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L^p BOUNDS FOR THE COMMUTATORS OF SINGULAR INTEGRALS AND MAXIMAL SINGULAR INTEGRALS WITH ROUGH KERNELS

YANPING CHEN AND YONG DING

ABSTRACT. The commutator of convolution type Calderon-Zygmund singular integral operators with rough kernels $p.v.\frac{\Omega(x)}{|x|^n}$ are studied. The authors established the $L^p\,(1 < p < \infty)$ boundedness of the commutators of singular integrals and maximal singular integrals with the kernel condition which is different from the condition $\Omega \in H^1(S^{n-1})$.

1. INTRODUCTION

The homogeneous singular integral operator T_Ω is defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy,$$

where $\Omega \in L^1(S^{n-1})$ satisfies the following conditions:

(a) Ω is homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, i.e.

$$\Omega(tx) = \Omega(x) \quad \text{for any } t > 0 \text{ and } \; x \in \mathbb{R}^n \setminus \{0\}. \tag{1.1}$$

(b) Ω has mean zero on S^{n-1}, the unit sphere in $\mathbb{R}^n$, i.e.

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0. \tag{1.2}$$

For a function $b \in L_{\rm loc}(\mathbb{R}^n)$, let A be a linear operator on some measurable function space. Then the commutator between A and b is defined by $[b, A]f(x) := b(x)Af(x) - A(bf)(x)$.

In 1965, Calderón [6] defined a commutator for the Hilbert transform H and a Lipshitz function b, which is connected closely the Cauchy integral along Lipschitz curves (see also [7]). Commutators have played an important role in harmonic analysis and PDE, for example in the theory of nondivergent elliptic equations with discontinuous coefficients (see [5], [8], [13], [14], [20]). Moreover, there is also an interesting connection between the nonlinear commutator, considered by Rochberg

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The first author is the corresponding author.

and Weiss in [16], and the Jacobian mapping of vector functions. They have been applied in the study of nonlinear partial differential equations (see [9], [27]).

In 1976, Coifman, Rochberg and Weiss [16] obtained a characterization of L^p-boundedness of the commutators $[b, R_j]$ generated by the Reisz transforms R_j ($j = 1, \cdots, n,$) and a BMO function b. As an application of this characterization, a decomposition theorem of the real Hardy space is given in this paper. Moreover, the authors in [16] proved also that if $\Omega \in \mathrm{Lip}(S^{n-1})$, then the commutator $[b, T_\Omega]$ for T_Ω and a BMO function b is bounded on L^p for $1 < p < \infty$, which is defined by

$$[b, T_\Omega]f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \, (b(x) - b(y))f(y)dy.$$

In the same paper, Coifman, Rochberg and Weiss [16] outlined a different approach, which is less direct but shows the close relationship between the weighted inequalities of the operator T and the weighted inequalities of the commutator $[b, T]$. In 1993, Alvarez, Bagby, Kurtz and Pérez [3] developed the idea of [16] and established a generalized boundedness criterion for the commutators of linear operators. The result of Alvarez, Bagby, Kurtz and Pérez (see [3], Theorem 2.13]) can be stated as follows.

Theorem A ([3])**.** *Let $1 < p < \infty$. If a linear operator T is bounded on $L^p(w)$ for all $w \in A_q$ $(1 < q < \infty)$, where A_q denote the weight class of Muckenhoupt, then for $b \in BMO$, $\|[b, T]f\|_{L^p} \le C\|b\|_{BMO}\|f\|_{L^p}$.*

Combining Theorem A with the well-known results by Duoandikoetxea [18] on the weighted L^p boundedness of the rough singular integral T_Ω, we know that if $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then $[b, T_\Omega]$ is bounded on L^p for $1 < p < \infty$. However, it is not clear up to now whether the operator T_Ω with $\Omega \in L^1 \setminus \bigcup_{q>1} L^q(S^{n-1})$ is bounded on $L^p(w)$ for $1 < p < \infty$ and all $w \in A_r$ $(1 < r < \infty)$. Hence, if $\Omega \in L^1 \setminus \bigcup_{q>1} L^q(S^{n-1})$, the L^p boundedness of $[b, T_\Omega]$ cannot be deduced from Theorem A.

The purpose of this paper is to give a sufficient condition which contains $\bigcup_{q>1} L^q(S^{n-1})$, such that the commutator of convolution operators are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This condition was introduced by Grafakos and Stefanov in [25], which is defined by

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y)| \left(\log \frac{1}{|\xi \cdot y|} \right)^{1+\alpha} d\sigma(y) < \infty, \tag{1.3}$$

where $\alpha > 0$ is a fixed constant. It is well known that

$$\bigcup_{q>1} L^q(S^{n-1}) \subset L \log^+ L(S^{n-1}) \subset H^1(S^{n-1}).$$

Let $F_\alpha(S^{n-1})$ denote the space of all integrable functions Ω on S^{n-1} satisfying (1.3). The examples in [25] show that there is the following relationship between $F_\alpha(S^{n-1})$ and $H^1(S^{n-1})$ (the Hardy space on S^{n-1}):

$$\bigcup_{q>1} L^q(S^{n-1}) \subset \bigcap_{\alpha>0} F_\alpha(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq \bigcup_{\alpha>0} F_\alpha(S^{n-1}).$$

Condition (1.3) above has been considered by many authors in the context of rough integral operators. One can consult [1], [2], [9], [10], [11], [12], [19], [26] among the numerous references, for its development and applications.

Now let us formulate our main results as follows.

Theorem 1. *Let* Ω *be a function in* $L^1(S^{n-1})$ *satisfying* (1.1) *and* (1.2). *If* $\Omega \in F_\alpha(S^{n-1})$ *for some* $\alpha > 1$, *then* $[b, T_\Omega]$ *extends to a bounded operator from* L^p *into itself for* $\frac{\alpha+1}{\alpha} < p < \alpha + 1$.

Corollary 1. *Let* Ω *be a function in* $L^1(S^{n-1})$ *satisfying* (1.1) *and* (1.2). *If* $\Omega \in \bigcap_{\alpha>1} F_\alpha(S^{n-1})$, *then* $[b, T_\Omega]$ *extends to a bounded operator from* L^p *into itself for* $1 < p < \infty$.

The proof of this result is in Section 4. In the proof of Theorem 1, we have used the Littlewood-Paley decomposition and interpolation theorem argument to prove $L^p\,(1 < p < \infty)$ norm inequalities for the rough commutator $[b, T_\Omega]$. These techniques have been used to prove the $L^p\,(1 < p < \infty)$ norm inequalities for rough singular integrals in [25] or [17]. They are very similar in spirit, though not in detail. In the following, we will point out the difference in the methods used to prove $L^p\,(1 < p < \infty)$ norm inequalities for rough commutators and rough singular integrals.

Let T be a linear operator; we may decompose $T = \sum_{l\in\mathbb{Z}} T_l$ by using the properties of Littlewood-Paley functions and Fourier transform, reducing T to a sequence of composition operators $\{T_l\}_{l\in\mathbb{Z}}$. Hence, to get the $L^p\,(1 < p < \infty)$ norm of T, it suffices to establish the delicate $L^p\,(1 < p < \infty)$ norm of each T_l with a summation convergence factor, which can be obtained by interpolating between the delicate L^2 norm of T_l, which has a summation convergent factor, and the $L^q\,(1 < q < \infty)$ norm of T_l, for each $l \in \mathbb{Z}$.

Let T be a rough singular integral. The delicate L^2 norm of each T_l can be obtained by using the Fourier transform, the Plancherel theorem and the Littlewood-Paley theory. The $L^q\,(1 < q < \infty)$ norm of each T_l can be obtained by the method of rotations, the $L^q\,(1 < q < \infty)$ bounds of the one dimensional case of the Hardy-Littlewood operator and the Littlewood-Paley theory.

On the other hand, if T is a rough commutator of singular integral, the delicate L^2 norm of each T_l can be obtained by using the L^2 norm of the commutators of Littlewood-Paley operators (see Lemma 3.3) and Lemma 3.4 in Section 3. With these techniques and lemmas, G. Hu [29] obtained the result in Theorem 1 for $p = 2$. Therefore, it reduces the $L^p\,(1 < p < \infty)$ norm of T to the $L^q\,(1 < q < \infty)$ norm of T_l for each $l \in \mathbb{Z}$. Unfortunately, since each T_l is generated by a BMO function and a composition operator, the method of rotations, which deals with the same problem in rough singular integrals, fails to treat this problem directly. Hence we need to look for a new idea. We find that the Bony paraproduct is the key technique to resolve the problem. In particular, it is worth pointing out that the main method used in this paper indeed gives a new application of the Bony paraproduct. It is well known that the Bony paraproduct is an important tool in PDE. However, the idea presented in this paper shows that the Bony paraproduct is also a powerful tool for handling the integral operators with rough kernels in harmonic analysis.

It is well known that maximal singular integral operators T^*_Ω play a key role in studying the almost everywhere convergence of the singular integral operators. The mapping properties of the maximal singular integrals with convolution kernels have been extensively studied (see [17], [25], [32], for example). Therefore, another aim of this paper is to give the $L^p(\mathbb{R}^n)$ boundedness of the maximal commutator $[b, T^*_\Omega]$

associated to the singular integral T_Ω, which is defined by

$$[b, T_\Omega^*]f(x) = \sup_{j\in\mathbb{Z}} \left| \int_{|x-y|>2^j} \frac{\Omega(x-y)}{|x-y|^n}(b(x)-b(y))f(y)\,dy \right|.$$

The following theorem is another main result given in this paper:

Theorem 2. *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (1.1) *and* (1.2). *If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 2$, then $[b, T_\Omega^*]$ extends to a bounded operator from L^p into itself for $\frac{\alpha}{\alpha-1} < p < \alpha$.*

Corollary 2. *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (1.1) *and* (1.2). *If $\Omega \in \bigcap_{\alpha>2} F_\alpha(S^{n-1})$, then $[b, T_\Omega^*]$ extends to a bounded operator from L^p into itself for $1 < p < \infty$.*

One will see that the maximal commutator $[b, T_\Omega^*]$ can be controlled pointwise by some composition operators of T_Ω, M, M_Ω and their commutators $[b, T_\Omega]$, $[b, M]$ and $[b, M_\Omega]$, where M is the standard Hardy-Littlewood maximal operator and M_Ω denotes the maximal operator with rough kernel, which is defined by

$$M_\Omega f(x) = \sup_{j\in\mathbb{Z}} \left| \int_{2^j<|x-y|\le 2^{j+1}} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right|.$$

The corresponding commutators $[b, M]$ and $[b, M_\Omega]$ are defined by

$$[b, M]f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |b(x)-b(y)||f(y)|\,dy$$

and

$$[b, M_\Omega]f(x) = \sup_{j\in\mathbb{Z}} \left| \int\limits_{2^j<|x-y|<2^{j+1}} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right|.$$

We give the following $L^p(\mathbb{R}^n)$ boundedness of the commutators $[b, M_\Omega]$:

Theorem 3. *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (1.1). *If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 1$, then $[b, M_\Omega]$ extends to a bounded operator from L^p into itself for $\frac{\alpha+1}{\alpha} < p < \alpha+1$.*

Corollary 3. *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (1.1). *If*

$$\Omega \in \bigcap_{\alpha>1} F_\alpha(S^{n-1}),$$

then $[b, M_\Omega]$ extends to a bounded operator from L^p into itself for $1 < p < \infty$.

Theorem 3 is actually a direct consequence of the $L^p(\mathbb{R}^n)$ boundedness of the commutator formed by a class of the Littlewood-Paley square operator with rough kernel and a BMO function. In fact, if $\widetilde{\Omega} = \Omega - \frac{A}{|S^{n-1}|}$ with $A = \int_{S^{n-1}} \Omega(x')d\sigma(x')$, then $\widetilde{\Omega}$ satisfies (1.2). It is easy to check that

$$\begin{aligned} [b, M_\Omega]f(x) &\le \sup_{j\in\mathbb{Z}} \left| \int\limits_{2^j<|x-y|<2^{j+1}} (b(x)-b(y)) \frac{\widetilde{\Omega}(x-y)}{|x-y|^n} f(y)\,dy \right| \\ &\quad + C[b, M]f(x) \\ &\le C([b, g_{\widetilde{\Omega}}]f(x) + [b, M]f(x)), \end{aligned} \tag{1.4}$$

where g_Ω and $[b, g_\Omega]$ denote the Littlewood-Paley square operator and its commutator, which are defined respectively by

$$g_\Omega f(x) = \left(\sum_{j\in\mathbb{Z}} \left| \int_{2^j<|x-y|\le 2^{j+1}} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right|^2 \right)^{1/2}$$

and

$$[b, g_\Omega]f(x) = \left(\sum_{j\in\mathbb{Z}} \left| \int_{2^j<|x-y|<2^{j+1}} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right|^2 \right)^{1/2}.$$

Thus, (1.4) shows that Theorem 3 will follow from the $L^p(\mathbb{R}^n)$ boundedness of the commutators $[b, g_\Omega]$ and $[b, M]$. Since the $L^p(\mathbb{R}^n)$ boundedness of the latter is well known (see [23]), we need only give the $L^p(\mathbb{R}^n)$ boundedness of the commutator $[b, g_\Omega]$, which can be stated as follows.

Theorem 4. *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (1.1) *and* (1.2). *If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 1$, then $[b, g_\Omega]$ extends to a bounded operator from L^p into itself for $\frac{\alpha+1}{\alpha} < p < \alpha + 1$.*

Corollary 4. *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (1.1) *and* (1.2). *If $\Omega \in \bigcap_{\alpha>1} F_\alpha(S^{n-1})$, then $[b, g_\Omega]$ extends to a bounded operator from L^p into itself for $1 < p < \infty$.*

In fact, Theorem 4 is a corollary of Theorem 1. Write $T_\Omega f(x) = \sum_{j\in\mathbb{Z}} K_j * f(x)$, where $K_j(x) = \frac{\Omega(x)}{|x|^n}\chi_{\{2^j<|x|\le 2^{j+1}\}}$. Define $T_j f(x) = K_j * f(x)$; then $[b, T_\Omega]f(x) = \sum_{j\in\mathbb{Z}}[b, T_j]f(x)$ and $[b, g_\Omega]f(x) = \left(\sum_{j\in\mathbb{Z}} |[b, T_j]f(x)|^2\right)^{1/2}$. Then we get the L^p boundedness of $[b, g_\Omega]$ by using Theorem 1, the Rademacher function and Khintchine's inequalities.

This paper is organized as follows. First, in Section 2, we give some important notation and tools, which will be used in the proofs of the main results. In Section 3, we give some lemmas which will be used in the proofs of the main results. In Section 4, we prove Theorem 1 by applying the lemmas in Section 3. Finally, we prove Theorem 2 by applying Theorem 3 and Theorem 4 in Section 5. Throughout this paper, the letter "C" will stand for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence.

2. Notation and preliminaries

Let us begin by giving some notation and important tools, which will be used in the proofs of our main results.

1. *Schwartz class and Fourier transform.* Denote by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ the Schwartz class and the space of tempered distributions, respectively. The notation "$\widehat{\ }$" and "$^\vee$" denote the Fourier transform and the inverse Fourier transform, respectively.

2. *Smooth decomposition of identity and multipliers.* Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ be a radial function satisfying $0 \le \varphi \le 1$ with its support in the unit ball and $\varphi(\xi) = 1$ for $|\xi| \le \frac{1}{2}$. The function $\psi(\xi) = \varphi(\frac{\xi}{2}) - \varphi(\xi) \in \mathscr{S}(\mathbb{R}^n)$ is supported by $\{\frac{1}{2} \le |\xi| \le 2\}$ and satisfies the identity $\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1$, for $\xi \ne 0$.

For $j \in \mathbb{Z}$, denote by Δ_j and G_j the convolution operators whose the symbols are $\psi(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$, respectively. That is, Δ_j and G_j are defined by $\widehat{\Delta_j f}(\xi) = \psi(2^{-j}\xi)\hat{f}(\xi)$ and $\widehat{G_j f}(\xi) = \varphi(2^{-j}\xi)\hat{f}(\xi)$ (see [30]). By the Littlewood-Paley theory, for $1 < p < \infty$ and $\{f_j\} \in L^p(l^2)$, the following vector-valued inequality holds (see [24], p. 343]):

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|\Delta_{j+k}f_j|^2\Big)^{1/2}\Big\|_{L^p} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^2\Big)^{1/2}\Big\|_{L^p}, \quad \text{for} \quad k \in [-10,10]. \tag{2.1}$$

3. *Homogeneous Triebel-Lizorkin space $\dot{F}_p^{s,q}(\mathbb{R}^n)$ and Besov space $\dot{B}_p^{s,q}(\mathbb{R}^n)$.* For $0 < p,\, q \le \infty$ ($p \ne \infty$) and $s \in \mathbb{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{s,q}(\mathbb{R}^n)$ is defined by

$$\dot{F}_p^{s,q}(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \ \|f\|_{\dot{F}_p^{s,q}} = \left\| \left(\sum_{j\in\mathbb{Z}} 2^{-jsq}|\Delta_j f|^q \right)^{1/q} \right\|_{L^p} < \infty \right\}$$

and the homogeneous Besov space $\dot{B}_p^{s,q}(\mathbb{R}^n)$ is defined by

$$\dot{B}_p^{s,q}(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \ \|f\|_{\dot{B}_p^{s,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{-jsq}\|\Delta_j f\|_{L^p}^q \right)^{1/q} < \infty \right\},$$

where $\mathscr{S}'(\mathbb{R}^n)$ denotes the tempered distribution class on $\mathbb{R}^n$.

4. *Sequence Carleson measures.* A sequence of positive Borel measures $\{v_j\}_{j\in\mathbb{Z}}$ is called a sequence Carleson measure in $\mathbb{R}^n \times \mathbb{Z}$ if there exists a positive constant $C > 0$ such that $\sum_{j\ge k} v_j(B) \le C|B|$ for all $k \in \mathbb{Z}$ and all Euclidean balls B with radius 2^{-k}, where $|B|$ is the Lebesgue measure of B. The norm of the sequence Carleson measure $v = \{v_j\}_{j\in\mathbb{Z}}$ is given by

$$\|v\| = \sup\left\{ \frac{1}{|B|}\sum_{j\ge k} v_j(B) \right\},$$

where the supremum is taken over all $k \in \mathbb{Z}$ and all balls B with radius 2^{-k}.

5. *Homogeneous BMO-Triebel-Lizorkin space.* For $s \in \mathbb{R}$ and $1 \le q < +\infty$, the homogeneous BMO -Triebel-Lizorkin space $\dot{F}_\infty^{s,q}$ is the space of all distributions b for which the sequence $\{2^{sjq}|\Delta_j(b)(x)|^q dx\}_{j\in\mathbb{Z}}$ is a Carleson measure (see [21]). The norm of b in $\dot{F}_\infty^{s,q}$ is given by

$$\|b\|_{\dot{F}_\infty^{s,q}} = \sup\left[\frac{1}{|B|}\sum_{j\ge k}\int_B 2^{sjq}|\Delta_j(b)(x)|^q dx \right]^{\frac{1}{q}},$$

where the supremum is taken over all $k \in \mathbb{Z}$ and all balls B with radius 2^{-k}. For $q = +\infty$, we set $\dot{F}_\infty^{s,\infty} = \dot{B}_\infty^{s,\infty}$. Moreover, $\dot{F}_\infty^{0,2} = BMO$ (see [21], [22]).

6. *Bony paraproduct and Bony decomposition.* The paraproduct of Bony [4] between two functions f, g is defined by

$$\pi_f(g) = \sum_{j\in\mathbb{Z}}(\Delta_j f)(G_{j-3}g).$$

At least formally, we have the following Bony decomposition:

$$fg = \pi_f(g) + \pi_g(f) + R(f,g) \quad \text{with} \quad R(f,g) = \sum_{i\in\mathbb{Z}}\sum_{|k-i|\le 2}(\Delta_i f)(\Delta_k g). \tag{2.2}$$

3. Lemmas

We first give some lemmas, which will be used in the proof of Theorem 1 and Theorem 2.

Riesz potential and its inverse. For $0 < \tau < n$, the Riesz potential I_τ of order τ is defined on $\mathscr{S}'(\mathbb{R}^n)$ by setting $\widehat{I_\tau f}(\xi) = |\xi|^{-\tau}\widehat{f}(\xi)$. Another expression of I_τ is

$$I_\tau f(x) = \gamma(\tau)\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\tau}}dy,$$

where $\gamma(\tau) = 2^{-\tau}\pi^{-n/2}\Gamma(\frac{n-\tau}{2})/\Gamma(\frac{\tau}{2})$. Moreover, for $0 < \tau < n$, the "*inverse operator*" I_τ^{-1} of I_τ is defined by $\widehat{I_\tau^{-1} f}(\xi) = |\xi|^{\tau}\widehat{f}(\xi)$, where $\wedge$ denotes the Fourier transform.

With the notation above, we show the following two facts:

Lemma 3.1. *For $0 < \tau < 1/2$, we have*

$$\gamma(\tau) \le C\tau, \tag{3.1}$$

where C is independent of τ.

Proof. Applying Stirling's formula, we have

$$\sqrt{2\pi}x^{x-1/2}e^{-x} \le \Gamma(x) \le 2\sqrt{2\pi}x^{x-1/2}e^{-x} \quad \text{for} \quad x > 1.$$

Thus, by the equation $s\Gamma(s) = \Gamma(s+1)$ for $s > 0$, we get

$$\Gamma(\tfrac{n-\tau}{2}) = \tfrac{2}{n-\tau}\Gamma(\tfrac{n-\tau}{2}+1) \le 2\sqrt{2\pi}\Big(\tfrac{n-\tau}{2}+1\Big)^{(\frac{n-\tau}{2}+\frac{1}{2})} e^{-\frac{(n-\tau)}{2}-1}\cdot \tfrac{2}{n-\tau} \le C \tag{3.2}$$

and

$$\Gamma(\tfrac{\tau}{2}) = \tfrac{2}{\tau}\Gamma(\tfrac{\tau}{2}+1) \ge \sqrt{2\pi}\Big(\tfrac{\tau}{2}+1\Big)^{(\frac{\tau}{2}+\frac{1}{2})} e^{-\frac{\tau}{2}-1}\cdot \tfrac{2}{\tau} \ge C/\tau. \tag{3.3}$$

Hence, (3.1) follows from (3.2) and (3.3). Obviously, the constant C in (3.1) is independent of τ.

Lemma 3.2. *For the multiplier G_k ($k \in \mathbb{Z}$), $b \in BMO(\mathbb{R}^n)$, and any fixed $0 < \tau < 1/2$, we have*

$$|G_k b(x) - G_k b(y)| \le C\frac{2^{k\tau}}{\tau}|x-y|^{\tau}\|b\|_{BMO}, \tag{3.4}$$

where C is independent of k and τ.

Proof. Note that $I_\tau(I_\tau^{-1}f) = f$; we have

$$G_k b(x) = \gamma(\tau)\int_{\mathbb{R}^n} \frac{I_\tau^{-1}(G_k b)(z)}{|x-z|^{n-\tau}}\,dz.$$

Hence

$$\begin{aligned} |G_k b(x) - G_k b(y)| &= \left|\gamma(\tau)\int_{\mathbb{R}^n} I_\tau^{-1}(G_k b)(z)\Big(\frac{1}{|x-z|^{n-\tau}} - \tfrac{1}{|y-z|^{n-\tau}}\Big)\,dz\right| \\ &\le \gamma(\tau)\|I_\tau^{-1}(G_k b)\|_{L^\infty}\int_{\mathbb{R}^n}\left|\frac{1}{|x-z|^{n-\tau}} - \tfrac{1}{|y-z|^{n-\tau}}\right|dz \\ &= \gamma(\tau)\|I_\tau^{-1}(G_k b)\|_{L^\infty}\int_{\mathbb{R}^n}\left|\frac{1}{|x-y+z|^{n-\tau}} - \tfrac{1}{|z|^{n-\tau}}\right|dz. \end{aligned} \tag{3.5}$$

We first show that

$$\left\| \frac{1}{|x-y+\cdot|^{n-\tau}} - \frac{1}{|\cdot|^{n-\tau}} \right\|_{L^1} \le C\tau^{-1}|x-y|^{\tau}. \tag{3.6}$$

In fact,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \frac{1}{|x-y+z|^{n-\tau}} - \tfrac{1}{|z|^{n-\tau}} \right| dz \\
&= \int_{|z|\le 2|x-y|} \left| \frac{1}{|x-y+z|^{n-\tau}} - \tfrac{1}{|z|^{n-\tau}} \right| dz \\
&\quad + \int_{|z|>2|x-y|} \left| \frac{1}{|x-y+z|^{n-\tau}} - \tfrac{1}{|z|^{n-\tau}} \right| dz \\
&\le \int_{|z|\le 3|x-y|} \frac{1}{|z|^{n-\tau}}\, dz + \int_{|z|\le 2|x-y|} \tfrac{1}{|z|^{n-\tau}}\, dz + C\int_{|z|>2|x-y|} \frac{|x-y|}{|z|^{n-\tau+1}}\, dz \\
&\le C\frac{|x-y|^{\tau}}{\tau},
\end{aligned}$$

where C is independent of τ. By (3.5), (3.6) and (3.1), we get

$$|G_k b(x) - G_k b(y)| \le C|x-y|^{\tau} \|I_\tau^{-1}(G_k b)\|_{L^\infty}, \tag{3.7}$$

where C is independent of τ. We now estimate $\|I_\tau^{-1}(G_k b)\|_{L^\infty}$. Since $G_k \Delta_u b = 0$ for $u \ge k+1$, we have

$$\begin{aligned}
\|I_\tau^{-1}(G_k b)\|_{L^\infty} &= \Big\| I_\tau^{-1} G_k \big(\sum_{u\in\mathbb{Z}} \Delta_u b \big) \Big\|_{L^\infty} \le \sum_{u\le k+1} \big\| G_k (I_\tau^{-1} \Delta_u b) \big\|_{L^\infty} \\
&\le \sum_{u\le k+1} \big\| I_\tau^{-1} \Delta_u b \big\|_{L^\infty}.
\end{aligned} \tag{3.8}$$

Take a radial function $\widetilde{\psi} \in \mathscr{S}(\mathbb{R}^n)$ such that $\operatorname{supp}(\widetilde{\psi}) \subset \{1/4 \le |x| \le 4\}$ and $\widetilde{\psi} = 1$ in $\{1/2 \le |x| \le 2\}$. Then we have

$$\widehat{I_\tau^{-1}\Delta_u} b(\xi) = 2^{u\tau} \widetilde{\psi}(2^{-u}\xi) |2^{-u}\xi|^{\tau} \widehat{\Delta_u b}(\xi).$$

Set a function h by $\widehat{h}(\xi) = \widetilde{\psi}(\xi)|\xi|^{\tau}$. Then

$$I_\tau^{-1}\Delta_u b(x) = 2^{u\tau} \int_{\mathbb{R}^n} 2^{un} h(2^u(x-y)) \Delta_u b(y)\, dy.$$

So we have

$$\|I_\tau^{-1}\Delta_u b\|_{L^\infty} \le 2^{u\tau} \|2^{un} h(2^u \cdot)\|_{L^1} \|\Delta_u b\|_{L^\infty} = 2^{u\tau} \|h\|_{L^1} \|\Delta_u b\|_{L^\infty}.$$

Thus, if there exists a constant $C > 0$, independent of τ, such that

$$\|h\|_{L^1} \le C, \tag{3.9}$$

then by (3.7)-(3.8), we have

$$\begin{aligned}
|G_k b(x) - G_k b(y)| &\le C|x-y|^{\tau} \textstyle\sum_{u\le k+1} 2^{u\tau} \|\Delta_u b\|_{L^\infty} \\
&\le C|x-y|^{\tau} 2^{k\tau} \sup_{u\in\mathbb{Z}} \|\Delta_u b\|_{L^\infty} \textstyle\sum_{u\le k+1} 2^{(u-k)\tau}.
\end{aligned}$$

Since for some $0 < \tau < 1$,

$$\sum_{u \le k+1} 2^{(u-k)\tau} = \sum_{j=-1}^{\infty} 2^{-j\tau} = \frac{2^\tau}{1-2^{-\tau}} = \frac{2^{2\tau}}{2^\tau - 1} = \frac{2^{2\tau}}{\tau 2^{\theta\tau}} < \frac{C}{\tau}, \quad \text{for} \quad 0 < \tau < 1/2,$$

where C is independent of τ. Using the fact (see [24], p. 615]) that

$$\sup_{u \in \mathbb{Z}} \|\Delta_u b\|_{L^\infty} \le C_n \|b\|_{BMO}, \tag{3.10}$$

we have

$$|G_k b(x) - G_k b(y)| \le C \tfrac{|x-y|^\tau 2^{k\tau}}{\tau} \|b\|_{BMO},$$

where C is independent of k and τ. Thus, to finish the proof of Lemma 3.2, it remains to show (3.9). In fact,

$$\|h\|_{L^1} = \int\limits_{|x|<1} |h(x)|dx + \int\limits_{|x|\ge 1} |h(x)|dx \le C_n\big(\|h\|_{L^2} + \||\cdot|^n h(\cdot)\|_{L^2}\big) := C_n(I_1 + I_2).$$

Since $\operatorname{supp}(\widetilde{\psi}) \subset \{1/4 \le |\xi| \le 4\}$ and $0 < \delta < 1/2$, we get

$$I_1 = \|\widetilde{\psi}(\xi)|\xi|^\tau\|_{L^2} \le C,$$

where C is independent of τ. Thus, to get (3.9), we need only verify that $I_2 \le C$. To do this, let us recall some notation about the multi-index. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, denote $\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$. By [24, p. 425], we know that

$$(1+|\xi|^2)^{n/2} = \sum_{|\alpha| \le n} \frac{n!}{\alpha_1! \dots \alpha_n!} \xi^\alpha \frac{\xi^\alpha}{(1+|\xi|^2)^{n/2}}$$

and the function $m_\alpha(\xi) = \frac{\xi^\alpha}{(1+|\xi|^2)^{n/2}}$ is an $L^p\,(1 < p < \infty)$ multiplier whenever $|\alpha| \le n$. Hence

$$\begin{aligned}\big((1+|\xi|^2)^{n/2} h(\xi)\big)^\vee &= \sum_{|\alpha| \le n} C_{\alpha,n} (m_\alpha(\xi) \xi^\alpha h(\xi))^\vee \\ &= C \sum_{|\alpha| \le n} C_{\alpha,n} (m_\alpha(\xi) \widehat{\partial^\alpha \check{h}}(\xi))^\vee,\end{aligned}$$

where $\vee$ denotes the inverse Fourier transform. Applying the equation above, we get

$$\begin{aligned} I_2 &\le C\|(1+|\xi|^2)^{n/2} h(\xi)\|_{L^2} = \|((1+|\xi|^2)^{n/2} h(\xi))^\vee\|_{L^2} \\ &\le C \sum\nolimits_{|\alpha| \le n} C_{\alpha,n} \|\widehat{\partial^\alpha \check{h}}\|_{L^2} \\ &= C \sum\nolimits_{|\alpha| \le n} C_{\alpha,n} \|\partial^\alpha \check{h}\|_{L^2} = C \sum\nolimits_{|\alpha| \le n} C_{\alpha,n} \|\partial^\alpha(\widetilde{\psi}(\xi)|\xi|^\tau)\|_{L^2}. \end{aligned}$$

Notice that

$$\partial^\alpha(\widetilde{\psi}(\xi)|\xi|^\tau) = \sum\nolimits_{\beta \le \alpha} C_{\alpha_1}^{\beta_1} \dots C_{\alpha_n}^{\beta_n} (\partial^\beta \widetilde{\psi}(\xi))(\partial^{\alpha-\beta}(|\xi|^\tau)), \tag{3.11}$$

where the sum in (3.11) is taken over all multi-indices β with $0 \le \beta_j \le \alpha_j$ for all $1 \le j \le n$. Trivial computations show that there exists $C > 0$, independent of

τ, such that $|\partial^{\alpha-\beta}(|\xi|^{\tau})| \le C$ for $1/4 < |\xi| < 4$ and $0 < \tau < 1/2$. Further, by $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^n)$, then $|\partial^\beta \widetilde{\psi}(\xi)| \le C$. So we get $|\partial^\alpha(\psi(\xi)|\xi|^\tau)| \le C$. From this we get

$$I_2 \le \sum\nolimits_{|\alpha|\le n} C_{\alpha,n} \|\partial^\alpha(\psi(\xi)|\xi|^\tau)\|_{L^\infty} \left(\int_{1/4\le|\xi|\le 4} d\xi \right)^{1/2} \le C,$$

where C is dependent only on n, but is independent of τ. This completes the estimate of (3.9) and Lemma 3.2 follows.

Lemma 3.3 (see [28])**.** *Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a radial function such that* $\operatorname{supp}\phi \subset \{1/2 \le |\xi| \le 2\}$ *and $\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1$ for $|\xi| \ne 0$. Define the multiplier operator S_l by $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi)$ and S_l^2 by $S_l^2 f = S_l(S_l f)$. For $b \in BMO(\mathbb{R}^n)$, denote by $[b, S_l]$ (respectively, $[b, S_l^2]$) the commutator of S_l (respectively, S_l^2). Then for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, we have*

(i) $\left\| \left(\sum_{l\in\mathbb{Z}} |[b, S_l](f)|^2 \right)^{1/2} \right\|_{L^p} \le C(n,p)\|b\|_{BMO}\|f\|_{L^p}$;

(ii) $\left\| \left(\sum_{l\in\mathbb{Z}} |[b, S_l^2](f)|^2 \right)^{1/2} \right\|_{L^p} \le C(n,p)\|b\|_{BMO}\|f\|_{L^p}$;

(iii) $\left\| | \sum_{l\in\mathbb{Z}} [b, S_l](f_l) | \right\|_{L^p} \le C(n,p)\|b\|_{BMO} \left\| \left(\sum_{l\in\mathbb{Z}} |f_l|^2 \right)^{1/2} \right\|_{L^p}$, $\{f_l\} \in L^p(l^2)$.

Lemma 3.4 (see [29])**.** *Let $m_\sigma \in C_0^\infty(\mathbb{R}^n)$ $(0 < \sigma < \infty)$ be a family of multipliers such that* $\operatorname{supp}(m_\sigma) \subset \{|\xi| \le 2\sigma\}$, *and for some constants C, $0 < A \le 1/2$, and $\alpha > 0$,*

$$\|m_\sigma\|_{L^\infty} \le C \min\{A\sigma, \log^{-\alpha-1}(2+\sigma)\}, \quad \|\nabla m_\sigma\|_{L^\infty} \le C.$$

Let T_σ be the multiplier operator defined by

$$\widehat{T_\sigma f}(\xi) = m_\sigma(\xi)\widehat{f}(\xi).$$

For $b \in BMO$, denote by $[b, T_\sigma]$ the commutator of T_σ. Then for any fixed $0 < v < 1$, there exists a positive constant $C = C(n, v)$ such that

$$\|[b, T_\sigma]f\|_{L^2} \le C(A\sigma)^v \log(1/A)\|b\|_{BMO}\|f\|_{L^2}, \text{ if } \sigma < 10/\sqrt{A},$$

$$\|[b, T_\sigma]f\|_{L^2} \le C \log^{-(\alpha+1)v+1}(2+\sigma)\|b\|_{BMO}\|f\|_{L^2}, \text{ if } \sigma \ge 10/\sqrt{A}.$$

Similar to the proof of Lemma 3.4, it is easy to get

Lemma 3.5. *Let $m_\sigma \in C_0^\infty(\mathbb{R}^n)$ $(0 < \sigma < \infty)$ be a family of multipliers such that* $\operatorname{supp}(m_\sigma) \subset \{|\xi| \le 2\sigma\}$, *and for some constants C, $0 < A \le 1/2$, and $\alpha > 0$, $j \in \mathbb{N}$,*

$$\|m_\sigma\|_{L^\infty} \le C \min\{A2^{-j}\sigma, \log^{-\alpha-1}(2+2^j\sigma)\}, \quad \|\nabla m_\sigma\|_{L^\infty} \le C2^j.$$

Let T_σ be the multiplier operator defined by

$$\widehat{T_\sigma f}(\xi) = m_\sigma(\xi)\widehat{f}(\xi).$$

For $b \in BMO$, denote by $[b, T_\sigma]$ the commutator of T_σ. Then for any fixed $0 < v < 1$, there exists a positive constant $C = C(n, v)$, $0 < \beta < 1$, such that

$$\|[b, T_\sigma]f\|_{L^2} \le C2^{-\beta j}(A\sigma)^v \log(1/A)\|b\|_{BMO}\|f\|_{L^2}, \text{ if } \sigma < 10/\sqrt{A},$$

$$\|[b, T_\sigma]f\|_{L^2} \le C \log^{-(\alpha+1)v+1}(2+2^j\sigma)\|b\|_{BMO}\|f\|_{L^2}, \text{ if } \sigma \ge 10/\sqrt{A}.$$

Lemma 3.6. *For any $j \in \mathbb{Z}$, let $K_j(x) = \frac{\Omega(x)}{|x|^n}\chi_{\{2^j<|x|\le 2^{j+1}\}}(x)$. Suppose $\Omega \in L^1(S^{n-1})$ satisfying (1.1). Then for $1 < p < \infty$, the vector-valued inequality*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}||K_j|*|f_j||^2\Big)^{1/2}\Big\|_{L^p} \le C_p\|\Omega\|_{L^1}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^2\Big)^{1/2}\Big\|_{L^p}$$

holds for any $\{f_j\}$ in $L^p(l^2)$.

Proof. Note that for $\Omega \in L^1(S^{n-1})$ and any local integrable function f on $\mathbb{R}^n$, we have

$$\sigma^*(f)(x) := \sup_{j\in\mathbb{Z}}||K_j|*f(x)| \le CM_\Omega f(x) \quad \text{for any} \quad x \in \mathbb{R}^n,$$

where

$$M_\Omega f(x) = \sup_{r>0}\frac{1}{r^n}\int_{|x-y|<r}|\Omega(x-y)||f(y)|\,dy. \tag{3.12}$$

By the L^q boundedness of M_Ω for all $q > 1$ with $\Omega \in L^1(S^{n-1})$, σ^* is also a bounded operator on $L^q(\mathbb{R}^n)$ for all $q > 1$ with $\Omega \in L^1(S^{n-1})$. Thus, by applying the lemma in [17, p.544], we know that, for $1 < p < \infty$, the vector-valued inequality (3.12) holds.

Lemma 3.7. *For any $j \in \mathbb{Z}$, define the operator T_j by $T_jf = K_j * f$, where $K_j(x) = \frac{\Omega(x)}{|x|^n}\chi_{\{2^j<|x|\le 2^{j+1}\}}(x)$. Denote by $[b, S_{l-j}T_jS^2_{l-j}]$ the commutator of $S_{l-j}T_jS^2_{l-j}$. Suppose $\Omega \in L^1(S^{n-1})$ satisfying (1.1). Then for any fixed $0 < \tau < 1/2$, $b \in BMO(\mathbb{R}^n)$, $1 < p < \infty$,*

$$\Big\|\sum_{j\in\mathbb{Z}}[b, S_{l-j}T_jS^2_{l-j}]f\Big\|_{L^p} \le C\|b\|_{BMO}\max\{\frac{2^{\tau l}}{\tau}, 2\}\|\Omega\|_{L^1}\|f\|_{L^p}, \tag{3.13}$$

where C is independent of τ and l.

Proof. For any $j, l \in \mathbb{Z}$, we may write

$$[b, S_{l-j}T_jS^2_{l-j}]f \quad = [b, S_{l-j}](T_jS^2_{l-j}f) + S_{l-j}[b, T_j](S^2_{l-j}f) + S_{l-j}T_j([b, S^2_{l-j}]f).$$

Thus,

$$\begin{aligned}\Big\|\sum\nolimits_{j\in\mathbb{Z}}[b, S_{l-j}T_jS^2_{l-j}]f\Big\|_{L^p} &\le \Big\|\sum\nolimits_{j\in\mathbb{Z}}[b, S_{l-j}](T_jS^2_{l-j}f)\Big\|_{L^p} \\ &\quad+\Big\|\sum\nolimits_{j\in\mathbb{Z}}S_{l-j}T_j([b, S^2_{l-j}]f)\Big\|_{L^p} \\ &\quad+\Big\|\sum\nolimits_{j\in\mathbb{Z}}S_{l-j}[b, T_j](S^2_{l-j}f)\Big\|_{L^p} \\ &:= L_1 + L_2 + L_3.\end{aligned} \tag{3.14}$$

Below we shall estimate L_i for $i = 1, 2, 3$, respectively. For L_1, by Lemma 3.3 (iii), Lemma 3.6 and the Littlewood-Paley theory, we have

$$\begin{aligned}L_1 &\le C\|b\|_{BMO}\Big\|\big(\sum\nolimits_{j\in\mathbb{Z}}|T_jS^2_{l-j}f|^2\big)^{1/2}\Big\|_{L^p} \\ &\le C\|\Omega\|_{L^1}\|b\|_{BMO}\Big\|\big(\sum\nolimits_{j\in\mathbb{Z}}|S^2_jf|^2\big)^{1/2}\Big\|_{L^p} \le C\|\Omega\|_{L^1}\|b\|_{BMO}\|f\|_{L^p}.\end{aligned}$$

Similarly, we have $L_2 \le C\|\Omega\|_{L^1}\|b\|_{BMO}\|f\|_{L^p}$.

Hence, by (3.14), to show (3.12) it remains to give the estimate of L_3. We will apply the Bony paraproduct to do this. By (2.2), we have

$$\begin{aligned}[b,T_j]S^2_{l-j}f(x) &= b(x)(T_jS^2_{l-j}f)(x) - T_j(bS^2_{l-j}f)(x)\\ &= [\pi_{(T_jS^2_{l-j}f)}(b)(x) - T_j(\pi_{(S^2_{l-j}f)}(b))(x)]\\ &\quad +[R(b,T_jS^2_{l-j}f)(x) - T_j(R(b,S^2_{l-j}f))(x)]\\ &\quad +[\pi_b(T_jS^2_{l-j}f)(x) - T_j(\pi_b(S^2_{l-j}f))(x)].\end{aligned}$$

Thus

$$\begin{aligned}L_3 &\le \Big\|\sum_{j\in\mathbb{Z}} S_{l-j}\big[\pi_{(T_jS^2_{l-j}f)}(b) - T_j(\pi_{(S^2_{l-j}f)}(b))\big]\Big\|_{L^p}\\ &\quad +\Big\|\sum_{j\in\mathbb{Z}} S_{l-j}\big[R(b,T_jS^2_{l-j}f) - T_j(R(b,S^2_{l-j}f))\big]\Big\|_{L^p}\\ &\quad +\Big\|\sum_{j\in\mathbb{Z}} S_{l-j}\big[\pi_b(T_jS^2_{l-j}f) - T_j(\pi_b(S^2_{l-j}f))\big]\Big\|_{L^p}\\ &:= M_1+M_2+M_3.\end{aligned}\tag{3.15}$$

(a) *The estimate of* M_1. For M_1, by $\Delta_iS_{l-j}g = 0$ for $g\in\mathscr{S}'(\mathbb{R}^n)$ when $|i-(l-j)|\ge 3$, we get

$$\begin{aligned}&\pi_{(T_jS^2_{l-j}f)}(b)(x) - T_j(\pi_{(S^2_{l-j}f)}(b))(x)\\ &= \sum_{|i-(l-j)|\le 2}\{(T_j\Delta_iS^2_{l-j}f)(x)(G_{i-3}b)(x) - T_j[(\Delta_iS^2_{l-j}f)(G_{i-3}b)](x)\}\\ &= \sum_{|i-(l-j)|\le 2}[G_{i-3}b,T_j](\Delta_iS^2_{l-j}f)(x).\end{aligned}\tag{3.16}$$

Note that

$$\begin{aligned}&[G_{i-3}b,T_j](\Delta_iS^2_{l-j}f)(x)|\\ &\quad= \Big|\int\limits_{2^j\le|x-y|<2^{j+1}}\frac{\Omega(x-y)}{|x-y|^n}(G_{i-3}b(x)-G_{i-3}b(y))\Delta_iS^2_{l-j}f(y)dy\Big|\\ &\quad\le C\int\limits_{2^j\le|x-y|<2^{j+1}}\frac{|\Omega(x-y)|}{|x-y|^n}|G_{i-3}b(x)-G_{i-3}b(y)||\Delta_iS^2_{l-j}f(y)|dy.\end{aligned}\tag{3.17}$$

By Lemma 3.2, we have

$$\begin{aligned}&|[G_{i-3}b,T_j]\Delta_iS^2_{l-j}f(x)|\\ &\quad\le C\frac{2^{i\tau}}{\tau}\|b\|_{BMO}\int\limits_{2^j\le|x-y|<2^{j+1}}\frac{|\Omega(x-y)|}{|x-y|^n}|x-y|^\tau\,|\Delta_iS^2_{l-j}f(y)|dy\\ &\quad\le C\frac{2^{(i+j)\tau}}{\tau}\|b\|_{BMO}\int\limits_{2^j\le|x-y|<2^{j+1}}\frac{|\Omega(x-y)|}{|x-y|^n}\,|\Delta_iS^2_{l-j}f(y)|dy\\ &\quad= C\frac{2^{(i+j)\tau}}{\tau}\|b\|_{BMO}T_{|\Omega|,j}(|\Delta_iS^2_{l-j}f|)(x),\end{aligned}\tag{3.18}$$

where

$$T_{|\Omega|,j}f(x) = \int\limits_{2^j\le|x-y|<2^{j+1}}\frac{|\Omega(x-y)|}{|x-y|^n}f(y)dy.$$

Then, by (3.16), (3.18) and applying Lemma 3.6, (2.1) and the Littlewood-Paley theory, we have that, for any fixed $0 < \tau < 1/2$,
(3.19)

$$\begin{aligned} M_1 &\le C\frac{2^{\tau l}}{\tau}\|b\|_{BMO}\sum\nolimits_{|k|\le 2}\Big\|\Big(\sum\nolimits_{j\in\mathbb{Z}}|T_{|\Omega|,j}(|\Delta_{l-j+k}S^2_{l-j}f|)|^2\Big)^{1/2}\Big\|_{L^p} \\ &\le C\frac{2^{\tau l}}{\tau}\|b\|_{BMO}\|\Omega\|_{L^1}\sum\nolimits_{|k|\le 2}\Big\|\big(\sum\nolimits_{j\in\mathbb{Z}}|\Delta_{j+k}S^2_j f|^2\big)^{1/2}\Big\|_{L^p} \\ &\le C\frac{2^{\tau l}}{\tau}\|b\|_{BMO}\|\Omega\|_{L^1}\Big\|\big(\sum\nolimits_{j\in\mathbb{Z}}|S_j f|^2\big)^{1/2}\Big\|_{L^p} \\ &\le C\frac{2^{\tau l}}{\tau}\|b\|_{BMO}\|\Omega\|_{L^1}\|f\|_{L^p}, \end{aligned}$$

where C is independent of l and τ.

(b) *The estimate of* M_2. Since $|k| \le 2$, $\Delta_{i+k}S_{l-j}g = 0$ for $g \in \mathscr{S}'(\mathbb{R}^n)$ when $|i-(l-j)| \ge 8$. Thus

$$\begin{aligned} &R(b, T_jS_{l-j}f) - T_j(R(b,S_{l-j}f))(x) \\ &\quad= \sum_{i\in\mathbb{Z}}\sum_{|k|\le 2}(\Delta_i b)(x)(T_j\Delta_{i+k}S_{l-j}f)(x) - T_j\Big(\sum_{i\in\mathbb{Z}}\sum_{|k|\le 2}(\Delta_i b)(\Delta_{i+k}S_{l-j}f)\Big)(x) \\ &\quad= \sum_{k=-2}^{2}\sum_{|i-(l-j)|\le 7}\Big((\Delta_i b)(x)(T_j\Delta_{i+k}S_{l-j}f)(x) - T_j\big((\Delta_i b)(\Delta_{i+k}S_{l-j}f)\big)(x)\Big) \\ &\quad= \sum_{k=-2}^{2}\sum_{|i-(l-j)|\le 7}[\Delta_i b, T_j](\Delta_{i+k}S_{l-j}f)(x). \end{aligned}$$

By the equality above and using Lemma 3.6, (2.1), (3.10) and the Littlewood-Paley theory, we have

$$\begin{aligned} M_2 &\le C\|\Omega\|_{L^1}\sup_{i\in\mathbb{Z}}\|\Delta_i(b)\|_{L^\infty}\sum_{|k|\le 7}\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_{j,|\Omega|}(|\Delta_{l-j+k}S_{l-j}f|)|^2\Big)^{1/2}\Big\|_{L^p} \\ &\le C\|b\|_{BMO}\|\Omega\|_{L^1}\Big\|\big(\sum_{j\in\mathbb{Z}}|S^2_j f|^2\big)^{1/2}\Big\|_{L^p} \\ &\le C\|b\|_{BMO}\|\Omega\|_{L^1}\|f\|_{L^p}. \end{aligned} \tag{3.20}$$

(c) *The estimate of* M_3. Finally, we give the estimate of M_3. Note that $S_{l-j}\big((\Delta_i g)(G_{i-3}h)\big) = 0$ for g, $h \in \mathscr{S}'(\mathbb{R}^n)$ if $|i-(l-j)| \ge 5$. Thus we get

$$\begin{aligned} &S_{l-j}\big(\pi_b(T_jS_{l-j}f) - T_j(\pi_b(S_{l-j}f))\big) \\ &= S_{l-j}\Big(\sum_{i\in\mathbb{Z}}(\Delta_i b)(G_{i-3}T_jS_{l-j}f) - T_j\big(\sum_{i\in\mathbb{Z}}(\Delta_i b)(G_{i-3}S_{l-j}f)\big)\Big)(x) \\ &= \sum_{|i-(l-j)|\le 4}\Big\{S_{l-j}\big((\Delta_i b)(G_{i-3}T_jS_{l-j}f)\big)(x) - S_{l-j}T_j\big((\Delta_i b)(G_{i-3}S_{l-j}f)\big)(x)\Big\} \\ &= \sum_{|i-(l-j)|\le 4}S_{l-j}\big([\Delta_i b, T_j](G_{i-3}S_{l-j}f)\big). \end{aligned}$$

Applying Proposition 5.1.4 in [24, p. 343], it is easy to see that

$$\Big\|\big(\sum_{j\in\mathbb{Z}}|G_{j+k}S_jf_j|^2\big)^{1/2}\Big\|_{L^p} \le \Big\|\big(\sum_{j\in\mathbb{Z}}|f_j|^2\big)^{1/2}\Big\|_{L^p} \quad \text{for} \quad k \in [-10,10].$$

Thus, by the Littlewood-Paley theory, Lemma 3.6 and (3.10) we get
(3.21)

$$\begin{aligned} M_3 &\le C\sup_{i\in\mathbb{Z}}\|\Delta_i(b)\|_{L^\infty}\sum_{|k|\le 4}\Big\|\Big(\sum\nolimits_{j\in\mathbb{Z}}|T_{|\Omega|,j}(|G_{l-j+k-3}S_{l-j}f|)|^2\Big)^{1/2}\Big\|_{L^p}\\ &\le C\|b\|_{BMO}\|\Omega\|_{L^1}\Big\|\big(\sum\nolimits_{j\in\mathbb{Z}}|S_jf|^2\big)^{1/2}\Big\|_{L^p}\\ &\le C\|b\|_{BMO}\|\Omega\|_{L^1}\|f\|_{L^p}. \end{aligned}$$

By (3.15), (3.19)-(3.21), we get

$$L_3\le C\max\{2,\frac{2^{\tau l}}{\tau}\}\|b\|_{BMO}\|\Omega\|_{L^1}\|f\|_{L^p}\quad \text{for}\quad l\in\mathbb{Z}.$$

Combining this with (3.14), we complete the proof of (3.13).

4. Proof of Theorem 1

Let $\phi\in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $0\le\phi\le 1$, supp$\phi\subset\{1/2\le|\xi|\le 2\}$ and

$$\sum_{l\in\mathbb{Z}}\phi^3(2^{-l}\xi)=1,\quad |\xi|\neq 0.$$

Define the multiplier operator S_l by

$$\widehat{S_lf}(\xi)=\phi(2^{-l}\xi)\widehat{f}(\xi).$$

Let $K_j(x)=\frac{\Omega(x)}{|x|^n}\chi_{\{2^j<|x|\le 2^{j+1}\}}$. Define the operator

$$T_jf(x)=K_j*f(x)=\int_{2^j<|y|\le 2^{j+1}}\frac{\Omega(y)}{|y|^n}f(x-y)\,dy,$$

and the multiplier T_j^l by $\widehat{T_j^lf}(\xi)=\widehat{T_jS_{l-j}}f(\xi)=\phi(2^{j-l}\xi)\widehat{K_j}(\xi)\widehat{f}(\xi)$. With the notation above, it is easy to see that

$$\begin{aligned}[][b,T_\Omega]f(x)&=\sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}[b,S_{l-j}T_jS_{l-j}^2]f(x)\\ &=\sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}[b,S_{l-j}T_j^lS_{l-j}]f(x):=\sum_{l\in\mathbb{Z}}V_lf(x),\end{aligned}$$

where $V_lf(x)=\sum_{j\in\mathbb{Z}}[b,S_{l-j}T_j^lS_{l-j}]f(x)$. Then by the Minkowski inequality, we get

$$\|[b,T_\Omega]f\|_{L^p}\le\Big\|\sum\nolimits_{l=-\infty}^{[\log\sqrt{2}]}V_lf\Big\|_{L^p}+\Big\|\sum\nolimits_{l=[\log\sqrt{2}]+1}^{\infty}V_lf\Big\|_{L^p}. \tag{4.1}$$

Now, we will estimate the two cases respectively.

Case 1. The estimate of $\Big\|\sum\nolimits_{l=-\infty}^{[\log\sqrt{2}]}V_lf\Big\|_{L^p}$.

Since $\Omega\in L^1(S^{n-1})$ satisfies (1.1) and (1.2), by a well-known Fourier transform estimate of Duoandikoetxea and Rubio de Francia (See [17, pp. 551-552]), it is easy to show that

$$|\widehat{K_j}(\xi)|\le C\|\Omega\|_{L^1}|2^j\xi|.$$

A trivial computation gives that

$$\|\nabla\widehat{K_j}\|_{L^\infty}\le C2^j\|\Omega\|_{L^1}.$$

Set $m_j(\xi)=\widehat{K}_j(\xi)$, $m_j^l(\xi)=m_j(\xi)\phi(2^{j-l}\xi)$, and recall T_j^l by

$$\widehat{T_j^l f}(\xi)=m_j^l(\xi)\widehat{f}(\xi).$$

Straightforward computations lead to

$$\|m_j^l(2^{-j}\cdot)\|_{L^\infty}\le C\|\Omega\|_{L^1}2^l,\quad \|\nabla m_j^l(2^{-j}\cdot)\|_{L^\infty}\le C\|\Omega\|_{L^1},$$

$$\mathrm{supp}\{m_j^l(2^{-j}\xi)\}\subset\{|\xi|\le 2^{l+2}\}.$$

Let $\widetilde{T}_j^l$ be the operator defined by

$$\widehat{\widetilde{T}_j^l f}(\xi)=m_j^l(2^{-j}\xi)\widehat{f}(\xi).$$

Denote $T_{j;b,1}^l f=[b,T_j^l]f$ and $T_{j;b,0}^l f=T_j^l f$. Similarly, denote $\widetilde{T}_{j;b,1}^l f=[b,\widetilde{T}_j^l]f$ and $\widetilde{T}_{j;b,0}^l f=\widetilde{T}_j^l f$. Thus via the Plancherel theorem and Lemma 3.4 we state that for any fixed $0<v<1$, $k\in\{0,1\}$,

$$\|\widetilde{T}_{j;b,k}^l f\|_{L^2}\le C\|b\|_{BMO}^k\|\Omega\|_{L^1}2^{vl}\|f\|_{L^2},\quad l\le[\log\sqrt{2}].$$

Dilation-invariance says that

$$\|T_{j;b,k}^l f\|_{L^2}\le C\|b\|_{BMO}^k\|\Omega\|_{L^1}2^{vl}\|f\|_{L^2},\quad l\le[\log\sqrt{2}]. \tag{4.2}$$

First, we will give the L^2 norm estimate of $V_l f$ by using inequality (4.2). Recalling that $V_l f(x)=\sum_{j\in\mathbb{Z}}[b,S_{l-j}T_j^l S_{l-j}]f(x)$, for any $j,l\in\mathbb{Z}$, we may write

$$[b,S_{l-j}T_j^l S_{l-j}]f\quad=[b,S_{l-j}](T_j^l S_{l-j}f)+S_{l-j}[b,T_j^l](S_{l-j}f)+S_{l-j}T_j^l([b,S_{l-j}]f).$$

Thus,

$$\begin{aligned}\|V_l f\|_{L^2}&\le\Big\|\sum_{j\in\mathbb{Z}}[b,S_{l-j}](T_j^l S_{l-j}f)\Big\|_{L^2}+\Big\|\sum_{j\in\mathbb{Z}}S_{l-j}T_j^l([b,S_{l-j}]f)\Big\|_{L^2}\\&\quad+\Big\|\sum_{j\in\mathbb{Z}}S_{l-j}[b,T_j^l](S_{l-j}f)\Big\|_{L^2}\\&:=Q_1+Q_2+Q_3.\end{aligned} \tag{4.3}$$

For Q_1, by Lemma 3.3(iii), (4.2) for $k=0$ and the Littlewood-Paley theory, we get

$$\begin{aligned}Q_1&\le C\|b\|_{BMO}\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_j^l S_{l-j}f|^2\Big)^{1/2}\Big\|_{L^2}\\&\le C\|b\|_{BMO}2^{vl}\|\Omega\|_{L^1}\Big\|\Big(\sum_{j\in\mathbb{Z}}|S_{l-j}f|^2\Big)^{1/2}\Big\|_{L^2}\\&\le C\|b\|_{BMO}2^{vl}\|\Omega\|_{L^1}\|f\|_{L^2}.\end{aligned} \tag{4.4}$$

For Q_2, by the Littlewood-Paley theory, (4.2) for $k=0$ and Lemma 3.3(i), we get

$$\begin{aligned}Q_2&\le C\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_j^l([b,S_{l-j}]f)|^2\Big)^{1/2}\Big\|_{L^2}\\&\le C2^{vl}\|\Omega\|_{L^1}\Big\|\Big(\sum_{j\in\mathbb{Z}}|[b,S_{l-j}]f|^2\Big)^{1/2}\Big\|_{L^2}\\&\le C\|b\|_{BMO}2^{vl}\|\Omega\|_{L^1}\|f\|_{L^2}.\end{aligned} \tag{4.5}$$

Regarding Q_3, by (4.2) for $k=1$ and the Littlewood-Paley theory, we have

$$\begin{aligned} Q_3 &\le C\Big\|\Big(\sum_{j\in\mathbb{Z}}|[b,T_j^l](S_{l-j}f)|^2\Big)^{1/2}\Big\|_{L^2} \\ &\le C2^{vl}\|\Omega\|_{L^1}\Big\|\Big(\sum_{j\in\mathbb{Z}}|S_{l-j}f|^2\Big)^{1/2}\Big\|_{L^2} \\ &\le C\|b\|_{BMO}2^{vl}\|\Omega\|_{L^1}\|f\|_{L^2}. \end{aligned} \tag{4.6}$$

Combining (4.4) with (4.5) and (4.6), we have

$$\|V_lf\|_{L^2}\le C\|b\|_{BMO}2^{vl}\|\Omega\|_{L^1}\|f\|_{L^2},\qquad l\le[\log\sqrt{2}]. \tag{4.7}$$

On the other hand, since $T_j^lf(x)=T_jS_{l-j}f(x)$, then

$$V_lf(x)=\sum_{j\in\mathbb{Z}}[b,S_{l-j}T_jS_{l-j}^2]f(x).$$

Applying Lemma 3.7, we get for $1<p<\infty$

$$\|V_lf\|_{L^p}\le C\|b\|_{BMO}\|\Omega\|_{L^1}\|f\|_{L^p},\qquad l\le[\log\sqrt{2}]. \tag{4.8}$$

Interpolating between (4.7) and (4.8), there exists a constant $0<\beta<1$, such that

$$\|V_lf\|_{L^p}\le C2^{\beta vl}\|\Omega\|_{L^1}\|b\|_{BMO}\|f\|_{L^p},\qquad l\le[\log\sqrt{2}]. \tag{4.9}$$

Then by the Minkowski inequality, we get for $1<p<\infty$

$$\begin{aligned} \Big\|\sum_{l=-\infty}^{[\log\sqrt{2}]}V_lf\Big\|_{L^p} &\le \sum_{l=-\infty}^{[\log\sqrt{2}]}\|V_lf\|_{L^p} \\ &\le C\sum_{l=-\infty}^{[\log\sqrt{2}]}2^{\beta vl}\|b\|_{BMO}\|\Omega\|_{L^1}\|f\|_{L^p} \\ &\le C\|b\|_{BMO}\|\Omega\|_{L^1}\|f\|_{L^p}. \end{aligned} \tag{4.10}$$

Case 2. The estimate of $\Big\|\sum_{l=1+[\log\sqrt{2}]}^{\infty}V_lf\Big\|_{L^p}$.

Recall that $V_lf(x)=\sum_{j\in\mathbb{Z}}[b,S_{l-j}T_jS_{l-j}^2]f(x)$. We will give the delicate L^2 norm of V_lf and the L^p $(1<p<\infty)$ norm of V_lf respectively. It is easy to see that if $\Omega\in F_\alpha(S^{n-1})$ for $\alpha>1$ satisfies (1.1) and (1.2),

$$|\widehat{K_j}(\xi)|\le C\log^{-\alpha-1}(|2^j\xi|+2),\qquad \|\nabla\widehat{K_j}\|_{L^\infty}\le C2^j.$$

Set $m_j(\xi)=\widehat{K_j}(\xi)$, $m_j^l(\xi)=\phi(2^{j-l}\xi)m_j(\xi)$. Let T_j^l be the operator defined by $\widehat{T_j^lf}(\xi)=m_j^l(\xi)\hat{f}(\xi)$. Straightforward computations lead to

$$\|m_j^l(2^{-j}\cdot)\|_{L^\infty}\le C\log^{-\alpha-1}(2+2^l),\quad \|\nabla m_j^l(2^{-j}\cdot)\|_{L^\infty}\le C,$$
$$\operatorname{supp}\{m_j^l(2^{-j}\xi)\}\subset\{|\xi|\le 2^{l+2}\}.$$

Let $\widetilde{T}_j^l$ be the operator defined by

$$\widehat{\widetilde{T}_j^lf}(\xi)=m_j^l(2^{-j}\xi)\hat{f}(\xi).$$

Denote $T_{j;b,1}^lf=[b,T_j^l]f$ and $T_{j;b,0}^lf=T_j^lf$. Similarly, denote $\widetilde{T}_{j;b,1}^lf=[b,\widetilde{T}_j^l]f$ and $\widetilde{T}_{j;b,0}^lf=\widetilde{T}_j^lf$. Thus via the Plancherel theorem and Lemma 3.4 with $\sigma=2^l$ we state that for any fixed $0<v<1$, $k\in\{0,1\}$,

$$\|\widetilde{T}_{j;b,k}^lf\|_{L^2}\le C\|b\|_{BMO}^kC\log^{(-\alpha-1)v+1}(2+2^l)\|f\|_{L^2},\quad l\ge 1+[\log\sqrt{2}]. \tag{4.11}$$

Dilation-invariance says that

$$\|T^l_{j;b,k}f\|_{L^2} \le C\|b\|^k_{BMO}\log^{(-\alpha-1)v+1}(2+2^l)\|f\|_{L^2}, \quad l \ge 1+[\log\sqrt{2}]. \tag{4.12}$$

Applying (4.12), Lemma 3.3 and the Littlewood-Paley theory, similar to the proof of (4.7), we get

$$\|V_l f\|_{L^2} \le C\|b\|_{BMO}\log^{(-\alpha-1)v+1}(2+2^l)\|f\|_{L^2}, \qquad l \ge 1+[\log\sqrt{2}]. \tag{4.13}$$

On the other hand, by Lemma 3.7, for any fixed $0<\tau<1/2$, $1<p<\infty$,

$$\|V_l f\|_{L^p} \le C\|b\|_{BMO}\frac{2^{\tau l}}{\tau}\|\Omega\|_{L^1}\|f\|_{L^p}, \qquad l \ge 1+[\log\sqrt{2}],$$

where C is independent of τ and l. Take $\tau = 1/l$; we get

$$\|V_l f\|_{L^p} \le Cl\|b\|_{BMO}\|\Omega\|_{L^1}\|f\|_{L^p}, \qquad l \ge 1+[\log\sqrt{2}],$$

where C is independent of l. This says that for any r satisfying $1<r<\infty$, we have

$$\|V_l f\|_{L^r} \le Cl\|b\|_{BMO}\|f\|_{L^r}, \quad l \ge 1+[\log\sqrt{2}]. \tag{4.14}$$

Now for any $p \ge 2$, we take r sufficient large such that $r>p$. Using the Riesz-Thorin interpolation theorem between (4.13) and (4.14), we have that for any $l \ge 1+[\log\sqrt{2}]$,

$$\|V_l f\|_{L^p} \le C\|b\|_{BMO} l^{1-\theta}\log^{((-\alpha-1)v+1)\theta}(2+2^l)\|f\|_{L^p},$$

where

$$\theta = \frac{2(r-p)}{p(r-2)}.$$

We can see that if $r \mapsto \infty$, then θ goes to $2/p$ and $\log^{((-\alpha-1)v+1)\theta}(2+2^l)$ goes to $\log^{((-\alpha-1)v+1)2/p}(2+2^l)$. Therefore, we get

$$\|V_l f\|_{L^p} \le C\|b\|_{BMO} l^{1-2/p}\log^{((-\alpha-1)v+1)\frac{2}{p}}(2+2^l)\|f\|_{L^p}, \quad l \ge 1+[\log\sqrt{2}], \quad p \ge 2. \tag{4.15}$$

Then by the Minkowski inequality, for $2 \le p < \alpha+1$, we get

$$\begin{aligned} &\Big\|\sum\nolimits_{l=1+[\log\sqrt{2}]}^{\infty} V_l f\Big\|_{L^p} \\ &\le C\|b\|_{BMO}\sum\nolimits_{l=1+[\log\sqrt{2}]}^{\infty} l^{1-2/p} l^{((-\alpha-1)v+1)\frac{2}{p}}\|f\|_{L^p} \\ &\le C\|b\|_{BMO}\|f\|_{L^p}. \end{aligned} \tag{4.16}$$

If $1<p<2$, by duality, we get for $p > \frac{\alpha+1}{\alpha}$

$$\Big\|\sum\nolimits_{l=1+[\log\sqrt{2}]}^{\infty} V_l f\Big\|_{L^p} \le C\|b\|_{BMO}\|f\|_{L^p}. \tag{4.17}$$

Combining (4.16) with (4.17), we get for $\frac{\alpha+1}{\alpha} < p < \alpha+1$,

$$\Big\|\sum\nolimits_{l=1+[\log\sqrt{2}]}^{\infty} V_l f\Big\|_{L^p} \le C\|b\|_{BMO}\|f\|_{L^p}.$$

This completes the proof of Theorem 1.

5. Proof of Theorem 2

Let $\alpha > 2$, K_j and the operator T_j be the same as in the proof of Theorem 1. Define

$$\begin{aligned}[b, T_\Omega^s]f(x) &= \int_{|x-y|>2^s} (b(x)-b(y))\frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy \\ &= \sum\nolimits_{j=s}^{\infty} \int_{2^j<|x-y|\le 2^{j+1}} (b(x)-b(y))\frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy \\ &= \sum\nolimits_{j=s}^{\infty} [b, T_j]f(x),\end{aligned}$$

where

$$[b, T_j]f(x) = \int_{2^j\le|x-y|\le 2^{j+1}} (b(x)-b(y))\frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy. \tag{5.1}$$

So, we get

$$\sup_{s>0} |[b, T_\Omega^s]f(x)| \quad \le \sup_{s\in\mathbb{Z}} \left| \sum\nolimits_{j=s}^{\infty} [b, T_j]f(x)\right|.$$

To prove Theorem 2, it suffices to estimate the L^p norm of $\sup_{s\in\mathbb{Z}}\left| \sum\nolimits_{j=s}^{\infty} [b, T_j]f(x)\right|$. Take a radial Schwartz function Φ such that $\widehat{\Phi}(\xi) = 1$ for $|\xi| \le 1$ and $\widehat{\Phi}(\xi) = 0$ for $|\xi| > 2$, and define Φ_s by $\widehat{\Phi_s}(\xi) = \widehat{\Phi}(2^s\xi)$. Write

$$\begin{aligned}\sum\nolimits_{j=s}^{\infty} [b, T_j]f(x) &= \left[\Phi_s * \left([b, T_\Omega]f - \sum\nolimits_{j=-\infty}^{s-1} [b, T_j]f\right)(x)\right] \\ &\quad + \left[\sum\nolimits_{j=s}^{\infty} [b, T_j]f(x) - \Phi_s * \left(\sum\nolimits_{j=s}^{\infty} [b, T_j]f\right)(x)\right] \\ &:= L_s f(x) + J_s f(x).\end{aligned}$$

Observed that

$$\Phi_s * \left(\sum_{j=-\infty}^{s-1} [b, T_j]f\right)(x) = [b, \Phi_s * \sum_{j=-\infty}^{s-1} K_j]f(x) - [b, W_s]\left(\sum_{j=-\infty}^{s-1} T_j f\right)(x),$$

where W_s is a convolution operator with its convolution kernel Φ_s. Observe that

$$\left|\Phi_s * \sum_{j=-\infty}^{s-1} K_j(x)\right| \le C\|\Omega\|_{L^1} 2^{-ns}/(1 + |2^{-s}x|^{n+1})$$

(see [17]) and $\sum\nolimits_{j=-\infty}^{s-1} T_j f(x) = T_\Omega f(x) - \sum_{j=s}^{\infty} T_j f(x)$. It follows that

$$\sup_{s\in\mathbb{Z}} |L_s f(x)| \le CM([b, T_\Omega]f)(x) + C[b, M]f(x) + [b, M](T_\Omega f)(x) + [b, M](T_\Omega^* f)(x).$$

Then by Theorem 1, the L^p ($\frac{\alpha}{\alpha-1} < p < \alpha$) boundedness of T_Ω, T_Ω^* with kernel function $\Omega \in F_\alpha$ for $\alpha > 2$ (see [25]) and $[b, M]$ (see [23]), we get for $\frac{\alpha}{\alpha-1} < p < \alpha$,

$$\|\sup_{s\in\mathbb{Z}} |L_s f|\|_{L^p} \le C\|b\|_{BMO}\|f\|_{L^p}. \tag{5.2}$$

To estimate $\sup_{s\in\mathbb{Z}}|J_sf(x)|$, write

$$\sum_{j=s}^{\infty}[b,T_j]f(x)-\Phi_s*\Big(\sum_{j=s}^{\infty}[b,T_j]f\Big)(x)=\sum_{j=s}^{\infty}[b,T_j]f(x)-[b,\Phi_s*\sum_{j=s}^{\infty}K_j]f(x)$$
$$+[b,W_s]\Big(\sum_{j=s}^{\infty}T_jf\Big)(x).$$

Thus we get

$$\sup_{s\in\mathbb{Z}}|J_sf(x)|\le\sup_{s\in\mathbb{Z}}\Big|\sum_{j=s}^{\infty}[b,(\delta-\Phi_s)*K_j]f(x)\Big|+[b,M](T_\Omega^*f)(x),$$

where δ is a Dirac mass at the origin. Since $\frac{\alpha}{\alpha-1}<p<\alpha$ (see [25]),

$$\|[b,M](T_\Omega^*f)\|_{L^p}\le C\|b\|_{BMO}\|f\|_{L^p}. \tag{5.3}$$

Thus, to give the estimate of the L^p norm for the term $\sup_{s\in\mathbb{Z}}|J_sf(x)|$, it suffices to give the estimate of the L^p norm for the term $\sup_{s\in\mathbb{Z}}\Big|\sum_{j=s}^{\infty}[b,(\delta-\Phi_s)*K_j]f(x)\Big|$. Note that

$$\sup_{s\in\mathbb{Z}}\Big|\sum_{j=s}^{\infty}[b,(\delta-\Phi_s)*K_j]f(x)\Big|\quad\le\sum_{j=0}^{\infty}\sup_{s\in\mathbb{Z}}|[b,(\delta-\Phi_s)*K_{j+s}]f(x)|.$$

Let $U_{s,j}f(x)=(\delta-\Phi_s)*K_{s+j}*f$ and $[b,U_{s,j}]f(x)=[b,(\delta-\Phi_s)*K_{s+j}]f$. Then

$$\sup_{s\in\mathbb{Z}}\Big|\sum_{j=s}^{\infty}[b,(\delta-\Phi_s)*K_j]f(x)\Big|\quad\le\sum_{j=0}^{\infty}\sup_{s\in\mathbb{Z}}|[b,U_{s,j}]f(x)|. \tag{5.4}$$

It is easy to see that

$$\begin{aligned}&\sup_{s\in\mathbb{Z}}|[b,U_{s,j}]f(x)|\\
&\le C\sup_{s\in\mathbb{Z}}|[b,T_{s+j}]f(x)|+C\sup_{s\in\mathbb{Z}}\Big(W_s|[b,T_{s+j}]f|+C[b,W_s](T_{s+j}f)\Big)(x)\\
&\le C\sup_{s\in\mathbb{Z}}|[b,T_{s+j}]f(x)|+CM(\sup_{s\in\mathbb{Z}}|[b,T_{s+j}]f|)(x)+C[b,M](M_\Omega f)(x)\\
&\le C[b,M_\Omega]f(x)+CM([b,M_\Omega]f)(x)+C[b,M](M_\Omega f)(x).\end{aligned}$$

Applying Theorem 3, the $L^p\,(1<p<\infty)$ boundedness of M, M_Ω with kernel function $\Omega\in L^1(S^{n-1})$ (see [24]) and $[b,M]$ (see [23]), we have for $\frac{\alpha}{\alpha-1}<p<\alpha$,

$$\|\sup_{s\in\mathbb{Z}}|[b,U_{s,j}]f|\|_{L^p}\quad\le C(\|[b,M_\Omega]f\|_{L^p}+\|b\|_{BMO}\|M_\Omega f\|_{L^p})\le C\|b\|_{BMO}\|f\|_{L^p}. \tag{5.5}$$

On the other hand, set

$$B_{s,j}(\xi)=(1-\widehat{\Phi}_s(\xi))\widehat{K}_{s+j}(\xi),\qquad B_{s,j}^l(\xi)=(1-\widehat{\Phi}_s(\xi))\widehat{K}_{s+j}(\xi)\phi(2^{s-l}\xi).$$

Define the operator $U^l_{s,j}$ by $\widehat{U^l_{s,j}f}(\xi)=\widehat{U_{s,j}f}(\xi)\phi(2^{s-l}\xi)$, and denote by $[b,U^l_{s,j}]$ the commutator of $U^l_{s,j}$. Then it is clear that

$$[b,U_{s,j}]f(x)=\sum_{l\in\mathbb{Z}}[b,U^l_{s,j}S^2_{l-s}]f(x).$$

By the Minkowski inequality, we get

$$\begin{aligned}\|\sup_{s\in\mathbb{Z}}|[b,U_{s,j}]f|\|_{L^2} &\le \Big\|\Big(\sum\nolimits_{s\in\mathbb{Z}}|[b,U_{s,j}]f|^2\Big)^{1/2}\Big\|_{L^2}\\ &\le \Big\|\Big(\sum\nolimits_{s\in\mathbb{Z}}|\sum\nolimits_{l\in\mathbb{Z}}[b,U^l_{s,j}S^2_{l-s}]f|^2\Big)^{1/2}\Big\|_{L^2}\\ &\le \sum\nolimits_{l\in\mathbb{Z}}\Big\|\Big(\sum\nolimits_{s\in\mathbb{Z}}|[b,U^l_{s,j}]S^2_{l-s}f|^2\Big)^{1/2}\Big\|_{L^2}\\ &\quad+\sum\nolimits_{l\in\mathbb{Z}}\Big\|\Big(\sum\nolimits_{s\in\mathbb{Z}}|U^l_{s,j}[b,S^2_{l-s}]f|^2\Big)^{1/2}\Big\|_{L^2}\\ &:=I_1+I_2.\end{aligned}\tag{5.6}$$

To complete the proof we will estimate each term separately. Denote $U^l_{s,j;b,1}f=[b,U^l_{s,j}]f$ and $U^l_{s,j;b,0}f=U^l_{s,j}f$. Obviously, if we can prove that for any $0<v<1$, $k\in\{0,1\}$, there exists a constant $0<\beta<1$, such that

$$\|U^l_{s,j;b,k}f\|_{L^2}\le C2^{-\beta j}\|b\|^k_{BMO}2^l\|f\|_{L^2},\qquad \text{for}\quad l\le[\log\sqrt{2}]\tag{5.7}$$

and

(5.7′)

$$\|U^l_{s,j;b,k}f\|_{L^2}\le C\|b\|^k_{BMO}\log^{(-\alpha-1)v+1}(2^{l+j}+2)\|f\|_{L^2},\qquad \text{for}\quad l\ge[\log\sqrt{2}],+1,$$

then we may finish the estimate of I_1 and I_2. We first consider I_1. In fact, by (5.7) and (5.7′) for $k=1$ and the Littlewood-Paley theory, we get

$$\begin{aligned}I_1 &\le \sum_{l=-\infty}^{[\log\sqrt{2}]}\Big\|\Big(\sum_{s\in\mathbb{Z}}|[b,U^l_{s,j}]S^2_{l-s}f|^2\Big)^{1/2}\Big\|_{L^2}\\ &\quad+\sum_{l=[\log\sqrt{2}]+1}^{\infty}\Big\|\Big(\sum_{s\in\mathbb{Z}}|[b,U^l_{s,j}]S^2_{l-s}f|^2\Big)^{1/2}\Big\|_{L^2}\\ &\le C2^{-\beta j}\|b\|_{BMO}\Big(\sum_{l=-\infty}^{[\log\sqrt{2}]}2^{l-1}\Big\|\Big(\sum_{s\in\mathbb{Z}}|S^2_{l-s}f|^2\Big)^{1/2}\Big\|_{L^2}\Big)\\ &\quad+C\|b\|_{BMO}\Big(\sum_{l=[\log\sqrt{2}]+1}^{\infty}\log^{(-\alpha-1)v+1}(2^{l+j}+2)\Big\|\Big(\sum_{s\in\mathbb{Z}}|S^2_{l-s}f|^2\Big)^{1/2}\Big\|_{L^2}\Big).\end{aligned}$$

Since $(l+j)^2\ge l(j+1)$, we get

$$I_1\le C(j+1)^{\frac{(-\alpha-1)v+1}{2}}\|b\|_{BMO}\|f\|_{L^2}.\tag{5.8}$$

We will now estimate I_2. By (5.7) for $k=0$, the Littlewood-Paley theory and Lemma 3.3 (ii), we get

(5.9)

$$\begin{aligned} I_2 &\le \sum_{l=-\infty}^{[\log\sqrt{2}]} \Big\| \Big(\sum_{s\in\mathbb{Z}} |U^l_{s,j}[b,S^2_{l-s}]f|^2 \Big)^{1/2} \Big\|_{L^2} \\ &\quad + \sum_{l=[\log\sqrt{2}]+1}^{\infty} \Big\| \Big(\sum_{s\in\mathbb{Z}} |U^l_{s,j}[b,S^2_{l-s}]f|^2 \Big)^{1/2} \Big\|_{L^2} \\ &\le C2^{-\beta j}\Big(\sum_{l=-\infty}^{[\log\sqrt{2}]} 2^{l-1} \Big\| \Big(\sum_{s\in\mathbb{Z}} |[b,S^2_{l-s}]f|^2 \Big)^{1/2} \Big\|_{L^2}\Big) \\ &\quad + C\Big(\sum_{l=[\log\sqrt{2}]+1}^{\infty} \log^{(-\alpha-1)v+1}(2^{l+j}+2) \Big\| \Big(\sum_{s\in\mathbb{Z}} |[b,S^2_{l-s}]f|^2 \Big)^{1/2} \Big\|_{L^2}\Big) \\ &\le C(j+1)^{\frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO}\|f\|_{L^2}. \end{aligned}$$

Combining I_1 with I_2, we get

$$\|\sup_{s\in\mathbb{Z}} |[b,U_{s,j}]f|\|_{L^2} \le C(j+1)^{\frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO}\|f\|_{L^2}. \tag{5.10}$$

Interpolating between (5.5) and (5.10), similar to the proof of (4.15), for $p\ge 2$, we get

$$\|\sup_{s\in\mathbb{Z}} |[b,U_{s,j}]f|\|_{L^p} \le C(j+1)^{\frac{2}{p}\frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO}\|f\|_{L^p}. \tag{5.11}$$

Then by (5.4), we get for $2\le p<\alpha$,

(5.12)

$$\begin{aligned} \Big\| \sup_{s\in\mathbb{Z}} \Big| \sum_{j=s}^{\infty} [b,(\delta-\Phi_s)*K_j]f(x) \Big| \Big\|_{L^p} &\le \sum_{j=0}^{\infty} (j+1)^{\frac{2}{p}\frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO}\|f\|_{L^p} \\ &\le C\|b\|_{BMO}\|f\|_{L^p}. \end{aligned}$$

Similarly, for $p<2$, we get

$$\|\sup_{s\in\mathbb{Z}} |[b,U_{s,j}]f|\|_{L^p} \le C(j+1)^{\frac{2}{p'}\frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO}\|f\|_{L^p}. \tag{5.13}$$

Then by (5.4), we get for $\frac{\alpha}{\alpha-1}<p<2$,

(5.14)

$$\begin{aligned} \Big\| \sup_{s\in\mathbb{Z}} \Big| \sum_{j=s}^{\infty} [b,(\delta-\Phi_s)*K_j]f(x) \Big| \Big\|_{L^p} &\le \sum_{j=0}^{\infty} (j+1)^{\frac{2}{p'}\frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO}\|f\|_{L^p} \\ &\le C\|b\|_{BMO}\|f\|_{L^p}. \end{aligned}$$

This completes the proof of Theorem 2. Hence it remains to prove (5.7) and (5.7′). To this end, define multiplier $\widetilde{U}^l_{s,j}$ by $\widehat{\widetilde{U}^l_{s,j}f}(\xi) = B^l_{s,j}(2^{-s}\xi)\widehat{f}(\xi)$, and denote by $[b,\widetilde{U}^l_{s,j}]$ the commutator of $\widetilde{U}^l_{s,j}$. Define $\widetilde{U}^l_{s,j;b,1}f=[b,\widetilde{U}^l_{s,j}]f$ and $\widetilde{U}^l_{s,j;b,0}f=\widetilde{U}^l_{s,j}f$. Recall that

$$B_{s,j}(\xi)=(1-\widehat{\Phi}_s(\xi))\widehat{K}_{s+j}(\xi), \qquad B^l_{s,j}(\xi)=(1-\widehat{\Phi}_s(\xi))\widehat{K}_{s+j}(\xi)\phi(2^{s-l}\xi).$$

It is easy to see that

$$|B_{s,j}(\xi)|\le C2^{-j}|2^s\xi| \quad \text{for} \quad |2^s\xi|\le 1,$$

$$|B_{s,j}(\xi)|\le C\log^{-\alpha-1}(|2^{s+j}\xi|+2) \quad \text{for} \quad |2^s\xi|>1,$$

$$|\nabla B_{s,j}(\xi)|\le C2^s2^j.$$

Since $\text{supp}(B^l_{s,j}(2^{-s}\xi)) \subset \{\xi : 2^{l-1} \leq |\xi| \leq 2^l\}$, we have the following estimates:

$$|B_{s,j}(2^{-s}\xi)| \leq C2^{l-j} \quad \text{for} \quad l \leq 0,$$

$$|B_{s,j}(2^{-s}\xi)| \leq C\log^{-\alpha-1}(2^{l+j}+2) \quad \text{for} \quad l > 0,$$

$$|\nabla B^l_{s,j}(2^{-s}\xi)| \leq C2^j.$$

Applying Lemma 3.5 with $\sigma = 2^l$, $A = 1/2$ and the Plancherel theory, there exists a constant $0 < \beta < 1$, such that for any fixed $0 < v < 1$, $k \in \{0, 1\}$,

$$\|\widetilde{U}^l_{s,j;b,k}f\|_{L^2} \leq C\|b\|^k_{BMO}2^{-\beta j}2^l\|f\|_{L^2}, \qquad \text{for} \quad l \leq [\log\sqrt{2}],$$

$$\|\widetilde{U}^l_{s,j;b,k}f\|_{L^2} \leq C\|b\|^k_{BMO}\log^{(-\alpha-1)v+1}(2^{l+j}+2)\|f\|_{L^2}, \qquad \text{for} \quad l \geq [\log\sqrt{2}]+1.$$

This implies that

$$\|U^l_{s,j;b,k}f\|_{L^2} \leq C\|b\|^k_{BMO}2^{-\beta j}2^l\|f\|_{L^2}, \qquad \text{for} \quad l \leq [\log\sqrt{2}],$$

$$\|U^l_{s,j;b,k}f\|_{L^2} \leq C\|b\|^k_{BMO}\log^{(-\alpha-1)v+1}(2^{l+j}+2)\|f\|_{L^2}, \qquad \text{for} \quad l \geq [\log\sqrt{2}]+1,$$

by dilation invariance. This establishes the proof of (5.7) and (5.7′).

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Department of Applied Mathematics, School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, The People's Republic of China
E-mail address: yanpingch@126.com

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems (BNU), Ministry of Education, Beijing 100875, The People's Republic of China
E-mail address: dingy@bnu.edu.cn