THE RESIDUAL SPECTRUM OF $\mathrm{Mp}_4(\mathbf{A}_k)$

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ABSTRACT. We compute the residual spectrum of the global metaplectic group $\mathrm{Mp}_4(\mathbf{A}_k)$ by using the theory of Eisenstein series. The residual spectra obtained are interpreted as near equivalence classes in the framework of the Arthur conjecture.

1. INTRODUCTION

Let Sp_4 be the symplectic group of rank 2 defined over a number field k with $\mathbf{A}_k$ its adele ring. The global metaplectic group $\mathrm{Mp}_4(\mathbf{A}_k)$ is a degree two central covering of $\mathrm{Sp}_4(\mathbf{A}_k)$. By the general theory of Eisenstein series extended to covering groups (cf. [MoW]), the Hilbert space $L^2_{\mathrm{gen}}(\mathrm{Sp}_4(k)\backslash\mathrm{Mp}_4(\mathbf{A}_k))$ of genuine functions has a spectral decomposition

$$\begin{aligned} L^2_{\mathrm{gen}}(\mathrm{Sp}_4(k)\backslash\mathrm{Mp}_4(\mathbf{A}_k)) &= L^2_{\mathrm{dis}} \oplus L^2_{\mathrm{cont}} \\ &= L^2_{\mathrm{cusp}} \oplus L^2_{\mathrm{res}} \oplus L^2_{\mathrm{cont}}, \end{aligned}$$

where L^2_{dis} is the closed invariant space spanned by irreducible subrepresentations of $L^2(\mathrm{Sp}_4(k)\backslash\mathrm{Mp}_4(\mathbf{A}_k))$, and it consists of the cuspidal part L^2_{cusp} and the residual part L^2_{res} .

The residual spectrum $L^2_{\mathrm{res}}(\mathrm{Sp}_4(k)\backslash\mathrm{Mp}_4(\mathbf{A}_k))$ can be further decomposed into

$$L^2_{\mathrm{res}}(\mathrm{Sp}_4(k)\backslash\mathrm{Mp}_4(\mathbf{A}_k)) = L^2_{\mathrm{d}}(\overline{P}_1) \oplus L^2_{\mathrm{d}}(\overline{P}_2) \oplus L^2_{\mathrm{d}}(\overline{B}),$$

where $\overline{P}_i, i = 1, 2$, are the maximal parabolic subgroups of $\mathrm{Mp}_4(\mathbf{A}_k)$ associated with the Siegel and non-Siegel maximal parabolic $P_i, i = 1, 2$, of Sp_4 respectively, while $\overline{B}$ is associated with the Borel subgroup B of Sp_4 . The three components in the decomposition together contribute exhaustively to the noncuspidal discrete spectrum, the so-called residual spectrum. They are obtained by taking the residues of Eisenstein series associated to cuspidal representations of the Levi factors of the parabolic subgroups $\overline{P}_1, \overline{P}_2, \overline{B}$ respectively.

Therefore a study of the residual spectrum inevitably involves an investigation of the poles of Eisenstein series, which in fact coincide with those of the constant terms. The latter can be expressed as sums of intertwining operators. The analysis of these intertwining operators thus leads to certain global L -functions associated to the aforementioned cuspidal representations, which in turn gives precise conditions for the existence of poles of Eisenstein series.

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The purpose of this paper is to explicate the process of analysis and computations and to determine L_{res}^2 for Mp_4 . For classical linear groups Sp_4 and G_2 such results have been obtained by H. Kim (cf. [Kim1], [Kim2]) and S. Zampera (cf. [Zam]). The interpretation in the framework of the Arthur conjecture (cf. [Art]), for the G_2 case for instance, has appeared in [GGJ]. General references on the theory of Eisenstein series include [MoW], [Lan]. Besides, we also find that the book [CKM] gives a very accessible summary on the theory of Eisenstein series, including background information on analytic properties of various L-functions, intertwining operators, etc.

2. PRELIMINARIES

2.1. Local metaplectic groups and parabolic subgroups. In view of the global considerations later, we fix a number field k henceforth as our base field. Let W be a 4-dimensional symplectic vector space over k . We obtain the associated symplectic group Sp_4 over k . For any finite place v of k , we denote by $\mathfrak{O}_v$ the valuation ring of k_v . Consider the groups $\text{Sp}_4(k)$ and $\text{Sp}_4(k_v)$ for a finite or infinite place v . In particular, when $k_v \neq \mathbf{C}$, the group $\text{Sp}_4(k_v)$ has the unique central extension group $\text{Mp}_4(k_v)$:

$$1 \longrightarrow \mu_2 \longrightarrow \text{Mp}_4(k_v) \longrightarrow \text{Sp}_4(k_v) \longrightarrow 1.$$

One can write this as the sets

$$\text{Mp}_4(k_v) = \text{Sp}_4(k_v) \times \{\pm 1\}$$

with group operation on the right given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 c(g_1, g_2)),$$

for some 2-cocycle on $\text{Sp}_4(k_v)$ with values in $\mu_2 = \{\pm 1\}$. The more precise description of the cocycle can be found in [Kud, sect. I.4], and [Rao, sect. 4 and 5].

For any subset $H \subseteq \text{Sp}_4(k_v)$, denote by $\overline{H}$ its preimage in $\text{Mp}_4(k_v)$, which may or may not be a split covering. For example, let $B = TU$ be the Borel subgroup in $\text{Sp}_4(k_v)$ with maximal split torus T :

$$T = \{t(a, b) = \text{diag}(a, b, b^{-1}, a^{-1})\}.$$

Accordingly, we choose $U \subseteq B$ to be the upper triangle matrices in $\text{Sp}_4(k_v)$. Then the metaplectic covering splits over U by the trivial section, and we have $\overline{U} = U \times \mu_2$ as groups. When the characteristic of the residue field of k_v is odd, the covering also splits over the maximal compact subgroup $\text{Sp}_4(\mathfrak{O}_v)$ (cf. [Szp, sect. 2.3]). Over the torus T , the covering map does not necessarily split, but one still knows that $\overline{T}$ is abelian.

Before we proceed, some notation needs to be set up first. Let α_1, α_2 be the short and long simple roots for the algebraic group Sp_4 , and denote $\alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = 2\alpha_1 + \alpha_2$ throughout this paper. Associated to the short and long roots are the fundamental weights $\beta_1 = \alpha_4/2$ and $\beta_2 = \alpha_3$. The Siegel and non-Siegel maximal parabolic subgroups P_1, P_2 are generated by α_1 and α_2 , with Levi-factors GL_2 and $\text{GL}_1 \times \text{SL}_2$ respectively. The Borel subgroup $B = TU$ is contained in both.

TABLE 1. Decomposition of Weyl elements and the actions on torus

w	α_1	α_2	α_3	α_4	$w^{-1}t(a, b)w$
1	α_1	α_2	α_3	α_4	$t(a, b)$
w_1	$-\alpha_1$	α_4	α_3	α_2	$t(b, a)$
w_2	α_3	$-\alpha_2$	α_1	α_4	$t(a, b^{-1})$
w_{12}	α_3	$-\alpha_4$	$-\alpha_1$	α_2	$t(b^{-1}, a)$
w_{21}	$-\alpha_3$	α_4	α_1	$-\alpha_2$	$t(b, a^{-1})$
w_{121}	$-\alpha_3$	α_2	$-\alpha_1$	$-\alpha_4$	$t(a^{-1}, b)$
w_{212}	α_1	$-\alpha_4$	$-\alpha_3$	$-\alpha_2$	$t(b^{-1}, a^{-1})$
w_{1212}	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$t(a^{-1}, b^{-1})$

As usual, we use $w_i, i = 1, 2$, to represent the simple reflection in the Weyl group associated to α_i . We choose representatives respectively for w_1 and w_2 as

$$w_1 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 1 \end{pmatrix}.$$

The blank entries are understood to be filled by zeros. For any place v , we can view w_i as an element in $\mathrm{Sp}_4(\mathfrak{O}_v)$. If k_v is of odd residual characteristic, the aforementioned splitting of $\mathrm{Sp}_4(\mathfrak{O}_v)$ gives $(w_i, 1)$ as the splitting of the representative w_i (cf. [Szp, sect. 2.3]). In this case we can write w_i for $(w_i, 1)$ without introducing any ambiguity.

For convenience and later reference, we list in Table 1 the reduced decomposition of Weyl group elements and their actions on positive roots and the torus. Throughout the paper, we will use the abbreviation w_{ijij} for $w_i w_j w_j w_i$, for example, i.e. w_{12} for $w_1 w_2$ and w_{121} for $w_1 w_2 w_1$.

Naturally, the local parabolic subgroups $\overline{P}_1(k_v), \overline{P}_2(k_v), \overline{B}(k_v)$ are defined to be the preimage of $P_1(k_v), P_2(k_v)$ and $B(k_v)$ under the projection $\mathrm{Mp}_4(k_v) \rightarrow \mathrm{Sp}_4(k_v)$. If $P(k_v) = M(k_v)N(k_v)$ is a parabolic subgroup of $\mathrm{Sp}(k_v)$, then $\overline{P}(k_v) = \overline{M}(k_v)N(k_v)$, since the unipotent radical splits trivially. We call $\overline{M}(k_v)$ the Levi of the parabolic subgroup $\overline{P}(k_v)$ in $\mathrm{Mp}_4(k_v)$. For example,

$$\overline{M}_1(k_v) \simeq \overline{\mathrm{GL}}_2(k_v) \text{ and } \overline{M}_2(k_v) \simeq (\overline{\mathrm{GL}}_1(k_v) \times \mathrm{Mp}_2(k_v))/\Delta\mu_2,$$

where Δ stands for the diagonal imbedding of $\mu_2 \rightarrow \mu_2 \times \mu_2$.

2.2. Local representations and parabolic induction. In order to define parabolic induction, one needs to understand the genuine representations of $\overline{T}(k_v)$,

$\overline{M}_1(k_v)$ and $\overline{M}_2(k_v)$. In fact, some observation shows that it suffices to understand those of $\overline{\mathrm{GL}}_n(k_v)$, $n = 1, 2$, and $\mathrm{Mp}_2(k_v)$.

For an arbitrary place v , let $\psi_v : k_v \rightarrow \mathbf{C}^\times$ be a nontrivial additive character. There is a natural genuine character $\overline{\chi}_{\psi_v}$ of $\overline{\mathrm{GL}}_n(k_v)$ derived from Weil's factor (cf. [Rao, p. 365]), such that the map

$$\tau \mapsto \overline{\tau}_v = \tau \otimes \overline{\chi}_{\psi_v}$$

gives a bijection between $\mathrm{Irr}(\mathrm{GL}_n(k_v))$ and the set $\mathrm{Irr}(\overline{\mathrm{GL}}_n(k_v))$ of genuine irreducible representations of $\overline{\mathrm{GL}}_n(k_v)$. Note that such bijection depends on the choice of the additive character.

For $\mathrm{Irr}(\mathrm{Mp}_2(k_v))$, we will refrain from giving the details but refer to [GaS] for descriptions.

Based on this, we can now introduce parabolic inductions from $\overline{P}_1(k_v)$, $\overline{P}_2(k_v)$ or $\overline{B}(k_v)$. For example, we consider the case of $\overline{P}_2(k_v)$. Let $\chi_v \in \mathrm{Irr}(\mathrm{GL}_1(k_v))$ and $\sigma_v \in \mathrm{Irr}(\mathrm{Mp}_2(k_v))$; then $\overline{\chi}_v \boxtimes \sigma_v$ is a representation of $\overline{\mathrm{GL}}_1(k_v) \times \mathrm{Mp}_2(k_v)$ which descends to be in $\mathrm{Irr}(\overline{M}_2(k_v))$. Therefore, one may define the normalized parabolic induction

$$I_{\overline{P}_2}(\overline{\chi}_v \boxtimes \sigma_v) = \mathrm{Ind}_{\overline{P}_2}^{\mathrm{Mp}_4} \overline{\chi}_v \boxtimes \sigma_v.$$

2.3. Rank one intertwining operator and coefficients. We collect some facts and computational results on local intertwining operators and the coefficients which appear in the unramified case. Let ψ_v be the additive character given above, and let χ_v, μ_v be two characters of $\mathrm{GL}_1(k_v)$. We assume that these characters are all unramified, and without loss of generality we also assume that the residual characteristics of k_v is odd. Consider the induced representation

$$I_{\overline{B}}(\overline{\chi}_v \boxtimes \overline{\mu}_v) = \mathrm{Ind}_{\overline{B}}^{\mathrm{Mp}_4} \overline{\chi}_v \boxtimes \overline{\mu}_v.$$

For the two Weyl group elements w_i associated to $\alpha_i, i = 1, 2$, we will compute the intertwining map

$$A(\overline{\chi}_v \boxtimes \overline{\mu}_v, w_i) : I_{\overline{B}}(\overline{\chi}_v \boxtimes \overline{\mu}_v) \longrightarrow I_{\overline{B}}(w_i(\overline{\chi}_v \boxtimes \overline{\mu}_v)).$$

Recall that by abuse of notation, we use $w_i \in \mathrm{Sp}_4(\mathfrak{O}_v)$ to represent the element $(w_i, 1)$ in $\mathrm{Mp}_4(k_v)$, though in general the splitting over $\mathrm{Sp}_4(\mathfrak{O}_v)$ is not trivial. The intertwining operator here essentially labels a rank one computation, and it is necessary to differentiate the case of short and long roots, i.e. w_1 or w_2 . For the short root α_1 , we have

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Mp}_4(k_v) & \longrightarrow & \mathrm{Sp}_4(k_v) \longrightarrow 1 \\ & & \uparrow = & & \uparrow & \nwarrow g \mapsto (g, 1) & \uparrow \phi_{\alpha_1} \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{SL}_2(k_v) \times \mu_2 & \longrightarrow & \mathrm{SL}(k_v) \longrightarrow 1, \end{array}$$

where ϕ_{α_1} is the embedding given by α_1 . That is to say, the top covering map splits over $\phi_{\alpha_1}(\mathrm{SL}_2(k_v))$ by the trivial section (cf. [Rao, sect. 5]).

However, for the long root α_2 , the $\mathrm{SL}_2(k_v)$ embedded by α_2 inherits a nontrivial degree two covering $\mathrm{Mp}_2(k_v)$. That is,

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Mp}_4(k_v) & \longrightarrow & \mathrm{Sp}_4(k_v) \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \phi_{\alpha_2} \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{SL}_2(k_v) \times \mu_2 & \longrightarrow & \mathrm{SL}(k_v) \longrightarrow 1. \end{array}$$

The computation of the local coefficients of intertwining operators for unramified representations reduces to an SL_2 and an Mp_2 case. Though the first case is familiar, we recall both to illustrate the difference. In this paper, we will use B_o to denote the Borel subgroup for either SL_2 or GL_2 , in which case the nontrivial Weyl element is written as w . No confusion will arise from the context.

First, let $f_v \in I_{B_o}^{\mathrm{SL}_2}(|^s \mu_v)$ be a normalized $\mathrm{Sp}_2(\mathfrak{O}_v)$ -fixed vector, and let $f'_v \in I_{B_o}^{\mathrm{SL}_2}(w(|^s \mu_v))$ be the corresponding vector. Then in the SL_2 case the intertwining operator $A(s, \mu_v)$ gives

$$A(s, \mu_v)f_v = \frac{L(s, \mu_v)}{L(s+1, \mu_v)}f'_v.$$

However, suppose $f_v \in I_{B_o}^{\mathrm{Mp}_2}(|^s \bar{\mu}_v)$ is a normalized $\mathrm{Sp}_2(\mathfrak{O}_v)$ -fixed vector, and let $f'_v \in I_{B_o}^{\mathrm{Mp}_2}(w(|^s \bar{\mu}_v))$ be the corresponding vector. Then the intertwining operator $A(s, \bar{\mu}_v)$, for the Mp_2 case, gives

$$A(s, \bar{\mu}_v)f_v = \frac{L(2s, \mu_v^2)}{L(2s+1, \mu_v^2)}f'_v.$$

We refer to [Szp, sect. 8] for the details of the computation.

2.4. Global metaplectic group. Keeping the notation as above, we give some basics on our underlying object of interest: the global metaplectic group $\mathrm{Mp}_4(\mathbf{A}_k)$. Fix the number field k , let $\mathrm{Mp}_4(k_v)$ be the local double cover of $\mathrm{Sp}_4(k_v)$ given before, and consider the restricted direct product $\prod'_v \mathrm{Mp}_4(k_v)$ with respect to the family of the split open compact subgroup $\mathrm{Sp}_4(\mathfrak{O}_v)$. This is a central extension of $\mathrm{Sp}_4(\mathbf{A}_k)$ by the restricted product $\prod'_v \mu_{2,v}$. Let

$$Z_o = \{(\epsilon_v) \in \prod'_v \mu_{2,v} : \prod_v \epsilon_v = 1\}$$

and set

$$\mathrm{Mp}_4(\mathbf{A}_k) = \left(\prod'_v \mathrm{Mp}_4(k_v) \right) / Z_o.$$

Thus we obtain the global metaplectic group which splits over $\mathrm{Sp}_4(k)$ from the exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Mp}_4(\mathbf{A}_k) \longrightarrow \mathrm{Sp}_4(\mathbf{A}_k) \longrightarrow 1.$$

Following this, we may define the space $\mathscr{A}(\mathrm{Mp}_4(\mathbf{A}_k))$ of genuine automorphic forms on $\mathrm{Sp}_4(k) \backslash \mathrm{Mp}_4(\mathbf{A}_k)$. Note that an irreducible genuine representation of $\mathrm{Mp}_4(\mathbf{A}_k)$ is of the form

$$\eta = \bigotimes \eta_v,$$

where η_v is an irreducible genuine representation of $\mathrm{Mp}_4(k_v)$. In particular, every irreducible automorphic representation is of this form. As alluded to in the introduction, this paper will focus on the square integrable genuine forms

$$L_{\mathrm{gen}}^2(\mathrm{Sp}_4(k) \backslash \mathrm{Mp}_4(\mathbf{A}_k)),$$

and in fact only on its residual spectrum L_{res}^2 obtained by taking residues of Eisenstein series. We refer to [MoW] and [GGP] for discussions of metaplectic forms on $\mathrm{Mp}_4(\mathbf{A}_k)$ and some conjectures on L_{disc}^2 in the spirit of Arthur conjectures for classical groups ([Art]) respectively.

3. DECOMPOSITION FOR THE SIEGEL PARABOLIC SUBGROUP

3.1. Eisenstein series on metaplectic groups. For this section, we have the decomposition $P_1 = M_1 N_1$ for the Siegel parabolic P_1 inside Sp_4 , with the Levi factor $M_1 \simeq \mathrm{GL}_2$. Consider the group of characters $X(M_1)$ of M_1 . Let $\mathfrak{a}_{P_1}^* = X(M_1) \otimes \mathbf{R} = \mathbf{R}\beta_2$ and $\rho_{P_1} = \frac{3}{2}\beta_2$ be the half sum of roots generating N_1 . In fact, we can view β_2 as the determinant map on M_1 and identify $s \in \mathbf{C}$ with $s\beta_2$.

Let $\bar{\tau} = \bigotimes_v \bar{\tau}_v$ be a cuspidal representation of $\overline{M}_1(\mathbf{A}_k) = \overline{\mathrm{GL}}_2(\mathbf{A}_k)$, where $\tau = \bigotimes_v \tau_v$ is a cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_k)$. The local component $\bar{\tau}_v$ is described as before. Let $\phi \in \mathrm{Ind}_{\overline{P}_1}^{\mathrm{Mp}_4} \bar{\tau}$ be an element in the representation space, identified with a certain $\mathbf{C}$ -valued function on $N_1(\mathbf{A}_k)M_1(k) \backslash \mathrm{Mp}_4(\mathbf{A}_k)$ (cf. [MoW, II.1]).

Define the Eisenstein series on $\mathrm{Mp}_4(\mathbf{A}_k)$ by

$$E(s, \bar{g}, \phi, \overline{P}_1) = \sum_{\gamma \in P_1(k) \backslash \mathrm{Sp}_4(k)} \phi(\gamma \bar{g}) \exp \langle s + \rho_{P_1}, H_{\overline{P}_1}(\gamma \bar{g}) \rangle,$$

where $H_{\overline{P}_1}(\bar{g}) = H_{P_1}(g)$ is the Harish-Chandra homomorphism. For convenience, we may let $\Phi_s(\bar{g}) = \phi(\bar{g}) \exp \langle s + \rho_{P_1}, H_{\overline{P}_1}(\bar{g}) \rangle$ and sometimes write $E(s, \bar{g}, \Phi_s, \overline{P}_1)$ for $E(s, \bar{g}, \phi, \overline{P}_1)$. Note that for fixed s the representation space generated by Φ_s is equivalent to $I_{\overline{P}_1}(s, \bar{\tau}) = I_{\overline{P}_1} \bar{\tau} \otimes \exp \langle s\beta_2, H_{\overline{P}_1}(\cdot) \rangle$.

For any Weyl group element w and any $f \in I_{\overline{P}_1}(s, \bar{\tau})$, the representation space of $\mathrm{Mp}_4(\mathbf{A}_k)$ generated by Φ_s , we consider the intertwining operator $M(s, \bar{\tau}, w)f(\bar{g})$. It is given by the integral

$$M(s, \bar{\tau}, w)f(\bar{g}) = \int_{N_1^w(\mathbf{A}_k)} f(w^{-1}n\bar{g})dn,$$

which converges for $\mathrm{Res} \gg 0$. In general we define $N_1^w = U \cap wN_1'w^{-1}$, and N_1' is the unipotent group opposite to N_1 .

We refer to [MoW, chap. IV] for general properties of the Eisenstein series and their constant terms. In particular, $E(s, \bar{g}, \phi, \overline{P}_1)$ converges for $\mathrm{Res} \gg 0$ and extends to a meromorphic function of $s \in \mathbf{C}$. The poles of the Eisenstein series coincide with those of its constant term along N_1 , which is given by

$$E_{P_1}(s, \bar{g}, \phi, \overline{P}_1) = \sum_{w \in \Omega_1} M(s, \bar{\tau}, w)\Phi_s(\bar{g}),$$

where $\Omega_1 = \{1, w_{212}\}$. The element $w_i \in \mathrm{Sp}_4(k)$ is considered to be in $\mathrm{Mp}_4(\mathbf{A}_k)$, since the group $\mathrm{Sp}_4(k)$ splits into the latter. We also write $n \in N_1(\mathbf{A}_k)$ for the splitting $(n, 1) \in \mathrm{Mp}_4(\mathbf{A}_k)$.

Here in this section we will fix $w = w_{212}$, the nontrivial element in Ω_1 . In this case, we have $N_1^w = N_1$. It now follows from the definition of $I_{\overline{P}_1}(s, \overline{\tau})$ as a restricted tensor product that

$$M(s, \overline{\tau}, w) = \bigotimes A(s, \overline{\tau}_v, w), \text{ with } A(s, \overline{\tau}_v, w)f_v(\overline{g}_v) = \int_{N_1^w(k_v)} f_v(w^{-1}n_v\overline{g}_v)dn,$$

where $f = \bigotimes f_v$ and f_v is the unique $\mathrm{Sp}_4(\mathfrak{O}_v)$ -fixed function with normalization $f_v(e_v) = 1$ for almost all v .

3.2. Computation of the local intertwining operator. By virtue of the tensor decomposition for $M(s, \overline{\tau}, w)$, we need to analyze both the local intertwining operator $A(s, \overline{\tau}_v, w)$ for almost all unramified places v and the other finite number of exceptions $v \in S$.

To facilitate subsequent computations for the non-Siegel and Borel cases, it is helpful to obtain the explicit formulas, at least in the unramified case. Therefore we assume $\overline{\tau}_v$ is unramified, i.e. $\overline{\tau}_v \hookrightarrow I_{\overline{B}_o}^{\mathrm{GL}_2} \overline{\chi}_v \boxtimes \overline{\mu}_v$ for unramified characters χ_v, μ_v of $\mathrm{GL}_1(k_v)$. By inducing in stages, $I_{\overline{P}_1}(s, \overline{\tau}_v) \subseteq I_{\overline{B}}(s\beta_2, \overline{\chi}_v \boxtimes \overline{\mu}_v)$. Then $A(s, \overline{\tau}_v, w) = A(s\beta_2, \overline{\chi}_v \boxtimes \overline{\mu}_v, w)$ on their common domain. Consider the unramified vector $f_v \in I_{\overline{B}}(s\beta_2, \overline{\chi}_v \boxtimes \overline{\mu}_v)$ with

$$f_v(\overline{t}nq) = (\overline{\chi}_v \boxtimes \overline{\mu}_v)(\overline{t})\exp\langle s\beta_2 + \rho_B, H_{\overline{B}}(\overline{t}) \rangle,$$

for $t \in \overline{T}(k_v), n \in N(k_v), q \in \mathrm{GL}_2(\mathfrak{O}_v)$.

We need to calculate

$$A(s, \overline{\chi}_v \boxtimes \overline{\mu}_v, w)f_v(e) = \int_{N_1^w(k_v)} f_v(w^{-1}n)dn.$$

The computation is reduced to the SL_2 and Mp_2 cases by using the cycle relation of intertwining operators. Let $\Lambda \in \mathfrak{a}_{\mathbf{C}}^* := \mathbf{C}\beta_1 \oplus \mathbf{C}\beta_2, \overline{\chi}_v \boxtimes \overline{\mu}_v$ a character of $\overline{T}(k)$, and let $A(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v, w)$ be the intertwining operator from $I_{\overline{B}}(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v)$ to $I_{\overline{B}}(w\Lambda, w(\overline{\chi}_v \boxtimes \overline{\mu}_v))$.

By using the result for SL_2 and Mp_2 intertwining operators of the last section, we have the following.

Proposition 3.1 (Gindikin-Karpelevich formula). *Let $\overline{\chi}_v \boxtimes \overline{\mu}_v$ be an unramified character of $\overline{T} \subseteq \mathrm{Mp}_4(k_v)$, and let $\Lambda \in \mathfrak{a}_{\mathbf{C}}^*$. Let f_v, f'_v be the unramified vectors in $I_{\overline{B}}(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v)$ and $I_{\overline{B}}(w\Lambda, w(\overline{\chi}_v \boxtimes \overline{\mu}_v))$ respectively with normalization $f_v(e) = f'_v(e) = 1$. Then*

$$A(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v, w)f_v = \prod_{\substack{\beta > 0, w\beta < 0 \\ \beta \text{ short}}} \frac{L(\langle \Lambda, \beta^\vee \rangle, (\chi_v \boxtimes \mu_v)\beta^\vee)}{L(1 + \langle \Lambda, \beta^\vee \rangle, (\chi_v \boxtimes \mu_v)\beta^\vee)} \prod_{\substack{\beta > 0, w\beta < 0 \\ \beta \text{ long}}} \frac{L(2\langle \Lambda, \beta^\vee \rangle, ((\chi_v \boxtimes \mu_v)\beta^\vee)^2)}{L(1 + 2\langle \Lambda, \beta^\vee \rangle, ((\chi_v \boxtimes \mu_v)\beta^\vee)^2)} f'_v.$$

Back to the Siegel parabolic case, consider $w = w_{212}$ the nontrivial element in Ω_1 . We have $\Lambda = s\beta_2 = s\alpha_3$. Substituting Λ into the above formula and combining all local intertwining operators, the nontrivial term in the constant term of Eisenstein

series along $\overline{P}_1$ is given by

$$M(s, \overline{\tau}, w_{212})f = \frac{L^S(2s, \tau, \text{Sym}^2)}{L^S(2s+1, \tau, \text{Sym}^2)} \bigotimes_{v \notin S} f'_v \otimes \bigotimes_{v \in S} A(s, \overline{\tau}_v, w_{212})f_v.$$

In order to determine the pole of $M(s, \overline{\tau}, w_{212})$, we could analyze the partial L -functions L^S and the local intertwining maps for the exceptional places. Instead, we first consider the complete $L(2s, \tau, \text{Sym}^2)$ with its local factors given by Shahidi. It is well known (cf. [CKM, pp. 187–188]) that the symmetric square L -function has a pole of order 1 if and only if

$$\tau \cong \tau^\vee \text{ with a nontrivial quadratic central character } \omega_\tau,$$

where the $\tau^\vee$ stands for the contragradient of τ .

Second, we carry out the analysis of $A(s, \overline{\tau}_v, w_{212})$, $v \in S$. We make the normalization

$$\begin{aligned} r(s, \overline{\tau}_v, w_{212}) &= \frac{L(2s, \tau_v, \text{Sym}^2)}{L(2s+1, \tau_v, \text{Sym}^2)\varepsilon(2s, \tau_v, \text{Sym}^2)}, \\ A(s, \overline{\tau}_v, w_{212}) &= r(s, \overline{\tau}_v, w_{212})R(s, \overline{\tau}_v, w_{212}). \end{aligned}$$

Lemma 3.2. *For each $v \in S$, $R(s, \overline{\tau}_v, w_{212})$ can be continued to a holomorphic function for $\text{Re}(s) \geq 0$.*

Proof. For $\text{Re}(s) \geq 0$, it suffices to show that the poles of $r(s, \overline{\tau}_v, w_{212})$ and $A(s, \overline{\tau}_v, w_{212})$ cancel each other.

If τ_v is tempered, it follows from [Sha, section 4], that the factor $L(2s, \tau_v, \text{Sym}^2)$ and therefore $r(s, \overline{\tau}_v, w_{212})$ both have no pole or zero for $\text{Re}(s) \geq 0$. Also in this case, the holomorphicity of $A(s, \overline{\tau}_v, w_{212})$ follows from [BoW, section IV. 4], and [Sil, sect. 5.4], with slight modification of the argument. Therefore we assume that τ_v is a complementary series, and without loss of generality of the form $\tau_v = \text{Ind}_{B_o}^{\text{GL}_2} \nu | \cdot|^a \times \nu | \cdot|^{-a}$, where ν is a unitary character and $a \in \mathbf{R}^+$. In fact, we know $a \in (0, 1/9)$ by [KiS, p. 178].

By the Langlands-Shahidi method and the multiplicativity of the γ -factor which is used to define $L(2s, \tau_v, \text{Sym}^2)$ (cf. [Sha, sect. 4]), we see that

$$L(2s, \tau_v, \text{Sym}^2) = L(2s - 2a, \nu^2)L(2s, \nu^2)L(2s + 2a, \nu^2).$$

Note that the notation we use for the local factor could be unifying, both for archimedean and nonarchimedean places. However, for elaboration purposes, we give details separately. Clearly for $\text{Re}(s) \geq 0$, $L(2s+1, \tau_v, \text{Sym}^2)$ has no pole or zero; therefore, it is reduced to show that the poles of $L(2s, \tau_v, \text{Sym}^2)$ and $A(s, \overline{\tau}_v, w_{212})$ cancel each other.

For a nonarchimedean place v , it follows from the above formula that for $\text{Re}(s) \geq 0$, the pole of $L(2s, \tau_v, \text{Sym}^2)$ occurs only at $s = a$ for $\nu^2 = \mathbb{1}$, or $s = 0$ for $\nu^2 \neq \mathbb{1}$. On the other hand, consider $A(s, \overline{\tau}_v, w_{212})$:

$$I_{\overline{B}}(\overline{\nu} | \cdot|^{a+s} \boxtimes \overline{\nu} | \cdot|^{-a+s}) \longrightarrow I_{\overline{B}}(\overline{\nu}^{-1} | \cdot|^{a-s} \boxtimes \overline{\nu}^{-1} | \cdot|^{-a-s}).$$

It factorizes as

$$\begin{aligned} I_{\overline{B}}(\overline{\nu} | \cdot|^{a+s} \boxtimes \overline{\nu} | \cdot|^{-a+s}) &\xrightarrow{A(w_2)} I_{\overline{B}}(\overline{\nu} | \cdot|^{a+s} \boxtimes \overline{\nu}^{-1} | \cdot|^{a+s}) \xrightarrow{A(w_1)} I_{\overline{B}}(\overline{\nu}^{-1} | \cdot|^{a-s} \boxtimes \overline{\nu} | \cdot|^{a+s}) \\ &\xrightarrow{A(w_2)} I_{\overline{B}}(\overline{\nu}^{-1} | \cdot|^{a-s} \boxtimes \overline{\nu}^{-1} | \cdot|^{-a-s}). \end{aligned}$$

The first and third operators are both Mp_2 rank one operators, while the middle is a GL_2 intertwining operator. The first operator $A(w_2)$ has a simple pole at $s = a$ when $\nu^2 = 1$, the second operator at $s = 0$ when $\nu^2 = 1$, and the third operator has a simple pole at $s = -a$ when $\nu^2 = 1$. Therefore for $\mathrm{Res} \geq 0$, the only pole for $A(s, \bar{\tau}_v, w)$ occurs at $s = a$ when $\nu^2 = 1$, or $s = 0$ when $\nu^2 = 1$.

For a real archimedean v , the character ν is either trivial or the sign character. The local L factor in terms of a Gamma function reads

$$L_\infty(2s, \tau_v, \mathrm{Sym}^2) = \pi^{-3s} \Gamma(s - a) \Gamma(s) \Gamma(s + a).$$

In either case we see that the only possible pole of the L -factor occurs at $s = a$ for $\nu^2 = 1$ or $s = 0$ for $\nu^2 = 1$, which agrees with that of the operator $A(s, \bar{\tau}_v, w_{212})$. The complex case follows from formulas similar to the real case. The proof is completed. $\square$

To determine the image of the normalized local intertwining operator we have

Lemma 3.3. *The images of $R(1/2, \bar{\tau}_v, w_{212})$ and $A(1/2, \bar{\tau}_v, w_{212})$ are both irreducible and nonzero for all places v of k .*

Proof. In all cases considered below, the normalizing factor $r(s, \bar{\tau}_v, w_{212})$ has no pole or zero at $s = 1/2$; therefore, it suffices to prove the lemma for the operator $A(1/2, \bar{\tau}_v, w_{212})$.

When τ_v is tempered, it follows from the Langlands classification theorem (cf. [BaJ, Thm. 4.1]). Now suppose that $\tau_v = \mathrm{Ind}_{B_o}^{\mathrm{GL}_2} \nu | \cdot |^a \boxtimes \nu | \cdot |^{-a}$ is a complementary series representation. We can assume $a \in (0, 1/9)$ by [KiS]. By inducing by steps,

$$\mathrm{Ind}_{\bar{P}_1}^{\mathrm{Mp}_4}(s\beta_2, \bar{\tau}_v) = \mathrm{Ind}_{\bar{B}}^{\mathrm{Mp}_4}(s\beta_2 + a\alpha_1, \bar{\nu} \boxtimes \bar{\nu}) = I_{\bar{B}}(\bar{\nu} | \cdot |^{s+a} \boxtimes \bar{\nu} | \cdot |^{s-a})$$

and $A(s\beta_2, \bar{\tau}_v, w_{212}) = A(s\beta_2 + a\alpha_1, \bar{\nu} \boxtimes \bar{\nu}, w_{212})$. Consider the chain of intertwining maps:

$$\begin{aligned} I_{\bar{B}}(\bar{\nu} | \cdot |^{s+a} \boxtimes \bar{\nu} | \cdot |^{s-a}) &\xrightarrow{A(s\beta_2 + a\alpha_1, w_{212})} I_{\bar{B}}(\bar{\nu}^{-1} | \cdot |^{a-s} \boxtimes \bar{\nu}^{-1} | \cdot |^{-s-a}) \\ &\xrightarrow{A(w_{212}(s\beta_2 + a\alpha_1), w_1)} I_{\bar{B}}(\bar{\nu}^{-1} | \cdot |^{-s-a} \boxtimes \bar{\nu}^{-1} | \cdot |^{a-s}). \end{aligned}$$

The last map $A(w_{212}(s\beta_2 + a\alpha_1), w_1)$ is the functorial lift of the rank one operator

$$I_{\bar{B}_o}^{\mathrm{GL}_2}(\bar{\nu}^{-1} | \cdot |^{a-s} \boxtimes \bar{\nu}^{-1} | \cdot |^{-s-a}) \longrightarrow I_{\bar{B}_o}^{\mathrm{GL}_2}(\bar{\nu}^{-1} | \cdot |^{-s-a} \boxtimes \bar{\nu}^{-1} | \cdot |^{a-s}),$$

which is an isomorphism. Therefore $A(w_{212}(s\beta_2 + a\alpha_1), w_1)$ is an isomorphism. On the other hand, the chain composition above yields $A(s\beta_2 + a\alpha_1, \bar{\nu} \boxtimes \bar{\nu}, w_{212}w_1)$, whose image by Langlands classification is irreducible when $s = 1/2$. Therefore, the image of $A(1/2, \bar{\tau}_v, w_{212})$ is isomorphic to the Langlands quotient and thus irreducible. In particular, it is nonzero. $\square$

This shows that the image of $R(1/2, \bar{\tau}_v, w_{212})$ is the Langlands quotient $J_{\bar{P}_1}(1/2, \bar{\tau}_v)$ with initial data described as in the lemma. Therefore, we have

$$\mathrm{Res}_{s=1/2} E_{P_1}(s, \bar{g}, \Phi_s, \bar{P}_1) = \bigotimes_v J_{\bar{P}_1}(1/2, \bar{\tau}_v).$$

Taking the residue of Eisenstein series commutes with taking the constant term, as in

$$\begin{array}{ccc}
 I_{\overline{P}_1}(s, \overline{\tau}) & \xrightarrow{\Phi_s \mapsto \text{Res}_{s=s_o} E(\Phi_s)} & \mathcal{A}^2(\text{Sp}_4(k) \backslash \text{Mp}_4(\mathbf{A}_k)) \\
 & \searrow \Phi_s \mapsto \text{Res}_{s=s_o} E_{\overline{P}_1}(\Phi_s) & \downarrow \text{take const. term} \\
 & & \mathcal{A}(N_1(\mathbf{A}_k)T(k) \backslash \text{Mp}_4(\mathbf{A}_k)).
 \end{array}$$

We know that the right vertical map is injective when restricted to the image of the top map $\text{Res}_{s=s_o} E(-)$; see [MoW, pg. 45] and [CKM, pg. 130]. The Eisenstein series $E(s, \overline{g}, \Phi_s, \overline{P}_1)$ has a pole at $s = 1/2$, and we consider the space spanned by its residues at $s = 1/2$. Since $w(\beta_2/2) = -\alpha_3/2$, it follows from Langlands' criterion (cf. [MoW, I.4]) that these residues are square integrable. By above injectivity, the residue $\text{Res}_{s=1/2} E(s, \overline{g}, \Phi_s, \overline{P}_1)$ can be identified with $\bigotimes_v J_{\overline{P}_1}(1/2, \overline{\tau}_v)$, which is irreducible and nonzero.

We will make an implicit identification of this kind for the non-Siegel and Borel cases later without mentioning the details again.

Hence, for the Siegel parabolic case we have shown

Theorem 3.4. *Let $\mathcal{A}_{\text{cusp}}(GL_2(\mathbf{A}_k))$ denote the cuspidal representations on $GL_2(\mathbf{A}_k)$ and let $\mathcal{S} \subseteq \mathcal{A}_{\text{cusp}}(GL_2(\mathbf{A}_k))$ be defined as*

$$\mathcal{S} = \{ \tau : \tau \cong \tau^\vee \text{ with quadratic central character } \omega_\tau \neq \mathbb{1} \}.$$

From above, we get $J_{\overline{P}_1}(1/2, \overline{\tau}_v)$, the irreducible representation as the image of $R(1/2, \overline{\tau}_v, w)$. Then the representation $J_{\overline{P}_1}(1/2, \overline{\tau}) = \bigotimes_v J_{\overline{P}_1}(1/2, \overline{\tau}_v)$ occurs in the residual spectrum of $\text{Mp}_4(\mathbf{A}_k)$. In fact,

$$L_d^2(\overline{P}_1) = \bigoplus_{\tau \in \mathcal{S}} J_{\overline{P}_1}(1/2, \overline{\tau}),$$

where τ runs over all representations in $\mathcal{S}$.

4. DECOMPOSITION FOR THE NON-SIEGEL MAXIMAL PARABOLIC SUBGROUP

In this section, we consider the non-Siegel maximal parabolic subgroup with decomposition $P_2 = M_2 N_2$ inside Sp_4 , where $M_2 \simeq \text{GL}_1 \times \text{SL}_2$. Recall $\beta_1 = \alpha_4/2$. Let $X(M_2)$ be the characters of M_2 , $\mathfrak{a}_{P_2}^* = X(M_2) \otimes \mathbf{R} = \mathbf{R}\beta_1$ and $\rho_{P_2} = 2\beta_1$ be the half sum of roots generating N . We identify $s \in \mathbf{C}$ with $s\beta_1$.

Let χ be a unitary Hecke character of $\text{GL}_1(\mathbf{A}_k)$ and σ be a cuspidal representation of $\text{Mp}_2(\mathbf{A}_k)$. Then it gives a genuine cuspidal representation $\overline{\chi} \boxtimes \sigma$ of $\overline{M}_2(\mathbf{A}_k)$, and conversely every genuine representation of $\overline{M}_2(\mathbf{A}_k)$ arises in this way.

Given a function $\phi \in \text{Ind}_{\overline{P}_2}^{\text{Mp}_4} \overline{\chi} \boxtimes \sigma$, we define the associated Eisenstein series on $\text{Mp}_4(\mathbf{A}_k)$ by

$$E(s, \overline{g}, \phi, \overline{P}_2) = \sum_{\gamma \in P_2(k) \backslash \text{Sp}_4(k)} \phi(\gamma \overline{g}) \exp \langle s + \rho_{P_2}, H_{\overline{P}_2}(\gamma \overline{g}) \rangle,$$

where $H_{\overline{P}_2}(\overline{g}) = H_{P_2}(g)$ as before is the Harish-Chandra homomorphism on $\overline{P}_2$. We write $\Phi_s = \phi(\overline{g}) \exp \langle s + \rho_{P_2}, H_{\overline{P}_2}(\overline{g}) \rangle$. The poles of the Eisenstein series

$E(s, \bar{g}, \phi, \bar{P}_2)$ coincide with those of its constant terms along N_2 , which is given by

$$E_{P_2}(s, \bar{g}, \phi, \bar{P}_2) = \sum_{w \in \Omega_2} M(s, \bar{\chi} \boxtimes \sigma, w) \Phi_s(\bar{g}),$$

where $\Omega_2 = \{1, w_{121}\}$. For any Weyl element w , any $f \in I_{\bar{P}_2}(s, \bar{\chi} \boxtimes \sigma)$ and $\mathrm{Re}(s) \gg 0$,

$$M(s, \bar{\chi} \boxtimes \sigma, w) f(\bar{g}) = \int_{N_2^w(\mathbf{A}_k)} f(w^{-1} n \bar{g}) dn$$

converges absolutely. Similar to the preceding section, we have $N_2^w = U \cap w N_2' w^{-1}$, where N_2' is the unipotent subgroup of U opposite to N_2 . We also note that here $I_{\bar{P}_2}(s, \bar{\chi} \boxtimes \sigma)$ is the representation space of $\mathrm{Mp}_4(\mathbf{A}_k)$ generated by Φ_s .

The tensor decomposition of $I_{\bar{P}_2}(s, \bar{\chi} \boxtimes \sigma)$ gives

$$M(s, \bar{\chi} \boxtimes \sigma, w) = \bigotimes A(s, \bar{\chi}_v \boxtimes \sigma_v, w), \text{ with} \\ A(s, \bar{\chi}_v \boxtimes \sigma_v, w) f_v(\bar{g}_v) = \int_{N_2^w(k_v)} f_v(w^{-1} n \bar{g}_v) dn,$$

where $f = \bigotimes f_v$ and f_v is the unique $\mathrm{Sp}_4(\mathfrak{O}_v)$ -fixed function with normalization $f_v(e_v) = 1$ for almost all v .

4.1. Cuspidal representation of $\mathrm{Mp}_2(\mathbf{A}_k)$. For details of this part, we refer to [GaS]. We recall a few facts on the cuspidal representations σ of $\mathrm{Mp}_2(\mathbf{A}_k)$. There are mainly two types to consider.

Let $\psi = \prod_v \psi_v$ be the fixed nontrivial additive character of $k \backslash \mathbf{A}_k$ given before for the Weil factor consideration. Then we have the decomposition

$$L_{\mathrm{cusp}}^2(\mathrm{Mp}_2(\mathbf{A}_k)) = (\mathrm{ETF}) \oplus \bigoplus_{\pi} L_{\pi}^2,$$

as π ranges over all cuspidal representations of $\mathrm{PGL}_2(\mathbf{A}_k)$.

We give a brief explanation of the space ETF , which can be decomposed as

$$\mathrm{ETF} = \bigoplus_{\eta} \mathrm{ETF}_{\eta} = \bigoplus_{\eta} \left(\bigoplus_S \omega_{\psi, \eta, S} \right).$$

The η is taken over all quadratic Hecke characters of $k^{\times} \backslash \mathbf{A}_k^{\times}$, and S over finite sets of positive even cardinality. Locally, for any place v of k , consider the even and odd Weil representations $\omega_{\psi_v, \eta_v}^+$ and $\omega_{\psi_v, \eta_v}^-$ (cf. [Kud, pg. 88]). Let $\bar{\chi}_{\psi_v}$ be the genuine character of $\bar{T}(k_v)$ attached to the Weil index. Then the even Weil representation $\omega_{\psi_v, \eta_v}^+$ lies in the short exact sequence

$$0 \rightarrow \omega_{\psi_v, \eta_v}^+ \rightarrow \mathrm{Ind}_{\bar{B}_o}^{\mathrm{Mp}_2} \bar{\chi}_{\psi_v}(\eta_v |^{-1/2}) \rightarrow st_{\psi_v, \eta_v} \rightarrow 0.$$

On the other hand, one knows that the odd Weil representations $\omega_{\psi_v, \eta_v}^-$ are supercuspidal. Now if S is any nonempty finite set of places of k with $|S|$ even, then the representation

$$\omega_{\psi, \eta, S} = \bigotimes_{v \in S} \omega_{\psi_v, \eta_v}^- \otimes \bigotimes_{v \notin S} \omega_{\psi_v, \eta_v}^+$$

is a cuspidal representation of $\mathrm{Mp}_2(\mathbf{A}_k)$. The representation space (ETF) is generated by all such $\omega_{\psi, \eta, S}$.

The complement of ETF is exhausted by L_π^2 , which is given by a theta correspondence with a cuspidal representation π on PGL_2 as input. The correspondence equates various arithmetic invariants, L -functions, γ factors, etc.

4.2. Analysis of local intertwining maps. As in the Siegel case, we analyze both the local intertwining operator $A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ for almost all unramified places v and the other finitely many exceptions.

We assume first that v is such that all ψ_v, χ_v and σ_v are unramified, and η_v as well in the ETF case. That is, we have $\sigma_v \hookrightarrow \mathrm{Ind}_{\bar{B}_o}^{\mathrm{Mp}_2} \bar{\mu}_v$ for an unramified character μ_v of $\mathrm{GL}_1(k_v)$. By applying the Gindikin-Karpelevich formula in Proposition 3.1, it follows that

$$M(s, \bar{\chi} \boxtimes \sigma, w_{121})f = \frac{L^S(s, \chi \times \sigma)}{L^S(s+1, \chi \times \sigma)} \frac{L^S(2s, \chi^2)}{L^S(2s+1, \chi^2)} \bigotimes_{v \notin S} f'_v \\ \otimes \bigotimes_{v \in S} A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})f_v.$$

The Rankin-Selberg product $L(s, \chi \times \sigma)$, or more precisely the local analog, is given as in [Szp, sect. 7]. In order to determine the pole of $M(s, \bar{\chi} \boxtimes \sigma, w_{121})$, we need to analyze the partial L -functions L^S and the local operators.

4.3. The ETF case. First we analyze $L^S(s, \chi \times \sigma)$. Let $\sigma = \omega_{\psi, \eta, S_\sigma} \in \mathrm{ETF}$ for some S_σ of even cardinality. Without loss of generality we assume $S_\sigma \subseteq S$. The partial L -function becomes $L^S(s, \chi \times \sigma) = L^S(s-1/2, \chi\eta)L^S(s+1/2, \chi\eta)$ since locally this holds by the description of even Weil representation. The operator $M(s, \bar{\chi} \boxtimes \sigma, w_{121})$ can thus be simplified as

$$M(s, \bar{\chi} \boxtimes \sigma, w_{121})f = \frac{L^S(s-1/2, \chi\eta)}{L^S(s+3/2, \chi\eta)} \frac{L^S(2s, \chi^2)}{L^S(2s+1, \chi^2)} \bigotimes_{v \notin S} f'_v \\ \otimes \bigotimes_{v \in S} A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})f_v.$$

As before, we normalize the intertwining operator by giving

$$(1) \quad r(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121}) = \frac{L(s, \chi_v \times \sigma_v)}{L(s+1, \chi_v \times \sigma_v)\varepsilon(s, \chi_v \times \sigma_v)} \frac{L(2s, \chi_v^2)}{L(2s+1, \chi_v^2)\varepsilon(2s, \chi_v^2)}, \\ A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121}) = r(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})R(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121}).$$

Regarding the holomorphicity of the normalized intertwining operator, we have

Lemma 4.1. *For $v \in S$, $R(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ can be continued to a holomorphic function for $\mathrm{Re}(s) \geq 0$.*

Proof. In all of the cases below, the denominator of the normalizing factor $r(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ has no zero or pole for $\mathrm{Re}(s) \geq 0$; therefore we can neglect it for our purpose of proving the lemma. It suffices to compare the poles of $L(s, \chi_v \times \sigma_v)L(2s, \chi_v^2)$ and $A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$.

If $v \in S_\sigma$, σ_v is supercuspidal and then we are in the tempered case. The local factor $L(s, \chi_v \times \sigma_v)$ has no poles, and the operator $A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ is holomorphic for $\mathrm{Res} \geq 0$.

Suppose $v \in S \setminus S_\sigma$; then σ_v is the even Weil representation $\omega_{\psi_v, \eta_v}^+$. In this case, $\mathrm{Ind}_{\overline{P}_2}(s, \overline{\chi}_v \boxtimes \sigma_v) \hookrightarrow \mathrm{Ind}_{\overline{B}} \overline{\chi}_v | \cdot^s \boxtimes \overline{\eta}_v | \cdot^{-1/2}$ and the intertwining operator $A(s\beta_1, \overline{\chi}_v \boxtimes \overline{\eta}_v | \cdot^{-1/2}, w_{121})$ for the latter factorizes as

$$I_{\overline{B}}(\overline{\chi}_v | \cdot^s \boxtimes \overline{\eta}_v | \cdot^{-1/2}) \xrightarrow{A(w_1)} I_{\overline{B}}(\overline{\eta}_v | \cdot^{-1/2} \boxtimes \overline{\chi}_v | \cdot^s) \\ \xrightarrow{A(w_2)} I_{\overline{B}}(\overline{\eta}_v | \cdot^{-1/2} \boxtimes \overline{\chi}_v^{-1} | \cdot^{-s}) \xrightarrow{A(w_1)} I_{\overline{B}}(\overline{\chi}_v^{-1} | \cdot^{-s} \boxtimes \overline{\eta}_v | \cdot^{-1/2}).$$

For $\mathrm{Res} \geq 0$, the first operator is holomorphic while the second has a pole at $s = 0$ when $\chi_v^2 = \mathbb{1}$. Also the last operator $A(w_1)$ gives a pole at $s = 1/2$ when $\chi_v = \eta_v$.

The factor $L(s, \chi_v \times \sigma_v)$ defined as in [Szp, sect. 7] is given by

$$L(s + 1/2, \chi_v \eta_v) L(s - 1/2, \chi_v \eta_v),$$

up to a sign. When v is nonarchimedean, we see already that the only possible pole of $L(s, \chi_v \times \sigma_v) L(2s, \chi_v^2)$ agrees with that of $A(s, \overline{\chi}_v \boxtimes \sigma_v, w_{121})$ for s nonnegative.

If v is real, then both χ_v, η_v are either trivial or the sign function. In this case the L -factor at a real infinite place is given by

$$L_\infty(s, \chi_v \times \sigma_v) = \begin{cases} \pi^{-s} \Gamma(\frac{s-1/2}{2}) \Gamma(\frac{s+1/2}{2}) & \text{if } \chi_v \eta_v = \mathbb{1}, \\ \pi^{-s-1} \Gamma(\frac{s+1/2}{2}) \Gamma(\frac{s+3/2}{2}) & \text{if } \chi_v \eta_v = \mathrm{sgn}. \end{cases}$$

The only possible pole is at $s = 1/2$ for $\chi_v = \eta_v = \mathrm{sgn}$. The complex case follows similarly. Therefore we see that in all cases, the poles of $L(s, \chi_v \times \sigma_v) L(2s, \chi_v^2)$ and $A(s, \overline{\chi}_v \boxtimes \sigma_v, w_{121})$ cancel each other, and hence the lemma is proved. $\square$

Now we are to consider the possible poles in different cases. There are three cases to consider:

- (1) $\chi = \eta, s = 3/2$;
- (2) $\chi = \eta, s = 1/2$;
- (3) $\chi \neq \eta, \chi^2 = \mathbb{1}, s = 1/2$.

FIRST, $s = 3/2, \chi = \eta$. The intertwining operator $M(s, \overline{\chi} \boxtimes \sigma, w_{121})$ has a simple pole at $s = 3/2$. We have the irreducibility of the image of the intertwining operator, as in Lemma 3.3

Lemma 4.2. *For $s = 3/2, \chi = \eta$, the image of $R(3/2, \overline{\chi}_v \boxtimes \sigma_v, w_{121})$ is irreducible and nonzero for all places v of k .*

Proof. The original operator $A(s, \overline{\chi}_v \boxtimes \sigma_v, w_{121})$ is holomorphic at $s = 3/2$, and thus its image is equal to that of $R(s, \overline{\chi}_v \boxtimes \sigma_v, w_{121})$. We will show the irreducibility of the image for the unnormalized operator.

If σ_v is an odd Weil representation, then it is tempered and the lemma follows from the Langlands classification theorem. Now assume $\sigma_v = \omega_{\psi_v, \eta_v}^+$ is the even Weil representation or equivalently the image of the rank one intertwining operator given by the nontrivial Weyl element:

$$I_{\overline{B}_o}^{\mathrm{Mp}_2}(\overline{\eta}_v | \cdot^{1/2}) \longrightarrow I_{\overline{B}_o}^{\mathrm{Mp}_2}(\overline{\eta}_v | \cdot^{-1/2}).$$

The image of $A(s\beta_1, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ in this case is the image of the following composition $A(s\beta_1 + 1/4\alpha_2, \bar{\chi}_v \boxtimes \bar{\eta}_v, w_{121}w_2)$:

$$\begin{array}{ccccc} I_{\bar{B}}(\bar{\chi}_v | \cdot^s \boxtimes \bar{\eta}_v | \cdot^{1/2}) & \xrightarrow{A(w_2)} & I_{\bar{B}}(\bar{\chi}_v | \cdot^s \boxtimes \bar{\eta}_v | \cdot^{-1/2}) & \xrightarrow{A(w_{121})} & I_{\bar{B}}(\bar{\chi}_v | \cdot^{-s} \boxtimes \bar{\eta}_v | \cdot^{-1/2}) \\ & \searrow & \uparrow & & \\ & & I_{\bar{P}_2}(\bar{\chi}_v | \cdot^s \boxtimes \sigma_v) & & \end{array}$$

For $s = 3/2$, $s\beta_1 + 1/4\alpha_2$ lies in the positive Weyl chamber, so by the Langlands classification, the image of $A(s\beta_1 + 1/4\alpha_2, \bar{\chi}_v \boxtimes \bar{\eta}_v, w_{121}w_2)$ is irreducible and the proof is completed. $\square$

We see that the image of $R(3/2, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ is the unique Langlands quotient with appropriate data. We denote it by $J_{\text{ETF}}(3/2, \bar{\chi}_v \boxtimes \sigma_v)$ and write

$$J_{\text{ETF}}(3/2, \bar{\chi} \boxtimes \sigma) = \bigotimes_v J_{\text{ETF}}(3/2, \bar{\chi}_v \boxtimes \sigma_v).$$

SECOND, $s = 1/2, \chi = \eta$. We combine the partial

$$L^S(s - 1/2, \chi\eta) \prod_{v \in S \setminus S_\sigma} L_v(s - 1/2, \chi\eta)$$

from the coefficient of normalized intertwining operator. We see that it has at least a simple zero at $s = 1/2$, since S_σ is nonempty of even cardinality. Therefore since $L^S(2s, \chi^2)$ has a simple pole at $s = 1/2$, we see that the intertwining operator cannot have any pole.

THIRD, $s = 1/2, \chi \neq \eta, \chi^2 = \mathbb{1}$. In this case, the operator $M(s, \bar{\chi} \boxtimes \sigma, w_{121})$ has a pole if and only if $L^{S_\sigma}(s - 1/2, \chi\eta)$ is nonvanishing, since we have for the global L -function $L(s - 1/2, \chi\eta) \neq 0$ for $s = 1/2$. Equivalently, we must have

$$\chi_v \neq \eta_v \text{ for all } v \in S_\sigma.$$

Meanwhile, under this condition

Lemma 4.3. *The image of $R(1/2, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ is irreducible and nonzero for all v .*

Proof. The argument is essentially the same as in the previous lemma. It suffices to consider σ_v being the even Weil representation case. We may assume $\chi_v = \eta_v$; otherwise the operator $A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ is holomorphic at $s = 1/2$ and the argument goes line by line as in the previous lemma.

If $\chi_v = \eta_v$, then the operator $A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ has a simple pole at $s = 1/2$ and the image of $R(1/2, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ is equal to the image of

$$\text{Res}_{s=1/2} A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$$

(the right diagonal map below). We have the commutative diagram

$$\begin{array}{ccccc} I_{\bar{B}}(\bar{\chi}_v | \cdot^s \boxtimes \bar{\eta}_v | \cdot^{1/2}) & \xrightarrow{A(w_2)} & I_{\bar{B}}(\bar{\chi}_v | \cdot^s \boxtimes \bar{\eta}_v | \cdot^{-1/2}) & \xrightarrow{\text{Res}_{s=1/2} A(w_{121})} & I_{\bar{B}}(\bar{\chi}_v | \cdot^{-s} \boxtimes \bar{\eta}_v | \cdot^{-1/2}) \\ & \searrow & \uparrow & \nearrow & \\ & & I_{\bar{P}_2}(\bar{\chi}_v | \cdot^s \boxtimes \sigma_v) & \xrightarrow{\text{Res}_{s=1/2} A(w_{121})} & \end{array}$$

It follows from the diagram that it suffices to consider the image of the top composition map

$$(\mathrm{Res}_{s=1/2} A(s\beta_1 - 1/4\alpha_2, \overline{\chi}_v \boxtimes \overline{\eta}_v, w_{121})) \cdot A(s\beta_1 + 1/4\alpha_2, \overline{\chi}_v \boxtimes \overline{\eta}_v, w_2),$$

which is equal to

$$(\mathrm{Res}_{s=1/2} A(-\alpha_4/4 - s\alpha_2/2, \overline{\eta}_v \boxtimes \overline{\chi}_v, w_1)) \cdot A(s\beta_1 + 1/4\alpha_2, \overline{\chi}_v \boxtimes \overline{\eta}_v, w_{212}).$$

When $s = 1/2$, $s\beta_1 + 1/4\alpha_2 = \alpha_3/2$. By induction in stages to the Siegel parabolic subgroup $\overline{P}_1$, $I_{\overline{B}}(\alpha_3/2, \overline{\chi}_v \boxtimes \overline{\eta}_v) = I_{\overline{P}_1}(\alpha_3/2, \mathrm{Ind}_{\overline{\chi}_v} \boxtimes \overline{\eta}_v)$. So the image of $A(\alpha_3/2, \overline{\chi}_v \boxtimes \overline{\eta}_v, w_{212})$ is irreducible by the Langlands classification theorem. Moreover, reduction to a simple rank one computation shows that the map

$$\begin{aligned} \mathrm{Res}_{s=1/2} A(-\alpha_4/4 - s\alpha_2/2, \overline{\eta}_v \boxtimes \overline{\chi}_v, w_1) : I_{\overline{B}}(\overline{\eta}_v |^{-1/2} \boxtimes \overline{\chi}_v |^{-s}) \\ \longrightarrow I_{\overline{B}}(\overline{\chi}_v |^{-s} \boxtimes \overline{\eta}_v |^{-1/2}) \end{aligned}$$

is a scaling map by a nonzero constant and thus an isomorphism. The proof of the lemma is now completed. $\square$

Based on this lemma, the image of $R(1/2, \overline{\chi}_v \boxtimes \sigma_v, w_{121})$ is a Langlands quotient, which we denote by $J_{\mathrm{ETF}}(1/2, \overline{\chi}_v \boxtimes \sigma_v)$. Define

$$J_{\mathrm{ETF}}(1/2, \overline{\chi} \boxtimes \sigma) = \bigotimes J_{\mathrm{ETF}}(1/2, \overline{\chi}_v \boxtimes \sigma_v).$$

We see that both $J_{\mathrm{ETF}}(3/2, \overline{\chi} \boxtimes \sigma)$ and $J_{\mathrm{ETF}}(1/2, \overline{\chi} \boxtimes \sigma)$ contribute to $L_d^2(\overline{P}_2)$.

4.4. The L_π^2 case. In this case, the local representation σ_v is the obtained from the local component π_v of a cuspidal representation π on $\mathrm{PGL}_2(\mathbf{A}_k)$ by Shimura correspondence. Therefore $L^S(s, \chi \times \sigma) = L^S(s, \chi \times \pi)$ and the latter is known to be entire over $\mathbf{C}$. The operator $M(s, \overline{\chi} \boxtimes \sigma, w_{121})$ has a pole at $s = 1/2$ provided $\chi^2 = 1$, $L(1/2, \chi \times \pi) \neq 0$ and that the normalized local operator for $v \in S$ is holomorphic and nonvanishing at $\mathrm{Re}(s) \geq 0$. More precisely, we use exactly the same formula (1) in the ETF case for the normalization. Then

Lemma 4.4. *For $v \in S$, the operator $R(s, \overline{\chi}_v \boxtimes \sigma_v, w_{121})$ can be continued to be a holomorphic function for $\mathrm{Re}(s) \geq 0$.*

Proof. The proof follows from a similar computation as in Lemma 4.1.

In this case σ_v is the theta lift of an irreducible representation π_v on $\mathrm{PGL}_2(k_v)$ or $\mathrm{PD}_v^\times$. We know that σ_v is tempered if and only if the same holds for π_v , and also $L(s, \chi_v \times \sigma_v) = L(s, \chi_v \times \pi_v)$. We refer to [GaS] for more properties for the theta lift and its consequences.

In view of this, the lemma holds in the tempered case. Now assume σ_v is not tempered; then we have $\pi_v = \mathrm{Ind}_{B_o} \nu |^a \boxtimes \nu |^{-a}$ with $\nu^2 = 1$ and $a \in (0, 1/9)$. Hence we have $\sigma_v = \mathrm{Ind}_{\overline{B}_o} \overline{\nu} |^a$. The factorization of the intertwining operator $A(s\beta_1, \overline{\chi}_v \boxtimes \overline{\nu} |^a, w_{121})$ gives

$$\begin{aligned} I_{\overline{B}}(\overline{\chi}_v |^s \boxtimes \overline{\nu} |^a) \xrightarrow{A(w_1)} I_{\overline{B}}(\overline{\nu} |^a \boxtimes \overline{\chi}_v |^s) \\ \xrightarrow{A(w_2)} I_{\overline{B}}(\overline{\nu} |^a \boxtimes \overline{\chi}_v^{-1} |^{-s}) \xrightarrow{A(w_1)} I_{\overline{B}}(\overline{\chi}_v^{-1} |^{-s} \boxtimes \overline{\nu} |^a). \end{aligned}$$

For $\mathrm{Re}(s) \geq 0$, the third operator is holomorphic, while the first operator $A(w_1)$ gives a pole at $s = a$ when $\chi_v = \nu$ and the second operator at $s = 0$ when $\chi_v^2 = 1$. On the other side, the factor $L(s, \chi_v \times \sigma_v)$ is given by $L(s + a, \chi_v \nu) L(s - a, \chi_v \nu)$,

whose pole is at $s = a$ for $\chi_v = \nu$ as well. Therefore the potential poles of the normalizing fact $r(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ and original operator $A(s, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$ cancel each other. $\square$

The image of $R(1/2, \bar{\chi}_v \boxtimes \sigma_v, w_{121})$, by the same argument as in Lemma 3.3, is a Langlands quotient with appropriate data. We denote the image by $J_\pi(1/2, \bar{\chi}_v \boxtimes \sigma_v)$ and let

$$J_\pi(1/2, \bar{\chi} \boxtimes \sigma) = \bigotimes J_\pi(1/2, \bar{\chi}_v \boxtimes \sigma_v),$$

which contributes to the residual spectrum $L_d^2(\bar{P}_2)$.

In conclusion, we have proved the following.

Theorem 4.5. *Let $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{A}_{cusp}(GL_1(\mathbf{A}_k)) \times \mathcal{A}_{cusp}(Mp_2(\mathbf{A}_k))$ be defined as the collections of pairs:*

$$\begin{aligned} \mathcal{P}_1 &= \{(\chi, \sigma) : \chi = \eta, \sigma \in ETF_\eta\}, \\ \mathcal{P}_2 &= \{(\chi, \sigma) : \chi^2 = \mathbb{1}, \sigma \in ETF_\eta, \chi_v \neq \eta_v \text{ if } v \in S_\sigma\}, \end{aligned}$$

where it is always assumed that $\eta^2 = \mathbb{1}$. Also, we let

$$\begin{aligned} \mathcal{P}_3 &= \{(\chi, \pi) : \chi^2 = \mathbb{1}, L(1/2, \chi \times \pi) \neq 0\} \\ &\subseteq \mathcal{A}_{cusp}(GL_1(\mathbf{A}_k)) \times \mathcal{A}_{cusp}(PGL_2(\mathbf{A}_k)). \end{aligned}$$

Then the residual spectrum of $Mp_4(\mathbf{A}_k)$ along $\bar{P}_2$ is given by

$$\begin{aligned} L_d^2(P_2) &= \bigoplus_{(\chi, \sigma) \in \mathcal{P}_1} J_{ETF}(3/2, \bar{\chi} \boxtimes \sigma) \oplus \bigoplus_{(\chi, \sigma) \in \mathcal{P}_2} J_{ETF}(1/2, \bar{\chi} \boxtimes \sigma) \\ &\quad \oplus \bigoplus_{(\chi, \pi) \in \mathcal{P}_3} \bigoplus_{\sigma \in L_\pi^2} J_\pi(1/2, \bar{\chi} \boxtimes \sigma). \end{aligned}$$

5. DECOMPOSITION FOR THE BOREL SUBGROUP

Given two unitary Hecke characters χ, μ of $GL_1(\mathbf{A}_k)$, we define a genuine character $\bar{\chi} \boxtimes \bar{\mu}$ of $\bar{T}(\mathbf{A}_k)$ as before. Consider the space $\text{Ind}_B^{Mp_4} \bar{\chi} \boxtimes \bar{\mu}$ of functions ϕ on $Mp_4(\mathbf{A}_k)$ satisfying $\phi(n\bar{t}\bar{g}) = (\bar{\chi} \boxtimes \bar{\mu})(\bar{t})\phi(g)$ for any $n \in U(\mathbf{A}_k), \bar{t} \in \bar{T}(\mathbf{A}_k)$ and $\bar{g} \in Mp_4(\mathbf{A}_k)$. For each $\Lambda \in \mathfrak{a}_{\mathbf{C}}^* = \mathbf{C}\beta_1 \oplus \mathbf{C}\beta_2$, the representation of $Mp_4(\mathbf{A}_k)$ on the space of functions of the form

$$\Phi_\Lambda : \bar{g} \mapsto \phi(\bar{g}) \exp \langle \Lambda + \rho_B, H_{\bar{B}}(\bar{g}) \rangle$$

is equivalent to $I_{\bar{B}}(\Lambda, \bar{\chi} \boxtimes \bar{\mu}) = \text{Ind}_B^{Mp_4}(\bar{\chi} \boxtimes \bar{\mu}) \otimes \exp \langle \Lambda, H_{\bar{B}}(\cdot) \rangle$, where ρ_B is the half-sum of positive roots, i.e. $\rho_B = \beta_1 + \beta_2$. We form the Eisenstein series as before:

$$E(\Lambda, \bar{g}, \phi, \bar{B}) = \sum_{\gamma \in B(k) \backslash Mp_4(k)} \Phi_\Lambda(\gamma \bar{g}).$$

Since $\Phi_\Lambda(\bar{g}) = \phi(\bar{g}) \exp \langle \Lambda + \rho_B, H_{\bar{B}}(\bar{g}) \rangle$, we may sometimes write $E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ for $E(\Lambda, \bar{g}, \phi, \bar{B})$ to emphasize the variation of functions in the representation space $I_{\bar{B}}(\Lambda, \bar{\chi} \boxtimes \bar{\mu})$. We know that the Eisenstein series satisfies the functional equation

$$E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B}) = E(w\Lambda, \bar{g}, M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w)\Phi_\Lambda, \bar{B})$$

for any Weyl group element w . Moreover, the constant term of $E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ along U is given by

$$E_B(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B}) = \sum_{w \in \Omega} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w) \Phi_\Lambda(\bar{g}),$$

where Ω is the full Weyl group. In general, for $w \in \Omega$ and $f \in I_{\bar{B}}(\Lambda, \bar{\chi} \boxtimes \bar{\mu})$ and sufficiently regular Λ ,

$$M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w) f(\bar{g}) = \int_{U^w(\mathbf{A}_k)} f(w^{-1} n \bar{g}) dn.$$

The symbol U^w represents $U \cap wU'w^{-1}$. It is known that

$$M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w) = \bigotimes A(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w), \text{ with}$$

$$A(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w) f_v(\bar{g}_v) = \int_{U^w(k_v)} f_v(w^{-1} n \bar{g}_v) dn,$$

where $f = \bigotimes f_v$ and f_v is the unique $\mathrm{Sp}_4(\mathfrak{O}_v)$ -fixed function with normalization $f_v(e_v) = 1$ for almost all v .

For almost all places, Proposition 3.1 gives the coefficients attached to the local operator $A(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w)$. For convenience, for $w \in \Omega$ we normalize $M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w)$ as follows.

First we introduce some notation to be used in the sequel. Consider any Hecke L -function attached to a grossencharacter $\theta = \bigotimes_v \theta_v$ of k and to a given nontrivial additive character $\delta = \bigotimes_v \delta_v$ of $k \backslash \mathbf{A}_k$. We let $\varepsilon(s, \theta) = \prod_v \varepsilon(s, \theta_v, \delta_v)$ denote the usual ε factor. For simplicity, we may write $\varepsilon(s, \theta_v) = \varepsilon(s, \theta_v, \delta_v)$. Then the functional equation

$$L(s, \theta) = \varepsilon(s, \theta) L(1 - s, \theta^{-1})$$

holds. We write

$$\zeta(s) = L(s, \mathbb{1}) = \frac{a_{-1}}{s-1} + a_0 + a_1(s-1) + \dots, \text{ with } a_{-1} = \mathrm{Res}_{s=1} \zeta(s).$$

Now the normalization is given by

$$\begin{aligned} r(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w) &= \prod_{\substack{\beta > 0, w\beta < 0 \\ \beta \text{ short}}} \frac{L(\langle \Lambda, \beta^\vee \rangle, (\chi_v \boxtimes \mu_v) \beta^\vee)}{L(1 + \langle \Lambda, \beta^\vee \rangle, (\chi_v \boxtimes \mu_v) \beta^\vee) \varepsilon(\langle \Lambda, \beta^\vee \rangle, (\chi_v \boxtimes \mu_v) \beta^\vee)} \\ &\quad \cdot \prod_{\substack{\beta > 0, w\beta < 0 \\ \beta \text{ long}}} \frac{L(2\langle \Lambda, \beta^\vee \rangle, ((\chi_v \boxtimes \mu_v) \beta^\vee)^2)}{L(1 + 2\langle \Lambda, \beta^\vee \rangle, ((\chi_v \boxtimes \mu_v) \beta^\vee)^2) \varepsilon(2\langle \Lambda, \beta^\vee \rangle, ((\chi_v \boxtimes \mu_v) \beta^\vee)^2)}, \\ A(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w) &= r(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w) R(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w). \end{aligned}$$

Regarding the holomorphicity of $R(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w)$, we have

Lemma 5.1. *The normalized operator $R(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w)$ is holomorphic for $\mathrm{Re}\langle \Lambda, \beta \rangle \neq -1$ or $-1/2$ when β is short or long, respectively. Here β is any positive root that appears in the expression for $R(\Lambda, \bar{\chi}_v \boxtimes \bar{\mu}_v, w)$ above.*

Proof. It follows from the same reasoning as in Lemma 4.1 or a modified argument in [Win]. To illustrate this, we consider, for example, the case $w = w_{121}$ in more detail. The others follow similarly. In this case, parameterize $\Lambda = s\beta_1 + t\beta_2$. Then

$I_{\overline{B}}(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v) = I_{\overline{B}}(\overline{\chi}_v | |^{s+t} \boxtimes \overline{\mu}_v | |^t)$ and the operator $A(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v, w_{121})$ factorizes as

$$\begin{aligned} I_{\overline{B}}(\overline{\chi}_v | |^{s+t} \boxtimes \overline{\mu}_v | |^t) &\longrightarrow I_{\overline{B}}(\overline{\mu}_v | |^t \boxtimes \overline{\chi}_v | |^{s+t}) \\ &\longrightarrow I_{\overline{B}}(\overline{\mu}_v | |^t \boxtimes \overline{\chi}_v^{-1} | |^{-s-t}) \longrightarrow I_{\overline{B}}(\overline{\chi}_v^{-1} | |^{-s-t} \boxtimes \overline{\mu}_v | |^t). \end{aligned}$$

Individually, the simple pole of each map in the chain occurs at $s = 0$ with $\chi_v = \mu_v$, $s + t = 0$ with $\chi_v^2 = 1$ and $s + 2t = 0$ with $\chi_v = \mu_v^{-1}$. On the other hand, we assume Λ is such that $\langle \Lambda, \alpha_1 \rangle > -1$, $\langle \Lambda, \alpha_3 \rangle > -1$ and $\langle \Lambda, \alpha_4 \rangle > -1/2$. Then the poles of $r(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v, w_{121})$ are among those of

$$\begin{aligned} &L(\langle \Lambda, \alpha_1^\vee \rangle, (\chi_v \boxtimes \mu_v) \alpha_1^\vee) L(\langle \Lambda, \alpha_3^\vee \rangle, (\chi_v \boxtimes \mu_v) \alpha_3^\vee) L(2\langle \Lambda, \alpha_4^\vee \rangle, ((\chi_v \boxtimes \mu_v) \alpha_4^\vee)^2) \\ &= L(s, \chi_v \mu_v^{-1}) L(s + 2t, \chi_v \mu_v) L(2s + 2t, \chi_v^2). \end{aligned}$$

It can be observed that the occurrence of the pole agrees with that of $A(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v, w_{121})$. $\square$

Note also that the normalized operator sends an unramified vector to an unramified one in the codomain space. It satisfies a similar cocycle relation

$$R(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v, w'w) = R(w\Lambda, w(\overline{\chi}_v \boxtimes \overline{\mu}_v), w') R(\Lambda, \overline{\chi}_v \boxtimes \overline{\mu}_v, w)$$

for w, w' any Weyl elements with the relation $l(ww') = l(w) + l(w')$. Here $l(w)$ is the total number of w_1 and w_2 that appear in the reduced decomposition of w .

We see that in order to determine the pole of $E_B(\Lambda, \overline{g}, \Phi_\Lambda, \overline{B})$, it is necessary to consider each $M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w)$, and therefore the several hyperplanes in the root space defined as

$$S_i = \begin{cases} \{\Lambda \in \mathfrak{a}_{\mathbf{C}}^* | \langle \Lambda, \alpha_i \rangle = 1\} & \text{if } \alpha_i \text{ is short,} \\ \{\Lambda \in \mathfrak{a}_{\mathbf{C}}^* | \langle \Lambda, \alpha_i \rangle = 1/2\} & \text{if } \alpha_i \text{ is long,} \end{cases}$$

where $i = 1, 2, 3, 4$. The hyperplanes are drawn in Figure 1.

We proceed first with a general discussion of $L_{\mathbf{d}}^2(\overline{B})$. Consider any entire function F on $\mathfrak{a}_{\mathbf{C}}^*$ of Paley-Wiener type such that $F(\Lambda) \in I_{\overline{B}}(\Lambda, \overline{\chi} \boxtimes \overline{\mu})$. According to Landlants' theory, $L^2(\overline{B})$ is generated by

$$\Theta_F(\overline{g}) = \frac{1}{(2\pi i)^2} \int_{\operatorname{Re} \Lambda = \Lambda_0} E(\Lambda, \overline{g}, F(\Lambda), \overline{B}) d\Lambda,$$

for all such $F(\Lambda)$. Here Λ_0 satisfies $\langle \Lambda_0 - \rho_B, \alpha^\vee \rangle > 0$ for all positive roots α (cf. [MoW, II.1]). In fact, in our case for Mp_4 , we can infer from the figure that ρ_B here can be replaced by $\alpha_1/2 + \alpha_3$. In order to obtain the discrete spectrum $L_{\mathbf{d}}^2(\overline{B})$, we have to deform the contour from $\operatorname{Re} \Lambda = \Lambda_0$ to $\operatorname{Re} \Lambda = 0$. A priori, the poles of the intertwining operators $M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w)$ lie on S_i given by real equations, and here we use the curly dotted line to represent the deformation process. The resulting integral at $\operatorname{Re} \Lambda = 0$,

$$\frac{1}{(2\pi i)^2} \int_{\operatorname{Re} \Lambda = 0} E(\Lambda, \overline{g}, F(\Lambda), \overline{B}) d\Lambda,$$

will contribute to the continuous spectrum of $L^2(\overline{B})$ of dimension 2. However, by the theory of several complex variables, such a deformation will enable us to pick

iterated residues along different hyperplanes:

$$\begin{aligned} & \operatorname{Res}_{\beta_1 + \frac{\beta_2}{2}} \operatorname{Res}_{S_1} E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B}), \quad \operatorname{Res}_{\beta_1} \operatorname{Res}_{S_1} E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B}), \\ & \operatorname{Res}_{\frac{\alpha_1}{2}} \operatorname{Res}_{S_1} E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B}), \quad \operatorname{Res}_{\frac{\beta_2}{2}} \operatorname{Res}_{S_2} E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B}), \\ & \operatorname{Res}_{\frac{\beta_2}{2}} \operatorname{Res}_{S_3} E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B}). \end{aligned}$$

If we choose a different path, then the iterated residue computed, for example at $\beta_1 + \beta_2/2$ along S_2 instead of S_1 , will differ by a sign. However, the function space spanned by such residues will be the same. The iterated residues as above depend on the characters. For instance, if $\chi = \mu = \mathbb{1}$, then S_2, S_3, S_4 are all hyperplanes of poles of the Eisenstein series. However, if $\chi^2 = \mu^2 = \mathbb{1}$ with $\chi \neq \mu$, then only S_2, S_4 occur as poles. More precisely, in the first case, both S_3 and S_4 contribute a pole to the function $\operatorname{Res}_{S_2} E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ on S_2 . In fact, the iterated residue $\operatorname{Res}_{\frac{\beta_2}{2}} \operatorname{Res}_{S_2} E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ at $\frac{\beta_2}{2}$ is from the combined contributions by both S_3 and S_4 .

Note that the function $\operatorname{Res}_{S_4} E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ defined on S_4 is holomorphic on the path from $\operatorname{Re} \Lambda = \Lambda_4$ to $\operatorname{Re} \Lambda = v_4$, i.e. no intersection with other hyperplanes along this path of deformation. Therefore no iterated residue will be generated in this way. The above list is exhaustive.

In the remaining part of this section, we fix a chosen path as indicated in the picture and will determine precisely which of the above iterated residues are zero, non-square integrable and square integrable. In fact, the nonsquare integrable residues cancel each other and the square integrable ones generated contribute to the residual spectrum associated with $\bar{B}$. To facilitate the computations of the above residues, we first prove a useful lemma.

Lemma 5.2. *For $\chi = \mu, \chi^2 = \mathbb{1}$, the equalities $w_1(\alpha_3) = \alpha_3, w_2(\alpha_4) = \alpha_4$ hold. Also, the character $\bar{\chi} \boxtimes \bar{\mu}$ is invariant under both w_1 and w_2 . Let $t \in \mathbf{R}$ be any real number; then the two intertwining operators*

$$\begin{aligned} R(t\alpha_3, \bar{\chi} \boxtimes \bar{\mu}, w_1) : I_{\bar{B}}(t\alpha_3, \bar{\chi} \boxtimes \bar{\mu}) &\longrightarrow I_{\bar{B}}(w_1(t\alpha_3), w_1(\bar{\chi} \boxtimes \bar{\mu})), \\ R(t\alpha_4, \bar{\chi} \boxtimes \bar{\mu}, w_2) : I_{\bar{B}}(t\alpha_4, \bar{\chi} \boxtimes \bar{\mu}) &\longrightarrow I_{\bar{B}}(w_2(t\alpha_4), w_2(\bar{\chi} \boxtimes \bar{\mu})) \end{aligned}$$

are both the identity maps.

Proof. By inducing in stages,

$$I_{\bar{B}}(t\alpha_3, \bar{\chi} \boxtimes \bar{\mu}) = I_{\bar{P}_1}(I_{\bar{B}_o}^{\overline{\operatorname{GL}}_2}(|{}^t\bar{\chi} \boxtimes |{}^t\bar{\mu})).$$

Since induction commutes with the intertwining operator, one is reduced to the rank one $\overline{\operatorname{GL}}_2$ case. However, the rank one intertwining operator is essentially a GL_2 operator. That is, it suffices to show that $R(\chi \boxtimes \chi, w) : I_{B_o}^{\operatorname{GL}_2}(\chi \boxtimes \chi) \longrightarrow I_{B_o}^{\operatorname{GL}_2}(\chi \boxtimes \mu)$ is the identity map, where w is the nontrivial Weyl reflection and $R(\chi \boxtimes \chi, w)$ is the intertwining operator

$$R(\chi \boxtimes \chi, w) = \lim_{s \rightarrow 0} R(s, \chi \boxtimes \chi, w) \text{ with } M(s, \chi \boxtimes \chi, w) = \frac{\zeta(s)}{\zeta(s+1)} R(s, \chi \boxtimes \chi, w).$$

However, we know $\lim_{s \rightarrow 0} M(\chi \boxtimes \chi, w) = -\text{identity}$ (cf. [KeS, prop. 6.3]), which gives $R(\chi \boxtimes \chi, w) = \text{identity}$.

For the second case, we have

$$I_{\bar{B}}(t\alpha_4, \bar{\chi} \boxtimes \bar{\mu}) = I_{\bar{P}_2}(|{}^{2t}\bar{\chi} \boxtimes I_{B_o}^{\operatorname{Mp}_2}\bar{\mu}).$$

Consider the un-normalized operator

$$M(\bar{\mu}, w) : \mathrm{Ind}_{\bar{B}_o}^{\mathrm{Mp}_2}(\bar{\mu}) \longrightarrow \mathrm{Ind}_{\bar{B}_o}^{\mathrm{Mp}_2}(w\bar{\mu}),$$

which appears in the constant term of the rank one Mp_2 Eisenstein series $E_{B_o}(s, \bar{g}, \phi)$ evaluated at $s = 0$. Here $\phi \in I_{\bar{B}_o}(\bar{\mu})$ and $\bar{g} \in \mathrm{Mp}_2$. By [Szp, sect. 9], the nonconstant term Fourier coefficient of $E(s, \bar{g}, \phi)$ vanishes at $s = 0$, which gives $E(0, \bar{g}, \phi) = E_{B_o}(0, \bar{g}, \phi)$.

Pick an arbitrary $n \in U(\mathbf{A}_k)$; we show that $E(0, \bar{g}n, \phi) = E(0, \bar{g}, \phi)$ holds for all $\bar{g} \in \mathrm{Mp}_2(\mathbf{A}_k)$. It is easy to check that for $\bar{b} \in \bar{B}_o(\mathbf{A}_k)$ the equality holds. From the definition of Eisenstein series we can write $E(0, \bar{g}n, \phi) = E(0, \bar{g}, n \cdot \phi)$, where $n \cdot \phi$ denotes the action of right translation. Since both $\phi, n \cdot \phi \in I_{\bar{B}_o}(\bar{\mu}) = \bigotimes_v I_{\bar{B}_o}(\bar{\mu}_v)$, they are spherical at almost all places. That is, there is a finite set S big enough (depending on n) such that $\phi, n \cdot \phi$ are invariant under $K^S = \prod_{v \notin S} \mathrm{SL}_2(\mathfrak{O}_v)$.

Note that both $E(0, \bar{g}, \phi)$ and $E(0, \bar{g}n, \phi)$ are left invariant under $\mathrm{SL}_2(k)$. Consider $\bar{g} = x\bar{b}k \in \mathrm{SL}_2(k) \cdot \bar{B}_o(\mathbf{A}_k) \cdot K^S$; we see that by above invariance properties

$$\begin{aligned} E(0, x\bar{b}kn, \phi) &= E(0, x\bar{b}, k \cdot n \cdot \phi) = E(0, \bar{b}, n \cdot \phi) = E(0, \bar{b}n, \phi) \\ &= E(0, \bar{b}, \phi) = E(0, x\bar{b}k, \phi). \end{aligned}$$

By the weak approximation theorem for SL_2 we know that $\mathrm{SL}_2(k) \cdot \bar{B}_o(\mathbf{A}_k) \cdot K^S$ is dense in $\mathrm{Mp}_2(\mathbf{A}_k)$. Therefore we conclude that $E(0, \bar{g}, \phi)$ is invariant under $U(\mathbf{A}_k)$ by right translation on the whole domain, which we assume belongs to $\bigotimes_v \pi_v$. Then for almost all v , π_v is an infinite-dimensional representation invariant under $U(k_v)$. This violates the Howe-Moore vanishing property at infinity of the matrix coefficients of π_v (cf. [HoM, Thm. 5.1]), unless $E(0, \bar{g}, \phi) = 0$.

It follows from $E(0, \bar{g}, \phi) = 0$ that

$$M(\bar{\mu}, w) = -\mathrm{identity} = \lim_{s \rightarrow 0} \frac{\zeta(2s)}{\zeta(2s+1)} R(\bar{\mu}, w) = -R(\bar{\mu}, w).$$

We also note that this argument in fact works for both cases above. The proof is completed. $\square$

The computation of iterated residues relies on the coordinate system that we use. In what follows, we will set as the basic one the coordinates

$$(2) \quad \Lambda = x\alpha_1/2 + y\alpha_3/2.$$

Therefore, the intertwining operators can be viewed as functions of x and y . In view of above lemma, one has immediately

Corollary 5.3. *With coordinates $\Lambda = x\alpha_1/2 + y\alpha_3/2$. Assume $\chi = \mu, \chi^2 = \mathbb{1}$, then for all $t \in \mathbf{R}$, the partial derivative $R_y(t\alpha_3, \bar{\chi} \boxtimes \bar{\mu}, w_1)$ along the y -direction vanishes.*

5.1. Residues along S_1 . The singularities of $E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ are the same as those of its constant term

$$E_B(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B}) = \sum_{w \in W} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w) \Phi_\Lambda.$$

In view of Proposition 3.1 on the intertwining map, we see that α_i appears only for $W_i = \{w \in W : w\alpha_i < 0\}$. More specifically, we have

$$W_1 = \{w_1, w_{21}, w_{121}, w_{1212}\}, \quad W_2 = \{w_2, w_{12}, w_{212}, w_{1212}\}, \\ W_3 = \{w_{12}, w_{121}, w_{212}, w_{1212}\}.$$

Therefore, to compute the residue $\text{Res}_{S_1} E_B(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ we only need to consider the terms $M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w)$, $w \in W_1$, and take the residues at the hyperplane S_1 and combine over all elements of W_1 . With respect to the (x, y) coordinates $\Lambda = x\alpha_1/2 + y\alpha_3/2$, taking the residue along S_1 amounts to considering the residue of intertwining operators when x approaches 1. It follows from Proposition 3.1 that $M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w)$, for all $w \in W_1$, has a pole at S_1 only when $(\chi \boxtimes \mu)\alpha_1^\vee = \mathbb{1}$, i.e. $\chi = \mu$. This gives

Lemma 5.4. *We use Λ to denote $\alpha_1/2 + y\alpha_3/2$. Then the residues along S_1 for the intertwining operators in $E_{\bar{B}}(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ are as follows:*

$$\begin{aligned} \text{Res}_{S_1} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_1) &= \frac{a-1}{\zeta(2)} R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_1), \\ \text{Res}_{S_1} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{21}) &= \frac{a-1}{\zeta(2)} \frac{L(y+1, \chi^2)}{L(y+2, \chi^2)\varepsilon(y+1, \chi^2)} R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{21}), \\ \text{Res}_{S_1} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{121}) &= \frac{a-1}{\zeta(2)} \frac{L(y, \chi^2)}{L(y+2, \chi^2)\varepsilon(y, \chi^2)\varepsilon(y+1, \chi^2)} R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{121}), \\ \text{Res}_{S_1} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}) &= \frac{a-1}{\zeta(2)} \frac{L(y-1, \chi^2)}{L(y+2, \chi^2)\varepsilon(y-1, \chi^2)\varepsilon(y, \chi^2)\varepsilon(y+1, \chi^2)} \\ &\quad \cdot R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}). \end{aligned}$$

We require $y \geq 0$ from the deformation for contour integrals of Eisenstein series. It follows that the pole of $\text{Res}_{S_1} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w)$, $w \in W_1$, appears only for the pairs $(\bar{\chi} \boxtimes \bar{\mu}, y)$ of the three types:

- (a) $\chi = \mu, \chi^2 = \mathbb{1}, y = 2$;
- (b) $\chi = \mu, \chi^2 = \mathbb{1}, y = 1$;
- (c) $\chi = \mu, \chi^2 = \mathbb{1}, y = 0$.

FIRST CASE (A), $\chi = \mu, \chi^2 = \mathbb{1}, y = 2$. In this case $\Lambda = \alpha_1/2 + \alpha_3$, and only $\text{Res}_{S_1} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{1212})$ contributes to the pole. Since the induced representations and local intertwining operators $R(\alpha_1/2 + \alpha_3, \bar{\chi}_v \boxtimes \bar{\mu}_v, w_{1212})$ are in the Langlands situation, it follows that the image of this local operator is the Langlands quotient $J_{\bar{B}}(\alpha_1/2 + \alpha_3, \bar{\chi}_v \boxtimes \bar{\mu}_v)$ with such prescribed data. In fact, we see it is the even Weil representation for the group $\text{Mp}_4(k_v)$. Now we can write

$$J_{\bar{B}}(\alpha_1/2 + \alpha_3, \bar{\chi} \boxtimes \bar{\mu}) = \bigotimes_v J_{\bar{B}}(\alpha_1/2 + \alpha_3, \bar{\chi}_v \boxtimes \bar{\mu}_v).$$

Since $w_{1212}(\alpha_1/2 + \alpha_3) = (-\alpha_1)/2 + (-\alpha_3)$, by Langlands' criterion the residue $\text{Res}_{\Lambda=\alpha_1/2+\alpha_3} \text{Res}_{S_1} E(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$, which we identify with $J_{\bar{B}}(\alpha_1/2 + \alpha_3, \bar{\chi} \boxtimes \bar{\mu})$, is square integrable. Let $\mathcal{B}_1 \subseteq \mathcal{A}(\text{GL}_1(\mathbf{A}_k)) \times \mathcal{A}(\text{GL}_1(\mathbf{A}_k))$ be given by

$$\mathcal{B}_1 = \{(\chi, \chi) : \chi^2 = \mathbb{1}\};$$

then $\bigoplus_{(\chi, \mu) \in \mathcal{B}_1} J_{\bar{B}}(\alpha_1/2 + \alpha_3, \bar{\chi} \boxtimes \bar{\mu})$ contributes to the residual spectrum $L_{\mathfrak{d}}^2(\bar{B})$.

SECOND CASE (B), $\chi = \mu, \chi^2 = \mathbb{1}, y = 1$. In this case $\Lambda = \alpha_4/2$, and simple computation gives

$$\begin{aligned} & \mathrm{Res}_{y=1} \mathrm{Res}_{S_1} E_B(\Lambda, \bar{g}, \phi, \bar{B}) \\ &= \frac{a_{-1}^2}{\zeta(2)\zeta(3)} R(\alpha_4/2, \bar{\chi} \boxtimes \bar{\mu}, w_{121}) \Phi_{\alpha_4/2} - \frac{a_{-1}^2}{\zeta(2)\zeta(3)} R(\alpha_4/2, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}) \Phi_{\alpha_4/2} \\ &= \frac{a_{-1}^2}{\zeta(2)\zeta(3)} \left(R(\alpha_4/2, \bar{\chi} \boxtimes \bar{\mu}, w_{121}) \Phi_{\alpha_4/2} - R(\alpha_4/2, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}) \Phi_{\alpha_4/2} \right). \end{aligned}$$

By the cocycle relation

$$R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}) = R(w_2 \Lambda, w_2(\bar{\chi} \boxtimes \bar{\mu}), w_{121}) R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_2),$$

the last term of above equality equals, up to a constant,

$$R(\alpha_4/2, \bar{\chi} \boxtimes \bar{\mu}, w_{121}) \left(\Phi_{\alpha_4/2} - R(\alpha_4/2, \bar{\chi} \boxtimes \bar{\mu}, w_2) \Phi_{\alpha_4/2} \right).$$

Note that this is valid since $w_2(\alpha_4/2) = \alpha_4/2$ and $w_2(\bar{\chi} \boxtimes \bar{\mu}) = \bar{\chi} \boxtimes \bar{\mu}$. It follows from Lemma 5.2 that $\mathrm{Res}_{z=1/2} \mathrm{Res}_{S_1} E_B(\Lambda, \bar{g}, \phi, \bar{B}) = 0$, namely that the point $\alpha_4/2$ does not contribute to the discrete spectrum of $L_d^2(\bar{B})$.

THIRD CASE (C), $\chi = \mu, \chi^2 = \mathbb{1}, y = 0$. In this case $\Lambda = \alpha_1/2$ and the residue is given by

$$\mathrm{Res}_{y=0} \mathrm{Res}_{S_1} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{21}) + \mathrm{Res}_{y=0} \mathrm{Res}_{S_1} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{121}),$$

which vanishes. The argument follows the same as for $y = 1$. Such vanishing yielded by Lemma 5.2 confirms the prediction from the general theory of Eisenstein series.

5.2. Residues along S_2 . We consider the possible poles of the intertwining operator for Λ along S_2 , which occur only if $2\langle \Lambda, \alpha_2^\vee \rangle = 1$ and $(\chi \alpha_2^\vee)^2 = \mu^2 = \mathbb{1}$. Set up another coordinates system (t, z) by

$$(3) \quad \Lambda = t\alpha_2/2 + z\alpha_4/2 \text{ with } \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix}.$$

The subset W_2 of the Weyl group where the intertwining operator can a pole at S_2 consists of $w_2, w_{12}, w_{212}, w_{1212}$, and we have

Lemma 5.5. *With respect to the coordinates (t, z) , at $t = 1/2$ the residues along S_2 for each intertwining operator in $E_{\bar{B}}(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ are given by:*

$$\begin{aligned} \mathrm{Res}_{S_2} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_2) &= \frac{a_{-1}}{2\zeta(2)} R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_2), \\ \mathrm{Res}_{S_2} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{12}) &= \frac{a_{-1}}{2\zeta(2)} \frac{L(z + 1/2, \chi\mu)}{L(z + 3/2, \chi\mu)\varepsilon(z + 1/2, \chi\mu)} R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{12}), \\ \mathrm{Res}_{S_2} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{212}) &= \frac{a_{-1}}{2\zeta(2)} \frac{L(z + 1/2, \chi\mu)}{L(z + 3/2, \chi\mu)\varepsilon(z + 1/2, \chi\mu)} \\ &\quad \cdot \frac{L(2z, \chi^2)}{L(2z + 1, \chi^2)\varepsilon(2z, \chi^2)} R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{212}), \\ \mathrm{Res}_{S_2} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}) &= \frac{a_{-1}}{2\zeta(2)} \frac{L(z - 1/2, \chi\mu)L(2z, \chi^2)}{L(z + 3/2, \chi\mu)L(2z + 1, \chi^2)} \\ &\quad \cdot \frac{R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{1212})}{\varepsilon(z - 1/2, \chi\mu)\varepsilon(z + 1/2, \chi\mu)\varepsilon(2z, \chi^2)}. \end{aligned}$$

From Figure 1, it follows that we need to consider the two cases where

$$\mathrm{Res}_{S_2} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w), \quad w \in W_2,$$

may possess a pole:

- (i) $\chi \neq \mu, \chi^2 = \mu^2 = \mathbb{1}, z = 1/2$;
- (ii) $\chi = \mu, \chi^2 = \mathbb{1}, z = 1/2$.

FIRST CASE (I), $\chi \neq \mu, \chi^2 = \mu^2 = \mathbb{1}, z = 1/2$. One simple pole occurs in this case and the residue of constant term along $\bar{U}$ is, up to scalar, given by

$$\begin{aligned} & \frac{L(1, \chi\mu)}{L(2, \chi\mu)\varepsilon(1, \chi\mu)} R(\alpha_3/2, \bar{\chi} \boxtimes \bar{\mu}, w_{212}) \Phi_{\alpha_3/2} \\ & + \frac{L(0, \chi\mu)}{L(2, \chi\mu)\varepsilon(0, \chi\mu)\varepsilon(1, \chi\mu)} R(\alpha_3/2, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}) \Phi_{\alpha_3/2} \\ & = \frac{L(1, \chi\mu)}{L(2, \chi\mu)\varepsilon(1, \chi\mu)} \left(R(\alpha_3/2, \bar{\chi} \boxtimes \bar{\mu}, w_{212}) \Phi_{\alpha_3/2} + R(\alpha_3/2, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}) \Phi_{\alpha_3/2} \right). \end{aligned}$$

To determine the image of the intertwining operator $R(\alpha_3/2, \bar{\chi} \boxtimes \bar{\mu}, w_{212}) + R(\alpha_3/2, \bar{\chi} \boxtimes \bar{\mu}, w_{1212})$, by the cocycle relation

$$R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}) = R(w_{212}\Lambda, w_{212}(\bar{\chi} \boxtimes \bar{\mu}), w_1) R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{212})$$

we see that locally

$$\begin{array}{ccccc} I_{\bar{B}}(\alpha_3/2, \bar{\chi}_v \boxtimes \bar{\mu}_v) & \xrightarrow{R(w_{212})} & I_{\bar{B}}(-\alpha_3/2, \bar{\mu}_v \boxtimes \bar{\chi}_v) & \xrightarrow{R(w_1)} & I_{\bar{B}}(-\alpha_3/2, \bar{\chi}_v \boxtimes \bar{\mu}_v) \\ & \searrow & \uparrow & & \\ & & J_{\bar{B}}(\alpha_3/2, \mathrm{Ind}_{\bar{B}}^{\mathrm{GL}_2} \bar{\chi}_v \boxtimes \bar{\mu}_v). & & \end{array}$$

Here the operator $R(w_1)$ is an isomorphism. Therefore, we see that by taking the sum of two iterated residues one obtains a representation space isomorphic to $\bigotimes_v J_{\bar{B}}(\alpha_3/2, \mathrm{Ind}_{\bar{B}}^{\mathrm{GL}_2} \bar{\chi}_v \boxtimes \bar{\mu}_v)$, which we write as $J_{\bar{B}}(\alpha_3/2, \bar{\chi} \boxtimes \bar{\mu})$. Let $\mathcal{B}_2 \subseteq \mathcal{A}(\mathrm{GL}_1(\mathbf{A}_k)) \times \mathcal{A}(\mathrm{GL}_1(\mathbf{A}_k))$ given by

$$\mathcal{B}_2 = \{(\chi, \mu) : \chi^2 = \mu^2 = \mathbb{1}, \chi \neq \mu\}.$$

Since $w_{212}(\alpha_3/2) = w_{1212}(\alpha_3/2) = -\alpha_3/2$, by Langlands' criterion, we know that $J_{\bar{B}}(\alpha_3/2, \bar{\chi} \boxtimes \bar{\mu})$ is square integrable. Therefore $\bigoplus_{(\chi, \mu) \in \mathcal{B}_2} J_{\bar{B}}(\alpha_3/2, \bar{\chi} \boxtimes \bar{\mu})$ contributes to $L_{\mathrm{d}}^2(\bar{B})$.

SECOND CASE (II), $\chi = \mu, \chi^2 = \mathbb{1}, z = 1/2$. The last three residues in the lemma all have poles, and we will obtain an explicit form of $\mathrm{Res}_{\alpha_3/2} \mathrm{Res}_{S_2} E_B(\Lambda, \bar{g}, \Phi_{\Lambda}, \bar{B})$. We will see that the residue obtained at $z = 1/2$ does not contribute to the discrete spectrum $L_{\mathrm{d}}^2(\bar{B})$. In fact, it follows from the computation along S_3 later that it will be canceled by $\mathrm{Res}_{\alpha_3/2} \mathrm{Res}_{S_3} E_B(\Lambda, \bar{g}, \Phi_{\Lambda}, \bar{B})$.

Note that the character $\overline{\chi} \boxtimes \overline{\mu}$ in this case is invariant under w_1 and w_2 ; for simplicity we may omit writing $\overline{\chi} \boxtimes \overline{\mu}$ repeatedly for the intertwining operators. First we consider the last two terms in the above lemma.

Proposition 5.6. *Write as before for the number field k its associated zeta function $\zeta(s) = \frac{a_{-1}}{s-1} + a_0 + a_1(s-1) + \dots$. Then with respect to the (t, z) coordinate system in (3) we have*

$$\begin{aligned} & \text{Res}_{z=1/2} \text{Res}_{S_2} M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{212}) + \text{Res}_{z=1/2} \text{Res}_{S_2} M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{1212}) \\ &= \frac{a_{-1}^2}{2\zeta^3(2)} \left(a_0 R(\alpha_3/2, w_{212}) - \frac{a_{-1}}{2} R_x(-\alpha_3/2, w_1) R(\alpha_3/2, w_{212}) \right). \end{aligned}$$

Here $R_x(-\alpha_3/2, w_1)$ is the partial derivative along the x direction of the operator $R(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_1)$. The coordinates (x, y) for Λ are given by (2).

Proof. Under the second case (ii), all the characters χ_μ, χ^2 that appear in the residues are trivial. Therefore, we are dealing with zeta functions. We can get easily that

$$\zeta(z + \frac{1}{2}) \zeta(2z) = \frac{a_{-1}^2}{2(z - \frac{1}{2})^2} + \frac{3a_0 a_{-1}}{2(z - \frac{1}{2})} + \dots,$$

and

$$\zeta(z - \frac{1}{2}) \zeta(2z) = -\frac{a_{-1}^2}{2(z - \frac{1}{2})^2} - \frac{a_0 a_{-1}}{2(z - \frac{1}{2})} + \dots$$

Moreover, from $\Lambda = \alpha_2/4 + z\alpha_4/2$ we can write the Taylor expansion

$$R(\Lambda, w_{212}) = R(\alpha_3/2, w_{212}) + \left(R_x(\alpha_3/2, w_{212}) + R_y(\alpha_3/2, w_{212}) \right) (z - 1/2) + \dots$$

and

$$R(w_{212}(\Lambda), w_1) = R(-\alpha_3/2, w_1) + \left(R_x(-\alpha_3/2, w_1) - R_y(-\alpha_3/2, w_1) \right) (z - 1/2) + \dots$$

Now Lemma 5.2 gives that $R(-\alpha_3/2, w_1)$ is the identity map and Corollary 5.3 that $R_y(-\alpha_3/2, w_1)$ is zero. Hence the cocycle relation gives the Taylor expansion

$$\begin{aligned} R(\Lambda, w_{1212}) &= R(\alpha_3/2, w_{212}) + \left(R_x(\alpha_3/2, w_{212}) + R_y(\alpha_3/2, w_{212}) \right. \\ &\quad \left. + R_x(-\alpha_3/2, w_1) R(\alpha_3/2, w_{212}) \right) (z - 1/2) + \dots \end{aligned}$$

Now the proposition follows easily from explicit computations of the Taylor expansion of $M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{212})$ and $M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{1212})$. $\square$

Corollary 5.7. *With respect to the (x, y) coordinate system, we have the iterated residue*

$$\begin{aligned} & \text{Res}_{\alpha_3/2} \text{Res}_{S_2} E_B(\Lambda, \overline{g}, \Phi_\Lambda, \overline{B}) = \frac{a_{-1}^2}{\zeta(2)^2} R(\alpha_3/2, w_{12}) \\ &+ \frac{a_{-1}^2}{\zeta(2)^3} \left(a_0 R(\alpha_3/2, w_{212}) - \frac{a_{-1}}{2} R_x(-\alpha_3/2, w_1) R(\alpha_3/2, w_{212}) \right). \end{aligned}$$

This residue is not square integrable and therefore does not contribute to the residual spectrum $L_d^2(\overline{B})$.

Proof. First, note that with respect to the (t, z) coordinates

$$\mathrm{Res}_{\alpha_3/2} \mathrm{Res}_{S_2} M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{12}) = \frac{a_{-1}^2}{2\zeta(2)^2} R(\alpha_3/2, w_{12}).$$

Now the transformation of coordinate systems from (t, z) to (x, y) is given by the Jacobian matrix

$$K = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Therefore, the residue computed in the (x, y) system is $\det K$ times the one in the (t, z) system. Thus the result follows from the previous proposition.

For the last assertion, we use the cocycle relation

$$R(\alpha_3/2, w_{212}) = R(w_{12}(\alpha_3/2), w_2) R(\alpha_3/2, w_{12}).$$

Let $f \in I_{\overline{B}}(\alpha_3/2, \overline{\chi} \boxtimes \overline{\mu})$. If $\mathrm{Res}_{z=1/2} \mathrm{Res}_{S_2} M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{12}) = 0$, then

$$\mathrm{Res}_{z=1/2} \mathrm{Res}_{S_2} M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{212}) f + \mathrm{Res}_{z=1/2} \mathrm{Res}_{S_2} M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{1212}) f = 0.$$

This refutes the square integrability of the iterated residue of Eisenstein series since $w_{12}(\alpha_3/2) = -\alpha_1/2$. The proof is completed. $\square$

Remark. We will see that such a nonsquare integrable function will be canceled by the residue computed from considering the hyperplane S_3 .

5.3. Residues along S_3 . Now we compute the iterated residue of Eisenstein series at α_3 along the hyperplane S_3 . We use the original (x, y) coordinate system with $\Lambda = x\alpha_1/2 + y\alpha_3/2$. Then S_3 being a singular hyperplane is equivalent to $y = 1$ and $(\chi \boxtimes \mu)\alpha_3^\vee = \mathbb{1}$, i.e. $\chi = \mu^{-1}$.

Furthermore, on S_3 , the point α_3 is a singular point for $\mathrm{Res}_{S_2} E_B(\Lambda, \overline{g}, \Phi_\Lambda, \overline{B})$, to which we deform and get a dimension one continuous spectrum by integration over the imaginary axis. By the general residue theory of a complex variable (cf. [Ahl, chap 4, sec. 5]), the residue obtained at $\alpha_3/2$ will be *half* the full residue of $\mathrm{Res}_{S_3} E_B(\Lambda, \overline{g}, \Phi_\Lambda, \overline{B})$ at $\alpha_3/2$. By full residue we mean the coefficient for the leading term in the Laurent expansion of $\mathrm{Res}_{S_3} E_B(\Lambda, \overline{g}, \Phi_\Lambda, \overline{B})$ at $\alpha_3/2$ with respect to x .

Recall that $W_3 = \{w : w\alpha_3 < 0\} = \{w_{12}, w_{121}, w_{212}, w_{1212}\}$. Now a simple computation gives the following.

Lemma 5.8. *With respect to the coordinates (x, y) , residues at $y = 1$, i.e. along S_3 , for intertwining operators in $E_B(\Lambda, \bar{g}, \phi, \bar{B})$ are given by:*

$$\begin{aligned} \mathrm{Res}_{S_3} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{12}) &= \frac{a_{-1}}{\zeta(2)} \frac{L(1-x, \mu^2)}{L(2-x, \mu^2)\varepsilon(1-x, \mu^2)} R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{12}), \\ \mathrm{Res}_{S_3} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{121}) &= \frac{a_{-1}}{\zeta(2)} \frac{L(x, \chi\mu^{-1})}{L(1+x, \chi\mu^{-1})\varepsilon(x, \chi\mu^{-1})} \\ &\quad \cdot \frac{L(1+x, \chi^2)}{L(2+x, \chi^2)\varepsilon(1+x, \chi^2)} R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{12}), \\ \mathrm{Res}_{S_3} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{212}) &= \frac{a_{-1}}{\zeta(2)} \frac{L(1-x, \mu^2)}{L(2-x, \mu^2)\varepsilon(1-x, \mu^2)} \\ &\quad \cdot \frac{L(1+x, \chi^2)}{L(2+x, \chi^2)\varepsilon(1+x, \chi^2)} R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{212}), \\ \mathrm{Res}_{S_3} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{1212}) &= \frac{a_{-1}}{\zeta(2)} \frac{L(x, \chi\mu^{-1})}{L(1+x, \chi\mu^{-1})\varepsilon(x, \chi\mu^{-1})} \\ &\quad \cdot \frac{L(1-x, \mu^2)L(1+x, \chi^2)}{L(2-x, \mu^2)\varepsilon(1-x, \mu^2)} \frac{R(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{212})}{L(2+x, \chi^2)\varepsilon(1+x, \chi^2)}. \end{aligned}$$

Here $\Lambda = x\alpha_1/2 + \alpha_3/2$.

We see that the meromorphic function $\mathrm{Res}_{S_3} E_B(\Lambda, \bar{g}, \Phi_\Lambda, \bar{B})$ has possibly a pole at $\alpha_3/2$ or equivalently $x = 0$ only when $\chi = \mu, \chi^2 = 1$. In this case, all L -functions are Dedekind zeta functions. Therefore

Proposition 5.9. *For $\chi = \mu, \chi^2 = 1$, we have the iterated residue*

$$\begin{aligned} \mathrm{Res}_{\alpha_3/2} \mathrm{Res}_{S_3} E_B(\Lambda, \bar{g}, \phi, \bar{B}) &= -\frac{a_{-1}^2}{\zeta(2)^2} R(\alpha_3/2, w_{12}) \\ &\quad + \frac{a_{-1}^2}{\zeta(2)^3} \left(\frac{a_{-1}}{2} R_x(-\alpha_3/2, w_1) R(\alpha_3/2, w_{212}) - a_0 R(\alpha_3/2, w_{212}) \right). \end{aligned}$$

In view of Corollary 5.7, we see that under the above condition, there is no residual spectrum generated at $\alpha_3/2$.

Proof. The proof relies on similar computation as in the S_2 case, and we may refrain from writing the character for the intertwining operators since it is invariant w_1, w_2 .

First, we have

$$\mathrm{Res}_{x=0} \mathrm{Res}_{S_3} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{12}) = -\frac{a_{-1}^2}{2\zeta(2)^2} R(\alpha_3/2, w_{12}).$$

Also, by applying the cocycle condition and Lemma 5.2, we get

$$\begin{aligned} \mathrm{Res}_{x=0} \mathrm{Res}_{S_3} M(\Lambda, \bar{\chi} \boxtimes \bar{\mu}, w_{121}) &= -\frac{a_{-1}^2}{2\zeta(2)^2} R(\alpha_3/2, w_{121}) \\ &= -\frac{a_{-1}^2}{2\zeta(2)^2} R(\alpha_3/2, w_{12}). \end{aligned}$$

For the last two terms, we first consider the zeta functions that contribute to the poles. Note that

$$\begin{aligned}\frac{\zeta(x)}{\zeta(1+x)} &= \frac{-\frac{a_{-1}}{x} + a_0 - a_1x + \dots}{\frac{a_{-1}}{x} + a_0 + a_1x + \dots} \\ &= -1 + \frac{2a_0}{a_{-1}}x + \dots\end{aligned}$$

Now the zeta functions that contribute to the pole for the last two intertwining operators are respectively

$$\zeta(1-x)\zeta(1+x) = \frac{-a_{-1}^2}{x^2} + \frac{0}{x} + \dots$$

and

$$\frac{\zeta(x)}{\zeta(1+x)}\zeta(1-x)\zeta(1+x) = \frac{a_{-1}^2}{x^2} - \frac{2a_0a_{-1}}{x} + \dots$$

For the normalized intertwining operator, we can write

$$\begin{aligned}R(\Lambda, w_{212}) &= R(\alpha_3/2, w_{212}) + xR_x(\alpha_3/2, w_{212}) + \dots, \\ R(w_{212}\Lambda, w_1) &= R(-\alpha_3/2, w_1) + xR_x(-\alpha_3/2, w_1) + \dots,\end{aligned}$$

which gives

$$\begin{aligned}R(\Lambda, w_{1212}) &= R(\alpha_3/2, w_{212}) \\ &\quad + x\left(R_x(\alpha_3/2, w_{212}) + R_x(-\alpha_3/2, w_1)R(\alpha_3/2, w_{212})\right) + \dots\end{aligned}$$

Now the proposition follows from combining the Taylor expansions, and the consideration that the actual residues $\text{Res}_{x=0}\text{Res}_{S_3}M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{212})$ and

$$\text{Res}_{x=0}\text{Res}_{S_3}M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{12})$$

are only half of the coefficients attached to the $1/x$ term in the Taylor expansions of $\text{Res}_{S_3}M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{212})$ and $\text{Res}_{S_3}M(\Lambda, \overline{\chi} \boxtimes \overline{\mu}, w_{12})$ respectively. $\square$

5.4. Conclusion. To summarize,

Theorem 5.10. *Keeping the notation as above, it follows from Langlands' theory of Eisenstein series that the discrete spectrum $L_d^2(\overline{B})$ associated to the Borel subgroup has the decomposition*

$$L_d^2(\overline{B}) = \bigoplus_{(\chi, \mu) \in \mathcal{B}_1} J_{\overline{B}}(\alpha_1/2 + \alpha_3, \overline{\chi} \boxtimes \overline{\mu}) \oplus \bigoplus_{(\chi, \mu) \in \mathcal{B}_2} J_{\overline{B}}(\alpha_3/2, \overline{\chi} \boxtimes \overline{\mu}).$$

6. INTERPRETATION WITH ARTHUR PARAMETERS

6.1. The Arthur conjecture for metaplectic groups. We refer to the paper [Art] for the Arthur conjecture for the discrete spectrum of classical groups. In this section, we depict coarsely how the residual spectrum computed can be interpreted in the framework of an Arthur conjecture corresponding to metaplectic groups (cf. [GGP, sect. 25]). More precisely, each residue that has appeared in L_{res}^2 will contribute to one near the equivalence class associated to an Arthur parameter ψ . We call two automorphic representations nearly equivalent if and only if they are isomorphic for almost all places.

Since the emphasis in this paper is not, even if possible, on the proof of Arthur conjectures on metaplectic groups, the following sketchy description is not completely unjustified. However, with further work, for example on the construction of local A-packets and proof of full near equivalence class, one can expect to make further progress on the conjecture.

We recall that a discrete A-parameter ψ valued in the Langlands dual group $\mathrm{Sp}_{2n}(\mathbf{C})$ of Mp_{2n} (cf. [GGP, sect. 25]) is

$$\psi : L_k \times \mathrm{SL}_2(\mathbf{C}) \rightarrow \mathrm{Sp}_{2n}(\mathbf{C}),$$

which takes the form

$$\psi = \phi_{n_1} \boxtimes S_{r_1} \oplus \dots \oplus \phi_{n_k} \boxtimes S_{r_k}.$$

Here L_k is the conjectural automorphic Langlands group. However, we do not need the precise knowledge on it for the Arthur conjecture. For a parameter ψ , we require that

- (a) ϕ_{n_i} is an irreducible n_i -dimensional representation of L_k ;
- (b) S_{r_i} is the r_i -dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{C})$;
- (c) $\sum_i n_i r_i = 2n$;
- (d) $L(s, \phi_{n_i}, \wedge^2)$ has a pole at $s = 1$ if r_i is odd, whereas $L(s, \phi_{n_i}, \mathrm{Sym}^2)$ has a pole at $s = 1$ if r_i is even;
- (e) ψ is called discrete if it is multiplicity-free, i.e. $(r_i, \phi_{n_i}) \neq (r_j, \phi_{n_j})$ if $i \neq j$. It is called tempered if $r_i = 1$ for all i .

We can assign a global component group to every discrete A-parameter ψ ,

$$\mathcal{S}_\psi = \prod_{i=1}^k \mathbf{Z}/2\mathbf{Z}a_i,$$

which is a vector space over $\mathbf{Z}/2\mathbf{Z}$ with distinguished basis index by the summands of ψ . Furthermore, every parameter ψ will give rise to local components by considering the composition

$$\psi_v : L_{k_v} \times \mathrm{SL}(\mathbf{C}) \hookrightarrow L_{k_v} \times \mathrm{SL}(\mathbf{C}) \longrightarrow \mathrm{Sp}_{2n}(\mathbf{C}),$$

where $L_{k_v} = \mathrm{WD}_v$, the Weil-Deligne group for k_v . We define the local component group

$$\mathcal{S}_{\psi_v} = \pi_0(\mathrm{Cent}_{\mathrm{Sp}_{2n}(\mathbf{C})}(\psi_v)),$$

to which one can attach conjecturally a finite set

$$A_{\psi_v} = \{\pi_{\xi_v} : \xi_v \in \mathrm{Irr}(\mathcal{S}_{\psi_v})\}$$

indexed by the irreducible characters of the finite component group A_{ψ_v} . One of the required conditions for A_{ψ_v} is that: for almost all v , π_{ξ_v} is irreducible and unramified if $\xi_v = \mathbb{1}_v$. In this case, $\pi_{\mathbb{1}_v}$ is the unramified representation whose Satake parameter is given by

$$s_{\psi_v} = \psi_v \left(\mathrm{Fr}_v \times \begin{pmatrix} |\varpi_v|^{1/2} & \\ & |\varpi_v|^{-1/2} \end{pmatrix} \right),$$

where Fr_v and ϖ_v denote the the Frobenius element and local uniformizer for the field k_v . In fact, this condition will enable us to decide the parameter ψ , associated to which the residual spectrum computed can be interpreted as a near equivalence class.

Based on the (conjectural) knowledge of the local packet, we may form the global A-packet

$$A_\psi = \left\{ \pi_\xi = \bigotimes \pi_{\xi_v} : \xi = \bigotimes \xi_v \in \text{Irr}(\mathcal{S}_{\psi, \mathbf{A}}) \right\},$$

which are unitary representations indexed by irreducible characters ξ of the compact group $\mathcal{S}_{\psi, \mathbf{A}} := \prod_v \mathcal{S}_{\psi_v}$. Now the Arthur conjecture can be stated as

Conjecture. Denote by Δ the diagonal map $\mathcal{S}_\psi \rightarrow \mathcal{S}_{\psi, \mathbf{A}}$. To an A-parameter ψ we can associate a quadratic character $\epsilon_\psi : \mathcal{S}_\psi \rightarrow \{\pm 1\}$, such that we can define the representation space

$$L_\psi^2 = \bigoplus_{\xi} \langle \Delta^*(\xi), \epsilon \rangle_{\mathcal{S}_\psi} \pi_\xi,$$

where ξ is taken over all characters of $\mathcal{S}_{\psi, \mathbf{A}}$. Then the discrete spectrum of Mp_{2n} , when ψ ranges over all A-parameters, has the decomposition

$$L_{disc}^2(Mp_{2n}) = \bigoplus_{\psi} L_\psi^2.$$

6.2. Residual spectrum and near equivalence classes. Now we specialize to the case $n = 2$ and interpret the residual spectrum of Mp_4 as near equivalence classes attached to proper A-parameters. Certainly the above description of the Arthur conjecture provides ample information and redundancy for this purpose, since an analysis of the Satake parameter in each case will be sufficient to give the answer. For our purpose, we have the following types of discrete A-parameters $\psi : L_k \rightarrow \text{Sp}_4(\mathbf{C})$ as from [Art2, pg. 78].

A) **Tempered case.** Either $\psi = \phi$ (stable case) or $\psi = \phi \oplus \phi'$ (unstable), where for the latter ϕ and ϕ' are irreducible 2-dimensional representations. This case will be impertinent to our residual spectrum.

B) **Soudry type.** $\psi = \phi \boxtimes S_2$, where ϕ is an irreducible representation such that $L(s, \phi, \text{Sym}^2)$ has a pole at 1. Suppose ϕ corresponds to a cuspidal representation τ of GL_2 . For almost all v , we know π_{1_v} is unramified with a Satake parameter

$$\begin{pmatrix} \chi_v(\varpi_v) |\varpi_v|^{1/2} & & & \\ & \mu_v(\varpi_v) |\varpi_v|^{1/2} & & \\ & & \chi_v(\varpi_v) |\varpi_v|^{-1/2} & \\ & & & \mu_v(\varpi_v) |\varpi_v|^{-1/2} \end{pmatrix},$$

where $\tau_v = \text{Ind}_{\chi_v} \boxtimes \mu_v$ is an unramified representation. Here ϖ_v is any chosen uniformizer of the local field k_v . One can see that for almost all v , we expect $\pi_{1_v} \simeq J_{\overline{P}_1}(1/2, \overline{\tau}_v)$, and therefore the representation $J_{\overline{P}_1}(1/2, \overline{\tau})$ in Theorem 3.4 belongs to the near equivalence class associated to an A-parameter of Soudry type.

C) **Saito-Kurokawa type.** In this case, we have $\psi = \phi_2 \boxtimes S_1 \oplus \chi \boxtimes S_2, \chi^2 = \mathbb{1}$. Let $\pi = \bigotimes \pi_v$ be the cuspidal representation of PGL_2 that corresponds to ϕ_2 .

For almost all v , the local representation $\pi_v = \text{Ind}_{\mu_v} \boxtimes \mu_v^{-1}$ is unramified and therefore π_{1_v} has Satake parameter

$$\begin{pmatrix} \chi_v(\varpi_v) |\varpi_v|^{1/2} & & & \\ & \mu_v(\varpi_v) & & \\ & & \mu_v^{-1}(\varpi_v) & \\ & & & \chi_v(\varpi_v) |\varpi_v|^{-1/2} \end{pmatrix}.$$

Therefore, for almost all v one expects $\pi_{\mathbb{1}_v} \simeq J_\pi(1/2, \overline{\chi}_v \boxtimes \sigma_v)$, and thus $J_\pi(1/2, \overline{\chi} \boxtimes \sigma)$ in Theorem 4.5 belongs to the near equivalence class with respect to such ψ .

D) **Howe-Piatetski-Shapiro type**, $\psi = \chi \boxtimes S_2 \oplus \mu \boxtimes S_2, \chi \neq \mu, \chi^2 = \mu^2 = \mathbb{1}$. For almost all v , the Satake parameter for $\pi_{\mathbb{1}_v}$ is given by

$$\begin{pmatrix} \chi_v(\varpi_v)|\varpi_v|^{1/2} & & & \\ & \mu_v(\varpi_v)|\varpi_v|^{1/2} & & \\ & & \mu_v^{-1}(\varpi_v)|\varpi_v|^{-1/2} & \\ & & & \chi_v(\varpi_v)|\varpi_v|^{-1/2} \end{pmatrix}.$$

Therefore for these places $\pi_{\mathbb{1}_v}$ is isomorphic to

$$J_{\overline{B}}(\alpha_3, \overline{\chi}_v \boxtimes \overline{\mu}_v) = J_{\mathrm{ETF}}(1/2, \overline{\chi}_v \boxtimes \sigma_v).$$

The equality of the latter two holds because we can assume σ_v is the even Weil representation without loss of generality. In view of this, the residual spectra $J_{\overline{B}}(\alpha_3, \overline{\chi} \boxtimes \overline{\mu})$ in Theorem 5.10 and $J_{\mathrm{ETF}}(1/2, \overline{\chi} \boxtimes \sigma)$ in Theorem 4.5 both belong to the near equivalence class for a parameter of Howe-Piatetski-Shapiro type.

E) **Principal type**, $\psi = \chi \boxtimes S_4, \chi^2 = \mathbb{1}$. For almost all v , the Satake parameter for $\pi_{\mathbb{1}_v}$ is given by

$$\begin{pmatrix} \chi_v(\varpi_v)|\varpi_v|^{3/2} & & & \\ & \chi_v(\varpi_v)|\varpi_v|^{1/2} & & \\ & & \chi_v(\varpi_v)|\varpi_v|^{-1/2} & \\ & & & \chi_v(\varpi_v)|\varpi_v|^{-3/2} \end{pmatrix}.$$

For such places, we know that $\pi_{\mathbb{1}_v}$ is isomorphic to

$$J_{\mathrm{ETF}}(3/2, \overline{\chi}_v \boxtimes \sigma_v) = J_{\overline{B}}(\alpha_1/2 + \alpha_3, \overline{\chi}_v \boxtimes \overline{\mu}_v).$$

The equality is due to the same reason as in D). Therefore, the residual spectra $J_{\mathrm{ETF}}(3/2, \overline{\chi} \boxtimes \sigma)$ in Theorem 4.5 and $J_{\overline{B}}(\alpha_1/2 + \alpha_3, \overline{\chi} \boxtimes \overline{\mu})$ in Theorem 5.10 both belong to the near equivalence class associated to ψ of principal type. In fact, we know they are both nearly equivalent to the Mp_4 -analog of elementary theta functions ETF introduced before which occur in the discrete spectrum of Mp_2 .

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