THE UNSTABLE SLICE FILTRATION

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ABSTRACT. The main goal of this paper is to construct an analogue of Voevodsky's slice filtration in the motivic unstable homotopy category. The construction is done via birational invariants; this is motivated by the existence of an equivalence of categories between the orthogonal components for Voevodsky's slice filtration and the birational motivic stable homotopy categories constructed by the author in 2013. Another advantage of this approach is that the slices appear naturally as homotopy fibres (and not as in the stable setting, where they are defined as homotopy cofibres) which behave much better in the unstable setting.

1. INTRODUCTION

Our main goal is to construct a tower in the motivic unstable homotopy category with similar properties to Voevodsky's slice tower [26] in the motivic stable homotopy category. In this section we recall Voevodsky's definition for the slice filtration and explain why our construction in the unstable setting is a natural analogue.

Definitions and notation. In this paper X will denote a Noetherian separated base scheme of finite Krull dimension, Sch_X the category of schemes of finite type over X and Sm_X the full subcategory of Sch_X consisting of smooth schemes over X regarded as a site with the Nisnevich topology. All the maps between schemes will be considered over the base X . Given $Y \in Sch_X$, all the closed subsets Z of Y will be considered as closed subschemes with the reduced structure.

Let $\mathcal{M}$ be the category of pointed simplicial presheaves on Sm_X equipped with the motivic Quillen model structure [21] constructed by Jardine [8, Thm. 1.1] which is Quillen equivalent to the one defined originally by Morel-Voevodsky [8, Thm. 1.2], taking the affine line $\mathbb{A}_X^1$ as an interval [13, p.86 Thm.3.2]. We will write $\mathcal{H}$ for its associated homotopy category. Given a map $f : Y \rightarrow W$ in Sm_X , we will abuse notation and denote by f the induced map $f : Y_+ \rightarrow W_+$ in $\mathcal{M}$ between the corresponding pointed simplicial presheaves represented by Y and W , respectively.

We define T in $\mathcal{M}$ as the pointed simplicial presheaf represented by $S^1 \wedge \mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplicative group $\mathbb{A}_X^1 - \{0\}$ pointed by 1 and S^1 denotes the simplicial circle. Given an arbitrary integer $r \geq 1$, let S^r (resp. $\mathbb{G}_m^r$) denote the iterated smash product of S^1 (resp. $\mathbb{G}_m$) with r -factors: $S^1 \wedge \cdots \wedge S^1$ (resp. $\mathbb{G}_m \wedge \cdots \wedge \mathbb{G}_m$); $S^0 = \mathbb{G}_m^0$ will be by definition equal to the pointed simplicial presheaf represented by the base scheme X .

We will use the following notation in all the categories under consideration: $*$ will denote the terminal object and $\cong$ will denote that a map is an isomorphism.

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Let $Spt(\mathcal{M})$ denote Jardine's category of symmetric T -spectra on $\mathcal{M}$ equipped with the motivic model structure defined in [8, theorem 4.15] and $\mathcal{SH}$ denote its homotopy category, which is triangulated. We will follow Jardine's notation [8, pp. 506-507] where F_n denotes the left adjoint to the n -evaluation functor

$$\begin{aligned} Spt(\mathcal{M}) &\xrightarrow{ev_n} \mathcal{M}, \\ (E^m)_{m \geq 0} &\longmapsto E^n. \end{aligned}$$

Notice that $F_0(A)$ is just the usual infinite suspension spectrum $\Sigma_T^\infty A$.

For every integer $q \in \mathbb{Z}$, we consider the following family of symmetric T -spectra:

$$(1.1) \quad C_{eff}^q = \{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \mid n, r, s \geq 0; s - n \geq q; U \in Sm_X\},$$

where U_+ denotes the simplicial presheaf represented by U with a disjoint base point. Let $\Sigma_T^q \mathcal{SH}^{eff}$ denote the smallest full triangulated subcategory of $\mathcal{SH}$ which contains C_{eff}^q and is closed under arbitrary coproducts. Voevodsky [26] defines the slice filtration in $\mathcal{SH}$ to be the following family of triangulated subcategories:

$$\cdots \subseteq \Sigma_T^{q+1} \mathcal{SH}^{eff} \subseteq \Sigma_T^q \mathcal{SH}^{eff} \subseteq \Sigma_T^{q-1} \mathcal{SH}^{eff} \subseteq \cdots.$$

It follows from the work of Neeman [14], [15] that the inclusion

$$i_q : \Sigma_T^q \mathcal{SH}^{eff} \rightarrow \mathcal{SH}$$

has a right adjoint $r_q : \mathcal{SH} \rightarrow \Sigma_T^q \mathcal{SH}^{eff}$ [18, Prop. 3.1.12] and that the functors

$$\begin{aligned} f_q : \mathcal{SH} &\rightarrow \mathcal{SH}, \\ s_{<q} : \mathcal{SH} &\rightarrow \mathcal{SH}, \\ s_q : \mathcal{SH} &\rightarrow \mathcal{SH} \end{aligned}$$

are triangulated, where f_q is defined as the composition $i_q \circ r_q$; and $s_{<q}, s_q$ are characterized by the fact that for every $E \in \mathcal{SH}$, there exist distinguished triangles in $\mathcal{SH}$ [18, Thms. 3.1.16, 3.1.18]:

$$(1.2) \quad f_q E \xrightarrow{\theta_q^E} E \xrightarrow{\pi_{<q}^E} s_{<q} E \longrightarrow S^1 \wedge f_q E,$$

$$(1.3) \quad f_{q+1} E \xrightarrow{\rho_q^E} f_q E \xrightarrow{\pi_q^E} s_q E \longrightarrow S^1 \wedge f_{q+1} E.$$

We will refer to $f_q E$ as the $(q-1)$ -connective cover of E , to $s_{<q} E$ as the q -orthogonal component of E , and to $s_q E$ as the q -slice of E . It follows directly from the definition that $s_{<q+1} E, s_q E$ satisfy that for every symmetric T -spectrum K in $\Sigma_T^{q+1} \mathcal{SH}^{eff}$,

$$\mathrm{Hom}_{\mathcal{SH}}(K, s_{<q+1} E) = \mathrm{Hom}_{\mathcal{SH}}(K, s_q E) = 0.$$

Given a symmetric T -spectrum E , we will say that it is q -orthogonal if for all $K \in \Sigma_T^q \mathcal{SH}^{eff}$,

$$\mathrm{Hom}_{\mathcal{SH}}(K, E) = 0.$$

Let $\mathcal{SH}^\perp(q)$ denote the full subcategory of $\mathcal{SH}$ consisting of the q -orthogonal objects.

Furthermore, the octahedral axiom for triangulated categories implies that the slices and the orthogonal components also fit in a natural distinguished triangle in $\mathcal{SH}$ [18, Prop. 3.1.19]:

$$(1.4) \quad s_q E \longrightarrow s_{<q+1} E \longrightarrow s_{<q} E \longrightarrow S^1 \wedge s_q E.$$

The slice filtration admits an alternative definition in terms of Quillen model categories [17, 18], namely:

Theorem 1.5. (1) *There exists a Quillen model category $R_{C_{\text{eff}}^q} \text{Spt}(\mathcal{M})$ which is defined as a right Bousfield localization of $\text{Spt}(\mathcal{M})$ with respect to the set of objects C_{eff}^q described in (1.1), such that its homotopy category $R_{C_{\text{eff}}^q} \mathcal{SH}$ is triangulated and naturally equivalent to $\Sigma_T^q \mathcal{SH}^{\text{eff}}$. Moreover, the functor f_q is canonically isomorphic to the following composition of triangulated functors:*

$$\mathcal{SH} \xrightarrow{R} R_{C_{\text{eff}}^q} \mathcal{SH} \xrightarrow{C_q} \mathcal{SH},$$

where R is a fibrant replacement functor in $\text{Spt}(\mathcal{M})$ and C_q is a cofibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}(\mathcal{M})$.

(2) *There exists a Quillen model category $L_{<q} \text{Spt}(\mathcal{M})$ which is defined as a left Bousfield localization of $\text{Spt}(\mathcal{M})$ with respect to the set of maps*

$$\{F_p(S^r \wedge \mathbb{G}_m^s \wedge U_+) \rightarrow * \mid F_p(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q\}$$

such that its homotopy category $L_{<q} \mathcal{SH}$ is triangulated and naturally equivalent to $\mathcal{SH}^\perp(q)$. Moreover, the functor $s_{<q}$ is canonically isomorphic to the following composition of triangulated functors:

$$\mathcal{SH} \xrightarrow{Q} L_{<q} \mathcal{SH} \xrightarrow{W_q} \mathcal{SH},$$

where Q is a cofibrant replacement functor in $\text{Spt}(\mathcal{M})$ and W_q is a fibrant replacement functor in $L_{<q} \text{Spt}(\mathcal{M})$.

(3) *There exists a Quillen model category $S^q \text{Spt}(\mathcal{M})$ which is defined as a right Bousfield localization of $L_{<q+1} \text{Spt}(\mathcal{M})$ with respect to the set of objects*

$$\{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \mid n, r, s \geq 0; s - n = q; U \in \text{Sm}_X\}$$

such that its homotopy category $S^q \mathcal{SH}$ is triangulated and the identity functor

$$\text{id} : R_{C_{\text{eff}}^q} \text{Spt}(\mathcal{M}) \rightarrow S^q \text{Spt}(\mathcal{M})$$

is a left Quillen functor. Moreover, the functor s_q is canonically isomorphic to the following composition of triangulated functors:

$$\mathcal{SH} \xrightarrow{R} R_{C_{\text{eff}}^q} \mathcal{SH} \xrightarrow{C_q} S^q \mathcal{SH} \xrightarrow{W_{q+1}} R_{C_{\text{eff}}^q} \mathcal{SH} \xrightarrow{C_q} \mathcal{SH}.$$

Proof. (1) and (3) follow directly from [18, Thms. 3.3.9, 3.3.25] and [18, Thms. 3.3.50, 3.3.68], respectively. Finally, (2) follows from Proposition 3.2.27(3) together with Theorem 3.3.26, Proposition 3.3.30 and Theorem 3.3.45 in [18]. $\square$

On the other hand, the model categories $L_{<q} \text{Spt}(\mathcal{M})$ admit an alternative description which is more suitable for an unstable interpretation. For this, we fix an arbitrary integer $n \geq 0$ and consider the following set of open immersions with a smooth closed complement of codimension at least $n + 1$:

$$(1.6) \quad WB_n = \{\iota_{U,Y} : U_+ \rightarrow Y_+ \text{ open immersion} \mid Y, Z = Y \setminus U \in \text{Sm}_X; Y \text{ irreducible}; (\text{codim}_Y Z) \geq n + 1\}.$$

Then we take the left Bousfield localization of the model category $Spt(\mathcal{M})$ with respect to the set of maps

$$(1.7) \quad sWB_n = \{F_p(\mathbb{G}_m^b \wedge \iota_{U,Y}) : \iota_{U,Y} \in WB_r; b, p, r \geq 0, r - p + b \geq n\}$$

which will be denoted by $WB_n Spt(\mathcal{M})$. The main result of [19] is the following comparison theorem:

Theorem 1.8 (see Theorem 3.6 in [19]). *Let $n \in \mathbb{Z}$ be an arbitrary integer. Then the identity functor*

$$id : WB_n Spt(\mathcal{M}) \rightarrow L_{<n+1} Spt(\mathcal{M})$$

is a Quillen equivalence.

Now, we proceed to describe how the unstable construction is carried out. In order to get an analogue for the slice filtration in the motivic unstable homotopy category it is necessary:

- (C1) To remove from the picture as much as possible the role played by the suspensions (with respect to S^1 and $\mathbb{G}_m$).
- (C2) That the slices appear naturally as homotopy fibres in the tower and not as homotopy cofibres, since the former behave much better in the unstable setting.

If we take (C2) as a guiding principle, then the distinguished triangle (1.4) and Theorem 1.5(2)-(3) lead us to consider analogues for the model structures $L_{<n} Spt(\mathcal{M})$ in the motivic unstable homotopy category $\mathcal{M}$. Once these model structures $WB_n \mathcal{M}$ have been defined, we obtain a tower of left Quillen functors:

$$\mathcal{M} \xrightarrow{id} \dots \xrightarrow{id} WB_{n+1} \mathcal{M} \xrightarrow{id} WB_n \mathcal{M} \xrightarrow{id} \dots \xrightarrow{id} WB_1 \mathcal{M} \xrightarrow{id} WB_0 \mathcal{M}.$$

Thus, the derived adjunctions induce for every simplicial presheaf $A \in \mathcal{M}$ a natural tower in $\mathcal{H}$:

$$A \longrightarrow \dots \xrightarrow{\nu_{n+1}^A} t_{n+1}(A) \xrightarrow{\nu_n^A} t_n(A) \xrightarrow{\nu_{n-1}^A} \dots \xrightarrow{\nu_2^A} t_2(A) \xrightarrow{\nu_1^A} t_1(A) \xrightarrow{\nu_0^A} *,$$

and the n -slice is then defined as the homotopy fibre of ν_n^A , i.e. the following is a homotopy fibre sequence in $\mathcal{H}$:

$$s_n(A) \rightarrow t_{n+1}(A) \rightarrow t_n(A).$$

Now, the only problem that remains is the construction of the model structures $WB_n \mathcal{M}$. For this, we consider (C1) and rely on Theorem 1.8. Hence, it is natural to define $WB_n \mathcal{M}$ as the left Bousfield localization of the model category $\mathcal{M}$ with respect to the set of maps WB_n described in (1.6).

We now describe the contents of the paper. In section 2, we construct the weakly birational motivic homotopy categories $WB_n \mathcal{M}$ as well as a natural generalization that we call birational motivic homotopy categories, and prove a comparison theorem between them when the base scheme is a perfect field (see Definitions 2.6, 2.11; Proposition 2.13 and Theorem 2.14). In section 3, we define the unstable slices and describe some of their properties (see Proposition 3.11 and Theorem 3.16). In section 4, we describe the unstable slice spectral sequence (see Theorem 4.3). In section 5, we study the behavior of the unstable slice tower with respect to transfers in the sense of Voevodsky, and construct an unstable slice tower for simplicial presheaves with transfers (see Proposition 5.2 and Theorem 5.7). In section 6, we

work over the complex numbers in order to study the behavior of the unstable slice tower with respect to the functor of complex points, and compare it with the classical Postnikov tower of algebraic topology (see Proposition 6.5, Theorem 6.9 and Remark 6.11). In section 7, we characterize the unstable slices, study the unstable slice tower for the infinite loop space of a stable slice, and carry out some computations of unstable slices (see Theorem 7.5; Propositions 7.8, 7.9; Example 7.10; and Theorems 7.13, 7.17, 7.20, 7.22, 7.25).

2. BIRATIONAL AND WEAKLY BIRATIONAL MOTIVIC HOMOTOPY CATEGORIES

In this section, we construct the birational and weakly birational motivic homotopy categories. These are defined as left Bousfield localizations of $\mathcal{M}$ with respect to maps which are induced by open immersions with a numerical condition in the codimension of the closed complement (which is assumed to be smooth in the weakly birational case). The left Bousfield localizations are constructed following Hirschhorn's approach [5]. In order to be able to apply Hirschhorn's techniques, it is necessary to know that $\mathcal{M}$ is *cellular* [5, Def. 12.1.1] and *proper* [5, Def. 13.1.1].

Theorem 2.1 (see [5, Thm. 4.1.1]). *Let $\mathcal{A}$ be a Quillen model category which is cellular and proper. Let L be a set of maps in $\mathcal{A}$. Then, the left Bousfield localization of $\mathcal{A}$ with respect to L exists.*

For details and definitions about Bousfield localization we refer the reader to Hirschhorn's book [5]. The following result guarantees the existence of left Bousfield localizations for the unstable motivic homotopy category $\mathcal{M}$.

Theorem 2.2. *The Quillen model category $\mathcal{M}$ is:*

- (1) cellular (see [6, Cor. 1.6] or [18, Thm. 2.3.2]);
- (2) proper (see [8, Thm. 1.1]).

Now we describe the open immersions which will be used to construct the left Bousfield localizations of $\mathcal{M}$.

Definition 2.3 (see [20, section 7.5]). Let $Y \in \text{Sch}_X$ and Z be a closed subscheme of Y . The *codimension* of Z in Y , $\text{codim}_Y Z$, is the infimum (over the generic points z_i of Z) of the dimensions of the local rings $\mathcal{O}_{Y, z_i}$.

Since X is Noetherian of finite Krull dimension and Y is of finite type over X , $\text{codim}_Y Z$ is always finite.

Definition 2.4. We fix an arbitrary integer $n \geq 0$ and consider the following set of open immersions which have a closed complement of codimension at least $n + 1$:

$$B_n = \{ \iota_{U,Y} : U_+ \rightarrow Y_+ \text{ open immersion} \mid \\ Y \in \text{Sm}_X; Y \text{ irreducible}; (\text{codim}_Y Y \setminus U) \geq n + 1 \}.$$

The letter B stands for birational.

Remark 2.5. The base scheme X is Noetherian and of finite Krull dimension. Therefore the smooth Nisnevich site Sm_X is essentially small, i.e. there exists a small category $\mathcal{S}$ equivalent to Sm_X . Hence, the collection of maps $\tilde{B}_n$ in $\mathcal{S}$ which correspond to B_n form a set. Since the Bousfield localization of $\mathcal{M}$ with respect to $\tilde{B}_n$ is identical to the localization with respect to B_n , we will abuse terminology and simply say that B_n is a set.

Now we consider the left Bousfield localization of $\mathcal{M}$ with respect to the set of open immersions B_n described above.

Definition 2.6. Let $n \geq 0$ be an arbitrary integer.

- (1) Let $B_n\mathcal{M}$ denote the left Bousfield localization of $\mathcal{M}$ with respect to the set of maps B_n .
- (2) Let $b_u^{(n)}$ denote its fibrant replacement functor and $\mathcal{H}(B_n)$ its associated homotopy category.

For $n \neq 0$ we will call $\mathcal{H}(B_n)$ the *codimension $n + 1$ -birational motivic homotopy category*, and for $n = 0$ we will call it the *birational motivic homotopy category*.

Lemma 2.7. Let $n \geq 0$ be an arbitrary integer. Then for every $a \geq 0$, the maps

$$S^a \wedge B_n = \{S^a \wedge \iota_{U,Y} : \iota_{U,Y} \in B_n\}$$

are weak equivalences in $B_n\mathcal{M}$.

Proof. Let $\iota_{U,Y} \in B_n$. We observe that U_+, Y_+ are cofibrant in $B_n\mathcal{M}$, since they are cofibrant in $\mathcal{M}$. By construction, $\iota_{U,Y}$ is a weak equivalence in $B_n\mathcal{M}$ and [5, Thm. 4.1.1.(4)] implies that $B_n\mathcal{M}$ is a simplicial model category. Thus, it follows from Ken Brown's lemma (see [7, Lem. 1.1.12]) that $S^a \wedge \iota_{U,Y}$ is also a weak equivalence in $B_n\mathcal{M}$ for every $a \geq 0$. $\square$

Proposition 2.8. Let A be an arbitrary simplicial presheaf in $\mathcal{M}$ and $n \geq 0$ an arbitrary integer. If A is fibrant in $B_n\mathcal{M}$, then the following conditions hold:

- (1) A is fibrant in $\mathcal{M}$.
- (2) For every $a \geq 0$ and every $\iota_{U,Y} \in B_n$, the induced map

$$\mathrm{Hom}_{\mathcal{H}}(S^a \wedge Y_+, A) \xrightarrow[\cong]{\iota_{U,Y}^*} \mathrm{Hom}_{\mathcal{H}}(S^a \wedge U_+, A)$$

is an isomorphism.

Proof. Since the identity functor

$$id : \mathcal{M} \rightarrow B_n\mathcal{M}$$

is a left Quillen functor, the conclusion follows from the derived adjunction

$$(id, b_u^{(n)}, \varphi) : \mathcal{H} \rightarrow \mathcal{H}(B_n)$$

together with Lemma 2.7. $\square$

Now we consider open immersions where the closed complement is smooth.

Definition 2.9. We fix an arbitrary integer $n \geq 0$, and consider the following set of open immersions with smooth closed complement of codimension at least $n + 1$:

$$WB_n = \{\iota_{U,Y} : U_+ \rightarrow Y_+ \text{ open immersion} \mid \\ Y, Z = Y \setminus U \in Sm_X; Y \text{ irreducible}; (\mathrm{codim}_Y Z) \geq n + 1\}.$$

Remark 2.10. As in Remark 2.5, the collection of maps $\widetilde{WB}_n$ in $\mathcal{S}$ which correspond to WB_n form a set. Since the Bousfield localization of $\mathcal{M}$ with respect to $\widetilde{WB}_n$ is identical to the localization with respect to WB_n , we will abuse terminology and simply say that WB_n is a set.

Now we describe the Bousfield localizations which will be used for the construction of the unstable slice tower.

Definition 2.11. Let $n \geq 0$ be an arbitrary integer.

- (1) Let $WB_n\mathcal{M}$ denote the left Bousfield localization of $\mathcal{M}$ with respect to the set of maps WB_n .
- (2) Let $wb_u^{(n)}$ denote its fibrant replacement functor and $\mathcal{H}(WB_n)$ its associated homotopy category.

For $n \neq 0$ we will call $\mathcal{H}(WB_n)$ the *codimension $n + 1$ -weakly birational motivic homotopy category*, and for $n = 0$ we will call it the *weakly birational motivic homotopy category*.

Proposition 2.12. Let A be an arbitrary simplicial presheaf in $\mathcal{M}$ and $n \geq 0$ an arbitrary integer. If A is fibrant in $WB_n\mathcal{M}$, then the following conditions hold:

- (1) A is fibrant in $\mathcal{M}$.
- (2) For every $a \geq 0$ and every $\iota_{U,Y} \in WB_n$, the induced map

$$\mathrm{Hom}_{\mathcal{H}}(S^a \wedge Y_+, A) \xrightarrow[\cong]{\iota_{U,Y}^*} \mathrm{Hom}_{\mathcal{H}}(S^a \wedge U_+, A)$$

is an isomorphism.

Proof. The proof is exactly the same as in Proposition 2.8. $\square$

Proposition 2.13. Let $n \geq 0$ be an arbitrary integer. Assume that the base scheme $X = \mathrm{Spec} k$, with k a perfect field. Then the Quillen adjunction

$$(id, id, \varphi) : WB_n\mathcal{M} \rightarrow B_n\mathcal{M}$$

is a Quillen equivalence.

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} & \mathcal{M} & \\ id \swarrow & & \searrow id \\ WB_n\mathcal{M} & \dashrightarrow \quad \overset{id}{\dashrightarrow} & B_n\mathcal{M} \end{array}$$

where the solid arrows are left Quillen functors. Clearly, $WB_n \subseteq B_n$; thus, every WB_n -local equivalence is a B_n -local equivalence. Therefore, the universal property of left Bousfield localizations (see [5, 3.3.19.(1) and 3.1.1.(1)]) implies that the horizontal arrow is also a left Quillen functor.

The universal property for left Bousfield localizations also implies that it is enough to show that all the maps $\iota_{U,Y}$ in B_n become weak equivalences in $WB_n\mathcal{M}$. We proceed by induction on the dimension of $Z = Y \setminus U$. If $\dim Z = 0$, then $Z \in Sm_X$ since k is a perfect field (and we are considering Z with the reduced scheme structure); hence $\iota_{U,Y} \in WB_n$ and then a weak equivalence in $WB_n\mathcal{M}$.

If $\dim Z > 0$, then we consider the singular locus Z_s of Z over X . We have that $\dim Z_s < \dim Z$ since k is a perfect field. Therefore, by induction on the dimension $\iota_{V,Y}$ is a weak equivalence in $WB_n\mathcal{M}$, where $V = Y \setminus Z_s$. On the other hand, $\iota_{U,V}$ is also a weak equivalence in $WB_n\mathcal{M}$, since it is in B_n and its closed complement $V \setminus U = Z \setminus Z_s$ is smooth over X , by construction of Z_s .

But $\iota_{U,Y} = \iota_{V,Y} \circ \iota_{U,V}$; so by the two out of three property for weak equivalences we conclude that $\iota_{U,Y}$ is a weak equivalence in $WB_n\mathcal{M}$. $\square$

Theorem 2.14. *Let $n \geq 0$ be an arbitrary integer. Then:*

(1) *The following diagram is a tower of left Quillen functors:*

$$\begin{array}{c}
 \mathcal{M} \\
 \swarrow id \quad \searrow id \\
 \cdots \xrightarrow{id} B_{n+1}\mathcal{M} \xrightarrow{id} B_n\mathcal{M} \xrightarrow{id} \cdots \xrightarrow{id} B_1\mathcal{M} \xrightarrow{id} B_0\mathcal{M}
 \end{array}$$

and

$$\begin{array}{c}
 \mathcal{H} \\
 \swarrow id \quad \searrow id \\
 \cdots \xrightarrow{id} \mathcal{H}(B_{n+1}) \xrightarrow{id} \mathcal{H}(B_n) \xrightarrow{id} \cdots \xrightarrow{id} \mathcal{H}(B_1) \xrightarrow{id} \mathcal{H}(B_0)
 \end{array}$$

$\begin{array}{c} \swarrow b_u^{(n+1)} \quad \searrow b_u^{(n)} \quad \swarrow b_u^{(1)} \quad \searrow b_u^{(0)} \end{array}$

is the corresponding tower of associated homotopy categories.

(2) *The following diagram is a tower of left Quillen functors:*

$$\begin{array}{c}
 \mathcal{M} \\
 \swarrow id \quad \searrow id \\
 \cdots \xrightarrow{id} WB_{n+1}\mathcal{M} \xrightarrow{id} WB_n\mathcal{M} \xrightarrow{id} \cdots \xrightarrow{id} WB_1\mathcal{M} \xrightarrow{id} WB_0\mathcal{M}
 \end{array}$$

and

$$\begin{array}{c}
 \mathcal{H} \\
 \swarrow id \quad \searrow id \\
 \cdots \xrightarrow{id} \mathcal{H}(WB_{n+1}) \xrightarrow{id} \mathcal{H}(WB_n) \xrightarrow{id} \cdots \xrightarrow{id} \mathcal{H}(WB_1) \xrightarrow{id} \mathcal{H}(WB_0)
 \end{array}$$

$\begin{array}{c} \swarrow wb_u^{(n+1)} \quad \searrow wb_u^{(n)} \quad \swarrow wb_u^{(1)} \quad \searrow wb_u^{(0)} \end{array}$

is the corresponding tower of associated homotopy categories.

Proof. Since the model categories $B_n\mathcal{M}$ are defined as left Bousfield localizations of $\mathcal{M}$ with respect to the set of maps B_n (see Definition 2.4), it follows that $id : \mathcal{M} \rightarrow B_n\mathcal{M}$ is a left Quillen functor. On the other hand, we observe that $B_{n+1} \subseteq B_n$; hence every B_{n+1} -local equivalence is also a B_n -local equivalence, and the universal property of left Bousfield localizations (see [5, 3.3.19.(1) and 3.1.1.(1)]) implies that $id : B_{n+1}\mathcal{M} \rightarrow B_n\mathcal{M}$ is a left Quillen functor. Finally, we observe that the identity functor is a cofibrant replacement functor in $\mathcal{M}$ since we are using the injective

model structure (see [18, Thms. 2.1.1, 2.3.1 and 2.3.2]). This proves (1). The proof of (2) is exactly the same. $\square$

3. THE UNSTABLE SLICE FILTRATION

In this section, we construct the unstable slice functors s_n (for $n \geq 0$) as well as their respective liftings $\tilde{s}_n$ to the level of Quillen model categories.

Lemma 3.1. *Let $0 \leq j \leq n$ be arbitrary integers and $f : A \rightarrow B$ be a map in $\mathcal{M}$ which is a weak equivalence in $WB_n\mathcal{M}$. Then f is also a weak equivalence in $WB_j\mathcal{M}$.*

Proof. Theorem 2.14 implies that the identity functor $id : WB_n\mathcal{M} \rightarrow WB_j\mathcal{M}$ is a left Quillen functor, and by construction both model categories are left Bousfield localizations of $\mathcal{M}$. Hence by the universal property of left Bousfield localizations (see [5, 3.3.19.(1) and 3.1.1.(1)]) it follows that f is also a weak equivalence in $WB_j\mathcal{M}$. $\square$

Definition 3.2. Let $n \geq 1$ be an arbitrary integer. We define inductively the functors

$$\tau_n : \mathcal{M} \rightarrow \mathcal{M}$$

together with natural transformations $\tau_{\leq n} : id \rightarrow \tau_n$ and $\nu_n : \tau_{n+1} \rightarrow \tau_n$ as follows:

- (1) τ_1 is just $wb_u^{(0)}$, i.e. the fibrant replacement functor in $WB_0\mathcal{M}$, and $\tau_{\leq 1}$ is the natural transformation induced by the functorial factorization in $WB_0\mathcal{M}$ of a map into a trivial cofibration followed by a fibration. Thus, every map $f : A \rightarrow B$ in $\mathcal{M}$ fits in the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\tau_{\leq 1}^A} & \tau_1 A & \longrightarrow & * \\ f \downarrow & & \downarrow \tau_1(f) & & \\ B & \xrightarrow{\tau_{\leq 1}^B} & \tau_1 B & \longrightarrow & * \end{array}$$

- (2) Assume that τ_n and $\tau_{\leq n}$ are already defined. Then, given an arbitrary simplicial presheaf $A \in \mathcal{M}$, we factor the natural map

$$\tau_{\leq n}^A : A \rightarrow \tau_n A$$

as a trivial cofibration (in $WB_n\mathcal{M}$) followed by a fibration (in $WB_n\mathcal{M}$)

$$\begin{array}{ccc} & A & \\ \tau_{\leq n+1}^A \swarrow & & \searrow \tau_{\leq n}^A \\ \tau_{n+1} A & \xrightarrow{\nu_n^A} & \tau_n A \end{array}$$

Remark 3.3. The functoriality of the factorizations in $WB_n\mathcal{M}$ implies that τ_{n+1} is a functor in $\mathcal{M}$ and that $\tau_{\leq n+1} : id \rightarrow \tau_{n+1}$, $\nu_n : \tau_{n+1} \rightarrow \tau_n$ are natural transformations.

Proposition 3.4. *Let $n \geq 0$ be an arbitrary integer and $A \in \mathcal{M}$ be an arbitrary simplicial presheaf. Then:*

(1) *The natural map*

$$\tau_{\leq n+1}^A : A \rightarrow \tau_{n+1}A$$

is a trivial cofibration in $WB_n\mathcal{M}$.

(2) *The natural map (for $n \geq 1$)*

$$\nu_n^A : \tau_{n+1}A \rightarrow \tau_nA$$

is a fibration in $WB_n\mathcal{M}$, and

$$\tau_1A \rightarrow *$$

is a fibration in $WB_0\mathcal{M}$.

(3) *The composition*

$$\tau_{n+1}A \xrightarrow{\nu_n^A} \tau_nA \xrightarrow{\nu_{n-1}^A} \cdots \longrightarrow \tau_2A \xrightarrow{\nu_1^A} \tau_1A \longrightarrow *$$

is a fibration in $\mathcal{M}$ and $WB_n\mathcal{M}$.

Proof. (1) and (2) follow directly from the construction of the functors τ_n , and the natural transformations $\tau_{\leq n}$, ν_n in Definition 3.2.

(3): By construction $WB_n\mathcal{M}$ is a left Bousfield localization of $\mathcal{M}$. Thus, the identity functor $id : WB_n\mathcal{M} \rightarrow \mathcal{M}$ is a right Quillen functor, so it is enough to show the result for $WB_n\mathcal{M}$. For this, it suffices to show that τ_1A is fibrant in $WB_n\mathcal{M}$ and that all the maps ν_n^A are fibrations in $WB_n\mathcal{M}$. But this follows from (2) above together with Theorem 2.14. $\square$

Corollary 3.5. *Let $n \geq 1$ be an arbitrary integer and $A \in \mathcal{M}$ be an arbitrary simplicial presheaf. Then, the natural map*

$$\nu_n^A : \tau_{n+1}A \rightarrow \tau_nA$$

is a weak equivalence in $WB_{n-1}\mathcal{M}$.

Proof. By construction, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\tau_{\leq n}^A} & \tau_nA \\ \tau_{\leq n+1}^A \downarrow & \nearrow \nu_n^A & \\ \tau_{n+1}A & & \end{array}$$

Combining Lemma 3.1 and Proposition 3.4(1) we deduce that the maps $\tau_{\leq n+1}^A, \tau_{\leq n}^A$ are weak equivalences in $WB_{n-1}\mathcal{M}$. Thus, by the two out of three property for weak equivalences we conclude that ν_nA is a weak equivalence in $WB_{n-1}\mathcal{M}$. $\square$

Definition 3.6. (1) Let $n \geq 1$ be an arbitrary integer, and A be an arbitrary simplicial presheaf in $\mathcal{M}$. Consider the natural map in $\mathcal{M}$ (see Definition 3.2):

$$\nu_n^A : \tau_{n+1}A \rightarrow \tau_nA$$

Let $\tilde{s}_n A$ denote the pullback in $\mathcal{M}$ of ν_n^A along the base point of $\tau_n A$:

$$(3.7) \quad \begin{array}{ccc} \tilde{s}_n A & \xrightarrow{i_n^A} & \tau_{n+1} A \\ \downarrow & & \downarrow \nu_n^A \\ * & \longrightarrow & \tau_n A \end{array}$$

(2) If $n = 0$, let $\tilde{s}_0 = \tau_1$.

Proposition 3.8. *Let $n \geq 1$ be an arbitrary integer. Then the construction given in Definition 3.6 defines a functor*

$$\tilde{s}_n : \mathcal{M} \rightarrow \mathcal{M}$$

and a natural transformation $i_n : \tilde{s}_n \rightarrow \tau_{n+1}$.

Proof. It follows immediately from the fact that ν_n is a natural transformation between functors in $\mathcal{M}$ and the universal property of pullbacks in $\mathcal{M}$. $\square$

Lemma 3.9. *Let $n \geq 1$ be an arbitrary integer. Then:*

(1) *The functor (see Definition 3.2)*

$$\tau_n : \mathcal{M} \rightarrow \mathcal{M}$$

maps weak equivalences in $WB_{n-1}\mathcal{M}$ to weak equivalences in $\mathcal{M}$.

(2) *The functor (see Proposition 3.8)*

$$\tilde{s}_n : \mathcal{M} \rightarrow \mathcal{M}$$

maps weak equivalences in $WB_n\mathcal{M}$ to weak equivalences in $\mathcal{M}$.

Proof. (1) Let $f : A \rightarrow B$ be a weak equivalence in $WB_{n-1}\mathcal{M}$. We consider the following commutative diagram in $WB_{n-1}\mathcal{M}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \tau_{\leq n}^A \downarrow & & \downarrow \tau_{\leq n}^B \\ \tau_n A & \xrightarrow{\tau_n(f)} & \tau_n B \end{array}$$

By hypothesis f is a weak equivalence in $WB_{n-1}\mathcal{M}$. On the other hand, it follows from Proposition 3.4(1) that the vertical maps are also weak equivalences in $WB_{n-1}\mathcal{M}$, and the two out of three property for weak equivalences implies that $\tau_n(f)$ is also a weak equivalence in $WB_{n-1}\mathcal{M}$. Finally, by Proposition 3.4(3) $\tau_n A$, $\tau_n B$ are both fibrant in $WB_{n-1}\mathcal{M}$. Thus [5, Thm. 3.2.13(1) and Prop. 3.4.1(1)] implies that $\tau_n(f)$ is a weak equivalence in $\mathcal{M}$.

(2) Let $f : A \rightarrow B$ be a weak equivalence in $WB_n\mathcal{M}$. By construction (see Definition 3.6 and Proposition 3.4(2)), the following are fibre sequences in $WB_n\mathcal{M}$:

$$\begin{array}{ccccc} \tilde{s}_n A & \longrightarrow & \tau_{n+1} A & \xrightarrow{\nu_n^A} & \tau_n A, \\ \tilde{s}_n B & \longrightarrow & \tau_{n+1} B & \xrightarrow{\nu_n^B} & \tau_n B. \end{array}$$

Now, Theorem 2.14 implies that they are also fibre sequences in $\mathcal{M}$. On the other hand, by naturality, f induces a map between these fibre sequences:

$$\begin{array}{ccccc} \tilde{s}_n A & \longrightarrow & \tau_{n+1} A & \xrightarrow{\nu_n^A} & \tau_n A \\ \downarrow \tilde{s}_n(f) & & \downarrow \tau_{n+1}(f) & & \downarrow \tau_n(f) \\ \tilde{s}_n B & \longrightarrow & \tau_{n+1} B & \xrightarrow{\nu_n^A} & \tau_n B \end{array}$$

Combining (1) above with Lemma 3.1, we deduce that $\tau_{n+1}(f)$, $\tau_n(f)$ are weak equivalences in $\mathcal{M}$. Thus, the result follows from [21, I.3 Prop. 5(iii)]. $\square$

Corollary 3.10. *Let $n \geq 1$ be an arbitrary integer. Then:*

(1) *The functor (see Definition 3.2)*

$$\tau_n : \mathcal{M} \rightarrow \mathcal{M}$$

maps weak equivalences in $\mathcal{M}$ to weak equivalences in $\mathcal{M}$.

(2) *The functor (see Proposition 3.8)*

$$\tilde{s}_n : \mathcal{M} \rightarrow \mathcal{M}$$

maps weak equivalences in $\mathcal{M}$ to weak equivalences in $\mathcal{M}$.

Proof. Let $f : A \rightarrow B$ be a weak equivalence in $\mathcal{M}$. By Lemma 3.9 it suffices to show that for every $n \geq 0$, f is also a weak equivalence in $WB_n\mathcal{M}$. But this follows from [5, Prop. 3.3.3(1)(a)], since $WB_n\mathcal{M}$ is defined as a left Bousfield localization of $\mathcal{M}$. $\square$

We will write $\mathrm{Ho} : \mathcal{M} \rightarrow \mathcal{H}$, for the homotopy functor associated to the Quillen model category $\mathcal{M}$.

Proposition 3.11. *Let $n \geq 1$ be an arbitrary integer. Then:*

(1) *There exists a unique functor*

$$t_n : \mathcal{H} \rightarrow \mathcal{H}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\tau_n} & \mathcal{M} \\ \mathrm{Ho} \downarrow & & \downarrow \mathrm{Ho} \\ \mathcal{H} & \xrightarrow{t_n} & \mathcal{H} \end{array}$$

(2) *There exists a unique functor*

$$s_n : \mathcal{H} \rightarrow \mathcal{H}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\tilde{s}_n} & \mathcal{M} \\ \mathrm{Ho} \downarrow & & \downarrow \mathrm{Ho} \\ \mathcal{H} & \xrightarrow{s_n} & \mathcal{H} \end{array}$$

Proof. It follows immediately from Corollary 3.10 together with the universal property of the homotopy functor Ho for a Quillen model category (see [21, I.1 Defs. 5, 6 and Thm. 1]). $\square$

Definition 3.12. We will call s_n the *unstable n -slice*, and $s_0 = t_1$ will be the *unstable zero slice*.

Proposition 3.13. Let $A \in \mathcal{H}$ be an arbitrary simplicial presheaf and $n \geq 1$ an arbitrary integer. Then the following is a fibre sequence in $\mathcal{H}$ (see [21, I.3 Def. 1]):

$$s_n A \xrightarrow{i_n^A} t_{n+1} A \xrightarrow{\nu_n^A} t_n A.$$

Furthermore, the map $s_n A \rightarrow *$ is a fibration in $WB_n \mathcal{M}$.

Proof. By construction (see Definition 3.6 and Proposition 3.4(2)) the following diagram is cartesian in $\mathcal{M}$:

$$\begin{array}{ccc} s_n A & \xrightarrow{i_n^A} & t_{n+1} A \\ \downarrow & & \downarrow \nu_n^A \\ * & \longrightarrow & t_n A \end{array}$$

and ν_n^A is a fibration in $WB_n \mathcal{M}$. Hence we conclude that $s_n A \rightarrow *$ is a fibration in $WB_n \mathcal{M}$ and that $s_n A \rightarrow t_{n+1} A \rightarrow t_n A$ is a fibre sequence in $\mathcal{H}(WB_n)$. On the other hand, by Theorem 2.14 we deduce that $s_n A \rightarrow t_{n+1} A \rightarrow t_n A$ is also a fibre sequence in $\mathcal{H}$. $\square$

Proposition 3.14. Let $A \in \mathcal{H}$ be an arbitrary simplicial presheaf and $n \geq j \geq 1$ arbitrary integers. Then the maps in the following commutative diagram are isomorphisms in $\mathcal{H}$:

$$\begin{array}{ccc} & t_j A & \\ t_j(\tau_{\leq n+1}^A) \swarrow & & \searrow t_j(\tau_{\leq n}^A) \\ t_j(t_{n+1} A) & \xrightarrow{t_j(\nu_n^A)} & t_j(t_n A) \end{array}$$

Proof. By Lemma 3.9(1) it suffices to show that the maps $\tau_{\leq n+1}^A$, $\tau_{\leq n}^A$ and ν_n^A are weak equivalences in $WB_{j-1} \mathcal{M}$. But this follows from Lemma 3.1 together with Proposition 3.4(1) and Corollary 3.5, since $n \geq j$. $\square$

Corollary 3.15. Let $A \in \mathcal{H}$ be an arbitrary simplicial presheaf and $n \geq j \geq 1$ arbitrary integers. Then $t_j A \cong t_j(t_n A)$ in $\mathcal{H}$.

Proof. This follows directly from Proposition 3.14. $\square$

The main result of this section is the following:

Theorem 3.16. *Let $A \in \mathcal{M}$ be an arbitrary simplicial presheaf and $n \geq 1$ an arbitrary integer. Then:*

(1) *The following tower is functorial in $\mathcal{M}$:*

$$\begin{array}{c} A \\ \swarrow \tau_{\leq n+1}^A \quad \downarrow \tau_{\leq n}^A \quad \searrow \tau_{\leq 2}^A \quad \swarrow \tau_{\leq 1}^A \\ \varprojlim \tau_n(A) \longrightarrow \cdots \xrightarrow{\nu_{n+1}^A} \tau_{n+1}(A) \xrightarrow{\nu_n^A} \tau_n(A) \xrightarrow{\nu_{n-1}^A} \cdots \xrightarrow{\nu_2^A} \tau_2(A) \xrightarrow{\nu_1^A} \tau_1(A) \longrightarrow * \end{array}$$

(2) *The following tower is functorial in $\mathcal{H}$:*

$$\begin{array}{c} A \\ \swarrow \tau_{\leq n+1}^A \quad \downarrow \tau_{\leq n}^A \quad \searrow \tau_{\leq 2}^A \quad \swarrow \tau_{\leq 1}^A \\ \operatorname{holim} t_n(A) \longrightarrow \cdots \xrightarrow{\nu_{n+1}^A} t_{n+1}(A) \xrightarrow{\nu_n^A} t_n(A) \xrightarrow{\nu_{n-1}^A} \cdots \xrightarrow{\nu_2^A} t_2(A) \xrightarrow{\nu_1^A} t_1(A) \longrightarrow * \end{array}$$

(3) $\tau_{\leq n}^A$ is a weak equivalence in $WB_{n-1}\mathcal{M}$ and ν_n^A (resp. $t_1(A) \rightarrow *$) is a fibration in $WB_n\mathcal{M}$ (resp. $WB_0\mathcal{M}$).

Proof. (1) follows directly from Definition 3.2, (2) follows from Definition 3.2 together with Proposition 3.11, and (3) follows from Proposition 3.4. $\square$

We will call the tower in (2) the *unstable slice tower*.

Remark 3.17. If the base scheme is of the form $X = \operatorname{Spec} k$, with k a perfect field, then an immediate consequence of Proposition 2.13 is the fact that all the constructions in this section are canonically isomorphic when they are carried out in the birational motivic categories $B_n\mathcal{M}$.

Remark 3.18. Notice that all the results in this section, and in particular Theorem 3.16, hold if we have the following data:

- (1) a model category $\mathcal{N}$ which is cellular and left proper.
- (2) for every $n \geq 0$, a set of maps L_n in $\mathcal{N}$ such that every weak equivalence in $L_{n+1}\mathcal{N}$ is also a weak equivalence in $L_n\mathcal{N}$, where $L_{n+1}\mathcal{N}$ (resp. $L_n\mathcal{N}$) is the left Bousfield localization of $\mathcal{N}$ with respect to L_{n+1} (resp. L_n).

Remark 3.19. In particular, if we choose a different Quillen model structure $\mathcal{M}'$ for the category of pointed simplicial presheaves on Sm_X such that $\mathcal{M}'$ and $\mathcal{M}$ are Quillen equivalent, then all the results in this section hold for $\mathcal{M}'$ as well, and the corresponding unstable slice towers of Theorem 3.16(2) for $\mathcal{M}'$ and $\mathcal{M}$ are canonically isomorphic.

4. THE UNSTABLE SLICE SPECTRAL SEQUENCE

In this section, we describe the spectral sequence associated to the unstable slice tower constructed in Theorem 3.16. We follow the approach of Bousfield-Kan [1, chapter IX].

In the rest of this section, A (resp. K) will be an arbitrary fibrant (resp. cofibrant) simplicial presheaf in $\mathcal{M}$. The set of morphisms between two objects in the motivic unstable homotopy category $\mathcal{H}$ will be denoted with square brackets $[-, -]$.

Definition 4.1. Given arbitrary integers $q \geq p \geq 0$, we consider:

- (1) $D_1^{p,q}(K, A) = [S^{q-p} \wedge K, t_{p+1}A]$,
- (2) $E_1^{p,q}(K, A) = [S^{q-p} \wedge K, s_pA]$,

and the following maps:

- (1) $f^{p,q}(K, A) : D_1^{p+1,q+1}(K, A) \rightarrow D_1^{p,q}(K, A)$ which is induced by ν_{p+1}^A (see Definition 3.2):

$$[S^{q-p} \wedge K, t_{p+2}A] \xrightarrow{(\nu_{p+1}^A)_*} [S^{q-p} \wedge K, t_{p+1}A].$$

- (2) $g^{p,q}(K, A) : D_1^{p,q}(K, A) \rightarrow E_1^{p+1,q}(K, A)$ which is only defined for $q-p \geq 1$:

$$[S^{q-p} \wedge K, t_{p+1}A] \cong [S^{q-p-1} \wedge K, \Omega_{S^1} t_{p+1}A] \longrightarrow [S^{q-p-1} \wedge K, s_{p+1}A]$$

and is induced by the boundary in the fibre sequence which characterizes the unstable $p+1$ -slice (see Proposition 3.13):

$$s_{p+1}A \rightarrow t_{p+2}A \rightarrow t_{p+1}A.$$

- (3) $h^{p,q}(K, A) : E_1^{p,q}(K, A) \rightarrow D_1^{p,q}(K, A)$ which is induced by i_p^A (see Proposition 3.8):

$$[S^{q-p} \wedge K, s_pA] \xrightarrow{(i_p^A)_*} [S^{q-p} \wedge K, t_{p+1}A]$$

for $p \geq 1$, and $h^{0,q}(K, A) = id$ is the identity.

Remark 4.2. Notice that $D_1^{p,q}(K, A), E_1^{p,q}(K, A)$ are just pointed sets for $p = q$, groups for $q = p+1$ and abelian groups for $q \geq p+2$.

Theorem 4.3. *The construction considered in Definition 4.1 forms an extended exact couple (see [1, section 4.2, chapter IX]):*

$$(4.4) \quad \begin{array}{ccc} D_1(K, A) & \xrightarrow{f} & D_1(K, A) \\ & \nwarrow h \quad \searrow g & \\ & E_1(K, A) & \end{array}$$

Proof. To construct the exact couple we consider the unstable slice tower for A (see Theorem 3.16(1)):

$$\begin{array}{c} \begin{array}{ccccccc} & & A & & & & \\ & \swarrow & \downarrow & \searrow & & & \\ \tau_{\leq n+1}^A & & \tau_{\leq n}^A & & \tau_{\leq 2}^A & & \tau_{\leq 1}^A \end{array} \\ \begin{array}{ccccccccccc} \varprojlim \tau_n(A) & \longrightarrow & \cdots & \xrightarrow{\nu_{n+1}^A} & \tau_{n+1}(A) & \xrightarrow{\nu_n^A} & \tau_n(A) & \xrightarrow{\nu_{n-1}^A} & \cdots & \xrightarrow{\nu_2^A} & \tau_2(A) & \xrightarrow{\nu_1^A} & \tau_1(A) & \longrightarrow & * \end{array} \end{array}$$

Combining Theorems 3.16(3) and 2.14(2) we deduce that the horizontal arrows in the diagram above are fibrations in $\mathcal{M}$. Since $\mathcal{M}$ is a simplicial model category

(see [18, Prop. 2.3.7]) and K is cofibrant in $\mathcal{M}$, we obtain the following tower of fibrations of pointed simplicial sets:

$$\begin{array}{c}
 \text{Map}_*(K, A) \\
 \swarrow \quad \downarrow (\tau_{\leq n}^A)_* \quad \searrow (\tau_{\leq 1}^A)_* \\
 \text{Map}_*(K, \varprojlim \tau_n(A)) \rightarrow \cdots \xrightarrow{(\nu_n^A)_*} \text{Map}_*(K, \tau_n(A)) \xrightarrow{(\nu_{n-1}^A)_*} \cdots \xrightarrow{(\nu_1^A)_*} \text{Map}_*(K, \tau_1(A)) \rightarrow *
 \end{array}$$

Finally, we obtain the extended exact couple using the construction of Bousfield-Kan [1, chapter IX] together with the following canonical isomorphisms (see [7, Lem. 6.1.2]) that exist in any simplicial model category with K cofibrant and B fibrant:

$$[S^r \wedge K, B] = \text{Hom}_{\mathcal{H}}(S^r \wedge K, B) \cong \pi_{r,\eta} \text{Map}_*(K, B),$$

where η is the base point of $\text{Map}_*(K, B)$ and for $r \geq 1$, $\pi_{r,\eta}$ denotes the r -th homotopy group considering η as the base point. $\square$

5. TRANSFERS AND KAHN-SUJATHA BIRATIONAL MOTIVES

In this section we study the unstable analogue of the Kahn-Sujatha construction of birational motives [2].

Let $\text{Cor}(X)$ denote the Suslin-Voevodsky category of finite correspondences over X , having the same objects as Sm_X , morphisms $c(Y, Z)$ given by the group of finite relative cycles on $Y \times_X Z$ over Y [24], and composition as in [27, p. 673 diagram (2.1)]. A presheaf with transfers is an additive contravariant functor from $\text{Cor}(X)$ to the category of abelian groups. Let $\mathcal{M}^{tr}$ denote the category of simplicial presheaves with transfers.

We will write Y^{tr} for the simplicial presheaf with transfers represented by Y in Sm_X . By taking the graph of a morphism in Sm_X , we obtain a functor $\Gamma : \text{Sm}_X \rightarrow \text{Cor}(X)$. This induces a forgetful functor:

$$\mathcal{U} : \mathcal{M}^{tr} \rightarrow \mathcal{M}$$

which admits a left adjoint

$$\mathbb{Z}_{tr} : \mathcal{M} \rightarrow \mathcal{M}^{tr}$$

such that $\mathbb{Z}_{tr}(Y_+) = Y^{tr}$ for $Y \in \text{Sm}_X$.

Moreover, the adjunction [22, Thm. 8 and Lem. 9]

$$(5.1) \quad (\mathbb{Z}_{tr}, \mathcal{U}, \varphi) : \mathcal{M} \rightarrow \mathcal{M}^{tr}$$

induces a Quillen model structure on $\mathcal{M}^{tr}$ (see Definition 5.9). We will abuse notation and write $\mathcal{M}^{tr}$ (resp. $\mathcal{H}^{tr}$) for the category of simplicial presheaves with transfers equipped with this particular Quillen model structure (resp. for the associated homotopy category).

Proceeding as in Definition 2.6 (resp. 2.11) we can consider for $n \geq 0$ the left Bousfield localization $B_n \mathcal{M}^{tr}$ (resp. $WB_n \mathcal{M}^{tr}$) of $\mathcal{M}^{tr}$ with respect to the set $\mathbb{Z}_{tr}(B_n)$ (resp. $\mathbb{Z}_{tr}(WB_n)$). Mimicking the same arguments as in sections 2 and 3, we deduce that $B_n \mathcal{M}^{tr}$ and $WB_n \mathcal{M}^{tr}$ are Quillen equivalent when the base scheme

is a perfect field (see Proposition 2.13), we obtain towers as in Theorems 2.14, 3.16 and finally an extended exact couple as in Definition 4.1 (see Remark 3.18). Thus, we conclude:

Proposition 5.2. *Given $A \in \mathcal{H}^{tr}$ we obtain a corresponding unstable slice tower with transfers, which is functorial in $\mathcal{H}^{tr}$:*

$$\begin{array}{c}
 A \\
 \swarrow \quad \downarrow \quad \searrow \\
 \rho_{\leq n+1}^A \quad \rho_{\leq n}^A \quad \rho_{\leq 2}^A \quad \rho_{\leq 1}^A \\
 \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 \text{holim } t_n^{tr}(A) \longrightarrow \cdots \xrightarrow{\mu_{n+1}^A} t_{n+1}^{tr}(A) \xrightarrow{\mu_n^A} t_n^{tr}(A) \xrightarrow{\mu_{n-1}^A} \cdots \xrightarrow{\mu_2^A} t_2^{tr}(A) \xrightarrow{\mu_1^A} t_1^{tr}(A) \longrightarrow *
 \end{array}$$

where $t_n^{tr}(A)$ is fibrant in $WB_{n-1}\mathcal{M}^{tr}$, $\rho_{\leq n}^A$ is a weak equivalence in $WB_{n-1}\mathcal{M}^{tr}$ and μ_n^A (resp. $t_1^{tr}(A) \rightarrow *$) is a fibration in $WB_n\mathcal{M}^{tr}$ (resp. $WB_0\mathcal{M}^{tr}$).

By construction, for every $n \geq 0$ the Quillen adjunction

$$(\mathbb{Z}_{tr}, \mathcal{U}, \varphi) : \mathcal{M} \rightarrow \mathcal{M}^{tr}$$

induces a pair of Quillen adjunctions between the corresponding birational motivic homotopy categories:

$$\begin{aligned}
 (5.3) \quad & (\mathbb{Z}_{tr}, \mathcal{U}, \varphi) : B_n\mathcal{M} \rightarrow B_n\mathcal{M}^{tr}, \\
 & (\mathbb{Z}_{tr}, \mathcal{U}, \varphi) : WB_n\mathcal{M} \rightarrow WB_n\mathcal{M}^{tr}.
 \end{aligned}$$

Passing to the associated homotopy categories, we obtain the corresponding derived adjunctions:

$$\begin{aligned}
 (5.4) \quad & (\mathbb{Z}_{tr}, b_{tr}^{(n)}, \varphi) : \mathcal{H}(B_n) \rightarrow \mathcal{H}(B_n)^{tr}, \\
 & (\mathbb{Z}_{tr}, wb_{tr}^{(n)}, \varphi) : \mathcal{H}(WB_n) \rightarrow \mathcal{H}(WB_n)^{tr},
 \end{aligned}$$

where $b_{tr}^{(n)}$ (resp. $wb_{tr}^{(n)}$) is a fibrant replacement functor in $B_n\mathcal{M}^{tr}$ (resp. $WB_n\mathcal{M}^{tr}$).

Remark 5.5. By the work of Röndigs and Østvær [22], $\mathcal{H}(B_0)^{tr}$ is the natural unstable analogue of the Kahn-Sujatha birational category of motives [2].

Since the maps in B_n and WB_n (see Definitions 2.4 and 2.9) are cofibrations in $\mathcal{M}$ and the model structure in $\mathcal{M}^{tr}$ is induced by the adjunction (5.1) (see Definition 5.9), we observe that the Quillen model structures $B_n\mathcal{M}^{tr}$, $WB_n\mathcal{M}^{tr}$ are induced by the adjunctions (5.3) (see [5, Prop. 3.1.12 and Thm. 11.3.2] and Definition 5.9). Thus, we conclude:

Proposition 5.6. *Let f be a map in $B_n\mathcal{M}^{tr}$ (resp. $WB_n\mathcal{M}^{tr}$). Then:*

- (1) *f is a weak equivalence in $B_n\mathcal{M}^{tr}$ (resp. $WB_n\mathcal{M}^{tr}$) if and only if $\mathcal{U}f$ is a weak equivalence in $B_n\mathcal{M}$ (resp. $WB_n\mathcal{M}$).*
- (2) *f is a fibration in $B_n\mathcal{M}^{tr}$ (resp. $WB_n\mathcal{M}^{tr}$) if and only if $\mathcal{U}f$ is a fibration in $B_n\mathcal{M}$ (resp. $WB_n\mathcal{M}$).*

Now we can state the main result of this section.

Theorem 5.7. *Let A in $\mathcal{H}^{tr}$ be an arbitrary simplicial presheaf with transfers. Then:*

- (1) *The forgetful functor $\mathcal{U} : \mathcal{M}^{tr} \rightarrow \mathcal{M}$ induces an isomorphism in $\mathcal{H}$ between the unstable slice tower with transfers for A (see Proposition 5.2) and the corresponding unstable slice tower for $\mathcal{U}(A)$ (see Theorem 3.16(2)).*

- (2) For $n \geq 0$, the unstable slices $s_n(\mathcal{U}(A))$ of $\mathcal{U}(A)$ admit a canonical structure of simplicial presheaves with transfers.

Proof. (1) By the functoriality of the unstable slice tower (see Theorem 3.16) and Lemma 3.9, it suffices to show that for $n \geq 0$ the maps

$$\rho_{\leq n}^A : A \rightarrow t_n^{tr}(A)$$

are weak equivalences in $WB_{n-1}\mathcal{M}$, i.e. after forgetting transfers. But this follows from Propositions 5.2 and 5.6.

(2) The case of the zero slice $s_0 = t_1$ follows from (1) above. If $n \geq 1$, let $s_n^{tr}(A)$ denote the unstable slice with transfers of A . Thus, by Proposition 3.13 the following is a fibre sequence in $\mathcal{H}^{tr}$:

$$s_n^{tr}(A) \longrightarrow t_{n+1}^{tr}(A) \xrightarrow{\mu_n^A} t_n^{tr}(A),$$

and by (1) above there is a commutative diagram where the rows are fibre sequences in $\mathcal{H}$ such that $t_n(\mathcal{U}(\rho_{\leq n}^A))$, $t_{n+1}(\mathcal{U}(\rho_{\leq n+1}^A))$ are isomorphisms in $\mathcal{H}$:

$$\begin{array}{ccccc} s_n(\mathcal{U}(A)) & \longrightarrow & t_{n+1}(\mathcal{U}(A)) & \xrightarrow{\nu_n^{\mathcal{U}(A)}} & t_n(\mathcal{U}(A)) \\ \alpha \downarrow & & t_{n+1}(\mathcal{U}(\rho_{\leq n+1}^A)) \downarrow & & \downarrow t_n(\mathcal{U}(\rho_{\leq n}^A)) \\ \mathcal{U}(s_n^{tr}(A)) & \longrightarrow & \mathcal{U}(t_{n+1}^{tr}(A)) & \xrightarrow{\mathcal{U}(\mu_n^A)} & \mathcal{U}(t_n^{tr}(A)) \end{array}$$

Hence, by [21, I.3 Prop. 5(iii)] we conclude that α is also an isomorphism in $\mathcal{H}$. $\square$

Remark 5.8. The unstable slices $s_n A$ of an arbitrary simplicial presheaf $A \in \mathcal{H}$ do not admit transfers in general. The example of Levine [11, §2] works as well in the unstable setting and shows that if our base scheme is a field k and C a smooth projective curve of genus $g > 0$ with no rational k -points, then $s_0(\mathbb{Z}_C)$ does not admit transfers, where $\mathbb{Z}_C$ is the presheaf of abelian groups represented by C .

Definition 5.9. Let $(F, G, \varphi) : \mathcal{A} \rightarrow \mathcal{B}$ be a Quillen adjunction between two Quillen model categories $\mathcal{A}, \mathcal{B}$. We say that the Quillen model structure on $\mathcal{B}$ is induced by the adjunction (F, G, φ) if a map f in $\mathcal{B}$ is a weak equivalence (resp. fibration) if and only if $G(f)$ is a weak equivalence (resp. fibration) in $\mathcal{A}$.

6. COMPARISON WITH THE CLASSICAL POSTNIKOV TOWER

In this section the base scheme X will be of the form $\text{Spec}(\mathbb{C})$, where $\mathbb{C}$ denotes the complex numbers, and Sm_X will consist of only smooth quasi-projective varieties. We will write $\mathcal{M}'$ for the category of pointed simplicial presheaves on Sm_X equipped with the Quillen model structure defined in [16, Thm. A.17], which is Quillen equivalent [16, Rmk. A.21] to $\mathcal{M}$. By Remarks 3.18 and 3.19, all the results in section 3 hold in $\mathcal{M}'$ and the constructions are canonically isomorphic after we pass to the respective homotopy categories. We will write $\mathcal{H}'$ for the homotopy category associated to $\mathcal{M}'$.

Our goal is to study the behavior of the unstable slice filtration (see Theorem 3.16(2)) in the motivic unstable homotopy category $\mathcal{H}'$ with respect to the functor of $\mathbb{C}$ -points in Sm_X and compare the unstable slice filtration with the classical Postnikov tower of algebraic topology.

We will write $\mathbf{Top}$ for the category of pointed compactly generated topological spaces [23] equipped with the model structure constructed by Quillen (see [21, Ch. 2, §3, Thm. 1] or [7, Thm. 2.4.23]), and $\mathbf{H}_{\mathbf{Top}}$ for its associated homotopy category. Notice that $\mathbf{Top}$ is a simplicial model category [21, Ch. 2, §2]; we will write $Map_{\mathbf{top}}(-, -)$ for the corresponding simplicial mapping space [5, 1.1.6].

For $n \geq 1$, we will write D^n for the unit disc contained in $\mathbb{R}^n$ and $d_n \in \mathbf{Top}$ will be the standard inclusion $\partial D_+^{n+1} \rightarrow D_+^{n+1}$ of the n -sphere into the $n+1$ -unit disc (both with a disjoint base point). Consider:

$$L_{n-1}\mathbf{T} = \{d_n\}.$$

We will write $L_{n-1}\mathbf{Top}$ for the left Bousfield localization of $\mathbf{Top}$ with respect to $L_{n-1}\mathbf{T}$, which exists by [5, 12.1.4] and Theorem 2.1.

Combining Propositions 1.5.2 and 1.5.4 in [5] we deduce that a map $f : K \rightarrow L$ in $\mathbf{Top}$ is a weak equivalence in $L_{n-1}\mathbf{Top}$ if and only if it induces isomorphisms on the homotopy groups $\pi_i(f) : \pi_i(K) \rightarrow \pi_i(L)$ for $i \leq n-1$ and every choice of base point in K . Then, by Remark 3.18 we can mimick the same arguments as for the unstable slice filtration to conclude:

Proposition 6.1. *Let $n \geq 1$ be an arbitrary integer. Then, there exist:*

- (1) *Functors $P_n : \mathbf{H}_{\mathbf{Top}} \rightarrow \mathbf{H}_{\mathbf{Top}}$.*
- (2) *Natural transformations:*

$$p_{\leq n} : id \rightarrow P_n, \quad p_n : P_{n+1} \rightarrow P_n$$

such that for every topological space $K \in \mathbf{Top}$ the following tower is functorial in $\mathbf{H}_{\mathbf{Top}}$:

$$\begin{array}{c} \begin{array}{c} K \\ \swarrow \quad \downarrow \quad \searrow \\ p_{\leq n+1}^K \quad p_{\leq n}^K \quad p_{\leq 2}^K \quad p_{\leq 1}^K \\ \swarrow \quad \downarrow \quad \searrow \end{array} \\ \text{holim} P_n(K) \xleftarrow{\quad} \cdots \xrightarrow[p_{n+1}^K]{} P_{n+1}(K) \xrightarrow[p_n^K]{} P_n(K) \xrightarrow[p_{n-1}^K]{} \cdots \xrightarrow[p_2^K]{} P_2(K) \xrightarrow[p_1^K]{} P_1(K) \longrightarrow * \end{array}$$

where $P_n(K)$ is fibrant in $L_{n-1}\mathbf{Top}$, $p_{\leq n}^K$ is a weak equivalence in $L_{n-1}\mathbf{Top}$ and p_n^K (resp. $P_1(K) \rightarrow *$) is a fibration in $L_n\mathbf{Top}$ (resp. $L_0\mathbf{Top}$).

The work of Hirschhorn [5] implies that the tower constructed in Proposition 6.1 coincides with the Postnikov tower in $\mathbf{Top}$:

Theorem 6.2 (Hirschhorn). *The tower constructed in Proposition 6.1 is the Postnikov tower of $K \in \mathbf{Top}$.*

Proof. Hirschhorn constructs the Postnikov tower in the context of unpointed topological spaces using Bousfield localizations with respect to $\partial D^{n+1} \rightarrow D^{n+1}$ (see 1.5.1, 1.5.2, 1.5.3 and 1.5.4 in [5]). Since the maps $\partial D^{n+1} \rightarrow D^{n+1}$ are cofibrations and the adjunction between unpointed and pointed topological spaces is enriched in simplicial sets, we deduce that a pointed space is local with respect to $\partial D_+^{n+1} \rightarrow D_+^{n+1}$ if and only if it is local (as an unpointed space) with respect to $\partial D^{n+1} \rightarrow D^{n+1}$ (see [5, Prop. 3.1.12 and Thm. 11.3.2]). This finishes the proof. $\square$

If $Y \in Sm_X$, we will write $Y(\mathbb{C})$ for the set of complex points of Y , which is a complex analytic manifold. Hence, in particular $Y(\mathbb{C})_+ \in \mathbf{Top}$.

Proposition 6.3 ([16]). *The functor $\mathbb{C}_R$ of complex points is the left adjoint in a Quillen adjunction:*

$$(\mathbb{C}_R, \text{Sing}, \varphi) : \mathcal{M}' \rightarrow \mathbf{Top},$$

where $\text{Sing}(K)$ is the simplicial presheaf

$$Y \in \text{Sm}_X \mapsto \text{Map}_{\mathbf{Top}}(Y(\mathbb{C})_+, K).$$

Proof. This follows from [16, Thm A.23]. $\square$

Now we are ready to establish a connection between our unstable slice filtration and the classical Postnikov tower in algebraic topology. Let $B_n(\mathbb{C})$ be the following set of maps in **Top**:

$$(6.4) \quad B_n(\mathbb{C}) = \{\mathbb{C}_R(\iota_{U,Y}) : U(\mathbb{C})_+ \rightarrow Y(\mathbb{C})_+ \mid \iota_{U,Y} \in B_n\}$$

where we are considering B_n as a set of maps in $\mathcal{M}'$ (see Definition 2.4). We define $B_n \mathbf{Top}$ to be the left Bousfield localization of **Top** with respect to the set $B_n(\mathbb{C})$, which exists by [5, 12.1.4] and Theorem 2.1. We will write $B_n \mathbf{H}_{\mathbf{Top}}$ for its associated homotopy category.

Then, mimicking the same arguments as for the unstable slice filtration (see Remark 3.18), we deduce:

Proposition 6.5. *Let $n \geq 1$ be and arbitrary integer. Then, there exist:*

- (1) *Functors $t_n^{\mathbb{C}} : \mathbf{H}_{\mathbf{Top}} \rightarrow \mathbf{H}_{\mathbf{Top}}$.*
- (2) *Natural transformations:*

$$r_{\leq n} : \text{id} \rightarrow t_n^{\mathbb{C}}, \quad r_n : t_{n+1}^{\mathbb{C}} \rightarrow t_n^{\mathbb{C}}$$

such that for every topological space $K \in \mathbf{Top}$ the following tower is functorial in $\mathbf{H}_{\mathbf{Top}}$:

$$\begin{array}{c} \begin{array}{ccccccc} & & & K & & & \\ & \swarrow & & \downarrow & \searrow & & \\ & r_{\leq n+1}^K & & r_n^K & & r_{\leq 2}^K & r_{\leq 1}^K \\ & \swarrow & & \downarrow & \searrow & & \\ \text{holim} t_n^{\mathbb{C}}(K) & \xrightarrow{\quad} \cdots \xrightarrow{r_{n+1}^K} & t_{n+1}^{\mathbb{C}}(K) & \xrightarrow{r_n^K} & t_n^{\mathbb{C}}(K) & \xrightarrow{r_{n-1}^K} \cdots \xrightarrow{r_2^K} & t_2^{\mathbb{C}}(K) & \xrightarrow{r_1^K} & t_1^{\mathbb{C}}(K) & \rightarrow * \end{array} \end{array}$$

where $t_n^{\mathbb{C}}(K)$ is fibrant in $B_{n-1} \mathbf{Top}$, $r_{\leq n}^K$ is a weak equivalence in $B_{n-1} \mathbf{Top}$ and r_n^K (resp. $t_1^{\mathbb{C}}(K) \rightarrow *$) is a fibration in $B_n \mathbf{Top}$ (resp. $B_0 \mathbf{Top}$).

Remark 6.6. By Proposition 2.13, the tower in Proposition 6.5 is identical if we replace B_n with WB_n (see Definition 2.9).

Notice that by construction, for every $n \geq 0$ the Quillen adjunction (see Proposition 6.3)

$$(\mathbb{C}_R, \text{Sing}, \varphi) : \mathcal{M}' \rightarrow \mathbf{Top}$$

induces a Quillen adjunction between the corresponding birational categories:

$$(6.7) \quad (\mathbb{C}_R, \text{Sing}, \varphi) : B_n \mathcal{M}' \rightarrow B_n \mathbf{Top},$$

and passing to the associated homotopy categories, we obtain the corresponding derived adjunction:

$$(6.8) \quad (\mathbb{C}_R, b_{\text{top}}^{(n)}, \varphi) : \mathcal{H}'(B_n) \rightarrow B_n \mathbf{H}_{\mathbf{Top}},$$

where $b_{\text{top}}^{(n)}$ is a fibrant replacement functor in $B_n \mathbf{Top}$.

The main result of this section is the following:

Theorem 6.9. *For every $n \geq 0$, the identity functor*

$$id : L_{2n}\mathbf{Top} \rightarrow B_n\mathbf{Top}$$

is a left Quillen functor and a Quillen equivalence.

Proof. We observe that there is a commutative diagram in $\mathbf{Top}$:

$$\begin{array}{ccc} \partial D^{2n+2} & \xrightarrow{d_{2n+1}} & D^{2n+2} \\ \downarrow & & \downarrow \\ \mathbb{C}^{n+1} \setminus 0 & \xrightarrow{\tilde{d}_n} & \mathbb{C}^{n+1} \end{array}$$

where the vertical arrows are weak equivalences in $\mathbf{Top}$. Thus, by construction the left Bousfield localization of $\mathbf{Top}$ with respect to d_{2n+1} and $\tilde{d}_n$ are identical [5, 4.1.1 and 4.1.2]. On the other hand, $\iota_n : \mathbb{A}_{\mathbb{C}}^{n+1} \setminus 0 \rightarrow \mathbb{A}_{\mathbb{C}}^{n+1}$ is in B_n , and $\mathbb{C}_R(\iota_n) = \tilde{d}_n$. By the universal property of left Bousfield localizations (see [5, 3.3.19.(1) and 3.1.1.(1)]), it follows that the identity functor is a left Quillen functor.

Using the universal property of left Bousfield localizations again, we observe that in order to conclude that we have a Quillen equivalence it is enough to show that every map in $B_n(\mathbb{C})$ (see (6.4)) is a weak equivalence in $L_{2n}\mathbf{Top}$. On the other hand, by Proposition 2.13 and Remark 6.6 it suffices to show that every map in $WB_n(\mathbb{C})$ is a weak equivalence in $L_{2n}\mathbf{Top}$. But this follows directly from Lemma 6.16. $\square$

Corollary 6.10. *Let K in $\mathbf{Top}$ be a topological space and $n \geq 0$ be an arbitrary integer. Then the following are equivalent:*

- (1) *K is fibrant in $L_{2n}\mathbf{Top}$,*
- (2) *K is fibrant in $B_n\mathbf{Top}$,*

Proof. This follows immediately from Theorem 6.9. $\square$

Remark 6.11. Combining Theorems 6.9 and 6.2 we observe that we have obtained an algebro-geometric construction of the odd part of the Postnikov tower (see Proposition 6.1):

$$\begin{array}{c} K \\ \swarrow \quad \downarrow \quad \searrow \\ p_{\leq 2n-1}^K \quad p_{\leq 2n+1}^K \quad p_{\leq 3}^K \quad p_{\leq 1}^K \\ \text{holim } P_{2n+1}(K) \rightarrow \cdots \rightarrow P_{2n+1}(K) \rightarrow P_{2n-1}(K) \rightarrow \cdots \rightarrow P_3(K) \rightarrow P_1(K) \rightarrow * \end{array}$$

In the rest of this section we will write Y for a quasi-projective variety in Sm_X with a fixed embedding into projective space, i.e. an open immersion $Y \rightarrow \bar{Y}$ and a closed embedding $\bar{Y} \rightarrow \mathbb{P}^N$ for a suitable N .

Given a map $\iota_{U,Y}$ in B_n and a hyperplane H in $\mathbb{P}^N$, we will write $e_1 : H \cap Y \rightarrow Y$, $e_2 : H \cap U \rightarrow U$ for the closed embedding of the corresponding hyperplane sections.

Lemma 6.12. *Let $Y \in Sm_X$ be a smooth irreducible quasi-projective variety of dimension $d \geq 2$ and $\iota_{U,Y} : U_+ \rightarrow Y_+$ be a map in B_n . Then there exists a*

hyperplane H in $\mathbb{P}^N$ such that in the commutative diagram

$$\begin{array}{ccc} H \cap U(\mathbb{C})_+ & \xrightarrow{\mathbb{C}_R(e_2)} & U(\mathbb{C})_+ \\ \mathbb{C}_R(h) \downarrow & & \downarrow \mathbb{C}_R(\iota_{U,Y}) \\ H \cap Y(\mathbb{C})_+ & \xrightarrow{\mathbb{C}_R(e_1)} & Y(\mathbb{C})_+ \end{array}$$

- (1) $\mathbb{C}_R(e_1)$ and $\mathbb{C}_R(e_2)$ are weak equivalences in $L_{d-2}\mathbf{Top}$,
- (2) $H \cap Y$ is smooth of dimension $d-1$,
- (3) if $n \leq d-2$, then $h : H \cap U_+ \rightarrow H \cap Y_+$ is in B_n , and
- (4) if $n = d-1$, then $h = id$, i.e. $H \cap (Y \setminus U) = \emptyset$.

Proof. By Proposition 1.5.2 in [5] in order to prove (1) it suffices to show that $\mathbb{C}_R(e_1)$ (resp. $\mathbb{C}_R(e_2)$) induces a bijection on the homotopy groups π_i for $i \leq d-2$ and for every choice of base point in $H \cap Y(\mathbb{C})_+$ (resp. $H \cap U(\mathbb{C})_+$).

Let $Z = Y \setminus U$ be the closed complement of U in Y , with codimension c in Y . Since $\iota_{U,Y} \in B_n$ we observe that $d \geq c \geq n+1$. Since Y is smooth, by the theorem of Bertini [9, Cor. 3.6] there exists a Zariski open dense subset Ω of the projective variety $\check{\mathbb{P}}^N$ of hyperplanes in $\mathbb{P}^N$ such that for every hyperplane H in Ω , (2) holds and the codimension of $H \cap Z$ in Y is $c+1$. Thus, we deduce that (3) and (4) also hold for any H in Ω .

Moreover, U and Y are smooth, so by the Zariski-Lefschetz theorem of Hamm-Lê [4, Thm. 1.1.3] we deduce that there exists a hyperplane H in Ω such that $Y(\mathbb{C})$ (resp. $U(\mathbb{C})$) has the homotopy type of a space obtained from $H \cap Y(\mathbb{C})$ (resp. $H \cap U(\mathbb{C})$) by attaching cells $\partial D^r \rightarrow D^r$ with $r \geq d$. Since all the spaces in the diagram above are CW -complexes, we deduce by the cellular approximation theorem that $\mathbb{C}_R(e_1)$ and $\mathbb{C}_R(e_2)$ induce a bijection on π_i for every choice of base point and $i \leq r-2$. Hence, we conclude that (1) holds since $d-2 \leq r-2$. $\square$

Lemma 6.13. *Let $Y \in Sm_X$ be a smooth irreducible quasi-projective variety of dimension d and $\iota_{U,Y} : U_+ \rightarrow Y_+$ be a map in B_n with $n > 0$. Then:*

$$\pi_1(\mathbb{C}_R(\iota_{U,Y})) : \pi_1(U(\mathbb{C})_+) \rightarrow \pi_1(Y(\mathbb{C})_+)$$

is an isomorphism for every choice of base point in $U(\mathbb{C})_+$.

Proof. Since the map $\iota_{U,Y}$ preserves the disjoint point $+$, we can assume that the base point is in $U(\mathbb{C})$. Moreover, Y is irreducible and U is open in Y , so we deduce that U is also irreducible and in particular $U(\mathbb{C})$ is path connected. Hence the result is independent of the choice of base point, and we will omit any reference to it in the rest of the proof. We proceed by induction on the dimension d of Y . We observe that $d \geq 2$, since $n > 0$.

Let $Z = Y \setminus U$ be the closed complement of U in Y , which has codimension $c \geq n+1$ in Y since $\iota_{U,Y} \in B_n$. So, if $d = 2$, then $\dim Z = 0$, since $n > 0$. Thus, Z is a disjoint union of closed points (since we are assuming that Z is reduced). We observe that it suffices to consider the case when Z consists of only one point, since $\iota_{U,Y}$ can be written as a composition of open immersions where the closed complement consists of only one point. If $Z = \text{Spec } \mathbb{C}$, then there exists a pushout

square of open immersions in **Top** of the form:

$$\begin{array}{ccc} D^4 \setminus 0 & \xrightarrow{\quad} & D^4 \\ \downarrow & & \downarrow \\ U(\mathbb{C}) & \xrightarrow{\mathbb{C}_R(\iota_{U,Y})} & Y(\mathbb{C}) \end{array}$$

By the theorem of Van Kampen we deduce that $\pi_1(\mathbb{C}_R(\iota_{U,Y}))$ is equal to

$$\pi_1(U(\mathbb{C})) \rightarrow \operatorname{colim} \left(\begin{array}{ccc} \pi_1(D^4 \setminus 0) = * & \xrightarrow{\quad} & \pi_1(D^4) = * \\ \downarrow & & \\ \pi_1(U(\mathbb{C})) & & \end{array} \right) \cong \pi_1(U(\mathbb{C})),$$

which is an isomorphism, so the claim holds in this case.

If $d \geq 3$, then by Lemma 6.12 there exists a hyperplane H in $\mathbb{P}^N$ such that in the commutative diagram

$$\begin{array}{ccc} H \cap U(\mathbb{C})_+ & \xrightarrow{\mathbb{C}_R(e_2)} & U(\mathbb{C})_+ \\ \mathbb{C}_R(h) \downarrow & & \downarrow \mathbb{C}_R(\iota_{U,Y}) \\ H \cap Y(\mathbb{C})_+ & \xrightarrow{\mathbb{C}_R(e_1)} & Y(\mathbb{C})_+ \end{array}$$

$\mathbb{C}_R(e_1)$ and $\mathbb{C}_R(e_2)$ are weak equivalences in $L_{d-2}\mathbf{Top}$. We observe that $d - 2 \geq 3 - 2 = 1$, so Proposition 1.5.2 in [5] implies that $\mathbb{C}_R(e_1)$ and $\mathbb{C}_R(e_2)$ induce a bijection on the corresponding fundamental groups π_1 .

If $n = d - 1$, then by Lemma 6.12(4), $\mathbb{C}_R(h)$ is the identity map; so we conclude that $\mathbb{C}_R(\iota_{U,Y})$ induces an isomorphism on π_1 . On the other hand, if $n \leq d - 2$, then by (2)-(3) in Lemma 6.12 we deduce that $h \in B_n$ and $H \cap Y$ is smooth of dimension $d - 1$. Thus, by induction on the dimension we conclude that $\mathbb{C}_R(h)$ induces an isomorphism on π_1 . Hence, it follows that $\mathbb{C}_R(\iota_{U,Y})$ also induces an isomorphism on π_1 , as we wanted. $\square$

Recall that a topological space $A \in \mathbf{Top}$ is simply connected if it is path connected (i.e. $\pi_0 A = *$) and $\pi_1(A) = *$ for every choice of base point in A .

Lemma 6.14. *Let $W' \rightarrow W$ be a closed embedding of real codimension $r \geq 2$ between real smooth manifolds. Assume that W and $U = W \setminus W'$ are simply connected. Then the open immersion*

$$\iota : U_+ \rightarrow W_+$$

is a weak equivalence in $L_{r-2}\mathbf{Top}$.

Proof. By Proposition 1.5.2 in [5] it suffices to show that ι induces a bijection on the homotopy groups π_i for $i \leq r - 2$ and for every choice of base point in U . Since U and W are simply connected, by Whitehead's theorem it suffices to show that the induced map $H_i(\iota)$ on singular homology with integral coefficients is an isomorphism for $0 \leq i \leq r - 2$ and an epimorphism for $i = r - 1$ (see [29, Thm. 4], [28, p. 181, Thm. 7.13]). But this is equivalent to

$$H_i(W/U) = * \quad \text{for every } i \leq r - 1,$$

where we write W/U for the homotopy cofibre of ι . Now, by excision and the tubular neighborhood theorem (cf. [12, p. 117, Cor. 11.2]),

$$H_i(W/U) \cong H_i(Th(N_{W',W})),$$

where $Th(N_{W',W})$ is the Thom space of the normal bundle of the closed embedding $W' \rightarrow W$ (see Definition 7.18). Since singular homology satisfies the Mayer-Vietoris property, it suffices to consider the case when $N_{W',W}$ is trivial. In this case, $Th(N_{W',W})$ is homotopy equivalent to $S^r \wedge W'_+$, since $N_{W',W}$ is a vector bundle of rank r over W' . Thus, we conclude that for $i \leq r-1$:

$$H_i(Th(N_{W',W})) \cong H_i(S^r \wedge W'_+) = *,$$

as we wanted. $\square$

Corollary 6.15. *Let $V' \rightarrow V$ be a closed embedding of complex codimension $c \geq 1$ between analytic manifolds. Assume that V and $U = V \setminus V'$ are simply connected. Then the open immersion*

$$\iota : U_+ \rightarrow V_+$$

is a weak equivalence in $L_{2c-2}\mathbf{Top}$.

Proof. This follows from Lemma 6.14, taking $r = 2c$. $\square$

Lemma 6.16. *Let $Y \in Sm_X$ be a smooth irreducible quasi-projective variety of dimension d and $\iota_{U,Y} : U_+ \rightarrow Y_+$ be a map in WB_n . Then*

$$\mathbb{C}_R(\iota_{U,Y}) : U(\mathbb{C})_+ \rightarrow Y(\mathbb{C})_+$$

is a weak equivalence in $L_{2n}\mathbf{Top}$.

Proof. By Proposition 1.5.2 in [5] it suffices to show that $\mathbb{C}_R(\iota_{U,Y})$ induces a bijection on the homotopy groups π_i for $i \leq 2n$ and for every choice of base point in $U(\mathbb{C})_+$. Since the map $\iota_{U,Y}$ preserves the disjoint point $+$, we can assume that the base point is in $U(\mathbb{C})$. Moreover, Y is irreducible and U is open in Y , so we deduce that U is also irreducible and in particular $U(\mathbb{C})$ is path connected. Hence the result is independent of the choice of base point, and we will omit any reference to it in the rest of the proof.

If $n = 0$, then it is enough to see that $\mathbb{C}_R(\iota_{U,Y})$ induces a bijection on π_0 . But this is clear since Y is irreducible.

Now we assume that $n > 0$, so $d \geq 2$. By Lemma 6.13, $\mathbb{C}_R(\iota_{U,Y})$ induces a bijection on the fundamental groups π_1 . Hence, it only remains to check that $\pi_i(\mathbb{C}_R(\iota_{U,Y}))$ is an isomorphism for $2 \leq i \leq 2n$.

Let $\widetilde{Y(\mathbb{C})}$ be the universal covering space of $Y(\mathbb{C})$, and consider the following cartesian square of topological spaces:

$$\begin{array}{ccc} p_Y^{-1}(U(\mathbb{C})) & \xrightarrow{\mathbb{C}_R(\widetilde{\iota_{U,Y}})} & \widetilde{Y(\mathbb{C})} \\ p_U \downarrow & & \downarrow p_Y \\ U(\mathbb{C}) & \xrightarrow{\mathbb{C}_R(\iota_{U,Y})} & Y(\mathbb{C}) \end{array}$$

Notice that $\widetilde{Y(\mathbb{C})}$ has the structure of an analytic manifold where p_Y is a local analytic isomorphism. Since $Y(\mathbb{C})$ is smooth we deduce that $\widetilde{Y(\mathbb{C})}$ is irreducible as an

analytic manifold. Thus, $p_Y^{-1}(U(\mathbb{C}))$ is irreducible and in particular path connected, since it is the open complement of the closed analytic submanifold $p_Y^{-1}(Z(\mathbb{C}))$ of $\widetilde{Y(\mathbb{C})}$, where $Z = Y \setminus U$. Hence, p_U is a covering map.

We claim that $p_Y^{-1}(U(\mathbb{C}))$ is the universal covering space of $U(\mathbb{C})$. In effect, it suffices to show that $\pi_1(p_Y^{-1}(U(\mathbb{C}))) = *$. Now, we observe that $\pi_1(p_U)$ is injective since p_U is a covering, and by Lemma 6.13, $\pi_1(\mathbb{C}_R(\iota_{U,Y}))$ is an isomorphism. Thus, by the commutativity of the diagram above we conclude that $\pi_1(p_Y) \circ \pi_1(\mathbb{C}_R(\iota_{U,Y}))$ is also injective, but this map is trivial since it factors through $\pi_1(\widetilde{Y(\mathbb{C})}) = *$. Thus, $\pi_1(p_Y^{-1}(U(\mathbb{C}))) = *$, as we wanted.

Hence, in order to show that $\pi_i(\mathbb{C}_R(\iota_{U,Y}))$ is an isomorphism for $2 \leq i \leq 2n$, it suffices to show that $\pi_i(\mathbb{C}_R(\iota_{U,Y}))$ is an isomorphism for $2 \leq i \leq 2n$. But this follows directly from Corollary 6.15 and [5, Prop. 1.5.2], since $\mathbb{C}_R(\iota_{U,Y})$ is an open immersion between analytic manifolds with smooth closed complement $p_Y^{-1}(Z(\mathbb{C}))$ of complex codimension $\geq n + 1$. $\square$

7. SOME COMPUTATIONS

In this section we study the behavior of unstable and stable slices with respect to the Quillen adjunction between the category of pointed simplicial presheaves on Sm_X and the category of symmetric T -spectra on $\mathcal{M}$:

$$(7.1) \quad (\Sigma_T^\infty, ev_0, \varphi) : \mathcal{M} \rightarrow Spt(\mathcal{M}).$$

We also carry out some computations of unstable slices using a method similar to [19, §4].

Definition 7.2. Let A in $\mathcal{H}$ be a pointed simplicial presheaf on Sm_X and $n \geq 0$ be an arbitrary integer. We will say that A is an n -slice if $A \cong s_n A$ in $\mathcal{H}$, i.e. the maps $\tau_{\leq n+1}^A$ and i_n^A in the following diagram are isomorphisms in $\mathcal{H}$:

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow \tau_{\leq n+1}^A & & \\ s_n A & \xrightarrow{i_n^A} & t_{n+1} A & \xrightarrow{\nu_n^A} & t_n A \end{array}$$

Recall that the functors, $t_n : \mathcal{H} \rightarrow \mathcal{H}$ were defined in Proposition 3.11.

Lemma 7.3. Let $n \geq 0$ be an arbitrary integer and $f : A \rightarrow B$ be a map in $\mathcal{M}$ which is a weak equivalence in $WB_n \mathcal{M}$. Then, for every integer $1 \leq j \leq n + 1$, $t_j(f)$ is an isomorphism in $\mathcal{H}$.

Proof. By Lemma 3.1 we deduce that f is a weak equivalence in $WB_{j-1} \mathcal{M}$. By Lemma 3.9(1), we conclude that $\tau_j(f)$ is a weak equivalence in $\mathcal{M}$. Hence, the result follows from Proposition 3.11(1). $\square$

Lemma 7.4. Let $n \geq 1$ be an arbitrary integer and $f : A \rightarrow B$ be a map in $\mathcal{H}$ such that $t_{n+1}(f)$ and $t_n(f)$ are isomorphisms in $\mathcal{H}$. Then the unstable n -slice of f , $s_n(f)$, is also an isomorphism in $\mathcal{H}$.

Proof. It follows from Proposition 3.13 that the rows in the following commutative diagram are fibre sequences in $\mathcal{H}$:

$$\begin{array}{ccccc} s_n(A) & \longrightarrow & t_{n+1}(A) & \longrightarrow & t_n(A) \\ s_n(f) \downarrow & & \downarrow t_{n+1}(f) & & \downarrow t_n(f) \\ s_n(B) & \longrightarrow & t_{n+1}(B) & \longrightarrow & t_n(B) \end{array}$$

Thus, by [21, I.3, Prop. 5(iii)] we conclude that $s_n(f)$ is also an isomorphism in $\mathcal{H}$. $\square$

Theorem 7.5. *Let A be in $\mathcal{H}$ and $n \geq 0$. Then A is an n -slice (see Definition 7.2) if and only if the following conditions hold:*

- (1) $t_j A \rightarrow *$ is an isomorphism in $\mathcal{H}$ for every $1 \leq j \leq n$,
- (2) the natural map $\tau_{\leq n+1}^A : A \rightarrow t_{n+1} A$ is an isomorphism in $\mathcal{H}$ (see Theorem 3.16).

Proof. ($\Rightarrow$): Assume that A is an n -slice. Thus, the maps $\tau_{\leq n+1}^A, i_n^A$ in the diagram below are isomorphisms in $\mathcal{H}$ (see Definition 7.2). Thus, the map ν_n^A is null homotopic in $\mathcal{H}$ (i.e. it factors through $*$), since the horizontal row in the following diagram is a fibre sequence in $\mathcal{H}$ by Proposition 3.13:

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow \tau_{\leq n+1}^A & & \\ s_n A & \xrightarrow{i_n^A} & t_{n+1} A & \xrightarrow{\nu_n^A} & t_n A \end{array}$$

Hence, the following diagram in $\mathcal{H}$ commutes:

$$\begin{array}{ccc} t_{n+1} A & \xrightarrow{\nu_n^A} & t_n A \\ & \searrow & \uparrow \\ & & * \end{array}$$

Applying the functor t_j , we obtain the following commutative diagram in $\mathcal{H}$:

$$\begin{array}{ccc} t_j A \cong t_j(t_{n+1} A) & \xrightarrow{t_j(\nu_n^A)} & t_j(t_n A) \cong t_j A \\ & \searrow & \uparrow \\ & & t_j(*) = * \end{array}$$

where $t_j A \cong t_j(t_{n+1} A)$ and $t_j(t_n A) \cong t_j A$ are isomorphic in $\mathcal{H}$ by Corollary 3.15. Now, Proposition 3.14 implies that $t_j(\nu_n^A)$ is an isomorphism in $\mathcal{H}$ since $n \geq j$. Thus, we conclude that $t_j A$ is a retract of $*$ in $\mathcal{H}$, so $t_j A \cong *$ in $\mathcal{H}$.

($\Leftarrow$): Assume that (1) and (2) hold. Then, by Proposition 3.13 we deduce that $s_n A \cong t_{n+1} A \cong A$ in $\mathcal{H}$. $\square$

Now, we will study the relation between the stable slice filtration (see Theorems 1.5, 1.8) and its unstable analogue (see Theorem 3.16). Consider the infinite suspension of the set of maps described in Definition 2.9:

$$(7.6) \quad \Sigma_T^\infty WB_n = \{ \Sigma_T^\infty \iota_{U,Y} \mid \iota_{U,Y} : U_+ \rightarrow Y_+ \text{ open immersion;} \\ Y, Z = Y \setminus U \in Sm_X; Y \text{ irreducible; } (\text{codim}_Y Z) \geq n+1 \}.$$

Proposition 7.7. *Let $n \geq 0$ be an arbitrary integer. In the following commutative diagram, all the arrows are left Quillen functors:*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Sigma_T^\infty} & Spt(\mathcal{M}) \\ \downarrow id & & \downarrow id \\ WB_n \mathcal{M} & \xrightarrow{\Sigma_T^\infty} & WB_n Spt(\mathcal{M}) \end{array}$$

Proof. It is clear that the diagram commutes and that the top horizontal arrow is a left Quillen functor. The vertical arrows are left Quillen functors since by construction $WB_n \mathcal{M}$ (resp. $WB_n Spt(\mathcal{M})$) is a left Bousfield localization of $\mathcal{M}$ (resp. $Spt(\mathcal{M})$).

Now, we observe that all the maps in (7.6) are contained in (1.7). Hence, we deduce that all the maps in (7.6) are weak equivalences in $WB_n Spt(\mathcal{M})$, since by construction it is the left Bousfield localization of $Spt(\mathcal{M})$ with respect to the maps in (1.7). Finally, by the universal property of left Bousfield localizations (see [5, 3.3.19(1) and 3.1.1(1)]) we conclude that the bottom horizontal arrow is also a left Quillen functor. $\square$

We will write $\mathcal{SH}(WB_n)$ for the homotopy category of $WB_n Spt(\mathcal{M})$ and

$$(\Sigma_T^\infty, \Omega_T^\infty, \varphi) : \mathcal{H} \rightarrow \mathcal{SH}$$

for the derived adjunction of (7.1).

Let $E \in Spt(\mathcal{M})$ be an arbitrary symmetric T -spectrum and $n \geq 0$ be an arbitrary integer. Consider the functors $s_n : \mathcal{SH} \rightarrow \mathcal{SH}$, $s_{<n+1} : \mathcal{SH} \rightarrow \mathcal{SH}$ as defined in Theorem 1.5.

Proposition 7.8. *$\Omega_T^\infty s_n E$ and $\Omega_T^\infty s_{<n+1} E$ are fibrant in $WB_n \mathcal{M}$.*

Proof. By Theorems 1.5(2) and 1.8 (resp. 1.5(2)-(3) and 1.8) we can assume that $\Omega_T^\infty s_{<n+1} E$ (resp. $\Omega_T^\infty s_n E$) is fibrant in $WB_n Spt(\mathcal{M})$. Hence, the result follows from Proposition 7.7. $\square$

On the other hand, combining (1.2) with Theorems 1.5(2) and 1.8 we deduce that:

$$f_{n+1} E \longrightarrow E \longrightarrow s_{<n+1} E$$

is a fibre sequence (and a distinguished triangle) in $\mathcal{SH}(WB_n)$ and $\mathcal{SH}$. Similarly, combining (1.4) with Theorems 1.5(2)-(3) and 1.8 we deduce that

$$s_n E \longrightarrow s_{<n+1} E \longrightarrow s_{<n} E$$

is a fibre sequence (and a distinguished triangle) in $\mathcal{SH}(WB_n)$ and $\mathcal{SH}$. Therefore, by Proposition 7.7 we conclude:

Proposition 7.9. *Let $E \in \text{Spt}(\mathcal{M})$ be an arbitrary symmetric T -spectrum and $n \geq 0$ be an arbitrary integer. Then the following are fibre sequences in $\mathcal{H}(WB_n)$ and $\mathcal{H}$:*

$$\begin{aligned} \Omega_T^\infty f_{n+1} E &\rightarrow \Omega_T^\infty E \rightarrow \Omega_T^\infty s_{<n+1} E, \\ \Omega_T^\infty s_n E &\rightarrow \Omega_T^\infty s_{<n+1} E \rightarrow \Omega_T^\infty s_{<n} E. \end{aligned}$$

By looking at Propositions 7.8 and 7.9, it is natural to ask if $\Omega_T^\infty s_n E$ is an unstable n -slice (see Definition 7.2). If $n = 0$, then by Theorem 7.5 we conclude that $\Omega_T^\infty s_0 E$ is an unstable zero-slice. However, if $n \geq 1$, then $\Omega_T^\infty s_n E$ is not an unstable n -slice in general.

Example 7.10. Let the base scheme $X = \text{Spec } k$, with k a perfect field and $\ell \geq 3$ a prime different from the characteristic of k . Consider the spectrum $\mathbf{H}_\ell$ which represents in $\mathcal{SH}$ motivic cohomology with $\mathbb{Z}/\ell$ -coefficients [25, §6.1]. Then, $\Omega_T^\infty s_{\ell-1}(S^{2-\ell} \wedge \mathbb{G}_m^{\ell-1} \wedge \mathbf{H}_\ell)$ is not an unstable $\ell - 1$ -slice.

In effect, Guillou and Weibel construct a non-trivial map in $\mathcal{H}$ (taking $r = 1$ in [3, Ex. 8.1] and using the grading for motivic cohomology introduced in [25, §6, p. 595]):

$$v : \Omega_T^\infty(S^{2-\ell} \wedge \mathbb{G}_m^{\ell-1} \wedge \mathbf{H}_\ell) \rightarrow \Omega_T^\infty(\mathbb{G}_m \wedge \mathbf{H}_\ell).$$

On the other hand, by the work of Levine [10, Lem. 10.4.1] $s_0 \mathbf{H}_\ell \cong \mathbf{H}_\ell$ in $\mathcal{SH}$. Thus, we conclude that in $\mathcal{SH}$,

$$\begin{aligned} s_{\ell-1}(S^{2-\ell} \wedge \mathbb{G}_m^{\ell-1} \wedge \mathbf{H}_\ell) &\cong S^{2-\ell} \wedge \mathbb{G}_m^{\ell-1} \wedge \mathbf{H}_\ell, \\ s_1(\mathbb{G}_m \wedge \mathbf{H}_\ell) &\cong \mathbb{G}_m \wedge \mathbf{H}_\ell. \end{aligned}$$

Now, by Theorem 3.16(3):

$$\begin{aligned} \tau_{\leq 2}^{\Omega_T^\infty s_{\ell-1}(S^{2-\ell} \wedge \mathbb{G}_m^{\ell-1} \wedge \mathbf{H}_\ell)} : \Omega_T^\infty s_{\ell-1}(S^{2-\ell} \wedge \mathbb{G}_m^{\ell-1} \wedge \mathbf{H}_\ell) \\ \rightarrow t_2(\Omega_T^\infty s_{\ell-1}(S^{2-\ell} \wedge \mathbb{G}_m^{\ell-1} \wedge \mathbf{H}_\ell)) \end{aligned}$$

is a weak equivalence in $WB_1\mathcal{M}$. Combining the adjunctions in Theorem 2.14(2) with the fact that $\Omega_T^\infty s_1(\mathbb{G}_m \wedge \mathbf{H}_\ell) \cong \Omega_T^\infty(\mathbb{G}_m \wedge \mathbf{H}_\ell)$ is fibrant in $WB_1\mathcal{M}$ (by Proposition 7.8) and the non-triviality of v , we deduce that

$$t_2(\Omega_T^\infty s_{\ell-1}(S^{2-\ell} \wedge \mathbb{G}_m^{\ell-1} \wedge \mathbf{H}_\ell)) \rightarrow *$$

is not an isomorphism in $\mathcal{H}$. Hence, by Theorem 7.5(1) it follows that

$$\Omega_T^\infty s_{\ell-1}(S^{2-\ell} \wedge \mathbb{G}_m^{\ell-1} \wedge \mathbf{H}_\ell)$$

is not an unstable $\ell - 1$ -slice.

Remark 7.11. From Example 7.10 we observe that the unstable slices are non-trivial and are finer invariants than the corresponding stable slices.

Now we proceed to describe some computations of unstable slices.

Remark 7.12 (see Remark 6.11). The following result is an unstable motivic analogue of the standard result in topology: $\pi_i(S^{2n}) = *$ for $i \leq 2n - 1$, i.e. the topological connectivity of the spheres.

We will write T^r for $S^r \wedge \mathbb{G}_m^r$, where $r \geq 0$ is an integer.

Theorem 7.13. *Let $r \geq 1$, $1 \leq m \leq r$, and $0 \leq n \leq r - 1$ be arbitrary integers. Then:*

- (1) *For every $Z \in Sm_X$, the natural map $t_m(T^r \wedge Z_+) \rightarrow *$ is an isomorphism in $\mathcal{H}$.*
- (2) *For every $Z \in Sm_X$, the natural map $s_n(T^r \wedge Z_+) \rightarrow *$ is an isomorphism in $\mathcal{H}$.*

Proof. By Lemma 7.4 and the definition of the zero slice (see Definition 3.12) it is enough to prove (1).

We claim that the natural map $T^r \wedge Z_+ \rightarrow *$ is a weak equivalence in $WB_{r-1}\mathcal{M}$. In effect, [13, Prop. 2.17(2)] implies that the following diagram is a cofibre sequence in $\mathcal{M}$:

$$(7.14) \quad (\mathbb{A}_Z^r \setminus Z) \xrightarrow{\alpha_{r,Z}} \mathbb{A}_Z^r \longrightarrow T^r \wedge Z_+,$$

where Z is embedded in $\mathbb{A}_Z^r$ via the zero section. Now, the identity functor $id : \mathcal{M} \rightarrow WB_{r-1}\mathcal{M}$ is a left Quillen functor, since $WB_{r-1}\mathcal{M}$ is a left Bousfield localization of $\mathcal{M}$. Hence diagram (7.14) is also a cofibre sequence in $WB_{r-1}\mathcal{M}$. Therefore, by the dual of [21, I.3, Prop. 5(iii)] it is enough to show that $\alpha_{r,Z}$ is a weak equivalence in $WB_{r-1}\mathcal{M}$. But this follows from the definition of $WB_{r-1}\mathcal{M}$ (see Definition 2.11), since the codimension of the closed complement of the open immersion $\alpha_{r,Z}$ is r .

Finally, the result follows from Lemma 7.3. $\square$

Given $Y \in Sm_X$, let $\mathbb{P}^n(Y)$ denote the trivial projective bundle of rank n over Y . Consider the following filtered diagram in $\mathcal{M}$:

$$(7.15) \quad \mathbb{P}^0(Y) \rightarrow \mathbb{P}^1(Y) \rightarrow \cdots \rightarrow \mathbb{P}^n(Y) \rightarrow \cdots,$$

given by the inclusions of the respective hyperplanes at infinity.

Remark 7.16 (see Remark 6.11). The following result is an unstable motivic analogue of the standard result in topology: $\pi_i(\mathbb{P}^n(\mathbb{C})) \cong \pi_i(\mathbb{P}^{n+1}(\mathbb{C}))$ for $i \leq 2n$.

Theorem 7.17. *(1) For any integer $1 \leq j \leq n + 1$, diagram (7.15) induces the following isomorphisms in $\mathcal{H}$:*

$$t_j(\mathbb{P}^n(Y)_+) \xrightarrow{\cong} t_j(\mathbb{P}^{n+1}(Y)_+) \xrightarrow{\cong} \cdots \xrightarrow{\cong} t_j(\mathbb{P}^{n+m}(Y)_+) \xrightarrow{\cong} \cdots.$$

- (2) *For any integer $0 \leq j \leq n$, diagram (7.15) induces the following isomorphisms in $\mathcal{H}$:*

$$s_j(\mathbb{P}^n(Y)_+) \xrightarrow{\cong} s_j(\mathbb{P}^{n+1}(Y)_+) \xrightarrow{\cong} \cdots \xrightarrow{\cong} s_j(\mathbb{P}^{n+m}(Y)_+) \xrightarrow{\cong} \cdots.$$

Proof. By Lemma 7.4 and the definition of the zero slice (see Definition 3.12) it is enough to prove (1).

Let $k > n$, and consider the closed embedding induced by diagram (7.15), $\lambda_n^k : \mathbb{P}^n(Y) \rightarrow \mathbb{P}^k(Y)$. It is possible to choose a linear embedding $\mathbb{P}^{k-n-1}(Y) \rightarrow \mathbb{P}^k(Y)$

such that its open complement $U_{k,n}$ contains $\mathbb{P}^n(Y)$ (embedded via the zero section σ_n^k) and has the structure of a vector bundle over $\mathbb{P}^n(Y)$:

$$\begin{array}{ccc} U_{k,n} & \xrightarrow{v_n^k} & \mathbb{P}^k(Y) \longleftarrow \mathbb{P}^{k-n-1}(Y) \\ \downarrow & \uparrow \sigma_n^k & \nearrow \lambda_n^k \\ \mathbb{P}^n(Y) & & \end{array}$$

By homotopy invariance $t_j(\sigma_n^k)$ is an isomorphism in $\mathcal{H}$ for every integer j . On the other hand, v_n^k is a weak equivalence in $WB_n\mathcal{M}$ since the codimension of its closed complement is $n+1$ (see Definition 2.11). Thus, Lemma 7.3 implies that if $1 \leq j \leq n+1$, then $t_j(v_n^k)$ is also an isomorphism in $\mathcal{H}$.

Therefore, $t_j(\lambda_n^k) = t_j(v_n^k) \circ t_j(\sigma_n^k)$ is an isomorphism in $\mathcal{H}$ for $1 \leq j \leq n+1$. $\square$

Definition 7.18. Given a closed embedding $Z \rightarrow Y$ in Sm_X , let $N_{Y,Z}$ denote the associated normal bundle, and given a vector bundle $\pi : V \rightarrow Y$ with $Y \in Sm_X$, let $Th(V)$ denote the Thom space of V , i.e. the pushout of the following diagram in $\mathcal{M}$:

$$\begin{array}{ccc} V \setminus \sigma_0(Y)_+ & \longrightarrow & V_+ \\ \downarrow & & \downarrow \\ X & \longrightarrow & Th(V) \end{array}$$

where $\sigma_0 : Y \rightarrow V$ is the zero section of V .

Remark 7.19 (see Remark 6.11). The following result is an unstable motivic analogue of the standard result in topology: If ξ is a complex vector bundle of rank r over a CW complex, then $\pi_k(Th(\xi)) = *$ for $k \leq 2r-1$ (see [12, p. 205, Lem. 18.1]).

Theorem 7.20. *Let $n \geq 0$, $0 \leq j \leq n$, and $1 \leq k \leq n+1$ be arbitrary integers, and let $\iota_{U,Y} \in WB_n$ with closed complement Z . Then:*

- (1) *The natural map $t_k(Th(N_{Y,Z})) \rightarrow *$ is an isomorphism in $\mathcal{H}$.*
- (2) *The natural map $s_j(Th(N_{Y,Z})) \rightarrow *$ is an isomorphism in $\mathcal{H}$.*

Proof. By Lemma 7.4 and the definition of the zero slice (see Definition 3.12) it is enough to prove (1).

Now, it follows from the Morel-Voevodsky homotopy purity theorem (see [13, Thm. 2.23]) that the following diagram is a cofibre sequence in $\mathcal{M}$

$$U_+ \xrightarrow{\iota_{U,Y}} Y_+ \longrightarrow Th(N_{Y,Z})$$

and Theorem 2.14 implies that the diagram above is also a cofibre sequence in $WB_n\mathcal{M}$. But $\iota_{U,Y}$ is a weak equivalence in $WB_n\mathcal{M}$, since it is defined as a left Bousfield localization of $\mathcal{M}$ with respect to WB_n . Hence, the dual of [21, I.3, Prop. 5(iii)] implies that the map

$$Th(N_{Y,Z}) \rightarrow *$$

is a weak equivalence in $WB_n\mathcal{M}$, and the result then follows from Lemma 7.3. $\square$

Remark 7.21 (see Remark 6.11). The following result is an unstable motivic analogue of the standard result in topology: If ξ is a complex vector bundle of rank r over a CW complex X , then $Th(\xi)$ has a cell decomposition given by the base

point and $(n + 2r)$ -cells which are in bijection with the n -cells of X , for $n > 0$ (see [12, p. 205, Lem. 18.1]).

In the case $U = Y$, the following result is due to Morel-Voevodsky (see Proposition 2.17(2) in [13]), and this particular case is an ingredient in our proof.

Theorem 7.22. *Let $n \geq 0$, $0 \leq j \leq n$, and $1 \leq k \leq n + 1$ be arbitrary integers, and let $\iota_{U,Y} \in WB_n$. Consider a vector bundle $\pi : V \rightarrow Y$ of rank r together with a trivialization $\lambda : \pi^{-1}(U) \rightarrow \mathbb{A}_U^r$ of its restriction to U . Then:*

(1) *There exists an isomorphism in $\mathcal{H}$*

$$t_k(Th(V)) \cong t_k(S^r \wedge \mathbb{G}_m^r \wedge Y_+).$$

(2) *There exists an isomorphism in $\mathcal{H}$*

$$s_j(Th(V)) \cong s_j(S^r \wedge \mathbb{G}_m^r \wedge Y_+).$$

Proof. Let $Z \in Sm_X$ be the closed complement of $\iota_{U,Y}$. Consider the following diagram in Sm_X , where all the squares are cartesian:

$$\begin{array}{ccccc} \pi^{-1}(Z) \cap (V \setminus \sigma_0(Y)) & \longrightarrow & V \setminus \sigma_0(Y) & \xleftarrow{\beta} & \pi^{-1}(U) \cap (V \setminus \sigma_0(Y)) \\ \downarrow & & \downarrow & & \downarrow \\ \pi^{-1}(Z) & \longrightarrow & V & \xleftarrow{\alpha} & \pi^{-1}(U) \\ \downarrow & & \downarrow \pi & & \downarrow \\ Z & \longrightarrow & Y & \xleftarrow{\iota_{U,Y}} & U \end{array}$$

and let $\gamma : Th(\pi^{-1}(U)) \rightarrow Th(V)$ be the induced map between the corresponding Thom spaces. We observe that α, β also belong to WB_n ; thus, α, β are weak equivalences in $WB_n\mathcal{M}$ (see Definition 2.11). We claim that

$$\gamma : Th(\pi^{-1}(U)) \rightarrow Th(V)$$

is also a weak equivalence in $WB_n\mathcal{M}$. In effect, by construction of the Thom spaces, we deduce that the rows in the following commutative diagram are in fact cofibre sequences in $\mathcal{M}$:

$$\begin{array}{ccccc} \pi^{-1}(U) \cap (V \setminus \sigma_0(Y))_+ & \longrightarrow & \pi^{-1}(U)_+ & \longrightarrow & Th(\pi^{-1}(U)) \\ \beta \downarrow & & \alpha \downarrow & & \gamma \downarrow \\ (V \setminus \sigma_0(Y))_+ & \longrightarrow & V_+ & \longrightarrow & Th(V) \end{array}$$

But, Theorem 2.14 implies that these rows are also cofibre sequences in $WB_n\mathcal{M}$. Since α, β are weak equivalences in $WB_n\mathcal{M}$, it follows from the dual of [21, I.3, Prop. 5(iii)] that γ is also a weak equivalence in $WB_n\mathcal{M}$.

Now, we use the trivialization λ to obtain the following commutative diagram in Sm_X where the rows are isomorphisms:

$$\begin{array}{ccc}
 \mathbb{A}_U^r \setminus U & \xleftarrow[\cong]{\bar{\lambda}} & \pi^{-1}(U) \cap (V \setminus \sigma_0(Y)) \\
 \downarrow & & \downarrow \\
 \mathbb{A}_U^r & \xleftarrow[\cong]{\lambda} & \pi^{-1}(U) \\
 & \searrow & \swarrow \pi_U \\
 & U &
 \end{array}$$

The same argument as above shows that there is a weak equivalence in $WB_n\mathcal{M}$,

$$\bar{\lambda} : Th(\pi^{-1}(U)) \rightarrow Th(\mathbb{A}_U^r).$$

On the other hand, [13, Prop. 2.17(2)] implies that there is a weak equivalence $w : Th(\mathbb{A}_U^r) \rightarrow S^r \wedge \mathbb{G}_m^r \wedge U_+$ in $\mathcal{M}$. Since $WB_n\mathcal{M}$ is a left Bousfield localization of $\mathcal{M}$, it follows from [5, Prop. 3.3.3(1)(a)] that w is also a weak equivalence in $WB_n\mathcal{M}$. Moreover, by construction $\iota_{U,Y}$ is a weak equivalence in $WB_n\mathcal{M}$ (see Definition 2.11), since we are assuming that $\iota_{U,Y} \in WB_n$, and Ken Brown's lemma (see [7, Lem. 1.1.12]) implies that $S^r \wedge \mathbb{G}_m^r \wedge \iota_{U,Y}$ is also a weak equivalence in $WB_n\mathcal{M}$.

Thus, all the maps in the following diagram are weak equivalences in $WB_n\mathcal{M}$:

$$\begin{array}{ccc}
 Th(\pi^{-1}(U)) & \xrightarrow{\bar{\lambda}} Th(\mathbb{A}_U^r) & \xrightarrow{w} S^r \wedge \mathbb{G}_m^r \wedge U_+ \\
 \gamma \downarrow & & \downarrow S^r \wedge \mathbb{G}_m^r \wedge \iota_{U,Y} \\
 Th(V) & & S^r \wedge \mathbb{G}_m^r \wedge Y_+
 \end{array}$$

Finally, the result follows from Lemmas 7.3 and 7.4. $\square$

Corollary 7.23. *Assume that the base scheme $X = \text{Spec } k$, with k a perfect field. Let $n \geq 0$, $0 \leq j \leq n$, and $1 \leq k \leq n+1$ be arbitrary integers, and let $\iota_{U,Y} \in B_n$. Consider a vector bundle $\pi : V \rightarrow Y$ of rank r together with a trivialization $\lambda : \pi^{-1}(U) \rightarrow \mathbb{A}_U^r$ of its restriction to U . Then:*

(1) *There exists an isomorphism in $\mathcal{H}$*

$$t_k(Th(V)) \cong t_k(S^r \wedge \mathbb{G}_m^r \wedge Y_+).$$

(2) *There exists an isomorphism in $\mathcal{H}$*

$$s_j(Th(V)) \cong s_j(S^r \wedge \mathbb{G}_m^r \wedge Y_+).$$

Proof. By Proposition 2.13 we can use the same argument as in Theorem 7.22 to prove the result. $\square$

Given a closed embedding $Z \rightarrow Y$ of smooth schemes over X , let $\mathcal{B}\ell_Z Y$ denote the blowup of Y with center in Z .

Remark 7.24 (see Remark 6.11). Combining Theorems 6.9 and 6.2 with the proof of Theorem 7.25 we obtain the following result in the classical topological setting, which although not surprising, it appears not to be in the literature: Let $Z \rightarrow Y$ be a closed embedding of smooth quasi-projective complex algebraic varieties, and

let $\mathcal{B}\ell_Z Y$ denote the blowup of Y with center in Z . Then the k -skeleton of Y is a retract of the k -skeleton of $\mathcal{B}\ell_Z Y$, where k is the codimension of Z in Y .

Theorem 7.25. *Let $n \geq 1$, $0 \leq j \leq n$, and $1 \leq k \leq n + 1$ be arbitrary integers, and let $\iota_{U,Y} \in WB_n$ with closed complement Z . Consider the following cartesian square in Sm_X :*

$$(7.26) \quad \begin{array}{ccccc} D & \xrightarrow{d} & \mathcal{B}\ell_Z Y & \xleftarrow{u} & U \\ q \downarrow & & \downarrow p & & \parallel \\ Z & \xrightarrow{i} & Y & \xleftarrow{\iota_{U,Y}} & U \end{array}$$

Then:

- (1) $t_1(u) = s_0(u)$ is an isomorphism in $\mathcal{H}$.
- (2) $t_k(\iota_{U,Y})$ is an isomorphism in $\mathcal{H}$, and the following diagram commutes:

$$(7.27) \quad \begin{array}{ccc} & \mathcal{B}\ell_Z Y_+ & \\ \tilde{r}_k \nearrow & \downarrow t_k(p) & \\ Y_+ & \xlongequal{\quad} & Y_+ \end{array}$$

where $\tilde{r}_k = t_k(u) \circ (t_k(\iota_{U,Y}))^{-1}$.

- (3) $s_j(\iota_{U,Y})$ is an isomorphism in $\mathcal{H}$, and the following diagram commutes:

$$(7.28) \quad \begin{array}{ccc} & \mathcal{B}\ell_Z Y_+ & \\ r_j \nearrow & \downarrow s_j(p) & \\ Y_+ & \xlongequal{\quad} & Y_+ \end{array}$$

where $r_j = s_j(u) \circ (s_j(\iota_{U,Y}))^{-1}$.

Proof. (1) We observe that $u \in WB_0$ (see Definition 2.9). Thus, u is a weak equivalence in $WB_0\mathcal{M}$ (see Definition 2.11), and the result follows from Lemma 7.3.

(2) We observe that $\iota_{U,Y}$ is a weak equivalence in $WB_n\mathcal{M}$ (see Definition 2.11). Then Lemma 7.3 implies that $t_k(\iota_{U,Y})$ is an isomorphism in $\mathcal{H}$. Thus, $\tilde{r}_k$ is well defined, and by functoriality diagram (7.27) commutes in $\mathcal{H}$.

(3) The result follows from (2) above together with Lemma 7.4 and the definition of the zero slice (see Definition 3.12). $\square$

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REFERENCES

- [1] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, vol. 304, Springer-Verlag, Berlin, 1972. MR0365573 (51 #1825)
- [2] Bruno Kahn and R. Sujatha, *Birational motives, I*. Preprint, October 30, 2002, K-theory Preprint Archives, <http://www.math.uiuc.edu/K-theory/0596/>.

- [3] B. Guillou and C. Weibel, Unstable operations in etale and motivic cohomology, *preprint*, 2013.
- [4] Helmut A. Hamm and Lê Dũng Tráng, *Lefschetz theorems on quasiprojective varieties* (English, with French summary), Bull. Soc. Math. France **113** (1985), no. 2, 123–142. MR820315 (87i:32017)
- [5] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR1944041 (2003j:18018)
- [6] Jens Hornbostel, *Localizations in motivic homotopy theory*, Math. Proc. Cambridge Philos. Soc. **140** (2006), no. 1, 95–114, DOI 10.1017/S030500410500890X. MR2197578 (2006k:14028)
- [7] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR1650134 (99h:55031)
- [8] J. F. Jardine, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–553 (electronic). MR1787949 (2002b:55014)
- [9] Steven L. Kleiman, *Geometry on Grassmannians and applications to splitting bundles and smoothing cycles*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 281–297. MR0265371 (42 #281)
- [10] Marc Levine, *The homotopy coniveau tower*, J. Topol. **1** (2008), no. 1, 217–267, DOI 10.1112/jtopol/jtm004. MR2365658 (2008j:14013)
- [11] Marc Levine, *Slices and transfers*, Doc. Math. Extra volume: Andrei A. Suslin sixtieth birthday (2010), 393–443. MR2804260 (2012d:14033)
- [12] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, no. 76. MR0440554 (55 #13428)
- [13] Fabien Morel and Vladimir Voevodsky, $\mathbf{A}^1$ -homotopy theory of schemes, Inst. Hautes Études Sci. Publ. Math. **90** (1999), 45–143 (2001). MR1813224 (2002f:14029)
- [14] Amnon Neeman, *The Grothendieck duality theorem via Bousfield’s techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236, DOI 10.1090/S0894-0347-96-00174-9. MR1308405 (96c:18006)
- [15] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001. MR1812507 (2001k:18010)
- [16] I. Panin, K. Pimenov, and O. Röndigs, On Voevodsky’s algebraic K -theory spectrum. In *Algebraic topology*, volume 4 of *Abel Symp.*, pages 279–330. Springer, Berlin, 2009.
- [17] Pablo Pelaez, *Mixed motives and the slice filtration* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **347** (2009), no. 9-10, 541–544, DOI 10.1016/j.crma.2009.02.028. MR2576905 (2010m:14030)
- [18] Pablo Pelaez, *Multiplicative properties of the slice filtration* (English, with English and French summaries), Astérisque **335** (2011), xvi+289. MR2807904 (2012d:14034)
- [19] P. Pelaez, Birational motivic homotopy theories and the slice filtration, *Doc. Math.*, 18:51–70, 2013.
- [20] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., vol. 341. MR0338129 (49 #2895)
- [21] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, no. 43, Springer-Verlag, Berlin, 1967. MR0223432 (36 #6480)
- [22] Oliver Röndigs and Paul Arne Østvær, *Modules over motivic cohomology*, Adv. Math. **219** (2008), no. 2, 689–727, DOI 10.1016/j.aim.2008.05.013. MR2435654 (2009m:14026)
- [23] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152. MR0210075 (35 #970)
- [24] Andrei Suslin and Vladimir Voevodsky, *Relative cycles and Chow sheaves*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 10–86. MR1764199
- [25] Vladimir Voevodsky, $\mathbf{A}^1$ -homotopy theory, Proceedings of the International Congress of Mathematicians, vol. I (Berlin, 1998), 1998, pp. 579–604 (electronic). MR1648048 (99j:14018)
- [26] Vladimir Voevodsky, *Open problems in the motivic stable homotopy theory. I*, Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), Int. Press Lect. Ser., vol. 3, Int. Press, Somerville, MA, 2002, pp. 3–34. MR1977582 (2005e:14030)
- [27] Vladimir Voevodsky, *Cancellation theorem*, Doc. Math. Extra volume: Andrei A. Suslin sixtieth birthday (2010), 671–685. MR2804268 (2012d:14035)

- [28] George W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York, 1978. MR516508 (80b:55001)
- [29] J. H. C. Whitehead, *Combinatorial homotopy. I*, Bull. Amer. Math. Soc. **55** (1949), 213–245. MR0030759 (11,48b)

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