CUBULAR TUBULAR GROUPS

DANIEL T. WISE

ABSTRACT. Let G split as a finite graph of free abelian groups with cyclic edge groups. We characterize when G acts freely on a $\text{CAT}(0)$ cube complex. We show that if G acts properly and semi-simply on a $\text{CAT}(0)$ space, then G acts freely on a $\text{CAT}(0)$ cube complex.

1. INTRODUCTION

1.1. Tubular groups. A *tubular group* G is a group that splits as a finite directed graph Γ of groups where each vertex group G_v is isomorphic to $\mathbb{Z}^2$ and each edge group G_e is isomorphic to $\mathbb{Z}$. We let $c_{\overleftarrow{e}}$ and $c_{\overrightarrow{e}}$ denote the image of a generator c_e of G_e in its vertex groups $G_{\overleftarrow{e}}$ and $G_{\overrightarrow{e}}$. Beautiful aspects of the isoperimetric geometry of tubular groups were investigated in [1]. Tubular groups were classified up to quasi-isometry in [4].

1.2. Statement of the main theorem. For elements $c, s \in \mathbb{Z}^2$, we let $\#[c, s]$ denote their *intersection number*, which is defined to be 0 when $\langle c, s \rangle$ is cyclic, and otherwise equals the index $[\mathbb{Z}^2 : \langle c, s \rangle]$. For a subset $S = \{s_1, \dots, s_r\}$ of $\mathbb{Z}^2$, we let $\#[c, S] = \sum_{i=1}^r \#[c, s_i]$.

An *equitable set* for G is a collection of finite subsets $\{S_v \subset G_v : v \in \Gamma^0\}$ such that S_v generates a finite index subgroup of G_v , and $\#[c_{\overleftarrow{e}}, S_{\overleftarrow{e}}] = \#[c_{\overrightarrow{e}}, S_{\overrightarrow{e}}]$ for each e .

Theorem 1.1. *A tubular group G acts freely on a $\text{CAT}(0)$ cube complex if and only if there is an equitable set of finite subsets of the vertex groups.*

A free action of G on a $\text{CAT}(0)$ cube complex is produced from an equitable set in Theorem 3.1. An equitable set is produced from a free action on a $\text{CAT}(0)$ cube complex in Theorem 5.1. In Section 4, we show that if an equitable set exists, then there exists an equitable set consisting of copies of generators of the edge groups at each vertex group G_v (together with an additional element in the exceptional case that they do not generate a finite index subgroup of G_v). We examine the nature of equitable sets in Section 4 and use this to give the following application in Theorem 4.3:

Theorem 1.2. *If G is tubular and G acts properly and semi-simply on a $\text{CAT}(0)$ metric space, then G acts freely on a $\text{CAT}(0)$ cube complex.*

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Remark 1.3 (Intersection numbers). There are two further equivalent ways of defining $\#[c, s]$ that we will use. First, expressing c and s in coordinates relative to some basis for $\mathbb{Z}^2$ we have the following well-known fact which we will use repeatedly:

$$\#[c, s] = \# \left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right] = \left| \det \begin{pmatrix} c_1 & s_1 \\ c_2 & s_2 \end{pmatrix} \right|.$$

Second, identify $\mathbb{Z}^2$ with $\pi_1 T$ for some torus T , and let s, t also denote closed based paths representing elements of $\pi_1 T$ thought of as maps $s : S^1 \rightarrow T$ and $t : S^1 \rightarrow T$. Then $\#[s, t]$ also equals the *geometric intersection number* of s, t , which is the minimal number of pairs of points $(u, v) \in S^1 \times S^1$ with $s'(u) = t'(v)$ where s', t' range over all maps homotopic to s, t . We will return to this point in Section 5 where we will employ oriented intersection numbers as well.

1.3. Some examples.

Example 1.4 (Gersten's group). We use the notation $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$. Consider the presentation $\langle a, b, c, t \mid a^t = a, b^t = ba, c^t = ca^2 \rangle$ for the group $G \cong F_3 \rtimes \mathbb{Z}$ studied by Gersten in [7]. Gersten showed that no group acting properly and cocompactly on a CAT(0) space can contain a subgroup isomorphic to G , and he used this to show that $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ are not CAT(0) groups for sufficiently large values of n .

Regarding b, c as stable letters of a splitting, Gersten's presentation can be rewritten as $\langle a, b, c, t \mid [a, t], (t^{-1})^b = at^{-1}, (t^{-1})^c = aat^{-1} \rangle$ and then re-presented as $\langle x, y, u, v \mid [x, y], (yx)^u = y, (yx)^v = yx^{-1} \rangle$ using the map: $a \mapsto x^{-1}, t \mapsto x^{-1}y^{-1}, b \mapsto u, c \mapsto v$. This splits G as a graph of groups where Γ is a bouquet of two circles, and the vertex group is $\langle x, y \mid [x, y] \rangle$ and the two stable letters are u, v .

Let $s_1 = yx$ and $s_2 = yx^{-1}$. Then

$$\sum_{i=1}^2 \#[yx, s_i] = \sum_{i=1}^2 \#[y, s_i] = \sum_{i=1}^2 \#[yx^{-1}, s_i] = 2.$$

Thus G acts freely on a CAT(0) cube complex by Theorem 1.1.

Example 1.5 (An "automatic" example). Consider the group G presented by $\langle a, b, u, v \mid ab = ba, a^u = aabb, b^v = aabb \rangle$. This group was shown to be CAT(0) and non-hopfian in [19]. An incorrect proof was given in [19] that G is automatic, and it remains an open problem whether or not G is automatic [6].

Let $\{s_1, s_2, s_3, s_4, s_5\} = \{ab, ab, ab, a, b\}$. Then $\#[a, ab] = 1$, $\#[b, ab] = 1$, $\#[a, b] = 1$, and of course $\#[x, x] = 0$. So we have $3\#[a, ab] + \#[a, a] + \#[a, b] = 3 + 0 + 1$, $3\#[b, ab] + \#[b, a] + \#[b, b] = 3 + 1 + 0$, $3\#[a^2b^2, ab] + \#[a^2b^2, a] + \#[a^2b^2, b] = 0 + 2 + 2$. The conditions of Theorem 3.1 are satisfied, so G acts freely on a CAT(0) cube complex.

G cannot act properly and cocompactly on a CAT(0) cube complex. We disappointedly verified this after realizing that the proof of automaticity in [19] was flawed. Sageev informed us that he and Bestvina have also verified this. The point is that the flat plane corresponding to the $\mathbb{Z}^2$ subgroup isomorphic to $\langle a, b \rangle$ would have at least three infinite families of pairwise crossing hyperplanes along $\langle a \rangle$, $\langle b \rangle$, and $\langle ab \rangle$. Thus cocompactness cannot hold as the cube complex would contain an isometric copy of $\mathbb{E}^3$, but the group does not contain a quasi-isometric copy of $\mathbb{E}^3$. We suspect that there is no finite dimensional free action. We characterize the tubular group that are fundamental groups of nonpositively curved cube complexes

in Theorem 5.8, and find that while Theorem 3.1 shows that there is a rich class of cubulated tubular groups, the class of compactly cubulated tubular groups is comparatively meagre.

We now describe some examples with no equitable sets:

Example 1.6. Consider the group G_n presented by $\langle a, b \mid [a, b], (a^n)^u = ab, (b^n)^v = ab \rangle$. When $n = 2$, the set $S = \{a, b\}$ is an equitable set, however, no equitable set of curves exists when $n \geq 3$. Indeed, for any $(p, q) \in \mathbb{Z}^2 \cong \langle a, b \rangle$ we have $\#[(p, q), (n, 0)] = |qn|$ and $\#[(p, q), (0, n)] = |pn|$ and $\#[(p, q), (1, 1)] = |p - q|$. Thus

$$\#[(p, q), (1, 1)] \leq \frac{1}{n} (\#[(p, q), (n, 0)] + \#[(p, q), (0, n)]),$$

and a similar inequality holds for any $S_i = \{s_1, \dots, s_r\}$ with $s_i = (p_i, q_i)$. Hence for $n > 2$ either $\sum_i \#[(p_i, q_i), (1, 1)] < \sum_i \#[(p_i, q_i), (n, 0)]$ or $\sum_i \#[(p_i, q_i), (1, 1)] < \sum_i \#[(p_i, q_i), (0, n)]$ or both.

Example 1.7 (Simple curve example). We now describe a similar family of examples where each edge group embeds as a direct factor of its vertex groups. Let G_n be the group presented by $\langle a, b, u, v \mid [a, b], (a^n b)^u = ab, (ab^n)^v = ab \rangle$.

For any (p, q) we have $\#[(n, 1), (p, q)] = |qn - p|$ and $\#[(1, n), (p, q)] = |np - q|$ but $\#[(1, 1), (p, q)] = |q - p|$. Consequently for $n \geq 3$ we have

$$\begin{aligned} \#[(1, 1), (p, q)] &= |q - p| \leq |q| + |p| < \frac{1}{2}(n - 1)(|q| + |p|) \\ &= \frac{1}{2}(|qn| - |p| + |pn| - |q|) \leq \frac{1}{2}\#[(n, 1), (p, q)] + \frac{1}{2}\#[(1, n), (p, q)]. \end{aligned}$$

It follows that when $n \geq 3$, for any nonempty collection $C = \{(p_i, q_i)\}$ of curves we have either $\#[(1, 1), C] < \#[(1, n), C]$ or $\#[(1, 1), C] < \#[(n, 1), C]$. More generally, Lemma 4.6 shows that there is no equitable collection when $n \geq 2$. Thus, Theorem 1.1 implies that G_n cannot act freely on a CAT(0) cube complex for $n \geq 2$.

The group G_2 is a single cyclic HNN extension of the Burns group $\langle a, b, x \mid [a, b], a^x = b \rangle$ and serves as an interesting counterexample to possible extensions of the main result in [18]. Note that each G_n is readily seen to be residually finite using $\mathbb{Z}_r^2$ quotients of the vertex group, and then the $\mathbb{Z}^2$ subgroup $\langle x, y \rangle$ is separable since it is maximal abelian, and the edge groups are separable.

Example 1.4 dispels concern that Gersten's example is evidence against the following:

Conjecture 1.8. *Every free-by-cyclic group $F_n \rtimes \mathbb{Z}$ acts freely on a CAT(0) cube complex.*

We expect that Conjecture 1.8 holds in a stronger form when the free-by-cyclic group is word-hyperbolic, as the action will be cocompact in that case. The method given in [18] cannot be applied since the edge group is not quasi-convex (in the word-hyperbolic case). In fact, the cubulation tool in [15] is focused on the setting when the edge groups are malnormal, which is not the case here, so a positive resolution of Conjecture 1.8 will push the envelope.

Conjecture 1.9. *Every [hyperbolic] ascending HNN extension of a free group and every one-relator group acts freely [and cocompactly] on a CAT(0) cube complex unless it has a Baumslag Solitar subgroup $BS(n, m)$ with $m \neq \pm n$.*

FIGURE 1. A highly simplified depiction of a cut-wall. Usually the walls cross each other.

2. BACKGROUND ON DUAL CUBE COMPLEXES

Wallspaces: A *wallspace* is a connected metric space X together with a collection $\mathcal{W}$ of walls. Each wall $W \in \mathcal{W}$ is a connected subspace of X such that $X - W$ consists of exactly two components which are called the *halfspaces* associated to W . A wall W *separates* p, q if they lie in distinct halfspaces of W . Finally, we shall always assume that for each pair of points p, q , there are only finitely many walls in $\mathcal{W}$ separating p, q . Wallspaces were introduced by Haglund-Paulin in [11]. The version of their ideas we describe here were called *geometric wallspaces* in [14]. The motivating case of a wallspace is where $X = \widetilde{M}$ is the universal cover of a 3-manifold, and each wall $W = \widetilde{S}$ is an embedded lift of the universal cover of an immersed 2-manifold $S \rightarrow M$.

Sageev's construction: An *orientation* of a wall is a choice of one of its two halfspaces (think of a choice of normal vector in the motivating case). The CAT(0) cube complex C that is *dual* to a wallspace $(X, \mathcal{W})$ is defined as follows: Each 0-cube of C is a choice of orientation for each wall in $\mathcal{W}$ so that:

- (1) The halfspaces of distinct walls intersect each other.
- (2) For each $p \in X$, all but finitely many halfspaces contain p .

Two 0-cubes are connected by a 1-cube precisely if their orientations differ exactly on that wall. The higher cubes are added precisely when their skeletons are there.

A group G *acts* on the wallspace $(X, \mathcal{W})$ provided that G acts on X and permutes the walls in the sense that $gW \in \mathcal{W}$ whenever $g \in G$ and $W \in \mathcal{W}$. We note that when G acts on $(X, \mathcal{W})$ there is an induced action of G on the dual cube complex. The dual cube complex was introduced in [16], and we refer to [14] for further background and details.

Let g be an automorphism of a wallspace X . A wall W *cuts* the element g if there is a g -invariant subspace A homeomorphic to and identified with $\mathbb{R}$ such that the subspaces of A corresponding to $(-\infty, 0)$ and $(0, \infty)$ lie in opposite halfspaces of W and such that $g^n W \cap A = \{n\}$ for each $n \in \mathbb{Z}$. See Figure 1.

Lemma 2.1. *Let G act on a wallspace $(X, \mathcal{W})$. Suppose each nontrivial element $g \in G$ is cut by a wall $W \in \mathcal{W}$. Then G acts freely on the cube complex C dual to $(X, \mathcal{W})$.*

Proof. Suppose $hp = p$ for some $h \in G$ and $p \in C$. Then $hc = c$, where p lies in the interior of a cube c . Then $g = h^m$ fixes c where $m = \dim(c)!$. Let v be a 0-cube in c .

By hypothesis, g is cut by a wall W , and let $A \hookrightarrow X$ be the associated embedded line. Observe that the walls $\{g^{2n}W\}$ are all oriented in the same A -direction by v ,

FIGURE 2. An immersed wall in the tubular space corresponding to $\langle a, b, s, t \mid [a, b], a^t = b^2, a^t = ab \rangle$. We used the equitable set $\{a, bbb, ab\}$.

since the halfspaces within A are preserved by g . But then we obtain an infinite sequence of walls $g^{2n}W$ that are oriented away from the same point in A . This contradicts the canonical component axiom, which requires that v orient all but finitely many walls toward some (and hence any) basepoint $p \in \tilde{X}$. $\square$

3. EQUITABLE SET $\Rightarrow$ FREE ACTION ON A CAT(0) CUBE COMPLEX

A *graph of spaces* X is associated to the following data: An underlying directed graph Γ and with *vertex space* X_v for each vertex v of Γ and *edge space* $X_e \times [-1, 1]$ for each edge e of Γ , and with π_1 -injective attaching maps $X_e \times \{-1\} \rightarrow X_{\bar{e}}$ and $X_e \times \{+1\} \rightarrow X_{\bar{e}}$. The space X is the quotient space associated to the above data obtained by gluing edge spaces to vertex spaces using the attaching maps. The group $G = \pi_1 X$ splits as a graph of groups whose vertex spaces are $G_v = \pi_1 X_v$ and edge groups are $G_e = \pi_1 X_e$. In general, we must keep track of basepoints unless the vertex groups are abelian.

We will focus on the case where each vertex space X_v is a copy of the torus $T = S^1 \times S^1$, each edge space $X_e \times [-1, 1]$ is a copy of the cylinder $S^1 \times [-1, 1]$, and where the attaching maps are closed local geodesics that we will denote by $c_{-i} \rightarrow T$ and $c_{+i} \rightarrow T$. The space X equipped with this structure is a *tubular space*, and we note that any tubular group G is isomorphic to $\pi_1 X$, where X is a tubular space corresponding to G .

For two homotopy classes α, β of closed curves in the torus T , we use the notation $\#[\alpha, \beta]$ for their *geometric intersection number* which is the nonnegative integer equal to the minimal number of intersections (counting multiplicity) between homotopic representatives α', β' in T .

Theorem 3.1. *Let X be a tubular space. Suppose that for each X_v there is a set of (not all parallel) closed geodesics $\sigma_{v1}, \dots, \sigma_{vr_v}$ in X_v with the following property for each e :*

$$\sum_{j=1}^{r_{\iota(e)}} \#[c_{-e}, \sigma_{\iota(e)j}] = \sum_{j=1}^{r_{\tau(e)}} \#[c_{+e}, \sigma_{\tau(e)j}].$$

Then there are codimension-1 subgroups $H_1, \dots, H_n$ of G and choices of H_i -invariant halfspace pairs, such that $G = \pi_1 X$ acts freely on the dual CAT(0) cube complex.

Proof. Immersed walls: We first describe immersed walls in X . The first type of such immersed walls are the circles $X_e \times \{0\}$. The second type consists of the

components of a graph built from the disjoint union of all the circles $\{\sigma_{vj} : v \in \Gamma^0, 1 \leq j \leq r_v\}$ together with connecting arcs traveling across $X_e \times [-1, 1]$ that join points in $c_{-e} \cap \sigma_{\iota(e)m}$ with points in $c_{+e} \cap \sigma_{\tau(e)m'}$.

The nature of the walls depends on a one-to-one correspondence between points in the geometric intersection of the attaching map of the cylinder with the immersed circles in its target edge space. There is also an extra parameter of leeway (the arc can twist around the circle on its way from one side to the other), but while this affects the structure of the cube complex a bit, it doesn't alter our conclusion.

Walls are trees that have connected intersection with each vertex and edge space: The universal covers of the immersed walls W we have constructed lift to embedded walls $\widetilde{W}$ in $\widetilde{X}$. This is obvious for the first type of *vertical* wall in $\widetilde{X}$ consisting of a line $\widetilde{X}_e \times \{0\} \subset \widetilde{X}_e \times [-1, 1]$.

For the second type, observe that $\widetilde{W}$ intersects a vertex space in a line (some $\tilde{\sigma}$) and then extends outwards through a *single* arc into an edge space whose attaching map intersects $\tilde{\sigma}$, and then exits a new vertex space which it intersects at line $\tilde{\sigma}'$, etc. In general, we see that $\widetilde{W}$ is a tree of spaces whose vertex spaces are lines of the form $\tilde{\sigma}$ and whose edge spaces are arcs, and that the map $\Gamma_{\widetilde{W}} \rightarrow \Gamma_{\widetilde{X}}$ is an embedding of graphs.

f.g. free groups: We have proven that $\widetilde{W}$ is a tree, and when the construction employs a finite amount of data (e.g. a finite graph of groups and finitely many circles), W is a finite graph and $\pi_1 W$ is a finitely generated free group.

Halfspace pairs: $\widetilde{W}$ has a product neighborhood $N \cong \widetilde{W} \times [-\epsilon, \epsilon]$ in $\widetilde{X}$, where $\widetilde{W}$ can be identified with $\widetilde{W} \times \{0\}$. It thus separates $\widetilde{X}$, and we use this to choose a halfspace pair in $\widetilde{X}$ associated to $\pi_1 W$ for each W .

Free action on dual cube complex: The proof that G acts freely divides into two cases depending on whether $g \in G$ is *hyperbolic* or *elliptic* with respect to the action of G on the Bass-Serre tree $\Gamma_{\widetilde{X}}$ of $\widetilde{X}$ induced by the graph of spaces structure on X . In each case we will find a cut-wall $\widetilde{W}$ for g and apply Lemma 2.1.

Let g be a hyperbolic element stabilizing an axis $\bar{A}$ in $\Gamma_{\widetilde{X}}$, and let A denote an axis for g that projects to $\bar{A}$. For instance, we can let $\tilde{x}$ denote a point in a vertex space of $\widetilde{X}$, let α denote an injective path from $\tilde{x}$ to $g\tilde{x}$, and let $A = \bigcup g^n \alpha$. Let $\widetilde{W}$ be a wall dual to a 1-cell in the axis of the action of g on $\Gamma_{\widetilde{X}}$. Then $g^i \widetilde{W}$ are nested with each, and so a strong form of the cut-wall property is satisfied. In particular, $g^i \widetilde{W} \cap A$ consists of a single point in $g^i \alpha$.

Let A be an axis in the plane $\widetilde{X}_v$ that is stabilized by g . Let $\widetilde{W}$ be a wall passing through $\widetilde{X}_v$ with the property that $\widetilde{W} \cap \widetilde{X}_v$ is a line that crosses A . It is here that we use the hypothesis that more than one parallelism class appears among $\sigma_{v1}, \dots, \sigma_{vr_v}$. Then since walls of $\widetilde{X}$ have connected intersection with vertex spaces, we see that $\widetilde{W}$ cuts the element g . $\square$

Remark 3.2. It seems that there are typically infinite dimensional cubes in $C(\widetilde{X})$. It would be interesting to determine exactly when this holds.

Lemma 2.1 does not prove that G acts *metrically properly* in the sense that finite radius balls intersect finitely many of their translates. Note that when $C(\widetilde{X})$ is locally finite, acting freely implies acting metrically properly.

Mark Hagen has observed that provided there is an equitable curve parallel to the attaching circle of each edge space within a vertex space, the cube complex

FIGURE 3. A single wall is depicted in $\tilde{X}$. There is a large plane with four infinite strips emerging upwards and three infinite strips emerging downwards and ending on three further planes. The wall cuts through each plane in a line and cuts through each strip in a segment.

$C(\tilde{X})$ of Theorem 3.1 fails to be locally finite precisely when it contains an infinite cube. The 1-cubes at a 0-cube of $C(\tilde{X})$ correspond to walls of $\tilde{X}$ that pairwise either cross or *osculate* in the sense that they don't cross but are not separated by a third wall. In the presence of the additional hypothesis above, one can verify that the number of pairwise osculating vertical walls is bounded by the valence of Γ . Also, the number of pairwise osculating nonvertical walls is bounded by 2, since a collection of pairwise osculating nonvertical walls must all cross some fixed plane in parallel lines, but the inner of three such walls would separate the other two, so they couldn't osculate. Consequently, the infinite Ramsey theorem implies that if $C(\tilde{X})$ is not locally finite, then there is an infinite collection of pairwise crossing hyperplanes at some 0-cube.

Problem 3.3. When does G act metrically properly on a CAT(0) cube complex?

Problem 3.4. When are the walls quasi-isometrically embedded?

Problem 3.5. Characterize when G acts freely on a finite dimensional CAT(0) cube complex.

We characterize those G acting freely and cocompactly in Theorem 5.8.

Remark 3.6. One way to recognize finite dimensionality is to show that (in a finite cover) each immersed wall embeds. Walls already embed if, for instance, there is one vertex space T , and the curves $\{\sigma_i\}$ are chosen so that $\#[\sigma_i, c_{-j}] = \#[\sigma_i, c_{+j}]$ for each i, j .

In examining the potential finiteness properties of dual cube complexes, one should have in mind the group $\langle a, b, s \mid [a, b], a^s = b \rangle$ given in [3] and its associate $\langle a, b, s, t \mid [a, b], a^s = a, b^t = b \rangle$. The point is that these are obviously cubulated, but have alternate cubulations that are nasty (by using inefficient choices in applying Theorem 3.1).

4. FINDING EQUITABLE SETS

An isomorphism between $\pi_1 T$ and $\mathbb{Z}^2 \subset \mathbb{R}^2$ allows us to identify (homotopy classes of) closed curves in T with integer vectors in $\mathbb{R}^2$. This will facilitate some computations in this section.

In Section 4.1 we explain that if an equitable collection of curves exists, then there is an equitable collection consisting of positive multiples (or copies) of the

attaching circles. Working over $\mathbb{R}$ instead of $\mathbb{Z}$ leads us to consider “equitable collections of vectors” that are not necessarily integer vectors and so do not correspond to closed curves. Nevertheless, in Section 4.2 we show that if an equitable collection of vectors exists, then an equitable collection of integer vectors exists, and so an equitable collection of closed curves exists. We will study the following variation on our problem about the existence of an equitable collection of curves for a tubular group. Consider a set $\{\vec{x}_1, \dots, \vec{x}_r\}$ of vectors in $\mathbb{R}^2$. An *equitable set of vectors* for $\{\vec{x}_1, \dots, \vec{x}_r\}$ is a spanning set $\{\vec{y}_1, \dots, \vec{y}_s\}$ such that $\sum_{i=1}^s |\det(\vec{x}_k, \vec{y}_i)|$ is independent of k .

Problem 4.1. When does a set $\{\vec{x}_1, \dots, \vec{x}_r\}$ of vectors in $\mathbb{R}^2$ have an equitable set of vectors?

As explained in Section 4.1, if an equitable set for $\{\vec{x}_i\}$ exists, then a set of the form $\{p_i \vec{x}_i\}$ exists with each $p_i \geq 0$.

Sometimes the problem about equitable collections of curves for a tubular group can be reformulated in terms of Problem 4.1. For example, consider the tubular group G arising as a multiple HNN extension of $\mathbb{Z}^2 = \langle a, b \rangle$:

$$\langle a, b, t_i : 1 \leq i \leq r \mid [a, b], U_i^{t_i} = V_i : U_i, V_i \in \mathbb{Z}^2 - 0, 1 \leq i \leq r \rangle.$$

The existence of an equitable collection of curves for G is equivalent to the existence of an equitable set of vectors for $\{n_i U_i, n_i V_i : 1 \leq i \leq r\}$ for some choice of $\{n_i > 0\}$.

In Section 4.3, we show that an equitable set of vectors exists for any set $\{\vec{x}_1, \dots, \vec{x}_r\}$ that lies on an ellipse in $\mathbb{R}^2$. We apply this in Section 4.4 to deduce that a tubular group that acts properly and cocompactly on a CAT(0) space also acts freely on a CAT(0) cube complex.

Finally, in Section 4.5 we use polar duals to resolve Problem 4.1 for a general set of vectors – not just a set that lies on an ellipse. This is done in Theorem 4.8, where we show that an equitable set exists if and only if $\{\pm \vec{x}_1, \dots, \pm \vec{x}_r\}$ lie on the boundary of their convex hull.

4.1. If they exist, equitable curves must also arise as multiples of the attaching circles. While a priori, it might appear necessary to use an arbitrary number of equitable curves, we now observe that the number of such curves is bounded by the number of distinct slopes of attached cylinders. Moreover, if $\vec{x}_1, \dots, \vec{x}_r$ are vectors corresponding to the attached cylinders in some torus T , then we can actually choose the equitable set to be of the form $\{p_s \vec{x}_s : s \in S\}$, where each p_i is a nonnegative integer and S is a subset of $\{1, \dots, r\}$.

Indeed, suppose $\{\vec{y}_k\}$ are vectors corresponding to curves in (the part of) an equitable set lying in a torus T . Let $\{\vec{x}_1, \dots, \vec{x}_r\}$ be the attachment curves (discard repeated vectors that span the same line). By possibly replacing $\vec{x}_i$ with $-\vec{x}_i$ we can assume that the corresponding rays all lie in the same halfspace H , and by letting $\vec{x}_{r+1}$ denote $-\vec{x}_1$, we note that H is bounded by the two rays $\vec{x}_1, \vec{x}_{r+1}$. By reordering, we can assume that the rays are in clockwise order within H . We refer to the closed cones between consecutive rays as *sectors*, and note that we have artificially added a sector bounded by $\vec{x}_r, \vec{x}_{r+1}$. Let $\{\vec{y}_1, \dots, \vec{y}_m\}$ denote the part of an equitable set appearing in T . By possibly replacing $\vec{y}_i$ with $-\vec{y}_i$, we can assume that each $\vec{y}_i$ lies in H .

Each $\vec{y}_i$ can be expressed as a rational positive linear combination $\vec{y}_i = a_i \vec{x}_{\overleftarrow{i}} + b_i \vec{x}_{\overrightarrow{i}}$. Observe that replacing each $\vec{y}_i$ by $\{a_i \vec{x}_{\overleftarrow{i}}, b_i \vec{x}_{\overrightarrow{i}}\}$ provides a new set of equitable vectors. Combining the positive linear multiples of each $\vec{x}_i$ to a single positive multiple $p_i \vec{x}_i$ leads to a new set of equitable vectors. Finally, replacing $p_{r+1} \vec{x}_{r+1}$ with $p_{r+1} \vec{x}_1$ does not affect the sum of intersection numbers, and so we complete the process by combining p_1 with p_{r+1} .

Note that keeping track of what happened for each torus, by clearing denominators, we can assume that we have used positive integer multiples.

When there is only one $\vec{x}_1$ the process we described is not sensible, but in that case, it is easy to see that we can use $p_1 \vec{x}_1$ together with $p_2 \vec{x}_2$, where $\vec{x}_2$ is an arbitrary vector outside $\text{span}\{\vec{x}_1\}$.

4.2. A real equitable set implies a rational equitable set. If there is an equitable set consisting of real vectors, then there is also an equitable set consisting of rational, and hence integer vectors. Indeed, a basis for each $\pi_1 X_v \cong \mathbb{Z}^2$ provides a basis for its extension to $\mathbb{R}^2$. An equitable set consisting of a finite subset S_v of vectors in the copy $\mathbb{R}_v^2$ of $\mathbb{R}^2$ at each X_v provides a solution to a system of linear equalities with integer coefficients but absolute values. There is one such equality $\#[c_{\overleftarrow{e}}, S_{\overleftarrow{e}}] = \#[c_{\overrightarrow{e}}, S_{\overrightarrow{e}}]$ for each edge space, and we express it as

$$\sum_{s \in S_{\overleftarrow{e}}} |\det(c_{\overleftarrow{e}}, s)| = \sum_{t \in S_{\overrightarrow{e}}} |\det(c_{\overrightarrow{e}}, t)|.$$

Use the notation $s = (x_s, y_s)$ and $t = (x_t, y_t)$ and $c_{\overleftarrow{e}} = (\overleftarrow{a}, \overleftarrow{b})$ and $c_{\overrightarrow{e}} = (\overrightarrow{a}, \overrightarrow{b})$. For each $|\ast| = |\det(\ast, \ast)|$ in the equation above, we replace $|\ast|$ by $\pm \ast$ when $\ast \neq 0$, and when $|\ast| = 0$ we add the additional equality $\ast = 0$ to our system of equations. We thus obtain a homogeneous system of linear equations with integer coefficients. Specifically, there is a long equation (of the first type below) for each $e \in \text{Edges}(\Gamma)$, and there is a short equation (of the second type) for each $|\ast| = 0$:

$$\sum_{s \in S_{\overleftarrow{e}}} \det((\overleftarrow{a}, \overleftarrow{b}), (x_s, y_s)) - \sum_{t \in S_{\overrightarrow{e}}} \pm \det((\overrightarrow{a}, \overrightarrow{b}), (x_t, y_t)) = 0,$$

$$\det((\overleftarrow{a}, \overleftarrow{b}), (x_s, y_s)) = 0 \quad \text{or} \quad \det((\overrightarrow{a}, \overrightarrow{b}), (x_t, y_t)) = 0.$$

In conclusion, as the coefficients are integers, a real solution to this homogenous system of linear equations implies a rational solution, and moreover, the rational nullspace is a dense subset of the real nullspace. Consequently, there are rational solutions arbitrarily close to a given real solution and, in particular, having at least two nonzero nonparallel equitable vectors in each $\pi_1 X_v$ persists, when we demand that these vectors actually lie in $\mathbb{Z}^2$.

4.3. On an ellipse towards cubulating nonpositively curved tubular groups. Our application towards nonpositively curved tubular groups relies on the following special case of Theorem 4.8 which is easier to motivate.

Theorem 4.2. *Suppose $\{\vec{p}_1, \dots, \vec{p}_k\}$ lie on an ellipse in $\mathbb{R}^2$ centered at $\vec{0}$. Then it has an equitable set of vectors.*

Proof. Without loss of generality, we assume the axes of the ellipse coincide with the cartesian axes, so the ellipse is parametrized by $\vec{u}(\theta) = (a \cos \theta, b \sin \theta)$ with

$a, b > 0$. We first show that for each i with $\vec{p}_i = (a \cos \theta_i, b \sin \theta_i)$ we have

$$\int_0^{2\pi} |\det(\vec{u}(\theta), \vec{p}_i)| d\theta = 4ab.$$

First note that

$$\begin{aligned} \det(\vec{u}(\theta), \vec{p}_i) &= \det((a \cos \theta, b \sin \theta), (a \cos \theta_i, b \sin \theta_i)) \\ &= ab(\cos \theta \sin \theta_i - \sin \theta \cos \theta_i) = -ab \sin(\theta - \theta_i). \end{aligned}$$

Observe that

$$\int_0^{2\pi} |\sin(\theta - \theta_i)| d\theta = \int_0^{2\pi} |\sin(\theta)| d\theta = \int_0^\pi \sin \theta d\theta - \int_\pi^{2\pi} \sin \theta d\theta = 2 + 2 = 4.$$

Consequently

$$\int_0^{2\pi} |\det(\vec{u}(\theta), \vec{p}_i)| d\theta = ab \int_0^{2\pi} |\sin(\theta - \theta_i)| d\theta = 4ab.$$

To simplify notation we assume that $\{\vec{p}_1, \dots, \vec{p}_k\}$ satisfies $P = -P$ and is in cyclic order around the ellipse. The equitable set consists of vectors of the form $\vec{w}_i = \int_{\theta_i}^{\theta_{i+1}} \vec{u}(\theta) d\theta$.

$\sum_i |\det(\vec{w}_i, \vec{p}_j)|$ is independent of j because

$$\begin{aligned} 4ab &= \int_0^{2\pi} |\det(\vec{u}(\theta), \vec{p}_j)| d\theta = \sum_k \int_{\theta_k}^{\theta_{k+1}} |\det(\vec{u}(\theta), \vec{p}_j)| d\theta \\ &= \sum_k \left| \int_{\theta_k}^{\theta_{k+1}} \det(\vec{u}(\theta), \vec{p}_j) d\theta \right| \\ &= \sum_k \left| \det \left(\int_{\theta_k}^{\theta_{k+1}} \vec{u}(\theta) d\theta, \vec{p}_j \right) \right| = \sum_k |\det(\vec{w}_k, \vec{p}_j)|. \quad \square \end{aligned}$$

4.4. Nice CAT(0) tubular groups are cubular. We now obtain the following consequence:

Theorem 4.3 (NPC $\Rightarrow$ cubulated). *Suppose the tubular space X has a metric of nonpositive curvature. So, there is a metric on each torus X_v such that cylinders are attached along closed geodesics of the same length on each side. Then $G = \pi_1 X$ acts freely on a CAT(0) cube complex.*

Proof. Each vertex space X_v is a flat torus that is a quotient of the Euclidean plane $\tilde{X}_v$ by a group of isometries generated by $\alpha_v, \beta_v \in \pi_1 X_v$. Choose a basepoint $\tilde{o}_v \in \tilde{X}_v$, and let $\vec{\alpha}_v$ and $\vec{\beta}_v$ denote the vectors from $\tilde{o}_v$ to $\alpha_v \tilde{o}_v$ and $\beta_v \tilde{o}_v$. We thus obtain a coordinate system $\mathbb{R}_v^2$ for $\tilde{X}_v$ relative to the basis $\{\vec{\alpha}_v, \vec{\beta}_v\}$. Note that the unit circle in $\tilde{X}_v$ corresponds to an ellipse in $\mathbb{R}_v^2$.

Consider the attaching circles $\{c_{\pm e} \rightarrow X_{\pm e} : e \in \text{Edges}(\Gamma)\}$ of the various cylinders C_e , where we use the notation $-e$ and $+e$ for the initial and terminal vertices of an edge e of the underlying graph Γ of the tubular space X . These attaching circles correspond to vectors $\{\vec{c}_{\pm e} \in \tilde{X}_{\pm e}\}$, and we note that the length of $c_{\pm e}$ equals the length of $\vec{c}_{\pm e}$.

For each $v \in \text{Vertices}(\Gamma)$ we choose unit vectors in $\tilde{X}_{\pm e}$ corresponding to the $\{\vec{c}_{\pm e} : \pm e = v\}$. If some X_v has all circles attached at the same slope, then we add an additional unit vector in some other rational direction. Let U_v denote the resulting set of unit length vectors in $\tilde{X}_v$, and let U'_v be the corresponding vectors in $\mathbb{R}_v^2$.

Applying Theorem 4.2 to the vectors U'_v on the ellipse in $\mathbb{R}_v^2$, we see that for each v there is an equitable set W'_v of vectors for the set U'_v of vectors, and hence an equitable set W_v for the unit vectors U_v of $\tilde{X}_v$. By possibly rescaling W_v , we can assume without loss of generality that $\# [W_v, \vec{u}] = 1$ for each $\vec{u} \in U_v$.

Observe that $\# [W_{-e}, \vec{c}_{-e}] = \# [W_{+e}, \vec{c}_{+e}]$ for each e , since the left and right sides equal the lengths of $\vec{c}_{-e}$ and $\vec{c}_{+e}$ respectively, and those equal the lengths of the attaching circles c_{-e} and c_{+e} respectively. It is here that we use that $\tilde{X}$ is $\text{CAT}(0)$ to know that the length of c_{-e} equals the length of c_{+e} .

Having found an equitable collection of vectors for the collection of vectors corresponding to the attaching circles in the various $\tilde{X}_v$, Section 4.2 shows that there is an equitable set of integer vectors with respect to our coordinate system, and we thus obtain an equitable collection of curves for X in the various tori X_v . We conclude that $G = \pi_1 X$ acts freely on a $\text{CAT}(0)$ cube complex by Theorem 3.1. $\square$

Lemma 4.4. *Let G be an f.g. tubular group. If G acts properly and semi-simply on a $\text{CAT}(0)$ space Y , then $G = \pi_1 X$ where X is compact npc and tubular.*

Proof. If G acts freely on a $\text{CAT}(0)$ space Y by semi-simple isometries, then each vertex group G_v stabilizes and acts cocompactly on a flat plane $\tilde{F}_v$ [2]. Since distinct axes for a generator of an edge group have the same translation length, this provides tori $G_v \backslash \tilde{F}_v$ and cylinders C_e from which to build a nonpositively curved tubular space X with $G \cong \pi_1 X$. $\square$

4.5. Equitable iff on the boundary of its convex hull.

Theorem 4.5. *Let $\{\pm p_1, \dots, \pm p_r\}$ be a set of vectors in $\mathbb{R}^2$. Then $\{\pm p_1, \dots, \pm p_r\}$ has an equitable set if and only if $\{\pm p_1, \dots, \pm p_r\}$ lies on the boundary of its convex hull in $\mathbb{R}^2$. (We do not insist that it is the vertex set of its hull.)*

We begin by observing that:

Lemma 4.6. *There is no equitable set if $\{\pm \vec{p}_1, \dots, \pm \vec{p}_r\}$ do not lie on the boundary of their convex hull.*

Proof. Suppose $\vec{p}_1 = s\vec{p}_2 + t\vec{p}_3$ where $s, t \geq 0$ but $s + t < 1$, and note that this means that $\vec{p}_1$ lies in the interior of the triangle whose vertices are at $\{\vec{0}, \vec{p}_2, \vec{p}_3\}$.

Let $Y = \bigsqcup_{i=1}^r a_i \vec{p}_i$ with $a_i \geq 0$. Then

$$\#[Y, \vec{p}_1] = \#[Y, s\vec{p}_2 + t\vec{p}_3] \leq \#[Y, s\vec{p}_2] + \#[Y, t\vec{p}_3] = s\#[Y, \vec{p}_2] + t\#[Y, \vec{p}_3].$$

Thus if there exists Y with $\#[Y, \vec{p}_i]$ all equal to the same positive number, then

$$\#[Y, \vec{p}_1] \leq s\#[Y, \vec{p}_2] + t\#[Y, \vec{p}_3] = s\#[Y, \vec{p}_1] + t\#[Y, \vec{p}_1] = (s+t)\#[Y, \vec{p}_1] < \#[Y, \vec{p}_1].$$

Note that the final inequality holds because $\#[Y, \vec{p}_1] > 0$ and $(s+t) < 1$. $\square$

The main direction in the proof of Theorem 4.5 is proven in Theorem 4.8 and depends upon the notion of a polar dual.

Definition 4.7 (Polar duals). The *polar dual* $\bar{M}$ of a convex polygon M in $\mathbb{R}^2$ that contains $\vec{0}$ is a polygon consisting of the following points:

$$\bar{M} = \{\vec{n} \in \mathbb{R}^2 : \vec{m} \cdot \vec{n} \leq 1 \ \forall \vec{m} \in M\}.$$

(Later we use $M, \bar{M}$ for the boundaries of polygons.) We refer to [8] for more on polar duals. The features of the polar dual that we will use are:

- (1) $\bar{M}$ is convex.
- (2) For a vertex $\vec{m}$ of M , there is an edge $\bar{m}$ in $\partial \bar{M}$ that is a subsegment of the line satisfying $\vec{m} \cdot \vec{x} = 1$.

Theorem 4.8. Let $\{\vec{p}_1, \dots, \vec{p}_{2k}\} = -\{\vec{p}_1, \dots, \vec{p}_{2k}\}$ be a set of vectors that lie on the boundary P of their convex hull. Then $\vec{p}_1, \dots, \vec{p}_{2k}$ has an equitable set.

Proof. Assume that $\{\vec{p}_1, \dots, \vec{p}_{2k}\}$ are arranged in counterclockwise order about P . Moreover, we can assume each $\vec{p}_i$ is a vertex of P – we explain how to deduce the general case at the end of the proof. Let $\bar{P}$ denote the polar dual of P . Let $2L$ denote the length of $\bar{P}$. Parametrize $\bar{P}$ by arc length, and let s_i be so that the point $\bar{P}(s_i)$ is the midpoint of the edge $\bar{p}_i$ for each i . We treat our indices and parameter modulo $2k$ and $2L$, so $\bar{P}(s_i + L)$ corresponds to $\bar{p}_{i+k} = -\bar{p}_i$. Let $T(s)$ denote the tangent vector to $\bar{P}$ at $\bar{P}(s)$ and let $N(s)$ denote the outward pointing normal vector at $\bar{P}(s)$. Note that $N(s)$ is piecewise constant and $\vec{p}_i \cdot \bar{P}(s_i) = 1$ for each i .

We claim that for each i ,

$$\int_0^{2L} |\det(\vec{p}_i, N(s))| ds = 4.$$

Note that $\bar{P}$ is convex since P is convex. Consequently, $\det(\vec{p}_i, N(s))$ is nonnegative for $s_i \leq s \leq s_i + L$ and is nonpositive for $s_i + L \leq s \leq s_i + 2L$.

Observe that

$$\begin{aligned} \int_{s_i}^{s_i+L} \det(\vec{p}_i, N(s)) ds &= \int_{s_i}^{s_i+L} -\vec{p}_i \cdot T(s) ds \\ &= \int_{[\bar{P}(s_i), \bar{P}(s_i+L)]} -\vec{p}_i \cdot d\bar{P} = \vec{p}_i \cdot \bar{P}(s_i) - \vec{p}_i \cdot \bar{P}(s_i + L) = 1 - (-1) = 2. \end{aligned}$$

Observe that

$$\begin{aligned} \int_0^{2L} |\det(\vec{p}_i, N(s))| ds &= \int_{s_i}^{s_i+2L} |\det(\vec{p}_i, N(s))| ds \\ &= \int_{s_i}^{s_i+L} \det(\vec{p}_i, N(s)) ds - \int_{s_i+L}^{s_i+2L} \det(\vec{p}_i, N(s)) ds = 2 - (-2) = 4. \end{aligned}$$

As in Theorem 4.2, the equitable set $\{\vec{w}_i\}$ is defined by: $\vec{w}_i = \int_{\bar{p}_i} N(s) ds$. Thus each $\vec{w}_i$ has the same direction as $\vec{p}_i$ but has length equal to the corresponding edge $\bar{p}_i$ of $\bar{P}$. Finally, when the vertices of P are rational, then the corresponding lines bounding $\bar{P}$ are rational, and so the vertices of $\bar{P}$ are rational. Consequently, the vectors $\vec{u}_i$ between consecutive vertices of $\bar{P}$ are rational, and hence the vectors $\vec{w}_i$ are rational since they are $\frac{\pi}{2}$ rotations of the $\vec{u}_i$.

We have found an equitable set under the assumption that $\{\pm \vec{p}_i\}$ are vertices of P , and, moreover, our equitable set consists of scalar multiples of these vertices. It follows that our equitable set is also an equitable set for any set contained in the boundary of P . Indeed (avoiding an integral argument), if $\vec{q} \in \partial P$, then $\vec{q} = t\vec{p}_i + (1-t)\vec{p}_{i+1}$ for some i and some $t \in [0, 1]$. Now

$$\begin{aligned} |\det(\vec{q}, \vec{p}_\ell)| &= |\det(t\vec{p}_i + (1-t)\vec{p}_{i+1}, \vec{p}_\ell)| \\ &= |t \det(\vec{p}_i, \vec{p}_\ell) + (1-t) \det(\vec{p}_{i+1}, \vec{p}_\ell)| \\ &= t |\det(\vec{p}_i, \vec{p}_\ell)| + (1-t) |\det(\vec{p}_{i+1}, \vec{p}_\ell)|, \end{aligned}$$

where the final equality holds because $\det(\vec{p}_i, \vec{p}_\ell)$ and $\det(\vec{p}_{i+1}, \vec{p}_\ell)$ are both nonpositive or both nonnegative. As each $\vec{w}_\ell$ is a positive multiple of $\vec{p}_\ell$, summing over $1 \leq \ell \leq 2k$ shows that $\#[\vec{q}, \{\vec{w}_1, \dots, \vec{w}_{2k}\}] = t\#[\vec{p}_i, \{\vec{w}_1, \dots, \vec{w}_{2k}\}] + (1-t)\#[\vec{p}_{i+1}, \{\vec{w}_1, \dots, \vec{w}_{2k}\}]$. The result is independent of $\vec{q} \in \partial P$, as claimed. $\square$

5. FREE ACTION ON A CAT(0) CUBE COMPLEX $\Rightarrow$ EQUITABLE SET

In this section we show the following converse to Theorem 3.1:

Theorem 5.1. *Suppose that G is a tubular group that acts freely on a CAT(0) cube complex $\tilde{Y}$. Then there is an equitable set of curves for the vertex tori of G .*

The proof hinges upon the following result whose proof is the focus of this section.

Main Lemma 5.2. *Let X be a square complex homeomorphic to a torus. Let $X \rightarrow Y$ be a π_1 -injective combinatorial map. Let $\{H_1, \dots, H_q\}$ denote representatives of the finitely many $\mathbb{Z}^2$ -orbits of hyperplanes in $\tilde{Y}$ that are “odd” in the sense that the preimage of H_i in $\tilde{X}$ contains a finite but odd number of infinite lines. For each i , let $\bar{d}_i$ denote the immersed dual curve in X that is the image of one of the infinite lines in $H_i \cap \tilde{X}$ quotiented by its stabilizer in $\mathbb{Z}^2$. Let $s \rightarrow X$ be a closed combinatorial path, and let $\gamma \rightarrow Y$ be a closed geodesic homotopic to the image $s \rightarrow Y$. Then*

$$|\gamma| = \sum_{i=1}^q \#[s, \bar{d}_i].$$

We use the term *dual curve* in a surface or disk diagram to refer to an “immersed hyperplane” consisting of a sequence of midcubes of squares. See Figure 4.

The *odd hyperplanes* are precisely those whose halfspace pair $\overleftarrow{H}_i, \overrightarrow{H}_i$ in $\tilde{Y}$ induces a deep halfspace pair $\overleftarrow{H}_i \cap \tilde{X}, \overrightarrow{H}_i \cap \tilde{X}$ in $\tilde{X}$.

Proof of Theorem 5.1. For each vertex group G_v , let $X_v \rightarrow Y$ be a combinatorial torus with $\pi_1 X_v$ mapping to G_v . The immersed circles $\{\bar{d}_{vi}\}$ of X_v will provide our equitable choice of dual curves. Note that there are at least two odd hyperplanes for each v , since $\pi_1 X_v$ acts freely.

For each edge group G_e embedded in G_v, G_u we choose closed combinatorial paths $s_v \rightarrow X_v$ and $s_u \rightarrow X_u$ representing the images of the generator g_e of G_e . Observe that $\sum_i \#[s_v, \bar{d}_{vi}] = \sum_j \#[s_u, \bar{d}_{uj}]$, since by Main Lemma 5.2, both sums equal the length of a closed geodesic γ_e representing the conjugacy class of g_e in $\pi_1 X$. $\square$

After subdividing Y we can assume that the 1-skeleton is oriented so that the 1-cells dual to immersed hyperplanes are all oriented consistently. This induces an orientation on the 1-skeleton of $\tilde{Y}$ that is preserved by the action of $\pi_1 Y$. Another convenience of this subdivision is provided by the following result from [10]:

Theorem 5.3 (Haglund). *Let G act on a $CAT(0)$ cube complex Y' and let Y be the cubical subdivision of Y' . For each $g \in G$, there is either a 0-cube p in Y with $gp = p$ or there is a combinatorial axis γ for g in the sense that γ is a combinatorial geodesic in Y and $g\gamma = \gamma$ and g acts by a nontrivial translation along γ .*

We sidestep the following issue whose positive resolution would have simplified matters:

Problem 5.4. Let Y be a nonpositively curved cube complex, and let $\mathbb{Z}^2$ be a subgroup of $\pi_1 Y$. Does there exist a square 2-complex X homeomorphic to a torus, and a combinatorial map $X \rightarrow Y$ with $\pi_1 X$ mapping to $\mathbb{Z}^2$, such that $\tilde{X} \rightarrow \tilde{Y}$ is injective on the set of hyperplanes? Does $\tilde{Y}$ contain a $\mathbb{Z}^2$ -cocompact isometrically embedded subcomplex?

Definition 5.5 (Oriented intersection number $\vec{\#}[s, d]$). For transverse oriented curves s, d in an orientable surface X , their oriented intersection number $\vec{\#}[s, d]$ counts the number of signed intersection points which are oriented according to the two possible types of oriented intersections. It is a homotopy invariant and, in fact, depends only on the homology classes [9].

We will take advantage of this invariance in our setting and compute using a combinatorial reformulation. Let X be an orientable surface that is a square complex. Let $d \rightarrow X$ be a closed immersed dual curve. Let $s \rightarrow X$ be a closed oriented combinatorial path. Choose a consistent orientation of the 1-cubes dual to d . Let $\#^{+1}[s, d]$ denote the number of times that s traverses d -dual 1-cubes positively, and, similarly, let $\#^{-1}[s, d]$ denote the number of negative traversals. Then we compute the *oriented intersection number* $\vec{\#}[s, d]$ as follows:

$$\vec{\#}[s, d] = \#^{+1}[s, d] - \#^{-1}[s, d].$$

When X is a torus, the paths s, d can be homotoped so that all their oriented intersection numbers have the same sign. We may thus record the following:

Lemma 5.6. *When X is a torus, we have $\vec{\#}[s, d] = |\vec{\#}[s, d]|$.*

Lemma 5.7. *Let $\mathbb{Z}^2$ act freely and without hyperplane inversions on $\tilde{Y}$. So $\mathbb{Z}^2$ preserves an orientation on the 1-skeleton that agrees with the positive and negative halfspace of each hyperplane. Let $g \in \mathbb{Z}^2$ be a nontrivial element stabilizing a geodesic $\tilde{\gamma} \subset \tilde{Y}$, and let H be a hyperplane crossed by $\tilde{\gamma}$. Then all 1-cells of $\tilde{\gamma}$ that are dual to $\mathbb{Z}^2$ translates of H are oriented in the same direction in $\tilde{\gamma}$.*

Proof. Since $\overleftarrow{H}, \overrightarrow{H}$ determines a deep halfspace pair for $\langle g \rangle$, we see that it also determines a deep halfspace pair for $\mathbb{Z}^2$. More precisely, let p be the initial vertex of the edge in $\tilde{\gamma}$ dual to H . The negative and positive halfspaces of $\langle g \rangle$ are those elements translating p to $\overleftarrow{H}$ and $\overrightarrow{H}$ respectively. We likewise (consistently) obtain halfspace pairs for $\mathbb{Z}^2$. These halfspaces for $\mathbb{Z}^2$ are not coarsely equivalent to each

other, since those for $\langle g \rangle$ are not coarsely equivalent. It is in this sense that we obtain a “deep” halfspace pair.

Translating by an element $k \in \mathbb{Z}^2$ coarsely preserves the halfspaces in $\mathbb{Z}^2$ and, in particular, positive and negative halfspaces of $\langle g \rangle$ are respectively translated by k into positive and negative halfspaces of $\mathbb{Z}^2$. Thus $k\vec{H} \cap \tilde{\gamma}$ is coarsely the same as $\vec{H} \cap \tilde{\gamma}$. We thus see that any dual 1-cube of kH that $\tilde{\gamma}$ passes through has the same direction as the 1-cube of H that $\tilde{\gamma}$ passes through. $\square$

FIGURE 4. Three dual curves in a disk diagram. If we identify opposite sides, we have three closed dual curves in a cubical torus.

FIGURE 5. Dual curves within the annular diagram A between s and γ .

Proof of Main Lemma 5.2. For each i , the preimage of H_i in $\tilde{X}$ consists of a collection of lines and circles. The circles have trivial stabilizers and the lines have co-compact stabilizers in $\mathbb{Z}^2$. We choose one representative in each $\text{Stabilizer}(H_i) \cap \mathbb{Z}^2$ orbit, and then quotienting each line by its stabilizer, we obtain a collection of immersed closed dual curves $\{d_{ij} \rightarrow X : 1 \leq j \leq m_i\}$. Note that each such immersed dual curve is 2-sided and has a determined positive normal direction.

Let $A \rightarrow Y$ be an annular diagram between $s \rightarrow Y$ and $\gamma \rightarrow Y$. See Figure 5. For a specific H_i , consider the dual curves in A corresponding to edges of s passing through $\bigsqcup_{j=1}^{m_i} d_{ij}$. A subset of these dual curves starts and ends on s , and corresponds to cancellations between pairs of edges where s traverses d_{ij} edges positively and negatively. The remaining dual curves end on γ and are all oriented in the same way by Lemma 5.7. Thus, the edges of γ are partitioned into classes according to the hyperplanes H_i , and as we have computed that the number of edges in the i -th class equals $|\sum_{j=1}^{m_i} \vec{\#}[s, d_{ij}]|$, we have

$$(1) \quad |\gamma| = \sum_{H_i} \left| \sum_{j=1}^{m_i} \vec{\#}[s, d_{ij}] \right|.$$

The infinite lines in $H_i \cap \tilde{X}$ are disjoint and $\mathbb{Z}^2$ acts by translations. Hence each line in $H_i \cap \tilde{X}$ has the same stabilizer in $\mathbb{Z}^2$. Consequently, the d_{ij} are all homotopic to each other in X . Observe that these lines in $\tilde{X}$ have alternating normal orientations inherited from H_i . Indeed, a path traveling between consecutive H_i lines might pass through H_i circles on its way, but passes through oppositely oriented dual edges in pairs for each such circle.

Let H_i be an *odd* hyperplane having an odd number of intersection lines with $\tilde{X}$. Let $\bar{d}_i$ denote an immersed dual curve that is the quotient of one of these infinite lines and note that $|\vec{\#}[s, \bar{d}_i]| = \#[s, \bar{d}_i]$ by Lemma 5.6. As $\vec{\#}[s, d_{ij}] = 0$ when d_{ij} is inessential, and as the odd number of remaining terms are of the same magnitude but alternate in sign, we have

$$(2) \quad \text{If } H_i \text{ is odd, then:} \quad \left| \sum_{j=1}^{m_i} \vec{\#}[s, d_{ij}] \right| = \#[s, \bar{d}_i].$$

There are two types of *even* hyperplanes H_k which we can ignore since $\sum_{j=1}^{m_k} \vec{\#}[s, d_{kj}] = 0$. These correspond to situations where the wall induced on $\tilde{X}$ is not essential, in the sense that at least one of the two halfspaces lies in a finite neighborhood of the preimage of H_i in $\tilde{X}$.

The first even type is when $H_k \cap \tilde{X}$ contains a *finite but even* number of lines. (This includes the possibility where there are no lines.) In this case the essential dual curves have alternating orientations but are homotopic. The sum is zero since its nonzero terms are of the same magnitude but alternate. Again, we note that the dual curves d_{kj} that are inessential have $\vec{\#}[s, d_{kj}] = 0$ for any s and so they do not contribute to the sum.

The second even type is when $H_k \cap \tilde{X}$ contains *infinitely many* lines. The orientations of these lines alternate, and there are an even number of resulting closed curves $\{d_{kj}\}$. While $\text{Stabilizer}(H_k)$ leaves each of the lines invariant in the case when $H_k \cap \tilde{X}$ has finitely many lines, in the case where there are infinitely many lines there is some minimal length translation shifting the sequence of lines but preserving their orientations. We again obtain an even number of alternatingly signed essential dual curves in $\{d_{kj}\}$:

$$(3) \quad \text{If } H_i \text{ is even, then:} \quad \left| \sum_{j=1}^{m_i} \vec{\#}[s, d_{ij}] \right| = 0.$$

In conclusion, combining equations (1), (2), and (3) we have

$$\begin{aligned} |\gamma| &= \sum_{H_i} \left| \sum_{j=1}^{m_i} \vec{\#}[s, d_{ij}] \right| = \sum_{H_i \text{ even}} \left| \sum_{j=1}^{m_i} \vec{\#}[s, d_{ij}] \right| + \sum_{H_i \text{ odd}} \left| \sum_{j=1}^{m_i} \vec{\#}[s, d_{ij}] \right| \\ &= \sum_{H_i \text{ odd}} \#[s, \bar{d}_i]. \end{aligned} \quad \square$$

5.1. Cocompactness. While the results of Section 3.1 indicate that a rich class of tubular groups acts freely on CAT(0) cube complexes, the family of such groups that act freely and cocompactly is comparatively meagre.

Theorem 5.8. *Let G be a tubular group that acts freely and cocompactly on a CAT(0) cube complex. Then G is the fundamental group of a nonpositively curved cube complex built from square tori vertex space by gluing product cylinder edge spaces along combinatorial local isometries.*

Proof. The flat torus theorem [2] provides a flat plane E_v in $\tilde{Y}$ with a free cocompact G_v action for each v . There are possibly finitely many hyperplanes of $\tilde{Y}$ that contain E_v , and all other hyperplanes intersect it in either a line or $\emptyset$. We refer to those

intersecting it in a line as *essential*. There are two parallelism classes of intersection lines in E_v , for if there were $n \geq 3$, then the G_v action would give n infinite families of pairwise crossing hyperplanes, and thus a quasi-isometrically embedded copy of $\mathbb{E}^n$ in $\tilde{Y}$ which is impossible since G is 2-dimensional.

Indeed, it was proven in [17] that if H embeds in G by a quasi-isometry, then $\text{cd}(H) \leq \text{cd}(G)$; see also [5]. A simple argument can be sketched in our case: $\mathbb{E}^n$ does not contain a deeply separating quasi-line for $n \geq 3$. Consequently, the image of $\mathbb{E}^n \rightarrow G$ must lie in a finite neighborhood of some Z^2 vertex group. But the growth of this neighborhood is x^2 , whereas the growth of $\mathbb{Z}^n$ is x^n .

There is a constant B such that intersection lines at distance $\geq B$ correspond to noncrossing hyperplanes. Indeed, otherwise we would be able to find two infinite families of intersection lines that are all parallel, so that the hyperplanes in one family cross the hyperplanes in the other. This would again give us an impossible quasi-isometric copy of $\mathbb{E}^3$ in $\tilde{Y}$ and thus G . In conclusion, the convex hull of E_v is a G_v cocompact subcomplex, as it contains finitely many G_v orbits of cubes since there are finitely many G_v orbits of pairwise crossing collections of hyperplanes.

For each G_v , there are arbitrarily large neighborhoods of E_v that are convex subcomplexes $\tilde{Y}_v^{+R}$. For adjacent u, v , there are thus cocompact thickenings $\tilde{Y}_u^{+R}$ and $\tilde{Y}_v^{+R}$ that intersect in a convex subcomplex stabilized cocompactly by the edge group G_e between G_u and G_v . By cocompactness, we see that the axis for (a generator of) this edge group passes through only one parallelism class of essential hyperplanes in each of $\tilde{Y}_u^{+R}$ and $\tilde{Y}_v^{+R}$. Indeed, otherwise the G_e -cocompact convex subcomplex $\tilde{Y}_u^{+R} \cap \tilde{Y}_v^{+R}$ contains two infinite families of pairwise crossing hyperplanes (and thus a 2-dimensional quasi-flat, which would contradict its G_e -cocompactness).

By Lemma 5.2 as these two parallelism classes represent the only possible odd hyperplanes, we see that the geodesic has the same combinatorial length on each side. Thus each G_v can be thought of as the fundamental group of the nonpositively curved square complex obtained from the essential parallelism classes, and that the G_e can be represented by a combinatorial local isometry of a circle on each side. (This last part of the argument repeats the construction proving Theorem 3.1.) $\square$

Corollary 5.9. *If the tubular group G is the fundamental group of a compact nonpositively curved cube complex, then G has a finite index subgroup G' that is special.*

We expect that X is virtually special for any compact nonpositively curved cube complex X with $G = \pi_1 X$. However, we do not prove this. Instead we show that the specific complex X constructed in the proof of Theorem 5.8 is virtually special.

Proof. The complex X is locally a $\mathcal{VH}$ -complex, in the sense that its vertex links are bipartite graphs. It has a finite $\mathcal{VH}$ double cover. There is a finite cover where all attaching maps are injective. Indeed, letting S_n^1 denote an n -edge circle, we choose finite product covers of all the tori vertex spaces that are products $S_n^1 \times S_n^1$, where n is the least common multiple of all lengths of attaching maps of edge spaces. It is then easy to see how to obtain a finite cover using this and corresponding covers of the edge spaces. The result is a clean $\mathcal{VH}$ -complex, which is then virtually special [12]. Alternatively, at this stage the result follows from the criterion for virtual specialness of graphs of groups given in [13]. $\square$

Corollary 5.10. *If there are more than two parallelism classes of edge groups in some vertex group G_v , then the tubular group G cannot act freely and cocompactly on a $CAT(0)$ cube complex.*

If there are exactly one or two parallelism classes of edge groups in each vertex group, then G acts freely and cocompactly unless it contains a Baumslag Solitar group $\langle a, t \mid (a^n)^t = a^m \rangle$ where $n \neq \pm m$.

Example 5.11. The tubular group $\langle a, b, s, t \mid [a, b], (aa)^s = ab, (bb)^t = ab \rangle$ has an equitable set with just two curves $\{a, b\}$, but it has more than two parallelism classes of edge groups at its vertex group.

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DEPARTMENT OF MATHEMATICS & STATISTICS, MCGILL UNIVERSITY, MONTREAL, QUEBEC,
CANADA H3A 0B9

E-mail address: wise@math.mcgill.ca