A SHARP TRUDINGER-MOSER TYPE INEQUALITY IN $\mathbb{R}^2$

MANASSÉS DE SOUZA AND JOÃO MARCOS DO Ó

ABSTRACT. In this paper, we establish a sharp Trudinger-Moser type inequality for a class of Schrödinger operators in $\mathbb{R}^2$. We obtain a result related to the compactness of the embedding of a subspace of $W^{1,2}(\mathbb{R}^2)$ into the Orlicz space $L_\phi(\mathbb{R}^2)$ determined by $\phi(t) = e^{\beta t^2} - 1$. Our result is similar to one obtained by Adimurthi and Druet for smooth bounded domains in $\mathbb{R}^2$, which is closely related to a compactness result proved by Lions. Furthermore, similarly to what has been done by Carleson and Chang, we prove the existence of an extremal function for this Trudinger-Moser inequality by performing a blow-up analysis. Trudinger-Moser type inequalities have a wide variety of applications to the study of nonlinear elliptic partial differential equations involving the limiting case of Sobolev inequalities and have received considerable attention in recent years.

1. INTRODUCTION

Let Ω be a smooth domain in $\mathbb{R}^n$, where $n \geq 2$, and let $W^{1,p}(\Omega)$ be the Sobolev space, that is, the set of all functions in $L^p(\Omega)$ whose distributional derivatives of the first order also belong to $L^p(\Omega)$ with the Sobolev norm

$$(1.1) \quad \|u\|_{1,p} := \left\{ \int_{\Omega} (|\nabla u|^p + |u|^p) \, dx \right\}^{1/p}.$$

We recall that $W^{1,p}(\Omega)$ is a Banach space. In the sequel, $\|\cdot\|_p$ denotes the L^p -norm with respect to the Lebesgue measure. Let $C_c^\infty(\Omega)$ be the space of smooth functions with compact support contained in Ω . We define $W_0^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

1.1. Trudinger-Moser inequality. The study of the limiting case of Sobolev inequalities has received considerable attention in recent years. First, we would like to mention a few properties of functions in $W_0^{1,p}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a smooth *bounded domain*, using the Dirichlet norm $\|\nabla u\|_p$ (equivalent to (1.1) in $W_0^{1,p}(\Omega)$).

We denote $p^* = np/(n-p)$ if $1 \leq p < n$. The classical Sobolev theorem [15] asserts that the following embedding is continuous:

$$(1.2) \quad W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for any } 1 < q \leq p^*.$$

Received by the editors June 22, 2011 and, in revised form, February 16, 2012.

2010 *Mathematics Subject Classification.* Primary 35J20; Secondary 35J60.

Key words and phrases. Trudinger-Moser inequality, blow-up analysis, extremal function.

The authors' research was partially supported by the National Institute of Science and Technology of Mathematics INCT-Mat, CAPES and CNPq grants 307400/2009-3, 620108/2008-8 and 142002/2006-2.

The second author was the corresponding author.

Although p^* tends to infinity as $p \nearrow n$ (formally), the space $W_0^{1,n}(\Omega)$ contains unbounded functions. Instead of an embedding into $L^\infty(\Omega)$, N. Trudinger [35] proved that $W_0^{1,n}(\Omega)$ is embedded in the Orlicz space $L_\phi(\Omega)$ determined by $\phi(t) = e^{\beta|t|^{n/(n-1)}} - 1$. More precisely,

Theorem A (Trudinger, 1967). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Then there exists some $\beta > 0$ (independent of u) such that*

$$\sup_{u \in W_0^{1,n}(\Omega) : \|\nabla u\|_n = 1} \int_{\Omega} e^{\beta|u|^{n/(n-1)}} \, dx \leq C|\Omega|.$$

Similar results had been obtained by S. I. Pohozaev [27] and V. I. Yudovich [36]. Hempel et al. [16] proved that this refinement of the Sobolev embedding, due to Trudinger, is optimal in the sense that $L_\phi(\Omega)$ above cannot be replaced by any smaller Orlicz space. After that Theorem A was extended into some directions; for more details we refer the reader to the review article on this subject by D. G. de Figueiredo et al. [10]. In the first direction Theorem A was sharpened by J. Moser [25] who proved

Theorem B (Moser, 1972). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, where ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. Then for any positive $\beta \leq \alpha_n$,*

$$(1.3) \quad \sup_{u \in W_0^{1,n}(\Omega) : \|\nabla u\|_n = 1} \int_{\Omega} e^{\beta|u|^{n/(n-1)}} \, dx \leq C|\Omega|.$$

Moreover, α_n is the best constant in the following sense: the integral on the left actually is finite for any positive β , but if $\beta > \alpha_n$ it can be made arbitrarily large by an appropriate choice of u and the supremum is $+\infty$.

Estimates like (1.3) are now called Trudinger-Moser type inequalities. Theorem B has a wide variety of applications to the study of nonlinear elliptic partial differential equations involving the limiting case of Sobolev inequalities and has received considerable attention in recent years. For works related to (1.3) and their applications, we refer, among many others, to [3, 9, 10, 12, 13, 34] and the references therein.

After Theorem B, a natural question arises: is the embedding $W_0^{1,n}(\Omega) \hookrightarrow L_\phi(\Omega)$ compact? In [23], P.-L. Lions proved that except for “small weak neighborhoods of 0” the embedding is compact and the best constant α_n may be improved. Among other results, P.-L. Lions [23] proved the following theorem.

Theorem C (Lions, 1985). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and (u_k) be a sequence of functions in $W_0^{1,n}(\Omega)$ with $\|\nabla u_k\|_n = 1$ such that $u_k \rightharpoonup u_0$ weakly in $W_0^{1,n}(\Omega)$. Then for any $p < 1/(1 - \|\nabla u_0\|_n^n)^{1/(n-1)}$ we have*

$$\limsup_{k \rightarrow \infty} \int_{\Omega} e^{\alpha_n p |u_k|^{n/(n-1)}} \, dx < \infty.$$

This result gives more precise information than Theorem B when $u_k \rightharpoonup u_0$ weakly in $W_0^{1,n}(\Omega)$ with $u_0 \not\equiv 0$. Motivated by this, Adimurthi and O. Druet [2] investigated possible extensions of (1.3) in the case $n = 2$. They provided valuable supplementary information on Theorem C giving extra information even when $u_k \rightharpoonup 0$ weakly in $W_0^{1,2}(\Omega)$. They obtain this complementary result by doing a

careful blow-up analysis of sequences of solutions to approximate elliptic equations with near critical Trudinger-Moser growth.

Theorem D (Adimurthi and Druet, 2002). *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $\bar{\lambda}_1(\Omega)$ the first eigenvalue of the Laplacian with Dirichlet boundary condition in Ω and*

$$(1.4) \quad C_\alpha(\Omega) = \sup_{u \in W_0^{1,2}(\Omega) : \|\nabla u\|_2 = 1} \int_{\Omega} e^{4\pi(1+\alpha\|u\|_2^2)u^2} dx.$$

Then we have

$$C_\alpha(\Omega) = \begin{cases} < \infty & \text{if } 0 \leq \alpha < \bar{\lambda}_1(\Omega), \\ +\infty & \text{if } \alpha \geq \bar{\lambda}_1(\Omega). \end{cases}$$

Note that if $\alpha = 0$ in (1.4), we have the classical Trudinger-Moser inequality given in (1.3). Theorem D was extended by Y. Yunyan [37] to arbitrary dimensions $n \geq 2$.

Inequalities (1.3) and (1.4) are valid only for bounded domains, and extensions of Trudinger-Moser inequality for unbounded domains were first considered by D. M. Cao [5] in $\mathbb{R}^2$ and for any dimension by J. M. do Ó [11]. See also [1, 26] for related results.

Theorem E (J. M. do Ó, 1997). *For any $u \in W^{1,n}(\mathbb{R}^n)$ and $\beta > 0$ it holds that*

$$\Phi_n(\beta|u|^{n/(n-1)}) \in L^1(\mathbb{R}^n), \quad \text{where} \quad \Phi_n(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}.$$

Moreover, if $\|\nabla u\|_n \leq 1$, $\|u\|_n \leq M < \infty$ and $\beta < \alpha_n$, then there exist $C = C(n, M, \beta)$ such that

$$(1.5) \quad \int_{\mathbb{R}^n} \Phi_n(\beta|u|^{n/(n-1)}) dx \leq C(n, M, \beta).$$

All these results in [5, 11] have been proved in the case of the unit ball and the supremum taken with respect to the Dirichlet norm $\|\nabla u\|_n$, and assuming (in some sense) a subcritical growth: $e^{\beta|u|^{n/(n-1)}}$ with $\beta < \alpha_n$. B. Ruf [30] proved that in the case $n = 2$ the result of J. Moser (1.3) can be fully extended to unbounded domains (and thus to all of $\mathbb{R}^2$) if the Dirichlet norm $\|\nabla u\|_2$ is replaced by the full Sobolev norm $\|u\|_{1,2}$ (see also [21] for an extension of this result to arbitrary dimensions $n \geq 3$). More precisely, one has:

Theorem F (Ruf, 2002). *Let Ω be any smooth domain in $\mathbb{R}^2$, and consider the Sobolev space $W_0^{1,2}(\Omega)$ endowed with the Sobolev norm $\|u\|_{1,2}$. Then there is a constant d independent of Ω such that*

$$(1.6) \quad \sup_{u \in W_0^{1,2}(\Omega) : \|u\|_{1,2} = 1} \int_{\Omega} (e^{4\pi u^2} - 1) dx \leq d.$$

1.2. Extremal functions for the Trudinger-Moser inequality. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and consider the best embedding constant for (1.2),

$$(1.7) \quad S_{p,q}(\Omega) := \sup_{u \in W_0^{1,p}(\Omega) : \|\nabla u\|_p = 1} \int_{\Omega} |u|^q dx.$$

It is well known that $S_{p,q}(\Omega)$ is attained if $1 \leq q < p^*$ and $S_{p,q}(\Omega)$ is not attained if $q = p^*$ and $\Omega \neq \mathbb{R}^n$. By contrast, in the situation of Trudinger-Moser inequality, L. Carleson and A. Chang in [6] surprisingly proved

Theorem G (Carleson and Chang, 1986). *If $\Omega \subset \mathbb{R}^n$ is the ball $B_1(0)$, then the supremum in (1.3) is achieved when $\alpha \leq \alpha_n$.*

M. Struwe [33] studied the existence of extremal functions for the nonsymmetric domain. In the case that $n = 2$, he obtained a sufficient condition for the existence of an extremal function, using blow-up analysis. M. Flucher [14] introduced another method, the conformal rearrangement, and derived an isoperimetric inequality which implies the existence of extremal functions when Ω is any smooth bounded domain in $\mathbb{R}^2$. Later, Lin [22] generalized the existence of an extremal function to any smooth bounded domain in $\mathbb{R}^n$ ($n \geq 2$). Recently, Y. Li [19, 20] obtained the existence of an extremal function for certain Trudinger-Moser inequalities on compact Riemannian manifolds with or without boundary. For a related result on the sphere $\mathbf{S}^2$ under a certain orthogonality condition we refer to [31], where Y. Sano found the best exponent.

On the existence of extremal functions for (1.4) the first results were obtained by Y. Yang [37] and G. Lu and Y. Yang [24]. B. Ruf [30] proved that supremum (1.6) is attained on balls B_R and $\mathbb{R}^2$ (see also [21] for an extension of this result to arbitrary dimensions $n > 2$).

1.3. Main results. Motivated by Adimurthi and O. Druet [2], in this paper we investigate possible extensions of Theorem D and Theorem F, and we complement the main result in [12] for a suitable subspace of $W^{1,2}(\mathbb{R}^2)$. Furthermore, motivated by [21, 24, 30, 37], we investigate the existence of an extremal function for this modified Trudinger-Moser inequality.

Our results can be applied to study the existence of solitary wave solutions for a class of semilinear Schrödinger equations involving critical growth of the form $-\Delta u + V(x)u = f(u)$ in $\mathbb{R}^2$, where $f(u) \sim e^{\alpha u^2} - 1$ at ∞ . Throughout this paper, we assume the following hypotheses for the potential V :

(V₁) $V \in C(\mathbb{R}^2, \mathbb{R})$ and $V(x) \geq 1$ for all $x \in \mathbb{R}^2$.

(V₂) V is radially symmetric, increasing and coercive, that is, $V(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$.

A typical example of this class of potentials is $V(x) = 1 + |x|^2$. Associated with the Schrödinger operator $Lu = -\Delta u + V(x)u$, we have the following subspace of $W^{1,2}(\mathbb{R}^2)$:

$$E = \left\{ u \in W^{1,2}(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2 \, dx < \infty \right\}.$$

It is well known that E , endowed with the inner product

$$(1.8) \quad \langle u, v \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv) \, dx,$$

is a Hilbert space to which there corresponds the norm $\|u\| = \langle u, u \rangle^{1/2}$. Moreover, for all $2 \leq s < \infty$,

$$(1.9) \quad E \hookrightarrow W^{1,2}(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$$

with continuous embedding, and $E \hookrightarrow L^s(\mathbb{R}^2)$ with compact embedding and

$$(1.10) \quad \lambda_1(V) := \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) \, dx}{\int_{\mathbb{R}^2} |u|^2 \, dx} \geq 1.$$

For more details on this topic, see [8] and [29].

First we state an extension of Theorem D for the space E .

Theorem 1.1. *Assume that the potential V satisfies (V_1) and (V_2) . Then for any $\alpha \in [0, \lambda_1(V))$, we have*

$$(1.11) \quad \ell(\alpha) := \sup_{u \in E, \|u\|=1} \int_{\mathbb{R}^2} \left[e^{4\pi(1+\alpha\|u\|_2^2)u^2} - 1 \right] \, dx < \infty.$$

Moreover, this inequality is sharp; more precisely, for any $\alpha \in [\lambda_1(V), +\infty)$, the supremum in (1.11) is $+\infty$.

Remark 1.2. This result is of a different nature from the one in Theorem 2.1. Indeed, when $u_k \rightharpoonup u_0$ weakly in E with $u_0 \not\equiv 0$, our set of inequalities is a consequence of Theorem 2.1. On the other hand, Theorem 1.1 gives new information when $u_k \rightharpoonup 0$ weakly in E . Clearly Theorem 1.1 is an extension of Theorem 2.1, but in our argument to prove Theorem 1.1 we use Theorem 2.1.

Now, we state the existence of an extremal function for $\ell(\alpha)$. Our next theorem is related to the main results in [6, 21, 24, 30, 37].

Theorem 1.3. *If (V_1) – (V_2) are satisfied, then for any $\alpha \in [0, \lambda_1(V))$ the supremum $\ell(\alpha)$ is attained, that is, there exists $u \in E$ such that $\|u\| = 1$ and*

$$\ell(\alpha) = \int_{\mathbb{R}^2} \left[e^{4\pi(1+\alpha\|u\|_2^2)u^2} - 1 \right] \, dx.$$

We organize this paper as follows: In Section 2, we prove an extension of Theorem C. In Sections 3 and 4, when $\alpha \in [0, \lambda_1(V))$ we prove the existence of a maximizers sequence $(u_k) \subset W_0^{1,2}(B_{R_k})$ for a subcritical functional satisfying suitable Euler-Lagrange equations. Moreover, we prove that $u_k \rightharpoonup u$ weakly in E and (u_k) is a maximizing sequence for $\ell(\alpha)$. Then we have two cases to analyze: (c_k) is a bounded sequence or $c_k \nearrow +\infty$ (up to subsequence), where $c_k = u_k(0)$ (maximum point). If (c_k) is bounded, then we prove that the function u is an extremal for $\ell(\alpha)$. If $c_k \nearrow +\infty$ the key point of the proof is to show that $c_k u_k \rightarrow G$ in $C_{loc}^{1,\theta}(\mathbb{R}^2 \setminus \{0\})$, where G is some Green function. This convergence result is based on a blow-up analysis of the sequence (u_k) , and for this we use the idea of M. Struwe in [34]. In Section 5, using a result due to Carleson-Chang in [6], we get

$$(1.12) \quad \ell(\alpha) \leq \pi e^{4\pi A_0 + 1}.$$

However, in Section 6 we construct a family of test functions $v_\epsilon \in E$, with $\|v_\epsilon\| = 1$ and satisfying

$$\int_{\mathbb{R}^2} \left[e^{4\pi(1+\alpha\|v_\epsilon\|_2^2)v_\epsilon^2} - 1 \right] \, dx > \pi e^{4\pi A_0 + 1},$$

for $\epsilon > 0$ sufficiently small, which is a contradiction with (1.12), and consequently the blow-up does not occur.

In Subsection 6.2 we prove the second point of Theorem 1.1, that is, $\ell(\alpha) = +\infty$ if $\alpha \in [\lambda_1(V), +\infty)$. The proof of this fact is based on computations of testing functions which are a combination of the eigenfunction associated to eigenvalue λ_1 with the Moser functions.

2. PROOF OF THEOREM 2.1

In this section we state an extension of Theorem C and of a related result due to J. M. do Ó et al. [12] for the space E .

Theorem 2.1. *Let (w_n) be in E with $\|w_n\| = 1$ and suppose that $w_n \rightharpoonup w_0$ in E . Then for all $0 < p < 4\pi(1 - \|w_0\|^2)^{-1}$ we have*

$$\sup_n \int_{\mathbb{R}^2} (e^{pw_n^2} - 1) \, dx < \infty.$$

Proof. Since (w_n) in E with $\|w_n\| = 1$ and $w_n \rightharpoonup w_0$ in E , we have $\|w_0\| \leq \liminf_{n \rightarrow +\infty} \|w_n\| = 1$. If $\|w_0\| = 1$, then $w_n \rightarrow w_0$ strongly in E , and using (1.9), we get $w_n \rightarrow w_0$ strongly in $W^{1,2}(\mathbb{R}^2)$. Thus, there exists $h \in W^{1,2}(\mathbb{R}^2)$ such that up to subsequence $|w_n(x)| \leq h(x)$ almost everywhere in $\mathbb{R}^2$. Therefore, by Theorem E we have

$$\sup_n \int_{\mathbb{R}^2} (e^{pw_n^2} - 1) \, dx \leq \int_{\mathbb{R}^2} (e^{ph^2} - 1) \, dx < \infty.$$

Next, we consider the case $\|w_0\| < 1$. Since $w_n \rightharpoonup w_0$ weakly in E and $\|w_n\| = 1$ holds,

$$\|w_n - w_0\|^2 = 1 - 2\langle w_n, w_0 \rangle + \|w_0\|^2 \rightarrow 1 - \|w_0\|^2 < \frac{4\pi}{p}.$$

Thus, for n sufficiently large, we have $p\|w_n - w_0\|^2 < \beta < 4\pi$ for some $\beta > 0$. Choosing $q > 1$, sufficiently close to 1, and $\epsilon > 0$ satisfying $q(1 + \epsilon^2)p\|w_n - w_0\|^2 < \beta$, by (1.5) we conclude that

$$(2.1) \quad \begin{aligned} \int_{\mathbb{R}^2} [e^{qp(1+\epsilon^2)(w_n-w_0)^2} - 1] \, dx &= \int_{\mathbb{R}^2} [e^{qp(1+\epsilon^2)\|w_n-w_0\|^2 \left(\frac{w_n-w_0}{\|w_n-w_0\|}\right)^2} - 1] \, dx \\ &\leq \int_{\mathbb{R}^2} [e^{\beta \left(\frac{|w_n-w_0|}{\|w_n-w_0\|}\right)^2} - 1] \, dx \leq C. \end{aligned}$$

Moreover, since

$$pw_n^2 \leq p(1 + \epsilon^2)(w_n - w_0)^2 + p(1 + \frac{1}{\epsilon^2})w_0^2,$$

it follows that

$$\begin{aligned} e^{pw_n^2} - 1 &\leq (e^{p(1+\epsilon^2)(w_n-w_0)^2} e^{p(1+1/\epsilon^2)w_0^2} - 1) \\ &\leq \frac{1}{q} \left(e^{qp(1+\epsilon^2)(w_n-w_0)^2} - 1 \right) + \frac{1}{r} \left(e^{rp(1+1/\epsilon^2)w_0^2} - 1 \right), \end{aligned}$$

where in the last estimate we have used the following elementary inequality:

$$ab - 1 \leq \frac{1}{q}(a^q - 1) + \frac{1}{r}(b^r - 1), \quad \text{for all } a, b > 0, \quad \text{where } r = \frac{q}{q-1}.$$

Therefore, by (2.1) and Theorem E we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} (e^{pw_n^2} - 1) \, dx &\leq \frac{1}{q} \int_{\mathbb{R}^2} [e^{qp(1+\epsilon^2)(w_n-w_0)^2} - 1] \, dx + \frac{1}{r} \int_{\mathbb{R}^2} [e^{rp(1+1/\epsilon^2)w_0^2} - 1] \, dx \\ &\leq C, \end{aligned}$$

and the proof is complete. $\square$

3. MAXIMIZERS OF SUBCRITICAL FUNCTIONALS

In this section we prove the existence of a radially symmetric and radially decreasing sequence for the supremum $\ell(\alpha)$ when $\alpha \in [0, \lambda_1(V))$.

Proposition 3.1. *For any $\alpha \in [0, \lambda_1(V))$, there exists $(u_k) \subset E$ such that $\|u_k\| = 1$ and u_k is a maximizing sequence for $\ell(\alpha)$, that is,*

$$(3.1) \quad \ell(\alpha) = \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2} (e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} - 1) \, dx,$$

where $\delta_k \searrow 0$ as $k \rightarrow +\infty$. Moreover, u_k may be chosen to be radially symmetric and radially decreasing, that is, $u_k(|x|) = u_k(|x'|)$ if $|x| = |x'|$ and $u_k(|x|) \geq u_k(|x'|)$ if $|x| \leq |x'|$.

We divide the proof of Proposition 3.1 into a sequence of lemmas. For $\delta \in (0, 4\pi)$, we consider the subcritical functional

$$(3.2) \quad J_{\delta, \alpha}(u) = \int_{\mathbb{R}^2} (e^{(4\pi - \delta)(1 + \alpha \|u\|_2^2)} - 1) \, dx, \quad u \in E.$$

By the terminology “subcritical functional” we just mean $\delta \in (0, 4\pi)$.

For Ω a smooth bounded domain in $\mathbb{R}^2$, we consider

$$(3.3) \quad \mu_1(\Omega) := \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla u|^2 + V(x)u^2) \, dx}{\int_{\Omega} u^2 \, dx} \geq 1.$$

Next we prove that for any bounded domain $\Omega \subset \mathbb{R}^2$, the supremum of the associated functional in the unit ball of $W_0^{1,2}(\Omega)$ is finite and attained for the subcritical case. Let us consider the space $W_0^{1,2}(\Omega)$ endowed with the equivalent norm

$$\|u\|_{\Omega} = \left\{ \int_{\Omega} (|\nabla u|^2 + V(x)u^2) \, dx \right\}^{1/2}.$$

Theorem 3.2. *For any $\alpha \in [0, \mu_1(\Omega))$ and $\delta \in (0, 4\pi)$, the supremum*

$$(3.4) \quad L_{\alpha, \delta}(\Omega) = \sup_{\{u \in W_0^{1,2}(\Omega) : \|u\|_{\Omega} = 1\}} \int_{\Omega} \left[e^{(4\pi - \delta)(1 + \alpha \|u\|_2^2)} - 1 \right] \, dx$$

is finite. Moreover, there exists a positive function $u \in C^{1,\theta}(\overline{\Omega})$, for some $\theta \in (0, 1)$, such that $\|u\|_{\Omega} = 1$ and

$$L_{\alpha, \delta}(\Omega) = \int_{\Omega} \left[e^{(4\pi - \delta)(1 + \alpha \|u\|_2^2)} - 1 \right] \, dx.$$

Proof. Let $(u_j) \subset W_0^{1,2}(\Omega)$ be a maximizing sequence for (3.4), that is, $\|u_j\|_{\Omega} = 1$ and

$$\lim_{j \rightarrow +\infty} \int_{\Omega} \left[e^{(4\pi - \delta)(1 + \alpha \|u_j\|_2^2)} - 1 \right] \, dx = L_{\alpha, \delta}(\Omega).$$

We can assume that (u_j) is a sequence of nonnegative functions. Since (u_j) is bounded in $W_0^{1,2}(\Omega)$, thus up to a subsequence, we have

$$\begin{aligned} u_j &\rightharpoonup u \quad \text{weakly in } W_0^{1,2}(\Omega), \\ u_j &\rightarrow u \quad \text{almost everywhere in } \Omega, \\ u_j &\rightarrow u \quad \text{strongly in } L^2(\Omega). \end{aligned}$$

Thus,

$$U_j = e^{(4\pi-\delta)(1+\alpha\|u_j\|_2^2)u_j^2} - 1 \rightarrow U = e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \text{ almost everywhere in } \Omega.$$

Assertion 1. *The function u is an extremal for $L_{\alpha,\delta}(\Omega)$.*

In order to prove Assertion 1, it is enough to verify that $U_j \rightarrow U$ strongly in $L^1(\Omega)$. Note that if $\|u\|_\Omega = 1$, the assertion follows by the same argument as in the proof of Theorem 2.1. In fact, we can see that this is the unique possibility. Indeed, assume by contradiction that $\|u\|_\Omega < 1$. Using $0 \leq \alpha < \mu_1(\Omega)$, the definition of $\mu_1(\Omega)$ and $1 + \xi \leq 1/(1 - \xi)$ if $0 \leq \xi < 1$, we obtain

$$1 + \alpha\|u\|_2^2 < 1 + \mu_1(\Omega)\|u\|_2^2 \leq 1 + \|u\|_\Omega^2 \leq \frac{1}{1 - \|u\|_\Omega^2}.$$

Thus,

$$\lim_{j \rightarrow +\infty} (4\pi - \delta)(1 + \alpha\|u_j\|_2^2) < \frac{4\pi}{1 - \|u\|_\Omega^2}.$$

Now choosing $q > s > 1$, sufficiently close to 1 such that

$$(4\pi - \delta)q(1 + \alpha\|u\|_2^2) < \frac{4\pi}{1 - \|u\|_\Omega^2},$$

we obtain by Theorem 2.1

$$(3.5) \quad \sup_j \int_\Omega \left[e^{(4\pi-\delta)(1+\alpha\|u_j\|_2^2)u_j^2} - 1 \right]^s dx \leq C \sup_j \int_\Omega \left[e^{q(4\pi-\delta)(1+\alpha\|u_j\|_2^2)u_j^2} - 1 \right] dx < \infty,$$

where in this last estimate we have used the following elementary inequality: Let $\beta > 0$ and $s > 1$; then for each $q > s$ there exists a positive constant $C = C(q)$ such that

$$(3.6) \quad (e^{\beta t^2} - 1)^s \leq C(e^{q\beta t^2} - 1) \quad \text{for all } t \in \mathbb{R}$$

(see [12]). Using (3.5) and the Hölder inequality, there exist $C_1 > 0$ such that

$$(3.7) \quad \int_\Omega u_j \left[e^{(4\pi-\delta)(1+\alpha\|u_j\|_2^2)u_j^2} - 1 \right] dx \leq C_1.$$

Since

$$[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1] \in L^1(\Omega),$$

given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$(3.8) \quad \int_A \left[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \right] dx \leq \epsilon \quad \text{if } |A| \leq \delta$$

for all measurable subsets A of Ω . Moreover, since $u \in L^1(\Omega)$, we can find $M_1 > 0$ such that

$$(3.9) \quad |\{x \in \Omega : u(x) \geq M_1\}| \leq \delta.$$

Taking $M = \max\{M_1, C_1/\epsilon\}$ we can write

$$\left| \int_\Omega \left[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \right] dx - \int_\Omega \left[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \right] dx \right| = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{\{x \in \Omega: u_j(x) \geq M\}} \left[e^{(4\pi-\delta)(1+\alpha\|u_j\|_2^2)u_j^2} - 1 \right] dx \\ I_2 &= \int_{\{x \in \Omega: u_j(x) < M\}} \left[e^{(4\pi-\delta)(1+\alpha\|u_j\|_2^2)u_j^2} - 1 \right] dx \\ &\quad - \int_{\{x \in \Omega: |u(x)| < M\}} \left[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \right] dx \\ I_3 &= \int_{\{x \in \Omega: u(x) \geq M\}} \left[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \right] dx. \end{aligned}$$

By (3.7), we have

$$\begin{aligned} I_1 &= \int_{\{x \in \Omega: u_j(x) \geq M\}} \left[e^{(4\pi-\delta)(1+\alpha\|u_j\|_2^2)u_j^2} - 1 \right] dx \\ &= \int_{\{x \in \Omega: u_j(x) \geq M\}} \frac{u_j \left[e^{(4\pi-\delta)(1+\alpha\|u_j\|_2^2)u_j^2} - 1 \right]}{u_j} dx \leq \frac{C_1}{M} \leq \epsilon. \end{aligned}$$

Thanks to the Lebesgue dominated convergence theorem,

$$\begin{aligned} I_2 &= \int_{\{x \in \Omega: u_j(x) < M\}} \left[e^{(4\pi-\delta)(1+\alpha\|u_j\|_2^2)u_j^2} - 1 \right] dx \\ &\quad - \int_{\{x \in \Omega: u(x) < M\}} \left[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \right] dx \rightarrow 0. \end{aligned}$$

Using (3.8) and (3.9) we have

$$I_3 = \int_{\{x \in \Omega: u(x) \geq M\}} \left[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \right] dx \leq \epsilon.$$

Thus

$$(3.10) \quad \int_{\Omega} \left[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \right] dx = L_{\alpha,\delta}(\Omega).$$

On the other hand, since $\|u\|_{\Omega} < 1$,

$$(4\pi - \delta)(1 + \alpha\|u\|_2^2)u^2 < (4\pi - \delta) \left(1 + \alpha \frac{\|u\|_2^2}{\|u\|_{\Omega}^2} \right) \frac{u^2}{\|u\|_{\Omega}^2},$$

and we obtain

$$\int_{\Omega} \left[e^{(4\pi-\delta)(1+\alpha\|u\|_2^2)u^2} - 1 \right] dx < \int_{\Omega} \left[e^{(4\pi-\delta) \left(1 + \alpha \frac{\|u\|_2^2}{\|u\|_{\Omega}^2} \right) \frac{u^2}{\|u\|_{\Omega}^2}} - 1 \right] dx,$$

which is a contradiction with (3.10). This completes the proof of Assertion 1.

The fact that $u \in C^{1,\theta}(\overline{\Omega})$ follows by standard regularity theory applied to the Euler-Lagrange equation verified by u . $\square$

For easy reference we introduce the following notation: Let $\delta_k \searrow 0$, B_{R_k} be the ball of $\mathbb{R}^2$ centered at the origin with radius R_k with $R_k \nearrow +\infty$ as $k \rightarrow +\infty$ and

$$S(R_k) = \left\{ v \in W_0^{1,2}(B_{R_k}) : \int_{B_{R_k}} (|\nabla v|^2 + V(x)|v|^2) dx = 1 \right\}.$$

Since any function in $S(R_k)$ can be extended by zero outside of B_{R_k} , we have $S(R_k) \subset S(R_{k+1}) \subset E$. Thus, $L_{\alpha, \delta_k}(B_{R_k}) \leq L_{\alpha, \delta_{k+1}}(B_{R_{k+1}})$. Therefore, using $\mu_1(B_{R_k}) \geq \lambda_1(V)$ and taking $\Omega = B_{R_k}$ in Theorem 3.2, we can state:

Corollary 3.3. *For any $\alpha \in [0, \lambda_1(V))$ and $k \in \mathbb{N}$, there exist a positive function $u_k \in S(R_k)$ such that $u_k \in C^{1, \theta}(\overline{B}_{R_k})$, for some $\theta \in (0, 1)$, and*

$$\begin{aligned} L_{\alpha, \delta_k}(B_{R_k}) &= \int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx \\ &= \int_{\mathbb{R}^2} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx. \end{aligned}$$

Moreover, the sequence $L_{\alpha, \delta_k}(B_{R_k})$ is increasing.

3.1. Proof of Proposition 3.1. Let η be a cut-off function such that $\eta \in C_0^\infty(\mathbb{R}^2, [0, 1])$ and

$$\eta(x) = \begin{cases} 1, & \text{if } x \in B_1, \\ 0, & \text{if } x \in \mathbb{R}^2 \setminus B_2. \end{cases}$$

Then given any $\varphi \in E$ with $\|\varphi\| = 1$, we have

$$\tau^2(L) := \int_{\mathbb{R}^2} \left(|\nabla(\eta(x/L)\varphi)|^2 + V(x) |\eta(x/L)\varphi|^2 \right) dx \rightarrow 1, \text{ as } L \rightarrow +\infty.$$

Indeed, using an argument of density, we can assume that $\varphi \in C_0^\infty(\mathbb{R}^2)$; hence there exists $R > 0$ such that $\text{supp}(\varphi) \subset B_R$, and consequently when $L > R$ we get $\tau^2(L) = 1$.

Next, for a fixed L and $R_k > 2L$, using the fact that $\|\eta(\frac{x}{L})\frac{\varphi}{\tau(L)}\| = 1$ and choosing u_k as an extremal function for $L_{\alpha, \delta_k}(B_{R_k})$, we get

$$\begin{aligned} &\int_{B_L} \left[e^{(4\pi - \delta_k)(1 + \alpha \|\frac{\varphi}{\tau(L)}\|_2^2)} |\frac{\varphi}{\tau(L)}|^2 - 1 \right] dx \\ &\leq \int_{B_{2L}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|\eta(\frac{x}{L})\frac{\varphi}{\tau(L)}\|_2^2)} |\eta(\frac{x}{L})\frac{\varphi}{\tau(L)}|^2 - 1 \right] dx \\ &\leq \int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx, \end{aligned}$$

which together with the Fatou Lemma implies

$$\int_{B_L} \left[e^{4\pi(1 + \alpha \|\frac{\varphi}{\tau(L)}\|_2^2)} |\frac{\varphi}{\tau(L)}|^2 - 1 \right] dx \leq \liminf_{k \rightarrow +\infty} \int_{\mathbb{R}^2} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx.$$

Then, letting $L \rightarrow +\infty$ by the Fatou Lemma, we get

$$\int_{\mathbb{R}^2} \left[e^{4\pi(1 + \alpha \|\varphi\|_2^2)} |\varphi|^2 - 1 \right] dx \leq \liminf_{k \rightarrow +\infty} \int_{\mathbb{R}^2} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx.$$

Hence

$$\ell(\alpha) \leq \liminf_{k \rightarrow +\infty} \int_{\mathbb{R}^2} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx,$$

thanks to

$$\limsup_{k \rightarrow +\infty} \int_{\mathbb{R}^2} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx \leq \ell(\alpha).$$

This completes the proof of (3.1).

For a fixed $k \in \mathbb{N}$, we consider V defined in the interval $[0, R_k]$ and we denote by $V_\#$ the increasing rearrangement of V . Using a similar argument as in the proof of

[17, Lemma 1.1.1], one can see that $V = V_{\#}$ almost everywhere in $[0, R_k]$, since the potential $V(x) = V(|x|)$ is continuous, nondecreasing and a positive function in the interval $[0, R_k]$. Thus by an inequality of Hardy-Littlewood ([17, Corollary 1.4.1]) we have

$$\int_{B_{R_k}} V(x) |u_k^*|^2 dx \leq \int_{B_{R_k}} V(x) |u_k|^2 dx.$$

Consequently, we have

$$(3.11) \quad \tau_k^2 := \int_{B_{R_k}} (|\nabla u_k^*|^2 + V(x) |u_k^*|^2) dx \leq \int_{B_{R_k}} (|\nabla u_k|^2 + V(x) |u_k|^2) dx = 1.$$

Thus, to complete the proof of Proposition 3.1, it is sufficient to show that $\tau_k = 1$, because we can assume that u_k is radially symmetric and radially decreasing, since

$$(3.12) \quad \int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k^*\|_2^2)} |u_k^*|^2 - 1 \right] dx = \int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx.$$

Using (3.11) and (3.12), we get

$$(3.13) \quad \int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k^*/\tau_k\|_2^2)} |u_k^*/\tau_k|^2 - 1 \right] dx \geq \int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx.$$

Since $u_k^*/\tau_k \in S(R_k)$, we have

$$(3.14) \quad \int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k^*/\tau_k\|_2^2)} |u_k^*/\tau_k|^2 - 1 \right] dx \leq \int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx.$$

From (3.12), (3.13) and (3.14), we get

$$\int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k^*/\tau_k\|_2^2)} |u_k^*/\tau_k|^2 - 1 \right] dx = \int_{B_{R_k}} \left[e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} u_k^2 - 1 \right] dx,$$

which implies that $\tau_k = 1$. Therefore, we can assume that u_k is a radially symmetric and radially decreasing function.

4. BLOW-UP ANALYSIS

In this section we analyze the behavior of the maximizers u_k described in Proposition 3.1 by using blow-up analysis in the case $\alpha \in [0, \lambda_1(V))$. Our analysis is composed of several lemmas.

Let $(u_k) \subset E$ be a maximizing sequence for $\ell(\alpha)$ as in Proposition 3.1. Up to a subsequence, we have $u_k \rightharpoonup u$ in E . Next we state the main result of this section.

Proposition 4.1. *The function u is an extremal for $\ell(\alpha)$, that is,*

$$(4.1) \quad \ell(\alpha) = \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2} (e^{(4\pi - \delta_k)(1 + \alpha \|u_k\|_2^2)} |u_k|^2 - 1) dx = \int_{\mathbb{R}^2} (e^{4\pi(1 + \alpha \|u\|_2^2)} u^2 - 1) dx.$$

Note that if $\|u\| = 1$, the proposition follows by the same argument as in the proof of Theorem 2.1. Thus, we assume from now on that $u_k \rightharpoonup u$ weakly in E and $\|u\| < 1$. In this case, the proof of (4.1) is very involved and will be done in several steps in the next subsections. We have to consider two cases: c_k is a bounded sequence or $c_k \nearrow +\infty$ (up to subsequence), where

$$(4.2) \quad c_k = u_k(0) \quad (\text{maximum point}).$$

Next, we summarize some results about the maximizing sequence (u_k) for $\ell(\alpha)$: u_k satisfies the Euler-Lagrange equation

$$(4.3) \quad \left\{ \begin{array}{ll} -\Delta u_k + V(x)u_k = \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} + \gamma_k u_k & \text{in } B_{R_k}, \\ \|u_k\| = 1, \quad u_k > 0 & \text{in } B_{R_k}, \\ \alpha_k = (4\pi - \delta_k)(1 + \alpha\|u_k\|_2^2), \\ \beta_k = \frac{1 + \alpha\|u_k\|_2^2}{1 + 2\alpha\|u_k\|_2^2}, \\ \gamma_k = \frac{\alpha}{1 + 2\alpha\|u_k\|_2^2}, \\ \lambda_k = \int_{B_{R_k}} u_k^2 e^{\alpha_k u_k^2} \, dx. \end{array} \right.$$

First, we need to prove that the sequence (λ_k) is bounded away from zero; more precisely:

Lemma 4.2. $\inf_k \lambda_k > 0$.

Proof. Assume by contradiction that $\lambda_k \rightarrow 0$. Thus

$$(4.4) \quad \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2} (e^{\alpha_k u_k^2} - 1) \, dx \leq 4\pi(1 + \alpha) \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2} u_k^2 e^{\alpha_k u_k^2} \, dx = 0,$$

where in the last estimate we have used the elementary inequality $(e^t - 1) \leq te^t$ for $t \geq 0$. Note that from (4.4) we have $\ell(\alpha) = 0$, which is impossible, because for $\phi \in C_0^\infty(B_1)$ with $\phi > 0$ in B_1 and $\|\phi\| = 1$, we have

$$\ell(\alpha) \geq \int_{B_1} (e^{4\pi(1+\alpha\|\phi\|_2^2)\phi^2} - 1) \, dx > 0.$$

□

4.1. The case: (c_k) is a bounded sequence. The aim of this subsection is to prove the first part of Theorem 1.1 if $\alpha \in [0, \lambda_1(V))$ and Theorem 1.3 in the case that (c_k) is bounded.

Lemma 4.3. *Suppose that (c_k) is bounded; then $\ell(\alpha)$ is attained.*

Proof. Let us denote

$$I_k = \left| \int_{\mathbb{R}^2} \left[e^{\alpha_k u_k^2} - 1 - \alpha_k u_k^2 \right] \, dx - \int_{\mathbb{R}^2} \left[e^{4\pi(1+\alpha\|u\|_2^2)u^2} - 1 - 4\pi(1 + \alpha\|u\|_2^2)u^2 \right] \, dx \right|.$$

Then

$$\begin{aligned} I_k &\leq \left| \int_{B_L} \left[e^{\alpha_k u_k^2} - 1 - \alpha_k u_k^2 \right] \, dx \right. \\ &\quad \left. - \int_{B_L} \left[e^{4\pi(1+\alpha\|u\|_2^2)u^2} - 1 - 4\pi(1 + \alpha\|u\|_2^2)u^2 \right] \, dx \right| \\ &\quad + \left| \int_{\mathbb{R}^2 \setminus B_L} \left[e^{\alpha_k u_k^2} - 1 - \alpha_k u_k^2 \right] \, dx \right| \\ &\quad + \left| \int_{\mathbb{R}^2 \setminus B_L} \left[e^{4\pi(1+\alpha\|u\|_2^2)u^2} - 1 - 4\pi(1 + \alpha\|u\|_2^2)u^2 \right] \, dx \right|. \end{aligned}$$

Since (c_k) is bounded, then using elliptic regularity, we have $u_k \rightarrow u$ strongly in $C^{1,\theta}(B_L)$ for some $\theta \in (0, 1)$, up to a subsequence. Moreover, using (1.9) we have $u_k \rightarrow u$ strongly in $L^2(\mathbb{R}^2)$ and

$$(4.5) \quad \lim_{k \rightarrow +\infty} \left| \int_{B_L} \left[e^{\alpha_k u_k^2} - 1 - \alpha_k u_k^2 \right] dx - \int_{B_L} \left[e^{4\pi(1+\alpha\|u\|_2^2)u^2} - 1 - 4\pi(1+\alpha\|u\|_2^2)u^2 \right] dx \right| = 0.$$

Since u_k is a radially decreasing function, we have

$$u_k^2(L)|B_L| \leq \int_{B_L} u_k^2 dx \leq 1.$$

Then given $\epsilon > 0$, we find L such that $u_k(x) \leq \epsilon$ for all $x \in \mathbb{R}^2 \setminus B_L$. Using the elementary inequality

$$e^{at^2} - 1 - at^2 \leq Ct^4 \quad \text{for any } t \in [0, \epsilon]$$

(where $C > 0$ and $\epsilon > 0$ is sufficiently small) we obtain

$$(4.6) \quad \int_{\mathbb{R}^2 \setminus B_L} \left[e^{\alpha_k u_k^2} - 1 - \alpha_k u_k^2 \right] dx \leq C\epsilon^2,$$

and for $L > 0$ sufficiently large,

$$(4.7) \quad \left| \int_{\mathbb{R}^2 \setminus B_L} \left[e^{4\pi(1+\alpha\|u\|_2^2)u^2} - 1 - 4\pi(1+\alpha\|u\|_2^2)u^2 \right] dx \right| \leq \epsilon.$$

From (4.5), (4.6) and (4.7), we obtain that $I_k \rightarrow 0$, which implies

$$(4.8) \quad \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2} \left(e^{\alpha_k u_k^2} - 1 \right) dx = \int_{\mathbb{R}^2} \left(e^{4\pi(1+\alpha\|u\|_2^2)u^2} - 1 \right) dx.$$

Thus, u is an extremal function for $\ell(\alpha)$. □

4.2. The case: $c_k \nearrow +\infty$. The aim of this section is to prove that the case

$$(4.9) \quad c_k \nearrow +\infty \quad \text{as } k \rightarrow \infty$$

does not occur and consequently by the previous sections follows the proof of the first part of Theorem 1.1 and Theorem 1.3.

In order to perform a blow-up procedure, we define

$$(4.10) \quad r_k^2 = \frac{\lambda_k}{\beta_k c_k^2 e^{\alpha_k c_k^2}}$$

and $\sigma_k = 4\pi - \delta_k$, where $\delta_k \searrow 0$ (see Proposition 3.1).

Lemma 4.4. *Assume that (4.9) holds; then $r_k^2 e^{\frac{\sigma_k}{2} c_k^2} \rightarrow 0$ and consequently $r_k \rightarrow 0$.*

Proof. By the definition of r_k and λ_k , we have

$$r_k^2 e^{\frac{\sigma_k}{2} c_k^2} = \beta_k^{-1} c_k^{-2} e^{-(\alpha_k - \frac{\sigma_k}{2}) c_k^2} \int_{\mathbb{R}^2} u_k^2 e^{\alpha_k u_k^2} dx.$$

Given $L > 0$, we write

$$(4.11) \quad \begin{aligned} r_k^2 e^{\frac{\sigma_k}{2} c_k^2} &= \beta_k^{-1} c_k^{-2} e^{-(\alpha_k - \frac{\sigma_k}{2}) c_k^2} \int_{B_L} u_k^2 e^{\alpha_k u_k^2} dx \\ &\quad + \beta_k^{-1} c_k^{-2} e^{-(\alpha_k - \frac{\sigma_k}{2}) c_k^2} \int_{\mathbb{R}^2 \setminus B_L} u_k^2 e^{\alpha_k u_k^2} dx. \end{aligned}$$

Note that

$$(4.12) \quad \int_{\mathbb{R}^2 \setminus B_L} u_k^2 e^{\alpha_k u_k^2} dx = \sum_{j=0}^{\infty} \alpha_k^j \int_{\mathbb{R}^2 \setminus B_L} \frac{|u_k|^{2j+2}}{j!} dx.$$

By the Radial Lemma (cf. [4] and [32]), we have

$$|u_k(\rho)| \leq \frac{1}{\sqrt{\pi}} \|u_k\|_2 \frac{1}{\rho}, \quad \text{for all } \rho > 0.$$

Hence

$$\int_{\mathbb{R}^2 \setminus B_L} |u_k|^{2j+2} dx \leq \|u_k\|_2^{2j+2} \frac{2}{\pi^j} \int_L^{+\infty} \rho^{-2j-1} d\rho \leq \frac{1}{\pi^j j L^{2j}},$$

which together with (4.12) implies

$$(4.13) \quad \int_{\mathbb{R}^2 \setminus B_L} u_k^2 e^{\alpha_k u_k^2} dx \leq \sum_{j=0}^{\infty} \frac{\alpha_k^j}{j! \pi^j j L^{2j}} \leq \sum_{j=0}^{\infty} \frac{\alpha_k^j}{j! \pi^j j L^{2j}} \leq C_1(L).$$

To estimate the first integral in (4.11), we use the fact that

$$-(\alpha_k - \frac{\sigma_k}{2}) c_k^2 \leq -(\alpha_k - \frac{\sigma_k}{2}) u_k^2(x) \quad \text{for all } x \in \mathbb{R}^2$$

and note that

$$\int_{B_L} u_k^2 e^{-(\alpha_k - \frac{\sigma_k}{2}) u_k^2} e^{\alpha_k u_k^2} dx = \int_{B_L} u_k^2 e^{\frac{\sigma_k}{2} u_k^2} dx.$$

Since $\int_{B_L} |\nabla(u_k - u_k(L))^+| dx \leq 1$, using the classical Trudinger-Moser inequality we get

$$(4.14) \quad \int_{B_L} e^{4\pi[(u_k - u_k(L))^+]^2} dx \leq C(L).$$

For any $p < 4\pi$ we can find a constant $C(p)$ such that

$$p u_k^2 \leq 4\pi[(u_k - u_k(L))^+]^2 + C(p),$$

which together with (4.14) implies that

$$\int_{B_L} e^{p u_k^2} dx \leq C(L, p).$$

Therefore, there exists $C_2 = C_2(L) > 0$ such that

$$\int_{B_L} u_k^2 e^{\frac{\sigma_k}{2} u_k^2} dx \leq C_2,$$

which together with (4.11) and (4.13) implies

$$r_k^2 e^{\frac{\sigma_k}{2} c_k^2} \leq C_2(L) \beta_k^{-1} c_k^{-2} + C_1(L) \beta_k^{-1} c_k^{-2} e^{-(\alpha_k - \frac{\sigma_k}{2}) c_k^2}.$$

Since β_k is bounded and $c_k \nearrow +\infty$, the proof is complete. $\square$

Lemma 4.5. *Suppose that (4.9) holds; then $u_k \rightharpoonup 0$ weakly in E and $u_k \rightarrow 0$ strongly in $L^t(\mathbb{R}^2)$ for all $t \geq 2$. Moreover, we have $\alpha_k \rightarrow 4\pi$, $\beta_k \rightarrow 1$ and $\gamma_k \rightarrow \alpha$.*

Proof. We know that

$$u_k \rightharpoonup u \quad \text{weakly in } E$$

and

$$u_k \rightarrow u \quad \text{strongly in } L^2(\mathbb{R}^2).$$

Suppose $u \not\equiv 0$. Then, using the characterization of $\lambda_1(V)$, for any $0 \leq \alpha < \lambda_1(V)$ we have

$$1 + \alpha \|u_k\|_2^2 \rightarrow 1 + \alpha \|u\|_2^2 \leq 1 + \|u\|^2 < \frac{1}{1 - \|u\|^2}.$$

By Theorem 2.1, $e^{\alpha_k u_k^2}$ is bounded in $L^r(\mathbb{R}^2)$ for some $r > 1$ sufficiently close to 1, which together with (4.3) implies that Δu_k is bounded in $L^{r_0}(B_R)$ for some $r_0 > 1$ sufficiently close to 1 and $R > 0$. Employing the elliptic estimates to (4.3), one gets that u_k is uniformly bounded, which contradicts (4.9). Therefore $u \equiv 0$, and consequently $\alpha_k \rightarrow 4\pi$, $\beta_k \rightarrow 1$ and $\gamma_k \rightarrow \alpha$. $\square$

In order to continue our analysis we define the blow-up functions

$$(4.15) \quad \begin{cases} \psi_k(x) = c_k^{-1} u_k(r_k x), \\ \varphi_k(x) = c_k(u_k(r_k x) - c_k), \end{cases}$$

where ψ_k and φ_k are defined on $\Omega_k = \{x \in \mathbb{R}^2 : r_k x \in B_1\}$. A direct computation gives

$$(4.16) \quad \begin{cases} -\Delta \psi_k(x) = c_k^{-2} \psi_k(x) e^{\alpha_k(u_k^2(r_k x) - c_k^2)} + [\gamma_k - V(x)] r_k^2 \psi_k(x) & \text{in } \Omega_k, \\ -\Delta \varphi_k(x) = \psi_k(x) e^{\alpha_k(u_k^2(r_k x) - c_k^2)} + [\gamma_k - V(x)] r_k^2 c_k^2 \psi_k(x) & \text{in } \Omega_k. \end{cases}$$

By Lemmas 4.4 and 4.5, applying the elliptic estimates (see, for instance, [15], Chapter 9) to (4.16), we obtain

$$(4.17) \quad \begin{cases} \psi_k \rightarrow 1, & \text{in } C_{loc}^{1,\theta}(\mathbb{R}^2), \\ \varphi_k \rightarrow \varphi, & \text{in } C_{loc}^{1,\theta}(\mathbb{R}^2). \end{cases}$$

Moreover, we can prove that φ satisfies

$$(4.18) \quad \begin{cases} -\Delta \varphi = e^{8\pi\varphi}, & \text{in } \mathbb{R}^2, \\ \varphi(0) = 0 = \sup \varphi, \\ \int_{\mathbb{R}^2} e^{8\pi\varphi} \, dx \leq 1. \end{cases}$$

To obtain the last inequality in (4.18), one can use the definition of λ_k and the fact that $\beta_k \rightarrow 1$

$$\int_{B_R} e^{8\pi\varphi} \, dx = \lim_{k \rightarrow +\infty} \frac{\beta_k}{\lambda_k} \int_{B_{r_k R}} u_k^2 e^{\alpha_k u_k^2} \, dx, \quad \text{for any } R > 0.$$

Furthermore, the uniqueness of the solutions of (4.18) due to Chen-Li [7] implies that

$$(4.19) \quad \begin{cases} \varphi(x) = -\frac{1}{4\pi} \log(1 + \pi|x|^2), \\ \int_{\mathbb{R}^2} e^{8\pi\varphi} \, dx = 1. \end{cases}$$

In the next lemmas we use the notation

$$u_k^A = \min \{u_k, c_k/A\} \quad \text{for } A > 1.$$

Lemma 4.6. *Assume that (4.9) holds; then*

$$(4.20) \quad \limsup_{k \rightarrow +\infty} \int_{\mathbb{R}^2} (|\nabla u_k^A|^2 + V(x)|u_k^A|^2) \, dx \leq \frac{1}{A}.$$

Proof. To obtain (4.20), we write

$$(4.21) \quad \int_{\mathbb{R}^2} (|\nabla u_k^A|^2 + V(x)|u_k^A|^2) \, dx = 1 - J_{1,k} + J_{2,k} - J_{3,k} + J_{4,k},$$

where

$$\begin{aligned} J_{1,k} &:= \left\{ \int_{\mathbb{R}^2} \left[\left| \nabla \left(u_k - \frac{c_k}{A} \right)^+ \right|^2 + V(x) \left(u_k - \frac{c_k}{A} \right)^+ u_k \right] \, dx \right\}, \\ J_{2,k} &:= \int_{\mathbb{R}^2} V(x) \left(u_k - \frac{c_k}{A} \right)^+ u_k \, dx, \\ J_{3,k} &:= \int_{[u_k > c_k/A]} V(x) u_k^2 \, dx, \\ J_{4,k} &:= \int_{[u_k > c_k/A]} V(x) |u_k^A|^2 \, dx. \end{aligned}$$

In order to estimate these integrals we note that

$$\left| \left\{ x \in \mathbb{R}^2 : u_k(x) \geq \frac{c_k}{A} \right\} \right| \left(\frac{c_k}{A} \right)^2 = \int_{[u_k \geq c_k/A]} \left(\frac{c_k}{A} \right)^2 \, dx \leq \int_{[u_k \geq c_k/A]} u_k^2 \, dx \leq 1,$$

which implies that

$$\left| \left\{ x \in \mathbb{R}^2 : u_k(x) \geq \frac{c_k}{A} \right\} \right| \leq \frac{A^2}{c_k^2} \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Thus, we can find a sequence $\rho_k \searrow 0$ such that

$$\left\{ x \in \mathbb{R}^2 : u_k(x) \geq \frac{c_k}{A} \right\} \subset B_{\rho_k}.$$

Using the fact that $u_k \rightharpoonup u$ weakly in E , then u_k converges strongly in $L_{loc}^p(\mathbb{R}^2)$ for any $p > 1$. Therefore, for any $R > 0$, there exists $h \in L^p(B_R)$ such that $u_k(x) \leq |h(x)|$ almost everywhere in B_R . Hence, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{k \rightarrow +\infty} \int_{[u_k > \frac{c_k}{A}]} V(x) |u_k^A|^p \, dx &= 0, \\ \lim_{k \rightarrow +\infty} \int_{[u_k > \frac{c_k}{A}]} u_k^p \, dx &= 0. \end{aligned}$$

Therefore, $J_{2,k} \rightarrow 0$, $J_{3,k} \rightarrow 0$ and $J_{4,k} \rightarrow 0$.

Now, testing equation (4.3) with $(u_k - c_k/A)^+$ and setting $L_k = r_k L$, we obtain

$$\begin{aligned} J_{1,k} &= \int_{\mathbb{R}^2} \left(u_k - \frac{c_k}{A} \right)^+ \left(\frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} + \gamma_k u_k \right) \, dx \\ &\geq \int_{B_{L_k}} \left(u_k - \frac{c_k}{A} \right)^+ \left(\frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} + \gamma_k u_k \right) \, dx. \end{aligned}$$

Performing the change of variables $x = r_k y$, we get by (4.10)

$$\begin{aligned} & \int_{B_{L_k}} \left(u_k - \frac{c_k}{A}\right)^+ \left(\frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} + \gamma_k u_k\right) dx \\ &= \int_{B_L} \left(u_k(r_k y) - \frac{c_k}{A}\right)^+ \left(\frac{\beta_k}{\lambda_k} u_k(r_k y) e^{\alpha_k u_k^2(r_k y)} + \gamma_k u_k(r_k y)\right) r_k^2 dy \\ &= \int_{B_L} \left(c_k \psi_k(y) - \frac{c_k}{A}\right)^+ \left(\frac{c_k^{-1}}{r_k^2} \psi_k(y) e^{\alpha_k (u_k^2(r_k y) - c_k^2)} + \gamma_k u_k(r_k y)\right) r_k^2 dy, \end{aligned}$$

which implies that

$$J_{1,k} \geq \int_{B_L} \frac{A\psi_k(x) - 1}{A} \psi_k e^{\alpha_k (\psi_k + 1)\varphi_k} dx + o_k(1),$$

where $o_k(1) \rightarrow 0$ as $k \rightarrow \infty$. This inequality together with (4.17) gives that

$$\liminf_{k \rightarrow +\infty} J_{1,k} \geq \frac{A-1}{A} \int_{B_L} e^{8\pi\varphi} dx.$$

Letting $L \rightarrow +\infty$, we get

$$(4.22) \quad \liminf_{k \rightarrow +\infty} J_{1,k} \geq \frac{A-1}{A}.$$

Using the fact that $J_{2,k} \rightarrow 0$, $J_{3,k} \rightarrow 0$, $J_{4,k} \rightarrow 0$, and (4.21) and (4.22), we get

$$\limsup_{k \rightarrow +\infty} \int_{\mathbb{R}^2} (|\nabla u_k^A|^2 + V(x)|u_k^A|^2) dx \leq 1 - \left(1 - \frac{1}{A}\right) + o_k(1) = \frac{1}{A} + o_k(1).$$

Hence, the proof is complete. $\square$

Corollary 4.7. Assume that (4.9) holds; then for any $c > 0$, up to subsequence,

$$\lim_{k \rightarrow +\infty} \int_{[u_k \leq c]} (|\nabla u_k|^2 + V(x)|u_k|^2) dx = 0.$$

Proof. Let $c > 0$; using the fact that $c_k \nearrow +\infty$ and $u_k^A = \min\{u_k, c_k/A\}$, there exists $k_0 \in \mathbb{N}$ such that $c_k/A > c$ for all $k \geq k_0$. Thus $u_k^A(x) = u_k(x)$ in $\{x \in \mathbb{R}^2 : u_k(x) \leq c\}$ for all $k \geq k_0$, and consequently

$$\lim_{k \rightarrow +\infty} \int_{[u_k \leq c]} (|\nabla u_k|^2 + V(x)|u_k|^2) dx = \lim_{k \rightarrow +\infty} \int_{[u_k \leq c]} (|\nabla u_k^A|^2 + V(x)|u_k^A|^2) dx \leq \frac{1}{A},$$

where in the last estimate we have used (4.20). Letting $A \rightarrow +\infty$, then for any c , we have

$$\lim_{k \rightarrow +\infty} \int_{[u_k \leq c]} (|\nabla u_k|^2 + |u_k|^2) dx = 0,$$

and the proof is complete. $\square$

Remark 4.8. Using the fact that u_k is radially symmetric and radially decreasing we see that

$$\{x \in \mathbb{R}^2 : u_k(x) \leq c\} = \mathbb{R}^2 \setminus B_{r(c)},$$

for some $r = r(c) > 0$. Thus, assuming (4.9), for any $\delta > 0$ it holds that

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2 \setminus B_\delta} (|\nabla u_k|^2 + V(x)|u_k|^2) dx = 0.$$

Lemma 4.9. *Suppose that (4.9) holds; then*

$$(4.23) \quad \limsup_{k \rightarrow +\infty} \int_{\mathbb{R}^2} (e^{\alpha_k u_k^2} - 1) \, dx \leq \limsup_{k \rightarrow +\infty} \frac{\lambda_k}{c_k^2}$$

and consequently

$$(4.24) \quad \frac{\lambda_k}{c_k} \rightarrow +\infty \text{ and } \sup_k \frac{c_k^2}{\lambda_k} < +\infty.$$

Proof. Notice that

$$(4.25) \quad \begin{aligned} \int_{\mathbb{R}^2} (e^{\alpha_k u_k^2} - 1) \, dx &= \int_{[u_k \leq \frac{c_k}{A}]} (e^{\alpha_k u_k^2} - 1) \, dx + \int_{[u_k > \frac{c_k}{A}]} (e^{\alpha_k u_k^2} - 1) \, dx \\ &\leq \int_{\mathbb{R}^2} (e^{\alpha_k |u_k^A|^2} - 1) \, dx + A^2 \frac{\lambda_k}{c_k^2} \int_{\mathbb{R}^2} \frac{u_k^2 e^{\alpha_k u_k^2}}{\lambda_k} \, dx \\ &= \int_{\mathbb{R}^2} (e^{\alpha_k |u_k^A|^2} - 1) \, dx + A^2 \frac{\lambda_k}{c_k^2}. \end{aligned}$$

Since u_k is radial and decreasing, we can find L such that $u_k \leq 1$ on $\mathbb{R}^2 \setminus B_L$; then $u_k^A = u_k$ on $\mathbb{R}^2 \setminus B_L$. Hence by Corollary 4.7 and the Radial Lemma (cf. [4] and [32]), given $p > 0$ there exists $C(p) > 0$ such that

$$(4.26) \quad \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2 \setminus B_L} (e^{p\alpha_k |u_k^A|^2} - 1) \, dx \leq \lim_{k \rightarrow +\infty} C(p) \int_{\mathbb{R}^2 \setminus B_L} u_k^2 \, dx = 0.$$

By Lemma 4.6, we have

$$\limsup_{k \rightarrow +\infty} \int_{\mathbb{R}^2} (|\nabla u_k^A|^2 + |u_k^A|^2) \, dx \leq \frac{1}{A} < 1,$$

which together with the Trudinger-Moser inequality (see (1.3)) and

$$\lim_{k \rightarrow +\infty} p' \alpha_k \|(u_k^A - u_k(L))^+\|^2 < 4\pi$$

implies

$$\sup_k \int_{B_L} e^{p' \alpha_k ((u_k^A - u_k(L))^+)^2} \, dx < +\infty \quad \text{for any } p' < A.$$

Therefore,

$$\sup_k \int_{B_L} (e^{p\alpha_k |u_k^A|^2} - 1) \, dx < +\infty \quad \text{for any } p < A,$$

since $p|u_k^A|^2 \leq p'|(u_k^A - u_k(L))|^2 + C(p, p')$.

Now, using the fact that $u_k^A \rightarrow 0$ almost everywhere in $\mathbb{R}^2$, by the same argument as in the proof of Theorem 3.2, we get

$$(4.27) \quad \lim_{k \rightarrow +\infty} \int_{B_L} (e^{\alpha_k |u_k^A|^2} - 1) \, dx = 0.$$

Hence from (4.25), (4.26) and (4.27), we get

$$\limsup_{k \rightarrow +\infty} \int_{\mathbb{R}^2} (e^{\alpha_k u_k^2} - 1) \, dx \leq \limsup_{k \rightarrow +\infty} A^2 \frac{\lambda_k}{c_k^2}.$$

Letting $A \rightarrow 1$, we see that (4.23) holds.

One can see that if $\frac{\lambda_k}{c_k}$ is bounded or $\sup_k \frac{c_k^2}{\lambda_k} = +\infty$, then by (4.23) we get

$$\ell(\alpha) \leq \frac{\lambda_k}{c_k^2} \rightarrow 0,$$

which is impossible and the proof of (4.24) is complete. $\square$

Lemma 4.10. *Suppose that (4.9) holds; then we have*

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2} \eta(x) \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} dx = \eta(0) \quad \text{for any } \eta \in C_0^\infty(\mathbb{R}^2).$$

Proof. Suppose $\text{supp}(\eta) \subset B_\rho$ and fixed $A > 1$. Setting $L_k = r_k L$, we split the integral

$$\int_{B_\rho} \eta(x) \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} dx = I_{1,k} + I_{2,k} + I_{3,k},$$

where

$$\begin{aligned} I_{1,k} &= \int_{[u_k \geq \frac{c_k}{A}] \setminus B_{L_k}} \eta(x) \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} dx, \\ I_{2,k} &= \int_{B_{L_k}} \eta(x) \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} dx, \\ I_{3,k} &= \int_{[u_k < \frac{c_k}{A}]} \eta(x) \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} dx. \end{aligned}$$

To complete the proof, it is enough to show that

$$\lim_{k \rightarrow \infty} I_{1,k} = \lim_{k \rightarrow \infty} I_{3,k} = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} I_{2,k} = \eta(0).$$

First we note that

$$|I_{1,k}| \leq \|\eta\|_\infty \int_{[u_k \geq \frac{c_k}{A}] \setminus B_{L_k}} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} dx.$$

Using the fact that $\lambda_k = \int_{B_{R_k}} u_k^2 e^{\alpha_k u_k^2} dx$, we obtain

$$|I_{1,k}| \leq A \beta_k \|\eta\|_\infty \left(1 - \int_{B_{L_k}} \frac{1}{\lambda_k} u_k^2 e^{\alpha_k u_k^2} dx \right),$$

which together with (4.10) and (4.15) implies

$$\begin{aligned} \int_{B_{L_k}} \frac{1}{\lambda_k} u_k^2 e^{\alpha_k u_k^2} dx &= \int_{B_L} \frac{1}{\lambda_k} u_k^2(r_k x) e^{\alpha_k u_k^2(r_k x)} r_k^2 dx \\ &= \frac{1}{\beta_k} \int_{B_L} \left(\frac{u_k(r_k x)}{c_k} \right)^2 e^{\alpha_k (u_k^2(r_k x) - c_k^2)} dx \\ &= \frac{1}{\beta_k} \int_{B_L} \psi_k^2(y) e^{\alpha_k (\psi_k(y) + 1) \varphi_k(y)} dy, \end{aligned}$$

where we have used the change of variables $y = r_k x$. Letting $k \rightarrow +\infty$ and using (4.17) and $\beta_k \rightarrow 1$, we get

$$|I_{1,k}| \leq A \|\eta\|_\infty \left(1 - \int_{B_L} e^{8\pi\varphi} dx \right),$$

which implies that

$$\lim_{k \rightarrow +\infty} |I_{1,k}| \leq A \|\eta\|_{\infty} \left(1 - \int_{B_L} e^{8\pi\varphi} \, dx \right).$$

Therefore, taking $L \rightarrow +\infty$ and using (4.19),

$$\lim_{k \rightarrow +\infty} |I_{1,k}| \leq \lim_{L \rightarrow +\infty} A \|\eta\|_{\infty} \left(1 - \int_{B_L} e^{8\pi\varphi} \, dx \right) = 0.$$

Now, using the change of variable $y = r_k x$ and using (4.10), (4.17) and (4.19), we have

$$\begin{aligned} I_{2,k} &= \int_{B_{L_k}} \eta(y) \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \, dy \\ &= \int_{B_L} \eta(r_k x) \frac{e^{-\alpha_k c_k^2}}{c_k^2 r_k^2} c_k u_k(r_k x) e^{\alpha_k u_k^2(r_k x)} r_k^2 \, dx \\ &= \eta(0) \int_{B_L} e^{8\pi\varphi} \, dx + o_k(1) = \eta(0) + o_k(1), \end{aligned}$$

where $o_k(1) \rightarrow 0$ as $k \rightarrow \infty$.

Notice that

$$\begin{aligned} I_{3,k} &= \int_{[u_k < \frac{c_k}{A}]} \eta(x) \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \, dx \\ &\leq \|\eta\|_{\infty} \beta_k \frac{c_k}{\lambda_k} \int_{[u_k < \frac{c_k}{A}]} u_k e^{\alpha_k u_k^2} \, dx \\ &\leq \|\eta\|_{\infty} \beta_k \frac{c_k}{\lambda_k} \int_{[u_k < \frac{c_k}{A}]} u_k^A e^{\alpha_k |u_k^A|^2} \, dx. \end{aligned}$$

By Lemma 4.6 and the Radial Lemma, we have

$$(4.28) \quad \sup_k \int_{[u_k < \frac{c_k}{A}]} u_k^A e^{\alpha_k p |u_k^A|^2} \, dx \leq C \quad \text{for any } p < A.$$

Then we get

$$I_{3,k} \leq C \|\eta\|_{\infty} \beta_k \frac{c_k}{\lambda_k}.$$

From (4.24) and (4.28), we conclude that $I_{3,k} \rightarrow 0$ as $k \rightarrow +\infty$, which completes the proof. $\square$

Using an argument of Struwe in [34], we have the following:

Lemma 4.11. *Suppose that (4.9) holds. Then for any $R > 0$ and $1 < p < 2$,*

$$c_k u_k \rightharpoonup G \quad \text{weakly in } W^{1,p}(B_R),$$

where G satisfies the equation

$$(4.29) \quad -\Delta G + V(x)G = \delta_0 + \alpha G,$$

in a distributional sense. Furthermore,

$$(4.30) \quad c_k u_k \rightarrow G \quad \text{strongly in } C_{loc}^{1,\theta}(\mathbb{R}^2 \setminus \{0\}).$$

Proof. Setting $U_k = c_k u_k$ and using (4.3), we have

$$(4.31) \quad -\Delta U_k + V(x)U_k = \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} + \gamma_k U_k \quad \text{in } B_{R_k}.$$

We first prove that for any $R > 0$ and $1 < p < 2$,

$$(4.32) \quad \int_{B_R} (|\nabla U_k|^p + |U_k|^p) \, dx \leq C(p, R).$$

For that matter we split the proof into two cases: $\alpha = 0$ and $0 < \alpha < \lambda_1(V)$.

Case 1 ($\alpha = 0$).

In this case, we have $\gamma_k = 0$ for all k and

$$-\Delta U_k + U_k \leq \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \quad \text{in } B_{R_k}.$$

Setting

$$\Omega_t = \{0 \leq U_k \leq t\} \quad \text{and} \quad U_k^t = \min\{U_k, t\} \in W_0^{1,2}(B_{R_k}),$$

for k sufficiently large, we obtain

$$(4.33) \quad \int_{\Omega_t} (|\nabla U_k^t|^2 + |U_k^t|^2) \, dx \leq \int_{B_{R_k}} U_k^t \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \, dx \leq t.$$

Indeed, let φ_R be a cut-off function such that $\varphi_R \in C_0^\infty(\mathbb{R}^2, [0, 1])$ and

$$\varphi_R(x) = \begin{cases} 1, & \text{if } x \in B_R, \\ 0, & \text{if } x \in \mathbb{R}^2 \setminus B_{2R}. \end{cases}$$

Using the Fatou Lemma, we have

$$\int_{B_{R_k}} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \, dx \leq \liminf_{R \rightarrow +\infty} \int_{B_{R_k}} \varphi_R \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \, dx.$$

Hence by Lemma 4.10,

$$\lim_{k \rightarrow +\infty} \int_{B_{R_k}} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \, dx \leq \liminf_{R \rightarrow +\infty} \varphi_R(0) = 1.$$

Consequently (4.33) holds for k sufficiently large.

Let η be a radially symmetric cut-off function such that $\eta \in C_0^\infty(\mathbb{R}^2, [0, 1])$ and

$$\eta(x) = \begin{cases} 1, & \text{if } x \in B_R, \\ 0, & \text{if } x \in \mathbb{R}^2 \setminus B_{2R}. \end{cases}$$

Now consider $t > 0$ sufficiently large such that $\mathbb{R}^2 \setminus \Omega_t \subset B_R$. Thus, there exists $C_1(R) > 0$ such that

$$\int_{B_{2R}} |\nabla(\eta U_k^t)|^2 \, dx \leq C_1(R)t.$$

Let $\rho_k > 0$ be such that $U_k(\rho_k) = t$. Then using the estimates above we have

$$(4.34) \quad \inf \left\{ \int_{B_{2R}} |\nabla v|^2 \, dx : v \in W_0^{1,2}(B_{2R}) \quad \text{and} \quad v|_{B_{\rho_k}} \equiv t \right\} \leq C_1(R)t.$$

Struwe proved in [34] that this infimum is attained by the function

$$V_k(x) = \begin{cases} \frac{-t \log\left(\frac{|x|}{2R}\right)}{\log\left(\frac{2R}{\rho_k}\right)} & \text{in } B_{2R} \setminus B_{\rho_k}, \\ t & \text{in } B_{\rho_k}. \end{cases}$$

Since

$$\nabla V_k(x) = \begin{cases} \frac{-tx}{\log\left(\frac{2R}{\rho_k}\right)|x|^2} & \text{in } B_{2R} \setminus B_{\rho_k}, \\ 0 & \text{in } B_{\rho_k}, \end{cases}$$

we obtain

$$\int_{B_{2R}} |\nabla V_k(x)|^2 dx = \frac{2\pi t^2}{\log\left(\frac{2R}{\rho_k}\right)},$$

which together with (4.34) implies

$$\rho_k \leq 2Re^{-2\pi t/C_1(R)},$$

and consequently

$$|\{x \in B_{2R} : U_k \geq t\}| = |B_{\rho_k}| \leq 4\pi R^2 e^{-A(R)t},$$

where $A(R) = 4\pi^2/C_1^2(R)$ is a positive constant depending only on R .

Let $m_1 \in \mathbb{N}$ such that $t \leq m_1$; then for any $\delta < A(R)$,

$$\begin{aligned} (4.35) \quad \int_{B_R} e^{\delta U_k} dx &\leq e^{\delta m_1} \pi R^2 + \sum_{m=m_1}^{\infty} |\{m \leq U_k \leq m+1\}| e^{\delta(m+1)} \\ &\leq e^{\delta m_1} \pi R^2 + 4\pi R^2 \sum_{m=m_1}^{\infty} e^{-(A(R)-\delta)m} e^{\delta} \leq C(R). \end{aligned}$$

Then, testing equation (4.3) with the function

$$\varphi(x) = \log \left[\frac{1 + 2(U_k(x) - U_k(R))^+}{1 + (U_k(x) - U_k(R))^+} \right],$$

we get

$$\begin{aligned} (4.36) \quad &\int_{B_R} \frac{|\nabla U_k|^2}{(1 + U_k - U_k(R))(1 + 2U_k - 2U_k(R))} dx \\ &\leq \log 2 \int_{B_R} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} dx - \int_{B_R} \log \left[\frac{1 + 2(U_k - U_k(R))^+}{1 + (U_k - U_k(R))^+} \right] U_k dx \leq C. \end{aligned}$$

Given $1 < p < 2$, by the Young inequality we have

$$\begin{aligned} &\int_{B_R} |\nabla U_k|^p dx \\ &\leq \int_{B_R} \left\{ \frac{|\nabla U_k|^2}{(1 + U_k - U_k(R))(1 + 2U_k - 2U_k(R))} + ((1 + U_k)(1 + 2U_k))^{2/(2-p)} \right\} dx \\ &\leq \int_{B_R} \left\{ \frac{|\nabla U_k|^2}{(1 + U_k - U_k(R))(1 + 2U_k - 2U_k(R))} + C e^{\delta U_k} \right\} dx, \end{aligned}$$

which together with (4.35) and (4.36) implies

$$(4.37) \quad \int_{B_R} |\nabla U_k|^p \, dx \leq C(p, R).$$

Claim: (U_k) is bounded in $L^1(\mathbb{R}^2)$. Suppose by contradiction that this is not the case. So, up to a subsequence, $\|U_k\|_1 \rightarrow +\infty$. Thus, setting $w_k = U_k/\|U_k\|_1$ and using (4.3) we obtain

$$(4.38) \quad -\Delta w_k \leq -\Delta w_k + V(x)w_k = \frac{1}{\|U_k\|_1} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \quad \text{in } B_R,$$

which together with Lemma 4.10 implies that $\Delta w_k \in L^1(B_R)$. Therefore, using the same argument as above, we can conclude that for any $R > 0$ and $1 < p < 2$,

$$\int_{B_R} |\nabla w_k|^p \, dx \leq C(p, R),$$

which together with the Gagliardo-Nirenberg inequality we can find $\theta(p) \in (0, 1)$ such that

$$\|w_k\|_p \leq C_1(p, R) \|w_k\|_1^{\theta(p)} \|\nabla w_k\|_p^{1-\theta(p)} \leq C(p, R).$$

Thus (w_k) is bounded in $W^{1,p}(B_R)$ and consequently, up to a subsequence, $w_k \rightharpoonup w$ weakly in $W^{1,p}(B_R)$ and $w_k \rightarrow w$ strongly in $L^1(B_R)$.

Assertion. *Up to a subsequence, it holds that*

$$(4.39) \quad \int_{B_R} \nabla w_k \nabla \phi \, dx \rightarrow \int_{B_R} \nabla w \nabla \phi \, dx, \quad \forall \phi \in W^{1,p}(B_R).$$

Using the fact that $\|w_k\|_{1,p} \leq C(R)$, it is clear that to prove (4.39), it is enough to obtain

$$\nabla w_k \rightarrow \nabla w \text{ almost everywhere in } B_R,$$

which is a consequence of the following result:

$$(4.40) \quad \int_{B_R \setminus B_{r_0}} |\nabla w_k - \nabla w|^2 \, dx \rightarrow 0, \quad \forall r_0 > 0.$$

In order to prove (4.40), consider $0 < r < r_0$ and let $\varphi \in C_0^\infty(\mathbb{R}^2, [0, 1])$ be such that $\varphi \equiv 1$ in $B_{1/2}$ and $\varphi \equiv 0$ in $\overline{B_R} \setminus B_1$. Taking $\psi_r(x) = 1 - \varphi(x/r)$, we have $0 \leq \psi_r \leq 1$, $\psi_r \equiv 1$ in $A_r = \overline{B_R} \setminus B_r$, $\psi_r \equiv 0$ in $B_{r/2}$ and $\psi_r w_k$ is a bounded sequence in $W^{1,p}(B_R)$. Using (4.3) with $\phi = \psi_r w_k$, we have

$$\int_{B_R} [\nabla w_k \nabla (\psi_r w_k) + V(x) w_k^2 \psi_r] \, dx - \frac{1}{\|U_k\|_1} \int_{B_R} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \psi_r w_k \, dx = 0.$$

Thus

$$(4.41) \quad \begin{aligned} \int_{B_R} [|\nabla w_k|^2 \psi_r + w_k \nabla w_k \nabla \psi_r] \, dx + \int_{B_R} V(x) w_k^2 \psi_r \, dx \\ - \frac{1}{\|U_k\|_1} \int_{B_R} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \psi_r w_k \, dx = 0. \end{aligned}$$

Now, using (4.3) with $v = -\psi_r w$, we have

$$(4.42) \quad \int_{B_R} [-\psi_r \nabla w_k \nabla w - w \nabla w_k \nabla \psi_r] \, dx - \int_{B_R} V(x) w_k w \psi_r \, dx + \frac{1}{\|U_k\|_1} \int_{B_R} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \psi_r w \, dx = 0.$$

Note that

$$0 \leq \int_{B_R \setminus B_{r_0}} |\nabla w_k - \nabla w|^2 \, dx \leq \int_{B_R} (\nabla w_k - \nabla w)(\nabla w_k - \nabla w) \psi_r \, dx.$$

Consequently, we have

$$(4.43) \quad 0 \leq \int_{B_R} [|\nabla w_k|^2 \psi_r - 2\psi_r \nabla w_k \nabla w + |\nabla w|^2 \psi_r] \, dx.$$

From (4.41), (4.42) and (4.43), we obtain

$$\begin{aligned} 0 &\leq \int_{B_R} [-|\nabla w_k|^2 \psi_r - w_k \nabla w_k \nabla \psi_r] \, dx - \int_{B_R} V(x) w_k^2 \psi_r \, dx \\ &\quad + \frac{1}{\|U_k\|_1} \int_{B_R} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \psi_r w_k \, dx \\ &\quad + \int_{B_R} [\psi_r \nabla w_k \nabla w + w \nabla w_k \nabla \psi_r] \, dx + \int_{B_R} V(x) w_k w \psi_r \, dx \\ &\quad - \frac{1}{\|U_k\|_1} \int_{B_R} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \psi_r w \, dx \\ &\quad + \int_{B_R} [|\nabla w_k|^2 \psi_r - 2\psi_r \nabla w_k \nabla w + |\nabla w|^2 \psi_r] \, dx. \end{aligned}$$

Therefore,

$$(4.44) \quad \begin{aligned} 0 &\leq \int_{B_R} \nabla w_k \nabla \psi_r (w_k - w) \, dx + \int_{B_R} \psi_r \nabla w (\nabla w - \nabla w_k) \, dx \\ &\quad + \int_{B_R} V(x) w_k (w - w_k) \psi_r \, dx + \frac{1}{\|U_k\|_1} \int_{B_R} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \psi_r (w_k - w) \, dx. \end{aligned}$$

Now we estimate each integral in (4.44) separately. First we have

$$\int_{B_R} \nabla w_k \nabla \psi_r (w_k - w) \, dx \leq C(R) \|w_k\|_{1,p} \|w_k - w\|_{p'} \rightarrow 0$$

and

$$\int_{B_R} V(x) w_k (w - w_k) \psi_r \, dx \leq C(R) \|w_k\|_{1,p} \|w_k - w\|_{p'} \rightarrow 0,$$

since $w_k \rightarrow w$ in $L^{p'}(B_R)$, where $p' = p/(p-1)$. Using the fact that $w_k \rightharpoonup w$ in $W^{1,p}(B_R)$, we have

$$\int_{B_R} \psi_r \nabla w (\nabla w - \nabla w_k) \, dx \rightarrow 0.$$

Using (4.51) we have that given $1 < p < 2$,

$$\int_{B_R} \left(\frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \psi_r \right)^p \, dx \leq C(R);$$

consequently, we get

$$\frac{1}{\|U_k\|_1} \int_{B_R} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \psi_r(w_k - w) \, dx \rightarrow 0.$$

Thus, from (4.44) we come to the conclusion that (4.40) holds, and hence Assertion (4.39) is proved.

Therefore, taking the limit in (4.38) and using (4.39) and Lemma 4.10, we get

$$\int_{B_R} [\nabla w \nabla \phi + V(x)w\phi] \, dx = 0, \quad \forall \phi \in W^{1,p}(B_R).$$

Since R is arbitrary, we have

$$(4.45) \quad \int_{\mathbb{R}^2} [\nabla w \nabla \phi + V(x)w\phi] \, dx = 0, \quad \forall \phi \in W^{1,p}(\mathbb{R}^2).$$

Consequently, we obtain $w = 0$, which contradicts the fact that $\|w\|_1 = 1$. Hence

$$(4.46) \quad \|U_k\|_1 \leq C.$$

Thus the claim is proved.

Now, using (4.37), (4.46) and the Gagliardo-Nirenberg inequality, we conclude that

$$\int_{B_R} |U_k|^p \, dx \leq C(p, R).$$

Thus (U_k) is bounded in $W^{1,p}(B_R)$.

Case 2 ($0 < \alpha < \lambda_1(V)$).

First we claim that (U_k) is bounded in $L^1(\mathbb{R}^2)$. Suppose by contradiction that this is not the case. So, up to a subsequence, $\|U_k\|_1 \rightarrow +\infty$. Thus, setting $w_k = U_k/\|U_k\|_1$ and using (4.3) we obtain

$$(4.47) \quad -\Delta w_k \leq -\Delta w_k + V(x)w_k = \frac{1}{\|U_k\|_1} \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} + \gamma_k w_k \quad \text{in } B_R,$$

which together with Lemma 4.10 implies that $\Delta w_k \in L^1(B_R)$. Therefore, using the same argument as in Case 1, we can conclude that for any $R > 0$ and $1 < p < 2$,

$$\int_{B_R} |\nabla w_k|^p \, dx \leq C(p, R).$$

Using the Gagliardo-Nirenberg inequality there exists $\theta(p) \in (0, 1)$ such that

$$\|w_k\|_p \leq C_1(p, R) \|w_k\|_1^{\theta(p)} \|\nabla w_k\|_p^{1-\theta(p)} \leq C(p, R).$$

Thus (w_k) is bounded in $W^{1,p}(B_R)$. Hence, up to a subsequence, $w_k \rightharpoonup w$ weakly in $W^{1,p}(B_R)$ and $w_k \rightarrow w$ strongly in $L^1(B_R)$. Then, using Lemma 4.10 and taking the limit in (4.47) and using the same argument as in Case 1, we obtain

$$\int_{B_R} [\nabla w \nabla \phi + V(x)w\phi] \, dx = \alpha \int_{B_R} w\phi \, dx, \quad \forall \phi \in W^{1,p}(B_R), \quad \forall R > 0.$$

Since R is arbitrary, we have

$$(4.48) \quad \int_{\mathbb{R}^2} [\nabla w \nabla \phi + V(x)w\phi] \, dx = \alpha \int_{\mathbb{R}^2} w\phi \, dx, \quad \forall \phi \in W^{1,p}(\mathbb{R}^2).$$

Since $\alpha < \lambda_1(V)$, one can derive from (1.10) and (4.48) that $w = 0$, which contradicts the fact that $\|w\|_1 = 1$. Hence

$$(4.49) \quad \|U_k\|_1 \leq C.$$

Therefore, using (4.31) and the same argument as in Case 1, we can conclude that for any $1 < p < 2$ there exists $C(p, R)$ such that

$$\int_{B_R} |\nabla U_k|^p \, dx \leq C(p, R),$$

which together with (4.49) and the Gagliardo-Nirenberg inequality implies

$$\int_{B_R} |U_k|^p \, dx \leq C(p, R).$$

Thus (U_k) is bounded in $W^{1,p}(B_R)$.

Therefore, combining Cases 1 and 2, we obtain (4.32). Thus, $U_k \rightharpoonup G$ weakly in $W^{1,p}(B_R)$ for any R and $1 < p < 2$.

Testing (4.31) by $\phi \in C_0^\infty(B_R)$, we have

$$\int_{B_R} (\nabla \phi \nabla (c_k u_k) + V(x) \phi c_k u_k) \, dx = \int_{B_R} \phi \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \, dx + \gamma_k \int_{B_R} \phi c_k u_k \, dx.$$

Letting $k \rightarrow +\infty$, we have by Lemma 4.10,

$$\int_{B_R} (\nabla \phi \nabla G + V(x) \phi G) \, dx = \phi(0) + \alpha \int_{B_R} \phi G \, dx.$$

Hence

$$-\Delta G + V(x) G = \delta_0 + \alpha G$$

in a distributional sense.

In order to prove the regularity of U_k , we start by fixing $r > 0$ such that $r < 2r < 3r < R$ and taking a cut-off function $\eta \in C_0^\infty(B_R \setminus B_r, [0, 1])$ such that $\eta \equiv 1$ on $B_{3r} \setminus B_{2r}$. Using Remark 4.8 and the estimate

$$\int_{B_R \setminus B_r} |\nabla(\eta u_k)|^2 \, dx = \int_{B_R \setminus B_r} |\eta \nabla u_k + u_k \nabla \eta|^2 \, dx \leq C \int_{B_R \setminus B_r} (|\nabla u_k|^2 + |u_k|^2) \, dx,$$

we obtain

$$(4.50) \quad \|\nabla(\eta u_k)\|_2 \rightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

which implies that $e^{\alpha_k \eta^2 u_k^2}$ is bounded in $L^s(B_R \setminus B_r)$ for any $s > 1$. Indeed, using (4.50) there exists $k_0 \in \mathbb{N}$ such that $s\alpha_k \|\nabla(\eta u_k)\|_2^2 < 4\pi$ for all $k \geq k_0$. Hence, by the Trudinger-Moser inequality we have

$$\int_{B_R \setminus B_r} e^{s\alpha_k (\eta u_k)^2} \, dx = \int_{B_R \setminus B_r} e^{s\alpha_k \|\nabla(\eta u_k)\|_2^2 \left(\frac{\eta u_k}{\|\nabla(\eta u_k)\|_2} \right)^2} \, dx \leq C(R).$$

Consequently,

$$(4.51) \quad e^{\alpha_k u_k^2} \quad \text{is bounded in } L^s(B_{3r} \setminus B_{2r}), \quad \forall s > 1.$$

Using (4.51), given $1 < p < 2$, we have

$$\begin{aligned} & \int_{B_{3r} \setminus B_{2r}} \left(\frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} \right)^p dx \\ & \leq C_1 \left(\int_{B_{3r} \setminus B_{2r}} (c_k u_k)^{sp} dx \right)^{1/s} \left(\int_{B_{3r} \setminus B_{2r}} e^{s' \alpha_k u_k^2} dx \right)^{1/s'} \leq C(R), \end{aligned}$$

where $1/s + 1/s' = 1$ with $sp < 2$ and $s > 1$.

Since

$$-\Delta(c_k u_k) = \frac{\beta_k}{\lambda_k} c_k u_k e^{\alpha_k u_k^2} - (V(x) - \gamma_k) c_k u_k \quad \text{in } B_{R_k},$$

we obtain

$$\begin{aligned} \int_{B_{3r} \setminus B_{2r}} |\Delta(c_k u_k)|^p dx & \leq 2^p \frac{\beta_k}{\lambda_k} \int_{B_{3r} \setminus B_{2r}} (c_k u_k e^{\alpha_k u_k^2})^p dx \\ & \quad + 2^p \int_{B_{3r} \setminus B_{2r}} |(V(x) - \gamma_k)|^p |c_k u_k|^p dx. \end{aligned}$$

Consequently, $\Delta(c_k u_k) \in L^p(B_{3r} \setminus B_{2r})$ for any $1 < p < 2$. Thus, applying the elliptic estimates there exists $C(R) > 0$ such that

$$\|c_k u_k\|_{C^{1,\gamma}(\overline{B_{3r}} \setminus B_{5r/2})} \leq C(R).$$

Hence, by the Ascoli-Arzelá Theorem we have $c_k u_k \rightarrow G$ in $C^{1,\gamma}(\overline{B_{3r}} \setminus B_{5r/2})$. Since R is arbitrary we conclude that (4.30) holds. $\square$

5. AN UPPER BOUND OF $\ell(\alpha)$

To derive an upper bound to $\ell(\alpha)$, similar to [18–20, 24], we need the following result due to Carleson and Chang (cf. [6]):

Lemma 5.1. *Let B be a ball in $\mathbb{R}^2$ and (v_k) a sequence in $W_0^{1,2}(B)$ with $\int_B |\nabla v_k|^2 dx = 1$. If $v_k \rightharpoonup 0$ weakly in $W_0^{1,2}(B)$, then*

$$\limsup_{k \rightarrow +\infty} \int_B \left(e^{4\pi v_k^2} - 1 \right) dx \leq |B|e.$$

Proposition 5.2. *Assume that (4.9) holds; then $\ell(\alpha) \leq \pi e^{4\pi A_0 + 1}$ for any $\alpha \in [0, \lambda_1(V))$, where A_0 is a constant depending only on α .*

Proof. By Lemma 4.11, we have $c_k u_k \rightarrow G$ in $C_{loc}^{1,\gamma}(\mathbb{R}^2 \setminus \{0\})$ and

$$-\Delta \left(G(x) + \frac{1}{2\pi} \log |x| \right) = [\alpha - V(x)]G(x) \in L_{loc}^r(\mathbb{R}^2), \quad \text{for all } r > 1.$$

Standard elliptic estimates yield that $G(x) + \frac{1}{2\pi} \log |x| \in C_{loc}^{1,\theta}(\mathbb{R}^2)$ (cf. [15], Chapter 9). Hence, similar to [18, 20, 24] and [37, Proposition 5.1], the Green function G can be represented by

$$G(x) = -\frac{1}{2\pi} \log |x| + A_0 + g_\alpha(x),$$

where A_0 is a constant depending only on α , $g_\alpha(0) = 0$, $g_\alpha(x)$ is continuous at zero, and $g_\alpha(x) \in C_{loc}^{1,\gamma}(\mathbb{R}^2 \setminus \{0\})$. Let ν be the unit outward normal to ∂B_δ ; then

$$(5.1) \quad \begin{aligned} \int_{\mathbb{R}^2 \setminus B_\delta} (|\nabla G(x)|^2 + V(x)|G(x)|^2) \, dx &= \alpha \int_{\mathbb{R}^2 \setminus B_\delta} |G(x)|^2 \, dx - \int_{\partial B_\delta} G(x) \frac{\partial G(x)}{\partial \nu} \, d\sigma \\ &= \alpha \|G\|_2^2 - G(\delta) \int_{\partial B_\delta} \frac{\partial G(x)}{\partial \nu} \, d\sigma + o_\delta(1). \end{aligned}$$

Integrating by parts on both sides of equation (4.29) over B_δ , we have

$$\int_{B_\delta} -\Delta G(x) \, dx = 1 + \int_{B_\delta} [\alpha - V(x)]G(x) \, dx,$$

and with the aid of the divergence theorem we get

$$- \int_{\partial B_\delta} \frac{\partial G(x)}{\partial \nu} \, d\sigma = 1 + \int_{B_\delta} [\alpha - V(x)]G(x) \, dx.$$

Thus,

$$(5.2) \quad \begin{aligned} \int_{\mathbb{R}^2 \setminus B_\delta} (|\nabla G(x)|^2 + V(x)|G(x)|^2) \, dx \\ = \alpha \|G\|_2^2 + G(\delta) + G(\delta) \int_{B_\delta} [\alpha - V(x)]G(x) \, dx + o_\delta(1) \\ = \alpha \|G\|_2^2 + G(\delta) + o_\delta(1). \end{aligned}$$

By the continuity of $g_\alpha(x)$, we have

$$\int_{\mathbb{R}^2 \setminus B_\delta} (|\nabla G(x)|^2 + V(x)|G(x)|^2) \, dx = -\frac{1}{2\pi} \log \delta + A_0 + \alpha \|G\|_2^2 + o_\delta(1),$$

which implies that

$$\int_{\mathbb{R}^2 \setminus B_\delta} (|\nabla u_k|^2 + V(x)|u_k|^2) \, dx = \frac{1}{c_k^2} \left(-\frac{1}{2\pi} \log \delta + A_0 + \alpha \|G\|_2^2 + o_\delta(1) + o_k(1) \right).$$

Therefore

$$\begin{aligned} \tau_k := \int_{B_\delta} |\nabla u_k|^2 \, dx &= 1 - \left(\int_{\mathbb{R}^2 \setminus B_\delta} (|\nabla u_k|^2 + V(x)|u_k|^2) \, dx \right) - \int_{B_\delta} V(x)u_k^2 \, dx \\ &= 1 - \left(\int_{\mathbb{R}^2 \setminus B_\delta} (|\nabla u_k|^2 + V(x)|u_k|^2) \, dx \right) + o_k(1) \\ &= 1 - \frac{1}{c_k^2} \left(-\frac{1}{2\pi} \log \delta + A_0 + \alpha \|G\|_2^2 + o_\delta(1) + o_k(1) \right), \end{aligned}$$

which together with Lemma 5.1 implies that

$$(5.3) \quad \limsup_{k \rightarrow +\infty} \int_{B_\delta} \left[e^{4\pi(w_k/\tau_k^{1/2})^2} - 1 \right] \, dx \leq \delta^2 \pi e,$$

where $b_k = u_k(\delta)$ and $w_k = (u_k - b_k)^+$. Note that $w_k \in W_0^{1,2}(B_\delta)$ and $w_k \rightharpoonup 0$ weakly in $W_0^{1,2}(B_\delta)$.

Using (4.17), we have $\varphi_k \rightarrow \varphi$ in $C_{loc}^{1,\theta}(\mathbb{R}^2)$, where $\varphi_k(x/r_k) = c_k(u_k(x) - c_k)$, which gives

$$(5.4) \quad \alpha_k u_k^2 \leq 4\pi w_k^2 / \tau_k - 2 \log \delta + 4\pi A_0 + o_k(1) \quad \text{on } B_{r_k L}.$$

Indeed, using the fact that $c_k u_k \rightarrow G$ in $C_{loc}^{1,\theta}(\mathbb{R}^2)$ and $\|u_k\|_2 \rightarrow 0$, we get

$$\begin{aligned}\alpha_k u_k^2 &\leq 4\pi(1 + \alpha\|u_k\|_2^2)(w_k + b_k)^2 \\ &= 4\pi(1 + \alpha\|u_k\|_2^2)(w_k^2 + 2w_k b_k + b_k^2) \\ &= 4\pi w_k^2 + 8\pi w_k b_k + 4\pi b_k^2 + 4\pi\alpha\|u_k\|_2^2 w_k^2 + 8\pi\alpha\|u_k\|_2^2 w_k b_k + 4\pi\alpha\|u_k\|_2^2 b_k^2.\end{aligned}$$

Since $b_k \leq c_k$ and $\varphi_k(x/r_k) = c_k(u_k(x) - c_k)$, we obtain

$$\begin{aligned}\alpha_k u_k^2 &\leq 4\pi w_k^2 + 8\pi(c_k - b_k)b_k + 4\pi b_k^2 + 12\pi\alpha\|u_k\|_2^2 [(u_k - c_k)c_k - \varphi_k(\delta/r_k)] + 4\pi\alpha\|u_k\|_2^2 c_k^2 \\ &= 4\pi w_k^2 + 8\pi b_k c_k - 4\pi b_k^2 + 12\pi\alpha\|u_k\|_2^2 [(u_k - c_k)c_k - \varphi_k(\delta/r_k)] + 4\pi\alpha\|c_k u_k\|_2^2 \\ &\leq 4\pi w_k^2 + 8\pi b_k c_k + 12\pi\alpha\|u_k\|_2^2 [(u_k - c_k)c_k - \varphi_k(\delta/r_k)] + 4\pi\alpha\|c_k u_k\|_2^2.\end{aligned}$$

Using the fact that $\varphi_k \rightarrow \varphi$ in $C_{loc}^{1,\theta}(\mathbb{R}^2)$ and $\|u_k\|_2 \rightarrow 0$, we can write

$$\alpha_k u_k^2 \leq 4\pi w_k^2 + 8\pi b_k c_k + 4\pi\alpha\|G\|_2^2 + o_k(1).$$

From the estimates above we get

$$(5.5) \quad 4\pi w_k^2 \leq 4\pi \left(\frac{w_k}{\tau_k^{1/2}} \right)^2 + 2\log \delta - 4\pi A_0 - 4\pi\alpha\|G\|_2^2 + o_\delta(1) + o_k(1).$$

Indeed, denote

$$H_k(\delta) = -\frac{1}{2\pi} \log \delta + A_0 + \alpha\|G\|_2^2 + o_\delta(1) + o_k(1)$$

and then

$$\tau_k = 1 - \frac{1}{c_k^2} H_k(\delta),$$

and since $0 \leq c_k^{-2} H_k^2(\delta)$, we obtain

$$-H_k(\delta) \leq -H_k(\delta) + c_k^{-2} H_k^2(\delta).$$

Thus

$$-H_k(\delta) \leq -H_k(\delta) (1 - c_k^{-2} H_k(\delta)).$$

Multiplying the last inequality by $c_k^{-2} w_k^2$ we get

$$-c_k^{-2} H_k(\delta) w_k^2 \leq -H_k(\delta) (1 - c_k^{-2} H_k(\delta)) c_k^{-2} w_k^2,$$

and by using the fact that $c_k^{-2} w_k^2 \leq 1$, we have

$$-c_k^{-2} H_k(\delta) w_k^2 \leq -H_k(\delta) (1 - c_k^{-2} H_k(\delta)).$$

Consequently, we get

$$w_k^2 (1 - c_k^{-2} H_k(\delta)) \leq w_k^2 - H_k(\delta) (1 - c_k^{-2} H_k(\delta)).$$

This inequality implies that

$$w_k^2 \leq \frac{w_k^2}{\tau_k} - H_k(\delta).$$

By the definition of τ_k the assertion follows.

Similarly, using the fact that $b_k c_k \rightarrow G(\delta)$, we have

$$(5.6) \quad 8\pi b_k c_k = -4\log \delta + 8\pi A_0 + o_\delta(1) + o(1).$$

Combining (5.5) and (5.6), we obtain (5.4).

Therefore, denoting $L_k = Lr_k$ we obtain by (5.4) that

$$\begin{aligned} \int_{B_{Lr_k}} e^{\alpha_k u_k^2} dx &\leq \delta^{-2} e^{4\pi A_0 + o(1)} \int_{B_{L_k}} e^{\alpha_k (w_k / \tau_k^{1/2})^2} dx \\ &= \delta^{-2} e^{4\pi A_0 + o(1)} \int_{B_{L_k}} \left(e^{\alpha_k (w_k / \tau_k^{1/2})^2} - 1 \right) dx + o_k(1) \\ &\leq \delta^{-2} e^{4\pi A_0 + o(1)} \int_{B_\delta} \left(e^{\alpha_k (w_k / \tau_k^{1/2})^2} - 1 \right) dx. \end{aligned}$$

It follows by (5.3) that

$$(5.7) \quad \limsup_{k \rightarrow +\infty} \int_{B_{L_k}} e^{\alpha_k u_k^2} dx \leq \pi e^{4\pi A_0 + 1}.$$

There exists $k_0 \in \mathbb{N}$ such that $\{x \in \mathbb{R}^2 : u_k(x) \geq c_k/A\} \subset B_{Lr_k}$ for $k \geq k_0$. Therefore from (4.27), we have

$$(5.8) \quad \limsup_{k \rightarrow +\infty} \int_{B_\delta \setminus B_{L_k}} (e^{\alpha_k u_k^2} - 1) dx = \limsup_{k \rightarrow +\infty} \int_{B_\delta \setminus B_{L_k}} (e^{\alpha_k |u_k^A|^2} - 1) dx = 0.$$

On the other hand, by the Radial Lemma (cf. [32]) there exists $C = C(\delta) > 0$ such that

$$\int_{\mathbb{R}^2 \setminus B_\delta} (e^{\alpha_k u_k^2} - 1) dx \leq C \|u_k\|_2^2.$$

Since $\|u_k\|_2^2 \rightarrow 0$ as $k \rightarrow +\infty$, we have

$$(5.9) \quad \limsup_{k \rightarrow +\infty} \int_{\mathbb{R}^2 \setminus B_\delta} (e^{\alpha_k u_k^2} - 1) dx = 0.$$

Note that for $\delta > Lr_k$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} (e^{\alpha_k u_k^2} - 1) dx &= \int_{B_{Lr_k}} (e^{\alpha_k u_k^2} - 1) dx + \int_{\mathbb{R}^2 \setminus B_{Lr_k}} (e^{\alpha_k u_k^2} - 1) dx \\ &\leq \int_{B_{Lr_k}} e^{\alpha_k u_k^2} dx + \int_{B_\delta \setminus B_{Lr_k}} (e^{\alpha_k u_k^2} - 1) dx + \int_{\mathbb{R}^2 \setminus B_\delta} (e^{\alpha_k u_k^2} - 1) dx. \end{aligned}$$

From (5.7), (5.8) and (5.9), we get

$$\ell(\alpha) \leq \pi e^{4\pi A_0 + 1}.$$

□

6. PROOF OF THEOREMS 1.1 AND 1.3

6.1. The supremum $\ell(\alpha)$ is finite and is attained if $\alpha \in [0, \lambda_1(V))$. We proved in Section 5 (Proposition 5.2) that if

$$c_k = \sup_{\mathbb{R}^2} u_k = u_k(0) \nearrow +\infty,$$

then

$$(6.1) \quad \ell(\alpha) \leq \pi e^{4\pi A_0 + 1}.$$

Using a similar argument in spirit with the one due to Y. Li and B. Ruf in [21] (see also [20]), we construct a family of test functions $v_\epsilon \in E$, with $\|v_\epsilon\| = 1$ and satisfying

$$(6.2) \quad \int_{\mathbb{R}^2} \left[e^{4\pi(1+\alpha\|v_\epsilon\|_2^2)v_\epsilon^2} - 1 \right] dx > \pi e^{4\pi A_0 + 1},$$

for $\epsilon > 0$ sufficiently small. This is a contradiction with (6.1), and consequently the blow-up does not occur.

We take

$$v_\epsilon(x) = \begin{cases} C - \frac{\log(1 + \pi|x/\epsilon|^2) + B}{4\pi C} & \text{if } |x| < \epsilon L, \\ \frac{G(|x|)}{C} & \text{if } |x| \geq \epsilon L, \end{cases}$$

where G is the Green function given in Lemma 4.11 and C, B, L are functions of ϵ (which will be defined later by (6.4), (6.5) and (6.7)) satisfying

- (i) $L \rightarrow +\infty, C \rightarrow +\infty$ and $\epsilon L \rightarrow 0$ as $\epsilon \rightarrow 0$;
- (ii) $C - \frac{\log(1 + \pi L^2) + B}{4\pi C} = \frac{G(\epsilon L)}{C}$;
- (iii) $\frac{\log L}{C^2} \rightarrow 0$ as $\epsilon \rightarrow 0$.

We use the normalization of v_ϵ to obtain information on B, C and L . Similar to (5.1) and (5.2) we get

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus B_{\epsilon L}} (|\nabla v_\epsilon|^2 + V(x)|v_\epsilon|^2) dx &= \frac{1}{C^2} \int_{\mathbb{R}^2 \setminus B_{\epsilon L}} (|\nabla G(x)|^2 + V(x)|G(x)|^2) dx \\ &= \frac{1}{C^2} \left(\alpha \|G\|_2^2 - \int_{\partial B_{\epsilon L}} G(\epsilon L) \frac{\partial G(x)}{\partial(-\nu)} d\sigma + o_\delta(1) \right) \\ &= \frac{1}{C^2} \left(\alpha \|G\|_2^2 + G(\epsilon L) + G(\epsilon L) \int_{B_{\epsilon L}} [\alpha - V(x)]G(x) dx + o_\delta(1) \right) \\ &= \frac{1}{C^2} (\alpha \|G\|_2^2 + G(\epsilon L) + o_\epsilon(1) + o_\delta(1)) \\ &= \frac{1}{4\pi C^2} (4\pi\alpha \|G\|_2^2 - \log(\epsilon L)^2 + 4\pi A_0 + o_\epsilon(1) + o_\delta(1)). \end{aligned}$$

By a standard calculation we get

$$\int_{B_{\epsilon L}} |\nabla v_\epsilon|^2 dx = \frac{1}{4\pi C^2} (\log(1 + \pi L^2) - 1) + O(L^{-2}C^{-2}).$$

It is easy to check that

$$\int_{B_{\epsilon L}} V(x)|v_\epsilon|^2 dx = O((\epsilon L)^2 C^2 \log L).$$

Consequently, we have

$$\begin{aligned} \int_{\mathbb{R}^2} (|\nabla v_\epsilon|^2 + V(x)|v_\epsilon|^2) dx &= \frac{1}{4\pi C^2} (-1 + 4\pi\alpha \|G\|_2^2 + 4\pi A_0 + \log(1 + \pi L^2) - \log(\epsilon L)^2 + \phi), \end{aligned}$$

where

$$\phi = O((\epsilon L)^2 C^2 \log L (\epsilon L)^2 \log^2(\epsilon L) + L^{-2} + o_\epsilon(1) + o_\delta(1)).$$

Setting $\int_{\mathbb{R}^2} (|\nabla v_\epsilon|^2 + V(x)|v_\epsilon|^2) \, dx = 1$, we obtain

$$(6.3) \quad 4\pi C^2 = -1 + 4\pi\alpha \|G\|_2^2 + 4\pi A_0 + \log(1 + \pi L^2) - \log(\epsilon L)^2 + \phi.$$

Thus

$$\begin{aligned} 4\pi C^2 &= -1 + 4\pi\alpha \|G\|_2^2 + 4\pi A_0 + \log(1 + \pi L^2) - \log L^2 - \log \epsilon^2 + \phi \\ &= -1 + 4\pi\alpha \|G\|_2^2 + 4\pi A_0 + \log \frac{(1 + \pi L^2)}{L^2} - \log \epsilon^2 + \phi. \end{aligned}$$

Hence

$$(6.4) \quad 4\pi C^2 = -1 + 4\pi\alpha \|G\|_2^2 + 4\pi A_0 + \log \pi - \log \epsilon^2 + \phi.$$

By (ii) we have

$$4\pi C^2 - \log(1 + \pi L^2) - \Lambda_\epsilon = 4\pi G(\epsilon L),$$

and then from (6.3) we get

$$(6.5) \quad \Lambda_\epsilon = -1 + 4\pi\alpha \|G\|_2^2 + \phi.$$

Now, we estimate the integral

$$\int_{B_{\epsilon L}} e^{4\pi(1+\alpha\|v_\epsilon\|_2^2)v_\epsilon^2} \, dx.$$

On $B_{\epsilon L}$ we can check that

$$4\pi(1 + \alpha\|v_\epsilon\|_2^2)v_\epsilon^2 \geq (4\pi C^2 - 2\Lambda_\epsilon) - 2\log(1 + \pi|x/\epsilon|^2) + 4\pi\alpha\|G\|_2^2 + O(C^{-2}\log L).$$

Indeed, first we have that

$$\|v_\epsilon\|_2^2 = C^{-2}\|G\|_2^2 + o_\epsilon(1),$$

and on $B_{\epsilon L}$,

$$\begin{aligned} &4\pi(1 + \alpha\|v_\epsilon\|_2^2)v_\epsilon^2 \\ &= 4\pi(1 + \alpha C^{-2}\|G\|_2^2) \left(C - \frac{\log(1 + \pi|x/\epsilon|^2) + \Lambda_\epsilon}{4\pi C} \right)^2 \\ &\geq 4\pi(1 + \alpha C^{-2}\|G\|_2^2) C^2 \left(1 - 2 \frac{\log(1 + \pi|x/\epsilon|^2) + \Lambda_\epsilon}{4\pi C^2} \right) \\ &= (1 + \alpha C^{-2}\|G\|_2^2) [(4\pi C^2 - 2\Lambda_\epsilon) - 2\log(1 + \pi|x/\epsilon|^2)] \\ &= (4\pi C^2 - 2\Lambda_\epsilon) - 2\log(1 + \pi|x/\epsilon|^2) + 4\pi\alpha\|G\|_2^2 \\ &\quad - 2\alpha C^{-2}\|G\|_2^2(\log(1 + \pi|x/\epsilon|^2) + \Lambda_\epsilon) \\ &= (4\pi C^2 - 2\Lambda_\epsilon) - 2\log(1 + \pi|x/\epsilon|^2) + 4\pi\alpha\|G\|_2^2 + O(C^{-2}\log^2 L). \end{aligned}$$

Then

$$\int_{B_{\epsilon L}} \left[e^{4\pi(1+\alpha\|v_\epsilon\|_2^2)v_\epsilon^2} - 1 \right] \, dx \geq \pi e^{4\pi A_0 + 1} + O[(\epsilon L)^2 C^2 \log L + L^{-2} + (\epsilon L)^2 \log^2 \epsilon L].$$

Moreover, on $\mathbb{R}^2 \setminus B_{\epsilon L}$ we have the estimates

$$\int_{\mathbb{R}^2 \setminus B_{\epsilon L}} \left[e^{4\pi(1+\alpha\|v_\epsilon\|_2^2)v_\epsilon^2} - 1 \right] \, dx \geq 4\pi \int_{\mathbb{R}^2 \setminus B_{\epsilon L}} \left| \frac{G(x)}{C} \right|^2 \, dx,$$

and thus we get

$$\begin{aligned}
 (6.6) \quad \int_{\mathbb{R}^2} \left[e^{4\pi(1+\alpha\|v_\epsilon\|_2^2)v_\epsilon^2} - 1 \right] dx &\geq \pi e^{4\pi A_0+1} + 4\pi \int_{\mathbb{R}^2 \setminus B_{\epsilon L}} \left| \frac{G(x)}{C} \right|^2 dx \\
 &\quad + O((\epsilon L)^2 C^2 \log L + L^{-2} + (\epsilon L)^2 \log^2 \epsilon L) \\
 &= \pi e^{4\pi A_0+1} + \frac{4\pi}{C^2} \int_{\mathbb{R}^2 \setminus B_{\epsilon L}} |G(x)|^2 dx \\
 &\quad + O[(\epsilon L)^2 C^4 \log L + C^2/L^2 + C^2(\epsilon L)^2 \log^2 \epsilon L].
 \end{aligned}$$

Now taking $L(\epsilon) = -\log \epsilon$, we get $\epsilon L(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. We then need to prove that there exists a $C = C(\epsilon)$ which solves equation (6.4). Setting

$$z(t) = -4\pi t^2 - 1 + 4\pi\alpha\|G\|_2^2 + 4\pi A_0 + \log \pi - \log \epsilon^2 + \phi$$

since, for ϵ small,

$$z\left(\left(-\frac{2}{4\pi} \log \epsilon^2\right)^{1/2}\right) = \log \epsilon^2 + o_\epsilon(1) + \log \pi + \phi < 0$$

and

$$z\left(\left(-\frac{1}{8\pi} \log \epsilon^2\right)^{1/2}\right) = -\frac{1}{2} \log \epsilon^2 + o_\epsilon(1) + \log \pi + \phi > 0,$$

there exists $t_\epsilon \in \left(\left(-\frac{1}{8\pi} \log \epsilon^2\right)^{1/2}, \left(-\frac{2}{4\pi} \log \epsilon^2\right)^{1/2}\right)$ such that $z(t_\epsilon) = 0$. Thus we defined C , and it satisfies

$$(6.7) \quad 4\pi C^2 = -\log \epsilon^2 + O(1).$$

Therefore, as $\epsilon \rightarrow 0$, we have

$$\frac{\log L}{C^2} \rightarrow 0,$$

and then

$$(\epsilon L)^2 C^4 \log L + C^2 L^{-2} + C^2 (\epsilon L)^2 \log^2 \epsilon L \rightarrow 0.$$

Therefore (i), (ii) and (iii) hold and we can conclude from (6.6) that for $\epsilon > 0$ sufficiently small

$$\int_{\mathbb{R}^2} \left[e^{4\pi(1+\alpha\|v_\epsilon\|_2^2)v_\epsilon^2} - 1 \right] dx > \pi e^{4\pi A_0+1}.$$

The contradiction between (6.2) and Proposition 5.2 implies that (c_k) is bounded.

6.2. The supremum $\ell(\alpha)$ is infinity if $\alpha \geq \lambda_1(V)$. We prove in this section the second point of Theorem 1.1, that is, $\ell(\alpha) = +\infty$ if $\alpha \geq \lambda_1(V)$. The proof of this fact is based on computations of testing functions which are a combination of the eigenfunction associated to eigenvalue λ_1 with the Moser functions.

Consider λ_1 as in (1.10) and ϕ_1 the positive eigenfunction associated to λ_1 normalized by $\|\phi_1\|_2 = 1$. Similar to Adimuthi and O. Druet (cf. [2]) we consider the Moser sequence:

$$w_k(x) = (2\pi)^{-1/2} \begin{cases} (\log k)^{1/2} & \text{if } |x| \leq 1/k, \\ \frac{\log(1/|x|)}{(\log k)^{1/2}} & \text{if } 1/k \leq |x| \leq 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Notice that $w_k \in W^{1,2}(\mathbb{R}^2)$, $\text{supp}(w_k) = \overline{B_1}$, $\int_{B_1} |\nabla w_k|^2 \, dx = 1$ and $w_k \rightarrow 0$ in $W_0^{1,2}(B_1)$.

Let $(t_k) \subset \mathbb{R}$ be a sequence such that

- (T₁) $t_k > 0$ for all $k \in \mathbb{N}$;
- (T₂) $t_k \rightarrow 0$ as $k \rightarrow +\infty$;
- (T₃) $\zeta_k = t_k(\log k)^{1/2} \rightarrow +\infty$ as $k \rightarrow +\infty$;
- (T₄) $t_k^2(\log k)^{1/2} \rightarrow 0$ as $k \rightarrow +\infty$.

We can take for example $t_k = (\log k)^{-1/3}$.

Setting

$$v_k = w_k + t_k \phi_1,$$

we are going to prove that

$$\int_{\mathbb{R}^2} \left[e^{4\pi \left(1 + \alpha \frac{\|v_k\|_2^2}{\|v_k\|^2} \right) \frac{v_k^2}{\|v_k\|^2}} - 1 \right] \, dx \rightarrow +\infty \quad \text{as } k \rightarrow +\infty.$$

For that purpose, we compute $\|v_k\|^2$ and $\|v_k\|_2^2$. First,

$$\begin{aligned} \|v_k\|^2 &= \|w_k\|^2 + 2t_k \int_{B_1} (\nabla w_k \nabla \phi_1 + V(x) w_k \phi_1) \, dx + t_k^2 \|\phi_1\|^2 \\ &= 1 + \int_{B_1} V(x) w_k^2 \, dx + 2\lambda_1 t_k \int_{B_1} \phi_1 w_k \, dx + \lambda_1 t_k^2. \end{aligned}$$

Let

$$Z(x) = \begin{cases} (2\pi)^{-1/2} \log(1/|x|) & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then by the Lebesgue dominated convergence theorem, we obtain

$$\|v_k\|^2 = 1 + \lambda_1 t_k^2 + 2\lambda_1 t_k^2 \zeta_k^{-1} \int_{B_1} \phi_1 Z(x) \, dx + o(t_k^2 \zeta_k^{-1}).$$

On the other hand,

$$\|v_k\|_2^2 = t_k^2 + 2t_k^2 \zeta_k^{-1} \int_{B_1} \phi_1 Z(x) \, dx + o(t_k^2 \zeta_k^{-1}).$$

Thus we have that

$$1 + \alpha \frac{\|v_k\|_2^2}{\|v_k\|^2} = 1 + \alpha t_k^2 + 2\alpha t_k^2 \zeta_k^{-1} \int_{B_1} \phi_1 Z(x) \, dx + o(t_k^2 \zeta_k^{-1}) + O(t_k^4)$$

and that

$$\begin{aligned} \left(1 + \alpha \frac{\|v_k\|_2^2}{\|v_k\|^2} \right) \frac{1}{\|v_k\|^2} &= 1 + (\alpha - \lambda_1) \left(t_k^2 + 2t_k^2 \zeta_k^{-1} \int_{B_1} \phi_1 Z(x) \, dx \right) \\ &\quad + o(t_k^2 \zeta_k^{-1}) + O(t_k^4). \end{aligned}$$

Since $\alpha \geq \lambda_1(V)$, we obtain that

$$\left(1 + \alpha \frac{\|v_k\|_2^2}{\|v_k\|^2} \right) \frac{1}{\|v_k\|^2} \geq 1 + o(t_k^2 \zeta_k^{-1}) + O(t_k^4).$$

This gives in particular that on $B_{1/k}$,

$$4\pi \left(1 + \alpha \frac{\|v_k\|_2^2}{\|v_k\|^2} \right) \frac{v_k^2}{\|v_k\|^2} \geq 2 \log k + 4(2\pi)^{1/2} \zeta_k \phi_1 + o(\zeta_k).$$

Hence by (T_3) and using that $\phi_1(0) > 0$, we obtain

$$\int_{\mathbb{R}^2} \left[e^{4\pi \left(1 + \alpha \frac{\|v_k\|_2^2}{\|v_k\|^2} \right) \frac{v_k^2}{\|v_k\|^2}} - 1 \right] dx \rightarrow +\infty \quad \text{as } k \rightarrow +\infty.$$

ACKNOWLEDGEMENT

The authors would like to thank the anonymous referee for valuable comments and suggestions to improve the quality of the paper.

REFERENCES

- [1] Shinji Adachi and Kazunaga Tanaka, *Trudinger type inequalities in $\mathbf{R}^N$ and their best exponents*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 2051–2057, DOI 10.1090/S0002-9939-99-05180-1. MR1646323 (2000m:46069)
- [2] Adimurthi and O. Druet, *Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality*, Comm. Partial Differential Equations **29** (2004), no. 1-2, 295–322, DOI 10.1081/PDE-120028854. MR2038154 (2005a:46064)
- [3] Adimurthi and Michael Struwe, *Global compactness properties of semilinear elliptic equations with critical exponential growth*, J. Funct. Anal. **175** (2000), no. 1, 125–167, DOI 10.1006/jfan.2000.3602. MR1774854 (2001g:35063)
- [4] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345, DOI 10.1007/BF00250555. MR695535 (84h:35054a)
- [5] D. M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbf{R}^2$* , Comm. Partial Differential Equations **17** (1992), no. 3-4, 407–435, DOI 10.1080/03605309208820848. MR1163431 (93f:35076)
- [6] Lennart Carleson and Sun-Yung A. Chang, *On the existence of an extremal function for an inequality of J. Moser* (English, with French summary), Bull. Sci. Math. (2) **110** (1986), no. 2, 113–127. MR878016 (88f:46070)
- [7] Wen Xiong Chen and Congming Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), no. 3, 615–622, DOI 10.1215/S0012-7094-91-06325-8. MR1121147 (93e:35009)
- [8] David G. Costa, *On a class of elliptic systems in $\mathbf{R}^N$* , Electron. J. Differential Equations (1994), No. 07, approx. 14 pp. (electronic). MR1292598 (95e:35065)
- [9] Djairo G. de Figueiredo, João Marcos do Ó, and Bernhard Ruf, *On an inequality by N. Trudinger and J. Moser and related elliptic equations*, Comm. Pure Appl. Math. **55** (2002), no. 2, 135–152, DOI 10.1002/cpa.10015. MR1865413 (2002j:35104)
- [10] Djairo G. de Figueiredo, João Marcos do Ó, and Bernhard Ruf, *Elliptic equations and systems with critical Trudinger-Moser nonlinearities*, Discrete Contin. Dyn. Syst. **30** (2011), no. 2, 455–476, DOI 10.3934/dcds.2011.30.455. MR2772124 (2011m:35083)
- [11] João Marcos B. do Ó, *N-Laplacian equations in $\mathbf{R}^N$ with critical growth*, Abstr. Appl. Anal. **2** (1997), no. 3-4, 301–315, DOI 10.1155/S1085337597000419. MR1704875 (2001h:35045)
- [12] João Marcos do Ó, Everaldo Medeiros, and Uberlandio Severo, *A nonhomogeneous elliptic problem involving critical growth in dimension two*, J. Math. Anal. Appl. **345** (2008), no. 1, 286–304, DOI 10.1016/j.jmaa.2008.03.074. MR2422651 (2009m:35161)
- [13] João Marcos do Ó and Yunyan Yang, *A quasi-linear elliptic equation with critical growth on compact Riemannian manifold without boundary*, Ann. Global Anal. Geom. **38** (2010), no. 3, 317–334, DOI 10.1007/s10455-010-9218-0. MR2721666 (2011k:58027)

- [14] Martin Flucher, *Extremal functions for the Trudinger-Moser inequality in 2 dimensions*, Comment. Math. Helv. **67** (1992), no. 3, 471–497, DOI 10.1007/BF02566514. MR1171306 (93k:58073)
- [15] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364 (2001k:35004)
- [16] J. A. Hempel, G. R. Morris, and N. S. Trudinger, *On the sharpness of a limiting case of the Sobolev imbedding theorem*, Bull. Austral. Math. Soc. **3** (1970), 369–373. MR0280998 (43 #6717)
- [17] S. Kesavan, *Symmetrization and applications*, Series in Analysis, vol. 3, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006. MR2238193 (2008a:35005)
- [18] Satyanad Kichenassamy and Laurent Véron, *Singular solutions of the p -Laplace equation*, Math. Ann. **275** (1986), no. 4, 599–615, DOI 10.1007/BF01459140. MR859333 (87j:35096)
- [19] Yuxiang Li, *Moser-Trudinger inequality on compact Riemannian manifolds of dimension two*, J. Partial Differential Equations **14** (2001), no. 2, 163–192. MR1838044 (2002h:58033)
- [20] Yuxiang Li, *Extremal functions for the Moser-Trudinger inequalities on compact Riemannian manifolds*, Sci. China Ser. A **48** (2005), no. 5, 618–648, DOI 10.1360/04ys0050. MR2158479 (2007a:53081)
- [21] Yuxiang Li and Bernhard Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^n$* , Indiana Univ. Math. J. **57** (2008), no. 1, 451–480, DOI 10.1512/iumj.2008.57.3137. MR2400264 (2009h:46071)
- [22] Kai-Ching Lin, *Extremal functions for Moser’s inequality*, Trans. Amer. Math. Soc. **348** (1996), no. 7, 2663–2671, DOI 10.1090/S0002-9947-96-01541-3. MR1333394 (96i:58043)
- [23] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana **1** (1985), no. 1, 145–201, DOI 10.4171/RMI/6. MR834360 (87c:49007)
- [24] Guozhen Lu and Yunyan Yang, *Sharp constant and extremal function for the improved Moser-Trudinger inequality involving L^p norm in two dimension*, Discrete Contin. Dyn. Syst. **25** (2009), no. 3, 963–979, DOI 10.3934/dcds.2009.25.963. MR2533985 (2010m:35011)
- [25] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092. MR0301504 (46 #662)
- [26] Ratikanta Panda, *On semilinear Neumann problems with critical growth for the n -Laplacian*, Nonlinear Anal. **26** (1996), no. 8, 1347–1366, DOI 10.1016/0362-546X(94)00360-T. MR1377667 (97d:35079)
- [27] S. I. Pohozaev, *The Sobolev embedding in the case $pl = n$* , Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964–1965. Mathematics Section, Moscov. Energet. Inst. (1965), 158–170.
- [28] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies, no. 27, Princeton Univ. Press, Princeton, 1951. MR0043486
- [29] Paul H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), no. 2, 270–291, DOI 10.1007/BF00946631. MR1162728 (93h:35194)
- [30] Bernhard Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$* , J. Funct. Anal. **219** (2005), no. 2, 340–367, DOI 10.1016/j.jfa.2004.06.013. MR2109256 (2005k:46082)
- [31] Yuji Sano, *The best constant of the Moser-Trudinger inequality on S^2* , Trans. Amer. Math. Soc. **356** (2004), no. 9, 3477–3482 (electronic), DOI 10.1090/S0002-9947-03-03483-4. MR2055742 (2005e:53052)
- [32] Walter A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), no. 2, 149–162. MR0454365 (56 #12616)
- [33] Michael Struwe, *Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces* (English, with French summary), Ann. Inst. H. Poincaré Anal. Non Linéaire **5** (1988), no. 5, 425–464. MR970849 (90c:35084)
- [34] Michael Struwe, *Positive solutions of critical semilinear elliptic equations on non-compactible planar domains*, J. Eur. Math. Soc. (JEMS) **2** (2000), no. 4, 329–388, DOI 10.1007/s100970000023. MR1796963 (2001h:35070)
- [35] Neil S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483. MR0216286 (35 #7121)

- [36] V. I. Judovič, *Some estimates connected with integral operators and with solutions of elliptic equations* (Russian), Dokl. Akad. Nauk SSSR **138** (1961), 805–808. MR0140822 (25 #4236)
- [37] Yunyan Yang, *A sharp form of Moser-Trudinger inequality in high dimension*, J. Funct. Anal. **239** (2006), no. 1, 100–126, DOI 10.1016/j.jfa.2006.06.002. MR2258218 (2007e:35109)

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE PERNAMBUCO, 50740-540 RECIFE, PE, BRAZIL

E-mail address: mxs@dmate.ufpe.br

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE PARAÍBA, 58051-900 JOÃO PESSOA, PB, BRAZIL

E-mail address: jmbo@pq.cnpq.br