}^N} g$ is given by

$$(-1)^m 2^m \int_{\mathbb{R}^{N-m}} \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_{-\infty}^{x_m} \cdots \int_{-\infty}^{x_1} f(y_1, y_2, \dots, y_m, x_{m+1}, \dots, x_N) dy dx,$$

where $dy = dy_1 \cdots dy_m$ and $dx = dx_1 \cdots dx_N$. Interchanging the order of integration between each x_j and y_j , for $1 \leq j \leq m$, we see that $\int_{\mathbb{R}^N} g$ is given by

$$\begin{aligned} (-1)^m 2^m \int_{\mathbb{R}^{N-m}} \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_{y_m}^0 \cdots \int_{y_1}^0 f(y_1, y_2, \dots, y_m, x_{m+1}, \dots, x_N) \\ \times dx_1 dx_2 \cdots dx_m dy dx', \end{aligned}$$

where $dx' = dx_{m+1} \cdots dx_N$. This equals

$$2^m \int_{\mathbb{R}^{N-m}} \int_{-\infty}^0 \cdots \int_{-\infty}^0 y_1 \cdots y_m f(y_1, y_2, \dots, y_m, x_{m+1}, \dots, x_N) dy dx',$$

which is precisely the right hand side of (4.4).

If we now set

$$g^\tau(x) = \tau^{-N/2} g\left(\frac{x}{\sqrt{\tau}}\right),$$

then clearly

$$g^\tau \rightarrow K_0 \delta$$

in $\mathcal{S}'(\mathbb{R}^N)$ as $\tau \rightarrow 0$, where K_0 is given by (4.7). Since $f_\tau = (-1)^m \partial_1 \partial_2 \cdots \partial_m g^\tau$, it follows that

$$f_\tau \rightarrow K_0 (-1)^m \partial_1 \partial_2 \cdots \partial_m \delta,$$

in $\mathcal{S}'(\mathbb{R}^N)$, as $\tau \rightarrow 0$. This implies (4.6). Finally, we prove that $|K_0| \leq K$. Indeed, since f is anti-symmetric with respect to the variables $x_1, \dots, x_m$ it follows that g is symmetric with respect to $x_1, \dots, x_m$. Thus

$$K_0 = 2^m \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_{\mathbb{R}^{N-m}} g(x_1, x_2, \dots, x_m, x') dx, \quad x = (x_1, x_2, \dots, x_m, x').$$

Using the fact that

$$|\Phi_0(t_0, x)| = \partial_1 \partial_2 \cdots \partial_m G_{t_0}(x), \quad t_0 > 0, \quad x \in \mathbb{R}^N, \quad x_i < 0,$$

for $i \leq m$, it follows that

$$\begin{aligned} |K_0| &\leq 2^m \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_{\mathbb{R}^{N-m}} |g(x_1, x_2, \dots, x_m, x')| dx_1 dx_2 \cdots dx_m dx' \\ &\leq 2^m K \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_{\mathbb{R}^{N-m}} \left[\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} |\Phi_0(t_0, y_1, y_2, \dots, y_m, x')| \right. \\ &\quad \left. \times \left(\prod_{i=1}^m dy_i \right) \right] dx \\ &= 2^m K \int_{\mathbb{R}^{N-m}} \int_{-\infty}^0 \cdots \int_{-\infty}^0 \left[\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} \partial_1 \cdots \partial_m G_{t_0}(y_1, \dots, y_m, x') \right. \\ &\quad \left. \times \left(\prod_{i=1}^m dy_i \right) \right] dx \\ &\leq K \int_{\mathbb{R}^{N-m}} 2^m \int_{-\infty}^0 \cdots \int_{-\infty}^0 G_{t_0}(x_1, x_2, \dots, x_m, x') \left(\prod_{i=1}^m dx_i \right) dx' \\ &\leq K \int_{\mathbb{R}^{N-m}} \int_{\mathbb{R}^m} G_{t_0}(x_1, x_2, \dots, x_m, x') \left(\prod_{i=1}^m dx_i \right) dx' = K. \end{aligned}$$

□

Remark 4.2. A class of examples of functions f as in the assumptions of Proposition 4.1 is not hard to construct. Let $f \in C^m(\mathbb{R}^N)$ with compact support and f be anti-symmetric with respect to the variables $x_1, \dots, x_m$. We claim that for any $t_0 > 0$, there exists $K > 0$ such that (4.1) holds on $\mathbb{R}^N$. To see this, we note that (3.4) implies that on any bounded set in $\mathbb{R}^N$, there exists $C_1 > 0$ such that $|\Phi_0(t_0, x)| \geq C_1 |x_1 x_2 \cdots x_m|$. Thus, the claim follows from Lemma 3.6. We may replace the condition on the compactness of the support of f by assuming that f has an appropriate exponential decaying with $\sup_{\mathbb{R}^N} |\partial_1 \partial_2 \cdots \partial_m f| < \infty$.

Proof of Theorem 1.2. Let $f \in C_0(\mathbb{R}^N)$ be as in the hypotheses of Theorem 1.2, i.e. as in the hypotheses of Proposition 4.1 with $K_0 \neq 0$. We assume for notational simplicity that $t_0 = 1$. (The general case can be treated by the same arguments, but with an additional factor of t_0 appearing in many formulas.) Also, we treat the case $K_0 > 0$. We consider the maximal solution in $C_0(\mathbb{R}^N)$ of (1.1) with initial value $u_0 = \lambda f$, for $\lambda > 0$. We wish to show that this solution blows up in finite time if $\lambda > 0$ is sufficiently small. By the standard invariance properties of solutions to (1.4) and (1.1), this is equivalent to showing that the maximal solution with initial value

$$u_{0,\tau}(x) = \lambda \tau^{-1/\alpha} f\left(\frac{x}{\sqrt{\tau}}\right)$$

blows up in finite time for any $\tau > 0$. For a given $\lambda > 0$ we let $\tau = \tau(\lambda)$ be such that

$$\lambda = \tau^{-\frac{N+m}{2} + \frac{1}{\alpha}}.$$

With this definition of τ , it follows that $u_{0,\tau} = f_\tau$, given by (4.5), and that as $\lambda \rightarrow 0$, so does $\tau \rightarrow 0$ (since $\alpha < 2/(N+m)$). It thus suffices to show that the maximal solution of (1.1) with initial value f_τ blows up in finite time, for $\tau > 0$ sufficiently small. The basic idea is to prove this using the fact that $f_\tau \rightarrow K_0(-1)^m \partial_1 \partial_2 \cdots \partial_m \delta$ in $\mathcal{S}'(\mathbb{R}^N)$, as $\tau \rightarrow 0$, and applying the machinery developed thus far in this paper. To do this, we start with the initial value $K_0(-1)^m \partial_1 \partial_2 \cdots \partial_m \delta$.

We let $u : (0, T] \rightarrow C_0(\mathbb{R}^N)$ be the local solution of the integral equation (1.1), constructed using Theorem 2.3, with $\Phi(t) = \Phi_0(t)$, and Proposition 3.3, with initial value

$$u_0 = K_0(-1)^m \partial_1 \partial_2 \cdots \partial_m \delta.$$

In other words, we apply Theorem 2.3 with $\varphi(t) = K_0 \Phi_0(t)$ in $C_0(\Omega)$. However, in the application of Theorem 2.3, we use the value of K above, rather than K_0 . This is possible since $\|\varphi(t)\|_X = |K_0| \leq K$. The constants K , M and T satisfy (2.4) and (2.5) and $|u(t)| \leq M|\Phi_0(t)|$ for all $t \in (0, T]$. Thus, by Proposition 3.3, we have

$$(4.8) \quad u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\sigma)\Delta} (|u(\sigma)|^\alpha u(\sigma)) d\sigma.$$

We suppress the “tilde” and use the same notation for a function in $C_0(\Omega)$ and its anti-symmetric extension to a function in $C_0(\mathbb{R}^N)$ so that $u(t)$, as an element of $C_0(\mathbb{R}^N)$, satisfies condition (3.5) and restricts to an element of $C_0(\Omega)$. This is essentially the same solution that was described in Theorem 1.1. Since $K_0 > 0$, it follows that $u(t) > 0$ on Ω .

Next, for each $\tau > 0$, we let $v_\tau : [0, T_\tau) \rightarrow C_0(\mathbb{R}^N)$ be the maximal solution of the integral equation (1.1) with initial value f_τ . We observe that $T < T_\tau$ since v_τ can also be constructed using Theorem 2.3 and Proposition 3.3, with the same values of K , M and T as just above. Indeed, $|f_\tau(x)| \leq K \Phi_0(\tau, x)$, and so we set $\Phi_\tau(t) = \Phi_0(t + \tau)$ and use the space X_{T, Φ_τ} in Theorem 2.3. Since $\|\Phi_\tau(t)\|_\infty \leq \|\Phi_0(t)\|_\infty$, the same values of K , M and T satisfy (2.4) and (2.5) with $\Phi(t) = \Phi_\tau(t)$ as well as $\Phi(t) = \Phi_0(t)$. Also, we use $\varphi_\tau(t) = e^{t\Delta} f_\tau$, and so (on Ω) $|\varphi(t)| \leq K e^{t\Delta} \Phi_0(\tau) = K \Phi_0(t + \tau) = K \Phi_\tau(t)$. This means that $\|\varphi_\tau\|_{X_{T, \Phi_\tau}} \leq K$. We therefore have

$$(4.9) \quad v_\tau(t) = e^{t\Delta} f_\tau + \int_0^t e^{(t-\sigma)\Delta} (|v_\tau(\sigma)|^\alpha v_\tau(\sigma)) d\sigma$$

and that $\|v_\tau\|_{X_{T,\Phi_\tau}} \leq M$. In particular, $|v_\tau(t, x)| \leq M\Phi_0(t+\tau, x)$ for all $0 < t \leq T$ and $x \in \Omega$, and since v_τ satisfies (3.5), $|v_\tau(t, x)| \leq M|\Phi_0(t+\tau, x)|$ for all $0 < t \leq T$ and $x \in \mathbb{R}^N$. [For the reader's information, if we had not chosen $t_0 = 1$, then in many of these formulas it would have been necessary to replace τ by τt_0 .]

We wish to show that for each $0 < t \leq T$, $v_\tau(t) \rightarrow u(t)$ in $C_0(\mathbb{R}^N)$ as $\tau \rightarrow 0$. To accomplish this, we will show, with the help of the dominated convergence theorem, that, passing to a subsequence, the v_τ converge to a solution of (4.8) in X_{T,Φ_0} . By the uniqueness part of Theorem 2.3 that limit must be the solution u above. Also, by uniqueness, the limit exists as $\tau \rightarrow 0$, not just along subsequences.

We fix $0 < t \leq T$ and let $h_\tau : [0, t] \rightarrow C_0(\mathbb{R}^N)$ and $H_\tau : [0, t] \rightarrow C_0(\mathbb{R}^N)$ be given by

$$h_\tau(\sigma) = e^{(t-\sigma)\Delta}(|v_\tau(\sigma)|^\alpha v_\tau(\sigma))$$

and

$$H_\tau(\sigma) = M^{\alpha+1}|\Phi_0(t+\tau)|\|\Phi_0(\sigma+\tau)\|_\infty^\alpha.$$

It is clear that $H_\tau \rightarrow H$ in $L^1((0, t); C_0(\mathbb{R}^N))$ where

$$H(\sigma) = M^{\alpha+1}|\Phi_0(t)|\|\Phi_0(\sigma)\|_\infty^\alpha.$$

Moreover, it follows from the construction of the v_τ that

$$h_\tau(\sigma) \leq H_\tau(\sigma)$$

for all $0 \leq \sigma \leq t$. To see this, it suffices to show the inequality on Ω , in which case we have

$$\begin{aligned} h_\tau(\sigma) &\leq (e^{(t-\sigma)\Delta_\Omega}|v_\tau(\sigma)|)|v_\tau(\sigma)|^\alpha \\ &\leq M^{\alpha+1}(e^{(t-\sigma)\Delta_\Omega}\Phi_0(\sigma+\tau))\|\Phi_0(\sigma+\tau)\|_\infty^\alpha \\ &\leq M^{\alpha+1}\Phi_0(t+\tau)\|\Phi_0(\sigma+\tau)\|_\infty^\alpha \\ &= H_\tau(\sigma). \end{aligned}$$

Next we fix $0 < \sigma_0 < t$. Since $|v_\tau(\sigma_0)| \leq M|\Phi_0(\sigma_0+\tau)|$, it follows that for all $0 < \tau < 1$,

$$|v_\tau(\sigma_0)| \leq \Theta,$$

where

$$\Theta = M \sup_{0 \leq \tau \leq 1} |\Phi_0(\sigma_0+\tau)| \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N).$$

By parabolic regularity, applied to the $v_\tau(\sigma_0)$, $0 < \tau < 1$, as initial values of the nonlinear heat equation (1.4), it follows that the $\partial_t v_\tau(\sigma)$ and $\nabla v_\tau(\sigma)$ are uniformly bounded in $C_0(\mathbb{R}^N)$ on any interval $[\sigma_1, t]$ with $\sigma_0 < \sigma_1 < t$. By the Arzela-Ascoli theorem and a standard diagonalization procedure, there exists a sequence $\tau_n \rightarrow 0$ and a function $w \in C((0, t) \times \mathbb{R}^N)$ such that

$$\lim_{n \rightarrow \infty} v_{\tau_n}(\sigma, x) = w(\sigma, x),$$

where the limit is uniform for σ in a closed subinterval of $(0, t)$ and x in any bounded set of $\mathbb{R}^N$. Since $|v_\tau(\sigma, x)| \leq M|\Phi_0(\sigma+\tau, x)|$ for all $0 < \sigma < t$ and $x \in \mathbb{R}^N$, it follows that $|w(\sigma, x)| \leq M|\Phi_0(\sigma, x)|$ and that the above limit takes place in $C_0(\mathbb{R}^N)$ for each $0 < \sigma < t$.

If we denote $w(\sigma) = w(\sigma, \cdot)$, it is clear that

$$h_{\tau_n}(\sigma) \rightarrow e^{(t-\sigma)\Delta}(|w(\sigma)|^\alpha w(\sigma))$$

in $C((0, t); C_0(\mathbb{R}^N))$, as $n \rightarrow \infty$. By the dominated convergence theorem and formulas (4.6) and (4.9), it now follows that

$$(4.10) \quad w(t) = K_0 \Phi_0(t) + \int_0^t e^{(t-\sigma)\Delta} (|w(\sigma)|^\alpha w(\sigma)) d\sigma.$$

As an element of $C_0(\mathbb{R}^N)$, $w(t)$ satisfies condition (3.5), since it is the limit of functions satisfying (3.5), and so restricts to an element of $C_0(\Omega)$. Thus, we may consider (4.10) as an integral equation with values in $C_0(\Omega)$, $w \in X_{T, \Phi_0}$ and $\|w\|_{X_{T, \Phi_0}} \leq M$. By the uniqueness part of Theorem 2.3, it follows that $w = u$ as elements of $w \in X_{T, \Phi_0}$ and therefore as functions $(0, T] \rightarrow C_0(\mathbb{R}^N)$. It now follows by standard arguments that

$$v_\tau \rightarrow u$$

in $C((0, T); C_0(\mathbb{R}^N))$ as $\tau \rightarrow 0$.

Since $\alpha < 2/(N + m)$, and since $u(t) > 0$ on Ω , it follows from the Fujita-type results on Ω ([16, 13, 1, 15]) that the solution u , extended to its maximal solution $u : (0, T_{\max}) \rightarrow C_0(\mathbb{R}^N)$, blows up in finite time. We complete the proof, as done by Dickstein [4], and later Ghoul [7], using the fact that in the Sobolev subcritical case all blow up is type I (see Giga, Matsui, and Sasayama [9, 10]) and that the set of initial values giving rise to type I blow up is open in $C_0(\mathbb{R}^N)$ (see Fermanian Kammerer, Merle, and Zaag [5]). Using $v_\tau(t_0)$ and $u(t_0)$ as initial values, for some fixed $0 < t_0 < T$, we deduce that v_τ blows up in finite time for $\tau > 0$ sufficiently small. This completes the proof. $\square$

Remark 4.3. The condition $\alpha < 2/(N + m)$ in Theorem 1.2 is optimal. If $\alpha > 2/(N + m)$ and if f is anti-symmetric in $x_1, x_2, \dots, x_m$, satisfying (1.7) for some $t_0 > 0$, then the solution of the integral equation (1.1) with initial value $u_0 = \lambda f$ is global for $\lambda > 0$ sufficiently small. To see this we apply Theorem 2.3 with $\Phi = \Phi_0(t_0 + \cdot)$. Thus, (2.3) is verified with $A = \infty$ since $\alpha > 2/(N + m)$. Let $K > 0$ and $M > 0$ be such that (2.4)-(2.5) are satisfied with $T = \infty$. (See Remark 2.6.) Let $\varphi = e^{t\Delta} u_0$. Then $|\varphi(t)| \leq K\Phi(t)$, $\forall t \geq 0$ and $\lambda > 0$ sufficiently small. That is, $\varphi \in X_\infty$ and $\|\varphi\| \leq K$. Then by Theorem 2.3, the solution of (1.1) with initial data u_0 is global. Of course, the case $m = 0$ is just the $\alpha > 2/N$ part of Fujita's original result.

5. GLOBAL EXISTENCE: PROOF OF THEOREM 1.3

In this section, we continue with the notation that $e^{t\Delta}$ denotes the heat semigroup on $\mathbb{R}^N$, $e^{t\Delta} f = G_t \star f$, where G_t is the Gaussian function given by (1.3). Recall that if $u_0 \in C_0(\mathbb{R}^N)$, then there exists a unique continuous solution $u : [0, T_{\max}(u_0)) \rightarrow C_0(\mathbb{R}^N)$ of (1.1) with $u(0) = u_0$. This solution is classical for $t > 0$, i.e. $u \in C^{1,2}((0, T_{\max}) \times \mathbb{R}^N)$.

Fix $\rho > 0$, and let $u_0 \in C_0(\mathbb{R}^N)$ be such that $|u_0| \leq KG_\rho$ for some $K > 0$. Then the solution u of (1.1) can also be constructed, at least on some smaller time interval, using Theorem 2.3. For this, we let $\Phi(t) = e^{t\Delta} G_\rho = G_{t+\rho}$ and $\varphi(t) = e^{t\Delta} u_0$. We clearly have that $|\varphi(t)| \leq K\Phi(t)$, i.e. $\|\varphi\|_{X_T} \leq K$ for all $T > 0$. It follows that if $T > 0$ and $M > 0$ satisfy (2.4) and (2.5), then the solution of (1.1) satisfies $|u(t)| \leq MG_{t+\rho}$ for all $t \in [0, T]$. In this case, it turns out that the first and second order spatial derivatives of $u(t)$ are likewise bounded by Gaussian functions for $t > 0$.

Proposition 5.1. *Let $\alpha > 0$, $\rho > 0$, and let $u_0 \in C_0(\mathbb{R}^N)$ be such that $|u_0| \leq KG_\rho$ for some $K > 0$. Set $\Phi(t) = e^{t\Delta}G_\rho = G_{t+\rho}$, and suppose that $T > 0$ and $M > 0$ satisfy (2.4) and (2.5). Let $u \in X_T$ be the resulting solution of (1.1), i.e. (2.6) with $\varphi(t) = e^{t\Delta}u_0$. It follows that for any $\epsilon > 0$ there exists $C > 0$, depending only on the various parameters: K, M, T, α, ρ , as well as ϵ , such that*

$$(5.1) \quad |\partial_k u(t)| \leq \frac{C}{\sqrt{t}} G_{(1+\epsilon)(t+\rho)},$$

$$(5.2) \quad |\partial_{jk} u(t)| \leq \frac{C}{t} G_{(1+\epsilon)(t+\rho)},$$

for all $t \in (0, T]$ and all indices $1 \leq j, k \leq N$.

Proof. As noted just above the statement of the proposition, we have $|u(t)| \leq MG_{t+\rho}$ for all $t \in [0, T]$. Further, we note that for any index $1 \leq k \leq N$,

$$\partial_k G_t(x) = \frac{1}{2\sqrt{t}} \frac{1}{(4\pi t)^{N/2}} \frac{(-x_k)}{\sqrt{t}} e^{-\frac{|x|^2}{4t}}.$$

On the other hand, for any $\epsilon > 0$, there exists C_ϵ such that

$$|y_k| e^{-\frac{|y|^2}{4}} \leq C_\epsilon e^{-\frac{|y|^2}{4(1+\epsilon)}},$$

from which we see that

$$(5.3) \quad |\partial_k G_t| \leq \frac{C_\epsilon}{\sqrt{t}} G_{(1+\epsilon)t},$$

where the value of C_ϵ may vary from line to line. Also, we note that

$$(G_t)^\beta = \frac{1}{\beta^{N/2}(4\pi t)^{N(\beta-1)/2}} G_{t/\beta} = \beta^{-N/2} \|G_t\|_\infty^{\beta-1} G_{t/\beta}.$$

For any index $1 \leq k \leq N$, we have

$$\partial_k u(t) = (\partial_k G_t) \star u_0 + \int_0^t (\partial_k G_{t-\sigma}) \star (|u(\sigma)|^\alpha u(\sigma)) d\sigma.$$

Therefore, for any $0 < \beta < \alpha + 1$, we have

$$\begin{aligned} |\partial_k u(t)| &\leq \frac{C_\epsilon}{\sqrt{t}} G_{(1+\epsilon)t} \star KG_\rho \\ &\quad + M^{\alpha+1} C_\epsilon \int_0^t (t-\sigma)^{-1/2} G_{(1+\epsilon)(t-\sigma)} \star (G_{\sigma+\rho})^{\alpha+1} d\sigma \leq \frac{KC_\epsilon}{\sqrt{t}} G_{(1+\epsilon)t+\rho} \\ &\quad + M^{\alpha+1} C_\epsilon \int_0^t (t-\sigma)^{-1/2} G_{(1+\epsilon)(t-\sigma)} \star (G_{\sigma+\rho})^\beta \|G_{\sigma+\rho}\|_\infty^{\alpha+1-\beta} d\sigma \\ &\quad = \frac{KC_\epsilon}{\sqrt{t}} G_{(1+\epsilon)t+\rho} \\ &\quad + M^{\alpha+1} C_\epsilon \beta^{-N/2} \\ &\quad \quad \times \int_0^t (t-\sigma)^{-1/2} G_{(1+\epsilon)(t-\sigma)} \star (G_{(\sigma+\rho)/\beta}) \|G_{\sigma+\rho}\|_\infty^{\beta-1} \|G_{\sigma+\rho}\|_\infty^{\alpha+1-\beta} d\sigma. \end{aligned}$$

We now choose $\beta = 1/(1+\epsilon)$ so that

$$G_{(1+\epsilon)(t-\sigma)} \star G_{(\sigma+\rho)/\beta} = G_{(1+\epsilon)(t-\sigma)} \star G_{(1+\epsilon)(\sigma+\rho)} = G_{(1+\epsilon)(t+\rho)}.$$

Inserting this into the previous calculation, we obtain (adjusting the value of C_ϵ as necessary)

$$\begin{aligned} |\partial_k u(t)| &\leq \frac{KC_\epsilon}{\sqrt{t}} G_{(1+\epsilon)t+\rho} + M^{\alpha+1} C_\epsilon \|G_\rho\|_\infty^\alpha G_{(1+\epsilon)(t+\rho)} \int_0^t (t-\sigma)^{-1/2} d\sigma \\ &= \frac{KC_\epsilon}{\sqrt{t}} G_{(1+\epsilon)t+\rho} + \sqrt{t} M^{\alpha+1} C_\epsilon \|G_\rho\|_\infty^\alpha G_{(1+\epsilon)(t+\rho)}. \end{aligned}$$

This yields

$$(5.4) \quad |\partial_k u(t)| \leq \frac{C}{\sqrt{t}} G_{(1+\epsilon)(t+\rho)},$$

where the constant C now depends (only) on K, M, T, ρ and ϵ .

We now note that for any indices $1 \leq j, k \leq N$,

$$\partial_{jk} G_t(x) = \frac{1}{t} \left(-\frac{\varepsilon_{jk}}{2} + \frac{x_j x_k}{4t} \right) \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}},$$

where $\varepsilon_{kk} = 1$ and $\varepsilon_{jk} = 0$ if $j \neq k$. By similar calculations as above, we see that

$$(5.5) \quad |\partial_{jk} G_t| \leq \frac{C_\epsilon}{t} G_{(1+\epsilon)t}.$$

For any two indices $1 \leq j, k \leq N$, we write

$$\begin{aligned} \partial_{jk} u(t) &= (\partial_{jk} G_t) \star u_0 + \int_0^t (\partial_{jk} G_{t-\sigma}) \star \partial_k (|u(\sigma)|^\alpha u(\sigma)) d\sigma \\ &= (\partial_{jk} G_t) \star u_0 + (\alpha + 1) \int_0^t (\partial_{jk} G_{t-\sigma}) \star ((\partial_k u(\sigma)) |u(\sigma)|^\alpha) d\sigma. \end{aligned}$$

Using the estimates already obtained and denoting all the constants by the same letter C , we see that

$$\begin{aligned} |\partial_{jk} u(t)| &\leq \frac{C}{t} G_{(1+\epsilon)t} \star G_\rho + C \int_0^t \frac{1}{\sqrt{t-\sigma}} \frac{1}{\sqrt{\sigma}} G_{(1+\epsilon)(t-\sigma)} \star G_{(1+\epsilon)(\sigma+\rho)} d\sigma \\ &\leq \frac{C}{t} G_{(1+\epsilon)(t+\rho)}. \end{aligned}$$

This concludes the proof. $\square$

Lemma 5.2. Assume that $f \in C^m(\mathbb{R}^N)$ and f is anti-symmetric with respect to the variables $x_1, \dots, x_m$; that is, f verifies (3.5). Then there exists $g \in C(\mathbb{R}^N)$ symmetric with respect to the variables $x_1, \dots, x_m$ such that

$$(5.6) \quad f(x) = x_1 x_2 \cdots x_m g(x), \quad \forall x \in \mathbb{R}^N.$$

Furthermore, if f and all its derivatives up through order m are bounded by Gaussian functions, then the function g is also bounded by a Gaussian function.

Proof. We give the proof only in the case $m = 2$, which is the case we need. The proof of the general case should be clear from this special case.

We define the function g by

$$g(x) = \begin{cases} \frac{1}{x_1 x_2} \int_0^{x_2} \int_0^{x_1} \partial_1 \partial_2 f(t_1, t_2, x_3, \dots, x_N) dt_1 dt_2, & x_1 \neq 0, x_2 \neq 0, \\ \frac{1}{x_1} \int_0^{x_1} \partial_1 \partial_2 f(t_1, 0, x_3, \dots, x_N) dt_1, & x_1 \neq 0, x_2 = 0, \\ \frac{1}{x_2} \int_0^{x_2} \partial_1 \partial_2 f(0, t_2, x_3, \dots, x_N) dt_2, & x_1 = 0, x_2 \neq 0, \\ \partial_1 \partial_2 f(0, 0, x_3, \dots, x_N), & x_1 = 0, x_2 = 0. \end{cases}$$

It is clear that $g \in C(\mathbb{R}^N)$, symmetric in x_1 and x_2 , and that $f(x) = x_1 x_2 g(x)$.

If we now suppose that f and all its derivatives up through order m are bounded by Gaussian functions in $\mathbb{R}^N$, then (also using the symmetry properties of f and its derivatives) it follows that g can also be written as

$$g(x) = \begin{cases} \frac{1}{x_1 x_2} \int_{x_2}^{\infty} \int_{x_1}^{\infty} \partial_1 \partial_2 f(t_1, t_2, x_3, \dots, x_N) dt_1 dt_2, & x_1 \neq 0, x_2 \neq 0, \\ -\frac{1}{x_1} \int_{x_1}^{\infty} \partial_1 \partial_2 f(t_1, 0, x_3, \dots, x_N) dt_1, & x_1 \neq 0, x_2 = 0, \\ -\frac{1}{x_2} \int_{x_2}^{\infty} \partial_1 \partial_2 f(0, t_2, x_3, \dots, x_N) dt_2, & x_1 = 0, x_2 \neq 0, \\ \partial_1 \partial_2 f(0, 0, x_3, \dots, x_N), & x_1 = 0, x_2 = 0. \end{cases}$$

In addition, note that

$$\int_{-\infty}^{\infty} \partial_1 \partial_2 f(t_1, t_2, x_3, \dots, x_N) dt_1 = \int_{-\infty}^{\infty} \partial_1 \partial_2 f(t_1, t_2, x_3, \dots, x_N) dt_2 = 0,$$

since $\partial_1 f$ and $\partial_2 f$ are bounded by Gaussian functions. It now follows easily that g is bounded by a Gaussian function. $\square$

Proof of Theorem 1.3. Let $\theta(x) = e^{|x|^2/4}$. By Theorem 1 in [8], there exists a function $f \in C_0(\mathbb{R}^N) \cap L^2(\theta dx) \cap H^1(\mathbb{R}^N, dx)$, anti-symmetric in x_1 and x_2 , verifying (1.8) with $m = 2$, such that the maximal solution $u : [0, T_{\max}) \rightarrow C_0(\mathbb{R}^N)$ of the integral equation (1.1) with initial value $u_0 = f$ is global in time. It is clear that for all $t > 0$, the solution $u(t, \cdot)$ remains anti-symmetric in x_1 and x_2 . (This was shown in [8].) Also, since $\alpha < 4/(N - 2)$ (the Sobolev critical index), (1.1) is well posed in $H^1(\mathbb{R}^N, dx)$, and so the solution remains in that space for all $t > 0$.

There is no guarantee, however, that f is bounded by a multiple of $|\partial_1 \partial_2 G_{t_0}|$ for some $t_0 > 0$. We will show this to be true if f is replaced by $u(t)$ with $t > 0$. If $t > 0$ is small enough, then condition (1.8) is also preserved.

To carry out this idea, we first look at the solution u transformed using self-similar variables. This is a standard technique and was used in particular in the papers [3, 8]. Hence our discussion will be brief. We set

$$v(s, y) = (1 + t)^{1/\alpha} u(t, x), \text{ where } s = \log(1 + t) \text{ and } y = x/\sqrt{1 + t}.$$

It follows that $v(0) = u(0) = f$ and that v satisfies the equation

$$(5.7) \quad v_s = -Lv + \frac{1}{\alpha} v + |v|^\alpha v.$$

The operator L is given by

$$Lv = -\Delta v - \frac{1}{2} y \cdot \nabla v$$

and is a positive self-adjoint operator on $L^2(\theta dx)$, with domain $H^2(\theta dx)$. The semigroup e^{-sL} is a contraction on all $L^q(\theta dx)$ (see [13, Corollary 2.2, p. 430]), and so by the Stein interpolation theorem (see for example Chapter III in [19]) the semigroup e^{-sL} is an analytic semigroup on each $L^q(\theta dx)$, $1 < q < \infty$. The domain of the operator L in $L^q(\theta dx)$ is $H^{2,q}(\theta dx)$. See [11, Theorem 2, p. 890].

As remarked in [8], using the abstract theory of [20], if $\alpha < 2/N$, the integral equation corresponding to (5.7) is locally well posed in all the spaces $L^q(\theta dx)$, $q > 1 + 2/N$. Since $f \in C_0(\mathbb{R}^N) \cap L^2(\theta dx)$, it follows that $f \in L^q(\theta dx)$ for all $2 \leq q < \infty$. For each such q , i.e. $q > 1 + 2/N$ and $2 \leq q < \infty$, the solution $v(s)$ is in the domain of L on $L^q(\theta dx)$, i.e. $H^{2,q}(\theta dx)$, for all $s > 0$. To see this, we first note that by Proposition 1.2 in [20], $v(s)$ is Hölder continuous into s in $L^q(\theta dx)$ on any interval of the form $[\epsilon, S]$. The same is therefore true for the function $\frac{1}{\alpha}v(s) + |v(s)|^\alpha v(s)$. That $v(s)$ is in the domain of the generator of the semigroup now follows from standard semigroup theory. See, for example, [12, Theorem 1.27, p. 491].

In other words, $v(s)$ and its spatial derivatives up through order 2 are all in $L^q(\theta dx)$, for $s > 0$. This implies that, for some sufficiently small $\gamma > 0$, perhaps depending on q , the function $e^{\gamma|\cdot|^2}v(s)$ is in $H^{2,q}(\mathbb{R}^N, dx)$ (i.e. with Lebesgue measure). Choosing $q < \infty$ large enough so that by the Sobolev embeddings we have that $H^{2,q}(\mathbb{R}^N, dx)$ is embedded in $C_0(\mathbb{R}^N)$, we see that, for any fixed $s > 0$, the function $e^{\gamma|\cdot|^2}v(s)$ is in $C_0(\mathbb{R}^N)$, and so $v(s)$ is bounded by some Gaussian function, $|v(s, y)| \leq Ce^{-\gamma|y|^2}$.

We return to the solution $u(t)$ in the original variables. It follows from the above considerations that for each $t > 0$, $u(t)$ is bounded by some Gaussian function. Next we apply Proposition 5.1, where we consider $u(t_0)$ as the initial value, for arbitrary $t_0 > 0$. It therefore follows that, for any fixed $t > 0$, $u(t)$ and all its partial derivatives up through order 2 are bounded by Gaussian functions. We now replace f by an appropriate $u(t)$, for $t > 0$ small. By the well posedness of the equation (5.7) in the $L^q(\theta dx)$ spaces, it follows that condition (1.8) with $m = 2$ is still true with this new f , as long as $t > 0$ is small enough.

Finally, Lemma 5.2 implies that $f(x) = x_1 x_2 g(x)$ for some continuous function g which is bounded by a Gaussian function. This implies that f is bounded by a multiple of $|\partial_1 \partial_2 G_{t_0}|$ for some $t_0 > 0$, and completes the proof of the theorem. $\square$

6. APPENDIX: SYMMETRY DECOMPOSITION OF FUNCTIONS

The purpose of this appendix is to give a decomposition of an arbitrary function of N variables into the sum of functions, each of which is either symmetric or anti-symmetric with respect to each of the variables. This extends the notion that a function of one variable can be expressed as the sum of an even and an odd function. While it is possible that the decomposition given here is known, the version we present is due to Robert Weissler [24].

For $1 \leq j \leq N$, let T_j be the operator on functions of N variables defined by

$$[T_j f](x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_N) = f(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_N).$$

If $U = \{j_1, \dots, j_k\} \subset \{1, 2, 3, \dots, N\}$, we let

$$T_U f = T_{j_1} \dots T_{j_k} f.$$

Of course, if U is the empty set, then $T_U f = f$. For notational simplicity, we set $S = \{1, 2, 3, \dots, N\}$.

Claim 1.

$$f = \frac{1}{2^N} \sum_{V \subset S} \sum_{U \subset S} (-1)^{|V \cap U|} T_U f,$$

where $|V \cap U|$ denotes the number of elements in $V \cap U$.

Proof. First consider the contribution to the sum where U is the empty set. Since in this case $T_U f = f$, and $|V \cap U| = 0$ for $V \subset S$, this contribution is precisely

$$\frac{1}{2^N} \sum_{V \subset S} f.$$

This contribution is clearly equal to f since there are 2^N subsets of S .

Thus, we need to show that all the other contributions add up to 0. Fix any nonempty subset $U \subset S$. The resulting contribution to the overall sum is

$$\frac{1}{2^N} \left(\sum_{V \subset S} (-1)^{|V \cap U|} \right) T_U f.$$

In fact, $\sum_{V \subset S} (-1)^{|V \cap U|} = 0$. To see this, fix some element $k \in U$, which is possible since U is nonempty. For a subset $V \subset S$ which does not contain the element k , let $\bar{V} = V \cup \{k\}$. In particular, if $k \notin V$, then (since $k \in U$) the sets $V \cap U$ and $\bar{V} \cap U$ differ by one element. It follows that

$$\sum_{V \subset S} (-1)^{|V \cap U|} = \sum_{V \subset S, k \notin V} \left[(-1)^{|V \cap U|} + (-1)^{|\bar{V} \cap U|} \right] = 0.$$

This shows that all the other contributions add up to 0 and proves the claim.

Now fix a subset $V \subset S$ and set

$$f_V = \sum_{U \subset S} (-1)^{|V \cap U|} T_U f$$

so that

$$(6.1) \quad f = \frac{1}{2^N} \sum_{V \subset S} f_V.$$

Claim 2. f_V is symmetric in x_j , i.e. $T_j f_V = f_V$, if and only if $j \notin V$. If $j \in V$, then $T_j f_V = -f_V$. In particular, if $V = S$, then f_V is anti-symmetric in all the variables, and if V is the empty set, then f_V is symmetric in all the variables.

Proof. Fix $j \in S$. We define the following map $\mathcal{R}$ between subsets of S :

$$\mathcal{R}(U) = U \cup \{j\}, \quad j \notin U,$$

$$\mathcal{R}(U) = U \setminus \{j\}, \quad j \in U.$$

$\mathcal{R}$ is clearly a bijection on the power set of S . Moreover, $T_j T_U = T_{\mathcal{R}(U)}$ for all subsets U of S . It follows that

$$T_j f_V = T_j \sum_{U \subset S} (-1)^{|V \cap U|} T_U f = \sum_{U \subset S} (-1)^{|V \cap U|} T_{\mathcal{R}(U)} f.$$

Now if $j \notin V$, then $V \cap U = V \cap \mathcal{R}(U)$, which proves that $T_j f_V = f_V$. On the other hand, if $j \in V$, then $V \cap U$ and $V \cap \mathcal{R}(U)$ differ by one element. This establishes the claim.

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