PIERI OPERATORS ON THE AFFINE NILCOXETER ALGEBRA

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ABSTRACT. We study a family of operators on the affine nilCoxeter algebra. We use these operators to prove conjectures of Lam, Lapointe, Morse, and Shimozono regarding strong Schur functions.

1. INTRODUCTION

The k -Schur functions of Lapointe, Lascoux and Morse [LLM03] first arose in the study of Macdonald polynomials. Since then, their study has flourished; see for instance [LM03, LM05, LM07, LS07, LLMS10, Lam10] and the references therein. This is due, in part, to an important geometric interpretation of the Hopf algebra $\Lambda_{(k)}$ of k -Schur functions and its dual Hopf algebra $\Lambda^{(k)}$: these algebras are isomorphic to the homology and cohomology of the affine Grassmannian in type A [Lam08]. Under this isomorphism, the k -Schur functions map to the Schubert basis of the homology and the dual k -Schur functions (also called the affine Schur functions) map to the Schubert basis of the cohomology.

An important problem in the theory of k -Schur functions is to find a k -Littlewood–Richardson rule, namely, a combinatorial interpretation for the (nonnegative) coefficients in the expansion

$$(1) \quad s_{\mu}^{(k)} s_{\nu}^{(k)} = \sum_{\lambda} c_{\mu, \nu}^{\lambda, (k)} s_{\lambda}^{(k)}.$$

The $c_{\mu, \nu}^{\lambda, (k)}$ are called the k -Littlewood–Richardson coefficients, and are of high relevance in combinatorics and geometry. It was proved by Lapointe and Morse [LM08] that special cases of these coefficients yield the 3-point Gromov–Witten invariants. These invariants are the structure constants of the quantum cohomology of the Grassmannian; they count the number of rational curves of a fixed degree in the Grassmannian. The 3-point Gromov–Witten invariants of flag varieties are also k -Littlewood–Richardson coefficients; see [LS10, LL12].

As an approach to finding the k -Littlewood–Richardson coefficients, Lam [Lam06] identified $\Lambda_{(k)}$ with the affine Fomin–Stanley subalgebra $\mathbb{B}$ of the affine nilCoxeter algebra $\mathbb{A}$ of the affine symmetric group W . Specifically, he constructed a family of elements $\mathfrak{s}_{\lambda}^{(k)} \in \mathbb{B}$ that map under this isomorphism to the k -Schur functions $s_{\lambda}^{(k)}$. Furthermore, he proved [Lam06, Proposition 42] that finding the k -Littlewood–Richardson rule is equivalent to finding the expansion of $\mathfrak{s}_{\lambda}^{(k)}$ in the “standard basis” $\mathbf{u}_w$ of $\mathbb{A}$. Explicitly, he proved that the coefficients in (1) appear

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as coefficients in the expansion

$$(2) \quad \mathfrak{s}_\lambda^{(k)} = \sum_{w \in W} d_\lambda^w \mathbf{u}_w.$$

In this article, we develop a family of operators on $\mathbb{A}$, which will facilitate the study of $\mathfrak{s}_\lambda^{(k)}$, and we prove certain conjectures regarding a family of functions that generalize the k -Schur functions $s_\lambda^{(k)}$. Each of these is described in more detail below.

1.1. The Pieri operators. Lam, Lapointe, Morse, and Shimozono [LLMS10] constructed a labelled directed graph $\mathcal{G}^\downarrow$ on the elements of W , which encompasses the strong order in W . In this article we study the operators on $\mathbb{A}$ induced by the Pieri operators of $\mathcal{G}^\downarrow$ in the spirit of [BMSvW00]. In Section 4, we develop the main properties of these operators. More specifically, in Theorem 4.9 we prove that these operators are determined by their restriction to $\mathbb{B}$, in Theorem 4.7 we determine this restriction, and in Theorem 4.10 we prove that the operators commute pairwise.

1.2. Properties of strong Schur functions. Lam, Lapointe, Morse, and Shimozono [LLMS10] generalized $s_\lambda^{(k)}$ to a larger set of functions called the strong Schur functions $\text{Strong}_{u/v}$, where u and v are any pair of elements in W . In Section 5, we use the Pieri operators to prove a series of conjectures of Lam, Lapointe, Morse, and Shimozono [LLMS10, Conjecture 4.18] regarding these functions. Specifically,

- (a) in Theorem 5.2 we prove that $\text{Strong}_{u/v}$ are symmetric functions;
- (b) in Theorem 5.4 we prove that they belong to the algebra $\Lambda_{(k)}$; and
- (c) in Theorem 5.5 we describe the coefficient of $s_\lambda^{(k)}$ in $\text{Strong}_{u/v}$, when u and v are 0-Grassmannian elements, in terms of the structure constants of the cohomology ring of the affine flag variety.

Note that (c) provides a combinatorial description of the *skew* k -Schur functions.

2. BACKGROUND AND NOTATION

2.1. Affine symmetric group. Fix a positive integer k . Let W denote the affine symmetric group with simple generators $s_0, s_1, \dots, s_k$. There is an interpretation of W as the group of permutations $w : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying $w(i+k+1) = w(i) + k+1$ for all $i \in \mathbb{Z}$ and $\sum_{i=1}^{k+1} w(i) = \sum_{i=1}^{k+1} i$. For two integers i and j that are not congruent modulo $k+1$, let $t_{i,j}$ be the element of W that interchanges i and j and fixes all integers not congruent to i or j modulo $k+1$.

Let W_0 denote the subgroup of W generated by $s_1, \dots, s_k$ and let W^0 denote the set of minimal length coset representatives of W/W_0 . Elements of W^0 are called *affine Grassmannian elements* or *0-Grassmannian elements*. There are bijections between 0-Grassmannian elements, k -bounded partitions, and $(k+1)$ -cores. We will not review these here, but refer the reader to [LM05]. For a k -bounded partition λ , we let w_λ denote the corresponding element of W^0 . Let $\mathcal{B}^{(k)}$ denote the set of k -bounded partitions.

2.2. Affine nilCoxeter algebra. Let $\mathbb{A}$ denote the *affine nilCoxeter algebra* of W . This is the algebra generated by $\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_k$ with relations

$$\begin{aligned}\mathbf{u}_i^2 &= 0 \text{ for all } i, \\ \mathbf{u}_i \mathbf{u}_{i+1} \mathbf{u}_i &= \mathbf{u}_{i+1} \mathbf{u}_i \mathbf{u}_{i+1} \text{ with } i+1 \text{ taken modulo } k+1, \\ \mathbf{u}_i \mathbf{u}_j &= \mathbf{u}_j \mathbf{u}_i \text{ if } i-j \neq \pm 1 \text{ modulo } k+1.\end{aligned}$$

It follows that a basis of $\mathbb{A}$ is given by the elements $\mathbf{u}_w = \mathbf{u}_{s_{i_1}} \mathbf{u}_{s_{i_2}} \cdots \mathbf{u}_{s_{i_l}}$, where $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ is a reduced word for $w \in W$. We define an inner product on $\mathbb{A}$ by $\langle \mathbf{u}_v, \mathbf{u}_w \rangle_{\mathbb{A}} = \delta_{u,v}$.

2.3. Affine Fomin–Stanley subalgebra. An element $w \in W$ is said to be *cyclically decreasing* if there exists a reduced factorization $s_{i_1} \cdots s_{i_j}$ of w satisfying: each letter occurs at most once; and, for all m , if s_m and s_{m+1} both appear in the reduced factorization, then s_{m+1} precedes s_m . If $D \subsetneq \{0, 1, \dots, k\}$, then there is a unique cyclically decreasing element w_D with letters $\{s_d : d \in D\}$. Let $\mathbf{u}_D = \mathbf{u}_{w_D}$ denote the corresponding basis element of $\mathbb{A}$. For $i \in \{0, 1, \dots, k\}$, let

$$\mathbf{h}_i = \sum_{\substack{D \subset I \\ |D|=i}} \mathbf{u}_D \in \mathbb{A}.$$

By a result of Thomas Lam [Lam06], the elements $\{\mathbf{h}_i\}_{i \leq k}$ commute and freely generate a subalgebra $\mathbb{B}$ of $\mathbb{A}$ called the *affine Fomin–Stanley subalgebra*. The elements $\mathbf{h}_\lambda = \mathbf{h}_{\lambda_1} \cdots \mathbf{h}_{\lambda_t}$, for all k -bounded partitions $\lambda = (\lambda_1, \dots, \lambda_t)$, form a basis of $\mathbb{B}$.

2.4. Symmetric functions. Let Λ denote the ring of symmetric functions. For a partition λ , we let m_λ , h_λ , e_λ , p_λ , s_λ denote the monomial, homogeneous, elementary, power sum and Schur symmetric functions, respectively, indexed by λ . Each of these families forms a basis of Λ . We recall the following change of bases formulae:

$$h_\mu = \sum_{\lambda} K_{\lambda, \mu} s_\lambda \quad \text{and} \quad s_\lambda = \sum_{\mu} K_{\lambda, \mu} m_\mu,$$

where $K_{\lambda, \mu}$, called the *Kostka number*, is the number of semistandard tableaux of shape λ and content μ [Sta].

Let $\Lambda_{(k)}$ denote the subalgebra of Λ generated by $h_0, h_1, \dots, h_k$. The elements h_λ with $\lambda_1 \leq k$ form a basis of $\Lambda_{(k)}$. Let $\Lambda^{(k)} = \Lambda / I_k$ denote the quotient of Λ by the ideal I_k generated by m_λ with $\lambda_1 > k$. The equivalence classes in $\Lambda^{(k)}$ of the elements m_λ with $\lambda_1 \leq k$ form a basis of $\Lambda^{(k)}$.

The *Hall inner product* of symmetric functions is defined by

$$\langle h_\lambda, m_\mu \rangle_{\Lambda} = \langle s_\lambda, s_\mu \rangle_{\Lambda} = \delta_{\lambda, \mu}.$$

Observe that every element of the ideal I_k is orthogonal to every element of $\Lambda_{(k)}$ with respect to this inner product. Hence, it induces a pairing $\langle \cdot, \cdot \rangle$ between $\Lambda_{(k)}$ and $\Lambda^{(k)}$. In particular, $\langle f, g \rangle = \langle f, \tilde{g} \rangle$ for $f \in \Lambda_{(k)}$, $g \in \Lambda^{(k)}$ and any preimage $\tilde{g}$ of g under the quotient map $\Lambda \rightarrow \Lambda^{(k)}$. For an element f in $\Lambda^{(k)}$, write $f^\perp : \Lambda_{(k)} \rightarrow \Lambda_{(k)}$ for the linear operator that is adjoint to multiplication by f with respect to $\langle \cdot, \cdot \rangle$.

2.5. Affine Schur functions. The affine Schur functions form a distinguished basis of $\Lambda^{(k)}$. For $w \in W$, the *affine Stanley symmetric function* is defined as

$$(3) \quad \tilde{F}_w = \sum_{\lambda \in \mathcal{B}^{(k)}} \langle \mathbf{h}_\lambda, \mathbf{u}_w \rangle_{\mathbb{A}} m_\lambda,$$

where m_λ is the monomial symmetric function indexed by λ . These functions are elements of $\Lambda^{(k)}$, but they are not linearly independent. For a k -bounded partition λ , let $\tilde{F}_\lambda = \tilde{F}_{w_\lambda}$, where w_λ denotes the 0-Grassmannian element corresponding to λ . The functions $\tilde{F}_\lambda$ are called *affine Schur functions* (or *dual k -Schur functions*), and they form a basis of $\Lambda^{(k)}$. See for instance [Lam06, LM08].

2.6. k -Schur functions. The k -Schur functions are a distinguished basis of $\Lambda_{(k)}$. They are defined as the duals of the affine Schur functions with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\Lambda_{(k)} \times \Lambda^{(k)}$. That is, they satisfy $\langle s_\lambda^{(k)}, \tilde{F}_\mu \rangle = \delta_{\lambda, \mu}$ for all k -bounded partitions λ and μ . Equivalently, they are uniquely defined by the *k -Pieri rule*:

$$h_i s_\lambda^{(k)} = \sum s_\nu^{(k)},$$

where the sum ranges over all k -bounded partitions ν such that $w_\nu w_\lambda^{-1}$ is cyclically decreasing of length i . It follows from duality and (3) that

$$h_\mu = \sum_{\lambda \in \mathcal{B}^{(k)}} \langle \mathbf{h}_\lambda, \mathbf{u}_{w_\mu} \rangle_{\mathbb{A}} s_\lambda^{(k)}.$$

2.7. Noncommutative k -Schur functions. The algebras $\Lambda_{(k)}$ and $\mathbb{B}$ are isomorphic with isomorphism given by $h_\lambda \mapsto \mathbf{h}_\lambda$. We denote by $\mathfrak{s}_\lambda^{(k)}$ the image of the k -Schur function $s_\lambda^{(k)}$ under this isomorphism. In the literature, $\mathfrak{s}_\lambda^{(k)}$ is called a *noncommutative k -Schur function*. They have the following expansion [Lam06, Proposition 42]:

$$(4) \quad \mathfrak{s}_\lambda^{(k)} = \sum_{w \in W} \langle s_\lambda^{(k)}, \tilde{F}_w \rangle \mathbf{u}_w.$$

That is, the coefficient of $\mathbf{u}_w$ in $\mathfrak{s}_\lambda^{(k)}$ is equal to the coefficient of $\tilde{F}_\lambda$ in $\tilde{F}_w$:

$$(5) \quad \langle s_\lambda^{(k)}, \tilde{F}_w \rangle = \langle \mathfrak{s}_\lambda^{(k)}, \mathbf{u}_w \rangle_{\mathbb{A}},$$

and so

$$\tilde{F}_w = \sum_{\lambda \in \mathcal{B}^{(k)}} \langle \mathfrak{s}_\lambda^{(k)}, \mathbf{u}_w \rangle_{\mathbb{A}} \tilde{F}_\lambda.$$

Consequently, $\mathfrak{s}_\lambda^{(k)}$ contains exactly one term $\mathbf{u}_w$ with $w \in W^0$ and its coefficient is 1. Furthermore, if $\sum_w c_w \mathbf{u}_w$ is known to lie in $\mathbb{B}$, then $\sum_w c_w \mathbf{u}_w = \sum_\lambda c_{w_\lambda} \mathfrak{s}_\lambda^{(k)}$.

3. DEFINITION OF THE OPERATORS

In this section, we define operators on the affine nilCoxeter algebra $\mathbb{A}$. The definitions are dependent upon the combinatorics introduced by Lam, Lapointe, Morse, and Shimozono in [LLMS10].

FIGURE 1. A subgraph of $\mathcal{G}^\uparrow$ for $k = 2$; cf. Figure 2.

3.1. Up operators. Define an edge-labelled oriented graph $\mathcal{G}^\uparrow$ with vertex set W : there is an edge from v to w labelled by i whenever $\ell(w) = \ell(v) + 1$ and $s_i v = w$. (See Figure 1.) So, $\mathcal{G}^\uparrow$ is the graph for the (*left*) *weak order* on W .

A *weak strip* of length j from w to v , denoted by $w \rightsquigarrow v$, is a pair of elements $w, v \in W$ such that w precedes v in weak order and vw^{-1} is a cyclically decreasing word of length j .

For any nonnegative integer j , define a linear operator $U_j : \mathbb{A} \rightarrow \mathbb{A}$ by

$$U_j(\mathbf{u}_w) = \sum_{\substack{w \rightsquigarrow v \\ \text{size}(w \rightsquigarrow v) = j}} \mathbf{u}_v = \mathbf{h}_j \mathbf{u}_w,$$

where the sum ranges over all weak strips of length j that begin at w . Equivalently, U_j is multiplication on the left by $\mathbf{h}_j$.

Example 3.1. With $k = 2$: $U_1(\mathbf{u}_0) = \mathbf{u}_2 \mathbf{u}_0 + \mathbf{u}_1 \mathbf{u}_0$ and $U_2(\mathbf{u}_0) = \mathbf{u}_0 \mathbf{u}_2 \mathbf{u}_0 + \mathbf{u}_2 \mathbf{u}_1 \mathbf{u}_0$.

3.2. Down operators. Define a second edge-labelled oriented graph $\mathcal{G}^\downarrow$, the *marked strong order* graph, with vertex set W : there is an edge from x to y labelled by $y(j) = x(i)$ whenever $\ell(x) = \ell(y) + 1$ and there exists $i \leq 0 < j$ such that $y t_{i,j} = x$.

Example 3.2 ($k = 2$). There are two edges from $x = s_0 s_1 s_2 s_0$ to $y = s_1 s_2 s_0$ since $y^{-1} x$ can be written as $t_{i,j}$ with $i \leq 0 < j$ in two ways: $y^{-1} x = t_{-4,1} = t_{-1,4}$. These edges are labelled by $y(1) = -2$ and $y(4) = 1$. See Figure 2.

Remark 3.3. The authors in [LLMS10] defined a similar graph except that they oriented their edges in the opposite direction and labelled the edges by the pair (i, j) : they wrote $y \xrightarrow{i,j} x$ whereas we write $x \xrightarrow{y(j)} y$; and they called our label $y(j)$ the *marking* of the edge.

A *strong strip* of length i from w to v , denoted by $w \rightarrow v$, is a path

$$w \xrightarrow{\ell_1} w_1 \xrightarrow{\ell_2} \cdots \xrightarrow{\ell_i} w_i = v$$

FIGURE 2. $\mathcal{G}^\downarrow$ for $k = 2$ truncated at the affine Grassmannian elements of length 4.

of length i in $\mathcal{G}^\downarrow$ with decreasing edge labels: $\ell_1 > \dots > \ell_i$. For nonnegative integers i , define $D_i : \mathbb{A} \rightarrow \mathbb{A}$ as

$$D_i(\mathbf{u}_w) = \sum_{\substack{w \rightarrow v \\ \text{size}(w \rightarrow v) = i}} \mathbf{u}_v,$$

where the sum ranges over all strong strips of length i that begin at w . In particular, the coefficient of $\mathbf{u}_v$ in $D_i(\mathbf{u}_w)$ is the number of strong strips of length i that begin at w and end at v .

Example 3.4. With $k = 2$, using the graph from Figure 2, one can verify that

$$D_1(\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_0) = 2\mathbf{u}_0\mathbf{u}_2\mathbf{u}_0 + \mathbf{u}_0\mathbf{u}_1\mathbf{u}_2 + 2\mathbf{u}_1\mathbf{u}_2\mathbf{u}_0 + \mathbf{u}_0\mathbf{u}_1\mathbf{u}_0,$$

$$D_2(\mathbf{u}_0\mathbf{u}_1\mathbf{u}_2\mathbf{u}_0) = \mathbf{u}_0\mathbf{u}_2 + \mathbf{u}_1\mathbf{u}_2 + \mathbf{u}_2\mathbf{u}_0 + \mathbf{u}_1\mathbf{u}_0.$$

More generally, we define an operator D_J for any composition J of positive integers; the operator D_i defined above is D_J for the composition $J = [i]$. We need some additional notation. The *ascent composition* of a sequence $\ell_1, \ell_2, \dots, \ell_m$ is the composition $[i_1, i_2 - i_1, \dots, i_j - i_{j-1}, m - i_j]$, where $i_1 < i_2 < \dots < i_j$ are the ascents of the sequence; that is, the elements in $\{1, \dots, m-1\}$ such that $\ell_{i_a} < \ell_{i_a+1}$. For example, the ascent composition of the sequence 3, 2, 0, 3, 4, 1 is $[3, 1, 2]$ since the ascents are in positions 3 and 4.

If $w_0 \xrightarrow{\ell_1} \dots \xrightarrow{\ell_m} w_m$ is a path in $\mathcal{G}^\downarrow$, then we let $\text{ascomp}(w_0 \xrightarrow{\ell_1} \dots \xrightarrow{\ell_m} w_m)$ denote the ascent composition of the sequence of labels $\ell_1, \dots, \ell_m$. It is a composition of the length of the path.

For a composition $J = [j_1, j_2, \dots, j_l]$ of positive integers, define

$$D_J(\mathbf{u}_w) = \sum_{\text{ascomp}\left(w \xrightarrow{\ell_1} w_1 \xrightarrow{\ell_2} \dots \xrightarrow{\ell_m} w_m\right) = J} \mathbf{u}_{w_m},$$

where the sum ranges over all paths in $\mathcal{G}^\downarrow$ of length $m = j_1 + \dots + j_l$ beginning at w whose sequence of labels has ascent composition J .

Example 3.5. With $k = 2$ one can verify using Figure 2 that

$$\begin{aligned} D_{[3]}(\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_1 \mathbf{u}_0) &= \mathbf{u}_2 + \mathbf{u}_0, \\ D_{[2,1]}(\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_1 \mathbf{u}_0) &= \mathbf{u}_2 + 2\mathbf{u}_0 + \mathbf{u}_1, \\ D_{[1,2]}(\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_1 \mathbf{u}_0) &= \mathbf{u}_2 + 2\mathbf{u}_0 + \mathbf{u}_1, \\ D_{[1,1,1]}(\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_1 \mathbf{u}_0) &= \mathbf{u}_0 + \mathbf{u}_1. \end{aligned}$$

For two compositions $I = [i_1, \dots, i_r]$ and $J = [j_1, \dots, j_s]$, let

$$\begin{aligned} I \boxplus J &= [i_1, \dots, i_{r-1}, i_r + j_1, j_2, \dots, j_s], \\ I \boxdot J &= [i_1, \dots, i_r, j_1, \dots, j_s]. \end{aligned}$$

Proposition 3.6. *If I and J are compositions, then*

$$D_I \circ D_J = D_{I \boxplus J} + D_{I \boxdot J}.$$

Proof. If $w \rightarrow \dots \rightarrow v$ and $v \rightarrow \dots \rightarrow u$ are two paths in $\mathcal{G}^\downarrow$ with ascent compositions J and I , respectively, then the path $w \rightarrow \dots \rightarrow v \rightarrow \dots \rightarrow u$ has ascent composition either $I \boxdot J$ or $I \boxplus J$. $\square$

Corollary 3.7. *Suppose $I = [i_1, \dots, i_r]$ is a composition. Then*

$$\sum_{J \preceq I} D_J = D_{i_1} \circ \dots \circ D_{i_r},$$

where $\preceq$ denotes reverse refinement order on compositions.¹

Proof. Proceed by induction on r . This is trivially true for $r = 1$. Suppose the result holds for compositions of length less than r . Then

$$D_{i_1} \circ \dots \circ D_{i_r} = D_{i_1} \left(\sum_{J' \preceq [i_2, \dots, i_r]} D_{J'} \right) = \sum_{J' \preceq [i_2, \dots, i_r]} (D_{[i_1, j_1, \dots, j_s]} + D_{[i_1 + j_1, \dots, j_s]}),$$

which is $\sum_{J \preceq I} D_J$ since the first part of a composition J that satisfies $J \preceq I$ is either i_1 or $i_1 + (i_2 + \dots + i_l)$ for some $l \geq 2$. $\square$

4. PROPERTIES OF THE OPERATORS

In this section we develop properties of the operators U_j and D_i .

4.1. Extensions of linear endomorphisms of $\mathbb{B}$ to $\mathbb{A}$. Since W^0 is a set of coset representatives of W_0 in W , every element w of W factors uniquely as $w = w^{(0)}w_{(0)}$ with $w^{(0)} \in W^0$ and $w_{(0)} \in W_0$. We call this the *0-Grassmannian factorization* of w . Since the elements of W^0 are in bijection with k -bounded partitions, we can write this factorization as $w = w_\lambda w_{(0)}$, and we let

$$\mathbf{b}_w = \mathfrak{s}_\lambda^{(k)} \mathbf{u}_{w_{(0)}}.$$

Proposition 4.1. *The set $\{\mathbf{b}_w : w \in W\}$ is a basis of $\mathbb{A}$.*

¹With respect to this order, the composition $[1, 1, \dots, 1]$ is the maximal element and the composition $[n]$ is the minimal element.

Proof. We will define a total order on the elements of W in such a way that the leading term of $\mathbf{b}_w$ is $\mathbf{u}_w$. Then, with respect to this ordering, the transition matrix from $\{\mathbf{b}_w\}$ to $\{\mathbf{u}_w\}$ is uni-triangular, from which the result follows.

Informally, we need an order in which v precedes u whenever $\ell(u) > \ell(v)$ or the “Grassmannian part” of u is bigger than that of v . Define v to precede u if: $\ell(u) > \ell(v)$; or $\ell(u) = \ell(v)$ and $\ell(u^{(0)}) > \ell(v^{(0)})$. Note that this is only a partial order, but any linear extension of this partial order will do the trick.

First we argue that the leading term of $\mathbf{b}_{w_\lambda} = \mathfrak{s}_\lambda^{(k)}$ is $\mathbf{u}_{w_\lambda}$. Indeed, $\mathfrak{s}_\lambda^{(k)}$ expanded in the basis $\{\mathbf{u}_v\}$ is a linear combination of terms $\mathbf{u}_v$ with the v all of the same length $|\lambda|$, and it contains exactly one term $\mathbf{u}_w$ with $w \in W^0$, namely w_λ (see §2.7).

Next, we prove that the leading term of $\mathbf{b}_w$ is $\mathbf{u}_w$. If $\mathbf{u}_v$ appears in $\mathbf{b}_w = \mathfrak{s}_\lambda^{(k)} \mathbf{u}_{w_{(0)}}$ with nonzero coefficient, then $\mathbf{u}_v = \mathbf{u}_{\tilde{v}} \mathbf{u}_{w_{(0)}}$ with $\mathbf{u}_{\tilde{v}}$ appearing in $\mathfrak{s}_\lambda^{(k)}$. It follows that $v = \tilde{v}w_{(0)}$ with $\ell(v) = \ell(\tilde{v}) + \ell(w_{(0)})$ and that $v^{(0)} = \tilde{v}^{(0)}$. Hence, to compare the terms $\mathbf{u}_v$ and $\mathbf{u}_u$ of $\mathbf{b}_w$, it suffices to compare the corresponding terms $\mathbf{u}_{\tilde{v}}$ and $\mathbf{u}_{\tilde{u}}$ of $\mathfrak{s}_\lambda^{(k)}$. So, the leading term of $\mathbf{b}_w$ is the leading term of $\mathfrak{s}_\lambda^{(k)}$ times $\mathbf{u}_{w_{(0)}}$, which is precisely $\mathbf{u}_{w_\lambda} \mathbf{u}_{w_{(0)}} = \mathbf{u}_w$. $\square$

Corollary 4.2. *The set $\{\mathbf{h}_\lambda \mathbf{u}_{w_{(0)}} : w = w_\lambda w_{(0)} \in W\}$ is a basis of $\mathbb{A}$.*

Proof. Follows from Proposition 4.1 and the fact that $\{\mathbf{h}_\lambda\}$ is a basis of $\mathbb{B}$. $\square$

The above results allow us to extend linear endomorphisms of $\mathbb{B}$ to linear endomorphisms of $\mathbb{A}$. Let $f : \mathbb{B} \rightarrow \mathbb{B}$ be a linear transformation of $\mathbb{B}$. Then we get a linear transformation $\widehat{f} : \mathbb{A} \rightarrow \mathbb{A}$ by defining $\widehat{f}$ on the basis $\{\mathbf{b}_w\}$ by

$$\widehat{f}(\mathbf{b}_w) = \widehat{f}\left(\mathfrak{s}_\lambda^{(k)} \mathbf{u}_{w_{(0)}}\right) = f\left(\mathfrak{s}_\lambda^{(k)}\right) \mathbf{u}_{w_{(0)}},$$

where $w = w_\lambda w_{(0)}$ is the 0-Grassmannian factorization of w .

4.2. Commutation relation. We prove a commutation relation between the operators U_j and D_i . This relation will allow us to bootstrap properties of D_1 and U_j to every operator D_i via an inductive argument.

Proposition 4.3 (Commutation relation).

$$D_i \circ U_j = \sum_{e \geq 0} U_{j-e} \circ D_{i-e}.$$

Proof. First note that the right-hand side is a finite sum. The coefficient of $\mathbf{u}_v$ in

$$(D_i \circ U_j)(\mathbf{u}_u) = \sum_{\substack{u \rightsquigarrow w \\ \text{size}(u \rightsquigarrow w) = j}} \sum_{\substack{w \rightarrow v \\ \text{size}(w \rightarrow v) = i}} \mathbf{u}_v$$

is the number of tuples $(u \rightsquigarrow w, w \rightarrow v)$ consisting of a weak strip $u \rightsquigarrow w$ of length j and a strong strip $w \rightarrow v$ of length i . The coefficient of $\mathbf{u}_v$ in

$$\left(\sum_{e \geq 0} U_{j-e} \circ D_{i-e} \right) (\mathbf{u}_u) = \sum_{e \geq 0} \sum_{\substack{u \rightarrow x \\ \text{size}(u \rightarrow x) = i-e}} \sum_{\substack{x \rightsquigarrow v \\ \text{size}(x \rightsquigarrow v) = j-e}} \mathbf{u}_v$$

is the number of triples $(e, u \rightarrow x, x \rightsquigarrow v)$ consisting of a nonnegative integer e , a strong strip $u \rightarrow x$ of length $i - e$ and a weak strip $x \rightsquigarrow v$ of length $j - e$.

By [LLMS10, Proposition 4.1], these two numbers are the same. Indeed, that proposition establishes a bijection between the sets:

$$\left\{ (W', S') : \begin{array}{l} W' \text{ is a weak strip beginning at } u, \\ S' \text{ is a strong strip ending at } v, \\ \text{with } W' \text{ ending where } S' \text{ begins.} \end{array} \right\} \\ \longleftrightarrow \left\{ (W, S, e) : \begin{array}{l} W \text{ is a weak strip ending at } v, \\ S \text{ is a strong strip beginning at } u, \\ e \geq 0 \text{ satisfies } \text{size}(W) + e \leq k, \\ \text{with } S \text{ ending where } W \text{ begins.} \end{array} \right\}$$

such that

$$\begin{aligned} \text{size}(S) &= \text{size}(S') - e, \\ \text{size}(W) &= \text{size}(W') - e. \end{aligned} \quad \square$$

Corollary 4.4 (Bracket).

$$D_i \circ U_j - U_j \circ D_i = D_{i-1} \circ U_{j-1}.$$

Proof.

$$D_i \circ U_j = \sum_{e \geq 0} U_{j-e} \circ D_{i-e} = U_j \circ D_i + \sum_{e \geq 1} U_{j-e} \circ D_{i-e} = U_j \circ D_i + D_{i-1} \circ U_{j-1}. \quad \square$$

4.3. D_i stabilizes $\mathbb{B}$. We use the commutation relation of the previous section to prove that $D_i(\mathbb{B}) \subseteq \mathbb{B}$. First we determine the image of $\mathbf{h}_r$ under D_i .

Lemma 4.5. *For $r \leq k$ and all i ,*

$$D_i(\mathbf{h}_r) = \mathbf{h}_{r-i}.$$

Proof. If $i > r$, then $D_i(\mathbf{h}_r) = 0$ since there are no strong strips of size i beginning at elements w of length r . Also, by definition, $\mathbf{h}_{r-i} = 0$. So suppose that $i \leq r$. Proceed by induction on i . If $i = 1$, then

$$D_1(\mathbf{h}_r) = (D_1 \circ U_r)(1_{\mathbb{A}}) = (U_r \circ D_1)(1_{\mathbb{A}}) + (D_0 \circ U_{r-1})(1_{\mathbb{A}}) = 0_{\mathbb{A}} + \mathbf{h}_{r-1}.$$

Suppose the result holds for $i - 1$. Then

$$\begin{aligned} D_i(\mathbf{h}_r) &= (D_i \circ U_r)(1_{\mathbb{A}}) = (U_r \circ D_i)(1_{\mathbb{A}}) + (D_{i-1} \circ U_{r-1})(1_{\mathbb{A}}) \\ &= 0_{\mathbb{A}} + D_{i-1}(\mathbf{h}_{r-1}) = \mathbf{h}_{r-i}. \end{aligned} \quad \square$$

Theorem 4.6. *Let J be a composition. Then D_J stabilizes $\mathbb{B}$; that is,*

$$D_J(\mathbb{B}) \subseteq \mathbb{B}.$$

Proof. It suffices to prove this for the operators D_i since D_J is a linear combination of compositions of these operators. Since $\mathbb{B}$ is spanned by the products $\mathbf{h}_{j_1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}$, it suffices to show that $D_i(\mathbf{h}_{j_1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) \in \mathbb{B}$. Proceed by induction on i and l . If $l = 1$, then by Lemma 4.5, $D_i(\mathbf{h}_{j_1}) = \mathbf{h}_{j_1-i} \in \mathbb{B}$. For $i = 1$ this was proved in [BSS12, Theorem 3.9]. If $l > 1$, then

$$\begin{aligned} D_i(\mathbf{h}_{j_1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) &= (D_i \circ U_{j_1})(\mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) \\ &= (U_{j_1} \circ D_i)(\mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) + (D_{i-1} \circ U_{j_1-1})(\mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) \\ &= \mathbf{h}_{j_1} D_i(\mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) + D_{i-1}(\mathbf{h}_{j_1-1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) \in \mathbb{B}. \end{aligned} \quad \square$$

Since the noncommutative k -Schur functions form a basis of $\mathbb{B}$, it is natural to ask for the expansion of $D_i(\mathfrak{s}_\lambda^{(k)})$ in terms of noncommutative k -Schur functions. We obtain the following combinatorial description in terms of strong strips. Recall that w_λ denotes the 0-Grassmannian element corresponding to the k -bounded partition λ under the bijection between $\mathcal{B}^{(k)}$ and W^0 .

Theorem 4.7.

$$D_i(\mathfrak{s}_\lambda^{(k)}) = \sum_{\text{size}(w_\lambda \rightarrow w_\mu) = i} \mathfrak{s}_\mu^{(k)}.$$

Proof. Since $D_i(\mathfrak{s}_\lambda^{(k)}) \in \mathbb{B}$, to compute its expansion in terms of k -Schur functions, it suffices to compute the coefficient of $\mathbf{u}_w$ for 0-Grassmannian elements w (see §2.7). This is the number of strong strips $v \rightarrow w$ of length i with $\mathbf{u}_v$ appearing as a term in $\mathfrak{s}_\lambda^{(k)}$. But a strong strip that ends at a 0-Grassmannian element necessarily begins at a 0-Grassmannian element [LLMS10, Proposition 2.6], and there is a unique term $\mathbf{u}_v$ appearing in $\mathfrak{s}_\lambda^{(k)}$ with v a 0-Grassmannian element, namely $\mathbf{u}_{w_\lambda}$. $\square$

4.4. Restriction to $\mathbb{B}$. We prove that D_J is determined by its restriction to $\mathbb{B}$ and we identify this restriction as a linear operator adjoint to multiplication by a symmetric function with respect to the pairing on $\Lambda_{(k)} \times \Lambda^{(k)}$.

Theorem 4.8. *Suppose $w \in W$ and $v \in W_0$. Then*

$$\begin{aligned} U_j(\mathbf{u}_w \mathbf{u}_v) &= U_j(\mathbf{u}_w) \mathbf{u}_v, \\ D_i(\mathbf{u}_w \mathbf{u}_v) &= D_i(\mathbf{u}_w) \mathbf{u}_v. \end{aligned}$$

Consequently, U_j and D_i are completely determined by their restriction to $\mathbb{B}$.

Proof. Since U_j is left-multiplication by $\mathbf{h}_j$, associativity implies that $U_j(\mathbf{u}_w \mathbf{u}_v) = U_j(\mathbf{u}_w) \mathbf{u}_v$, establishing the first equality. By Corollary 4.2, it suffices to show that $D_i(\mathbf{h}_{j_1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l} \mathbf{u}_v) = D_i(\mathbf{h}_{j_1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) \mathbf{u}_v$. Proceed by induction. The case $i = 1$ was proved in [BSS12, Theorem 3.10]. Suppose the result holds for D_{i-1} . We prove the result also holds for D_i by induction on l . If $l = 1$, then

$$D_i(\mathbf{h}_{j_1} \mathbf{u}_v) = (D_i \circ U_{j_1})(\mathbf{u}_v) = (U_{j_1} \circ D_i)(\mathbf{u}_v) + (D_{i-1} \circ U_{j_1-1})(\mathbf{u}_v).$$

Note that $D_i(\mathbf{u}_v) = 0$ because there is no strong strip starting from $v \in W_0$. Also, since the result holds for D_{i-1} , we have

$$(D_{i-1} \circ U_{j_1-1})(\mathbf{u}_v) = D_{i-1}(\mathbf{h}_{j_1-1} \mathbf{u}_v) = D_{i-1}(\mathbf{h}_{j_1-1}) \mathbf{u}_v = D_i(\mathbf{h}_{j_1}) \mathbf{u}_v.$$

For $l > 1$, use the identity $D_i \circ U_{j_1} = U_{j_1} \circ D_i + D_{i-1} \circ U_{j_1-1}$ to write

$$D_i(\mathbf{h}_{j_1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l} \mathbf{u}_v) = (U_{j_1} \circ D_i)(\mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l} \mathbf{u}_v) + D_{i-1}(\mathbf{h}_{j_1-1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l} \mathbf{u}_v).$$

Since the product $\mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}$ involves less than l terms, by induction we have that

$$(U_{j_1} \circ D_i)(\mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l} \mathbf{u}_v) = U_{j_1}(D_i(\mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) \mathbf{u}_v) = (U_{j_1} \circ D_i)(\mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) \mathbf{u}_v.$$

Since the result holds for D_{i-1} , we have that

$$D_{i-1}(\mathbf{h}_{j_1-1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l} \mathbf{u}_v) = D_{i-1}(\mathbf{h}_{j_1-1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) \mathbf{u}_v.$$

Hence, $D_i(\mathbf{h}_{j_1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l} \mathbf{u}_v) = D_i(\mathbf{h}_{j_1} \mathbf{h}_{j_2} \cdots \mathbf{h}_{j_l}) \mathbf{u}_v$, as desired. $\square$

We next identify the restriction of D_J to $\mathbb{B}$. For a composition J , let s_J denote the *ribbon Schur function* indexed by J (for a good introduction to ribbon Schur functions, see for instance [BTvW06]) and let $\overline{s_J}$ denote its image in $\Lambda^{(k)}$. Recall that $\overline{s_J}^\perp : \Lambda_{(k)} \rightarrow \Lambda_{(k)}$ is the linear operator adjoint to multiplication by $\overline{s_J}$ in $\Lambda^{(k)}$. We also denote the corresponding linear operator on $\mathbb{B}$ by $\overline{s_J}^\perp$.

Theorem 4.9. *The restriction of D_J to $\mathbb{B}$ is $\overline{s_J}^\perp$. Consequently, D_J is the extension to $\mathbb{A}$, as defined in §4.1, of $\overline{s_J}^\perp : \mathbb{B} \rightarrow \mathbb{B}$.*

Proof. In the following, let $D_J(s_\lambda^{(k)})$ denote the image $D_J(\mathfrak{s}_\lambda^{(k)})$ under the isomorphism $\mathbb{B} \rightarrow \Lambda_{(k)}$. We will prove, for all $s_\lambda^{(k)}$ and $\tilde{F}_\mu$,

$$\left\langle D_J \left(s_\lambda^{(k)} \right), \tilde{F}_\mu \right\rangle = \left\langle s_\lambda^{(k)}, \overline{s_J} \tilde{F}_\mu \right\rangle.$$

Proceed by induction on the length of $J = [j_1, j_2, \dots, j_l]$. Suppose $l = 1$. Then it suffices to prove that

$$\left\langle D_j \left(s_\lambda^{(k)} \right), \tilde{F}_\mu \right\rangle = \left\langle s_\lambda^{(k)}, \overline{h_j} \tilde{F}_\mu \right\rangle.$$

But this follows immediately from Theorem 4.7 and the Pieri rule: $\overline{h_j} \tilde{F}_{w_\mu} = \sum \tilde{F}_{w_\lambda}$ with the sum running over all strong strips $w_\lambda \rightarrow w_\mu$ of size j (see [LLMS10, Theorem 4.13]).

Now suppose the result holds for compositions of length less than l . Let $J = [j_1, j_2, \dots, j_l]$. Observe that

$$D_J = D_{j_1} \circ D_{[j_2, \dots, j_l]} - D_{[j_1+j_2, \dots, j_l]},$$

so by induction and the product rule for ribbon Schur functions [Mac, §169],

$$\begin{aligned} D_J &= \overline{s_{j_1}}^\perp \circ \overline{s_{[j_2, \dots, j_l]}}^\perp - \overline{s_{[j_1+j_2, \dots, j_l]}}^\perp \\ &= \overline{s_{j_1} s_{[j_2, \dots, j_l]} - s_{[j_1+j_2, \dots, j_l]}}^\perp = \overline{s_J}^\perp. \end{aligned}$$

□

Theorem 4.10. *The operators D_J and D_K commute.*

Proof. $\mathbb{A}$ is spanned by elements of the form $\mathbf{b} \mathbf{u}_w$ with $\mathbf{b} \in \mathbb{B}$ and $w \in W_0$ (Proposition 4.1), so it suffices to prove this for these elements. Combining Theorems 4.8 and 4.9, we have

$$\begin{aligned} (D_K \circ D_J)(\mathbf{b} \mathbf{u}_w) &= (D_K \circ D_J)(\mathbf{b}) \mathbf{u}_w = (\overline{s_K}^\perp \circ \overline{s_J}^\perp)(\mathbf{b}) \mathbf{u}_w \\ &= (\overline{s_J}^\perp \circ \overline{s_K}^\perp)(\mathbf{b}) \mathbf{u}_w = (D_J \circ D_K)(\mathbf{b}) \mathbf{u}_w = (D_J \circ D_K)(\mathbf{b} \mathbf{u}_w), \end{aligned}$$

where the third equality comes from the commutation of symmetric functions. □

Remark 4.11. Theorem 4.10 implies that there is a bijection between strong strips $w \rightarrow u$ with ascent composition $I \boxplus J$ or $I \boxminus J$ and strong strips $w \rightarrow u$ with ascent composition $J \boxplus I$ or $J \boxminus I$. We do not know of an explicit bijection.

5. STRONG SCHUR FUNCTIONS

Lam, Lapointe, Morse, and Shimozono [LLMS10] generalized the k -Schur functions to a larger set of functions called the strong Schur functions. We use the properties of the operators developed in the previous section to prove a series of their conjectures [LLMS10, Conjecture 4.18] regarding these functions.

5.1. Strong Schur functions are symmetric functions. For $u, v \in W$, define the *strong Schur function*

$$\text{Strong}_{u/v} = \sum_{u \rightarrow \cdots \rightarrow v \in \mathcal{G}^\downarrow} L_{\text{ascomp}(u \rightarrow \cdots \rightarrow v)},$$

where L_J denotes the fundamental quasi-symmetric function indexed by the composition J . In [LLMS10], it was shown that $\text{Strong}_{u/\text{id}}$ is a symmetric function and that when u is 0-Grassmannian, it is a k -Schur function.

Remark 5.1. The definition given here is a reformulation of that in [LLMS10]. They defined $\text{Strong}_{u/v}$ as the generating function of “strong tableaux”; the above definition is obtained from theirs by lumping together tableaux of the same “weight”, yielding the expansion in terms of monomial quasi-symmetric functions below.

Theorem 5.2 ([LLMS10, Conjecture 4.18(1)]). *$\text{Strong}_{u/v}$ is a symmetric function. Furthermore, it expands positively in the monomial basis m_λ of Λ :*

$$\text{Strong}_{u/v} = \sum_{\lambda} \langle D^\lambda(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} m_\lambda,$$

where $D^\lambda = D_{\lambda_1} \circ \cdots \circ D_{\lambda_\ell}$.

Proof. The coefficient of the fundamental quasi-symmetric function L_J in $\text{Strong}_{u/v}$ is the number of paths in $\mathcal{G}^\downarrow$ from u to v with ascent composition equal to J . This is precisely the coefficient of $\mathbf{u}_v$ in $D_J(\mathbf{u}_u)$. Hence,

$$\text{Strong}_{u/v} = \sum_{J \models \ell(u) - \ell(v)} \langle D_J(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} L_J.$$

Recall that $L_J = \sum_{I \succeq J} M_I$, where M_I denotes the monomial quasi-symmetric function indexed by the composition $I = [i_1, \dots, i_r]$. Thus,

$$\begin{aligned} \text{Strong}_{u/v} &= \sum_J \langle D_J(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} \sum_{I \succeq J} M_I \\ &= \sum_I \left(\sum_{J \preceq I} \langle D_J(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} \right) M_I \\ &= \sum_I \langle D^I(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} M_I, \end{aligned}$$

where $D^I = D_{i_1} \circ \cdots \circ D_{i_r}$. Since the operators D_i and D_j commute for all i and j , the operator D^I depends only on the underlying partition $\lambda(I)$ of I . Hence,

$$\begin{aligned} \text{Strong}_{u/v} &= \sum_{\lambda} \sum_{\lambda(I)=\lambda} \langle D^I(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} M_I \\ &= \sum_{\lambda} \langle D^\lambda(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} \sum_{\lambda(I)=\lambda} M_I \\ &= \sum_{\lambda} \langle D^\lambda(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} m_\lambda, \end{aligned}$$

where m_λ is the monomial symmetric function. In particular, $\text{Strong}_{u/v} \in \Lambda$. $\square$

If u and v are 0-Grassmannian elements, we write $\text{Strong}_{\mu/\nu}$ instead of $\text{Strong}_{u/v}$, where μ and ν are the k -bounded partitions corresponding to u and v , respectively. It follows from §2.7 (as in the proof of Theorem 4.7) that the coefficient of $\mathbf{u}_v$ in $D^\lambda(\mathbf{u}_u)$ is the coefficient of $s_\nu^{(k)}$ in the expansion of $D^\lambda(s_\mu^{(k)})$ in terms of k -Schur functions. Thus,

$$\langle D^\lambda(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} = \langle D^\lambda(s_\mu^{(k)}), \tilde{F}_\nu \rangle = \langle s_\mu^{(k)}, \overline{h_\lambda} \tilde{F}_\nu \rangle,$$

where the last equality follows from the fact that the restriction of D^λ to $\mathbb{B}$ is the adjoint to the multiplication by $\overline{h_\lambda}$ (Theorem 4.9).

Corollary 5.3. *If u and v are 0-Grassmannian elements corresponding to the k -bounded partitions μ and ν , respectively, then*

$$\text{Strong}_{\mu/\nu} = \sum_{\lambda} \langle s_\mu^{(k)}, \overline{h_\lambda} \tilde{F}_\nu \rangle m_\lambda.$$

5.2. Strong Schur functions belong to $\Lambda_{(k)}$. Next we verify the second part of Conjecture 4.18 from [LLMS10]. Recall that for a linear operator f on $\mathbb{B}$, we denote by $\hat{f}$ its extension to $\mathbb{A}$ as defined in §4.1.

Theorem 5.4. *Let $u, v \in W$. The strong Schur function $\text{Strong}_{u/v}$ lies in $\Lambda_{(k)}$. Furthermore, we have the expansion in homogeneous symmetric functions:*

$$\text{Strong}_{u/v} = \sum_{\lambda \in \mathcal{B}^{(k)}} \langle \widehat{m_\lambda^\perp}(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} h_\lambda.$$

Proof. Since the m_μ form a basis of Λ , there exist coefficients $\mathbf{c}_{\lambda,\mu}$ for which $h_\lambda = \sum_{\mu} \mathbf{c}_{\lambda,\mu} m_\mu$. Hence,

$$\begin{aligned} \text{Strong}_{u/v} &= \sum_{\lambda} \langle D^\lambda(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} m_\lambda \\ &= \sum_{\lambda} \langle \widehat{h_\lambda^\perp}(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} m_\lambda \\ &= \sum_{\lambda, \mu} \langle \mathbf{c}_{\lambda,\mu} \widehat{m_\mu^\perp}(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} m_\lambda \\ &= \sum_{\mu} \langle \widehat{m_\mu^\perp}(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} \sum_{\lambda} \mathbf{c}_{\lambda,\mu} m_\lambda \\ &= \sum_{\mu} \langle \widehat{m_\mu^\perp}(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} h_\mu. \end{aligned}$$

Since $m_\mu^\perp = 0$ for any partition μ that is not k -bounded, the above summation runs over k -bounded partitions. $\square$

5.3. Expansions of strong Schur functions. Since $\text{Strong}_{u/v}$ lies in $\Lambda_{(k)}$, it has an expansion in terms of k -Schur functions. The third part of Conjecture 4.18 of [LLMS10] deals with the coefficients in this expansion.

Theorem 5.5. *Let $u, v \in W$. Then*

$$\text{Strong}_{u/v} = \sum_{\lambda \in \mathcal{B}^{(k)}} \langle \widehat{\tilde{F}_\lambda^\perp}(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} s_\lambda^{(k)}.$$

Proof. By using the expansions $h_\lambda = \sum_\tau K_{\tau,\lambda}^{(k)} s_\tau^{(k)}$ and $\tilde{F}_\tau = \sum_\lambda K_{\tau,\lambda}^{(k)} m_\lambda$,

$$\begin{aligned} \text{Strong}_{u/v} &= \sum_{\lambda \in \mathcal{B}^{(k)}} \left\langle \widehat{m}_\lambda^\perp(\mathbf{u}_u), \mathbf{u}_v \right\rangle_{\mathbb{A}} h_\lambda \\ &= \sum_{\lambda \in \mathcal{B}^{(k)}} \left\langle \widehat{m}_\lambda^\perp(\mathbf{u}_u), \mathbf{u}_v \right\rangle_{\mathbb{A}} \sum_{\tau \in \mathcal{B}^{(k)}} K_{\tau,\lambda}^{(k)} s_\tau^{(k)} \\ &= \sum_{\tau \in \mathcal{B}^{(k)}} \left\langle \sum_{\lambda \in \mathcal{B}^{(k)}} K_{\tau,\lambda}^{(k)} \widehat{m}_\lambda^\perp(\mathbf{u}_u), \mathbf{u}_v \right\rangle_{\mathbb{A}} s_\tau^{(k)} \\ &= \sum_{\tau \in \mathcal{B}^{(k)}} \left\langle \widehat{\tilde{F}_\tau}^\perp(\mathbf{u}_u), \mathbf{u}_v \right\rangle_{\mathbb{A}} s_\tau^{(k)}. \quad \square \end{aligned}$$

The third part of Conjecture 4.18 of [LLMS10] states that the coefficients in the above summation are nonnegative; more explicitly, that they are the structure constants of the Schubert classes for the cohomology of the affine flag variety. We prove this conjecture for 0-Grassmannian elements; the general case is still open.

Corollary 5.6 ([LLMS10, Conjecture 4.18(3)]). *Let μ and ν be k -bounded partitions. The coefficient of $s_\lambda^{(k)}$ in $\text{Strong}_{\mu/\nu}$ is the coefficient of $\tilde{F}_\mu$ in $\tilde{F}_\lambda \tilde{F}_\nu$:*

$$\text{Strong}_{\mu/\nu} = \sum_{\lambda \in \mathcal{B}^{(k)}} \left\langle s_\mu^{(k)}, \tilde{F}_\lambda \tilde{F}_\nu \right\rangle s_\lambda^{(k)}.$$

In particular, the strong Schur functions $\text{Strong}_{\mu/\nu}$ are k -Schur positive.

Proof. If u and v are 0-Grassmannian, with $u = w_\mu$ and $v = w_\lambda$, then the coefficient in Theorem 5.5 reduces to

$$\left\langle \widehat{\tilde{F}_\lambda}^\perp(\mathbf{u}_u), \mathbf{u}_v \right\rangle_{\mathbb{A}} = \left\langle \tilde{F}_\lambda^\perp(s_\mu^{(k)}), \tilde{F}_\nu \right\rangle = \left\langle s_\mu^{(k)}, \tilde{F}_\lambda \tilde{F}_\nu \right\rangle.$$

These coefficients are known to be nonnegative because they are the structure constants of the Schubert classes in the cohomology of the affine Grassmannian [Lam08]. $\square$

Corollary 5.7. *Let μ and ν be k -bounded partitions. Then the skew k -Schur function is*

$$s_{\mu/\nu}^{(k)} := \tilde{F}_\nu^\perp(s_\mu^{(k)}) = \text{Strong}_{\mu/\nu}.$$

Proof. By Corollary 5.6, we have $\text{Strong}_\mu = s_\mu^{(k)}$. Thus, the coefficient of $s_\lambda^{(k)}$ in the left-hand side is $\langle \tilde{F}_\nu^\perp(s_\mu^{(k)}), \tilde{F}_\lambda \rangle = \langle s_\mu^{(k)}, \tilde{F}_\nu \tilde{F}_\lambda \rangle$, which is the coefficient of $s_\lambda^{(k)}$ in $\text{Strong}_{\mu/\nu}$. $\square$

Consequently, we obtain an explicit combinatorial description of the skew k -Schur function $s_{\lambda/\mu}^{(k)}$ since the strong Schur function $\text{Strong}_{\mu/\nu}$ has an explicit combinatorial description in terms of “strong tableaux” (see [LLMS10] for details).

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REFERENCES

- [BSS12] Chris Berg, Franco Saliola, and Luis Serrano, *The down operator and expansions of near rectangular k -Schur functions*, 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012), Discrete Math. Theor. Comput. Sci. Proc., AR, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012, pp. 433–444 (English, with English and French summaries). MR2958018
- [BMSvW00] Nantel Bergeron, Stefan Mykytiuk, Frank Sottile, and Stephanie van Willigenburg, *Noncommutative Pieri operators on posets*, J. Combin. Theory Ser. A **91** (2000), no. 1-2, 84–110, DOI 10.1006/jcta.2000.3090. In memory of Gian-Carlo Rota. MR1779776 (2002d:05122)
- [BTvW06] Louis J. Billera, Hugh Thomas, and Stephanie van Willigenburg, *Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions*, Adv. Math. **204** (2006), no. 1, 204–240, DOI 10.1016/j.aim.2005.05.014. MR2233132 (2007b:05207)
- [Lam10] T. Lam, *Stanley symmetric functions and Peterson algebras* (2010). ArXiv e-prints.
- [Lam06] Thomas Lam, *Affine Stanley symmetric functions*, Amer. J. Math. **128** (2006), no. 6, 1553–1586. MR2275911 (2008b:05178)
- [Lam08] Thomas Lam, *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. **21** (2008), no. 1, 259–281, DOI 10.1090/S0894-0347-06-00553-4. MR2350056 (2009a:05207)
- [LLMS10] Thomas Lam, Luc Lapointe, Jennifer Morse, and Mark Shimozono, *Affine insertion and Pieri rules for the affine Grassmannian*, Mem. Amer. Math. Soc. **208** (2010), no. 977, xii+82, DOI 10.1090/S0065-9266-10-00576-4. MR2741963 (2012f:05314)
- [LS10] Thomas Lam and Mark Shimozono, *Quantum cohomology of G/P and homology of affine Grassmannian*, Acta Math. **204** (2010), no. 1, 49–90, DOI 10.1007/s11511-010-0045-8. MR2600433 (2011h:14082)
- [LS07] Thomas F. Lam and Mark Shimozono, *Dual graded graphs for Kac-Moody algebras*, Algebra Number Theory **1** (2007), no. 4, 451–488, DOI 10.2140/ant.2007.1.451. MR2368957 (2009a:05209)
- [LLM03] L. Lapointe, A. Lascoux, and J. Morse, *Tableau atoms and a new Macdonald positivity conjecture*, Duke Math. J. **116** (2003), no. 1, 103–146, DOI 10.1215/S0012-7094-03-11614-2. MR1950481 (2004c:05208)
- [LM03] L. Lapointe and J. Morse, *Schur function analogs for a filtration of the symmetric function space*, J. Combin. Theory Ser. A **101** (2003), no. 2, 191–224, DOI 10.1016/S0097-3165(02)00012-2. MR1961543 (2004c:05209)
- [LM05] Luc Lapointe and Jennifer Morse, *Tableaux on $k+1$ -cores, reduced words for affine permutations, and k -Schur expansions*, J. Combin. Theory Ser. A **112** (2005), no. 1, 44–81, DOI 10.1016/j.jcta.2005.01.003. MR2167475 (2006j:05214)
- [LM07] Luc Lapointe and Jennifer Morse, *A k -tableau characterization of k -Schur functions*, Adv. Math. **213** (2007), no. 1, 183–204, DOI 10.1016/j.aim.2006.12.005. MR2331242 (2008c:05187)
- [LM08] Luc Lapointe and Jennifer Morse, *Quantum cohomology and the k -Schur basis*, Trans. Amer. Math. Soc. **360** (2008), no. 4, 2021–2040, DOI 10.1090/S0002-9947-07-04287-0. MR2366973 (2009d:14072)

- [LL12] Naichung Conan Leung and Changzheng Li, *Gromov-Witten invariants for G/B and Pontryagin product for ΩK* , Trans. Amer. Math. Soc. **364** (2012), no. 5, 2567–2599, DOI 10.1090/S0002-9947-2012-05438-9. MR2888220 (2012m:14107)
- [Mac] P.A. MacMahon, *Combinatory analysis*, Combinatory Analysis, no. v. 2. The University Press, 1916.
- [SCc12] The Sage-Combinat community, *Sage-Combinat: enhancing sage as a toolbox for computer exploration in algebraic combinatorics*, The Sage Development Team (2012).
- [Sta] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR1676282 (2000k:05026)
- [Ste82] E. M. Stein, *The development of square functions in the work of A. Zygmund*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 359–376, DOI 10.1090/S0273-0979-1982-15040-6. MR663787 (83i:42001)

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