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MULTIFRACTAL ANALYSIS FOR EXPANDING INTERVAL MAPS WITH INFINITELY MANY BRANCHES

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ABSTRACT. In this paper we investigate multifractal decompositions based on values of Birkhoff averages of functions from a class of symbolically continuous functions. This will be done for an expanding interval map with infinitely many branches and is a generalisation of previous work for expanding maps with finitely many branches. We show that there are substantial differences between this case and the setting where the expanding map has only finitely many branches.

1. SETTING

Let (X, d) be a metric space and $T : X \to X$ be a piecewise continuous transformation. Let $\phi : X \to \mathbb{R}$ be a real-valued function (called a potential). The Birkhoff average of ϕ is defined by

$$A_n\phi(x) := \frac{1}{n}\sum_{j=0}^{n-1}\phi(T^j x).$$

With respect to an ergodic measure, for a measurable potential ϕ, the Birkhoff averages $A_n\phi(x)$ almost surely converge to the integral of ϕ. However, since for an expanding map there is a large family of ergodic measures, the Birkhoff averages can take a wide variety of values. From the point of view of multifractal analysis, one considers the size (Hausdorff dimension) of the level sets of the limit of the Birkhoff averages. That is, for a given level $\alpha \in \mathbb{R}$, the Hausdorff dimension of the set

$$\Big\{x \in X : \lim_{n\to\infty} A_n\phi(x) = \alpha\Big\}.$$

There has been a substantial amount of works on this multifractal analysis, especially for expanding interval maps with finitely many branches. The first example we know where a problem of this type was studied is the work of Besicovitch in [Bes35] on the Hausdorff dimension of sets determined by the frequency of the digits in dyadic expansions. This can be viewed as a multifractal analysis of the Birkhoff averages of the indicator functions for the doubling map. This work

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was subsequently extended by Eggleston [Egg49] and many others [BS00, Caj81, Dur97, Oli98, Oli00, Ols02, Ols03b, OW03, PS07, Vol58]. For a continuous potential, the case of mixing subshift of finite type is studied in several papers including [BS01, BSS02b, BSS02a, FF00, FFW01, FLW02, Ols03a, Oli99, OW07, PW01, Tem01]. In [FLW02] Feng, Lau and Wu proved a conditional variational principle for continuous potentials in the setting of general conformal expanding maps and in [BS01] Barreira and Saussol showed that this conditional variational principle varies analytically for Hölder potentials. In [TV03] Takens and Verbitzkiy considered systems with specification property and calculated the topological entropy of the level sets. In [Hof10], Hofbauer studied the entropy of the level set of Birkhoff averages for piecewise monotone interval maps. It is also possible to study a countable family of piecewise continuous potentials. This case was investigated by Olsen [Ols03a], Olsen and Winter [OW07] for subshifts of finite type and conformal iterated function systems and by Fan, Liao and Peyrière [FLP08], in terms of topological entropy, for systems satisfying the specification property.

In particular in [Ols03a], the following situation is considered. Let $T : [0,1] \to [0,1]$ be a C^1 expanding map and for $i \in \mathbb{N}$ let $\phi_i : [0,1] \to \mathbb{R}$ be continuous functions. For a vector $\underline{\alpha} \in \mathbb{R}^{\mathbb{N}}$ let

$$X_{\underline{\alpha}} := \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi_i(T^j x) = \alpha_i \text{ for all } i \in \mathbb{N} \right\}.$$

It is shown that if $X_{\underline{\alpha}} \neq \emptyset$ there exists a T-invariant measure μ such that $\int \phi_i \mathrm{d}\mu = \alpha_i$ for all $i \in \mathbb{N}$ and

$$\dim X_{\underline{\alpha}} = \sup \left\{ \frac{h(\mu)}{\lambda(\mu)} : \int \phi_i \mathrm{d}\mu = \alpha_i \text{ for all } i \in \mathbb{N} \right\}.$$

Here dim stands for the Hausdorff dimension, $h(\mu)$ denotes the measure theoretic entropy of μ and $\lambda(\mu) = \int \log(|T'(x)|)\mathrm{d}\mu$ is the Lyapunov exponent of μ.

The aim of this paper is to look at expanding maps T on a non-compact space where T has a countable number of inverse branches. While much of the same theory still holds there are also substantial differences.

In the setting of expanding maps with a countable number of branches, there have been several papers looking at multifractal analysis. Most of these papers concentrate on the local dimension of Gibbs' measures or specific examples of continuous potentials, for example $\log |T'|$ concerning the Lyapunov exponent. Of particular relevance to our work are the papers [FLM10] and [FLMW10] which consider the frequency of digits for certain maps with a countable number of branches. This can be viewed as an example of multifractal analysis for Birkhoff averages of a specific family of potentials. A notable feature of these papers is that frequencies of digits which sum to less than 1 still yield positive dimension. However such sets cannot be related to an invariant measure. There is also a preprint [IJ10], which considers the case of one piecewise continuous potential with certain properties. Our main aim is to generalize these results to more general families of potentials and more general countable expanding maps with a countable number of branches. Related questions are also studied in certain non-conformal settings [Ree11, KR12].

Let $\{I_i\}_{i=1}^{\infty}$ be a countable collection of disjoint subintervals of $[0,1]$. Let $T_i : \overline{I_i} \to [0,1]$ be a bijective C^1 map such that $|T_i'(x)| \geq \xi > 1$. By this we will mean that T_i can be extended to a C^1 diffeomorphism from an open neighbourhood of

I_i to an open neighbourhood of $[0,1]$ which maps $\overline{I_i}$ to $[0,1]$. We define the map $T : \bigcup \overline{I_i} \to [0,1]$ as follows. If x is not a common end point of two intervals, define

$$T(x) = T_i(x) \quad \text{if } x \in \overline{I_i}.$$

Otherwise we simply set $T(x) = T_l(x)$ where $l = \min\{j : x \in I_j\}$.

Consider the full shift (Σ, σ) with $\Sigma = \mathbb{N}^{\mathbb{N}}$ and the natural projection $\Pi : \Sigma \to [0,1]$ defined by

$$\Pi(\underline{i}) = \lim_{n\to\infty} T_{i_1}^{-1} \circ \cdots \circ T_{i_n}^{-1}([0,1]).$$

Let

$$\Lambda = \Pi(\Sigma).$$

Then (Λ, T) defines a dynamical system. We remark that the space Λ is not necessarily compact and it could also be a Cantor type set. We will denote

$$E := \{x \in \Lambda : \#\Pi^{-1}(x) \geq 2\}$$

and note that this set is at most countable and so for any set $\Omega \subset \Lambda$ we have that $\dim \Omega = \dim \Omega \backslash E$. To avoid confusion with the notion of the derivative of T we will adopt the convention that for $x \in \Lambda$ $T'(x) = T_l'(x)$ where $l = \min\{j : x \in I_j\}$. We will also assume that the variations of $\log |T'|$ converge uniformly to 0 (defined precisely in Section 2; see Definition 2.1).

Let $\mathcal{M}(T)$ be the set of T-invariant probability measures on Λ and note that they must assign 0 measure to E. Thus Π gives a bijection between the set of shift invariant probability measures and T-invariant probability measures. To avoid complications when we refer to weak* limits of a sequence of measures we will always mean weak* limits of the measures in the symbolic space.

Given a sequence of functions $\phi_i : \Lambda \to \mathbb{R}$ $(i \in \mathbb{N})$, which satisfy that the variations tend uniformly to 0 (again see Definition 2.1), we will denote the Birkhoff averages

$$A_n\phi_i(x) = \frac{1}{n}\sum_{j=0}^{n-1} \phi_i(T^j x).$$

We wish to study the possible limit points in $\mathbb{R}^{\mathbb{N}}$ of the Birkhoff average sequences $\{A_n\phi_i(x)\}_{n\in\mathbb{N}}$ by investigating the sets of the form

$$\Lambda_{\underline{\gamma}} = \{x \in \Lambda : \lim_{n\to\infty} A_n\phi_i(x) = \gamma_i \text{ for all } i \in \mathbb{N}\}, \quad \underline{\gamma} \in \mathbb{R}^{\mathbb{N}}.$$

The following sets will describe the possible limits of the Birkhoff averages. Let

$$Z_0 = \left\{\underline{\gamma} \in \mathbb{R}^{\mathbb{N}} : \exists \mu \in \mathcal{M}(T), \forall i \in \mathbb{N}, \int \phi_i \mathrm{d}\mu = \gamma_i\right\}.$$

We will denote by Z the closure of Z_0 in the pointwise limit topology. That is to say, $\underline{\gamma} \in Z$ means that for any $\varepsilon > 0$ and any $k \in \mathbb{N}$ there exists a T-invariant probability measure μ such that

$$\forall 1 \leq i \leq k, \quad \left|\int \phi_i \mathrm{d}\mu - \gamma_i\right| \leq \varepsilon.$$

For a T-invariant probability measure μ let $h(\mu)$ and $\lambda(\mu)$ denote the measure theoretic entropy and the Lyapunov exponent of μ respectively. See Section 2 for formal definitions.

Our aim is to find the Hausdorff dimension of $\Lambda_{\underline{\gamma}}$ and consider how the dimension varies with $\underline{\gamma}$. The known results for dynamical systems of finite branches suggest three natural candidates in the infinite case. Given $\underline{\gamma} \in Z$, let

$$\alpha_1(\underline{\gamma}) = \lim_{\varepsilon\to 0}\lim_{k\to\infty}\sup_{\mu\in\mathcal{M}(T)}\left\{\frac{h(\mu)}{\lambda(\mu)} : \left|\int \phi_i \mathrm{d}\mu - \gamma_i\right| < \varepsilon\ \forall i \le k,\ h(\mu) < \infty\right\}.$$

Let α_2 be a similar function, the difference being that the supremum is taken over ergodic measures ($\mathcal{M}_{\mathcal{E}}(T)$):

$$\alpha_2(\underline{\gamma}) = \lim_{\varepsilon\to 0}\lim_{k\to\infty}\sup_{\mu\in\mathcal{M}_{\mathcal{E}}(T)}\left\{\frac{h(\mu)}{\lambda(\mu)} : \left|\int \phi_i \mathrm{d}\mu - \gamma_i\right| < \varepsilon\ \forall i \le k,\ h(\mu) < \infty\right\}.$$

Finally, for $\underline{\gamma} \in Z_0$ we will define

$$\alpha_3(\underline{\gamma}) = \sup_{\mu\in\mathcal{M}(T)}\left\{\frac{h(\mu)}{\lambda(\mu)} : \int \phi_i \mathrm{d}\mu = \gamma_i\ \forall i \in \mathbb{N},\ h(\mu) < \infty\right\}.$$

We can now state our main theorems.

Theorem 1.1. *For $\underline{\gamma} \notin Z$, we have $\Lambda_{\underline{\gamma}} = \emptyset$. For all $\underline{\gamma} \in Z$, we have*

$$\dim \Lambda_{\underline{\gamma}} = \alpha_1(\underline{\gamma}) = \alpha_2(\underline{\gamma}).$$

We would like to state the spectrum using the function α_3, too (hence, without the awkward limits in k and ε). However, as shown in [FLM10] and [FLMW10], the spectrum is not necessarily equal to $\alpha_3(\underline{\gamma})$. One particular problem is that there might be points in $Z \setminus Z_0$ that are limits of the Birkhoff averages of ϕ_i for some $x \in \Lambda$ (while, not belonging to Z_0, they are not averages of potentials ϕ_i for any invariant measure). Another problem is that even for points in Z_0 the spectrum needs not to be the supremum of h/λ over invariant measures with given averages of ϕ_i.

We are only able to present the "exact" type statement for bounded potentials, and the proof involves more steps than for the "approximate" type statements of Theorem 1.1. We also need to introduce the quantity,

$$s_\infty = \inf\Big\{s \ge 0 : \sum_{i\in\mathbb{N}} \operatorname{diam}(I_i)^s < \infty\Big\}.$$

Observe that $0 \le s_\infty \le 1$. The exponent s_∞ will play an important role.

Theorem 1.2. *If the potentials ϕ_i are all bounded, then for all $\underline{\gamma} \in Z_0$ we have*

$$\dim \Lambda_{\underline{\gamma}} = \max\left\{s_\infty,\ \alpha_3(\underline{\gamma})\right\},$$

while for all $\underline{\gamma} \in Z \setminus Z_0$ we have

$$\dim \Lambda_{\underline{\gamma}} = s_\infty.$$

The rest of the paper is structured in the following way. In Section 2 we give some results based on the distortion of the functions and the topological pressure. Next we use Section 3 to introduce the main tools we will use to prove Theorems 1.1 and 1.2. Section 4 gives the proof of Theorem 1.1 and the proof of Theorem 1.2 is given in Sections 5 and 6. Finally in Section 7 we give some examples of our results, including frequency of digits, harmonic averages for continued fractions and multifractal spectra with flat regions.

We now give a list of the notation which will be used in this paper.

- $\Sigma = \mathbb{N}^{\mathbb{N}}$: the full shift with the shift transformation σ.
- $\Sigma_q = \{1, \dots, q\}^{\mathbb{N}}$: the symbolic space of q symbols.
- $[\omega_1, \cdots, \omega_n]$: nth level cylinder set in Σ.
- $C_n(\omega) = C_n(x) = C_n(\omega_1 \cdots \omega_n)$ with $x = \Pi\omega, \omega \in [\omega_1, \cdots, \omega_n]$: nth level basic interval in Λ.
- ϕ, ϕ_i : functions on Λ; $f = \phi \circ \Pi, f_i = \phi_i \circ \Pi$: the corresponding functions on Σ.
- $A_n\phi_i(x) = \frac{1}{n}\sum_{j=0}^{n-1} \phi_i(T^j x)$: Birkhoff averages of ϕ_i.
- μ, μ_j : measures on Λ; ν, ν_j, η, η_j : measures on Σ.
- λ_i : the maximal contraction ratio of map T_i.
- $\psi_k(x) = \frac{1}{k}\sup_{y\in C_k(x)} \log |(T^k)'(y)|$.
- $\xi_k(\mu) = \int \psi_k \mathrm{d}\mu$.
- $\dim A$: Hausdorff dimension of a set A.
- $h(\mu)$: entropy of μ.
- $\lambda(\mu)$: Lyapunov exponent of μ.
- For $(\omega_1, \dots, \omega_n) \in \mathbb{N}^n$, $\overline{(\omega_1, \dots, \omega_n)}$ denotes the periodic point $\tau \in \Sigma$ such that for any $a \in \mathbb{N}$ and $1 \le b \le n$ $\tau_{an+b} = \omega_b$.

2. Topological pressure and distortion

We first introduce several useful quantities (including entropy, Lyapunov exponent, pressure) and a variational condition on potentials which ensures a distortion result.

We start by defining cylinders and basic intervals in our setting. Let $\omega \in \Sigma$. Denote by $[\omega_1, \cdots, \omega_n]$ the nth level cylinder. The nth level basic interval determined by ω is

$$C_n(\omega) = C_n(\omega_1, \dots, \omega_n) = T_{\omega_1}^{-1} \circ \cdots \circ T_{\omega_n}^{-1}([0,1])\backslash E.$$

Sometimes, we also write this basic interval by $C_n(x)$ with $x = \Pi\omega$.

Two key concepts for this paper will be the measure theoretic entropy and the Lyapunov exponent of an invariant measure. For a T-invariant probability measure μ we define its entropy ([MU03], pages 292-293) by

$$h(\mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{(\omega_1\dots,\omega_n)\in\mathbb{N}^n} \mu(C_n(\omega_1, \dots, \omega_n)) \cdot \log \mu(C_n(\omega_1, \dots, \omega_n))$$

and its Lyapunov exponent by

$$\lambda(\mu) = \int \log |T'(x)| \mathrm{d}\mu(x).$$

It is well known that $h(\mu) \le \lambda(\mu)$. However it is possible that they could both be infinite.

We now consider the regularity conditions we will need our potential functions ϕ_i to satisfy. For $\phi : \Lambda \to \mathbb{R}$ define its nth variation by

$$\mathrm{var}_n(\phi) = \sup\{|\phi(x) - \phi(y)| : \ x, y \in C_n(\omega), \omega = (\omega_1, \dots, \omega_n) \in \mathbb{N}^n\}.$$

It is clear that $\mathrm{var}_n(\phi)$ decreases as n tends to ∞ and that $\lim_n \mathrm{var}_n(\phi) = 0$ means $f := \phi \circ \Pi$ is uniformly continuous on Σ when Σ is equipped with the usual metric.

Definition 2.1. Let $\phi : \Lambda \to \mathbb{R}$. We say that ϕ has variations uniformly converging to 0 if $\mathrm{var}_1(\phi) < \infty$ and $\lim_{n\to\infty} \mathrm{var}_n(\phi) = 0$.

Given a basic interval $C_n(\omega_1, \ldots, \omega_n)$ we define

$$M^*\phi(\omega_1, \ldots, \omega_n) = \sup_{x \in C_n(\omega_1, \ldots, \omega_n)} A_n\phi(x),$$
$$M_*\phi(\omega_1, \ldots, \omega_n) = \inf_{x \in C_n(\omega_1, \ldots, \omega_n)} A_n\phi(x).$$

Lemma 2.2. *Let $\phi : \Lambda \to \mathbb{R}$ have variations uniformly tending to* 0*. Then*

$$\lim_{n\to\infty} \sup_{(\omega_1,\ldots,\omega_n)\in\mathbb{N}^n} M^*\phi(\omega_1, \ldots, \omega_n) - M_*\phi(\omega_1, \ldots, \omega_n) = 0.$$

Proof. The result immediately follows from the estimation: for fixed $n \in \mathbb{N}$ we have

$$|M^*\phi(\omega_1, \ldots, \omega_n) - M_*\phi(\omega_1, \ldots, \omega_n)| \leq \frac{1}{n}\sum_{j=1}^{n} \operatorname{var}_j \phi = o(1).$$

□

Since we are assuming that $\log|T'(x)|$ has variations uniformly tending to 0, this lemma has an immediate consequence on the size of basic intervals.

Lemma 2.3. *For any $\omega \in \Sigma$*

$$|\log(\operatorname{diam}(C_n(\omega))) - nA_n(-\log|T' \circ \Pi(\omega)|)| = o(n).$$

Proof. This can be proved straightforwardly since by the mean value theorem we have

$$\log(\operatorname{diam}(C_n(\omega)) = nA_n(-\log|T' \circ \Pi(\tau)|)$$

for some $\tau \in \Sigma$ such that $(\tau_1, \ldots, \tau_n) = (\omega_1, \ldots, \omega_n)$. We can then apply Lemma 2.2 to $\phi = \log|T'|$ which was assumed to have variations tending uniformly to 0. □

Now it is time to refer to the notion of pressure of a potential. If $\phi : \Lambda \to \mathbb{R}$ is a function with variations uniformly tending to 0, then we define its pressure by

$$P(\phi) = \sup_{\mu\in\mathcal{M}(T)} \left\{ h(\mu) + \int \phi \mathrm{d}\mu : \int \phi \mathrm{d}\mu > -\infty \right\}.$$

This can be alternatively stated as (see [MU03], p. 7)

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} e^{S_n(\phi\circ\Pi(\overline{\omega}))}. \tag{2.1}$$

Notice that it is possible that $P(\phi) = \infty$.

Finally we prove some important results regarding the relationship between the topological pressure and s_∞. Observe that $t \mapsto P(-t\log|T'|)$ is decreasing because $\log|T'(x)| > 0$.

Lemma 2.4. *We have*

$$s_\infty = \inf\{t \geq 0 : P(-t\log|T'|) < \infty\}.$$

Proof. For convenience we will let

$$\psi(x) = -\log|T_l'(x)|, \quad G(x) = \log\operatorname{diam}(I_l), \text{ where } l = \min\{j : x \in I_j\}.$$

To complete the proof simply note that if $P(tG) < \infty$ or $P(t\psi) < \infty$, then by (2.1) we have

$$|P(tG) - P(t\psi)| \leq t\operatorname{var}_1(\psi).$$

□

Lemma 2.5. *There exists a sequence of T-invariant probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty}\lambda(\mu_n)=\infty,\quad \lim_{n\to\infty}\frac{h(\mu_n)}{\lambda(\mu_n)}=s_\infty.$$

Proof. We suppose $s_\infty>0$ and leave the easy case $s_\infty=0$ to the readers. We start by fixing $\varepsilon>0$ and noting that for any T-invariant measure μ such that $\frac{h(\mu)}{\lambda(\mu)}\geq s_\infty+2\varepsilon$ we have

$$\begin{aligned} P(-(s_\infty+\varepsilon)\log|T'|) &\geq h(\mu)-(s_\infty+\varepsilon)\lambda(\mu)\\ &\geq \varepsilon\lambda(\mu)\end{aligned}$$

and so $\lambda(\mu)\leq\frac{P(-(s_\infty+\varepsilon)\log|T'|)}{\varepsilon}$.

We now take two sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{k_n\}_{n\in\mathbb{N}}$ such that for each n, $t_n<s_\infty$, $\lim_{n\to\infty}t_n=s_\infty$ and $\lim_{n\to\infty}k_n=\infty$. Since for all n we have that $P(-t_n\log|T'|)=\infty$, by variational principle, we can find a sequence of T-invariant measures μ_n such that

$$h(\mu_n)-t_n\lambda(\mu_n)\gg 1,$$

and hence $\frac{h(\mu_n)}{\lambda(\mu_n)}>t_n$. Furthermore, by the fact that $\lambda(\mu)\geq h(\mu)$, we can have

$$\lambda(\mu_n)\geq k_n.$$

However, for any $\varepsilon>0$, if $k_n\geq\frac{P(-(s_\infty+\varepsilon)\log|T'|)}{\varepsilon}$, then $\frac{h(\mu_n)}{\lambda(\mu_n)}\leq s_\infty+2\varepsilon$. So,

$$\lim_{n\to\infty}\frac{h(\mu_n)}{\lambda(\mu_n)}=s_\infty.$$

□

3. Tools

It will be useful for us to describe in some detail the main tools we are going to use. They are already used in the literature in the finite symbolic case, but in this paper we are working with infinitely many symbols and this introduces some minor changes. We remind the reader that (Σ,σ) is the full shift on one-sided symbolic space over an infinite alphabet.

3.1. Bernoulli approximation.

In this section we will present a process of using sets of cylinders to define Bernoulli type ergodic measures. This is a similar idea to Misurewicz's proof of the variational principle but here we also exploit the structure of the symbolic space. Since we are in a non-compact setting, an added complication is that weak* limits of measures will not always exist.

Let $\phi:\Sigma\to\mathbb{R}$ have variations uniformly tending to 0. Let $f=\phi\circ\Pi$. We prove the following result.

Proposition 3.1. *Let $\varepsilon>0$ and $n\in\mathbb{N}$ be fixed. Suppose that*

$$\mathrm{var}_n(A_n\phi)\leq\varepsilon,\quad \mathrm{var}_n(A_n\log|T'|)\leq\varepsilon.$$

For any set $J\subseteq\mathbb{N}^n$ and any probability vector $\{p_j\}_{j\in J}$ ($0<p_j<1$, $\sum_{j\in J}p_j=1$), we can find an ergodic T-invariant measure μ such that

1. $\int\phi\mathrm{d}\mu\in(\gamma_1-\varepsilon,\gamma_1+\varepsilon)$,
2. $\lambda(\mu)\in(\gamma_2-\varepsilon,\gamma_2+\varepsilon)$,
3. $h(\mu)=1/n\sum_{j\in J}p_j\log p_j$,

where

$$\gamma_1 = \frac{1}{n}\sum_{j\in J} p_j S_n f(\bar{j}), \qquad \gamma_2 = \frac{1}{n}\sum_{j\in J} p_j \log \operatorname{diam}(\Pi(j)).$$

Proof. For convenience define $\Psi : \Sigma \to \mathbb{R}$ by

$$\Psi(\omega) = \log |(T^n)'(\Pi\omega)|.$$

Each j in J defines a cylinder. We start by defining a σ^n-invariant Bernoulli measure ν_n on Σ by assigning each cylinder $j \in J$ the weight p_j. This measure will satisfy

(1) $\frac{1}{n}\int S_n f \mathrm{d}\nu_n \in (\gamma_1 - \varepsilon, \gamma_1 + \varepsilon)$,
(2) $\frac{1}{n}\int S_n(\log T' \circ \Pi)\mathrm{d}\nu_n \in (\gamma_2 - \varepsilon, \gamma_2 + \varepsilon)$,
(3) $h(\nu_n, \sigma^n) = -\sum_{j\in J} p_j \log p_j$.

Then define a σ-invariant measure

$$\nu = \frac{1}{n}\sum_{l=0}^{n-1} \nu_n \circ \sigma^{-l}.$$

Since the measure ν_n is σ^n-ergodic, ν is σ-ergodic. By straightforward calculations and Abramov's formula for entropy (see [PU10], Theorem 2.4.6 page 32), the above three formulas can be written for ν as

(1) $\int f \mathrm{d}\nu \in (\gamma_1 - \varepsilon, \gamma_1 + \varepsilon)$,
(2) $\int \log T' \circ \Pi \mathrm{d}\nu \in (\gamma_2 - \varepsilon, \gamma_2 + \varepsilon)$,
(3) $h(\nu, \sigma) = -\frac{1}{n}\sum_{j\in J} p_j \log p_j$.

To finish the proof we simply let

$$\mu = \nu \circ \Pi^{-1}.$$

□

We will use this proposition in two ways. One is to construct measures from sets of cylinders where the Birkhoff averages for certain potentials will be the same. The other is to approximate invariant measures with ergodic measures.

Let $k \in \mathbb{N}$. For $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}^k$, denote by $\Sigma(\gamma)$ the following set of cylinders in Σ

$$\big\{[\omega_1, \dots, \omega_n] : A_n\phi_i(\Pi\omega) \in (\gamma_i - \varepsilon, \gamma_i + \varepsilon), \forall \omega \in [\omega_1, \dots, \omega_n], \forall 1 \le i \le k\big\}.$$

Corollary 3.2. *Fix $k \in \mathbb{N}$, $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}^k$ and $n \in \mathbb{N}$. If there exists s such that*

$$\sum_{\Sigma(\gamma)} \operatorname{diam}([\omega_1, \dots, \omega_n])^s = 1,$$

and

$$K := -\sum_{\Sigma(\gamma)} \operatorname{diam}([\omega_1, \dots, \omega_n])^s \log \operatorname{diam}([\omega_1, \dots, \omega_n]) < \infty,$$

then there exists a T-invariant ergodic measure μ such that

$$\int \phi_i \mathrm{d}\mu \in (\gamma_i - \varepsilon, \gamma_i + \varepsilon),\ \forall 1 \le i \le k, \quad \textit{and} \quad \left|\frac{h(\mu)}{\lambda(\mu)} - s\right| \le \frac{\varepsilon}{K - \varepsilon}.$$

Proof. We simply apply Proposition 3.1 with $J = \Sigma(\gamma)$ and with $\operatorname{diam}([\omega_1, \dots, \omega_n])^s$ as probabilities. □

Corollary 3.3. *If there exists a T-invariant measure μ and a vector $\gamma \in \mathbb{R}^k$ ($k \in \mathbb{N}$) such that*

$$\lambda(\mu) < \infty; \quad \forall 1 \le i \le k, \int \phi_i \, d\mu = \gamma_i,$$

then there exist strictly increasing sequences of integers $\{q_\ell\}, \{n_\ell\}$, and a sequence of T^{n_ℓ}-invariant Bernoulli measures $\{\mu_\ell\}$ supported on $\Pi(\Sigma_{q_\ell})$ such that

(1) $\lim_{n\to\infty} \int A_n\phi_i \, d\mu_\ell = \gamma_i$ *for* $1 \le i \le k$,
(2) $\lim_{\ell\to\infty} h(\mu_\ell, T^{n_\ell}) = h(\mu)$,
(3) $\lim_{\ell\to\infty} \lambda(\mu_\ell, T^{n_\ell}) = \lambda(\mu)$.

Proof. Take such an invariant measure μ. For any $\varepsilon > 0$ we can find $N \in \mathbb{N}$ and $q \in \mathbb{N}$ such that for any $n \ge N$

(1) $\mathrm{var}_n\{A_n(\log T' \circ \Pi)\} \le \varepsilon$,
(2) for each $1 \le i \le k$, $\max_i\{\mathrm{var}_n(A_n\phi_i)\} \le \varepsilon$,
(3) for each $1 \le i \le k$,

$$\left| \sum_{\omega_1,\dots,\omega_n} \tilde{\mu}(\Pi[\omega_1,\dots,\omega_n]) A_n\phi_i(\Pi(\overline{\omega_1,\dots,\omega_n})) - \gamma_i \right| \le \varepsilon,$$

(4) $\left|\sum_{\omega_1,\dots,\omega_n} \tilde{\mu}(\Pi[\omega_1,\dots,\omega_n]) \log \mathrm{diam}([\omega_1,\dots,\omega_n]) - n\lambda(\mu)\right| \le n\varepsilon$,
(5) $\left|\sum_{\omega_1,\dots,\omega_n} \tilde{\mu}(\Pi[\omega_1,\dots,\omega_n]) \log \tilde{\mu}(\Pi[\omega_1,\dots,\omega_n]) - nh(\mu)\right| \le n\varepsilon$,

where in points (3)-(5) the sums are taken over all words $\omega_1 \dots \omega_n \in \{1,\dots,q\}^n$ and

$$\tilde{\mu}(\Pi[\omega_1,\dots,\omega_n]) = \frac{\mu(\Pi[\omega_1,\dots,\omega_n])}{\sum_{\omega_1,\dots,\omega_n} \mu(\Pi[\omega_1,\dots,\omega_n])}.$$

We can now apply the first part of the proof of Proposition 3.1 to construct our sequence of measures. We could go on to get a sequence of T-ergodic measures. However, these T^{n_ℓ}-ergodic measures μ_ℓ will actually be more useful for our purposes. □

3.2. **W-measures.** The main tool to prove the lower bound of our theorems will be to use the technique of w-measures used in [GR09]. This involves using a sequence of ergodic measures to define a new measure which we will use to calculate the lower bound for the dimension.

Theorem 3.4. *Let $\{\mu_j\}_{j=1}^\infty$ be a sequence of T-invariant measures of finite entropy such that the following limits exist:*

$$\gamma_i = \lim_{j\to\infty} \int \phi_i \, d\mu_j, \quad \forall i \in \mathbb{N}.$$

Then for $\underline{\gamma} = (\gamma_i)_{i\in\mathbb{N}}$ we have

$$\dim \Lambda_{\underline{\gamma}} \ge \limsup \frac{h(\mu_j)}{\lambda(\mu_j)}.$$

Proof. This statement is analogous to the one proven in [GR09, Proposition 9, Theorem 3] in the special case: it was a finite iterated function system, the measures μ_j were Gibbs and there was only one potential $\phi = \log|T'|$. The proof of the general statement is analogous, but there are some changes so we rewrite it.

By choosing a subsequence we can freely assume that $h(\mu_j)/\lambda(\mu_j)$ have a limit.

To begin, we are not going to use the measures μ_j directly. Fix a sequence $\varepsilon_j \to 0$. By Corollary 3.3, for each j, there exist an integer n_j and a Gibbs (even Bernoulli) T^{n_j}-invariant measure μ'_j such that

(1) $\left|\int A_n\phi_i \mathrm{d}\mu'_j - \gamma_i\right| < \varepsilon_j/2$ for $1 \le i \le j$,
(2) $|h(\mu'_j, T^{n_j}) - h(\mu_j)| < \varepsilon_j/2$,
(3) $|\lambda(\mu'_j, T^{n_j}) - \lambda(\mu_j)| < \varepsilon_j/2$.

Then let

$$\eta_j = \frac{1}{n_j}\sum_{l=0}^{n_j-1} \mu'_j \circ \Pi \circ \sigma^{-l}. \tag{3.1}$$

The family $\{\eta_j\}_{j=1}^{\infty}$ has the following properties:

– $h(\eta_j) = \frac{1}{n_j} h(\mu'_j; \sigma^{n_j})$,
– each measure η_j is supported on a symbolic space Σ_{q_j} with only finitely many symbols, the sequence $\{q_j\}$ is in general unbounded (note that Σ_{q_j} is compact, hence each $f_i = \phi_i \circ \Pi$ is bounded on Σ_{q_j}),
–
$$\left|\frac{h(\eta_j)}{\lambda(\eta_j)} - \frac{h(\mu_j)}{\lambda(\mu_j)}\right| \le \varepsilon_j,$$
– for all $1 \le i \le j$
$$\left|\int f_i \mathrm{d}\eta_j - \int \phi_i \mathrm{d}\mu_j\right| \le \varepsilon_j.$$

Let $\{m_j\}$ be a fast increasing sequence of integers (in the following we will provide further conditions). We will construct a probability measure η supported on Σ by defining it on a family of cylinders, which has a product structure.

First, on all cylinders of level m_1 we define

$$\eta([\omega_1, \ldots, \omega_{m_1}]) = \eta_1([\omega_1, \ldots, \omega_{m_1}]).$$

Then, in an inductive step, having the measure defined on cylinders of level m_{j-1}, we subdivide it on their subcylinders of level m_j by the following formula:

$$\eta([\omega_1, \ldots, \omega_{m_j}]) = \eta([\omega_1, \ldots, \omega_{m_{j-1}}]) \cdot \eta_j([\omega_{m_{j-1}+1}, \ldots, \omega_{m_j}]).$$

We assume that

$$m_1 \gg n_1, \qquad (m_j - m_{j-1}) \gg n_j.$$

Note that at each step of construction the measure is defined on a symbolic space with finitely many symbols. Denote

$$L_n(\omega) = \frac{1}{n}\log|(T^n)'(\Pi\omega)| \quad \text{and} \quad M_n(\omega) = -\frac{1}{n}\log \eta([\omega_1, \ldots, \omega_n]).$$

We claim the following: we can choose $\{m_j\}$ such that

$$-\log \lambda_{q_{j+1}} < \varepsilon_{j+1} m_j \tag{3.2}$$

where λ_j is the maximal contraction ratio of map T_j and that the following properties are satisfied for any j and for all points ω in a positive η-measure set $A \subset \Sigma$: for all $1 \le i \le j$ and $m_j \le n < m_{j+1}$ we have

$$M^* f_i(\omega_1, \ldots, \omega_n) - M_* f_i(\omega_1, \ldots, \omega_n) \le \varepsilon_j, \tag{3.3}$$

$$\left|A_n f_i(\omega) - \frac{m_j}{n}\int f_i \mathrm{d}\eta_j - \frac{n - m_j}{n}\int f_i \mathrm{d}\eta_{j+1}\right| \le \varepsilon_j, \tag{3.4}$$

(3.5) $$\left|L_n(\omega) - \frac{m_j}{n}\lambda(\eta_j) - \frac{n-m_j}{n}\lambda(\eta_{j+1})\right| \le \varepsilon_j,$$

(3.6) $$\left|M_n(\omega) - \frac{m_j}{n}h(\eta_j) - \frac{n-m_j}{n}h(\eta_{j+1})\right| \le \varepsilon_j.$$

Let us prove the last four expressions. Formula (3.3) follows from Lemma 2.2 provided all m_j are big enough. The other three expressions are the main part. Note that (3.5) and (3.6) are actually special cases of (3.4). $L_n(\omega)$ is (by bounded distortion) approximately a partial Cesaro average of the function $\log|T'|$. Similarly, while η_j is not a Gibbs measure, μ'_j is (for T^{n_j}). Hence, $\frac{1}{n}(nM_n(\omega) - m_j M_{m_j}(\omega))$ is (by Gibbs property) approximately a partial Cesaro average of the potential of the Gibbs measure μ'_{j+1} (average under iterations of $T^{n_{j+1}}$). For this reason, we will provide a detailed proof of the formula (3.4) only and the formulas (3.5) and (3.6) can be proved analogously.

Applying the Birkhoff Ergodic Theorem to the measure η_1 and the function f_1, we get that

(3.7) $$\left|A_{m_1}f_1(\omega) - \int f_1 \mathrm{d}\eta_1\right| \le \frac{\varepsilon_1}{2}$$

on a set of η_1-measure $1-\delta_1$, where δ_1 can be chosen arbitrarily small if m_1 is sufficiently big. The next statement we will need is that

(3.8) $$n\left|A_n f_1(\sigma^{m_1}(\omega)) - \int f_1 \mathrm{d}\eta_2\right| \le \frac{m_1\varepsilon_1}{2} + n\varepsilon_2$$

for all $n \ge 1$ for a set of ω of η_2-measure $1-\tilde{\delta}_1$ (more precisely, we will only need this statement for $1 \le n \le m_2 - m_1$, but it is important that we can choose arbitrarily big m_2 and the statement will still be true). It follows from the Central Limit Theorem (see [PU10, Thm 5.7.1]) for the measure η_2 that for any continuous f and for big n

$$\left|A_n f(\omega) - \int f \mathrm{d}\eta_2\right| < \varepsilon$$

for all ω except a subset of measure approximately $\exp(-cn\varepsilon^2)$. Hence, $\tilde{\delta}_1$ can be chosen arbitrarily small, provided $m_1\varepsilon_1$ is big enough (how big is big enough will depend on ε_2).

We continue in an inductive way. By the Birkhoff Ergodic Theorem we have

(3.9) $$\left|A_{m_j}f_i(\omega) - \int f_i \mathrm{d}\eta_j\right| \le \frac{\varepsilon_j}{2}$$

for all $1 \le i \le j$ on a set of η-measure $1-\delta_j$, where δ_j can be chosen arbitrarily small provided m_j is sufficiently big and sufficiently big in comparison with m_{j-1}. By the Central Limit Theorem

(3.10) $$n\left|A_n f_i(\sigma^{m_j}(\omega)) - \int f_i \mathrm{d}\eta_{j+1}\right| \le \frac{m_j\varepsilon_j}{2} + n\varepsilon_{j+1}$$

for all $1 \le i \le j$ and $n \ge 1$ for a set of ω of η_{j+1}-measure $1-\tilde{\delta}_j$, where $\tilde{\delta}_j$ can be chosen arbitrarily small, provided $m_j\varepsilon_j$ is big enough. Combining (3.7), (3.8), (3.9) and (3.10) we get (3.4) true on a set A of η-measure at least $1-\sum\delta_j - \sum\tilde{\delta}_j$, which can be chosen arbitrarily close to 1.

Let η_A be the restriction of η to A. By (3.3) and (3.4), we have

$$A \subset \Lambda_{\underline{\gamma}}.$$

On the other hand, for all $m_j < n \le m_{j+1}$, A is contained in a union of nth level cylinders, each of size at least

$$r_n := \exp\left(-m_j\lambda(\eta_j) - (n-m_j)\lambda(\eta_{j+1}) - n\varepsilon_j\right)$$

(by (3.5)) and of μ-measure at most

$$c_n := \exp\left(-m_j h(\eta_j) - (n-m_j)h(\eta_{j+1}) + n\varepsilon_j\right)$$

(by (3.6)). According to (3.2), we have

$$|\log r_{n+1} - \log r_n| \le \varepsilon_j |\log r_n|/n.$$

For any $\omega \in A$, the ball $B_{r_n}(\omega)$ intersects A at most two nth level cylinders. Hence

$$\eta_A(B_{r_n}(\omega)) \le 2c_n$$

By Frostman's Lemma,

$$\dim \Pi(A) \ge \liminf \frac{h(\eta_j)}{\lambda(\eta_j)} = \liminf \frac{h(\mu_j)}{\lambda(\mu_j)}.$$

Recall that at the beginning, we assume that $h(\mu_j)/\lambda(\mu_j)$ have a limit. The proof is then completed. $\square$

4. Proof of Theorem 1.1

The proof is divided into the following three propositions. Recall that

$$\Lambda_{\underline{\gamma}} = \{x \in \Lambda : \lim_{n\to\infty} A_n\phi_i(x) = \gamma_i \text{ for all } i \in \mathbb{N}\}.$$

Proposition 4.1. *If $\underline{\gamma} \notin Z$, then $\Lambda_{\underline{\gamma}} = \emptyset$.*

Proof. Given $\underline{\gamma}$, assume there exists $x \in \Lambda$ such that $\lim_{n\to\infty} A_n\phi_i(x) = \gamma_i$ for all $i \in \mathbb{N}$. Let $\omega \in \Sigma$ satisfy $\Pi\omega = x$. If we fix $\varepsilon > 0$ and $k \in \mathbb{N}$, then we can find N such that for all $n \ge N$ we have

$$\sup_{1\le i\le k} |A_n\phi_i(x) - \gamma_i| \le \varepsilon/2,$$

$$\sup_{1\le i\le k} \sup_{x,y\in\Pi([\omega_1,\ldots,\omega_n])} |A_n\phi_i(x) - A_n\phi_i(y)| \le \varepsilon/2.$$

We then let ν be the shift invariant measure on Σ defined on the periodic orbit $\overline{(\omega_1,\ldots,\omega_n)}$. If we let $\mu = \nu\circ\Pi$, then we have that μ is T-invariant and that $\left|\int \phi_i d\mu - \gamma_i\right| \le \varepsilon$ for each $1 \le i \le k$. This completes the proof. $\square$

In what follows, we will restrict ourselves to the case $\underline{\gamma} \in Z$.

Proposition 4.2. *If $\underline{\gamma} \in Z$, then* $\dim \Lambda_{\underline{\gamma}} \ge \alpha_1(\gamma)$.

Proof. It follows immediately from Theorem 3.4. $\square$

Proposition 4.3. *If* $\underline{\gamma} \in Z$*, then* $\dim \Lambda_{\underline{\gamma}} \le \alpha_2(\gamma)$.

Proof. Let $\tilde{s} = \dim \Lambda_{\underline{\gamma}} = \dim(\Lambda_{\underline{\gamma}} \backslash E)$. Given $\varepsilon > 0$, for any covering of $\Lambda_{\underline{\gamma}} \backslash E$ with intervals E_j of lengths $|E_j| < \delta$ we will have

$$\sum |E_j|^{\tilde{s}-\varepsilon} > N(\delta)$$

with $N(\delta) \to \infty$ as $\delta \to 0$. In particular, if we choose a covering of $\Lambda_{\underline{\gamma}}$ with nth level basic intervals, the corresponding sum $\sum |\Pi[\omega_1, \dots, \omega_n]|^{\tilde{s}-\varepsilon}$ will be greater than 1 provided n is big enough. If this summand is infinite, we can choose a finite subfamily of nth level basic intervals intersecting $\Lambda_{\underline{\gamma}}$ such that the sum of their diameters in power $\tilde{s} - \varepsilon$ is still greater than 1. We can then choose a different exponent $s > \tilde{s} - \varepsilon$ for which this sum is equal to 1.

By Lemma 2.2, for any k for sufficiently big n if an nth level cylinder intersects $\Lambda_{\underline{\gamma}}$ then

$$|A_n \phi_i(\omega) - \gamma_i| < \varepsilon$$

for all $i \le k$ and for all ω in this cylinder.

We can now apply Proposition 3.1 and Corollary 3.2 to construct an ergodic measure ν with respect to the shift acting on finitely many symbols (hence, of finite entropy), and then a T-invariant ergodic measure μ satisfying

$$\left| \int \phi_i \mathrm{d}\mu - \gamma_i \right| < 2\varepsilon, \quad \left| \frac{h(\mu)}{\lambda(\mu)} - s \right| \le \frac{2\varepsilon}{K - 2\varepsilon}$$

for all $1 \le i \le k$. By the formula $\dim \mu = h(\mu)/\lambda(\mu)$ the proof of the upper bound in Theorem 1.1 is completed. $\square$

5. Proof of Theorem 1.2

From now on we will assume that each function ϕ_i is bounded above and below. Recall that

$$\alpha_1(\underline{\gamma}) = \lim_{\varepsilon \to 0} \lim_{k \to \infty} \sup_{\mu \in \mathcal{M}(T)} \left\{ \frac{h(\mu)}{\lambda(\mu)} : \left| \int \phi_i \mathrm{d}\mu - \gamma_i \right| < \varepsilon \ \forall i \le k, \ h(\mu) < \infty \right\},$$

$$\alpha_3(\underline{\gamma}) = \sup_{\mu \in \mathcal{M}(T)} \left\{ \frac{h(\mu)}{\lambda(\mu)} : \int \phi_i \mathrm{d}\mu = \gamma_i \ \forall i \in \mathbb{N}, \ h(\mu) < \infty \right\}.$$

We will first show that for all $\underline{\gamma} \in Z$ we have (see Lemma 5.1)

$$\alpha_1(\underline{\gamma}) \ge s_\infty.$$

As Theorem 1.1 is already proven, we then have

$$\dim \Lambda_\gamma = \max \left\{ s_\infty, \ \alpha_1(\underline{\gamma}) \right\}.$$

Then we will show that (see Proposition 5.2)

$$\alpha_1(\underline{\gamma}) > s_\infty \Rightarrow \alpha_1(\underline{\gamma}) = \alpha_3(\underline{\gamma}).$$

It will follow that $\dim \Lambda_\gamma = \max \left\{ s_\infty, \ \alpha_3(\underline{\gamma}) \right\}$.

Let us first prove the following lemma.

Lemma 5.1. *Let* $\underline{\gamma} \in Z$, $k \in \mathbb{N}$, $\varepsilon > 0$ *and* $\mu \in \mathcal{M}(T)$ *such that*

$$\lambda(\mu) < \infty, \qquad \sup_{1 \le i \le k} \left| \int \phi_i \mathrm{d}\mu - \gamma_i \right| \le \varepsilon.$$

There then exists a measure $\nu \in \mathcal{M}(T)$ *such that*

$$\frac{h(\nu)}{\lambda(\nu)} \ge s_\infty - \varepsilon, \qquad \sup_{1 \le i \le k} \left| \int \phi_i \mathrm{d}\mu - \gamma_i \right| \le 2\varepsilon.$$

Proof. Let $A = \sup_{1 \le i \le k} \sup_{x \in \Lambda} |\phi_i(x)|$. By Lemma 2.5 we can find a sequence of T-invariant measures μ_n such that $\lim_{n\to\infty} \lambda(\mu_n) = \infty$ and $\frac{h(\mu_n)}{\lambda(\mu_n)} \ge s_\infty - \frac{\varepsilon}{2}$ for each n. Consider the measure

$$\nu_n = (1 - \frac{\varepsilon}{A})\mu + \frac{\varepsilon}{A}\mu_n.$$

Then we have that for each $1 \le i \le k$,

$$\left| \int \phi_i \mathrm{d}\nu_n - \gamma_i \right| \le \left| \int \phi_i \mathrm{d}\mu - \gamma_i \right| + \left| \int \phi_i \mathrm{d}\mu - \int \phi_i \mathrm{d}\nu_n \right| \le 2\varepsilon.$$

Furthermore,

$$\begin{aligned} \liminf_{n\to\infty} \frac{h(\nu_n)}{\lambda(\nu_n)} &= \liminf_{n\to\infty} \frac{(1-\varepsilon/A)h(\mu) + \varepsilon/A h(\mu_n)}{(1-\varepsilon/A)\lambda(\mu) + \varepsilon/A\lambda(\mu_n)} \\ &= \liminf_{n\to\infty} \frac{h(\mu_n)}{\lambda(\mu_n)} \\ &\ge s_\infty - \frac{\varepsilon}{2}. \end{aligned}$$

This completes the proof. □

Thus we can conclude that for all $\underline{\gamma} \in Z$, we have $\alpha_1(\underline{\gamma}) \ge s_\infty$.

Proposition 5.2. *Let* $\underline{\gamma} \in Z$. *If* $\alpha_1(\underline{\gamma}) > s_\infty$, *we have* $\alpha_1(\underline{\gamma}) = \alpha_3(\underline{\gamma})$.

The proof of the proposition is lengthy and it is presented in the next section.

6. Proof of Proposition 5.2

The assertion of Proposition 5.2 will follow immediately from the following statement.

Proposition 6.1. *Given* $\underline{\gamma} \in Z$ *and a sequence of invariant measures* μ_j *such that*

- $h(\mu_j)/\lambda(\mu_j) > s_\infty + \delta$ *for some* $\delta > 0$,
- $\int \phi_i \mathrm{d}\mu_j \to \gamma_i$ *for all* $i \in \mathbb{N}$,

there exists an invariant measure μ *satisfying*

$$\frac{h(\mu)}{\lambda(\mu)} = \limsup \frac{h(\mu_j)}{\lambda(\mu_j)} \quad \text{and} \quad \int \phi_i \mathrm{d}\mu = \gamma_i \ \forall i \in \mathbb{N}.$$

To prove the statement we will consider the locally constant potentials ψ_k defined by

$$\psi_k(x) = \frac{1}{k} \sup_{y \in C_k(x)} \log |(T^k)'(y)|.$$

We then have the following straightforward lemma.

Lemma 6.2. *For any $\mu \in \mathcal{M}(T)$ such that $\lambda(\mu) < \infty$ we have*

$$\left|\lambda(\mu) - \int \psi_k \mathrm{d}\mu\right| = o(1).$$

Proof. This follows simply because the variations of $\log|T'(x)|$ tend uniformly to 0. $\square$

We will first prove an analogous statement to Proposition 6.1 for $\psi_k(x)$ and then use Lemma 6.2 to deduce Proposition 6.1. For convenience for $\mu \in \mathcal{M}(T)$ we will let $\xi_k(\mu) = \int \psi_k \mathrm{d}\mu$.

Lemma 6.3. *Fix any $k \in \mathbb{N}$. Given $\underline{\gamma} \in Z$ and a sequence of invariant measures μ_j such that*

- *$h(\mu_j)/\xi_k(\mu_j) > s_\infty + \delta$ for some $\delta > 0$,*
- *$\int \phi_i \, d\mu_j \to \gamma_i$ for all $i \in \mathbb{N}$,*

there exists an invariant measure μ satisfying

$$\frac{h(\mu)}{\xi_k(\mu)} = \limsup_j \frac{h(\mu_j)}{\xi_k(\mu_j)} \quad \textit{and} \quad \int \phi_i \, d\mu = \gamma_i \ \forall i \in \mathbb{N}.$$

Note that to prove Lemma 6.3 it suffices to prove the statement for $k = 1$ since the statement for general k can then be deduced by considering the map T^k. The proof of Lemma 6.3 will now follow by a series of technical lemmas.

Lemma 6.4. *For any $\delta > 0$ there is $K(\delta) > 0$ such that if μ is a T-invariant measure and $\frac{h(\mu)}{\xi_1(\mu)} > s_\infty + \delta$, then $h(\mu) \le \xi_1(\mu) \le K(\delta)$.*

Proof. We fix $t \in \mathbb{R}$ such that $s_\infty < t < s_\infty + \delta$. By the methods from Lemma 2.4 we get $P(-t\psi_1) < \infty$. So by the variational principle we get $h(\mu) - t\xi_1(\mu) \le P(-t\psi_1)$. Since $\frac{h(\mu)}{\xi_1(\mu)} > s_\infty + \delta$, we have

$$P(-t\psi_1) \ge (s_\infty + \delta - t)\xi_1(\mu).$$

So,

$$\xi_1(\mu) \le \frac{P(-t \log T')}{s_\infty + \delta - t}.$$

$\square$

Therefore if the hypothesis of Lemma 6.3 holds, then we can deduce that the sequence of measures $\{\mu_j\}$ is tight and so will have at least one limit point μ which will be a T-invariant probability measure. Moreover by the lower-semi continuity of $\xi_1(\mu_j)$ (see Lemma 1 in [JMU05]), by the simple fact that $h(\mu) \le \lambda(\mu)$ and the fact that $\lambda(\mu) \le \xi_1(\mu)$ we know that $h(\mu) \le \xi_1(\mu) \le K$. To finish the proof of Proposition 6.1 we would only need entropy to be upper semicontinuous.

Unfortunately, the entropy is not upper semicontinuous on $\mathcal{M}(T)$. We have, however, a limited form of semicontinuity when we consider entropy divided by Lyapunov exponent, and this will be enough:

Lemma 6.5. *Let $\{\mu_j\}_{j \in \mathbb{N}}$ be a sequence of measures converging weakly to μ and satisfying that $h(\mu_j)/\xi_1(\mu_j) > s_\infty + \delta$ for some $\delta > 0$ and all $j \in \mathbb{N}$. We have*

$$\frac{h(\mu)}{\xi_1(\mu)} \ge \limsup \frac{h(\mu_j)}{\xi_1(\mu_j)}.$$

Proof. Denote by η_j the measure on Σ such that $\mu_j = \eta_j \circ \Pi^{-1}$. We start by choosing a subsequence of η_j such that $h(\eta_j)/\xi_1(\eta_j)$ converges to the maximal possible limit.

Given q, consider the projection $\pi_q : \Sigma \to \Sigma_q$ obtained by replacing in a sequence $\omega_1, \omega_2, \ldots$ all symbols $q+1, q+2, \ldots$ by symbol q. The projection of a measure ν under π_q will be denoted by $\nu|_q$.

Let us denote

$$c_{j,q} = \sum_{k>q} \eta_j([k]),$$

$$\tilde{\lambda}_q := |\log \inf_{x \in \cup_{l=q}^{\infty} I_l} \{|T'(x)|\}|.$$

Note that $c_{j,q}$ is uniformly (in j) converging to 0 as q increases. Consider the two partitions:

$$\mathcal{A} = \Big\{[1], [2], \ldots, [q-1], \bigcup_{k=q}^{\infty}[k]\Big\}, \quad \mathcal{B} = \Big\{ \bigcup_{k=1}^{q}[k], [q+1], [q+2], \ldots \Big\}.$$

We have

$$h(\eta_j) = h(\eta_j|\mathcal{A} \vee \mathcal{B}) \leq h(\eta_j|\mathcal{A}) + h(\eta_j|\mathcal{B}).$$

The former summand is $h(\eta_j|_q)$. The latter can be bounded from above by the entropy of the corresponding Bernoulli measure. It has one atom with measure $1 - c_{j,q}$ and the other atoms are cylinders $[k]$ $(k > q)$. Hence,

$$\begin{aligned} h(\eta_j|\mathcal{B}) &\leq (1-c_{j,q})|\log(1-c_{j,q})| + c_{j,q}|\log c_{j,q}| + c_{j,q}h(\nu_{j,q}) \\ &\leq c_{j,q}h(\nu_{j,q}) + \varepsilon_0(q), \end{aligned} \tag{6.1}$$

where $\nu_{j,q}$ is the Bernoulli measure obtained by assigning on each symbol $k > q$ probability $\eta_j([k])/c_{j,q}$, and $\varepsilon_0(q)$ converges to 0 as $q \to \infty$. We know that

$$\xi_1(\nu_{j,q}) \geq \log \inf_{x \in \bigcup_{l=q}^{\infty} I_l} \{|T'(x)|\} = \tilde{\lambda}_q$$

which must tend to ∞ as q goes to ∞. Thus by Lemma 6.4

$$\frac{h(\nu_{j,q})}{\xi_1(\nu_{j,q})} \leq s_\infty + \varepsilon_1(q) \tag{6.2}$$

for some $\varepsilon_1(q)$ converging to 0 as $q \to \infty$. At the same time,

$$\lambda(\eta_j) \geq \sum_k \eta_j([k])\psi_1(\Pi(\overline{k})) = \lambda(\eta_j|_q) + c_{j,q}(\lambda(\nu_{j,q}) - \psi_1(\Pi(\overline{q}))). \tag{6.3}$$

As $\xi_1(\eta_j) < \infty$, $c_{j,q}\psi_1(\Pi(\overline{q}))$ must converge to 0, but this convergence is not uniform. Still, from the sequence $\{\eta_j\}$ we can choose a subsequence η_{j_k}, a sequence q_l and a sequence $\varepsilon_2(q_l) \to 0$ such that for each q_l we have

$$\limsup_{j_k} c_{j_k,q_l}\psi_1(\Pi(\overline{q_l})) < \varepsilon_2(q_l).$$

Indeed, otherwise we would be able to choose a sequence η_{j_k} such that for some $c > 0$ and for any sufficiently big q we would have

$$\liminf_{j_k} c_{j_k,q}\psi_1(\Pi(\overline{q})) > c$$

and that would imply that $\xi_1(\eta_j) = \infty$.

So, finally we get by (6.1), (6.2) and (6.3) and Lemma 6.2 that given l, for all k big enough we have

$$h(\eta_{j_k}) - h(\eta_{j_k}|_{q_l}) < s_\infty \cdot K(j_k, q_l) + \varepsilon_3(q_l, \delta) \tag{6.4}$$

and

$$\xi_1(\eta_{j_k}) - \xi_1(\eta_{j_k}|_{q_l}) > K(j_k, q_l) - \varepsilon_3(q_l, \delta), \tag{6.5}$$

where $K(j,q) = c_{j,q}\xi_1(\nu_{j,q}) > 0$.

Now consider the following diagram:

$$\begin{tikzcd} \eta_{j_k} \arrow[r, dashed] \arrow[d] & \eta \\ \eta_{j_k}|_{q_l} \arrow[r, dashed] & \eta|_{q_l} \arrow[u] \end{tikzcd}$$

By (6.4) and (6.5), given l, for k big enough

$$\frac{h(\eta_{j_k}|_{q_l})}{\xi_1(\eta_{j_k}|_{q_l})} \geq \frac{h(\eta_{j_k})}{\xi_1(\eta_{j_k})} - \varepsilon(q_l, \delta).$$

The convergence of $\eta_{j_k}|_{q_l}$ to $\eta|_{q_l}$ takes place in space of invariant measures of (Σ_{q_l}, σ), where entropy (and hence h/ξ_1) is upper semicontinuous. Finally, $h(\eta) = \lim h(\eta|_{q_l})$. Taking $\mu = \eta \circ \Pi^{-1}$, we have

$$\frac{h(\mu)}{\xi_1(\mu)} > \lim \frac{h(\eta_{j_k})}{\xi_1(\eta_{j_k})} - \varepsilon(q_l).$$

As we can choose arbitrarily big q_l, $\varepsilon(q_l)$ is arbitrarily small. We are done. $\square$

The statement of Lemma 6.3 now follows.

To complete the proof of Proposition 6.1 choose a sequence of T-invariant measures μ_j such that

- $h(\mu_j)/\lambda(\mu_j) > s_\infty + \delta$ for some $\delta > 0$,
- $\int \phi_i \mathrm{d}\mu_j \to \gamma_i$ for all $i \in \mathbb{N}$.

We choose $\varepsilon > 0$ sufficiently small such that $h(\mu_j)/(\lambda(\mu_j) + \varepsilon) > s_\infty + \delta/2$. We then choose k sufficiently large such that $\mathrm{var}_k(\log|T'(x)|) < \varepsilon$ and so in particular $\xi_k(\mu_j) - \lambda(\mu_j) < \varepsilon$. Thus $h(\mu_j)/\xi_k(\mu_j) > s_\infty + \delta/2$ and we may apply Lemma 6.3 to show that there exists a T-invariant measure μ such that $h(\mu)/\xi_k(\mu) = \limsup_{j\to\infty} h(\mu_j)/\xi_k(\mu_j)$ and $\int \phi_i \mathrm{d}\mu = \gamma_i$ for all $i \in \mathbb{N}$. Moreover

$$\begin{aligned} \limsup_{j\to\infty} \frac{h(\mu_j)}{\lambda(\mu_j)} &\geq \limsup_{j\to\infty} \frac{h(\mu_j)}{\xi_k(\mu_j)} = h(\mu)/\xi_k(\mu) \\ &\geq \frac{h(\mu)}{\lambda(\mu) + \varepsilon} \geq \frac{h(\mu)}{\lambda(\mu)} + \frac{\varepsilon h(\mu)}{\lambda(\mu)^2 + \varepsilon\lambda(\mu)} \end{aligned}$$

and Proposition 6.1 now easily follows.

This completes the proof of Theorem 1.2.

7. Examples

We now look at some examples where our results can be applied. We will consider an application to frequency of digits which applies the fact that our level sets are defined using countably many functions. We then consider two cases which look at possible behaviour when the level set is just determined by one bounded function.

7.1. **Frequency of digits.** There have been many papers on the Hausdorff dimension of sets determined by the frequency of digits for various types of expansion; see for example [Bes35], [BSS02a], [Egg49], [FLM10], [FLMW10], [Ols04]. Here we show how our results can be applied to give results in this direction in the setting of expanding maps with countably many branches. We take a partition $\{I_i\}_{i\in\mathbb{N}}$ and a map T as in the first section. We define ϕ_i to be the characteristic function for the interval I_i, that is,

$$\phi_i(x) = \chi_{I_i}(x) := \begin{cases} 1 & \text{if} \quad x \in I_i, \\ 0 & \text{if} \quad x \notin I_i. \end{cases}$$

For an infinite vector $\underline{p} = (p_1, p_2, \ldots)$ where $\sum_{i=1}^{\infty} p_i \leq 1$ let

$$\Lambda_{\underline{p}} = \{x \in \Lambda : \lim_{n\to\infty} A_n\phi_i(x) = p_i \text{ for all } i \in \mathbb{N}\}.$$

The assumptions of Theorem 1.2 are all satisfied and it is easy to see that all such $\underline{p}$ belong to Z. Therefore

$$\dim \Lambda_{\underline{p}} = \max\left\{s_\infty,\ \alpha_3(\underline{p})\right\}$$

where

$$\alpha_3(\underline{p}) = \sup_{\mu\in\mathcal{M}(T)} \left\{\frac{h(\mu)}{\lambda(\mu)} : \mu(I_i) = p_i \ \forall i \in \mathbb{N},\ h(\mu) < \infty\right\}.$$

We refer to these sets $\Lambda_{\underline{p}}$ as "sets of digit frequency". This is because in the case where T is the Gauss map, $T(x) = 1/x \mod 1$, $A_n\phi_i(x)$ gives the frequency of i in the first n terms of the continued fraction expansion of x. In particular our work shows that the dimension of such a set is always bounded below by s_∞ even if the frequencies sum to less than 1. Note that $s_\infty = 1/2$ when T is the Gauss map. This problem has already been studied in the setting of continued fractions ([FLM10]), and in the countable state symbolic space ([FLMW10]). Our work shows that this phenomenon extends to more general countable branch expanding maps. We should also point out that there was a step missing from the proof in [FLM10] where the argument of how to go from the statement of Theorem 1.1 to Theorem 1.2 was not given. The section on the proof of Theorem 1.2 shows how this can be done.

7.2. **Harmonic averages for continued fractions.** For another example we again let T be the Gauss map. If we just take one potential $\phi : [0,1]\backslash\mathbb{Q} \to \mathbb{R}$ defined by $\phi(x) = \frac{1}{a_1(x)}$ where $a_1(x)$ is the first digit in the continued expansion of x, then Theorem 1.2 is still applicable. In particular, if for $\alpha \in [0,1]$ we let

$$\Lambda_\alpha = \left\{x \in [0,1]\backslash\mathbb{Q} : \lim_{n\to\infty} \frac{\frac{1}{a_1(x)} + \frac{1}{a_2(x)} + \cdots + \frac{1}{a_n(x)}}{n} = \alpha\right\},$$

then we have

$$\dim \Lambda_\alpha = \max\left\{\frac{1}{2}, \sup_{\mu\in\mathcal{M}(T)} \left\{\frac{h(\mu)}{\lambda(\mu)} : \int \phi \mathrm{d}\mu = \alpha,\ h(\mu) < \infty\right\}\right\}.$$

From this we can deduce that

$$\dim \Lambda_0 = \Lambda_1 = \frac{1}{2}.$$

For Λ_1, note that the Dirac measure on the point $\frac{\sqrt{5}-1}{2}$ is the only T-invariant measure ν with $\int \phi \mathrm{d}\nu = 1$. However, despite the fact that this measure clearly has dimension 0, the set Λ_1 still has dimension $\frac{1}{2}$.

Furthermore, in this case we can show that the only points where the dimension achieves the lower bound $\frac{1}{2}$ are the end points of the spectrum.

Proposition 7.1. *For all $\alpha \in (0,1)$ $\dim \Lambda_\alpha > \frac{1}{2}$.*

Proof. Fix $\alpha \in (0,1)$. Consider the set of irrationals x for which the continued fraction expansion $a_1(x), a_2(x), \ldots$ satisfies that for some $N \in \mathbb{N}$ $a_i(x) > N$ for all $i \in \mathbb{N}$. We will denote this set E_N and note that if we consider the restriction of the Gauss map T to the union of the intervals I_j $(j \geq N)$, then E_N is its attractor and the corresponding value of s_∞ is still $\frac{1}{2}$. In [JK10] it is shown that

$$\dim E_N \sim \frac{1}{2} + \frac{\log \log N}{2 \log N}.$$

Since $\frac{1}{2} < \dim E_N$, we can deduce that E_N admits an ergodic measure of maximal dimension μ_N with $h(\mu_N) < \infty$. Note that for N sufficiently large we have that $\lambda(\mu_N) \geq \log N$.

Take δ_1 to be the Dirac measure at $\frac{\sqrt{5}-1}{2}$. Then δ_1 is ergodic and $\int \phi \mathrm{d}\delta_1 = 1$. Now consider measures of the form

$$\nu_p = p\mu_N + (1-p)\delta_1.$$

If we choose $p > \frac{\lambda(\delta_1) 4 \log N}{\log \log N \lambda(\mu_N)}$, then we have

$$\begin{aligned} h(\nu_p) &= ph(\mu_N) \geq p\left(\frac{1}{2} + \frac{\log \log N}{4 \log N}\right)\lambda(\mu_N) \\ &= \frac{p}{2}\lambda(\mu_N) + p\frac{\log \log N}{4 \log N}\lambda(\mu_N) \\ &> \frac{1}{2}(p\lambda(\mu_N) + (1-p)\lambda(\delta_1)) = \frac{1}{2}\lambda(\nu_p). \end{aligned}$$

Thus $\frac{h(\nu_p)}{\lambda(\nu_p)} > \frac{1}{2}$. Furthermore, since $\lim_{N\to\infty} \frac{\lambda(\delta(1)) 4 \log N}{\log \log N \lambda(\mu_N)} = 0$ and $\lim_{N\to\infty} \int \phi \mathrm{d}\mu_N = 0$, we can choose q such that $\frac{h(\nu_p)}{\lambda(\nu_p)} > \frac{1}{2}$ for all $p > q$ and $\alpha = \int \phi \mathrm{d}\nu_p$ for some $p > q$. $\square$

It is straightforward to adapt this argument to the case where T is the Gauss map and where ϕ is a bounded function with variations uniformly tending to 0. This will show that the interior of the spectrum is strictly greater than $\frac{1}{2}$. However this is not always the case for alternative choices of T. A simple counterexample is when $P(-s_\infty \log |T'|) \leq 0$ and ϕ is any bounded potential. In this case $\dim \Lambda_\alpha = s_\infty$ for all

$$\alpha \in \left[\inf_{\mu \in \mathcal{M}(T)} \left\{\int \phi \mathrm{d}\mu\right\}, \sup_{\mu \in \mathcal{M}(T)} \left\{\int \phi \mathrm{d}\mu\right\}\right].$$

7.3. **Locally flat spectrum.** Here we look at single functions where the multifractal spectrum will have interesting phase transitions. These are examples where the function $\alpha \to \dim \Lambda_\alpha$ has flat regions but for which the whole spectrum is not flat. Let T be a piecewise linear map defined using a partition (similar maps are studied in [KMS12]) as follows. We consider a set of disjoint closed intervals $\{I_i\}_{i=1}^{\infty}$. Denote s_∞ as before and let

$$K = \mathrm{diam}(I_1)^{s_\infty} \text{ and } C = \sum_{i=2}^{\infty} \mathrm{diam}(I_i)^{s_\infty}.$$

We will assume that $C < 1$, $K+C > 1$. (These can be easily satisfied. For example, take $|I_n| \approx n^{-2}(\log n)^{-4}$.) Define T to be the piecewise linear map which maps each interval I_i bijectively to the interval $[0,1]$. These conditions will ensure that

$$\dim \Lambda > s_\infty, \qquad P(-s_\infty \log |T'|) < \infty.$$

We will take $\phi = \chi_{I_1}$, that is, the characteristic function for the interval I_1. We will prove the following result.

Theorem 7.2. *There exist* $0 < \alpha_* < \alpha^* < 1$ *such that* $\dim \Lambda_\alpha = s_\infty$ *for* $\alpha \in [0, \alpha_*] \cup [\alpha^*, 1]$ *and* $\dim \Lambda_\alpha > s_\infty$ *for* $\alpha \in (\alpha_*, \alpha^*)$.

Proof. We will prove Theorem 7.2 by a series of propositions and lemmas. We start with the following general proposition.

Proposition 7.3. *Let* $\phi : \Lambda \to \mathbb{R}$ *have variations uniformly converging to* 0. *For any* $\alpha \in \mathbb{R}$ *if there exist* q, δ *such that*

$$P(q(\phi - \alpha) - \delta \log |T'|) \leq 0,$$

then

$$\sup_{\mu \in \mathcal{M}(T)} \left\{ \frac{h(\mu)}{\lambda(\mu)} : \int \phi d\mu = \alpha \text{ and } \lambda(\mu) < \infty \right\} \leq \delta.$$

Proof. Let $\mu \in \mathcal{M}(T)$ such that $\int \phi \mathrm{d}\mu = \alpha$ and $\lambda(\mu) < \infty$. By the variational principle, we have

$$h(\mu) + \int (q(\phi - \alpha) - \delta \log |T'|) \mathrm{d}\mu \leq 0.$$

So,

$$h(\mu) - \delta \lambda(\mu) \leq 0.$$

Thus $h(\mu)/\lambda(\mu) \leq \delta$ which completes the proof. □

Therefore, for our specific choice of T and ϕ if we can find $q > 0$ and $\alpha^* \in (0,1)$ such that $P(q(\phi - \alpha^*) - s_\infty \log |T'|) = 0$, then $\dim \Lambda_\alpha = s_\infty$ for all $\alpha \in (\alpha^*, 1)$. Similarly if we can find $q < 0$ and $\alpha_* \in (0,1)$ such that $P(q(\phi - \alpha_*) - s_\infty \log |T'|) = 0$, then $\dim \Lambda_\alpha = s_\infty$ for all $\alpha \in (0, \alpha_*)$.

We are going to show that we can indeed find such α_*, α^*. We can calculate

$$P(q(\phi - \alpha) - s_\infty \log |T'|) = \log(Ke^q + C) - \alpha q.$$

By solving the equation $P(q(\phi - \alpha) - s_\infty \log |T'|) = 0$, we have

$$\alpha(q) = \frac{\log(Ke^q + C)}{q}, \quad q \neq 0.$$

We then have the following lemma.

Lemma 7.4. *Such* α_*, α^* *do exist.*

Proof. The function $\alpha(q)$ has the following properties:

(1) The function $\alpha(q)$ is real analytic on both $(-\infty, 0)$ and $(0, \infty)$.

(2) $\lim_{q\to\infty}\alpha(q)=1$ and $\lim_{q\to-\infty}\alpha(q)=0$.
(3) $\lim_{q\to 0+}\alpha(q)=+\infty$ and $\lim_{q\to 0-}\alpha(q)=-\infty$.
(4) Under our conditions $K+C>1$ and $C<1$, $\alpha(q)<1$ for $q<0$ and $\alpha(q)>0$ for $q>0$ and the equation $\alpha(q)=0$ admits only one solution

$$q=q_-=\log\frac{1-C}{K}<0$$

and the equation $\alpha(q)=1$ admits only one solution

$$q=q_+=\log\frac{C}{1-K}>0.$$

From the above properties, one can see the minimum and maximum of the following can be obtained:

$$\alpha^*=\inf_{q>0}\alpha(q)=\inf_{q>q_+}\alpha(q)\quad\text{and}\quad\alpha_*=\inf_{q<0}\alpha(q)=\inf_{q<q_-}\alpha(q).$$

These are what we want. □

Therefore we have that for any $\alpha\in[0,\alpha_*]\cup[\alpha^*,1]$ there exists q such that $P(q(\phi-\alpha)-s_\infty\log|T'|)\le 0$ and so by Proposition 7.3 we have

$$\dim\Lambda_\alpha=s_\infty,\quad\forall\alpha\in[0,\alpha_*]\text{ and }\forall\alpha\in[\alpha^*,1].$$

Now we need to show

$$\forall\alpha\in(\alpha_*,\alpha^*),\quad\dim_H\Lambda_\alpha>s_\infty.$$

For $t\in[s_\infty,\dim\Lambda]$, denote

$$K(t)=|I_1|^t\quad\text{and}\quad C(t)=\sum_{i=2}^{\infty}|I_i|^t.$$

Let $f(t,q)=P(q\phi-t\log|T'|)$. Then the dimension of the set Λ_α is the first component $t(\alpha)$ of the solution $(t(\alpha),q(\alpha))$ to the following system (see [FLWW09]):

$$\begin{cases} f(t,q)=q\alpha,\\ \dfrac{\partial f}{\partial q}(t,q)=\alpha\end{cases}\tag{7.1}$$

whenever such a solution exists. By a simple calculation we have

$$f(t,q)=\log(K(t)e^q+C(t)).$$

For a fixed t, let $f_t(q)=f(t,q)$.

Lemma 7.5. *For $\alpha\in(\alpha_*,\alpha^*)$ we have that $P(q(\phi-\alpha)-s_\infty\log|T'|)>0$ for all q and that $P(q(\phi-\alpha)-(\dim\Lambda)\cdot\log|T'|)\le 0$ for some $q\in\mathbb{R}$.*

Proof. The function $q\mapsto f_t(q)$ has the following properties:

(1) For $t\in(s_\infty,\dim\Lambda)$, the function $f_t(q)$ has two asymptotic lines $y=\log C(t)$ for $q\to-\infty$ and $y=x+\log K(t)$ for $q\to\infty$. In particular note that for any $\alpha\in(0,1)$ there exists $q(\alpha,t)$ such that $f_t'(q(\alpha,t))=\alpha$.
(2)
$$\alpha^*=\inf_{q>0}\frac{f_{s_\infty}(q)}{q}<1\quad\text{and}\quad\alpha_*=\inf_{q<0}\frac{f_{s_\infty}(q)}{q}>0.$$
(3) If $\alpha\in(\alpha_*,\alpha^*)$, then $f_{s_\infty}(q)=\alpha q$ has no solution.

By property (3) and property (2) we can thus deduce that for $\alpha \in (\alpha_*, \alpha^*)$ and for any $q \in \mathbb{R}$

$$P(q(\phi - \alpha) - s_\infty \log |T'|) = f_{s_\infty}(q) - \alpha q > 0,$$

which is the first part of the lemma.

By property (1) if we let $s = \dim \Lambda$, then there exists $q(\alpha, s)$ such that $f'_s(q(\alpha, s)) = \alpha$. It then follows that there will be an equilibrium state $\mu_{q,s}$ such that $\int \phi \mathrm{d}\mu_{q,s} = \alpha$ and

$$f_s(q(\alpha, s)) = \alpha q - s\lambda(\mu_{q,s}) + h(\mu_{q,s}) \leq \alpha q.$$

Thus the second part of the lemma follows. $\square$

Due to the fact that $f(t, q)$ depends analytically on t, q in the region $t > s_\infty, q \in \mathbb{R}$, we can now assert that for $\alpha \in (\alpha_*, \alpha^*)$ there exists $t(\alpha) \in (s_\infty, \dim \Lambda)$ which is the first coordinate of the solution $(t(\alpha), q(\alpha))$ to (7.1) and thus $\dim \Lambda_\alpha = t(\alpha)$. This completes the proof of Theorem 7.2. $\square$

We can also deduce that if μ_{SRB} is the equilibrium state for the potential $-(\dim \Lambda) \cdot \log |T'|$ and $\tilde{\alpha} = \int \phi \mathrm{d}\mu_{SRB}$, then the function $\alpha \to \dim \Lambda_\alpha$ is strictly increasing on $(\alpha_*, \tilde{\alpha})$ and strictly decreasing on $(\tilde{\alpha}, \alpha_*)$ and by the implicit function theorem varies analytically in the region (α_*, α^*).

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