

EISENSTEIN SERIES ON LOOP GROUPS

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ABSTRACT. Based on Garland’s work, in this paper we construct the Eisenstein series on the adelic loop groups over a number field, induced from either a cusp form or a quasi-character which is assumed to be unramified. We compute the constant terms and prove their absolute and uniform convergence under the affine analog of Godement’s criterion. For the case of quasi-characters the resulting formula is an affine Gindikin-Karpelevich formula. Then we prove the convergence of Eisenstein series themselves in certain analogs of Siegel subsets.

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1. INTRODUCTION

One of the most important tools to study automorphic forms is the theory of Eisenstein series. In the fundamental work of R. Langlands [31], he showed how to

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get automorphic L -functions from the constant terms of the Eisenstein series. This method, which was further developed by F. Shahidi and known as the Langlands-Shahidi method, has been applied to the Ramanujan conjecture and Langlands functoriality [26, 27, 35]. On the other hand, H. Garland [10–12] has made very important generalizations to loop groups. He considered the Eisenstein series induced from a character and proved the absolute convergence of constant terms first and then the Eisenstein series itself, under certain affine analog of Godement's criterion. His work lays the foundation of this field and gives the first example of automorphic forms on infinite dimensional groups.

Based on the methods of Garland, in this paper we study the Eisenstein series defined on adelic loop groups over a number field, induced from either a cusp form or a quasi-character which is unramified. We prove the absolute and uniform convergence of these series and analyze their constant terms and Fourier coefficients.

Given an untwisted affine Kac-Moody Lie algebra $\tilde{\mathfrak{g}}$ associated to a complex simple Lie algebra \mathfrak{g} , we have the affine root system $\tilde{\Phi}$ and the set of simple roots $\tilde{\Delta} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ such that $\Delta = \{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots of \mathfrak{g} . The affine Weyl group \tilde{W} is isomorphic to the semi-direct product $W \ltimes Q^\vee$ where W is the Weyl group of \mathfrak{g} and Q^\vee is the coroot lattice. Associated to $\tilde{\mathfrak{g}}$ we first construct the central extension

$$1 \rightarrow F^\times \rightarrow \hat{G}(F((t))) \rightarrow G(F((t))) \rightarrow 1$$

and then form the semi-direct product

$$\tilde{G}(F((t))) = \hat{G}(F((t))) \ltimes \sigma(F^\times),$$

where F is any field and $\sigma(\mathfrak{q})$, $\mathfrak{q} \in F^\times$, acts on $F((t))$ as the automorphism $t \mapsto \mathfrak{q}t$. There are two methods to construct the central extension. One is via the tame symbol (3.14) on $F((t))^\times$, and the other is to use a rational representation of G and the method of determinant bundles (see [1]). In Theorem 3.18 we give the explicit relation between these two constructions. More precisely, we obtain a homomorphism between loop groups, which is identity after modulo the center, and when restricted on the center is to the power of the Dynkin index of the rational representation.

For a number field F with adele ring \mathbb{A} and idele group \mathbb{I} , we may form the adelic loop group $\tilde{G}(\mathbb{A}\langle t \rangle) = \hat{G}(\mathbb{A}\langle t \rangle) \ltimes \sigma(\mathbb{I})$, where $\mathbb{A}\langle t \rangle = \prod_v' F_v((t))$ is the restricted product with respect to $\mathcal{O}_v((t))$ for all finite places v . The “ F -rational points” of the loop group is $\tilde{G}(F\langle t \rangle)$, where $F\langle t \rangle = F((t)) \cap \mathbb{A}\langle t \rangle$. We have defined the subgroups \hat{B}_v and \hat{K}_v of $\hat{G}(F_v((t)))$ for each place v , which are analogues of the Borel subgroup and maximal compact subgroup respectively. More concretely, \hat{B}_v is the preimage of the Borel subgroup of $G(F_v)$ under the map

$$\hat{G}(F_v[[t]]) \longrightarrow G(F_v[[t]]) \xrightarrow{t=0} G(F_v).$$

We have the Iwasawa decomposition $\hat{G}(F_v((t))) = \hat{B}_v \hat{K}_v$. The group $\hat{G}(F_v[[t]])$ can be interpreted as the maximal parabolic subgroup of $\hat{G}(F_v((t)))$ corresponding to Δ . It can be shown that the central extension splits over $G(F_v[[t]])$, i.e. we may realize $G(F_v[[t]])$ as a subgroup of $\hat{G}(F_v((t)))$ canonically. The corresponding results for the adelic groups are formulated in an obvious way. There is also the Bruhat

decomposition $\widehat{G}(F_v(\langle t \rangle)) = \widehat{B}_v \widehat{W} \widehat{B}_v$. These results can be proved by using the standard theory of Tits systems.

Fix $\mathfrak{q} \in \mathbb{I}$. If f is an unramified cusp form on $G(\mathbb{A})$, $s \in \mathbb{C}$, we define a function \widetilde{f}_s on $\widehat{G}(\mathbb{A}(\langle t \rangle)) \rtimes \sigma(\mathfrak{q})$ by

$$\widetilde{f}_s(g) = |c|^s f(p_0),$$

where $g = cp\sigma(\mathfrak{q})k$ is the Iwasawa decomposition such that $c \in \mathbb{I}$, $p \in G(\mathbb{A}(\langle t \rangle)_+)$ with $\mathbb{A}(\langle t \rangle)_+ = \mathbb{A}(\langle t \rangle) \cap \mathbb{A}[[t]]$, $k \in \widehat{K}$, and p_0 is the image of p under the projection $G(\mathbb{A}(\langle t \rangle)_+) \xrightarrow{t=0} G(\mathbb{A})$. This function is well defined and we construct the Eisenstein series defined on $\widehat{G}(\mathbb{A}(\langle t \rangle)) \rtimes \sigma(\mathfrak{q})$ as

$$E(s, f, g) = \sum_{\gamma \in \widehat{G}(F(\langle t \rangle)_+) \backslash \widehat{G}(F(\langle t \rangle))} \widetilde{f}_s(\gamma g),$$

where $F(\langle t \rangle)_+ = F(\langle t \rangle) \cap F[[t]]$. The Eisenstein series is left invariant under $\widehat{G}(F(\langle t \rangle))$ and right invariant under \widehat{K} . Similar construction applies for an unramified quasi-character $\chi_{\widehat{T}}$ on $\widehat{T}(\mathbb{A})/\widehat{T}(F)$ where \widehat{T} is the maximal torus of \widehat{B} .

The unipotent radical \widehat{U} of \widehat{B} is the subgroup corresponding to the set of all the positive roots of $\widetilde{\Phi}$. It can be proved that $\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})$ is compact and inherits the product measure from that of \mathbb{A}/F . We define the constant term of $E(s, f, g)$ along \widehat{B} by

$$E_{\widehat{B}}(s, f, g) = \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} E(s, f, ug) du.$$

The following theorem generalizes Garland's results in [11].

Theorem 1.1. (i) Suppose that $g \in \widehat{G}(\mathbb{A}(\langle t \rangle)) \rtimes \sigma(\mathfrak{q})$ with $\mathfrak{q} \in \mathbb{I}$ and $|\mathfrak{q}| > 1$, $s \in H = \{z \in \mathbb{C} | \Re z > h + h^\vee\}$, where h (resp. h^\vee) is the Coxeter (resp. dual Coxeter) number. Then $E(s, f, ug)$, as a function on $\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})$, converges absolutely outside a subset of measure zero and is measurable.

(ii) For any $\varepsilon, \eta > 0$, let $H_\varepsilon = \{z \in \mathbb{C} | \Re z > h + h^\vee + \varepsilon\}$, $\sigma_\eta = \{\sigma(\mathfrak{q}) | \mathfrak{q} \in \mathbb{I}, |\mathfrak{q}| > 1 + \eta\}$. The integral defining $E_{\widehat{B}}(s, f, g)$ converges absolutely and uniformly for $s \in H_\varepsilon$, $g \in \widehat{U}(\mathbb{A})\Omega\sigma_\eta\widehat{K}$, where Ω is a compact subset of $T(\mathbb{A})$.

(iii) Replace \widetilde{f}_s by the height function $h_s = |c|^s$, and denote the resulting series by $E(s, h, g)$. Then for $a \in \widehat{T}(\mathbb{A})$,

$$(1.1) \quad E_{\widehat{B}}(s, h, a\sigma(\mathfrak{q})) = \sum_{w \in W \backslash \widetilde{W}} (a\sigma(\mathfrak{q}))^{\tilde{\rho} - w^{-1}\tilde{\rho} + w^{-1}sL} c_w(s),$$

where the summation is taken over representatives of minimal length of the cosets $W \backslash \widetilde{W}$, L is the fundamental weight corresponding to α_0 , $\tilde{\rho} \in \tilde{\mathfrak{h}}^*$ satisfies $\langle \tilde{\rho}, \alpha_i^\vee \rangle = 1$, $i = 0, 1, \dots, n$, and

$$(1.2) \quad c_w(s) = \prod_{\beta \in \tilde{\Phi}_+ \cap w\tilde{\Phi}_-} \frac{\Lambda_F(\langle sL - \tilde{\rho}, \beta^\vee \rangle)}{\Lambda_F(\langle sL - \tilde{\rho}, \beta^\vee \rangle + 1)},$$

with Λ_F the normalized Dedekind zeta function.

The formula (1.1) is an affine analogue of the Gindikin-Karpelevich formula. The condition $\Re z > h + h^\vee$ is an affine analogue of Godement's criterion. Similarly,

if we consider the Eisenstein series $E(\chi_{\widehat{T}}, g)$ induced from an unramified quasi-character $\chi_{\widehat{T}}$ on $\widehat{T}(\mathbb{A})/\widehat{T}(F)$, then under the condition $|\mathfrak{q}| > 1$ and $\Re(\chi_{\widehat{T}}\alpha_i^\vee) > 2$, $i = 0, 1, \dots, n$, the constant term $E_{\widehat{B}}(\chi_{\widehat{T}}, a\sigma(\mathfrak{q}))$, $a \in \widehat{T}$, is given by

$$(1.3) \quad E_{\widehat{B}}(\chi_{\widehat{T}}, a\sigma(\mathfrak{q})) = \sum_{w \in \widehat{W}} (a\sigma(\mathfrak{q}))^{\widetilde{\rho} + w^{-1}(\chi_{\widehat{T}} - \widetilde{\rho})} c_w(\chi_{\widehat{T}}),$$

where

$$(1.4) \quad c_w(\chi_{\widehat{T}}) = \prod_{\beta \in \widetilde{\Phi}_+ \cap w\widetilde{\Phi}_-} |\Delta_F|^{-\frac{1}{2}} \frac{L(-\langle \widetilde{\rho}, \beta^\vee \rangle, \chi_{\widehat{T}}\beta^\vee)}{L(1 - \langle \widetilde{\rho}, \beta^\vee \rangle, \chi_{\widehat{T}}\beta^\vee)}.$$

Here Δ_F is the discriminant of F and $L(s, \chi)$ is the Hecke L -function.

We have also considered the Fourier coefficients of our Eisenstein series. To obtain a general formula would be quite difficult and non-trivial. But at least for \widetilde{SL}_2 we have computed everything explicitly, and the formulas are given in Section 4.4.

Following Garland's approach in [12], we also prove some results on the absolute convergence of the Eisenstein series themselves instead of the constant terms. For example, we establish uniform convergence over certain analogues of Siegel sets. The proof is technical and involves the systematic use of Demazure modules together with estimations of some norms for both archimedean and non-archimedean cases. Let us only state the main results along this direction.

Theorem 1.2. *Fix $\mathfrak{q} \in \mathbb{I}$, $|\mathfrak{q}| > 1$. There exists a constant $c_{\mathfrak{q}} > 0$ depending on \mathfrak{q} , such that for any $\varepsilon > 0$ and compact subset Ω of $T(\mathbb{A})$, $E(s, f, g)$ and $E(s, h, g)$ converge absolutely and uniformly for $s \in \{z \in \mathbb{C} | \Re z > \max(h + h^\vee + \varepsilon, c_{\mathfrak{q}})\}$ and $g \in \widehat{U}(\mathbb{A})\Omega\sigma(\mathfrak{q})\widehat{K}$.*

Theorem 1.3. *There exist constants $c_1, c_2 > 0$ which depend on the number field F , such that for any $\varepsilon > 0$ and compact subset Ω of $T(\mathbb{A})$, $E(s, f, g)$ and $E(s, h, g)$ converge absolutely and uniformly for $s \in \{z \in \mathbb{C} | \Re z > \max(h + h^\vee + \varepsilon, c_1 h^\vee)\}$ and $g \in \widehat{U}(\mathbb{A})\Omega\sigma_{c_2}\widehat{K}$.*

We conjecture that Theorem 1.3 is true for $c_1 = 1$ (in which case the first condition reads $s \in H_\varepsilon$) and arbitrary $c_2 > 0$. In other words, we conjecture that the domain of uniform convergence for the constant term $E_{\widehat{B}}(s, f, g)$ in Theorem 1.1 also applies for $E(s, f, g)$ itself. We again interpret this as the analogue of Godement's criterion. We have proved the conjecture for $F = \mathbb{Q}$. For the geometric analogue we know that the conjecture is true for $F = \mathbb{F}_q(T)$, the function field of $\mathbb{P}_{\mathbb{F}_q}^1$.

The theory of Eisenstein series on infinite dimensional groups is far from complete. Besides the above conjecture, let us propose some other related open problems.

(A) Build the foundations of representation theory and harmonic analysis for infinite dimensional algebraic groups. Since we are dealing with groups which are not locally compact, we do not have Haar measures. One should also be concerned with induction from ramified representations, in contrast to what we do in this paper where we only consider induction from unramified cusp forms or quasi-characters.

(B) Generalize the theory of Eisenstein series further to all Kac-Moody groups (see [28] for the theory of Kac-Moody groups) and also non-split infinite dimensional

groups. Compute the constant and non-constant coefficients to see if there are any new L -functions [17, 36].

(C) Establish the Maass-Selberg relations [13–16] and as applications prove the analytic continuation and functional equations for the Eisenstein series. This project, together with (B), would be crucial for the generalization of the Langlands-Shahidi method.

(D) In his thesis [37], M. Patnaik investigated the geometric meaning of Eisenstein series on loop groups over a function field, where he used the concept of ribbons [24]. It would be interesting to consider this problem for the number field case.

2. AFFINE KAC-MOODY LIE ALGEBRAS

In this section we review the theory of affine Kac-Moody Lie algebras. The basic references are [7, 23, 42].

2.1. Definition. Let \mathfrak{g} be a complex simple finite dimensional Lie algebra. Let $(,)$ denote an invariant symmetric bilinear form on \mathfrak{g} , normalized such that the square length of a long root is equal to 2. Following [7] we call it the standard bilinear form. The affine Lie algebra $\widetilde{\mathfrak{g}}$ is a complex infinite dimensional Lie algebra constructed as follows.

Let $\mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in the indeterminate t over \mathbb{C} . For a Laurent polynomial $P = \sum c_i t^i$ the residue is defined by $\text{Res } P = c_{-1}$. Consider the complex infinite dimensional Lie algebra $\overline{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$. The invariant form on \mathfrak{g} can be extended naturally to a bilinear $\mathbb{C}[t, t^{-1}]$ -valued form on $\overline{\mathfrak{g}}$, which we again denote by $(,)$. Any derivation D of $\mathbb{C}[t, t^{-1}]$ can be extended to a derivation of $\overline{\mathfrak{g}}$ by $D(P \otimes g) = D(P) \otimes g$.

Define a \mathbb{C} -valued bilinear form ψ on $\overline{\mathfrak{g}}$ by

$$\psi(x, y) = \text{Res} \left(\frac{dx}{dt}, y \right).$$

ψ satisfies the properties:

- (i) $\psi(x, y) = -\psi(y, x)$ and
- (ii) $\psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) = 0$.

Then ψ is a 2-cocycle and we define $\hat{\mathfrak{g}}$ to be the corresponding one-dimensional central extension of $\overline{\mathfrak{g}}$. The affine algebra $\widetilde{\mathfrak{g}}$ is obtained by adding to $\hat{\mathfrak{g}}$ a derivation d which acts on $\overline{\mathfrak{g}}$ as $t \frac{d}{dt}$ and acts on the center as 0.

More precisely, $\widetilde{\mathfrak{g}}$ is the complex vector space

$$\widetilde{\mathfrak{g}} = (\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with the Lie bracket

$$[x_1 \oplus \alpha_1 c \oplus \beta_1 d, x_2 \oplus \alpha_2 c \oplus \beta_2 d] = \left([x_1, x_2] + \beta_1 t \frac{dx_2}{dt} - \beta_2 t \frac{dx_1}{dt} \right) \oplus \psi(x_1, x_2)c.$$

Here $x_i \in \overline{\mathfrak{g}}$, $[x_1, x_2]$ is the bracket in the Lie algebra $\overline{\mathfrak{g}}$ and $\alpha_i, \beta_i \in \mathbb{C}$.

We introduce a \mathbb{C} -valued bilinear form $(,)$ on $\widetilde{\mathfrak{g}}$ by

$$(x_1 \oplus \alpha_1 c \oplus \beta_1 d, x_2 \oplus \alpha_2 c \oplus \beta_2 d) = \text{Res} (t^{-1} (x_1, x_2)) + \alpha_1 \beta_2 + \alpha_2 \beta_1.$$

It is easy to check that this bilinear form is symmetric, non-degenerate and invariant. Note that the restriction of the form $(,)$ to the subalgebra $\mathfrak{g} \subset \widetilde{\mathfrak{g}}$ induces

the standard bilinear form on \mathfrak{g} . Following [7] we also call the form $(,)$ on $\tilde{\mathfrak{g}}$ the standard bilinear form.

2.2. Root system of $\tilde{\mathfrak{g}}$ and subalgebras in $\tilde{\mathfrak{g}}$. Let \mathfrak{h} denote a Cartan subalgebra of \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$ be the root space decomposition of \mathfrak{g} with respect to \mathfrak{h} ; here $\Phi \subset \mathfrak{h}^*$ is the system of roots. We fix a choice of positive roots $\Phi_+ \subset \Phi$; let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the subset of simple roots and let $\tilde{\alpha}$ be the highest root.

Define the following subalgebra in $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

This is a maximal abelian diagonalizable subalgebra in $\tilde{\mathfrak{g}}$ and is called a Cartan subalgebra of $\tilde{\mathfrak{g}}$. For $\alpha \in \tilde{\mathfrak{h}}^*$ the attached root space is

$$\tilde{\mathfrak{g}}_{\alpha} = \{x \in \tilde{\mathfrak{g}} \mid [h, x] = \alpha(h)x, h \in \tilde{\mathfrak{h}}\},$$

and α is called a root if $\tilde{\mathfrak{g}}_{\alpha} \neq 0$. We extend any linear function $\lambda \in \mathfrak{h}^*$ to a linear function on $\tilde{\mathfrak{h}}$, which we still denote by λ , by setting $\lambda(c) = \lambda(d) = 0$. Let $\delta \in \tilde{\mathfrak{h}}^*$ be defined by $\delta|_{\mathfrak{h} + \mathbb{C}c} = 0$, $\delta(d) = 1$. Similarly, define $L \in \mathfrak{h}^*$ by $L|_{\mathfrak{h} + \mathbb{C}d} = 0$, $L(c) = 1$.

The decomposition of $\tilde{\mathfrak{g}}$ into a sum of root spaces with respect to $\tilde{\mathfrak{h}}$ is

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \sum_{\alpha \in \Phi, i \in \mathbb{Z}} (t^i \otimes_{\mathbb{C}} \mathfrak{g}_{\alpha}) \oplus \sum_{i \in \mathbb{Z} \setminus \{0\}} (t^i \otimes_{\mathbb{C}} \mathfrak{h}).$$

Therefore the root system of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$ is

$$\tilde{\Phi} = \{\alpha + i\delta \mid \alpha \in \Phi, i \in \mathbb{Z}\} \cup \{i\delta \mid i \in \mathbb{Z} \setminus \{0\}\}.$$

A root $\beta = \alpha + i\delta$ with $\alpha \in \Phi$ is called real and a root $\beta = i\delta$, $i \in \mathbb{Z} \setminus \{0\}$, is called imaginary. The multiplicity $\dim \tilde{\mathfrak{g}}_{\beta}$ of a root $\beta \in \tilde{\Phi}$ is 1 if β is real and is n otherwise.

The following properties of the standard form on $\tilde{\mathfrak{g}}$ can be deduced from the corresponding properties of the standard form on \mathfrak{g} :

$$\begin{aligned} (,)|_{\tilde{\mathfrak{h}}} &\text{ is non-degenerate;} \\ (,)|_{\tilde{\mathfrak{g}}_{\beta} \oplus \tilde{\mathfrak{g}}_{-\beta}} &\text{ is non-degenerate;} \\ (\tilde{\mathfrak{g}}_{\beta}, \tilde{\mathfrak{g}}_{\gamma}) &= 0 \text{ if } \beta + \gamma \neq 0. \end{aligned}$$

Let $\nu : \tilde{\mathfrak{h}} \simeq \tilde{\mathfrak{h}}^*$ be the isomorphism induced from the standard bilinear form, and we still write $(,)$ for the induced bilinear form on $\tilde{\mathfrak{h}}^*$. Moreover, we denote by \langle, \rangle the canonical pairing $\tilde{\mathfrak{h}}^* \times \tilde{\mathfrak{h}} \rightarrow \mathbb{C}$. Note that

$$(\alpha_1 + i_1\delta + j_1L, \alpha_2 + i_2\delta + j_2L) = (\alpha_1, \alpha_2) + i_1j_2 + i_2j_1$$

and that a root $\alpha \in \tilde{\Phi}$ is real if and only if $\langle \alpha, \alpha \rangle \neq 0$, in which case $\langle \alpha, \alpha \rangle > 0$. Let us write

$$\tilde{\Phi} = \tilde{\Phi}_{re} \cup \tilde{\Phi}_{im}$$

for the decomposition of $\tilde{\Phi}$ into real roots and imaginary roots.

Define a subsystem of positive roots $\tilde{\Phi}_+$ by

$$\tilde{\Phi}_+ = \{\alpha + i\delta \mid \text{either } i > 0, \text{ or } i = 0, \alpha \in \Phi_+\}.$$

Then $\tilde{\Phi} = \tilde{\Phi}_+ \cup (-\tilde{\Phi}_+)$, and the corresponding system of simple roots is

$$\tilde{\Delta} = \{\alpha_0 = \delta - \tilde{\alpha}, \alpha_1, \dots, \alpha_n\}.$$

The following subalgebras of $\tilde{\mathfrak{g}}$ are the analogues of the maximal nilpotent and the Borel subalgebras of \mathfrak{g} :

$$\tilde{\mathfrak{n}}_+ = \bigoplus_{\beta \in \tilde{\Phi}_+} \tilde{\mathfrak{g}}_\beta, \quad \tilde{\mathfrak{n}}_- = \bigoplus_{\beta \in \tilde{\Phi}_+} \tilde{\mathfrak{g}}_{-\beta}, \quad \tilde{\mathfrak{b}} = \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+.$$

2.3. Affine Weyl group of $\tilde{\mathfrak{g}}$. For a real root $\beta \in \tilde{\Phi}_{re}$, let $\beta^\vee \in \tilde{\mathfrak{h}}$ be the coroot.

Lemma 2.1. *Let $\alpha^\vee \in \mathfrak{h}$ be the coroot of the root $\alpha \in \Phi$; then the coroot of $\beta = i\delta + \alpha$ is $\beta^\vee = \frac{i}{2}(\alpha^\vee, \alpha^\vee)c + \alpha^\vee$. In particular, the coroot of α_0 is $c - \tilde{\alpha}^\vee$.*

Proof. Let $x_\alpha, x_{-\alpha}$ and α^\vee be a standard basis of $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + \mathbb{C}\alpha^\vee \simeq \mathfrak{sl}_2$; then

$$\begin{aligned} \beta^\vee &= [t^i \otimes x_\alpha, t^{-i} \otimes x_{-\alpha}] \\ &= [x_\alpha, x_{-\alpha}] + i(x_\alpha, x_{-\alpha})c \\ &= \alpha^\vee + \frac{i}{2}([\alpha^\vee, x_\alpha], x_{-\alpha})c \\ &= \alpha^\vee + \frac{i}{2}(\alpha^\vee, [x_\alpha, x_{-\alpha}])c \\ &= \alpha^\vee + \frac{i}{2}(\alpha^\vee, \alpha^\vee)c. \end{aligned}$$

One can easily check that

$$[\beta^\vee, t^i \otimes x_\alpha] = 2t^i \otimes x_\alpha, \quad [\beta^\vee, t^{-i} \otimes x_{-\alpha}] = -2t^{-i} \otimes x_{-\alpha}.$$

□

Let $\rho \in \mathfrak{h}^*$ be the half sum of all the positive roots in Φ ; then $\langle \rho, \alpha_i^\vee \rangle = 1$ for $i = 1, \dots, n$. Let $\tilde{\rho} = \rho + (1 + \langle \rho, \tilde{\alpha}^\vee \rangle)L \in \tilde{\mathfrak{h}}^*$; then $\langle \tilde{\rho}, \alpha_i^\vee \rangle = 1$ for $i = 0, 1, \dots, n$. The number $h^\vee := 1 + \langle \rho, \tilde{\alpha}^\vee \rangle$ is called the dual Coxeter number of the root system Φ . Therefore $\tilde{\rho} = \rho + h^\vee L$.

For a real root $\beta \in \tilde{\Phi}_{re}$, let r_β be the reflection whose action on $\tilde{\mathfrak{h}}^*$ is given by

$$r_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta, \quad \lambda \in \tilde{\mathfrak{h}}^*,$$

and whose action on $\tilde{\mathfrak{h}}$ is given by

$$r_\beta(h) = h - \langle \beta, h \rangle \beta^\vee, \quad h \in \tilde{\mathfrak{h}}.$$

The two actions are dual to each other:

$$\langle r_\beta(\lambda), r_\beta(h) \rangle = \langle \lambda, h \rangle, \quad h \in \tilde{\mathfrak{h}}, \lambda \in \tilde{\mathfrak{h}}^*.$$

The group $\tilde{W} \subset GL(\tilde{\mathfrak{h}}^*)$ generated by r_β 's over all real roots $\beta \in \tilde{\Phi}_{re}$ is called the affine Weyl group of $\tilde{\mathfrak{g}}$. The form $(,)|_{\tilde{\mathfrak{h}}^*}$ is \tilde{W} -invariant. Note that any real root is a \tilde{W} -conjugate of a simple root and the line $\mathbb{C}\delta$ is the fixed point set for \tilde{W} . Write r_i instead of r_{α_i} , $i = 0, 1, \dots, n$. Then the group \tilde{W} is generated by r_i 's.

Let W be the Weyl group of \mathfrak{g} , which can be identified with the subgroup of \tilde{W} generated by the reflections r_1, \dots, r_n .

Let Q be the root lattice of \mathfrak{g} , i.e., the \mathbb{Z} -lattice generated by Δ , and let Q^\vee denote the coroot lattice, i.e., the lattice generated by $\alpha_i^\vee, i = 1, 2, \dots, n$. It is known that $\alpha^\vee \in Q^\vee$ for all $\alpha \in \Phi$.

Theorem 2.2. *The affine Weyl group \widetilde{W} is isomorphic to $W \ltimes Q^\vee$. Write T_h for $h \in Q^\vee$ as an element in $W \ltimes Q^\vee$. Then the isomorphism is given by, for $\alpha \in \Phi$,*

$$r_\alpha \mapsto r_\alpha, \quad r_{i\delta - \alpha} r_\alpha \mapsto T_{i\alpha^\vee}.$$

See [7, 23, 28]. The following two lemmas give the explicit action of \widetilde{W} on $\widetilde{\mathfrak{h}}$ and $\widetilde{\mathfrak{h}}^*$.

Lemma 2.3. *The \widetilde{W} -action on $\widetilde{\mathfrak{h}}$ fixes c . The action is given by the formula: for $\alpha \in \Phi$, $\gamma \in Q^\vee$, $h \in \mathfrak{h}$, and $i \in \mathbb{Z}$,*

$$\begin{aligned} r_\alpha(h + id) &= r_\alpha(h) + id, \\ T_\gamma(h + id) &= h + id + i\gamma - \left((h, \gamma) + \frac{i}{2} (\gamma, \gamma) \right) c \end{aligned}$$

or, equivalently,

$$T_\gamma(x) = x + (x, c) \gamma - \left((h, \gamma) + \frac{1}{2} (\gamma, \gamma) (x, c) \right) c, \quad \forall x \in \widetilde{\mathfrak{h}}.$$

Proof. Since $(c, \beta) = 0$ for every real root β , c is fixed by \widetilde{W} . We prove the second formula for $\gamma = \alpha^\vee$ with $\alpha \in \Phi$. The general case can be reduced to this one:

$$\begin{aligned} T_{\alpha^\vee}(h + kd) &= r_{\delta - \alpha^\vee} r_\alpha(h + id) \\ &= r_{\delta - \alpha}(r_\alpha(h) + id) \\ &= r_\alpha(h) + id - (r_\alpha(h) + id, \delta - \alpha) \left(\frac{(\alpha^\vee, \alpha^\vee)}{2} c - \alpha^\vee \right) \\ &= r_\alpha(h) + id + (i + (h, \alpha)) \left(\alpha^\vee - \frac{(\alpha^\vee, \alpha^\vee)}{2} c \right) \\ &= h + id + i\alpha^\vee - \left((h, \alpha^\vee) + \frac{i}{2} (\alpha^\vee, \alpha^\vee) \right) c. \end{aligned}$$

□

Lemma 2.4. *The \widetilde{W} -action on $\widetilde{\mathfrak{h}}^*$ fixes δ . The action is given by the formula: for $\alpha \in \Phi$, $\gamma \in Q^\vee$, $\lambda \in \mathfrak{h}^*$, and $i \in \mathbb{Z}$,*

$$\begin{aligned} r_\alpha(\lambda + iL) &= r_\alpha(\lambda) + iL, \\ T_\gamma(\lambda + iL) &= \lambda + iL + i\nu(\gamma) - \left(\langle \lambda, \gamma \rangle + \frac{i}{2} (\gamma, \gamma) \right) \delta \end{aligned}$$

or, equivalently,

$$T_\gamma(x) = x + (x, \delta) \nu(\gamma) - \left(\langle x, \gamma \rangle + \frac{1}{2} (\gamma, \gamma) (x, \delta) \right) \delta, \quad \forall x \in \widetilde{\mathfrak{h}}^*.$$

In particular,

$$T_\gamma(\beta) = \beta - \langle \beta, \gamma \rangle \delta, \quad \beta \in \widetilde{\Phi}.$$

The proof of this lemma is similar to that of Lemma 2.3 and can be reduced to the case $\gamma = \alpha^\vee$ with $\alpha \in \Phi$.

3. CONSTRUCTIONS OF LOOP GROUPS

We shall construct the loop groups associated with complex simple Lie algebras and obtain central extensions of loop groups by using tame symbols. Then we discuss the highest weight representations of loop groups. Another construction of loop groups starting from a linear algebraic group and a rational representation of this group will be given, and we will see the relationships between these two constructions. We also construct adelic loop groups and review some fundamental results of H. Garland [8] on arithmetic quotients.

3.1. First construction of loop groups. We first recall the definition of Chevalley groups. The main references are [33, 38]. Let \mathfrak{g} be a complex simple Lie algebra, and we use the same notation as in Section 2. Fix a Chevalley basis of \mathfrak{g} . The universal Chevalley group associated to \mathfrak{g} is a simply connected affine group scheme G over \mathbb{Z} , and for any field F the F -rational points $G(F)$ of G are generated by the elements $x_\alpha(u)$, $\alpha \in \Phi$, $u \in F$ subject to relations (3.1)-(3.3) if $\text{rank } \mathfrak{g} \geq 2$, or relations (3.1), (3.3) and (3.4) if $\mathfrak{g} = \mathfrak{sl}_2$.

For $\alpha \in \Phi$, $u, v \in F$,

$$(3.1) \quad x_\alpha(u)x_\alpha(v) = x_\alpha(u+v).$$

For $\alpha, \beta \in \Phi$, $\alpha \neq -\beta$, $u, v \in F$,

$$(3.2) \quad x_\alpha(u)x_\beta(v)x_\alpha(u)^{-1}x_\beta(v)^{-1} = \prod_{i,j \in \mathbb{Z}^+, i\alpha+j\beta \in \Phi} x_{i\alpha+j\beta}(c_{ij}^{\alpha\beta} u^i v^j),$$

where the order of the right-hand side is given by some fixed order, and the coefficients $c_{ij}^{\alpha\beta}$ are integers which depend on this order and the Chevalley basis of \mathfrak{g} , but not on the field F or on u, v . For $\alpha \in \Phi$, $u \in F^\times$ we set

$$w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u), \quad h_\alpha(u) = w_\alpha(u)w_\alpha(1)^{-1}.$$

Then for $u, v \in F^\times$,

$$(3.3) \quad h_\alpha(u)h_\alpha(v) = h_\alpha(uv).$$

If $\mathfrak{g} = \mathfrak{sl}_2$, there are only two roots $\pm\alpha$, and the relation (3.2) above is replaced by

$$(3.4) \quad w_\alpha(u)x_\alpha(v)w_\alpha(-u) = x_{-\alpha}(-u^2v), \quad u \in F^\times, v \in F.$$

The universal Steinberg group $G'(F)$ is generated by $\tilde{x}_\alpha(u)$, $\alpha \in \Phi$, $u \in F$ subject to relations (3.1) and (3.2) if $\text{rank } \mathfrak{g} \geq 2$, or (3.1) and (3.4) if $\mathfrak{g} = \mathfrak{sl}_2$. Here for $\alpha \in \Phi$, $u \in F^\times$ we define

$$\tilde{w}_\alpha(u) = \tilde{x}_\alpha(u)\tilde{x}_{-\alpha}(-u^{-1})\tilde{x}_\alpha(u), \quad \tilde{h}_\alpha(u) = \tilde{w}_\alpha(u)\tilde{w}_\alpha(1)^{-1}.$$

Let $\pi : G'(F) \rightarrow G(F)$ be the homomorphism defined by $\pi(\tilde{x}_\alpha(u)) = x_\alpha(u)$ for all $\alpha \in \Phi$, $u \in F$. Steinberg ([38, p.78] Theorem 10) proved that if $|F| > 4$, and $|F| \neq 9$ when $\mathfrak{g} = \mathfrak{sl}_2$, then (π, G') is a universal central extension of G . Recall from [38, p.74] that a central extension (π, E) of a group G is universal if for any central extension (π', E') of G there exists a unique homomorphism $\varphi : E \rightarrow E'$ such that $\pi'\varphi = \pi$, i.e. the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ & \searrow \pi \quad \swarrow \pi' & \\ & G & \end{array}$$

Let $C = \text{Ker } \pi$. Matsumoto [33] and Moore [34] (cf. [38, Theorem 12, pp.86-87]) proved that if $|F| > 4$, then C is isomorphic to the abstract group generated by the symbols $c(u, v)$ ($u, v \in F^\times$) subject to the relations

$$(3.5) \quad c(u, v)c(uv, w) = c(u, vw)c(v, w), \quad c(1, u) = c(u, 1) = 1,$$

$$(3.6) \quad c(u, v)c(u, -v^{-1}) = c(u, -1),$$

$$(3.7) \quad c(u, v) = c(v^{-1}, u),$$

$$(3.8) \quad c(u, v) = c(u, -uv),$$

$$(3.9) \quad c(u, v) = c(u, (1 - u)v),$$

and in the case Φ is not of type C_n ($n \geq 1$), the additional relation

$$(3.10) \quad c \text{ is bimultiplicative.}$$

In this case the relations (3.5)-(3.9) may be replaced by (3.10) and

$$(3.11) \quad c \text{ is skew,}$$

$$(3.12) \quad c(u, -u) = 1,$$

$$(3.13) \quad c(u, 1 - u) = 1.$$

The isomorphism is given by $c(u, v) \mapsto \tilde{h}_\alpha(u)\tilde{h}_\alpha(v)\tilde{h}_\alpha(uv)^{-1}$, where α is a fixed long root. For a field F and an abelian group A , a map $c : F^\times \times F^\times \rightarrow A$ is called a Steinberg symbol on $F^\times \times F^\times$ with values in A if it satisfies the relations (3.5)-(3.9), and it is said to be bilinear if it also satisfies (3.10).

In the Steinberg group G' let $c_\alpha(u, v) = \tilde{h}_\alpha(u)\tilde{h}_\alpha(v)\tilde{h}_\alpha(uv)^{-1}$, $\alpha \in \Phi$, $u, v \in F^\times$.

Lemma 3.1 ([33, Lemma 5.4. p.26]).

(a) $c_\alpha(u, v) = c_{-\alpha}(v, u)^{-1}$, $\forall \alpha \in \Phi$.

(b) If there exists $w \in W$ such that $\beta = w\alpha$, then c_β equals c_α or $c_{-\alpha}$.

(c) For $\alpha, \beta \in \Phi$,

$$\tilde{h}_\alpha(u)\tilde{h}_\beta(v)\tilde{h}_\alpha(u)^{-1}\tilde{h}_\beta(v)^{-1} = c_\alpha(u, v^{\langle \alpha, \beta^\vee \rangle}) = c_\beta(v, u^{\langle \beta, \alpha^\vee \rangle})^{-1}.$$

(d) The Steinberg symbol c_α is bilinear except when the case G is symplectic and α is a long root.

Suppose that $c : F^\times \times F^\times \rightarrow A$ is a Steinberg symbol. By [33, Théorème 5.10, p.30], there exists a central extension of $G(F)$ by A such that $c_\alpha = c$ for any long root α , if either c is bilinear or G is symplectic. In fact, the symbol gives a homomorphism of abelian groups $\phi : C \rightarrow A$. We may assume that ϕ is surjective. Then from the universal central extension $1 \rightarrow C \rightarrow G'(F) \rightarrow G(F) \rightarrow 1$ we obtain

$$1 \longrightarrow \frac{C}{\text{Ker } \phi} \longrightarrow \frac{G'(F)}{\text{Ker } \phi} \longrightarrow G(F) \longrightarrow 1.$$

Then $G'(F)/\text{Ker } \phi$ is the required central extension. If c is bilinear, then $c_\alpha = c^{\frac{2}{\langle \alpha, \alpha \rangle}}$, $\forall \alpha \in \Phi$. This can be proved by using Lemma 3.1 (c) and checking the Dynkin diagrams. Recall that the square length of a long root equals 2.

The following lemma follows from [39]; see [33, Lemme 5.1 and Lemme 5.2, pp.23-24].

Lemma 3.2. In a central extension of G by a Steinberg symbol we have the following relations for $\alpha, \beta \in \Phi$:

(a) $\tilde{w}_\alpha(u)\tilde{x}_\beta(v)\tilde{w}_\alpha(u)^{-1} = \tilde{x}_{r_{\alpha\beta}}(\eta_{\alpha,\beta}u^{-\langle \beta, \alpha^\vee \rangle}v)$, where $\eta_{\alpha,\beta}$ are integers equal to ± 1 given by [33, Lemme 5.1 (c)].

$$\begin{aligned}
(b) \quad & \tilde{w}_\alpha(u)\tilde{h}_\beta(v)\tilde{w}_\alpha(u)^{-1} = \tilde{h}_{r_\alpha\beta}(\eta_{\alpha,\beta}u^{-\langle\beta,\alpha^\vee\rangle}v)\tilde{h}_{r_\alpha\beta}(\eta_{\alpha,\beta}u^{-\langle\beta,\alpha^\vee\rangle})^{-1}, \text{ i.e.} \\
& \tilde{w}_\alpha(u)\tilde{h}_\beta(v)\tilde{w}_\alpha(u)^{-1} = c_{r_\alpha\beta}(v, \eta_{\alpha,\beta}u^{-\langle\beta,\alpha^\vee\rangle})^{-1}\tilde{h}_{r_\alpha\beta}(v), \\
(c) \quad & \tilde{h}_\alpha(u)\tilde{x}_\beta(v)\tilde{h}_\alpha(u)^{-1} = \tilde{x}_\beta(u^{\langle\beta,\alpha^\vee\rangle}v), \\
(d) \quad & \tilde{h}_\alpha(u)\tilde{w}_\beta(v)\tilde{h}_\alpha(u)^{-1} = \tilde{w}_\beta(u^{\langle\beta,\alpha^\vee\rangle}v), \\
(e) \quad & \tilde{w}_\alpha(1)\tilde{h}_\beta(u)\tilde{w}_\alpha(1)^{-1} = \tilde{h}_\beta(u)\tilde{h}_\alpha(u^{-\langle\alpha,\beta^\vee\rangle}), \\
(f) \quad & \tilde{w}_\alpha(u) = \tilde{w}_{-\alpha}(-u^{-1}), \tilde{h}_\alpha(u) = \tilde{h}_{-\alpha}(u)^{-1}, \tilde{w}_\alpha(1)\tilde{h}_\alpha(u)\tilde{w}_\alpha(1)^{-1} = \tilde{h}_\alpha(u^{-1}), \\
(g) \quad & \tilde{w}_\alpha(1)^{-1}\tilde{x}_\alpha(u)\tilde{w}_\alpha(1) = \tilde{x}_{-\alpha}(-u) = \tilde{x}_\alpha(-u^{-1})\tilde{w}_\alpha(u^{-1})\tilde{x}_\alpha(-u^{-1}), u \neq 0.
\end{aligned}$$

The tame symbol defined for the field of Laurent power series $F((t))$ is the map $(\cdot)_{tame} : F((t))^\times \times F((t))^\times \rightarrow F^\times$ given by

$$(3.14) \quad (x, y)_{tame} = (-1)^{v(x)v(y)} \left. \frac{x^{v(y)}}{y^{v(x)}} \right|_{t=0},$$

where v is the valuation on $F((t))$ normalized such that $v(t^i) = i$. Note that tame symbol is trivial on $F^\times \times F^\times$.

Since the tame symbol is a bilinear Steinberg symbol, we obtain a central extension of $G(F((t)))$ by F^\times , associated to the inverse of the tame symbol. Let us denote this central extension by $\widehat{G}(F((t)))$. It is generated by $\tilde{x}_\alpha(u)$ with $\alpha \in \Phi$, $u \in F((t))$ and F^\times , subject to the relations (3.1), (3.2) and (3.15) below if $\text{rank } \mathfrak{g} \geq 2$, or the relations (3.1), (3.4) and (3.15) if $\mathfrak{g} = \mathfrak{sl}_2$. By previous remarks, for each $\alpha \in \Phi$,

$$(3.15) \quad \tilde{h}_\alpha(x)\tilde{h}_\alpha(y)\tilde{h}_\alpha(xy)^{-1} = (x, y)_{tame}^{-\frac{2}{\langle\alpha,\alpha\rangle}}, \quad x, y \in F((t))^\times.$$

Then we have the following exact sequence for $G(F((t)))$:

$$1 \longrightarrow F^\times \longrightarrow \widehat{G}(F((t))) \xrightarrow{\pi} G(F((t))) \longrightarrow 1,$$

where π is given by $\tilde{x}_\alpha(u) \mapsto x_\alpha(u)$.

For a real root $\beta = \alpha + i\delta \in \Phi_{re}$, and $u \in F$, $v \in F^\times$, we define for $G(F((t)))$,

$$(3.16) \quad \begin{cases} x_\beta(u) = x_\alpha(ut^i), \\ w_\beta(v) = x_\beta(v)x_{-\beta}(-v^{-1})x_\beta(v) = w_\alpha(vt^i), \\ h_\beta(v) = w_\beta(v)w_\beta(1)^{-1} = h_\alpha(v). \end{cases}$$

For $\widehat{G}(F((t)))$ we can define the elements by the same formula with x, w, h replaced by $\tilde{x}, \tilde{w}, \tilde{h}$. From the definition we have

$$\begin{aligned}
(3.17) \quad \tilde{h}_\beta(u) &= \tilde{w}_\alpha(ut^i)\tilde{w}_\alpha(t^i)^{-1} = \tilde{w}_\alpha(ut^i)\tilde{w}_\alpha(1)^{-1}(\tilde{w}_\alpha(t^i)\tilde{w}_\alpha(1)^{-1})^{-1} \\
&= \tilde{h}_\alpha(ut^i)\tilde{h}_\alpha(t^i)^{-1} = (u, t^i)_{tame}^{-\frac{2}{\langle\alpha,\alpha\rangle}}\tilde{h}_\alpha(u) \\
&= u^{\frac{2i}{\langle\alpha,\alpha\rangle}}\tilde{h}_\alpha(u) = u^{\frac{2i}{\langle\beta,\beta\rangle}}\tilde{h}_\alpha(u).
\end{aligned}$$

It is clear that $\{x_\beta(u)|u \in F\}$ (resp. $\{\tilde{x}_\beta(u)|u \in F\}$) forms a subgroup isomorphic to the additive group $\mathbb{G}_a(F)$. We call it the root subgroup associated to β , and denote it by U_β (resp. \tilde{U}_β).

For a positive imaginary root $\beta = i\delta \in \Phi_{im+}$, $i \in \mathbb{N}$, we define the root subgroup U_β (resp. \tilde{U}_β) as follows, which is isomorphic to \mathbb{G}_a^n . The map

$$\exp : tF[[t]] \longrightarrow 1 + tF[[t]]$$

is a bijection with inverse map \log . The root subgroup U_β is given by

$$(3.18) \quad U_\beta = \{h_{\alpha_1}(\exp(u_1 t^i)) \cdots h_{\alpha_n}(\exp(u_n t^i)) | u_1, \dots, u_n \in F\}.$$

We define \tilde{U}_β similarly, with h replaced by \tilde{h} .

Lemma 3.3. *For each real root $\beta = \alpha + i\delta \in \tilde{\Phi}_{re}$, there is a unique group homomorphism $\varphi_\beta : SL_2(F) \rightarrow G(F((t)))$ (resp. $\hat{G}(F((t)))$) such that*

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mapsto x_\beta(u) \text{ (resp. } \tilde{x}_\beta(u)), \quad \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \mapsto x_{-\beta}(u) \text{ (resp. } \tilde{x}_{-\beta}(u)).$$

Proof. For $G(F((t)))$ we have

$$w_\beta(u) = x_\alpha(ut^i)x_{-\alpha}(-u^{-1}t^{-i})x_{-\alpha}(ut^i) = w_\alpha(ut^i),$$

$$h_\beta(u) = h_\alpha(ut^i)h_\alpha(t^i)^{-1} = h_\alpha(u),$$

and therefore

$$h_\beta(u)h_\beta(v)h_\beta(uv)^{-1} = h_\alpha(u)h_\alpha(v)h_\alpha(uv)^{-1} = 1.$$

For $\hat{G}(F((t)))$, we only need to verify the last equation above since others are similar. However, from (3.17) it follows that

$$\tilde{h}_\beta(u)\tilde{h}_\beta(v)\tilde{h}_\beta(uv)^{-1} = \tilde{h}_\alpha(u)\tilde{h}_\alpha(v)\tilde{h}_\alpha(uv)^{-1} = 1.$$

This verifies that $x_\beta(u)$ (resp. $\tilde{x}_\beta(u)$) and $x_{-\beta}(u)$ (resp. $\tilde{x}_{-\beta}(u)$) satisfy the relations of $SL_2(F)$. \square

Apply (3.17) and use properties of the tame symbol; we can translate Lemma 3.1 and Lemma 3.2 into the data of affine root system $\tilde{\Phi}$. Assume that $\alpha = \alpha_0 + i\delta$, $\beta = \beta_0 + j\delta \in \tilde{\Phi}_{re}$, where $\alpha_0, \beta_0 \in \Phi$, $i, j \in \mathbb{Z}$. One should not confuse α_0 with the simple root $\alpha_0 = \delta - \tilde{\alpha}$. Let $\eta_{\alpha, \beta} = \eta_{\alpha_0, \beta_0}$.

Corollary 3.4. *We have the following relations in $\hat{G}(F((t)))$ for $\alpha, \beta \in \tilde{\Phi}_{re}$:*

- (a) $\tilde{x}_\alpha(u)\tilde{x}_\beta(v)\tilde{x}_\alpha(u)^{-1}\tilde{x}_\beta(v)^{-1} = \prod_{m, n \in \mathbb{Z}^+, m\alpha + n\beta \in \tilde{\Phi}_{re}} \tilde{x}_{m\alpha + n\beta}(c_{mn}^{\alpha_0\beta_0} u^m v^n)$, $u, v \in F$,
- (b) $\tilde{h}_\alpha(u)\tilde{h}_\alpha(v) = \tilde{h}_\alpha(uv)$, $u, v \in F^\times$,
- (c) $\tilde{h}_\alpha(u)\tilde{h}_\beta(v) = \tilde{h}_\beta(v)\tilde{h}_\alpha(u)$, $u, v \in F^\times$,
- (d) $\tilde{w}_\alpha(u)\tilde{x}_\beta(v)\tilde{w}_\alpha(u)^{-1} = \tilde{x}_{r_\alpha\beta}(\eta_{\alpha, \beta} u^{-\langle \beta, \alpha^\vee \rangle} v)$, $u \in F^\times$, $v \in F$,
- (e) $\tilde{w}_\alpha(u)\tilde{h}_\beta(v)\tilde{w}_\alpha(u)^{-1} = \tilde{h}_{r_\alpha\beta}(v)$, $u, v \in F^\times$,
- (f) $\tilde{h}_\alpha(u)\tilde{x}_\beta(v)\tilde{h}_\alpha(u)^{-1} = \tilde{x}_\beta(u^{\langle \beta, \alpha^\vee \rangle} v)$, $u \in F^\times$, $v \in F$,
- (g) $\tilde{h}_\alpha(u)\tilde{w}_\beta(v)\tilde{h}_\alpha(u)^{-1} = \tilde{w}_\beta(u^{\langle \beta, \alpha^\vee \rangle} v)$, $u, v \in F^\times$,
- (h) $\tilde{w}_\alpha(1)\tilde{h}_\beta(v)\tilde{w}_\alpha(1)^{-1} = \tilde{h}_\beta(u)\tilde{h}_\alpha(u^{-\langle \alpha, \beta^\vee \rangle})$, $u \in F^\times$,
- (i) $\tilde{w}_\alpha(u) = \tilde{w}_{-\alpha}(-u^{-1})$, $\tilde{h}_\alpha(u)^{-1} = \tilde{h}_\alpha(u^{-1})$, $\tilde{w}_\alpha(1)\tilde{h}_\alpha(u)\tilde{w}_\alpha(1)^{-1} = \tilde{h}_\alpha(u^{-1})$, $u \in F^\times$,
- (j) $\tilde{w}_\alpha(1)^{-1}\tilde{x}_\alpha(u)\tilde{w}_\alpha(1) = \tilde{x}_{-\alpha}(-u) = \tilde{x}_\alpha(-u^{-1})\tilde{w}_\alpha(u^{-1})\tilde{x}_\alpha(-u^{-1})$, $u \in F^\times$.

Proof. (a) By (3.2),

$$\begin{aligned}
 & \tilde{x}_\alpha(u) \tilde{x}_\beta(v) \tilde{x}_\alpha(u)^{-1} \tilde{x}_\beta(v)^{-1} \\
 &= \tilde{x}_{\alpha_0}(ut^i) \tilde{x}_{\beta_0}(vt^j) \tilde{x}_{\alpha_0}(ut^i)^{-1} \tilde{x}_{\beta_0}(vt^j)^{-1} \\
 &= \prod_{m, n \in \mathbb{Z}^+, m\alpha_0 + n\beta_0 \in \Phi} \tilde{x}_{m\alpha_0 + n\beta_0}(c_{mn}^{\alpha_0\beta_0}(ut^i)^m(vt^j)^n) \\
 &= \prod_{m, n \in \mathbb{Z}^+, m\alpha + n\beta \in \tilde{\Phi}_{re}} \tilde{x}_{m\alpha + n\beta}(c_{mn}^{\alpha_0\beta_0}u^m v^n).
 \end{aligned}$$

(b) Since the tame symbol is trivial on $F^\times \times F^\times$,

$$\begin{aligned}
 \tilde{h}_\alpha(u) \tilde{h}_\alpha(v) &= u^{\frac{2i}{\langle \alpha, \alpha \rangle}} v^{\frac{2i}{\langle \alpha, \alpha \rangle}} \tilde{h}_{\alpha_0}(u) \tilde{h}_{\alpha_0}(v) \\
 &= (uv)^{\frac{2i}{\langle \alpha, \alpha \rangle}} \tilde{h}_{\alpha_0}(uv) = \tilde{h}_\alpha(uv).
 \end{aligned}$$

(c) By Lemma 3.1 (c),

$$\begin{aligned}
 \tilde{h}_\alpha(u) \tilde{h}_\beta(v) &= u^{\frac{2i}{\langle \alpha, \alpha \rangle}} v^{\frac{2j}{\langle \beta, \beta \rangle}} \tilde{h}_{\alpha_0}(u) \tilde{h}_{\beta_0}(v) \\
 &= u^{\frac{2i}{\langle \alpha, \alpha \rangle}} v^{\frac{2j}{\langle \beta, \beta \rangle}} \tilde{h}_{\beta_0}(v) \tilde{h}_{\alpha_0}(u) \\
 &= \tilde{h}_\beta(v) \tilde{h}_\alpha(u).
 \end{aligned}$$

(d) By Lemma 3.2 (a),

$$\begin{aligned}
 \tilde{w}_\alpha(u) \tilde{x}_\beta(v) \tilde{w}_\alpha(u)^{-1} &= \tilde{w}_{\alpha_0}(ut^i) \tilde{x}_{\beta_0}(vt^j) \tilde{w}_{\alpha_0}(ut^i)^{-1} \\
 &= \tilde{x}_{r_{\alpha_0}\beta_0}((ut^i)^{-\langle \beta_0, \alpha_0^\vee \rangle} vt^j) \\
 &= \tilde{x}_{r_\alpha\beta}(u^{-\langle \beta, \alpha^\vee \rangle} v),
 \end{aligned}$$

where the last equality follows from the formulas

$$\langle \beta, \alpha^\vee \rangle = \langle \beta_0, \alpha_0^\vee \rangle, \quad r_\alpha\beta = r_{\alpha_0}\beta_0 + (j - \langle \beta_0, \alpha_0^\vee \rangle i)\delta.$$

(e) By Lemma 3.2 (b),

$$\begin{aligned}
 & \tilde{w}_\alpha(u) \tilde{h}_\beta(v) \tilde{w}_\alpha(u)^{-1} \\
 &= v^{\frac{2j}{\langle \beta, \beta \rangle}} \tilde{w}_{\alpha_0}(ut^i) \tilde{h}_{\beta_0}(v) \tilde{w}_{\alpha_0}(ut^i)^{-1} \\
 &= v^{\frac{2j}{\langle \beta, \beta \rangle}} \left(v, \eta_{\alpha, \beta}(ut^i)^{-\langle \beta, \alpha^\vee \rangle} \right)_{tame} \tilde{h}_{r_{\alpha_0}\beta_0}(v) \\
 &= v^{\frac{2(j - \langle \beta, \alpha^\vee \rangle i)}{\langle \beta, \beta \rangle}} \tilde{h}_{r_{\alpha_0}\beta_0}(v) \\
 &= \tilde{h}_{r_\alpha\beta}(v).
 \end{aligned}$$

(f) and (g) are easy consequences of (3.16) and (3.17). By (d) the left-hand side of (h) equals $\tilde{h}_{r_\alpha\beta}(u)$. By Lemma 3.2 (e), the right-hand side of (h) equals

$$\begin{aligned}
 & u^{\frac{2j}{\langle \beta, \beta \rangle} - \frac{2i\langle \alpha, \beta^\vee \rangle}{\langle \alpha, \alpha \rangle}} \tilde{h}_{\beta_0}(u) \tilde{h}_{\alpha_0}(u^{-\langle \alpha, \beta^\vee \rangle}) \\
 &= u^{\frac{2(j - \langle \beta, \alpha^\vee \rangle i)}{\langle \beta, \beta \rangle}} \tilde{w}_{\alpha_0}(1) \tilde{h}_{\beta_0}(u) \tilde{w}_{\alpha_0}(1)^{-1} \\
 &= u^{\frac{2(j - \langle \beta, \alpha^\vee \rangle i)}{\langle \beta, \beta \rangle}} \tilde{h}_{r_{\alpha_0}\beta_0}(u) \\
 &= \tilde{h}_{r_\alpha\beta}(u).
 \end{aligned}$$

This proves (h). (i) follows from (a) and (g). (j) follows from (d) and the fact $\eta_{\alpha, \alpha} = \eta_{\alpha, -\alpha} = 1$; see [33, Lemme 5.1 (c), p.24]. \square

For each $\alpha \in \Phi_+$, the subgroup of $G(F)$ generated by $x_\alpha(u)$ and $x_{-\alpha}(u)$ ($u \in F$) is isomorphic to $SL_2(F)$ by the map

$$(3.19) \quad x_\alpha(u) \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad x_{-\alpha}(u) \mapsto \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

The subgroup $B(F)$ generated by $x_\alpha(u)$ ($\alpha \in \Phi_+$) and $h_\alpha(u)$ is a Borel subgroup, and the subgroup generated by $x_\alpha(u)$ ($\alpha \in \Phi_+$) is the unipotent radical of $B(F)$. When F is a local field, $G(F)$ is a locally compact topological group. We choose a maximal compact subgroup K of $G(F)$ as follows. We first choose for $SL_2(F)$ a maximal compact subgroup. If $F = \mathbb{R}$ or \mathbb{C} we choose $SO_2(\mathbb{R})$ or $SU_2(\mathbb{C})$. If F is non-archimedean, we choose $SL_2(\mathcal{O}_F)$, where \mathcal{O}_F is the ring of integers of F . Using (3.19) we obtain a maximal compact subgroup in the $SL_2(F)$ corresponding to each positive root α . Let K be the subgroup generated by these subgroups. Then we have the Iwasawa decomposition $G(F) = B(F)K$.

Let \widehat{B}_0 be the preimage of $B(F)$ of the canonical projection $G(F[[t]]) \rightarrow G(F)$. It is easy to prove that \widehat{B}_0 is generated by the elements $x_\alpha(u)$ where either $\alpha \in \Phi_+$, $u \in F[[t]]$ or $\alpha \in \Phi_-$, $u \in tF[[t]]$, and the elements $h_\alpha(u)$, $\alpha \in \Phi$, $u \in F[[t]]^\times$. The subgroup \widehat{B}_0 plays the role of Borel subgroup for $G(F((t)))$. Let \widehat{N}_0 be the group generated by $w_\alpha(u)$ with $\alpha \in \Phi$, $u \in F((t))^\times$.

Lemma 3.5. *The subgroup $\widehat{H}_0 = \widehat{B}_0 \cap \widehat{N}_0$ is generated by elements $h_\alpha(u)$ where $\alpha \in \Phi$, $u \in F[[t]]^\times$, and is normal in \widehat{N}_0 .*

Let $w_\alpha = w_\alpha(1)$, $\tilde{w}_\alpha = \tilde{w}_\alpha(1)$. Let $\widehat{S}_0 = \{w_{\alpha_0}\widehat{H}_0, \dots, w_{\alpha_n}\widehat{H}_0\} \subset \widehat{N}_0/\widehat{H}_0$.

Theorem 3.6. *$(G(F((t))), \widehat{B}_0, \widehat{N}_0, \widehat{S}_0)$ is a Tits system, and its Weyl group is isomorphic to \widetilde{W} . Moreover, $w_{\alpha_i}\widehat{H}_0 \mapsto r_i$ gives an isomorphism.*

Theorem 3.6 follows from [22, Theorems 2.22 and 2.24, pp.37-38]. Now consider the central extension $\widehat{G}(F((t)))$.

Theorem 3.7. *There exists a lifting $G(F[[t]]) \rightarrow \widehat{G}(F((t)))$ given by $x_\alpha(u) \mapsto \tilde{x}_\alpha(u)$, where $\alpha \in \Phi$, $u \in F[[t]]$.*

The proof of Theorem 3.7 requires the theory of highest weight representations of loop groups and will be given in the next section. Assume its validity at the moment; we may regard $G(F[[t]])$ and its subgroups as subgroups of $\widehat{G}(F((t)))$. In particular, we may identify \widetilde{U}_β with U_β for each $\beta \in \widetilde{\Phi}_+$. Let \widehat{B} be the preimage of \widehat{B}_0 in $\widehat{G}(F((t)))$ under the canonical map $\widehat{G}(F((t))) \rightarrow G(F((t)))$. Then $\widehat{B} = \widehat{B}_0 \times F^\times$ is a subgroup of $G(F[[t]]) \times F^\times$. Let \widehat{N} be the subgroup of $\widehat{G}(F((t)))$ generated by the center F^\times and the elements $\tilde{w}_\alpha(u)$ with $\alpha \in \Phi$, $u \in F((t))^\times$.

Lemma 3.8. *The subgroup $\widehat{H} = \widehat{B} \cap \widehat{N}$ is generated by the center F^\times and the elements $\tilde{h}_\alpha(u)$ with $\alpha \in \Phi$, $u \in F[[t]]^\times$, and is normal in \widehat{N} .*

Let $\widehat{S} = \{\tilde{w}_{\alpha_0}\widehat{H}, \dots, \tilde{w}_{\alpha_n}\widehat{H}\} \subset \widehat{N}/\widehat{H}$.

Theorem 3.9. *$(\widehat{G}(F((t))), \widehat{B}, \widehat{N}, \widehat{S})$ is a Tits system, and its Weyl group is isomorphic to \widetilde{W} under the isomorphism given by $\tilde{w}_{\alpha_i}\widehat{H} \mapsto r_i$.*

Lemma 3.8 and Theorem 3.9 are immediate consequences of Lemma 3.5 and Theorem 3.6. We shall always identify the quotient $\widehat{N}/\widehat{B} \cap \widehat{N}$ with the affine Weyl group \widetilde{W} using the isomorphism in the theorem.

Lemma 3.10. *For every $\alpha \in \Phi$, $\tilde{h}_\alpha(t^i) \in \hat{N}$ and it maps to $T_{-i\alpha^\vee} \in \widetilde{W}$.*

Proof. $\tilde{h}_\alpha(t^i) = \tilde{w}_\alpha(t^i)\tilde{w}_\alpha^{-1} \in \hat{N}$. The isomorphism in Theorem 3.9 maps $\tilde{w}_{\delta-\alpha}\tilde{w}_\alpha$ to $r_{\delta-\alpha}r_\alpha = T_{\alpha^\vee}$. On the other hand,

$$\begin{aligned}\tilde{w}_{\delta-\alpha}\tilde{w}_\alpha &= \tilde{w}_{-\alpha}(t)\tilde{w}_\alpha = \tilde{w}_\alpha(-t^{-1})\tilde{w}_\alpha \\ &= \tilde{h}_\alpha(-t^{-1})\tilde{w}_\alpha^2 = \tilde{h}_\alpha(-t^{-1})\tilde{h}_\alpha(-1) = \tilde{h}_\alpha(t^{-1}),\end{aligned}$$

where the 2nd equality used Lemma 3.2(f), the 2nd last equality used [33, Théorème 6.3(b), pp.34-35], and the last equality used Corollary 3.4(b). Therefore $\tilde{h}_\alpha(t^{-1})$ corresponds to $T_{\alpha^\vee} \in \widetilde{W}$. \square

The standard results about the Tits system implies the Bruhat decomposition

$$(3.20) \quad \widehat{G}(F((t))) = \bigcup_{w \in \widetilde{W}} \widehat{B}w\widehat{B}.$$

The general Bruhat decomposition with respect to parabolic subgroups also applies to the loop groups, where the notion of parabolic subgroups are explained below.

For any $\theta \subset \hat{\Delta} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$, there corresponds a parabolic subgroup P_θ of $\widehat{G}(F((t)))$ such that $P_{\theta_1} \subset P_{\theta_2}$ if and only if $\theta_1 \subset \theta_2$. Let $P_\theta = M_\theta N_\theta$ be the Levi decomposition where M_θ is the Levi subgroup and N_θ is the unipotent radical.

For example, we have $P_{\hat{\Delta}} = \widehat{G}(F((t)))$, $P_\emptyset = \widehat{B}$, $M_\emptyset = T \times F^\times$, where $T \simeq \mathbb{G}_m^n$ is generated by $\tilde{h}_\alpha(u)$ with $\alpha \in \Phi$ and $u \in F^\times$. An important example is the maximal parabolic subgroup $P_\Delta = G(F[[t]]) \times F^\times$ with $M_\Delta = G(F) \times F^\times$. In fact these are the subgroups from which we induce the Eisenstein series in Section 4.

Let $\widehat{U} = N_\emptyset$ be the unipotent radical of $\widehat{B} = P_\emptyset$, i.e. \widehat{U} is generated by the elements $\tilde{x}_\alpha(u)$ where either $\alpha \in \Phi_+$, $u \in F[[t]]$ or $\alpha \in \Phi_-$, $u \in tF[[t]]$. Let U^+ be the subgroup generated by the elements $\tilde{x}_\alpha(u)$ where $\alpha \in \Phi_+$ and $u \in F[[t]]$, and U^- be the subgroup generated by the elements $\tilde{x}_\alpha(u)$ where $\alpha \in \Phi_-$ and $u \in tF[[t]]$, and $D = \widehat{B} \cap \widehat{N}$ be the subgroup generated by the center F^\times and the elements $\tilde{h}_\alpha(u)$ where $\alpha \in \Phi$, $u \in F[[t]]^\times$, and D^1 be the subgroup of D generated by $\tilde{h}_\alpha(u)$ where $\alpha \in \Phi$ and $u \in 1 + tF[[t]]$. Let $\widehat{T} = T \times F^\times$; then both \widehat{T} and D^1 are stable under the conjugation of \widetilde{W} .

Lemma 3.11 ([22, Proposition 2.1, p.29]). *We have unique factorizations*

$$\begin{aligned}D^1 &= \prod_{\beta \in \widetilde{\Phi}_{im+}} U_\beta, \quad U^-U^+ = \prod_{\beta \in \widetilde{\Phi}_{re+}} U_\beta, \quad D = \widehat{T}D^1, \\ \widehat{B} &= \widehat{T}\widehat{U} = U^-DU^+, \quad \widehat{U} = U^-D^1U^+ = \prod_{\beta \in \Phi_+} U_\beta.\end{aligned}$$

In general let $N_\theta^\pm = U^\pm \cap N_\theta$; then

$$(3.21) \quad N_\theta = N_\theta^- D^1 N_\theta^+.$$

Note that if Φ_θ is the subsystem of Φ generated by θ , then

$$(3.22) \quad N_\theta = \prod_{\alpha \in \Phi_+ - \Phi_\theta} U_\alpha.$$

Let W_θ be the subgroup of \widetilde{W} generated by $\{r_i | \alpha_i \in \theta\}$. The following result is also standard; see [3].

Theorem 3.12. For $\theta_1, \theta_2 \subset \widetilde{\Delta}$, there is the Bruhat decomposition into disjoint unions

$$\widehat{G}(F((t))) = \bigcup_{W_{\theta_1} \backslash \widetilde{W} / W_{\theta_2}} P_{\theta_1} w P_{\theta_2},$$

where w runs over a set of representatives of the double cosets in $W_{\theta_1} \backslash \widetilde{W} / W_{\theta_2}$. The following is such a set of double coset representatives:

$$W(\theta_1, \theta_2) = \{w \in \widetilde{W} | w^{-1}\theta_1 \subset \widetilde{\Phi}_+, w\theta_2 \subset \widetilde{\Phi}_+\}.$$

In the case $\theta_1 = \theta$ and $\theta_2 = \emptyset$, for each $w \in W(\theta, \emptyset)$ there is a bijection

$$P_{\theta} w P_{\emptyset} \simeq P_{\theta} \times \{w\} \times U_w,$$

where

$$U_w = \prod_{\alpha > 0, w\alpha < 0} U_{\alpha}.$$

Assume now that F is a local field. For a real root $\beta \in \widetilde{\Phi}_{re}$, we denote K_{β} as the image of the standard maximal compact subgroup of $SL_2(F)$ under the map φ_{β} in Lemma 3.3. Let \widehat{K} denote the subgroup of $\widehat{G}(F((t)))$ generated as follows:

$$(3.23) \quad \widehat{K} = \begin{cases} \langle K_{\beta}, \beta \in \widetilde{\Phi}_{re}, \pm 1 \in \mathbb{R}^{\times} \rangle, & \text{if } F = \mathbb{R}, \\ \langle K_{\beta}, \beta \in \widetilde{\Phi}_{re}, S^1 \subset \mathbb{C}^{\times} \rangle, & \text{if } F = \mathbb{C}, \\ \langle K_{\beta}, \beta \in \widetilde{\Phi}_{re}, \mathcal{O}_F^{\times}, G(\mathcal{O}_F[[t]]) \rangle, & \text{if } F \text{ is } p\text{-adic.} \end{cases}$$

The standard method using the Tits system shows that there is the Iwasawa decomposition

$$\widehat{G}(F((t))) = \widehat{B}\widehat{K}.$$

For all local fields F and all real roots β , $\widetilde{w}_{\beta} = \widetilde{w}_{\beta}(1) \in \widehat{K}$; therefore \widetilde{W} has a set of representatives in \widehat{K} . Denote the image of \widehat{K} in $G(F((t)))$ by \widehat{K}_0 .

Now let us construct the full loop group $\widetilde{G}(F((t)))$. The reparametrization group of $F((t))$ is

$$\text{Aut}_F F((t)) = \left\{ \sum_{i=1}^{\infty} u_i t^i \in F[[t]] | u_1 \neq 0 \right\},$$

where $\sigma(t) \in \text{Aut}_F F((t))$ acts on $F((t))$ by $u(t) \mapsto u(\sigma(t))$, and the group law is $(\sigma_1 * \sigma_2)(t) = \sigma_2(\sigma_1(t))$. This induces an action of $\text{Aut}_F F((t))$ on $G(F((t)))$ as automorphisms. It is easy to check that the action of $\text{Aut}_F F((t))$ preserves the tame symbol, therefore it acts on $\widehat{G}(F((t)))$ as automorphisms. More precisely, we have

$$\sigma(t) \cdot \widetilde{x}_{\alpha}(u(t)) = \widetilde{x}_{\alpha}(u(\sigma(t))),$$

and the action on the center F^{\times} is trivial. It is also clear that the subgroup $G(F)$ is fixed under this action. We have the semi-direct product group

$$\widehat{G}(F((t))) \rtimes \text{Aut}_F F((t))$$

on which there is the standard relation

$$\sigma(t)g\sigma^{-1}(t) = \sigma(t) \cdot g.$$

We shall only consider the subgroup $\sigma(F^\times) \subset \text{Aut}_F F((t))$ which consists of the elements $\sigma(\mathfrak{q}) = \mathfrak{q}t$ ($\mathfrak{q} \in F^\times$). It is clear that $\sigma(F^\times)$ is isomorphic to $\mathbb{G}_m(F)$. We form the semi-direct product group

$$(3.24) \quad \tilde{G}(F((t))) = \hat{G}(F((t))) \rtimes \sigma(F^\times).$$

Consider the tori in $\tilde{G}(F((t)))$,

$$(3.25) \quad T \hookrightarrow \hat{T} \hookrightarrow \tilde{T},$$

where $\tilde{T} = \hat{T} \times \sigma(F^\times)$. The torus $\tilde{T} \simeq \mathbb{G}_m^{n+2}$ will play the role of a maximal torus for $\tilde{G}(F((t)))$. Then we have the cocharacter lattices

$$\begin{array}{ccccc} X_*(T) & \xrightarrow{\subset} & X_*(\hat{T}) & \xrightarrow{\subset} & X_*(\tilde{T}) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ Q^\vee & \xrightarrow{\subset} & Q^\vee \oplus \mathbb{Z}c & \xrightarrow{\subset} & Q_{\text{aff}}^\vee \end{array}$$

where $Q_{\text{aff}}^\vee = Q^\vee \oplus \mathbb{Z}c \oplus \mathbb{Z}d$ is called the affine coroot lattice. It has a basis $\{\alpha_1^\vee, \dots, \alpha_n^\vee, c, d\}$, and $\{\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_n^\vee, d\}$ is also a basis. The identification of Q_{aff}^\vee with $X_*(\tilde{T})$ is, for $\lambda = \alpha^\vee + ic + jd \in Q_{\text{aff}}^\vee$, where $\alpha \in \Phi$, $i, j \in \mathbb{Z}$, the corresponding cocharacter is $\lambda : \mathbb{G}_m \rightarrow \tilde{T}$ given by

$$\lambda(u) = \tilde{h}_\alpha(u)u^i\sigma(u^j).$$

It is clear that \hat{N} normalizes \tilde{T} , and therefore \tilde{W} acts on \tilde{T} . On the other hand \tilde{W} acts on $\tilde{\mathfrak{h}}$ by the formula in Lemma 2.3, and the lattice Q_{aff}^\vee is stable under the action. We have

Lemma 3.13. *The cocharacter map*

$$Q_{\text{aff}}^\vee \times \mathbb{G}_m \rightarrow \tilde{T}, \quad (\lambda, u) \mapsto \lambda(u)$$

is \tilde{W} -equivariant.

Proof. The lemma is equivalent to: for every $w \in \tilde{W}$, let $\tilde{w} \in \hat{N}$ be a representative; then

$$(3.26) \quad \tilde{w}\lambda(u)\tilde{w}^{-1} = (w \cdot \lambda)(u).$$

It is clear that (3.26) is true for $w \in W$. We now prove it for $w = T_{-\alpha^\vee}$. By Lemma 3.10, $\tilde{w} = \tilde{h}_\alpha(t)$ is a lifting of w . For $\lambda = \beta^\vee \in Q^\vee$, (3.26) is a special case of the following identity:

$$\tilde{h}_\alpha(t^i)\tilde{h}_\beta(u)\tilde{h}_\alpha(t^i)^{-1} = u^{i(\alpha^\vee, \beta^\vee)}\tilde{h}_\beta(u),$$

which follows from Lemma 3.1 (c). It remains to prove (3.26) for $\lambda = d$, for which the left-hand side of (3.26) is

$$(3.27) \quad \begin{aligned} \tilde{h}_\alpha(t)\sigma(u)\tilde{h}_\alpha(t)^{-1} &= \sigma(u)\tilde{h}_\alpha(u^{-1}t)\tilde{h}_\alpha(t)^{-1} \\ &= \sigma(u)u^{-\frac{2}{(\alpha, \alpha)}}\tilde{h}_\alpha(u^{-1}), \end{aligned}$$

where we have used (3.15). By Lemma 2.3,

$$T_{-\alpha^\vee}(d) = -\alpha^\vee - \frac{(\alpha^\vee, \alpha^\vee)}{2}c + d = -\alpha^\vee - \frac{2}{(\alpha, \alpha)}c + d.$$

Therefore $T_{-\alpha^\vee}(d)(u)$ is equal to the right-hand side of (3.27). \square

Finally, define $\widetilde{B} = \widehat{B} \rtimes \sigma(F^\times)$. If F is a local field, we also define

$$\widetilde{K} = \widehat{K} \rtimes \sigma(\mathcal{M}_F),$$

where \mathcal{M}_F is the maximal compact subgroup of F^\times , i.e.

$$(3.28) \quad \mathcal{M}_F = \begin{cases} \{\pm 1\}, & \text{if } F = \mathbb{R}, \\ S^1, & \text{if } F = \mathbb{C}, \\ \mathcal{O}_F^\times, & \text{if } F \text{ is } p\text{-adic.} \end{cases}$$

3.2. Highest weight representations of loop groups. Let $\lambda \in \widetilde{\mathfrak{h}}^*$ be a dominant integral weight, i.e. $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$, $i = 0, \dots, n$, and $\langle \lambda, d \rangle \in \mathbb{Z}$. Let V_λ be the corresponding irreducible highest weight representation of $\widetilde{\mathfrak{g}}$, and v_λ be a highest weight vector. A vector $v \in V_\lambda$ is said to be homogeneous of weight μ if it lies in a weight space $V_{\lambda, \mu}$. Every vector $v \in V_\lambda$ is a sum of homogeneous elements (called components of v). There is a lattice $V_{\lambda, \mathbb{Z}} \subset V_\lambda$ which is preserved by the action of $\frac{1}{j!}(X_\alpha \otimes t^i)^j \in \mathcal{U}(\widetilde{\mathfrak{g}})$ for every positive integer j and basis vector $X_\alpha \in \mathfrak{g}_\alpha$ in the Chevalley basis of \mathfrak{g} . Moreover,

$$V_{\lambda, \mathbb{Z}} = \bigoplus_{\mu} V_{\lambda, \mu, \mathbb{Z}},$$

where μ runs over all the weights of V_λ , and $V_{\lambda, \mu, \mathbb{Z}} = V_{\lambda, \mathbb{Z}} \cap V_{\lambda, \mu}$. Assume that $V_{\lambda, \lambda, \mathbb{Z}} = \mathbb{Z}v_\lambda$.

For any local field F , $V_{\lambda, F} = V_{\lambda, \mathbb{Z}} \otimes_{\mathbb{Z}} F$ is a representation of $\widetilde{G}(F((t)))$, with the action of $\widetilde{x}_\alpha(ut^i)$ given by

$$\widetilde{x}_\alpha(ut^i)v = \sum_{j=0}^{\infty} \frac{1}{j!} u^j (X_\alpha \otimes t^i)^j v$$

for every $v \in V_{\lambda, F}$. Since $(X_\alpha \otimes t^i)^j v = 0$ for j large enough, the sum above is finite. Since the operators $X_\alpha \otimes t^i$ ($i \in \mathbb{Z}$) are commutative, and $(X_\alpha \otimes t^i)v = 0$ for i large enough, for $u = \sum_{i=N}^{\infty} u_i t^i \in F((t))$ the product $\prod_{i=N}^{\infty} \widetilde{x}_\alpha(u_i t^i)v$ is finite, and we define the action of $\widetilde{x}_\alpha(u)$ as $\prod_{i=N}^{\infty} \widetilde{x}_\alpha(u_i t^i)$. The action can be extended to an action of $\widetilde{G}(F((t)))$ by setting

$$(3.29) \quad \sigma(\mathfrak{q})v = \mathfrak{q}^{(\mu, d)}v$$

for each $v \in V_{\lambda, \mu, F}$.

Theorem 3.14 (Garland [8]). *There is an action of $\widetilde{G}(F((t)))$ on $V_{\lambda, F}$ defined as above. The action of $u \in F^\times$ on $V_{\lambda, F}$ is the scalar $u^{\langle \lambda, c \rangle}$, and the action of $\widetilde{h}_\alpha(u)$ ($\alpha \in \Phi$) on $V_{\lambda, \mu, F}$ is $u^{\langle \mu, \alpha^\vee \rangle}$.*

If $F = \mathbb{R}$ or \mathbb{C} , there is a hermitian inner product $(,)$ on $V_{\lambda, F}$ such that

- (i) $(v_\lambda, v_\lambda) = 1$,
- (ii) homogeneous vectors with different weights are orthogonal,
- (iii) there is a homogeneous orthonormal basis contained in $V_{\lambda, \mathbb{Z}}$,
- (iv) elements of \widehat{K} act as unitary operators,
- (v) $X_\alpha \otimes t^i$ and $X_{-\alpha} \otimes t^{-i}$ are adjoint operators.

In particular, the norm $\|v\| = (v, v)^{\frac{1}{2}} \geq 1$ for all $v \in V_{\lambda, \mathbb{Z}}$, $v \neq 0$. If F is a p -adic field with \mathcal{O}_F the ring of integers and $\pi \in \mathcal{O}_F$ a uniformizer, we let

$$V_{\lambda, \mathcal{O}_F} = V_{\lambda, \mathbb{Z}} \otimes \mathcal{O}_F$$

and define a norm on $V_{\lambda, F}$ by setting $\|0\| = 0$ and $\|v\| = |\pi^l|$ for $v \neq 0$, where l is the largest integer such that $v \in \pi^l V_{\lambda, \mathcal{O}_F}$. Recall that the normalized absolute value on F is defined by $|\pi| = q^{-1}$, where q is the cardinality of the residue field $\mathcal{O}_F/\pi\mathcal{O}_F$. Since the action of \widehat{K} preserves $V_{\lambda, \mathcal{O}_F}$, this norm is preserved by \widehat{K} . We also have $\|xv\| = |x|\|v\|$ for $x \in F$, $x \in V_{\lambda, F}$, and $\|v_1 + v_2\| \leq \max(\|v_1\|, \|v_2\|)$.

Now we are ready to prove Theorem 3.7. Let \mathcal{H} be the subgroup of $\widehat{G}(F((t)))$ generated by the elements $\tilde{x}_\alpha(u)$ with $\alpha \in \Phi$, $u \in F[[t]]$. Write π for the projection $\widehat{G}(F((t))) \rightarrow G(F((t)))$. It suffices to prove the following lemma.

Lemma 3.15. $\pi : \mathcal{H} \rightarrow G(F[[t]])$ is an isomorphism.

Proof. Using $\mathcal{U}(\tilde{\mathfrak{g}}) = \mathcal{U}_-(\tilde{\mathfrak{g}})\mathcal{U}_+(\tilde{\mathfrak{g}})$, we have a decomposition

$$V_\lambda = V_\lambda(0) \oplus V_\lambda(1) \oplus \cdots,$$

where

$$V_\lambda(d) = \text{Span}\{(X_{\alpha_1} \otimes t^{-d_1}) \cdots (X_{\alpha_l} \otimes t^{-d_l})v_\lambda \mid d_i \geq 0, \sum_{i=1}^l d_i = d\}.$$

It is clear that $V_\lambda(0)$ is the highest weight module of \mathfrak{g} with highest weight $\lambda|_{\mathfrak{h}}$. Then $V_\lambda(0)$ is a representation of $G(F)$ and becomes an \mathcal{H} -module via the following diagram:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\pi} & G(F[[t]]) \\ & \searrow & \downarrow t=0 \\ & & G(F) \longrightarrow GL(V_\lambda(0)) \end{array}$$

Assume $u \in \text{Ker}(\pi|_{\mathcal{H}}) \subset F^\times$; then u acts on $V_\lambda(0)$ trivially. On the other hand, by Theorem 3.14 u acts by the scalar $u^{\langle \lambda, c \rangle}$. Therefore $u^{\langle \lambda, c \rangle} = 1$ for any dominant integral weight λ . It follows that $u = 1$. \square

Using Theorem 3.14 we can also prove the following lemma for p -adic loop groups.

Lemma 3.16. If F is p -adic, then

$$\text{Ker}(\pi|_{\widehat{K}}) = \mathcal{O}_F^\times.$$

Proof. If $u \in \text{Ker}(\pi|_{\widehat{K}}) \subset F^\times$, since \widehat{K} preserves $V_{\lambda, \mathcal{O}_F}$ we obtain $u^{\langle \lambda, c \rangle} V_{\lambda, \mathcal{O}_F} = V_{\lambda, \mathcal{O}_F}$ by Theorem 3.14. It follows that $u^{\langle \lambda, c \rangle} \in \mathcal{O}_F^\times$ for any dominant integral weight λ . This implies $u \in \mathcal{O}_F^\times$. \square

3.3. Second construction of loop groups. We start from the example $G = GL_n$. A lattice L of an n -dimensional $F((t))$ -vector space V is a free $F[[t]]$ -submodule of rank n . In other words L is an $F[[t]]$ -span of a basis of V . Any two lattices L_1, L_2 in V are *commensurable*, which means that the quotients $L_1/(L_1 \cap L_2)$ and $L_2/(L_1 \cap L_2)$ are finite dimensional over F . For example, any lattice in $F((t))^n$ is commensurable with $F[[t]]^n$.

Let L_0 be the lattice $F[[t]]^n$, and $g \in GL_n(F((t)))$. Since $gL_0/(L_0 \cap gL_0)$ is finite dimensional over F , we can define the top wedge power $\bigwedge^{\text{top}}(gL_0/L_0 \cap gL_0)$, which is a one-dimensional vector space over F . Let $\det(L_0, gL_0)$ be the tensor product

$$\bigwedge^{\text{top}}(gL_0/L_0 \cap gL_0) \otimes_F \bigwedge^{\text{top}}(L_0/L_0 \cap gL_0)^{-1},$$

where $(-)^{-1} = \text{Hom}_F(-, F)$ denotes the dual vector space. Let $\det(L_0, gL_0)^\times$ be the set of non-zero vectors in $\det(L_0, gL_0)$ which form an F^\times -torsor. Now define the group

$$\widehat{GL}_n(F((t)))_{\text{st}} = \{(g, \omega_g) | g \in GL_n(F((t))), \omega_g \in \det(L_0, gL_0)^\times\}.$$

Here the subscript “st” stands for the standard representation of GL_n . The multiplication in the group is given by

$$(g, \omega_g)(h, \omega_h) = (gh, \omega_g \wedge g\omega_h),$$

where $g\omega_h$ is the image of ω_h under the natural map $\det(L_0, hL_0) \xrightarrow{g} \det(gL_0, ghL_0)$, and $\omega_g \wedge g\omega_h$ is defined by the isomorphism $\det(L_0, gL_0) \wedge \det(gL_0, ghL_0) \rightarrow \det(L_0, ghL_0)$.

$\sigma(\mathbf{q}) \in \text{Aut}_F F((t))$ with $\mathbf{q} \in F^\times$ preserves L_0 , and hence induces the maps $gL_0/L_0 \cap gL_0 \rightarrow (\sigma(\mathbf{q}) \cdot g)L_0/L_0 \cap (\sigma(\mathbf{q}) \cdot g)L_0$, $L_0/L_0 \cap gL_0 \rightarrow L_0/L_0 \cap (\sigma(\mathbf{q}) \cdot g)L_0$. Therefore $\sigma(\mathbf{q})$ induces the map $\det(L_0, gL_0) \rightarrow \det(L_0, (\sigma(\mathbf{q}) \cdot g)L_0)$, and hence acts on the group $\widehat{GL}_n(F((t)))_{\text{st}}$. We can form the semi-direct product group

$$\widehat{GL}_n(F((t)))_{\text{st}} = \widehat{GL}_n(F((t)))_{\text{st}} \rtimes \sigma(F^\times).$$

Suppose that G is a linear reductive algebraic group over F and that (ρ, V) is a (faithful) rational representation of G . Then $G(F((t)))$ acts on $V_{F((t))} = V \otimes_F F((t))$. Let $V_0 = V_{F[[t]]} = V \otimes_F F[[t]]$; then V_0 is a lattice of $V_{F((t))}$. Define the following loop group:

$$\widehat{G}(F((t)))_\rho = \{(g, \omega_g) | g \in G(F((t))), \omega_g \in \det(V_0, \rho(g)V_0)^\times\}.$$

The group law is defined similarly to the GL_n case. It is clear that $\widehat{G}(F((t)))_\rho$ is a central extension of $G(F((t)))$ by F^\times . $\sigma(\mathbf{q})$ acts on $V_{F((t))}$ as $\text{id} \otimes \sigma(\mathbf{q})$, which preserves V_0 . Therefore we can form the full loop group:

$$\widetilde{G}(F((t)))_\rho = \widehat{G}(F((t)))_\rho \rtimes \sigma(F^\times).$$

From the construction it is seen that this notion only depends on the equivalence class of ρ . Namely, if (ρ, V) and (ρ', V') are equivalent representations of G , then $\widetilde{G}(F((t)))_\rho$ and $\widetilde{G}(F((t)))_{\rho'}$ are isomorphic. To show this, let $f: V \rightarrow V'$ be any intertwining linear bijection; then $f(\rho(g)v) = \rho'(g)f v$ for all $v \in V$, $g \in G$. The action of f extends to $V_{F((t))} = V \otimes_F F((t))$ by scalar extension. Then $f(V_0) = V'_0$, $f(\rho(g)V_0) = \rho'(g)V'_0$, and f induces an isomorphism of F^\times -torsors $f_g: \det(V_0, \rho(g)V_0)^\times \rightarrow \det(V'_0, \rho'(g)V'_0)^\times$, which commutes with the action of $\sigma(F^\times)$. Let us identify f_g with a scalar in F^\times . Then

$$\widetilde{f}: \widetilde{G}(F((t)))_\rho \rightarrow \widetilde{G}(F((t)))_{\rho'}, \quad (g, \omega_g) \rtimes \sigma(u) \mapsto (g, f_g \omega_g) \rtimes \sigma(u)$$

is a group isomorphism, which can be easily checked. Write \hat{f} for the restriction of \widetilde{f} to $\widehat{G}(F((t)))_\rho$; then the the following diagram with exact rows is commutative:

$$\begin{array}{ccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & \widehat{G}(F((t)))_\rho & \xrightarrow{\pi_\rho} & G(F((t))) \longrightarrow 1 \\ & & \downarrow \text{id} & & \downarrow \hat{f} & & \downarrow \text{id} \\ 1 & \longrightarrow & F^\times & \longrightarrow & \widehat{G}(F((t)))_{\rho'} & \xrightarrow{\pi_{\rho'}} & G(F((t))) \longrightarrow 1 \end{array}$$

It is also clear that $\widetilde{f_2} \circ \widetilde{f_1} = \widetilde{f_2 \circ f_1}$ if $V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$ are the equivalence of representations of G .

Lemma 3.17. *If G is a connected and simply connected semi-simple algebraic group split over F , then the isomorphism $\hat{f} : \widehat{G}(F((t)))_\rho \rightarrow \widehat{G}(F((t)))_{\rho'}$ does not depend on the choice of the intertwining map f .*

Proof. It is equivalent to prove that if $\rho = \rho'$ and $f : V \rightarrow V$ is an intertwining map, then $\hat{f} = \text{id}$. In other words, we have to show that $f_g : \det(V_0, \rho(g)V_0)^\times \rightarrow \det(V_0, \rho(g)V_0)^\times$ is the identity map for all $g \in G(F((t)))$. We may assume that $V = mV_\tau$, where V_τ is irreducible. Then Schur's lemma implies that f is given by an element $a_f \in GL_m(F)$. Recall that for any two lattices L_1, L_2 in $V_{F((t))}$, we have the notion of relative dimension:

$$\dim(L_1, L_2) = \dim_F \frac{L_1}{L_1 \cap L_2} - \dim_F \frac{L_2}{L_1 \cap L_2}.$$

It is easy to see that

- (i) $\dim(hL_1, hL_2) = \dim(L_1, L_2)$ for any $h \in GL(V_{F((t))})$,
- (ii) $\dim(L_1, L_3) = \dim(L_1, L_2) + \dim(L_2, L_3)$.

Since f commutes with $\rho(g)$, we have $f_g = (\det a_f)^{\dim(V_{\tau 0}, \rho(g)V_{\tau 0})}$, where $V_{\tau 0} = V_\tau \otimes_F F[[t]]$. Therefore we only have to show that $\dim(V_{\tau 0}, \rho(g)V_{\tau 0}) = 0$. We have the decomposition $G = P_\Delta Q^\vee P_\Delta$ since G is simply connected. Using this together with the fact that $\rho(P_\Delta)$ preserves $V_{\tau 0}$, by (i) above we may assume that $g \in \widetilde{W}$ and is mapped into Q^\vee , e.g. g is a product of elements of the form $h_\alpha(t)$, $\alpha \in \Phi$. Using (i) and (ii) repeatedly we get the formula

$$\dim(L, h_1 h_2 \cdots h_n L) = \dim(L, h_1 L) + \cdots + \dim(L, h_n L).$$

Hence we are reduced to proving that $\dim(V_{\tau 0}, h_\alpha(t)V_{\tau 0}) = 0$. However, this follows from the fact that $h_\alpha(u)$ acts on the weight space $V_{\tau, \lambda}$ of V_τ of weight λ by the scalar $u^{\langle \lambda, \alpha^\vee \rangle}$, $\langle r_\alpha \lambda, \alpha^\vee \rangle = -\langle \lambda, \alpha^\vee \rangle$, and $\dim V_{\tau, w\lambda} = \dim V_{\tau, \lambda}$, $\forall w \in W$. \square

Let us denote by $[\rho]$ the equivalence class of representations of ρ . Under the condition of Lemma 3.17, $\widehat{G}(F((t)))_{\rho_1}$ and $\widehat{G}(F((t)))_{\rho_2}$ are canonically isomorphic for any $\rho_1, \rho_2 \in [\rho]$, and our loop group can be written as $\widehat{G}(F((t)))_{[\rho]}$.

To see the relations with the construction in Section 3.1, let us assume that G is a connected and simply connected simple linear algebraic group split over F . Let \mathfrak{g}_F be the (simple) Lie algebra of G , and $\mathfrak{g} = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ be the complex simple Lie algebra, where $\mathfrak{g}_\mathbb{Z}$ is the lattice spanned by a Chevalley basis of \mathfrak{g}_F . Let $\widehat{G}(F((t)))$ be the central extension of $G(F((t)))$ constructed in Section 3.1. Let (ρ, V) be a rational representation of G .

Theorem 3.18. *There exists a group homomorphism $\phi_\rho : \widehat{G}(F((t))) \rightarrow \widehat{G}(F((t)))_\rho$ such that the following diagram is commutative:*

$$\begin{array}{ccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & \widehat{G}(F((t))) & \xrightarrow{\pi} & G(F((t))) \longrightarrow 1 \\ & & \downarrow d_\rho & & \downarrow \phi_\rho & & \downarrow \text{id} \\ 1 & \longrightarrow & F^\times & \longrightarrow & \widehat{G}(F((t)))_\rho & \xrightarrow{\pi_\rho} & G(F((t))) \longrightarrow 1 \end{array}$$

where d_ρ is the Dynkin index of the representation ρ and $F^\times \xrightarrow{d_\rho} F^\times$ is the d_ρ -th power.

The Dynkin index of a representation, introduced to the theory of G -bundles over a curve by Faltings [6] and Kumar et al. [29], is defined as follows. By abuse of notation we also write the representation $\rho : \mathfrak{g} \rightarrow \mathfrak{sl}(V)$. Let

$$\text{ch } V = \sum_{\lambda} n_{\lambda} e^{\lambda}$$

be the formal character of V . Then the Dynkin index of ρ is defined to be

$$d_{\rho} = \frac{1}{2} \sum_{\lambda} n_{\lambda} \langle \lambda, \tilde{\alpha}^{\vee} \rangle^2.$$

[32] contains a Lie algebra version of this theorem, which is much easier to prove. The minimal Dynkin index $d_{\mathfrak{g}}$ is defined to be $\min d_{\rho}$, where ρ runs over all representations $\rho : \mathfrak{g} \rightarrow \mathfrak{sl}(V)$. For a dominant weight λ , let ρ_{λ} be the irreducible \mathfrak{g} -module with highest weight λ . The following table is given in [32]:

Type of \mathfrak{g}	A_n	$B_n, n \geq 3$	C_n	$D_n, n \geq 4$	E_6	E_7	E_8	F_4	G_2
$d_{\mathfrak{g}}$	1	2	1	2	6	12	60	6	2
λ s.t. $d_{\rho_{\lambda}} = d_{\mathfrak{g}}$	ϖ_1	ϖ_1	ϖ_1	ϖ_1	ϖ_6	ϖ_7	ϖ_8	ϖ_4	ϖ_1

Proof of the theorem. By the existence of the canonical lifting (Theorem 3.7)

$$G(F[[t]]) \hookrightarrow \widehat{G}(F((t))),$$

and such that $\rho(G(F[[t]]))$ preserves V_0 , we first define $\phi_{\rho}(\tilde{x}_{\alpha}(u)) = (x_{\alpha}(u), 1)$ for $\alpha \in \Phi, u \in F[[t]]$. For general $u \in F((t))$, choose $\beta \in \Phi$ with $\langle \alpha, \beta^{\vee} \rangle \neq 0$. There exists $v \in F((t))$ such that $v^{-\langle \alpha, \beta^{\vee} \rangle} u \in F[[t]]$. Let $\bar{h}_{\beta}(v) = (h_{\beta}(v), \omega)$ be an element in the preimage of $h_{\beta}(v)$ under the projection $\widehat{G}(F((t)))_{\rho} \rightarrow \widehat{G}(F((t)))$. We define

$$(3.30) \quad \phi_{\rho}(\tilde{x}_{\alpha}(u)) = \bar{h}_{\beta}(v) \phi_{\rho}(\tilde{x}_{\alpha}(v^{-\langle \alpha, \beta^{\vee} \rangle} u)) \bar{h}_{\beta}(v)^{-1}.$$

Let us check that this is well defined, namely, the right-hand side of (3.30) does not depend on the choice of $\bar{h}_{\beta}(v)$. But we have

$$\begin{aligned} (3.31) \quad & \bar{h}_{\beta}(v) \phi_{\rho}(\tilde{x}_{\alpha}(v^{-\langle \alpha, \beta^{\vee} \rangle} u)) \bar{h}_{\beta}(v)^{-1} \\ &= (h_{\beta}(v), \omega)(x_{\alpha}(v^{-\langle \alpha, \beta^{\vee} \rangle} u), 1)(h_{\beta}(v)^{-1}, h_{\beta}(v)^{-1} \omega^{-1}) \\ &= (x_{\alpha}(u), \omega \wedge x_{\alpha}(u) \omega^{-1}). \end{aligned}$$

Then we have to prove that for $\omega' \in \det(V_0, h_{\beta'}(v')V_0)^{\times}$ with $v'^{-\langle \alpha, \beta'^{\vee} \rangle} u \in F[[t]]$,

$$\omega \wedge x_{\alpha}(u) \omega^{-1} = \omega' \wedge x_{\alpha}(u) \omega'^{-1}.$$

Let $\eta = \omega^{-1} \omega' \in \det(h_{\beta}(v)V_0, h_{\beta'}(v')V_0)^{\times}$. Since

$$x_{\alpha}(u) h_{\beta}(v) V_0 = h_{\beta}(v) x_{\alpha}(v^{-\langle \alpha, \beta^{\vee} \rangle} u) V_0 = h_{\beta}(v) V_0$$

and similarly $x_{\alpha}(u) h_{\beta'}(v') V_0 = h_{\beta'}(v') V_0$, it is clear that $x_{\alpha}(u)$ acts on the finite dimensional spaces

$$\frac{h_{\beta}(v) V_0}{h_{\beta}(v) V_0 \cap h_{\beta'}(v') V_0}, \quad \frac{h_{\beta'}(v') V_0}{h_{\beta}(v) V_0 \cap h_{\beta'}(v') V_0}$$

unipotently. Taking the top wedge product we see that $x_{\alpha}(u)$ fixes η . This proves (3.30) is well defined.

To show that ϕ_ρ is the required homomorphism, we need to verify that

(i) $\phi_\rho(\tilde{x}_\alpha(u))$, $\alpha \in \Phi$, $u \in F((t))$ satisfy (3.1) and (3.2) if $\text{rank } \mathfrak{g} \geq 2$, or (3.1) and (3.4) if $\mathfrak{g} = \mathfrak{sl}_2$.

(ii) $\phi_\rho(\tilde{h}_\alpha(u))\phi_\rho(\tilde{h}_\alpha(v))\phi_\rho(\tilde{h}_\alpha(uv))^{-1} = (u, v)_{tame}^{-\frac{2}{(\alpha, \alpha)}d_\rho}$, $\alpha \in \Phi$, $u, v \in F((t))^\times$.

(3.1) is trivial. (3.2) is clearly true when $u, v \in F[[t]]$, and to prove the general case we need a lemma.

Lemma 3.19. *Suppose that \mathfrak{g} is a simple complex Lie algebra with root system Φ , and α, β are positive roots in Φ . Then there exists $\gamma \in \Phi$ such that $\langle \alpha, \gamma^\vee \rangle \langle \beta, \gamma^\vee \rangle > 0$.*

The lemma can be verified for the Lie algebra \mathfrak{g} of type A, B, \dots, G separately. Since we cannot find this lemma in the literature, let us sketch procedures of the proof. If $(\alpha, \beta) > 0$, then it is obvious. If $(\alpha, \beta) < 0$, it is reduced to checking the lemma for all irreducible root systems of rank 2, namely A_2, B_2 and G_2 . For $(\alpha, \beta) = 0$ we only have a case-by-case proof, and we omit the details here.

We continue to prove (3.2). Since $\alpha + \beta \neq 0$, there exists $w \in W$ such that $w\alpha, w\beta > 0$. By Lemma 3.19, we can find $\gamma \in \Phi$ satisfying $\langle \alpha, \gamma^\vee \rangle \langle \beta, \gamma^\vee \rangle > 0$. Then there exists $a \in F((t))^\times$ such that $a^{-\langle \alpha, \gamma^\vee \rangle}u, a^{-\langle \beta, \gamma^\vee \rangle}v \in F[[t]]$. As before let $\bar{h}_\gamma(a) = (h_\gamma(a), \omega) \in \widehat{G}(F((t)))_\rho$. It follows that

$$\begin{aligned} & \phi_\rho(\tilde{x}_\alpha(u))\phi_\rho(\tilde{x}_\beta(v))\phi_\rho(\tilde{x}_\alpha(u))^{-1}\phi_\rho(\tilde{x}_\beta(v))^{-1} \\ &= \bar{h}_\gamma(a)\phi_\rho(\tilde{x}_\alpha(a^{-\langle \alpha, \gamma^\vee \rangle}u))\phi_\rho(\tilde{x}_\beta(a^{-\langle \beta, \gamma^\vee \rangle}v)) \\ & \quad \times \phi_\rho(\tilde{x}_\alpha(a^{-\langle \alpha, \gamma^\vee \rangle}u))^{-1}\phi_\rho(\tilde{x}_\beta(a^{-\langle \beta, \gamma^\vee \rangle}v))^{-1}\bar{h}_\gamma(a)^{-1} \\ &= \bar{h}_\gamma(a) \prod_{i,j \in \mathbb{Z}^+, i\alpha+j\beta \in \Phi} \phi_\rho(\tilde{x}_{i\alpha+j\beta}(c_{ij}^{\alpha\beta}(a^{-\langle \alpha, \gamma^\vee \rangle}u)^i(a^{-\langle \beta, \gamma^\vee \rangle}v)^j))\bar{h}_\gamma(a)^{-1} \\ &= \prod_{i,j \in \mathbb{Z}^+, i\alpha+j\beta \in \Phi} \phi_\rho(\tilde{x}_{i\alpha+j\beta}(c_{ij}^{\alpha\beta}u^i v^j)). \end{aligned}$$

This proves (3.2). The same trick also applies to the proof of (3.4).

Let us compute the 2-cocycle and prove (ii) above. It suffices to treat the case $\mathfrak{g} = \mathfrak{sl}_2$. In fact, if we define

$$e_\alpha = \frac{1}{2} \sum_\lambda n_\lambda \langle \lambda, \alpha^\vee \rangle^2$$

for $\alpha \in \Phi$, then e_α is proportional to $1/(\alpha, \alpha)$. So assume $\mathfrak{g} = \mathfrak{sl}_2$, α is the simple root, and we shall prove that

$$\phi_\rho(\tilde{h}_\alpha(ut^i))\phi_\rho(\tilde{h}_\alpha(vt^j))\phi_\rho(\tilde{h}_\alpha(uvt^{i+j}))^{-1} = (ut^i, vt^j)_{tame}^{-d_\rho},$$

where $u, v \in F^\times$, $i, j \in \mathbb{Z}$. Let us restrict ourselves to the case $i, j \geq 0$. Other cases can be treated similarly. We can further assume that ρ is irreducible, say, of highest weight m . Then V has the weight space decomposition

$$V = \bigoplus_{\lambda=m, m-2, \dots, -m} V_\lambda,$$

and $d_\rho = \frac{1}{2} \sum_\lambda \lambda^2 = \sum_{\lambda > 0} \lambda^2$. We first study the element $\phi_\rho(\tilde{x}_{-\alpha}(ct^{-i}))$ where $c \in F^\times$, $i \in \mathbb{Z}_{\geq 0}$. By (3.30) and (3.31) we have

$$\phi_\rho(\tilde{x}_{-\alpha}(ct^{-i})) = (x_{-\alpha}(ct^{-i}), \omega \wedge x_{-\alpha}(ct^{-i})\omega^{-1}),$$

where $\omega \in \det(V_0, h_\alpha(t^i)V_0)^\times$. Let $\{v_\lambda | \lambda = m, m-2, \dots, -m\}$ be a basis of V such that $X_{-\alpha}v_\lambda = v_{\lambda-2}$. Since $h_\alpha(t^i)$ acts on $V_{\lambda, F((t))}$ as the scalar $t^{i\lambda}$, we can choose $\omega = \bigwedge_{\lambda=m, m-2, \dots, -m} \omega_\lambda$, where

$$(3.32) \quad \omega_\lambda = \begin{cases} \bigwedge_{l=-i\lambda}^{-1} t^l v_\lambda, & \text{if } \lambda < 0, \\ \left(\bigwedge_{l=0}^{i\lambda-1} t^l v_\lambda \right)^{-1}, & \text{if } \lambda > 0. \end{cases}$$

Recall that the action of $x_{-\alpha}(ct^{-i})$ is given by

$$x_{-\alpha}(ct^{-i})v_\lambda = \sum_{j=0}^{\infty} \frac{c^j t^{-ij}}{j!} X_{-\alpha}^j v_\lambda = \sum_{j=0}^{\infty} \frac{c^j t^{-ij}}{j!} v_{\lambda-2j},$$

from which it is seen that $x_\alpha(ct^{-i})$ preserves the F -span of $\{t^l v_\lambda | \lambda < 0, i\lambda \leq l \leq -1\}$ and acts unipotently. Write $\omega = \omega_+ \wedge \omega_-$, where $\omega_+ = \bigwedge_{\lambda>0} \omega_\lambda$, $\omega_- = \bigwedge_{\lambda<0} \omega_\lambda$, with the λ 's in decreasing order. Then $x_{-\alpha}(ct^{-i})$ fixes ω_- , and

$$\omega \wedge x_{-\alpha}(ct^{-i})\omega^{-1} = \omega_+ \wedge x_{-\alpha}(ct^{-i})\omega_+^{-1}.$$

Now we have

$$\begin{aligned} & \phi_\rho(\tilde{h}_\alpha(ut^i)) \\ &= \phi_\rho(w_\alpha(ut^i))\phi_\rho(w_\alpha(1)) \\ &= (x_\alpha(ut^i), 1)\phi_\rho(\tilde{x}_{-\alpha}(-u^{-1}t^{-i}))(x_\alpha(ut^i), 1)(w_\alpha(1), 1) \\ &= (x_\alpha(ut^i), 1)(x_{-\alpha}(-u^{-1}t^{-i}), \omega_+ \wedge x_{-\alpha}(-u^{-1}t^{-i})\omega_+^{-1})(x_\alpha(ut^i)w_\alpha(1), 1) \\ &= (h_\alpha(ut^i), \tilde{\omega}), \end{aligned}$$

where

$$\begin{aligned} \tilde{\omega} &= x_\alpha(ut^i)(\omega_+ \wedge x_{-\alpha}(-u^{-1}t^{-i})\omega_+^{-1}) \\ &= x_\alpha(ut^i)\omega_+ \wedge x_\alpha(ut^i)x_{-\alpha}(-u^{-1}t^{-i})\omega_+^{-1} \\ &= \omega_+ \wedge w_\alpha(ut^i)x_\alpha(-ut^i)\omega_+^{-1} \\ &= \omega_+ \wedge w_\alpha(ut^i)\omega_+^{-1}. \end{aligned}$$

From the action of w_α on V it is seen that

$$\begin{aligned} w_\alpha(ut^i)\omega_+^{-1} &= w_\alpha(ut^i) \bigwedge_{\lambda>0} \bigwedge_{l=0}^{i\lambda-1} t^l v_\lambda \\ &= \bigwedge_{\lambda>0} \bigwedge_{l=0}^{i\lambda-1} (-1)^{\frac{\lambda+m}{2}} u^{-\lambda} t^{l-i\lambda} v_{-\lambda} \\ &= \bigwedge_{\lambda>0} (-1)^{\frac{i\lambda(\lambda+m)}{2}} u^{-i\lambda^2} \bigwedge_{l=-i\lambda}^{-1} t^l v_{-\lambda} \\ &= \left(\prod_{\lambda>0} (-1)^{\frac{i\lambda(\lambda+m)}{2}} u^{-i\lambda^2} \right) \bigwedge_{\lambda>0} \omega_{-\lambda}. \end{aligned}$$

In summary, we can write

$$(3.33) \quad \phi_\rho(\tilde{h}_\alpha(ut^i)) = (h_\alpha(ut^i), \omega_{u,i}),$$

where

$$\omega_{u,i} = \prod_{\lambda > 0} (\varepsilon_\lambda^i u^{-i\lambda^2}) \bigwedge_{\lambda > 0} \omega_{i,\lambda} \wedge \bigwedge_{\lambda > 0} \omega_{i,-\lambda},$$

$\varepsilon_\lambda = (-1)^{\frac{\lambda(\lambda+m)}{2}}$, and $\omega_{i,\lambda}$ is given by (3.32). Let us apply this to prove (ii). We have

$$\phi_\rho(\tilde{h}_\alpha(ut^i))\phi_\rho(\tilde{h}_\alpha(vt^j)) = (h_\alpha(uvt^{i+j}), \omega_{u,i} \wedge h_\alpha(ut^i)\omega_{v,j})$$

and

$$\begin{aligned} & \omega_{u,i} \wedge h_\alpha(ut^i)\omega_{v,j} \\ &= \prod_{\lambda > 0} (\varepsilon_\lambda^{i+j} u^{-i\lambda^2} v^{-j\lambda^2}) \bigwedge_{\lambda > 0} \left(\bigwedge_{l=0}^{i\lambda-1} t^l v_\lambda \right)^{-1} \bigwedge_{\lambda > 0} \left(\bigwedge_{l=-i\lambda}^{-1} t^l v_{-\lambda} \right) \\ & \quad \wedge \bigwedge_{\lambda > 0} \left(\bigwedge_{l=0}^{j\lambda-1} u^\lambda t^{i\lambda+l} v_\lambda \right)^{-1} \bigwedge_{\lambda > 0} \left(\bigwedge_{l=-j\lambda}^{-1} u^{-\lambda} t^{-i\lambda+l} v_{-\lambda} \right) \\ &= \prod_{\lambda > 0} (u^{-j\lambda^2} v^{i\lambda^2}) \prod_{\lambda > 0} (\varepsilon_\lambda^{i+j} (xy)^{-(i+j)\lambda^2}) \bigwedge_{\lambda > 0} \left(\bigwedge_{l=0}^{i\lambda-1} t^l v_\lambda \right)^{-1} \bigwedge_{\lambda > 0} \left(\bigwedge_{l=-i\lambda}^{-1} t^l v_{-\lambda} \right) \\ & \quad \wedge \bigwedge_{\lambda > 0} \left(\bigwedge_{l=i\lambda}^{(i+j)\lambda-1} t^l v_\lambda \right)^{-1} \bigwedge_{\lambda > 0} \left(\bigwedge_{l=-(i+j)\lambda}^{-i\lambda-1} t^l v_{-\lambda} \right) \\ &= \prod_{\lambda > 0} (u^{-j\lambda^2} v^{i\lambda^2} (-1)^{ij\lambda^2}) \omega_{uv,i+j} \\ &= (xt^i, yt^j)_{\text{tame}}^{-d_\rho} \omega_{uv,i+j}, \end{aligned}$$

where the second-to-last equality is obtained from a direct counting. This finishes the proof of (ii), hence ϕ_ρ is the required homomorphism. \square

3.4. Adelic loop groups and arithmetic quotients. Let F be a number field, and for each place v let F_v be the completion of F at v . For each local field F_v , we have the local loop groups $G(F_v((t)))$, $\widehat{G}(F_v((t)))$ and $\widetilde{G}(F_v((t)))$ constructed in Section 3.1 which correspond to a complex simple Lie algebra \mathfrak{g} . We add the subscript v to indicate the corresponding local subgroups. So we have

$$\widehat{B}_{0v} \hookrightarrow \widehat{B}_v \hookrightarrow \widetilde{B}_v, \quad \widehat{K}_{0v} \hookrightarrow \widehat{K}_v \hookrightarrow \widetilde{K}_v, \quad T_v \hookrightarrow \widehat{T}_v \hookrightarrow \widetilde{T}_v.$$

For example, in $\widetilde{G}(F_v((t)))$ we have $\widetilde{T}_v = \widehat{T}_v \times \sigma(F_v^\times)$. We form the restricted direct product group $\prod'_v G(F_v((t)))$ (resp. $\prod'_v \widehat{G}(F_v((t)))$, $\prod'_v \widetilde{G}(F_v((t)))$) with respect to the \widehat{K}_{0v} (resp. \widehat{K}_v , \widetilde{K}_v)'s.

Let \mathbb{A} and \mathbb{I} be the adèle group and the idele group of F respectively. We let $\mathbb{A}\langle t \rangle$ be the restricted product $\prod'_v F_v((t))$ with respect to the $\mathcal{O}_v((t))$'s for finite places v . In other words,

$$\mathbb{A}\langle t \rangle = \{(x_v)_v | x_v \in F_v((t)), \text{ and } x_v \in \mathcal{O}_v((t)) \text{ for almost all finite places } v\}.$$

Note that we do not require that the (x_v) 's in $(x_v)_v$ have bounded poles, so the ring $\mathbb{A}\langle t \rangle$ is not a subring of $\mathbb{A}((t))$. Let

$$F\langle t \rangle = F((t)) \cap \mathbb{A}\langle t \rangle,$$

i.e. $F\langle t \rangle$ is the subset of the elements $x \in F((t))$ such that $x \in \mathcal{O}_v((t))$ for almost all finite places v . We also define

$$F\langle t \rangle_+ = F\langle t \rangle \cap F[[t]], \quad \mathbb{A}\langle t \rangle_+ = \mathbb{A}\langle t \rangle \cap \mathbb{A}[[t]].$$

Lemma 3.20. $F\langle t \rangle$ is a subfield of $F((t))$.

Proof. The only thing we need to check is that if $x \in F\langle t \rangle$ and $x \neq 0$, then $x^{-1} \in F\langle t \rangle$. We can assume that $x = 1 + x_1t + x_2t^2 + \cdots$; then the coefficients in x^{-1} are polynomials of the x_n 's. Therefore x^{-1} also lies in $F\langle t \rangle$. \square

We shall denote $\prod'_v \widehat{G}(F_v((t)))$ by $\widehat{G}(\mathbb{A}\langle t \rangle)$, and $\prod'_v G(F_v((t)))$ by $G(\mathbb{A}\langle t \rangle)$. For $\alpha \in \Phi$, $u \in F((t))$, we also denote by $\tilde{x}_\alpha(u)$ the element in $\prod'_v \widehat{G}(F_v((t)))$ whose v -component is $\tilde{x}_\alpha(u)$ in $\widehat{G}(F_v((t)))$. If $u \in F\langle t \rangle$, then $\tilde{x}_\alpha(u) \in \widehat{G}(\mathbb{A}\langle t \rangle)$. We denote the subgroup generated by $\tilde{x}_\alpha(u)$ ($\alpha \in \Phi, u \in F\langle t \rangle$) and $G(F\langle t \rangle_+)$ by $\widehat{G}(F\langle t \rangle)$. It is clear that $\widehat{G}(F\langle t \rangle)/F^\times$ is isomorphic to $G(F\langle t \rangle)$. We have the following diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & \widehat{G}(F\langle t \rangle) & \longrightarrow & G(F\langle t \rangle) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{I} & \longrightarrow & \widehat{G}(\mathbb{A}\langle t \rangle) & \longrightarrow & G(\mathbb{A}\langle t \rangle) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{I}/F^\times & \longrightarrow & \widehat{G}(\mathbb{A}\langle t \rangle)/F^\times & \longrightarrow & G(\mathbb{A}\langle t \rangle) & \longrightarrow & 1 \end{array}$$

where $F^\times \hookrightarrow \mathbb{I}$ is the diagonal subgroup.

By abuse of notation, we also use $\widehat{T}, \widehat{B}, \dots$ to denote the following adelic subgroups of $\widehat{G}(\mathbb{A}\langle t \rangle)$:

$$\begin{aligned} \widehat{T} &= \widehat{T}(\mathbb{A}) = \prod'_v \widehat{T}_v, & \widehat{U} &= \widehat{U}(\mathbb{A}) = \prod'_v \widehat{U}_v, \\ \widehat{B} &= \widehat{B}(\mathbb{A}) = \prod'_v \widehat{B}_v, & \widehat{K} &= \widehat{K}(\mathbb{A}) = \prod'_v \widehat{K}_v, \end{aligned}$$

where the restricted products are defined with respect to the corresponding analogues of maximal compact subgroups in the finite dimensional case. For example, \widehat{U} is generated by $\tilde{x}_\alpha(u)$ where either $\alpha \in \Phi_+$, $u \in \mathbb{A}\langle t \rangle_+$ or $\alpha \in \Phi_-$, $u \in t\mathbb{A}\langle t \rangle_+$. Then every element $g \in \widehat{G}(\mathbb{A}\langle t \rangle)$ can be written as $g = u_g a_g k_g$ with $u_g \in \widehat{U}$, $a_g \in \widehat{T}$ and $k_g \in \widehat{K}$. The local actions of $\sigma(F_v^\times)$ on $\widehat{G}(F_v((t)))$ piece together to form an action of the group $\sigma(\mathbb{I})$ on $\widehat{G}(\mathbb{A}\langle t \rangle)$. Define the semi-direct product group

$$(3.34) \quad \widetilde{G}(\mathbb{A}\langle t \rangle) = \widehat{G}(\mathbb{A}\langle t \rangle) \rtimes \sigma(\mathbb{I}).$$

Similarly we can set the subgroups of $\widetilde{G}(\mathbb{A}\langle t \rangle)$:

$$\begin{aligned} \widetilde{T} &= \widehat{T} \times \sigma(\mathbb{I}) = \prod'_v \widetilde{T}_v, & \widetilde{B} &= \widehat{B} \rtimes \sigma(\mathbb{I}) = \prod'_v \widetilde{B}_v, \\ \widetilde{K} &= \widehat{K} \rtimes \prod_v \sigma(\mathcal{M}_{F_v}) = \prod_v \widetilde{K}_v. \end{aligned}$$

Since $\sigma(\mathbb{I})$ normalizes \tilde{B} , any $g \in \tilde{G}(\mathbb{A}\langle t \rangle)$ can be written as $g = u_g a_g k_g$ with $u_g \in \tilde{U}$, $a_g \in \tilde{T}$ and $k_g \in \tilde{K}$.

Let $\tilde{G}(F\langle t \rangle) = \tilde{G}(F\langle t \rangle) \rtimes \sigma(F^\times) \hookrightarrow \tilde{G}(\mathbb{A}\langle t \rangle)$. By abuse of notation, for any subgroup H of $\tilde{G}(F\langle t \rangle)$, we still denote by H the subgroup $H \cap \tilde{G}(F\langle t \rangle)$ of $\tilde{G}(F\langle t \rangle)$. For example we have the subgroup $\hat{U}(F)$ of $\hat{G}(F\langle t \rangle)$, which is generated by $\tilde{x}_\alpha(u)$, where either $\alpha \in \Phi$, $u \in F\langle t \rangle_+$ or $\alpha \in \Phi_-$, $u \in tF\langle t \rangle_+$.

By Lemma 3.13 we have a \tilde{W} -equivariant map

$$Q^\vee \times \mathbb{I} \rightarrow \tilde{T}$$

given by, for $\lambda = \alpha^\vee + ic + jd \in Q_{\text{aff}}^\vee$ where $\alpha \in \Phi$, $u \in \mathbb{I}$,

$$(\lambda, u) \mapsto \lambda(u) = \tilde{h}_\alpha(u) u^i \sigma(u^j).$$

For a dominant integral weight λ , we have a representation of V_{λ, F_v} of $\tilde{G}(F_v\langle t \rangle)$ for each place v . Form the restricted product

$$V_{\lambda, \mathbb{A}} = \prod'_v V_{\lambda, F_v}$$

with respect to the lattices $V_{\lambda, \mathcal{O}_v}$ which are defined for all finite places. Denote by $v_\lambda \in V_{\lambda, \mathbb{A}}$ the element with v -component v_λ for each place v . Note that $V_{\lambda, F}$ embeds diagonally into $V_{\lambda, \mathbb{A}}$.

We define a map $|\cdot| : V_{\lambda, \mathbb{A}} \rightarrow \mathbb{R}_{\geq 0}$ as follows. Recall that we have defined a norm on V_{λ, F_v} for each place v in Section 3.2. For $(u_v)_v \in V_{\lambda, \mathbb{A}}$, if v is real or p -adic, let $|u_v| = \|u_v\|$; if v is complex, let $|u_v| = \|u_v\|^2$. Then define $|(u_v)_v| = \prod_v |u_v|$. Note that almost all $|u_v|$'s are less than or equal to 1, hence the product is finite. If $u \in V_{\lambda, \mathbb{A}}$ and $k \in \tilde{K}$, then $|ku| = |u|$; and if $x = (x_v)_v \in \mathbb{I}$, then $|xu| = |x||u|$, where $|x| = \prod_v |x_v|_v$. In particular, $|xu| = |u|$ for $x \in F^\times$ by the Artin product formula.

For $\mu \in \tilde{\mathfrak{h}}^*$, define a quasi-character $\mu : \tilde{T}(F) \backslash \tilde{T}(\mathbb{A}) \rightarrow \mathbb{C}^\times$ by, for $g = \tilde{h}_\alpha(x) y \sigma(z) \in \tilde{T}$ where $x, y, z \in \mathbb{I}$,

$$\mu(g) = |x|^{\langle \mu, \alpha^\vee \rangle} |y|^{\langle \mu, c \rangle} |z|^{\langle \mu, d \rangle}.$$

In particular, for $\lambda \in Q_{\text{aff}}^\vee$ we have

$$\lambda(x)^\mu = |x|^{\langle \lambda, \mu \rangle}.$$

However, sometimes we also use the following notation: if $a = (a_v)_v \in \tilde{T}(\mathbb{A})$, $\beta \in X^*(\tilde{T})$, the character lattice of \tilde{T} spanned by Φ , δ and L , then write

$$a^\beta = (a_v^\beta)_v \in \mathbb{I}.$$

Interpretations of the notation we shall use depend on the situation and would not cause any confusion.

Lemma 3.21. *For each $g \in \tilde{G}(\mathbb{A}\langle t \rangle)$ with decomposition $g = u_g a_g k_g$, and $v_\lambda \in V_{\lambda, \mathbb{A}_F}$ the highest weight vector as above, then*

$$|g^{-1} v_\lambda| = a_g^{-\lambda}.$$

Proof. Note that $u_g v_\lambda = v_\lambda$, and since \tilde{K} preserves $|\cdot|$, it follows that

$$|g^{-1} v_\lambda| = |a_g^{-1} v_\lambda| = a_g^{-\lambda}.$$

□

In the rest of this section we collect some lemmas on the arithmetic quotients of loop groups based on [8, 42].

Lemma 3.22. *For each $i = 0, 1, \dots, n$, and each integer $l > 0$, the set of weights of V_λ of the form*

$$\lambda - \sum_{j=0}^n l_j \alpha_j$$

with $l_i \leq l$, is finite.

Lemma 3.23. *Let $a \in \widehat{T}$ and $a\sigma(\mathfrak{q}) \in \widetilde{T}$ with $\mathfrak{q} \in \mathbb{I}$, $|\mathfrak{q}| < 1$. Then there exists $w \in \widetilde{W}$ such that $(wa\sigma(\mathfrak{q}))^{\alpha_i} \leq 1$ for all $i = 0, 1, \dots, n$.*

Proof. It is easy to see that the lemma can be reduced to $\widehat{G}(\mathbb{R}((t)))$. It follows from the well-known fact that for each $h = h_0 + ic + jd \in \widetilde{\mathfrak{h}}_{\mathbb{R}}$ with $j > 0$, there exists $w \in \widetilde{W}$ such that $(wh, \alpha_i) \geq 0$ for $i = 0, 1, \dots, n$. \square

Lemma 3.24. *Assume the conditions of the previous lemma. Moreover, suppose that $(a\sigma(\mathfrak{q}))^{\alpha_i} \leq 1$ for $i = 0, 1, \dots, n$. Then there exists $0 \leq j \leq n$ such that*

$$(a\sigma(\mathfrak{q}))^{\alpha_j} < 1.$$

Lemma 3.25. *For any $g \in \widehat{G}(\mathbb{A}\langle t \rangle) \rtimes \sigma(\mathfrak{q})$ with $\mathfrak{q} \in \mathbb{I}$, $|\mathfrak{q}| < 1$, there exists $\gamma_0 \in \widehat{G}(F\langle t \rangle)$ such that*

$$|g\gamma_0 v_\lambda| \leq |g\gamma v_\lambda|$$

for all $\gamma \in \widehat{G}(F\langle t \rangle)$.

Proof. The lemma is essentially an adelic formulation of Lemma 17.15 in [8]. Write $g = ka\sigma(\mathfrak{q})u$. First note that for any positive number C , we may choose a finite set of weights w_C of V_λ such that $|(a\sigma(\mathfrak{q}))^\mu| > C$ for any weight μ of V_λ which is not in w_C . Enlarge w_C if necessary; we may assume that if $\mu \in w_C$, then all the weights of V_λ with depth less than the depth of μ are also in w_C . Recall that the depth of a weight $\lambda - \sum_{i=0}^n l_i \alpha_i$ is $\sum_{i=0}^n l_i$. Then Lemma 3.22 guarantees the finiteness of w_C . Now set $C = (a\sigma(\mathfrak{q}))^\lambda$ and divide $\widehat{G}(F((t)))$ into two parts,

$$\widehat{G}(F((t))) = G_1 \cup G_2,$$

where G_1 consists of all the γ 's such that some component of γv_λ has weight not in w_C , and G_2 consists of other γ 's, i.e. all the γ 's such that all components of γv_λ have weights in w_C . If $\gamma \in G_1$, let $\mu \notin w_C$ be a maximal weight of γv_λ , and let $v' \neq 0$ be the μ -component of γv_λ . It is clear that $v' \in V_{\mu, F}$, and the μ -component of $a\sigma(\mathfrak{q})u\gamma v_\lambda$ is $(a\sigma(\mathfrak{q}))^\mu v'$. Then

$$|g\gamma v_\lambda| = |a\sigma(\mathfrak{q})u\gamma v_\lambda| \geq (a\sigma(\mathfrak{q}))^\mu |v'| = (a\sigma(\mathfrak{q}))^\mu > C.$$

For $\gamma \in G_2$, γv_λ lies in the finite dimensional F -space $\sum_{\mu \in w_C} V_{\lambda, \mu, F}$. Consequently $a\sigma(\mathfrak{q})u\gamma v_\lambda$ lies in a finite dimensional F -space in $\sum_{\mu \in w_C} V_{\lambda, \mu, \mathbb{A}}$, so there exists $\gamma_0 \in G_2$ such that

$$(3.35) \quad |g\gamma_0 v_\lambda| \leq |g\gamma v_\lambda|$$

for all $\gamma \in G_2$. In particular $e \in G_2$; hence by Lemma 3.21

$$|g\gamma_0 v_\lambda| \leq |gv_\lambda| = (a\sigma(\mathfrak{q}))^\lambda = C.$$

Therefore (3.35) holds for all $\gamma \in \widehat{G}(F\langle t \rangle)$. \square

Consider the partial order on Φ_+ such that $\alpha < \beta$ if $\beta - \alpha$ is a sum of positive roots. Fix a total order on Φ_+ which extends this partial order, and induce the corresponding order on Φ_- by identifying Φ_- with Φ_+ via $\alpha \mapsto -\alpha$.

Lemma 3.26. *We have unique factorizations*

$$\widehat{U}(\mathbb{A}) = U^+(\mathbb{A})D^1(\mathbb{A})U^-(\mathbb{A}) = \prod_{\alpha \in \Phi_+} U_\alpha(\mathbb{A}\langle t \rangle_+) \prod_{i=1}^n \widetilde{h}_{\alpha_i}(1+t\mathbb{A}\langle t \rangle_+) \prod_{\alpha \in \Phi_-} U_\alpha(t\mathbb{A}\langle t \rangle_+),$$

$$\widehat{U}(F) = U^+(F)D^1(F)U^-(F) = \prod_{\alpha \in \Phi_+} U_\alpha(F\langle t \rangle_+) \prod_{i=1}^n \widetilde{h}_{\alpha_i}(1+tF\langle t \rangle_+) \prod_{\alpha \in \Phi_-} U_\alpha(tF\langle t \rangle_+),$$

where the product is taken with respect to the above fixed orders on Φ_+ and Φ_- .

Let $\mathcal{D} \subset \mathbb{A}$ be a fundamental domain of \mathbb{A}/F . We shall take

$$(3.36) \quad \mathcal{D} = \mathcal{D}_\infty \times \prod_{v < \infty} \mathcal{O}_v,$$

where \mathcal{D}_∞ is a fundamental domain of $(\prod_{v|\infty} F_v)/\mathcal{O}_F = (F \otimes_{\mathbb{Q}} \mathbb{R})/\mathcal{O}_F$ constructed as follows. Let $\omega_1, \dots, \omega_N$ be a basis of \mathcal{O}_F over \mathbb{Z} , where $N = [F : \mathbb{Q}]$. The diagonal embedding $\mathcal{O}_F \hookrightarrow \prod_{v|\infty} F_v$ identifies \mathcal{O}_F with a lattice in $F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^N$. Let \mathcal{D}_∞ be the following subset of $\prod_{v|\infty} F_v$:

$$\mathcal{D}_\infty = \left\{ \sum_{i=1}^N t_i \omega_i \mid 0 \leq t_i < 1 \right\}.$$

For example, if $F = \mathbb{Q}$, then $\mathcal{D}_\infty = [0, 1)$ is a fundamental domain of \mathbb{R}/\mathbb{Z} . We define

$$\mathcal{D}\langle t \rangle = \left\{ \sum_i u_i t^i \in \mathbb{A}\langle t \rangle \mid u_i \in \mathcal{D}, \forall i \right\}, \quad \mathcal{D}\langle t \rangle_+ = \mathcal{D}\langle t \rangle \cap \mathbb{A}\langle t \rangle_+.$$

Let $\widehat{U}_{\mathcal{D}}$ be a subset of $\widehat{U}(\mathbb{A})$:

$$(3.37) \quad \widehat{U}_{\mathcal{D}} = \prod_{\alpha \in \Phi_+} U_\alpha(\mathcal{D}\langle t \rangle_+) \prod_{i=1}^n \widetilde{h}_{\alpha_i}(1+t\mathcal{D}\langle t \rangle_+) \prod_{\alpha \in \Phi_-} U_\alpha(t\mathcal{D}\langle t \rangle_+),$$

where the product is taken with respect to the order in Lemma 3.26.

Lemma 3.27. *Every $u \in \widehat{U}(\mathbb{A})$ can be written as $u = \gamma_u u_{\mathcal{D}}$ (or $u_{\mathcal{D}} \gamma_u$) for some $\gamma_u \in \widehat{U}(F)$ and $u_{\mathcal{D}} \in \widehat{U}_{\mathcal{D}}$.*

4. EISENSTEIN SERIES AND THEIR COEFFICIENTS

Let G be the Chevalley group associated to a complex simple Lie algebra \mathfrak{g} , and let F be a number field. In this section we construct the Eisenstein series $E(s, f, g)$ defined on $\widehat{G}(\mathbb{A}\langle t \rangle) \rtimes \sigma(\mathfrak{q})$, where $s \in \mathbb{C}$ and f is an unramified cusp form on $G(\mathbb{A})$. We always make the assumption that $\mathfrak{q} \in \mathbb{I}$ and $|\mathfrak{q}| > 1$. We establish the absolute convergence of the constant terms of $E(s, f, g)$ along \widehat{U} , under the condition that $\Re s$ is large enough. The proof makes use of the Gindikin-Karpelevich formula, which will be used frequently. The same method gives the values of constant terms and Fourier coefficients of $E(s, f, g)$ along unipotent radicals of parabolic subgroups.

4.1. Definition of the Eisenstein series. Let $G(\mathbb{A})$ be the restricted product $\prod'_v G(F_v)$ with respect to K_v , where K_v is a maximal compact subgroup of $G(K_v)$. If v is finite we take K_v to be $G(\mathcal{O}_v)$. Then

$$G(\mathbb{A}) = \varprojlim_S \prod_{v \in S} G(F_v) \prod_{v \notin S} K_v,$$

where the inverse limit is taken over all finite sets of places.

Let $f \in L^2(G(F) \backslash G(\mathbb{A}))$ be an unramified cuspidal automorphic form, i.e.

- (i) f is invariant under right translation of $K = \prod_v K_v$,
- (ii) f is an eigenform for all p -adic Hecke operators,
- (iii) f is an eigenform for all invariant differential operators at all infinite places,
- (iv) the constant term of f along the unipotent radical of any parabolic subgroup of G is zero, i.e.

$$\int_{U_P(F) \backslash U_P(\mathbb{A})} f(ug) du = 0,$$

where P is any parabolic subgroup of G and U_P is the unipotent radical of P .

Associated to f and $s \in \mathbb{C}$, we shall define a function \tilde{f}_s on $\tilde{G}(\mathbb{A}\langle t \rangle)$. Suppose $g \in \tilde{G}(\mathbb{A}\langle t \rangle)$ has a decomposition

$$g = cp\sigma(\mathfrak{q})k,$$

where $c, \mathfrak{q} \in \mathbb{I}$, $p \in G(\mathbb{A}\langle t \rangle_+)$, $k \in \widehat{K} = \prod_v \widehat{K}_v$. Write p_0 for the image of p under the projection

$$G(\mathbb{A}\langle t \rangle_+) \rightarrow G(\mathbb{A}).$$

Then we define

$$\tilde{f}_s(g) = |c|^s f(p_0).$$

We have to check that \tilde{f}_s is well defined, namely, $\tilde{f}_s(g)$ does not depend on the choice of the decomposition of g . In fact, if $cp\sigma(\mathfrak{q})k = c'p'\sigma(\mathfrak{q})k'$, then $c'^{-1}c\sigma(\mathfrak{q})^{-1} \cdot (p'^{-1}p) = k'k^{-1} \in \widehat{B} \cap \widehat{K}$, which implies that $|c'c^{-1}| = 1$, $p_0 = p'_0k_0$ for some $k_0 \in K$. Note that $(\sigma(\mathfrak{q}) \cdot p)_0 = p_0$. This proves that \tilde{f}_s is well defined since f is right K -invariant. \tilde{f}_s has the following invariance properties:

- Lemma 4.1.** (i) \tilde{f}_s is right \widehat{K} -invariant,
(ii) \tilde{f}_s is left $G(F\langle t \rangle_+)$ -invariant and left $\sigma(\mathbb{I})$ -invariant,
(iii) $\tilde{f}_s(cg) = |c|^s \tilde{f}_s(g)$, and \tilde{f}_s is F^\times -invariant.

Proof. (i) By definition \tilde{f}_s is right \widehat{K} -invariant. It is also \widehat{K} -invariant since $\sigma(\mathcal{M}_{F_v})$ normalizes \widehat{K}_v for each place v , where \mathcal{M}_{F_v} is given by (3.28). (ii) follows from the fact that f is left $G(F)$ -invariant, and that $(\sigma(\mathfrak{q}) \cdot p)_0 = p_0$, as noted above. (iii) follows from the definition and the Artin product formula. \square

Let $\widehat{G}(F\langle t \rangle_+) = G(F\langle t \rangle_+) \times F^\times$; then \tilde{f}_s is left $\widehat{G}(F\langle t \rangle_+)$ -invariant by the previous lemma. We define the Eisenstein series $E(s, f, g)$ on $\widehat{G}(\mathbb{A}\langle t \rangle)$ by

$$(4.1) \quad E(s, f, g) = \sum_{\gamma \in \widehat{G}(F\langle t \rangle_+) \backslash \widehat{G}(F\langle t \rangle)} \tilde{f}_s(\gamma g).$$

It is clear that the right-hand side is a countable sum, right \tilde{K} -invariant and left $\widehat{G}(F\langle t \rangle)$ -invariant. $E(s, f, g)$ is also left $\sigma(F^\times)$ -invariant, hence left $\tilde{G}(F\langle t \rangle)$ -invariant. To see this, let $\mathfrak{q} \in F^\times$. By Lemma 4.1 (ii),

$$E(s, f, \sigma(\mathfrak{q})g) = \sum_{\gamma} \tilde{f}_s(\gamma\sigma(\mathfrak{q})g) = \sum_{\gamma} \tilde{f}_s(\sigma(\mathfrak{q}^{-1})\gamma\sigma(\mathfrak{q})g) = E(s, f, g).$$

Note that in this case $\sigma(\mathfrak{q})$ acts on $\widehat{G}(F\langle t \rangle)$ as an automorphism and preserves $\widehat{G}(F\langle t \rangle_+)$.

For completeness let us construct an Eisenstein series induced from cusp forms on other parabolic subgroups. Let P be a parabolic subgroup of \widehat{G} with Levi decomposition $P = M_P N_P$. Then M_P is a finite dimensional split reductive group. Let f_{M_P} be an unramified cusp form on $M_P(\mathbb{A})$, and ν be an unramified quasi-character of $M_P(\mathbb{A})$. If $g \in \tilde{G}(\mathbb{A}\langle t \rangle)$ decomposes as $g = mn\sigma(\mathfrak{q})k$, where $m \in M_P(\mathbb{A})$, $n \in N_P(\mathbb{A})$ and $k \in \tilde{K}$, then we define a function $\tilde{f}_{M_P, \nu}$ on $\tilde{G}(\mathbb{A}\langle t \rangle)$ associated to f_{M_P} and ν by

$$(4.2) \quad \tilde{f}_{M_P, \nu}(g) = \nu(m)f_{M_P}(m).$$

Then we form the Eisenstein series

$$(4.3) \quad E(\nu, f_{M_P}, g) = \sum_{\gamma \in P(F) \backslash \tilde{G}(F\langle t \rangle)} \tilde{f}_{M_P, \nu}(\gamma g).$$

Similarly one may verify invariance properties of $\tilde{f}_{M_P, \nu}$ and $E(\nu, f_{M_P}, g)$, and check that they are well defined.

For later use, we shall specialize to the case that P is maximal. We follow the treatment in [35]. Assume $P = P_\theta$ and that α_P is the corresponding simple root. Let A_P denote the (split) torus in the center of M_P . For any group H defined over F , let $X(H)_F$ be the group of F -rational characters of H . Set

$$\mathfrak{a}_P = \text{Hom}(X(M_P)_F, \mathbb{R})$$

as the real Lie algebra of A_P . Then

$$\mathfrak{a}_P^* = X(M_P)_F \otimes_{\mathbb{Z}} \mathbb{R} = X(A_P)_F \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $H_P : M_P(\mathbb{A}) \rightarrow \mathfrak{a}_P$ be the homomorphism defined in [35]. Let ρ_{M_P} be the half sum of the roots in $\Phi_{\theta+}$, and $\rho_P = \tilde{\rho} - \rho_{M_P}$. Then $\tilde{\alpha}_P = \langle \rho_P, \alpha_P^\vee \rangle^{-1} \rho_P$ belongs to $X(\hat{T})_F$ and thus by restriction to A_P can be viewed as an element in \mathfrak{a}_P^* . We shall now identify $s \in \mathbb{C}$ with $s\tilde{\alpha}_P \in \mathfrak{a}_{P, \mathbb{C}}^*$, and with

$$(4.4) \quad s\nu_P = \exp\langle s\tilde{\alpha}_P, H_P(\cdot) \rangle$$

which is an unramified quasi-character of $M_P(\mathbb{A})$. Then we set $\tilde{f}_{M_P, s} = \tilde{f}_{M_P, s\nu_P}$ and $E(s, f_{M_P}, g) = E(s\nu_P, f_{M_P}, g)$, which are defined by (4.2) and (4.3). Finally we remark that this definition is compatible with the previous definition of $E(s, f, g)$, but different from the usual one which uses the quasi-character

$$\exp\langle s\tilde{\alpha}_P + \rho_P, H_P(\cdot) \rangle.$$

For the case $P = P_\Delta$ the latter corresponds to shifting s by h^\vee .

4.2. Absolute convergence of constant terms. The constant term of $E(s, f, g)$ along the unipotent radical \widehat{U} of \widehat{B} is defined to be the following integral:

$$(4.5) \quad E_{\widehat{B}}(s, f, g) = \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} E(s, f, ug) du.$$

We have to specify the topology of $\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})$ and the measure du . By Lemma 3.26 it suffices to define topologies and measures on $\mathbb{A}\langle t \rangle_+ / F\langle t \rangle_+$ and $(1 + t\mathbb{A}\langle t \rangle_+) / (1 + tF\langle t \rangle_+)$.

Lemma 4.2. *The natural map $\varphi : \mathbb{A}\langle t \rangle_+ / F\langle t \rangle_+ \longrightarrow \mathbb{A}[[t]] / F[[t]] \xrightarrow{\sim} \prod_{i=0}^{\infty} (\mathbb{A}/F)_i$ is an isomorphism of abelian groups.*

Proof. φ is clearly injective. Let S_{∞} be the set of all infinite places of F , and let

$$\mathbb{A}_{S_{\infty}} = \prod_{v \in S_{\infty}} F_v \times \prod_{v \notin S_{\infty}} \mathcal{O}_v;$$

then $F + \mathbb{A}_{S_{\infty}} = \mathbb{A}$. Hence for any $u = \sum_{i=0}^{\infty} u_i t^i \pmod{F[[t]]} \in \mathbb{A}[[t]] / F[[t]]$, we may assume that $u_i \in \mathbb{A}_{S_{\infty}}$ for each i . Then $u \in \mathbb{A}\langle t \rangle_+$, and therefore φ is surjective. \square

Lemma 4.3. *The natural map*

$$\tau : (1 + t\mathbb{A}\langle t \rangle_+) / (1 + tF\langle t \rangle_+) \longrightarrow (1 + t\mathbb{A}[[t]]) / (1 + tF[[t]]) \xrightarrow{\log} t\mathbb{A}[[t]] / tF[[t]]$$

is an isomorphism of abelian groups.

Proof. Again it is clear that τ is injective. To prove τ is surjective, we have to show that for any $x \in 1 + t\mathbb{A}[[t]]$ there exists $y \in 1 + tF[[t]]$ such that $z = xy \in 1 + t\mathbb{A}\langle t \rangle_+$. Write $x = 1 + \sum_{i=1}^{\infty} x_i t^i$, $y = 1 + \sum_{i=1}^{\infty} y_i t^i$; then $z = 1 + \sum_{i=1}^{\infty} z_i t^i$ with

$$z_i = x_i + x_{i-1}y_1 + \cdots + x_1y_{i-1} + y_i.$$

Applying $F + \mathbb{A}_{S_{\infty}} = \mathbb{A}$ repeatedly, we can find a sequence of $y_i \in F$ such that $z_i \in \mathbb{A}_{S_{\infty}}$ for each i . This finishes the proof. \square

Since \mathbb{A}/F is compact, Lemma 4.2 and Lemma 4.3 imply that $\mathbb{A}\langle t \rangle_+ / F\langle t \rangle_+$ and $(1 + t\mathbb{A}\langle t \rangle_+) / (1 + tF\langle t \rangle_+)$ are compact and inherit the product measure from that of \mathbb{A}/F , which will be defined as follows.

We first specify the self-dual Haar measure on the local field F_v with respect to a non-trivial additive character ψ_{F_v} of F_v . If $F_v = \mathbb{R}$, we take

$$(4.6) \quad \psi_{\mathbb{R}}(x) = e^{2\pi i x},$$

and dx is the usual Lebesgue measure on \mathbb{R} ; if $F_v = \mathbb{C}$, we take

$$(4.7) \quad \psi_{\mathbb{C}}(z) = e^{2\pi i \operatorname{tr} z} = e^{4\pi i \Re z},$$

and $dzd\bar{z} = 2dx dy$ is twice the usual Lebesgue measure on \mathbb{C} ; if F_v is a finite extension of \mathbb{Q}_p , then we first take the character ψ_p of \mathbb{Q}_p given by

$$(4.8) \quad \psi_p(x) = e^{-2\pi i (\text{fractional part of } x)},$$

and define ψ_{F_v} by

$$(4.9) \quad \psi_{F_v}(x) = \psi_p(\operatorname{tr}_{F_v/\mathbb{Q}_p} x).$$

The self-dual Haar measure on \mathbb{Q}_p satisfies $\text{vol}(\mathbb{Z}_p) = 1$. Let \mathcal{O}_v be the ring of integers of F_v , and $\delta_{F_v}^{-1}$ be the inverse different

$$\delta_{F_v}^{-1} = \{x \in F_v \mid \psi(x\mathcal{O}_{F_v}) = 1\} = \{x \in F_v \mid \text{tr}(x\mathcal{O}_v) \in \mathbb{Z}_p\}.$$

Then the self-dual Haar measure on F_v satisfies $\text{vol}(\mathcal{O}_v) = N(\delta_{F_v})^{-\frac{1}{2}}$. If $\varpi_v \in \mathcal{O}_v$ is a uniformizer, then $\delta_{F_v} = \varpi_v^e \mathcal{O}_v$ for some non-negative integer e , and $N(\delta_{F_v}) = q_v^e$, where q_v is the cardinality of the residue field $\mathcal{O}_v/\mathfrak{p}_v$.

Let us compute $\text{vol}(\mathbb{A}/F)$ under the above self-dual measures. Recall from (3.36) that \mathbb{A}/F has a fundamental domain $\mathcal{D} = \mathcal{D}_\infty \times \prod_{i < \infty} \mathcal{O}_v$. Therefore

$$\text{vol}(\mathbb{A}/F) = \text{vol}((F \otimes_{\mathbb{Q}} \mathbb{R})/\mathcal{O}_F) \times \prod_{v < \infty} \text{vol}(\mathcal{O}_v).$$

F is unramified at almost all places v , hence the right-hand side is a finite product. It is known that $\text{vol}((F \otimes_{\mathbb{Q}} \mathbb{R})/\mathcal{O}_F) = |\Delta_F|^{\frac{1}{2}}$, where Δ_F is the discriminant of F . On the other hand

$$\prod_{v < \infty} \text{vol}(\mathcal{O}_v) = \prod_{v < \infty} N(\delta_{F_v})^{-\frac{1}{2}} = \prod_{v < \infty} |\Delta_{F_v}|^{-\frac{1}{2}} = |\Delta_F|^{-\frac{1}{2}},$$

where Δ_{F_v} is the relative discriminant $\Delta_{F_v/\mathbb{Q}_p}$ with $v|p$. We conclude that $\text{vol}(\mathbb{A}/F) = 1$. It follows that the quotient spaces $U_\alpha(F) \backslash U_\alpha(\mathbb{A})$ ($\alpha \in \tilde{\Phi}_+$) and $\hat{U}(F) \backslash \hat{U}(\mathbb{A})$ are compact with volume equal to 1.

The main result of this section is the following theorem.

Theorem 4.4. (i) Suppose that $g \in \hat{G}(\mathbb{A}\langle t \rangle) \rtimes \sigma(\mathfrak{q})$ with $\mathfrak{q} \in \mathbb{I}$ and $|\mathfrak{q}| > 1$, $s \in H = \{z \in \mathbb{C} \mid \Re z > h + h^\vee\}$, where h (resp. h^\vee) is the Coxeter (resp. dual Coxeter) number. Then $E(s, f, ug)$, as a function on $\hat{U}(F) \backslash \hat{U}(\mathbb{A})$, converges absolutely outside a subset of measure zero and is measurable,

(ii) For any $\varepsilon, \eta > 0$, let $H_\varepsilon = \{z \in \mathbb{C} \mid \Re z \geq h + h^\vee + \varepsilon\}$, $\sigma_\eta = \{\sigma(\mathfrak{q}) \mid \mathfrak{q} \in \mathbb{I}, |\mathfrak{q}| \geq 1 + \eta\}$. The integral (4.5) defining $E_{\hat{B}}(s, f, g)$ converges absolutely and uniformly for $s \in H_\varepsilon$, $g \in \hat{U}(\mathbb{A})\Omega\sigma_\eta\hat{K}$, where Ω is a compact subset of $T(\mathbb{A})$.

Define the height function h_s , $s \in \mathbb{C}$, on $\tilde{G}(\mathbb{A}\langle t \rangle)$ by

$$(4.10) \quad h_s(cp\sigma(\mathfrak{q})k) = |c|^s$$

if $c, \mathfrak{q} \in \mathbb{I}$, $p \in G(\mathbb{A}\langle t \rangle_+)$, $k \in \hat{K}$. Then h_s has the same invariance properties as those of \tilde{f}_s . Also note that the restriction of h_s on \tilde{T} can be expressed as $h_s(a) = a^{sL}$. Let us define

$$(4.11) \quad E(s, h, g) = \sum_{\gamma \in \hat{G}(F\langle t \rangle_+) \backslash \hat{G}(F\langle t \rangle)} h_s(\gamma g)$$

and

$$(4.12) \quad E_{\hat{B}}(s, h, g) = \int_{\hat{U}(F) \backslash \hat{U}(\mathbb{A})} E(s, h, ug) du.$$

Lemma 4.5. Theorem 4.4 is true for $E(s, h, g)$. Moreover, for s, \mathfrak{q} satisfying the conditions of the theorem and $a \in \hat{T}$, one has

$$(4.13) \quad E_{\hat{B}}(s, h, a\sigma(\mathfrak{q})) = \sum_{w \in W(\Delta, \emptyset)} (a\sigma(\mathfrak{q}))^{\tilde{\rho} + w^{-1}(sL - \tilde{\rho})} c_w(s),$$

where

$$(4.14) \quad c_w(s) = \prod_{\beta \in \tilde{\Phi}_+ \cap w\tilde{\Phi}_-} \frac{\Lambda_F(\langle sL - \tilde{\rho}, \beta^\vee \rangle)}{\Lambda_F(\langle sL - \tilde{\rho}, \beta^\vee \rangle + 1)},$$

with Λ_F the normalized Dedekind zeta function of F defined below.

Let r_1 (resp. r_2) be the number of real (resp. complex) places of F , and let

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s),$$

where $\Gamma(s)$ is the Gamma function. Then Λ_F is given by

$$\Lambda_F(s) = |\Delta_F|^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_F(s),$$

where ζ_F is the Dedekind zeta function of F and has an Euler product over all prime ideals \mathcal{P} of \mathcal{O}_F :

$$\zeta_F(s) = \prod_{\mathcal{P} \subset \mathcal{O}_F} \frac{1}{1 - N_{F/\mathbb{Q}}(\mathcal{P})^{-s}}, \quad \Re s > 1.$$

E. Hecke proved that ζ_F has a meromorphic continuation to the complex plane with only a simple pole at $s = 1$. Moreover Λ_F satisfies the functional equation

$$\Lambda_F(s) = \Lambda_F(1 - s).$$

Proof of Lemma 4.5. From the Bruhat decomposition (Theorem 3.12), it follows that

$$\widehat{G}(F\langle t \rangle) = \bigcup_{w \in W(\Delta, \emptyset)} \widehat{G}(F\langle t \rangle_+) w U_w(F).$$

Note that $U_w = \prod_{\alpha > 0, w\alpha < 0} U_\alpha$ is finite dimensional (of dimension $l(w)$) and hence lies in $G(F\langle t \rangle_+)$. Then for $u \in \widehat{U}(\mathbb{A})$,

$$E(s, h, ug) = \sum_{w \in W(\Delta, \emptyset)} \sum_{\gamma \in U_w(F)} h_s(w\gamma ug) =: \sum_{w \in W(\Delta, \emptyset)} H_w(s, ug).$$

We first prove that each inner sum $H_w(s, ug)$ is a measurable function on $\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})$. Let us introduce

$$(4.15) \quad U'_w(\mathbb{A}) = G(\mathbb{A}\langle t \rangle_+) \cap \prod_{\alpha > 0, w\alpha > 0} U_\alpha(\mathbb{A})$$

and

$$(4.16) \quad U'_w(F) = G(F\langle t \rangle_+) \cap \prod_{\alpha > 0, w\alpha > 0} U_\alpha(F).$$

Then $\widehat{U} = U'_w U_w$ and $w U'_w w^{-1} \subset \widehat{U}$; therefore

$$H_w(s, ug) = \sum_{\gamma \in U'_w(F) \backslash \widehat{U}(F)} h_s(w\gamma ug),$$

which is left $\widehat{U}(F)$ -invariant. Since $\dim U_w = l(w) < \infty$, applying Lemma 3.2 (c) and Corollary 3.4 (a) it is easy to see that there exists $i_w \in \mathbb{N}$ such that the

commutator $[U_w, U_\beta] \subset U'_w$ for each $\beta = \alpha + i\delta$ with $\alpha \in \Phi \cup \{0\}$ and $i \geq i_w$. If we define

$$(4.17) \quad U''_w(\mathbb{A}) = G(\mathbb{A}\langle t \rangle_+) \cap \prod_{\alpha \in \Phi \cup \{0\}, i \geq i_w} U_{\alpha+i\delta}(\mathbb{A}),$$

$$(4.18) \quad U''_w(F) = G(F\langle t \rangle_+) \cap \prod_{\alpha \in \Phi \cup \{0\}, i \geq i_w} U_{\alpha+i\delta}(F),$$

then U''_w is of finite codimension in \widehat{U} and $H_w(s, ug)$ is left $U''_w(F) \backslash U''_w(\mathbb{A})$ -invariant. This proves that $H_w(s, ug)$ is measurable.

The lemma can be reduced to the case $s \in \mathbb{R}$. Indeed, since $|h_s| = h_{\Re s}$ we have $|H_w(s, ug)| \leq H_w(\Re s, ug)$. By Fubini's theorem,

$$E_{\widehat{B}}(\Re s, h, g) = \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} E(\Re s, h, ug) du = \sum_{w \in W(\Delta, \emptyset)} \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} H_w(\Re s, ug) du.$$

If we can show that $E_{\widehat{B}}(\Re s, h, g)$ is finite, then $E(\Re s, h, g)$ converges almost everywhere and hence is measurable. It follows that $E(s, h, ug)$ converges absolutely almost everywhere and is measurable. Using the Lebesgue dominated convergence theorem we get

$$E_{\widehat{B}}(s, h, g) = \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} E(s, h, ug) du = \sum_{w \in W(\Delta, \emptyset)} \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} H_w(s, ug) du.$$

So we may assume that $s \in \mathbb{R}$. Let us evaluate $E_{\widehat{B}}(s, h, g)$ and prove its finiteness and uniform convergence. The computation for general s is the same.

Since $E_{\widehat{B}}$ is left \widehat{U} -invariant and right \widehat{K} -invariant, by the Iwasawa decomposition we may also assume that $g = a\sigma(\mathbf{q})$ with $a \in T$ and $|\mathbf{q}| > 1$. Let us prove (4.13) and show that the summation is finite. By previous discussion we have

$$\begin{aligned} (4.19) \quad E_{\widehat{B}}(s, h, g) &= \sum_{w \in W(\Delta, \emptyset)} \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} H_w(s, ug) du \\ &= \sum_{w \in W(\Delta, \emptyset)} \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} \sum_{\gamma \in U'_w(F) \backslash \widehat{U}(F)} h_s(w\gamma ug) du \\ &= \sum_{w \in W(\Delta, \emptyset)} \int_{U'_w(F) \backslash U'_w(\mathbb{A})} \int_{U_w(\mathbb{A})} h_s(wu'ug) du du' \\ &= \sum_{w \in W(\Delta, \emptyset)} \int_{U_w(\mathbb{A})} h_s(wug) du, \end{aligned}$$

where the last equality follows from the fact that $\text{vol}(U'_w(F) \backslash U'_w(\mathbb{A})) = 1$ and that h_s is left $G(\mathbb{A}\langle t \rangle_+)$ -invariant.

To evaluate (4.19) let us introduce some notation. Let $w = r_{i_1} \cdots r_{i_l}$ be the reduced expression of w , where $l = l(w)$. Let

$$\widetilde{\Phi}_w = \widetilde{\Phi}_+ \cap w\widetilde{\Phi}_- = \{\beta_1, \dots, \beta_l\},$$

where $\beta_j = r_{i_1} \cdots r_{i_{j-1}} \alpha_{i_j}$. Then

$$\widetilde{\Phi}_{w^{-1}} = \widetilde{\Phi}_+ \cap w^{-1}\widetilde{\Phi}_- = \{\gamma_1, \dots, \gamma_l\},$$

where $\gamma_j = -w^{-1}\beta_j = r_{i_l} \cdots r_{i_{j+1}}\alpha_{i_j}$. Note that

$$\beta_1 + \cdots + \beta_l = \tilde{\rho} - w\tilde{\rho}, \quad \gamma_1 + \cdots + \gamma_l = \tilde{\rho} - w^{-1}\tilde{\rho}.$$

Recall that we have assumed $s \in \mathbb{R}$, and $g = a\sigma(\mathbf{q})$, $a \in T$. We have

$$\begin{aligned} (4.20) \quad & \int_{U_w(\mathbb{A})} h_s(wua\sigma(\mathbf{q}))du \\ &= \int_{U_w(\mathbb{A})} h_s(wa\sigma(\mathbf{q})\text{Ad}(a\sigma(\mathbf{q}))^{-1}(u))du \\ &= \int_{\mathbb{A}^l} h_s(wa\sigma(\mathbf{q})\tilde{x}_{\gamma_1}((a\sigma(\mathbf{q}))^{-\gamma_1}u_1) \cdots \tilde{x}_{\gamma_l}((a\sigma(\mathbf{q}))^{-\gamma_l}u_l))du_1 \cdots du_l \\ &= (a\sigma(\mathbf{q}))^{\gamma_1 + \cdots + \gamma_l + w^{-1}sL} \int_{\mathbb{A}^l} h_s(w\tilde{x}_{\gamma_1}(u_1) \cdots \tilde{x}_{\gamma_l}(u_l))du_1 \cdots du_l \\ &= (a\sigma(\mathbf{q}))^{\tilde{\rho} - w^{-1}\tilde{\rho} + w^{-1}sL} \int_{\mathbb{A}^l} h_s(\tilde{x}_{-\beta_1}(u_1) \cdots \tilde{x}_{-\beta_l}(u_l))du_1 \cdots du_l. \end{aligned}$$

By the Iwasawa decomposition we have

$$\tilde{x}_{-\beta_l}(u_l) = n(u_l)a(u_l)k(u_l),$$

where $a(u_l) \in T_{\beta_l} = \{\tilde{h}_{\beta_l}(u)|u \in \mathbb{I}\}$, $n(u_l) \in U_{\beta_l}$, $k(u_l) \in K_{\beta_l}$. Let $w' = r_{i_1} \cdots r_{i_{l-1}}$; then $\{\beta_1, \dots, \beta_{l-1}\} = \tilde{\Phi}_+ \cap w'\tilde{\Phi}_-$. Consider the decomposition

$$(4.21) \quad \hat{U} = U'_{w'}U_{w'},$$

where $U'_{w'}$ is given by (4.15), (4.16) with w replaced by w' . Then

$$U_{-\beta_1} \cdots U_{-\beta_{l-1}} = w'U_{w'}w'^{-1}.$$

Using (4.21) we can define the projection

$$\pi : w'\hat{U}w'^{-1} \longrightarrow w'U_{w'}w'^{-1}.$$

Since $U_{-\beta_1}, \dots, U_{-\beta_{l-1}}, U_{\beta_l} \subset w'\hat{U}w'^{-1}$, we have the map

$$\pi \circ \text{Ad}(n(u_l)) : w'U_{w'}w'^{-1} \longrightarrow w'U_{w'}w'^{-1},$$

which is unimodular. From this fact, together with invariance properties of h_s and Corollary 3.4 (f), we get

$$\begin{aligned} (4.22) \quad & \int_{\mathbb{A}^l} h_s(\tilde{x}_{-\beta_1}(u_1) \cdots \tilde{x}_{-\beta_l}(u_l))du_1 \cdots du_l \\ &= \int_{\mathbb{A}^l} h_s(\tilde{x}_{-\beta_1}(u_1) \cdots \tilde{x}_{-\beta_{l-1}}(u_{l-1})a(u_l))du_1 \cdots du_l \\ &= \int_{\mathbb{A}^l} h_s(a(u_l)\tilde{x}_{-\beta_1}(a(u_l)^{\beta_1}u_1) \cdots \tilde{x}_{-\beta_{l-1}}(a(u_l)^{\beta_{l-1}}u_{l-1}))du_1 \cdots du_l \\ &= \int_{\mathbb{A}} a(u_l)^{sL - \beta_1 - \cdots - \beta_{l-1}}du_l \int_{\mathbb{A}^{l-1}} h_s(\tilde{x}_{-\beta_1}(u_1) \cdots \tilde{x}_{-\beta_{l-1}}(u_{l-1}))du_1 \cdots du_{l-1} \\ &= \int_{\mathbb{A}} a(u_l)^{sL - \tilde{\rho} + w'\tilde{\rho}}du_l \int_{\mathbb{A}^{l-1}} h_s(\tilde{x}_{-\beta_1}(u_1) \cdots \tilde{x}_{-\beta_{l-1}}(u_{l-1}))du_1 \cdots du_{l-1}. \end{aligned}$$

From the Gindikin-Karpelevich formula [19, 31], the first integral in (4.22) equals

$$\frac{\Lambda_F(z_l)}{\Lambda_F(z_l + 1)},$$

where

$$z_l = \langle sL - \tilde{\rho} + w'\tilde{\rho}, \beta_l^\vee \rangle - 1 = \langle sL - \tilde{\rho}, \beta_l^\vee \rangle.$$

Note that $\langle w'\tilde{\rho}, \beta_l^\vee \rangle = \langle \tilde{\rho}, w'^{-1}\beta_l \rangle = \langle \tilde{\rho}, \alpha_{i_l}^\vee \rangle = 1$. By induction on l it is clear that (4.22) equals $c_w(s)$.

Now we prove that the right-hand side of (4.13) is finite. We first prove that $s > h + h^\vee$ implies that $\langle sL - \tilde{\rho}, \beta^\vee \rangle > 1$ for each $\beta \in \Phi_w = \tilde{\Phi}_+ \cap w\tilde{\Phi}_-$. In fact, since $w \subset W(\Delta, \emptyset)$, $w^{-1}\Delta \subset \tilde{\Phi}_+$. It follows that $\beta = i\delta + \alpha$ with $i > 0$, $\alpha \in \Phi$. By Lemma 2.1, $\beta^\vee = jc + \alpha^\vee$ with $j > 0$. Then

$$\langle sL - \tilde{\rho}, \beta^\vee \rangle = j(s - h^\vee) - \langle \rho, \alpha^\vee \rangle \geq s - h^\vee - h + 1 > 1.$$

By standard results on zeta functions, for every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that whenever $\Re z \geq 1 + \varepsilon$ we have

$$\left| \frac{\Lambda_F(z)}{\Lambda_F(z+1)} \right| < c_\varepsilon.$$

It follows that $c_w(s) \leq c_\varepsilon^{l(w)}$ for $s > h + h^\vee + \varepsilon$.

Next consider $(a\sigma(\mathbf{q}))^{\tilde{\rho} - w^{-1}\tilde{\rho} + w^{-1}sL}$. Write $w^{-1} = T_\lambda w_0$, where $\lambda \in Q^\vee$, $w_0 \in W$. By Lemma 2.4,

$$\begin{aligned} w^{-1}L &= L + \lambda - \frac{1}{2}(\lambda, \lambda)\delta, \\ w^{-1}\tilde{\rho} &= w_0\rho - \langle w_0\rho, \lambda \rangle\delta + h^\vee w^{-1}L. \end{aligned}$$

Let $\|\lambda\| = (\lambda, \lambda)^{\frac{1}{2}}$. If we write $\lambda = \sum_{i=1}^n l_i \alpha_i^\vee$, then there exists a constant $c_1 > 0$ which does not depend on λ such that $\sum_{i=1}^n |l_i| \leq c_1 \|\lambda\|$. Then

$$|\langle w_0\rho, \lambda \rangle| = \left| \sum_{i=1}^n l_i \langle \rho, w_0^{-1}\alpha_i^\vee \rangle \right| \leq c_1 h \|\lambda\|.$$

Combining the above equations we obtain

$$(4.23) \quad \begin{cases} \sigma(\mathbf{q})^{w^{-1}L} = |\mathbf{q}|^{-\frac{1}{2}\|\lambda\|^2}, \\ \sigma(\mathbf{q})^{-w^{-1}\tilde{\rho}} = |\mathbf{q}|^{\langle w_0\rho, \lambda \rangle + \frac{h^\vee}{2}\|\lambda\|^2} \leq |\mathbf{q}|^{c_1 h \|\lambda\| + \frac{h^\vee}{2}\|\lambda\|^2}. \end{cases}$$

Let $c_a = \max_{\alpha \in \Phi} |a^\alpha|$; then

$$(4.24) \quad a^{\tilde{\rho} - w^{-1}\tilde{\rho} + w^{-1}sL} \leq c_a^{l(w)} a^{s\lambda} \leq c_a^{l(w) + c_1 s \|\lambda\|}.$$

But we have

$$(4.25) \quad l(w) \leq l(T_\lambda) + l(w_0) \leq \sum_{i=1}^n |l_i| l(T_{\alpha_i^\vee}) + |\Phi_+| \leq c_2 \|\lambda\| + |\Phi_+|,$$

where $c_2 = c_1 \max_{1 \leq i \leq n} l(T_{\alpha_i^\vee})$. In summary we obtain

$$\begin{aligned} E_{\hat{B}}(s, h, a\sigma(\mathbf{q})) &= \sum_{w \in W(\Delta, \emptyset)} (a\sigma(\mathbf{q}))^{\tilde{\rho} + w^{-1}(sL - \tilde{\rho})} c_w(s) \\ (4.26) \quad &\leq |W|(c_\varepsilon c_a)^{|\Phi_+|} \sum_{\lambda \in Q^\vee} (c_\varepsilon^{c_2} c_a^{c_1 s + c_2})^{\|\lambda\|} |\mathbf{q}|^{c_1 h \|\lambda\| - \frac{s - h^\vee}{2} \|\lambda\|^2} \end{aligned}$$

for $s \geq h + h^\vee + \varepsilon$. It is clear that the last series in (4.26) is finite and satisfies the required uniform convergence properties. \square

The proof of Theorem 4.4 is similar to that of Lemma 4.5, and we only need the following two observations: (1) f is bounded by cuspidality, (2) consider

$$E(s, f, ug) = \sum_{w \in W(\Delta, \emptyset)} F_w(s, ug) = \sum_{w \in W(\Delta, \emptyset)} \sum_{\gamma \in U_w(F)} \tilde{f}_s(w\gamma ug).$$

We can modify the definition of U_w'' in (4.17) and (4.18) by taking i_w large enough such that $[U_w, U_\beta] \subset w^{-1}N_\Delta w \subset U'_w$ whenever $\beta = \alpha + i\delta$ with $\alpha \in \Phi \cup \{0\}$ and $i \geq i_w$. Here by our convention for any $\theta \subset \tilde{\Delta}$ we let

$$(4.27) \quad N_\theta(\mathbb{A}) = G(\mathbb{A}\langle t \rangle_+) \cap \prod_{\alpha \in \tilde{\Phi}_+ - \Phi_\theta} U_\alpha(\mathbb{A}),$$

$$(4.28) \quad N_\theta(F) = G(F\langle t \rangle_+) \cap \prod_{\alpha \in \tilde{\Phi}_+ - \Phi_\theta} U_\alpha(F).$$

Then U_w'' is again of finite codimension in \hat{U} , and $F_w(s, ug)$ is left $U_w''(F) \backslash U_w''(\mathbb{A})$ -invariant.

Corollary 4.6. *If $\Re s > h + h^\vee$ and $|\mathfrak{q}| > 1$, then $E(s, f, ua\sigma(\mathfrak{q}))$ and $E(s, h, ua\sigma(\mathfrak{q}))$, defined by (4.1) and (4.11), as functions on $\hat{U}(F) \backslash \hat{U}(\mathbb{A}) \times \hat{T}$, are measurable and converge absolutely outside a subset of measure zero.*

Let us state the Gindikin-Karpelevich formula in the case of $SL_2(F_v)$ where F_v is the completion of F at place v , whose proof is well known. Let χ_v be an unramified character of F_v^\times . Let $f_{s, \chi_v} \in \text{Ind}_B^{SL_2}(|\cdot|^s \otimes \chi_v)$, $s \in \mathbb{C}$, $\Re s > 0$ be the unique spherical function satisfying

$$(4.29) \quad f_{s, \chi_v} \left(\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} k \right) = \chi_v(a) |a|_v^{s+1},$$

where $k \in K$, the standard maximal compact subgroup of $SL_2(F_v)$. Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Define

$$(4.30) \quad c(s, \chi_v) = \int_{F_v} f_{s, \chi_v} \left(w^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx.$$

Proposition 4.7. *Suppose F_v is p -adic. Let ϖ_v be a uniformizer of $\mathfrak{p}_v \subset \mathcal{O}_v$, and q_v be the cardinality of the residue field $\mathcal{O}_v/\mathfrak{p}_v$. Then*

$$c(s, \chi_v) = \text{vol}(\mathcal{O}_v) \frac{1 - \chi_v(\varpi_v) q_v^{-s-1}}{1 - \chi_v(\varpi_v) q_v^{-s}} = \frac{|\Delta_v|^{\frac{s}{2}} L(s, \chi_v)}{|\Delta_v|^{\frac{s+1}{2}} L(s+1, \chi_v)}.$$

Recall that Δ_v is the relative discriminant $\Delta_{F_v/\mathbb{Q}_p}$ satisfying $\text{vol}(\mathcal{O}_v) = |\Delta_v|^{-\frac{1}{2}}$. In particular, if χ_v is trivial, then $c_v(s, \chi_v)$ contributes the local factor of $\frac{\Lambda_F(s)}{\Lambda_F(s+1)}$ at v . In the case that $F_v = \mathbb{R}$ or \mathbb{C} , unramified characters of F_v are of the form $|\cdot|_v^{s_0}$ for s_0 purely imaginary. Then one has the following

Proposition 4.8. *Suppose $F_v = \mathbb{R}$ or \mathbb{C} . Then*

$$c(s, |\cdot|_v^{s_0}) = \frac{\Gamma_{F_v}(s + s_0)}{\Gamma_{F_v}(s + s_0 + 1)}.$$

4.3. Constant terms and Fourier coefficients. In this section we shall compute the constant terms of Eisenstein series along unipotent radicals of parabolic subgroups of \widehat{G} . In the classical theory [31], if $P = MN$ is a parabolic subgroup of G and f is a cusp form on $M(\mathbb{A})$, then the L -functions associated to f and the representations of ${}^L M$ on ${}^L \mathfrak{n}$ will appear in certain constant terms of the Eisenstein series induced by f . Unfortunately for loop groups the constant terms are all trivial. However, we will obtain certain non-trivial higher Fourier coefficients of the Eisenstein series.

Let $P = P_\theta$ and $Q = P_{\theta'}$ be two maximal parabolics of \widehat{G} with Levi decompositions $P = M_P N_P$ and $Q = M_Q N_Q$. Let α_P and α_Q be the corresponding simple roots. Let f_{M_P} be an unramified cusp form on $M_P(\mathbb{A})$ and $E(s, f_{M_P}, g)$ be the Eisenstein series defined at the end of Section 4.1. The constant term of $E(s, f_{M_P}, g)$ along N_Q is given by

$$(4.31) \quad E_Q(s, f_{M_P}, g) = \int_{N_Q(F) \backslash N_Q(\mathbb{A})} E(s, f_{M_P}, ng) dn.$$

Using similar method to that in the proof of Theorem 4.4, one can show that the integral (4.31) converges absolutely for $|\mathfrak{q}| > 1$ and $\Re s \gg 0$. One can easily find the precise range of convergence for $\Re(s)$, and we will not give it here.

Theorem 4.9. $E_Q(s, f_{M_P}, g) = 0$ unless $P = Q$, in which case

$$E_P(s, f_{M_P}, g) = (s\nu_P)(m) f_{M_P}(m),$$

where $g = mn\sigma(\mathfrak{q})k$, $m \in M_P(\mathbb{A})$, $n \in N_P(\mathbb{A})$ and $k \in \widehat{K}$.

Proof. We follow the arguments of Langlands [31]. We write

$$E_Q(s, f_{M_P}, g) = \sum_{\gamma \in P(F) \backslash \widehat{G}(F(t))/N_Q(F)} \int_{\gamma^{-1}P(F) \cap \gamma N_Q(F) \backslash N_Q(\mathbb{A})} \widetilde{f}_{M_P, s}(\gamma ng) dn.$$

From the Bruhat decomposition we may assume that each γ is of the form $\gamma = w\gamma'$ with $w \in W(\theta_1, \theta_2)$ and $\gamma' \in M_Q(F)$. Then up to a scalar depending on γ' , a typical integral equals

$$\int_{w^{-1}P(F) \cap w N_Q(F) \backslash N_Q(\mathbb{A})} \widetilde{f}_{M_P, s}(wn\gamma'g) dn.$$

Since $w \in W(\theta_1, \theta_2)$ we have

$$U_w \simeq w^{-1}Pw \cap N_Q \backslash N_Q.$$

Let $w^{-1}Pw \cap N_Q = N_1 N_2$, where $N_1 = w^{-1}M_P w \cap N_Q$, $N_2 = w^{-1}N_P w \cap N_Q$; then the above integral equals

$$\int_{U_w(\mathbb{A})} \int_{N_2(F) \backslash N_2(\mathbb{A})} \int_{N_1(F) \backslash N_1(\mathbb{A})} \widetilde{f}_{M_P, s}(wn_1 n_2 u \gamma' g) dn_1 dn_2 du.$$

Since f_{M_P} is a cusp form on M_P , the most inner integral vanishes unless $N_1 = 1$. Then $w^{-1}\theta \subset \Phi_{\theta'+}$, which forces that $w^{-1}\theta = \theta'$. By the following lemma, which is essentially Shahidi's Lemma [17, Lemma 4.1], we have $w = 1$ and $P = Q$. In this case we can take $\gamma' = 1$, and thus $E_P(s, f_{M_P}, g) = \widetilde{f}_{M_P, s}(g) = (s\nu_P)(m) f_{M_P}(m)$. \square

Lemma 4.10. *In the above settings if there exists $w \in \widetilde{W}$ such that $w\theta = \theta'$, then $w = 1$ and $P = Q$.*

Proof. For convenience let us enumerate $\alpha_P = \alpha_i$, $\alpha_Q = \alpha_j$. It is enough to show that $w\alpha_i > 0$. For the contrary suppose $w\alpha_i < 0$. Let w_0^θ be the longest element in W_θ ; then $ww_0^\theta\theta = -\theta'$. We may write

$$w_0^\theta\alpha_i = \alpha_i + \sum_{k \neq i} n_k \alpha_k.$$

Then $ww_0^\theta\alpha_i$ has an expression

$$ww_0^\theta\alpha_i = w\alpha_i + \sum_{k \neq j} n'_k \alpha_k$$

since $w\theta = \theta'$. Let us write $w\alpha_i = \sum_{k=0}^n b_k \alpha_k$. If $b_j = 0$, then $w\alpha_i \in \Phi_{\theta'}$, which is impossible. Therefore $b_j < 0$, which further implies $ww_0^\theta\alpha_i < 0$. Thus we obtain $ww_0^\theta\tilde{\Delta} < 0$, a contradiction. \square

In general, for a connected reductive algebraic group G which is split over F (e.g. $G = GL_n$), the theory of a generalized Tits system [21, 22] implies that $G(F((t))) = \widehat{B}_0 \widehat{W}' \widehat{B}_0$, where $\widehat{W}' = W \ltimes X_*(T)$. The proof of Theorem 4.9 together with certain variants of Lemma 4.10 suggests that the triviality of constant terms should also hold for \widehat{G} .

Now let us define and compute the Fourier coefficients of $E(s, f_{M_P}, g)$. Let ψ be a character of $\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})$; then $\psi = \prod_{\alpha \in \tilde{\Delta}} \psi_\alpha$, where ψ_α is a character of $U_\alpha(F) \backslash U_\alpha(\mathbb{A})$. This follows from the fact that

$$\widehat{U}/[\widehat{U}, \widehat{U}] \simeq \prod_{\alpha \in \tilde{\Delta}} U_\alpha.$$

Define the ψ -th Fourier coefficient of $E(s, f_{M_P}, g)$ along \widehat{B} by

$$E_{\widehat{B}, \psi}(s, f_{M_P}, g) = \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} E(s, f_{M_P}, g) \overline{\psi}(u) du.$$

Then $E_{\widehat{B}, \psi}(s, f_{M_P}, g)$ is a Whittaker function on $\widehat{G}(\mathbb{A} \langle t \rangle) \rtimes \sigma(\mathfrak{q})$, i.e. a function W satisfying the relation $W(ug) = \psi(u)W(g)$, $\forall u \in \widehat{U}(\mathbb{A})$. Let $U_P = M_P \cap \widehat{U}$ be the unipotent radical of $M_P \cap \widehat{B} \subset M_P$. We say that $\psi|_{U_P}$ is generic if ψ_α is non-trivial for each $\alpha \in \theta$, and that ψ is generic if ψ_α is non-trivial for each $\alpha \in \tilde{\Delta}$.

Theorem 4.11. (i) Assume that ψ is generic. Then $E_{\widehat{B}, \psi}(s, f_{M_P}, g) = 0$.

(ii) Assume that ψ_{α_P} is trivial. Then $E_{\widehat{B}, \psi}(s, f_{M_P}, g) = 0$ unless $\psi|_{U_P}$ is generic, in which case

$$E_{\widehat{B}, \psi}(s, f_{M_P}, g) = (s\nu_P)(m) \int_{U_P(F) \backslash U_P(\mathbb{A})} f_{M_P}(um) \overline{\psi}(u) du,$$

where $g = mn\sigma(\mathfrak{q})k$, $m \in M_P(\mathbb{A})$, $n \in N_P(\mathbb{A})$ and $k \in \widehat{K}$.

Proof. (i) Since ψ is $\widehat{U}(F)$ -invariant, similarly as before we have

$$\begin{aligned} E_{\widehat{B}, \psi}(s, f_{M_P}, g) &= \sum_{w \in W(\theta, \emptyset)} \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} \sum_{\gamma \in U_w(F)} \widetilde{f}_{M_P, s}(w\gamma ug) \overline{\psi}(u) du \\ &= \sum_{w \in W(\theta, \emptyset)} \int_{w^{-1}P(F)w \cap \widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} \widetilde{f}_{M_P, s}(wug) \overline{\psi}(u) du. \end{aligned}$$

For each $w \in W(\theta, \emptyset)$, $w^{-1}\theta \subset \tilde{\Phi}_+$, therefore $w^{-1}U(F)w \subset w^{-1}P(F)w \cap \hat{U}(F)$. If we write $\hat{U} = N_{2w}N_{1w}U_w$, where $N_{1w} = w^{-1}U_Pw$, $N_{2w} = w^{-1}N_Pw \cap \hat{U}$, then a typical integral equals

$$\int_{U_w(\mathbb{A})} \int_{N_{1w}(F) \backslash N_{1w}(\mathbb{A})} \int_{N_{2w}(F) \backslash N_{2w}(\mathbb{A})} \tilde{f}_{M_P,s}(wn_2n_1ug) \overline{\psi}(n_2n_1u) dn_2 dn_1 du.$$

By definition of $\tilde{f}_{M_P,s}$ the function $\tilde{f}_{M_P,s}(wn_2n_1ug)$ is constant on N_{2w} . Hence we are reduced to showing that $\psi|_{N_{2w}}$ is non-trivial, provided that ψ is generic. In this case $\psi|_{N_{2w}}$ is trivial if and only if $U_\alpha \not\subset w^{-1}N_Pw$ for any $\alpha \in \tilde{\Delta}$. Equivalently we are looking for a w satisfying the conditions $w\tilde{\Delta} \subset \tilde{\Phi}_- \cup \Phi_{\theta+}$ and $w^{-1}\theta > 0$. For such w in fact one has

$$w\tilde{\Delta} \subset (\tilde{\Phi}_- - \Phi_{\theta-}) \cup \Phi_{\theta+}.$$

From this it is easy to deduce that $w\tilde{\Delta} \cap \Phi_{\theta+}$ forms a system of simple roots for Φ_θ , hence $w\tilde{\Delta} \cap \Phi_{\theta+} = \theta$. By Lemma 4.10 we must have $w = 1$, which is obviously a contradiction. Therefore we have proved that such a w does not exist, hence $\psi|_{N_{2w}}$ is non-trivial.

(ii) We reverse the order of N_{1w} and N_{2w} to rewrite the integral as

$$\int_{N_{2w}(F) \backslash N_{2w}(\mathbb{A})} \int_{N_{1w}(F) \backslash N_{1w}(\mathbb{A})} \tilde{f}_{M_P,s}(wn_1n_2ug) \overline{\psi}(n_1n_2u) dn_1 dn_2 du.$$

Consider the subset $\theta_w = \theta \cap w\Phi_{\theta+}$ of θ and let P_w be the corresponding parabolic subgroup of M_P . If $\theta_w \neq \theta$, then P_w has a non-trivial unipotent radical $U_{P_w} \subset U_P$. From the definition of θ_w it follows that $w^{-1}U_{P_w}w \subset N_\theta$, and therefore by our assumption ψ is trivial on $w^{-1}U_{P_w}w$. Since \tilde{f}_{M_P} is a cusp form on $M_P(\mathbb{A})$, it follows that the most inner integral vanishes. If $\theta_w = \theta$, then $w^{-1}\theta \subset \Phi_{\theta+}$, from which we deduce that $w^{-1}\theta = \theta$ and hence $w = 1$ by Lemma 4.10. It follows that

$$\begin{aligned} E_{\hat{B},\psi}(s, f_{M_P}, g) &= \int_{\hat{U}(F) \backslash \hat{U}(\mathbb{A})} \tilde{f}_{M_P,s}(ug) \overline{\psi}(u) du \\ &= (s\nu_P)(m) \int_{U_P(F) \backslash U_P(\mathbb{A})} f_{M_P}(um) \overline{\psi}(u) du, \end{aligned}$$

which vanishes unless $\psi|_{U_P}$ is generic, again by the cuspidality of f_{M_P} . \square

In the rest of this section we consider the Eisenstein series induced from a quasi-character on \hat{T} . More precisely, let $\chi_{\hat{T}} = \bigotimes_v \chi_{\hat{T}_v} : \hat{T}(\mathbb{A})/\hat{T}(F) \rightarrow \mathbb{C}^\times$ be a quasi-character such that $\chi_{\hat{T}_v}$ is unramified for each place v . Extend $\chi_{\hat{T}}$ to $\tilde{T} = \hat{T} \times \sigma(\mathbb{I})$ such that $\chi_{\tilde{T}}|_{\sigma(\mathbb{I})}$ is trivial. For each $\alpha \in \tilde{\Phi}_{re}$, $\chi_{\tilde{T}}\alpha^\vee : \mathbb{I}/F^\times \rightarrow \mathbb{C}^\times$ is the Hecke quasi-character such that $\chi_{\tilde{T}}\alpha^\vee(x) = \chi_{\tilde{T}}(\tilde{h}_\alpha(x))$. In particular, $\chi_{\tilde{T}_v}\alpha^\vee$ is unramified for each place v .

In general, if $\chi : \mathbb{I}/F^\times \rightarrow \mathbb{C}^\times$ is a Hecke quasi-character, we may write χ as $\mu|\cdot|^{s_0}$, where μ is unitary and $s_0 \in \mathbb{C}$. Define $\Re\chi = \Re s_0$, which is called the exponent of χ . Recall that $L(s, \chi) = \prod_v L(s, \chi_v)$ is the Hecke L -function of χ . We may twist χ to make it unitary. Then $L(s, \chi)$ is holomorphic in $\{s \in \mathbb{C} : \Re s > 1\}$, admits meromorphic continuation to the entire complex plane, and satisfies the functional equation

$$L(1-s, \chi^\vee) = \varepsilon(s, \chi) L(s, \chi),$$

where $\chi^\vee = \chi^{-1}|\cdot|$ is the shifted dual of χ .

We define the Eisenstein series on $\widetilde{G}(\mathbb{A}\langle t \rangle)$ induced from $\chi_{\widehat{T}}$ by

$$(4.32) \quad E(\chi_{\widehat{T}}, g) = \sum_{\gamma \in \widehat{B}(F) \backslash \widehat{G}(F\langle t \rangle)} \widetilde{\chi}_{\widehat{T}}(\gamma g),$$

where $\widetilde{\chi}_{\widehat{T}}$ is given by

$$(4.33) \quad \widetilde{\chi}_{\widehat{T}}(cb\sigma(\mathfrak{q})k) = \chi_{\widehat{T}}(cb_0),$$

where $c, \mathfrak{q} \in \mathbb{I}$, $b \in \widehat{B}_0(\mathbb{A})$, $b_0 \in B(\mathbb{A})$ is the image of b under the projection $\widehat{B}_0(\mathbb{A}) \rightarrow B(\mathbb{A})$, and $k \in \widehat{K}$. Then $\widetilde{\chi}_{\widehat{T}}$ is well defined, right \widehat{K} -invariant and left $\widehat{B}(F)$ -invariant. We define the constant term and Fourier coefficients of $E(\chi, g)$ along \widehat{B} by

$$\begin{aligned} E_{\widehat{B}}(\chi_{\widehat{T}}, g) &= \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} E(\chi_{\widehat{T}}, ug) du, \\ E_{\widehat{B}, \psi}(\chi_{\widehat{T}}, g) &= \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} E(\chi_{\widehat{T}}, ug) \overline{\psi}(u) du. \end{aligned}$$

Theorem 4.12. (i) Suppose that $g \in \widehat{G}(\mathbb{A}\langle t \rangle) \rtimes \sigma(\mathfrak{q})$ with $|\mathfrak{q}| > 1$, and $\Re(\chi_{\widehat{T}} \alpha_i^\vee) > 2$, $i = 0, 1, \dots, n$. Then $E(\chi_{\widehat{T}}, ug)$, as a function on $\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})$, converges absolutely outside a subset of measure zero and is measurable.

(ii) For any $\varepsilon > 0$, let

$$\mathcal{H}_\varepsilon = \left\{ \chi_{\widehat{T}} : \widehat{T}(\mathbb{A}) / \widehat{T}(F) \rightarrow \mathbb{C}^\times \mid \chi_{\widehat{T}} \text{ unramified, } \Re(\chi_{\widehat{T}} \alpha_i^\vee) > 2 + \varepsilon, i = 0, 1, \dots, n \right\}.$$

The integral defining $E_{\widehat{B}}(\chi_{\widehat{T}}, g)$ converges absolutely and uniformly for $\chi_{\widehat{T}} \in \mathcal{H}_\varepsilon$, $g \in \widehat{U}(\mathbb{A}) \Omega \sigma_\eta \widehat{K}$, where $\eta > 0$ and Ω is a compact subset of $T(\mathbb{A})$. More explicitly, for $a \in \widehat{T}$ one has

$$(4.34) \quad E_{\widehat{B}}(\chi_{\widehat{T}}, a\sigma(\mathfrak{q})) = \sum_{w \in \widetilde{W}} (a\sigma(\mathfrak{q}))^{\widetilde{\rho} + w^{-1}(\chi_{\widehat{T}} - \widetilde{\rho})} c_w(\chi_{\widehat{T}}),$$

where

$$(4.35) \quad c_w(\chi_{\widehat{T}}) = \prod_{\beta \in \widetilde{\Phi}_+ \cap w\widetilde{\Phi}_-} |\Delta_F|^{-\frac{1}{2}} \frac{L(-\langle \widetilde{\rho}, \beta^\vee \rangle, \chi_{\widehat{T}} \beta^\vee)}{L(1 - \langle \widetilde{\rho}, \beta^\vee \rangle, \chi_{\widehat{T}} \beta^\vee)}.$$

Proof. The proof follows exactly the line for proving Lemma 4.5 if we apply analytic properties of Hecke L -functions together with Propositions 4.7 and 4.8. With the analog of Godement's criterion,

$$(4.36) \quad \Re(\chi_{\widehat{T}} \alpha_i^\vee) > 2, \quad i = 0, 1, \dots, n,$$

we have the following two observations, which suffice for the convergence of the Eisenstein series: (1) $\Re(\chi_{\widehat{T}} \beta^\vee) - \langle \widetilde{\rho}, \beta^\vee \rangle > 1$ for any $\beta \in \widetilde{\Phi}_{re+}$, which is precisely (4.36) when β is a simple root. (2) Consider the factor $\sigma(\mathfrak{q})^{w^{-1}(\chi_{\widehat{T}} - \widetilde{\rho})} = (w \cdot \sigma(\mathfrak{q}))^{\chi_{\widehat{T}} - \widetilde{\rho}}$. Write $w = T_\lambda w_0 \in \widetilde{W}$ such that $\lambda \in Q^\vee$, $w_0 \in W$. Then from

$$wd = T_\lambda d = d + \lambda - \frac{\|\lambda\|^2}{2} c,$$

we see that

$$|(w \cdot \sigma(\mathfrak{q}))^{\chi_{\widehat{T}} - \widetilde{\rho}}| = |\mathfrak{q}|^{\Re(\chi_{\widehat{T}} \lambda) - \langle \rho, \lambda \rangle + (h^\vee - \Re(\chi_{\widehat{T}} c)) \frac{\|\lambda\|^2}{2}}.$$

The coefficient of the quadratic term is negative. Indeed,

$$\Re(\chi_{\widehat{T}}c) = \Re(\chi_{\widehat{T}}(\alpha_0^\vee + \widetilde{\alpha}^\vee)) > 2 + \langle \widetilde{\rho}, \alpha_0^\vee + \widetilde{\alpha}^\vee \rangle = 2 + h^\vee.$$

Let us remark that due to the last equation, for our second consideration Godement's criterion is much stronger than required. \square

4.4. Explicit computations for \widetilde{SL}_2 . It would be interesting to investigate the Fourier coefficients of our Eisenstein series in full generality. To obtain an explicit formula would be a quite difficult and non-trivial problem, as suggested by the paper of W. Casselman and J. Shalika [5] where they gave a formula for finite dimensional groups. The reason their method does not work here is that we do not have a longest element in the affine Weyl group, as opposed to the classical case. However, in the case of SL_2 we do have explicit computations and everything is known.

Let $\alpha_0 = \delta - \alpha$, $\alpha_1 = \alpha$ be the simple roots of $\widetilde{\mathfrak{sl}}_2$, and $\psi = \psi_0\psi_1$ be a character on $\widehat{U}(F)\backslash\widehat{U}(\mathbb{A})$ where ψ_i corresponds to α_i , $i = 0, 1$. Let f be an unramified cusp form on $SL_2(\mathbb{A})$.

Proposition 4.13. *Assume that ψ is non-trivial. Then $E_{\widehat{B},\psi}(s, f, g)$ vanishes unless ψ equals ψ_1 and is non-trivial, in which case it equals the Fourier coefficient of f ,*

$$E_{\widehat{B},\psi}(s, f, g) = |c|^s \int_{F\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} p_0\right) \overline{\psi}_1(u) du,$$

where $g = cp\sigma(\mathfrak{q})k$, $c \in \mathbb{I}$, $p \in SL_2(\mathbb{A}\langle t \rangle_+)$, $k \in \widehat{K}$.

Proof. The case $\psi = \psi_1$ is known due to Theorem 4.11; hence we only need to show the vanishing of $E_{\widehat{B},\psi}$ in other cases. As usual let us write

$$E_{\widehat{B},\psi}(s, f, g) = \sum_{w \in \widetilde{W}, w^{-1}\alpha > 0} \int_{w^{-1}\widehat{G}(F\langle t \rangle_+)w \cap \widehat{U}(F)\backslash\widehat{U}(\mathbb{A})} \widetilde{f}_s(wug)\overline{\psi}(u) du.$$

For $w \in \widetilde{W}$ such that $w^{-1}\alpha > 0$, if $w \neq 1$, then $w^{-1}\alpha \neq \alpha_0, \alpha_1$, which implies that ψ is trivial on the root subgroup $U_{w^{-1}\alpha} = w^{-1}U_\alpha w$. On the other hand, the integration of \widetilde{f}_s along $U_\alpha(F)\backslash U_\alpha(\mathbb{A})$ is zero since f is cuspidal. Therefore we arrive at

$$E_{\widehat{B},\psi}(s, f, g) = \int_{\widehat{U}(F)\backslash\widehat{U}(\mathbb{A})} \widetilde{f}_s(ug)\overline{\psi}(u) du.$$

Since by definition \widetilde{f}_s is left invariant under the root subgroup U_{α_0} , $E_{\widehat{B},\psi}$ vanishes unless ψ_0 is trivial, i.e. ψ equals ψ_1 and is non-trivial. \square

Now suppose that $\chi_{\widehat{T}}$ is an unramified character on $\widehat{T}(\mathbb{A})/\widehat{T}(F)$ such that

$$(4.37) \quad \chi_{\widehat{T}}(c \cdot a) = \chi_0(c)\chi_1(a), \quad c \in \mathbb{I}, \quad a \in T(\mathbb{A}) \simeq \mathbb{I},$$

where χ_0 and χ_1 are unramified Hecke quasi-characters on \mathbb{I} . We define $E(\chi_{\widehat{T}}, g)$ by (4.32) and the Fourier coefficient $E_{\widehat{B},\psi}(\chi, g)$. Similarly as before we have

$$E_{\widehat{B},\psi}(\chi_{\widehat{T}}, g) = \sum_{w \in \widetilde{W}} \int_{w^{-1}\widehat{G}(F\langle t \rangle_+)w \cap \widehat{U}(F)\backslash\widehat{U}(\mathbb{A})} \widetilde{\chi}_{\widehat{T}}(wug)\overline{\psi}(u) du.$$

For any $w \in \widetilde{W}$, at least one of $w\alpha_i$, $i = 0, 1$, is positive. Since $\widetilde{\chi}_{\widehat{T}}$ is left invariant under any root subgroup of a positive root, we see that $E_{\widehat{B},\psi}$ does not vanish unless

one of ψ_i , $i = 0, 1$, is trivial, i.e. $\psi = \psi_0$ or ψ_1 . Let us work out the explicit formula for $E_{\widehat{B}, \psi_i}(\chi_{\widehat{T}}, g)$ at $g = \sigma(q)$, $i = 0, 1$. For simplicity let us assume that

$$(4.38) \quad \psi_i(\widetilde{x}_{\alpha_i}(u)) = \psi_F(u), \quad u \in \mathbb{A}, \quad i = 0, 1,$$

where $\psi_F = \bigotimes_v \psi_{F_v}$ and ψ_{F_v} is the standard character of F_v defined by (4.6)-(4.9). As preliminary computations let us give the local coefficients for $SL_2(F_v)$. Define

$$(4.39) \quad W(s, \chi_v) = \int_{F_v} f_{s, \chi_v} \left(w^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \overline{\psi}_{F_v}(x) dx,$$

where f_{s, χ_v} is given by (4.29). The following proposition is well known.

Proposition 4.14. *Suppose F_v is p -adic; then*

$$W(s, \chi_v) = \text{vol}(\mathcal{O}_v) L(s+1, \chi_v)^{-1}.$$

We also have the local coefficients at archimedean places. Assume $\chi_v = |\cdot|_v^{s_v}$, $v|\infty$. If $F_v = \mathbb{R}$, then

$$(4.40) \quad \begin{aligned} W(s, \chi_v) &= 2 \int_0^\infty (1+x^2)^{-\frac{s+s_v+1}{2}} e^{-2\pi i x} dx \\ &= 2\Gamma_{\mathbb{R}}(s+1, \chi_v)^{-1} K_{\frac{s+s_v}{2}}(2\pi). \end{aligned}$$

If $F_v = \mathbb{C}$, then

$$(4.41) \quad \begin{aligned} W(s, |\cdot|_{\mathbb{C}}^{s_0}) &= 2 \int_{\mathbb{R}^2} (1+x^2+y^2)^{-s-s_0-1} e^{-4\pi i x} dx dy \\ &= B\left(s+s_0+\frac{1}{2}, \frac{1}{2}\right) \int_0^\infty (1+x^2)^{-s-s_0} e^{-4\pi i x} dx \\ &= \frac{1}{\sqrt{2}} \Gamma_{\mathbb{C}}(s+1, \chi_v)^{-1} B(s+s_v, \frac{1}{2})^{-1} K_{s+s_v-\frac{1}{2}}(4\pi). \end{aligned}$$

In the above, $B(\cdot, \cdot)$ is the Beta function and $K_s(y)$ is the K -Bessel function, also known as the *Macdonald Bessel function*, defined by

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-y(t+t^{-1})/2} t^s \frac{dt}{t}.$$

We have used the formulas in [2, pp.66-67] to obtain (4.40) and (4.41). Define

$$(4.42) \quad W(s, \chi) = \prod_v W(s, \chi_v).$$

Then $W(s, \chi)$ can be written as $|\Delta_F|^{-\frac{1}{2}} W'_\infty(s, \chi) L(s+1, \chi)^{-1}$, where $W'_\infty(s, \chi)$ is a product involving Bessel functions and Beta functions. Now we are ready to give the formula for the Fourier coefficients $E_{\widehat{B}, \psi_i}(\chi_{\widehat{T}}, \sigma(q))$. We do the case $\psi = \psi_1$. The other case is similar and is only a little bit more involved.

Proposition 4.15. *Assume $|\mathbf{q}| > 1$ and Godement's criterion*

$$\Re \chi_1 > 2, \quad \Re(\chi_0 - \chi_1) > 2.$$

Then

$$\begin{aligned} E_{\widehat{B}, \psi_1}(\chi_{\widehat{T}}, \sigma(\mathbf{q})) &= \sum_{n=1}^{\infty} |\Delta_F|^{-n+\frac{1}{2}} |\mathbf{q}|^{n-2n^2} \left(\chi_1^n \chi_0^{-n^2} \right) (\mathbf{q}) W(1-4n, \chi_0^{2n} \chi_1^{-1}) \prod_{i=1}^{2n-1} \frac{L(1-2i, \chi_0^i \chi_1^{-1})}{L(2-2i, \chi_0^i \chi_1^{-1})} \\ &+ \sum_{n=0}^{\infty} |\Delta_F|^{-n} |\mathbf{q}|^{-n-2n^2} \left(\chi_1^{-n} \chi_0^{-n^2} \right) (\mathbf{q}) W(-1-4n, \chi_0^{2n} \chi_1) \prod_{i=1}^{2n} \frac{L(1-2i, \chi_0^{i-1} \chi_1)}{L(2-2i, \chi_0^{i-1} \chi_1)}. \end{aligned}$$

Proof. By previous reasonings, we have

$$E_{\widehat{B}, \psi_1}(\chi_{\widehat{T}}, g) = \sum_{w \in \widetilde{W}, w\alpha < 0} \int_{w^{-1}\widehat{G}(F\langle t \rangle_+) w \cap \widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} \widetilde{\chi}_{\widehat{T}}(wu\sigma(q)) \overline{\psi}(u) du.$$

For $w \in \widetilde{W}$, recall our notation

$$\widetilde{\Phi}_w = \widetilde{\Phi}_+ \cap w\widetilde{\Phi}_- = \{\beta_1, \dots, \beta_l\},$$

$$\widetilde{\Phi}_{w^{-1}} = \widetilde{\Phi}_+ \cap w^{-1}\widetilde{\Phi}_- = \{\gamma_1, \dots, \gamma_l\}.$$

It is clear from the formula for β_i, γ_i that if $w\alpha < 0$, then $\gamma_l = \alpha$. Following the arguments of (4.19), (4.20) we obtain

$$\begin{aligned} (4.43) \quad & \int_{w^{-1}\widehat{G}(F\langle t \rangle_+) w \cap \widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} \widetilde{\chi}_{\widehat{T}}(wu\sigma(q)) \overline{\psi}(u) du \\ &= \int_{U_w(\mathbb{A})} \widetilde{\chi}_{\widehat{T}}(wu\sigma(q)) \overline{\psi}(u) du \\ &= \int_{\mathbb{A}^l} \widetilde{\chi}_{\widehat{T}}(w\sigma(\mathbf{q}) \widetilde{x}_{\gamma_1}(\sigma(\mathbf{q})^{-\gamma_1} u_1) \cdots \widetilde{x}_{\gamma_l}(\sigma(\mathbf{q})^{-\gamma_l} u_l)) \overline{\psi}_F(u_l) du_1 \cdots du_l \\ &= \sigma(\mathbf{q})^{\gamma_1 + \cdots + \gamma_l + w^{-1}\chi_{\widehat{T}}} \int_{\mathbb{A}^l} \widetilde{\chi}_{\widehat{T}}(w\widetilde{x}_{\gamma_1}(u_1) \cdots \widetilde{x}_{\gamma_l}(u_l)) \overline{\psi}_F(u_l) du_1 \cdots du_l \\ &= \sigma(\mathbf{q})^{w^{-1}(\chi_{\widehat{T}} - \widetilde{\rho})} \int_{\mathbb{A}^l} \widetilde{\chi}_{\widehat{T}}(\widetilde{x}_{-\beta_1}(u_1) \cdots \widetilde{x}_{-\beta_l}(u_l)) \overline{\psi}_F(u_l) du_1 \cdots du_l. \end{aligned}$$

Notice that $\sigma(\mathbf{q})^{\gamma_l} = \sigma(\mathbf{q})^{\alpha} = 1$. Similarly as in (4.22),

$$\begin{aligned} & \int_{\mathbb{A}^l} \widetilde{\chi}_{\widehat{T}}(\widetilde{x}_{-\beta_1}(u_1) \cdots \widetilde{x}_{-\beta_l}(u_l)) \overline{\psi}_F(u_l) du_1 \cdots du_l \\ &= \int_{\mathbb{A}} a(u_l)^{\chi_{\widehat{T}} - \widetilde{\rho} + w' \widetilde{\rho}} \overline{\psi}_F(u_l) du_l \int_{\mathbb{A}^{l-1}} \widetilde{\chi}_{\widehat{T}}(\widetilde{x}_{-\beta_1}(u_1) \cdots \widetilde{x}_{-\beta_{l-1}}(u_{l-1})) du_1 \cdots du_{l-1}. \end{aligned}$$

By applying the formula of Fourier coefficients for SL_2 , together with the Gindikin-Karpelevich formula, we see that the last equation equals

$$|\Delta_F|^{-\frac{l-1}{2}} W(-\langle \widetilde{\rho}, \beta_l^\vee \rangle, \chi_{\widehat{T}} \beta_l^\vee) \prod_{i=1}^{l-1} \frac{L(-\langle \widetilde{\rho}, \beta_i^\vee \rangle, \chi_{\widehat{T}} \beta_i^\vee)}{L(1 - \langle \widetilde{\rho}, \beta_i^\vee \rangle, \chi_{\widehat{T}} \beta_i^\vee)}.$$

There are two cases for $w \in \widetilde{W}$ such that $w\alpha < 0$.

Case 1: $w = T_{n\alpha^\vee}$, $n > 0$. Then $l = 2n$, $\beta_i = i\delta - \alpha$ and $\gamma_i = (2n - i)\delta + \alpha$. In this case we obtain

$$\begin{aligned}\langle \tilde{\rho}, \beta_i^\vee \rangle &= \langle \frac{1}{2}\alpha + 2L, ic - \alpha^\vee \rangle = 2i - 1, \\ wd &= T_{n\alpha^\vee}d = d + n\alpha^\vee - n^2c, \\ \chi_{\widehat{T}}\beta_i^\vee &= \chi_0^i\chi_1^{-1}, \quad \chi_{\widehat{T}}wd = \chi_1^n\chi_0^{-n^2}, \quad \langle \tilde{\rho}, wd \rangle = n - 2n^2.\end{aligned}$$

Case 2: $w = T_{-n\alpha^\vee}r_\alpha$, $n \geq 0$. Then $l = 2n + 1$, $\beta_i = (i - 1)\delta + \alpha$, $\gamma_i = (2n + 1 - i)\delta + \alpha$. Similarly we obtain

$$\begin{aligned}\langle \tilde{\rho}, \beta_i^\vee \rangle &= \langle \frac{1}{2}\alpha + 2L, (i - 1)c + \alpha^\vee \rangle = 2i - 1, \\ wd &= T_{-n\alpha^\vee}r_\alpha d = T_{-n\alpha^\vee}d = d - n\alpha^\vee - n^2c, \\ \chi_{\widehat{T}}\beta_i^\vee &= \chi_0^{i-1}\chi_1, \quad \chi_{\widehat{T}}wd = \chi_1^{-n}\chi_0^{-n^2}, \quad \langle \tilde{\rho}, wd \rangle = -n - 2n^2.\end{aligned}$$

Combining contributions from these two cases, we get the formula. \square

5. ABSOLUTE CONVERGENCE OF THE EISENSTEIN SERIES

Under the conditions of Theorem 4.4 we have proved that $E(s, f, ug)$ converges absolutely almost everywhere on $\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})$, by proving the finiteness of the constant term $E_{\widehat{B}}(s, h, g)$. The main result of this section is the absolute and uniform convergence of $E(s, f, g)$ for g in a certain Siegel set and with $\Re s$'s large enough. By boundedness of the cusp form f , it is enough to prove the absolute convergence of $E(s, h, g)$. The main ingredient of the proof is the systematic use of Demazure modules together with some technical estimations. We follow Garland's idea in [12]. Our arguments in the adelic settings also involve a property of algebraic number fields, which is analogous to the Riemann-Roch theorem for algebraic curves.

5.1. Demazure modules. Recall that for any dominant integral weight λ , we have the irreducible highest weight module V_λ and the highest weight vector v_λ such that $V_{\lambda, \lambda, \mathbb{Z}} = \mathbb{Z}v_\lambda$. There is a map $|\cdot| : V_{\lambda, \mathbb{A}} \rightarrow \mathbb{R}_{\geq 0}$ defined in Section 3.4. The highest weight vector v_λ embeds into $V_{\lambda, \mathbb{A}}$ diagonally. Recall that L is the fundamental weight such that $\langle L, \alpha_i^\vee \rangle = 0$, $i = 1, \dots, n$, and $\langle L, \alpha_0^\vee \rangle = \langle L, c - \tilde{\alpha}^\vee \rangle = 1$. Then by Lemma 3.21 it follows that the height function h_s can be defined as

$$(5.1) \quad h_s(g) = |g^{-1}v_L|^{-s},$$

where v_L is the highest weight vector in $V_{L, \mathbb{A}}$ as above.

The simple equation (5.1) plays a crucial role in the proof of the absolute convergence of the Eisenstein series. In this section we shall collect some basic results on the Demazure module $V_\lambda(w)$, which is associated with a dominant integral weight λ and $w \in \widehat{W}$, and is a submodule of V_λ defined below.

Recall that $\tilde{\mathfrak{n}}_+ = \bigoplus_{\alpha \in \tilde{\Phi}_+} \mathfrak{g}_\alpha$ is the Lie algebra of \widehat{U} . Let $\mathcal{U}^{\mathbb{Z}}(\tilde{\mathfrak{n}}_+)$ be a \mathbb{Z} -form of the universal enveloping algebra $\mathcal{U}(\tilde{\mathfrak{n}}_+)$ of $\tilde{\mathfrak{n}}_+$. We define

$$(5.2) \quad V_\lambda(w)_{\mathbb{Z}} = \mathcal{U}^{\mathbb{Z}}(\tilde{\mathfrak{n}}_+) \cdot w \cdot v_\lambda.$$

Let $\Lambda(w)$ be the subset of all weights μ of V_λ such that $\mu \geq w\lambda$, i.e.

$$\mu - w\lambda = \sum_{i=0}^n l_i \alpha_i, \quad l_i \in \mathbb{Z}_{\geq 0}.$$

Then it is clear that

$$(5.3) \quad V_\lambda(w)_\mathbb{Z} \subset \bigoplus_{\mu \in \Lambda(w)} V_{\lambda, \mu, \mathbb{Z}}.$$

For any field F we define the Demazure module over F corresponding to λ and w by $V_\lambda(w)_F = V_\lambda(w)_\mathbb{Z} \otimes F$. Let $V_\lambda(w)_\mathbb{A} \subset V_{\lambda, \mathbb{A}}$ be the restricted product

$$\prod'_v V_\lambda(w)_{F_v}$$

with respect to $V_\lambda(w)_{\mathcal{O}_v} = V_\lambda(w)_\mathbb{Z} \otimes \mathcal{O}_v$ defined for all finite places v . If $\phi = \otimes_v \phi_v$ such that ϕ_v is a linear operator on V_{λ, F_v} and ϕ_v preserves $V_{\lambda, \mathcal{O}_v}$ for almost all finite places v , then we define

$$(5.4) \quad \|\phi_v\| = \sup_{x \in V_{\lambda, F_v}, \|x\|=1} \|\phi_v x\|, \quad \|\phi\| = \prod_v \|\phi_v\|$$

and

$$(5.5) \quad |\phi_v| = \sup_{x \in V_{\lambda, F_v}, \|x\|=1} |\phi_v x|, \quad |\phi| = \prod_v |\phi_v|.$$

Note that $|\phi_v| = \|\phi_v\|^2$ or $\|\phi_v\|$ according to whether or not v is complex. If ϕ_v preserves $V_\lambda(w)_{F_v}$ for each v , then we define $\|\cdot\|_w$ and $|\cdot|_w$ similarly by restriction. In particular, $\|\phi_v\|_w \leq \|\phi_v\|$, $|\phi_v|_w \leq |\phi_v|$.

Let F be a local field. We shall estimate the norm of $\tilde{x}_\beta(u)$ acting on $V_\lambda(w)_F$, where $\beta \in \tilde{\Phi}_{re+}$, $u \in F$. By Lemma 3.3 we have the group homomorphism $\varphi_\beta : SL_2(F) \rightarrow \widehat{G}(F((t)))$ such that

$$\varphi_\beta \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \tilde{x}_\beta(u).$$

Then it follows that

$$\varphi_\beta \left(\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \right) = \tilde{h}_\beta(u), \quad u \in F^\times,$$

and φ_β maps the maximal compact subgroup of $SL_2(F)$ into \widehat{K} .

If $F = \mathbb{R}$ or \mathbb{C} , consider the Cartan decomposition

$$(5.6) \quad \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{u} & 1 \end{pmatrix} = k \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k^{-1},$$

where k lies in the maximal compact subgroup of $SL_2(F)$ and $a \geq a^{-1} > 0$. Comparing the trace and determinant of both sides it is easy to obtain

$$(5.7) \quad a = a(u) = \frac{|u|^2 + 2 + |u|\sqrt{|u|^2 + 4}}{2}.$$

Lemma 5.1 ([12, Lemma 4.1]). *Suppose $F = \mathbb{R}$ or \mathbb{C} , $\beta \in \tilde{\Phi}_{re+}$, $u \in F$. The square norm $\|\tilde{x}_\beta(u)\|_w^2$ of $\tilde{x}_\beta(u)$ restricted on $V_\lambda(w)_F$ is bounded by*

$$\sup_{\mu \in \text{weights of } V_\lambda(w)} a(u)^{\langle \mu, \beta^\vee \rangle},$$

where $a(u)$ is given by (5.7).

Proof. The adjoint operator of $\tilde{x}_\beta(u)$ is $\tilde{x}_{-\beta}(\bar{u})$, which follows from the fact that $X_\alpha \otimes t^i$ and $X_{-\alpha} \otimes t^{-i}$ ($\alpha \in \Phi$) are adjoint operators and that the inner product is hermitian. Then by the above discussions we get

$$\begin{aligned} \|\tilde{x}_\beta(u)\|_w^2 &= \sup_{v \in V_\lambda(w)_F, \|v\|=1} \|\tilde{x}_\beta(u)v\|^2 \\ &= \sup_{v \in V_\lambda(w)_F, \|v\|=1} (\tilde{x}_{-\beta}(\bar{u})\tilde{x}_\beta(u)v, v) \\ &= \sup_{v \in V_\lambda(w)_F, \|v\|=1} (k_\beta \tilde{h}_\beta(a) k_\beta^{-1} v, v) \\ &= \sup_{v \in V_\lambda(w)_F, \|v\|=1} (\tilde{h}_\beta(a)v, v) \\ &\leq \sup_{\mu \in \text{weights of } V_\lambda(w)} a^{\langle \mu, \beta^\vee \rangle}, \end{aligned}$$

where $a = a(u)$ and $k_\beta = \varphi_\beta(k) \in \widehat{K}$, with k the element appearing in (5.6). \square

Lemma 5.2. *Suppose F is a p -adic field and $\beta = \alpha + i\delta \in \tilde{\Phi}_{re+}$, $u \in F$. Then $|\tilde{x}_\beta(u)|_w = 1$ if $|u| \leq 1$, and it is bounded by*

$$\sup_{\mu \in \text{weights of } V_\lambda(w)} |u|^{2|\langle \mu, \alpha^\vee \rangle|}$$

if $|u| > 1$.

Proof. If $|u| \leq 1$ the lemma is clear since $\tilde{x}_\beta(u) \in \widehat{K}$, which preserves the norm on $V_{\lambda,F}$. Assume that $|u| > 1$. Then

$$\begin{aligned} |\tilde{x}_\beta(u)|_w &= |\tilde{h}_\alpha(u)\tilde{x}_\beta(u^{-1})\tilde{h}_\alpha(u^{-1})|_w \\ &\leq |\tilde{h}_\alpha(u)|_w |\tilde{h}_\alpha(u^{-1})|_w \\ &\leq \sup_{\mu \in \text{weights of } V_\lambda(w)} |u|^{2|\langle \mu, \alpha^\vee \rangle|}. \end{aligned}$$

\square

Consider the real Cartan subalgebra and its dual,

$$\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \oplus \mathbb{R}c \oplus \mathbb{R}d, \quad \tilde{\mathfrak{h}}_{\mathbb{R}}^* = \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta \oplus \mathbb{R}L.$$

For any $\mu \in \tilde{\mathfrak{h}}_{\mathbb{R}}^*$, write μ as the decomposition

$$(5.8) \quad \mu = \mu_0 - \kappa_\mu \delta + \sigma_\mu L,$$

where $\mu_0 \in \mathfrak{h}_{\mathbb{R}}^*$, $\kappa_\mu, \sigma_\mu \in \mathbb{R}$. If μ is a weight of $V_\lambda(w)$, then $\mu \geq w\lambda$ and $\kappa_\mu \leq \kappa_{w\lambda}$. We impose the condition $\kappa_\lambda = 0$ in order that λ be a dominant integral. The following lemmas are due to Garland [12]. For the reader's convenience we shall sketch the proof.

Lemma 5.3. *Let $\lambda \in \tilde{\mathfrak{h}}_{\mathbb{R}}^*$ be a dominant integral weight λ . Then there exists a constant $\kappa_0 > 0$ such that for all $w \in \widetilde{W}$,*

$$\kappa_{w\lambda} \leq \kappa_0 l(w)^2,$$

and for any weight μ of $V_\lambda(w)$, $w \neq 1$,

$$(\mu_0, \mu_0) \leq \kappa_0 l(w)^2.$$

Proof. We only prove the first inequality. See [12] for the proof of the second one. Let us write $\lambda = \lambda_0 + \sigma_\lambda L$, where $\lambda_0 \in \mathfrak{h}^*$, $\sigma_\lambda \in \mathbb{N}$. Write $w = T_\gamma w_0$, where $\gamma \in Q^\vee$, $w_0 \in W$. Then by Lemma 2.4

$$w\lambda = w_0\lambda_0 + \sigma_\lambda L + \sigma_\lambda \gamma - \left(\langle \lambda_0, \gamma \rangle + \frac{\sigma_\lambda}{2} (\gamma, \gamma) \right) \delta.$$

It follows from Lemma 5.6 below that

$$\kappa_{w\lambda} = \langle \lambda_0, \gamma \rangle + \frac{\sigma_\lambda}{2} (\gamma, \gamma) = O(1)\|\gamma\|^2 = O(1)l(T_\gamma)^2 \leq O(1)(l(w) + |\Phi_+|)^2.$$

From this the first inequality is clear. Here we denote by $O(1)$ a bounded term. \square

As a consequence of this lemma we obtain the following result.

Corollary 5.4. *Given $\kappa > 0$, there exists $\kappa_1 > 0$ such that for all $w \in \widetilde{W}$, $w \neq 1$, and for all $\beta \in \widetilde{\Phi}$ of the form*

$$\beta = \alpha + i\delta, \quad 0 \leq i \leq \kappa l(w), \quad \alpha \in \Phi,$$

we have $\langle \mu, \beta^\vee \rangle \leq \kappa_1 l(w)$ for any weight μ of $V_\lambda(w)$.

Proof. Write $\mu = \mu_0 - \kappa_\mu \delta + \sigma_\lambda L$. By Lemma 2.1, $\beta^\vee = \frac{i}{2} (\alpha^\vee, \alpha^\vee) c + \alpha^\vee$. Then from Lemma 5.3 it follows that

$$\langle \mu, \beta^\vee \rangle = \langle \mu_0, \alpha^\vee \rangle + \frac{i}{2} (\alpha^\vee, \alpha^\vee) \sigma_\lambda \leq \left(\kappa_0^{\frac{1}{2}} \|\alpha^\vee\| + \frac{\kappa}{2} \|\alpha^\vee\|^2 \sigma_\lambda \right) l(w).$$

Let κ_1 be the maximum of the above coefficient of $l(w)$ over $\alpha \in \Phi$. \square

The condition on β in Corollary 5.4 is satisfied in the following case.

Lemma 5.5. *There exists $\kappa \in \mathbb{N}$ such that for all $w \in \widetilde{W}$ and $\beta = \alpha + i\delta \in \widetilde{\Phi}_w = \widetilde{\Phi}_+ \cap w\widetilde{\Phi}_-$ ($\alpha \in \Phi, i \in \mathbb{Z}_{\geq 0}$), we have $0 \leq i < \kappa l(w)$.*

Proof. Write $w^{-1} = T_\gamma w_0$, where $\gamma \in Q^\vee$, $w_0 \in W$. Then

$$w^{-1}\beta = w_0\alpha + (i - \langle w_0\alpha, \gamma \rangle)\delta < 0,$$

which implies that $i \leq \langle w_0\alpha, \gamma \rangle \leq \|\gamma\|\|\alpha\|$. The proof follows from Lemma 5.6 below, together with the inequality $l(T_\gamma) \leq l(w) + l(w_0) \leq l(w) + |\Phi_+|$. \square

Lemma 5.6. *There exists $\tilde{\kappa} > 0$ such that $\|\gamma\| \leq \tilde{\kappa} l(T_\gamma)$ for all $\gamma \in Q^\vee$.*

Proof. From [22, Proposition 1.23], we get

$$l(T_\gamma) = \sum_{\alpha \in \Phi_+} |\langle \alpha, \gamma \rangle| \geq \sum_{i=1}^n |\langle \alpha_i, \gamma \rangle|.$$

γ can be written as a linear combination of the fundamental coweights, with coefficients $\langle \alpha_i, \gamma \rangle$, $1 \leq i \leq n$. The lemma is clear from this observation. \square

The Demazure module $V_\lambda(w)_F$ is preserved by the action of the elements $\tilde{x}_\alpha(u)$, $\alpha \in \Phi$, $u \in tF[[t]]$, and $\tilde{h}_\alpha(u)$, $\alpha \in \Phi$, $u \equiv 1 \pmod{tF[[t]]}$.

Lemma 5.7. *Let κ be given as in Lemma 5.5. For any $w \in \widetilde{W}$, the elements $\tilde{x}_\alpha(u)$, $\alpha \in \Phi$, $u \in t^{\kappa l(w)} F[[t]]$ and $\tilde{h}_\alpha(u)$, $\alpha \in \Phi$, $u \equiv 1 \pmod{t^{\kappa l(w)}}$ act as identity on the Demazure module $V_\lambda(w)_F$.*

Proof. Let $\mathfrak{u}_{w^{-1}} = \bigoplus_{\alpha \in \tilde{\Phi}_w} \mathfrak{g}_\alpha$ be the Lie algebra of $U_{w^{-1}}$; then

$$V_\lambda(w)_\mathbb{Z} = \mathcal{U}^\mathbb{Z}(\mathfrak{u}_{w^{-1}})wv_\lambda.$$

Let $\tilde{\Phi}_w = \{\beta_1, \dots, \beta_l\}$ with $l = l(w)$. The PBW Theorem implies that the monomials $X_{\beta_1}^{i_1} \cdots X_{\beta_l}^{i_l}$, $i_1, \dots, i_l \in \mathbb{Z}_{\geq 0}$, form a basis of $\mathcal{U}(\mathfrak{u}_{w^{-1}})$. To prove the lemma, it suffices to show that $U_\beta(F)$ acts on $V_\lambda(w)_F$ trivially for each $\beta = \alpha + i\delta$ with $\alpha \in \Phi \cup \{0\}$, $i \geq \kappa l(w)$. Then we are further reduced to show that $\mathfrak{g}_\beta = \mathfrak{g}_{\alpha+i\delta}$ with $\alpha \in \Phi \cup \{0\}$, $i \geq \kappa l(w)$ acts on $V_\lambda(w)_F$ as zero. We prove by induction on $i_1 + \cdots + i_l$ that

$$(5.9) \quad \mathfrak{g}_\beta X_{\beta_1}^{i_1} \cdots X_{\beta_l}^{i_l} wv_\lambda = 0.$$

Since $w^{-1}\mathfrak{g}_\beta w = \mathfrak{g}_{w^{-1}\beta}$ and $w^{-1}\beta > 0$ by Lemma 5.5, we have $\mathfrak{g}_\beta wv_\lambda = 0$. Consider $[\mathfrak{g}_\beta, \mathfrak{g}_{\beta_i}]$, which is zero if $\beta + \beta_i \notin \tilde{\Phi}$ and equals $\mathfrak{g}_{\beta+\beta_i}$ otherwise. In each case the induction follows and (5.9) is proved. \square

5.2. Estimations of some norms. Let F be a local field. In this section we shall apply the results of the previous section to estimate the norms of elements in $\hat{U}(F)$ acting on $V_\lambda(w)_F$, under certain conditions.

Lemma 5.8 ([12, pp.228-232]). *Suppose that $F = \mathbb{R}$ or \mathbb{C} , $\tilde{x}_\alpha(u) \in \hat{U}(F)$, where $\alpha \in \Phi$, $u = \sum_{i=0}^\infty u_i t^i \in F[[t]]$ ($\in tF[[t]]$ if $\alpha \in \Phi_-$) such that $|u_i| \leq M\tau^i$, $i = 0, 1, \dots$, for some $M > 0$ and $0 < \tau < 1$. Then $\|\tilde{x}_\alpha(u)\|_w \leq \exp(\kappa_{M,\tau} l(w))$ for some constant $\kappa_{M,\tau}$ only depending on M and τ .*

Proof. Consider $\beta = \alpha + i\delta \in \tilde{\Phi}_{re+}$, $u_i \in F$. Let $\kappa \in \mathbb{N}$ be the constant in Lemma 5.5. If $i \geq \kappa l(w)$, then $\tilde{x}_\beta(u_i)$ acts on $V_\lambda(w)_F$ trivially; if $i < \kappa l(w)$, by Lemma 5.1 and Corollary 5.4 we have

$$\|\tilde{x}_\beta(u_i)\|_w^2 \leq \sup_{\mu \in \text{weights of } V_\lambda(w)} a(u)^{\langle \mu, \beta^\vee \rangle} \leq a(u_i)^{\kappa l(w)},$$

where $a(u_i) \geq 1$ is given by (5.7). It is easy to show that there exists $c_M > 0$ depending on M such that $a(u_i) \leq 1 + c_M \tau^i$. Then by Lemma 5.7 we have

$$\begin{aligned} \|\tilde{x}_\alpha(u)\|_w^2 &= \left\| \prod_{i=0}^{\kappa l(w)-1} \tilde{x}_{\alpha+i\delta}(u_i) \right\|_w^2 \leq \prod_{i=0}^{\kappa l(w)-1} \|\tilde{x}_{\alpha+i\delta}(u_i)\|_w^2 \\ &\leq \prod_{i=0}^{\kappa l(w)-1} (1 + c_M \tau^i)^{\kappa l(w)} \leq \exp \left(\kappa_1 c_M l(w) \sum_{i=0}^{\kappa l(w)-1} \tau^i \right) \\ &\leq \exp \left(\frac{\kappa_1 c_M}{1-\tau} l(w) \right). \end{aligned}$$

The lemma follows if we set $\kappa_{M,\tau} = \frac{\kappa_1 c_M}{2(1-\tau)}$. \square

Lemma 5.9 ([12, pp.233-240]). *Suppose that $F = \mathbb{R}$ or \mathbb{C} , $u = 1 + \sum_{j=1}^\infty u_j t^j \in 1 + tF[[t]]$ such that $|u_j| < M\tau^j$, $j = 1, 2, \dots$, for some $M > 0$ and $0 < \tau < 1$. Then $\|\tilde{h}_{\alpha_i}(u)\|_w \leq \exp(\tilde{\kappa}_{M,\tau} l(w))$, $i = 0, 1, \dots, n$, for some constant $\tilde{\kappa}_{M,\tau}$ only depending on M and τ .*

Proof. Let us consider the following two cases:

Case 1: $w^{-1}\alpha_i < 0$. Let $w' = \tilde{w}_{\alpha_i}w$; then $l(w) = l(w') + 1$. Moreover, if we write $\tilde{\Phi}_{w'} = \tilde{\Phi}_+ \cap w'\tilde{\Phi}_- = \{\beta_1, \dots, \beta_{l-1}\}$, then $\tilde{\Phi}_w = \{\alpha_i, r_i\beta_1, \dots, r_i\beta_{l-1}\}$. We shall prove the following:

$$(5.10) \quad X_{-\alpha_i}V_\lambda(w)_F \subset V_\lambda(w)_F,$$

$$(5.11) \quad V_\lambda(w')_F \subset V_\lambda(w)_F.$$

To prove (5.10) it suffices to prove that for $j, j_1, \dots, j_{l-1} \in \mathbb{Z}_{\geq 0}$,

$$X_{-\alpha_i}X_{\alpha_i}^jX_{r_i\beta_1}^{j_1}\cdots X_{r_i\beta_{l-1}}^{j_{l-1}}wv_\lambda \in V_\lambda(w)_F.$$

We use induction on $j + j_1 + \dots + j_{l-1}$. It is clear by assumption that $X_{-\alpha_i}wv_\lambda = 0$. The induction follows since $[X_{-\alpha_i}, X_{\alpha_i}], [X_{-\alpha_i}, X_{r_i\beta_1}], \dots, [X_{-\alpha_i}, X_{r_i\beta_{l-1}}] \in \tilde{\mathfrak{b}} = \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$. $V_\lambda(w)_F$ is also preserved by X_{α_i} ; hence it is preserved by w_{α_i} . Then (5.11) follows from

$$X_{\beta_1}^{j_1}\cdots X_{\beta_{l-1}}^{j_{l-1}}w'v_\lambda = w_{\alpha_i}(w_{\alpha_i}X_{\beta_1}w_{\alpha_i})^{j_1}\cdots(w_{\alpha_i}X_{\beta_{l-1}}w_{\alpha_i})^{j_{l-1}}wv_\lambda \in V_\lambda(w)_F.$$

Since $w_{\alpha_i} = w_{\alpha_i}(1) \in \hat{K}$, which preserves the norm, by Lemma 5.8 and its proof we obtain

$$\begin{aligned} \|\tilde{h}_{\alpha_i}(u)\|_w &= \|w_{\alpha_i}(u)w_{\alpha_i}(1)^{-1}\|_w = \|w_{\alpha_i}(u)\|_w \\ &\leq \|\tilde{x}_{\alpha_i}(u)\|_w \|\tilde{x}_{-\alpha_i}(-u^{-1})\|_w \|\tilde{x}_{\alpha_i}(u)\|_w \leq \exp(3\kappa_{M,\tau}l(w)). \end{aligned}$$

Case 2: $w^{-1}\alpha_i > 0$. Then thanks to (5.10) and (5.11), $V_\lambda(w')_F$ is preserved by $X_{-\alpha_i}$ and \tilde{w}_{α_i} , and we have $V_\lambda(w)_F \subset V_\lambda(w')_F$. Applying the result in Case 1 we get

$$\|\tilde{h}_{\alpha_i}(u)\|_w \leq \|\tilde{h}_{\alpha_i}(u)\|_{w'} \leq \exp(3\kappa_{M,\tau}(l(w) + 1)).$$

Combining the two cases, the lemma holds for $\tilde{\kappa}_{M,\tau} = 3\kappa_{M,\tau} + 1$. \square

We have the following p -adic analog of Lemma 5.8.

Lemma 5.10. *Suppose that F is a p -adic field and $\tilde{x}_\alpha(u) \in \hat{U}$, where $\alpha \in \Phi$, $u = \sum_{i=0}^\infty u_i t^i \in F[[t]]$ ($\in tF[[t]]$ if $\alpha \in \Phi_-$) such that $|u_i| < M$, $i = 0, 1, \dots$, for some $M > 0$. Then $|\tilde{x}_\alpha(u)|_w \leq \exp(\kappa_M l(w))$ for a constant κ_M only depending on M .*

Proof. We may assume that $M > 1$. By Lemma 5.3 we have

$$|\langle \mu, \alpha^\vee \rangle| = |\langle \mu_0, \alpha^\vee \rangle| \leq \kappa_0^{\frac{1}{2}} \|\alpha^\vee\| l(w) \leq \kappa_1 l(w)$$

for any weights μ of $V_\lambda(w)$. Following the proof of Lemma 5.2 we obtain

$$\begin{aligned} |\tilde{x}_\alpha(u)|_w &= |\tilde{h}_\alpha(M)\tilde{x}_\alpha(M^{-2}u)\tilde{h}_\alpha(M^{-1})|_w \leq |\tilde{h}_\alpha(M)|_w |\tilde{h}_\alpha(M^{-1})|_w \\ &\leq \sup_{\mu \in \text{weights of } V_\lambda(w)} |M|^{2|\langle \mu, \alpha^\vee \rangle|} \leq |M|^{2\kappa_1 l(w)}; \end{aligned}$$

note that $\tilde{x}_\alpha(M^{-2}u) \in \hat{K}$. The lemma follows by setting $\kappa_M = 2\kappa_1 \log M$. \square

Let F be any local field. Recall from (3.29) that $\sigma(\mathfrak{q})$, $\mathfrak{q} \in F^\times$ acts on $V_{\lambda,F}$ by

$$\sigma(\mathfrak{q})v = \mathfrak{q}^{(\mu,d)}v$$

for each $v \in V_{\lambda, \mu, F}$. Then

$$\|\sigma(\mathbf{q})\|_w \leq \sup_{\mu \in \text{weights of } V_\lambda(w)} |\mathbf{q}|^{\langle \mu, d \rangle} \leq \sup_{\mu \in \Lambda(w)} |\mathbf{q}|^{\langle \mu, d \rangle}.$$

If $\lambda = L$, then $\langle \mu, d \rangle \leq 0$ for any $\mu \in \Lambda(w)$. In this case we obtain the following:

Lemma 5.11. *Assume $\lambda = L$, $\mathbf{q} \in F^\times$. Then*

$$\|\sigma(\mathbf{q})\|_w = \begin{cases} 1, & \text{if } |\mathbf{q}| \geq 1, \\ |\sigma(\mathbf{q})|^{wL}, & \text{if } |\mathbf{q}| \leq 1. \end{cases}$$

From this lemma we can get the following global result.

Corollary 5.12. *Assume $\lambda = L$, $\mathbf{q} = (\mathbf{q}_v)_v \in \mathbb{I}$. Then $|\sigma(\mathbf{q})|_w = 1$ if $|\mathbf{q}_v|_v \geq 1$ for each v , and $|\sigma(\mathbf{q})|_w = \sigma(\mathbf{q})^{wL}$ if $|\mathbf{q}_v|_v \leq 1$ for each v .*

Proof. This is clear from the fact that $|\sigma(\mathbf{q})|_w = \prod_v |\sigma(\mathbf{q}_v)|_w$. \square

5.3. Convergence of the Eisenstein series. In this section we prove the absolute convergence of the Eisenstein series everywhere, whenever the conditions of Theorem 4.4 are satisfied. The uniform convergence of the Eisenstein series over a certain analog of Siegel set will be established. The main result is the following:

Theorem 5.13. *Fix $\mathbf{q} \in \mathbb{I}$, $|\mathbf{q}| > 1$. There exists a constant $c_{\mathbf{q}} > 0$ depending on \mathbf{q} , such that for any $\varepsilon > 0$ and compact subset Ω of $T(\mathbb{A})$, $E(s, f, g)$ and $E(s, h, g)$ converge absolutely and uniformly for $s \in \{z \in \mathbb{C} | \Re z > \max(h + h^\vee + \varepsilon, c_{\mathbf{q}})\}$ and $g \in \widehat{U}(\mathbb{A})\Omega\sigma(\mathbf{q})\widehat{K}$.*

Proof. We only need to prove the theorem for $E(s, h, g)$. Let us write $g = u_g a_g \sigma(\mathbf{q}) k_g$, where $u_g \in \widehat{U}(\mathbb{A})$, $a_g \in \Omega$, $|\mathbf{q}| > 1$ and $k_g \in \widehat{K}$. Since $E(s, h, g)$ is left $\widehat{G}(F\langle t \rangle)$ -invariant, by Lemma 3.27 we may assume that $u_g \in \widehat{U}_{\mathcal{D}}$. We may also assume that $k_g = 1$. Recall from (5.1) that h_s is the height function such that $h_s(g) = |g^{-1}v_L|^{-s}$, and

$$E(s, h, g) = \sum_{w \in W(\Delta, \emptyset)} \sum_{\gamma \in U_w(F)} h_s(w\gamma g).$$

Let $C > 1$ be a constant which will be determined later. Write $\mathbf{q} = \mathbf{q}_1 \mathbf{q}_2^{-1}$ such that

- (i) $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{I}_+ := \{x = (x_v)_v \in \mathbb{I} | |x_v|_v \geq 1, \forall v\}$,
- (ii) $|\mathbf{q}_{1v}| \geq C$ for each $v | \infty$.

By Corollary 5.12, $|\sigma(\mathbf{q}_2^{-1})|_{w^{-1}} = \sigma(\mathbf{q}_2^{-1})^{w^{-1}L}$. Assume $\gamma \in U_w(F)$. Let $g_1 = u_g a_g \sigma(\mathbf{q}_1) = g\sigma(\mathbf{q}_2)$. Since $(w\gamma g_1)^{-1}v_L \in V_L(w^{-1})_{\mathbb{A}}$, we have

$$\begin{aligned} (5.12) \quad h_1(w\gamma g) &= |(w\gamma g)^{-1}v_L|^{-1} = |\sigma(\mathbf{q}_2)(w\gamma g_1)^{-1}v_L|^{-1} \\ &\leq |\sigma(\mathbf{q}_2^{-1})|_{w^{-1}} |(w\gamma g_1)^{-1}v_L|^{-1} = \sigma(\mathbf{q}_2^{-1})^{w^{-1}L} h_1(w\gamma g_1). \end{aligned}$$

Similarly for any $u \in \widehat{U}(\mathbb{A})$ we have $(w\gamma u g_1)^{-1}v_L \in V_L(w^{-1})_{\mathbb{A}}$. Note that g_1 acts on $V_L(w^{-1})_{\mathbb{A}}$; therefore

$$\begin{aligned} (5.13) \quad h_1(w\gamma g_1) &= |g_1^{-1} \gamma^{-1} w^{-1} v_L|^{-1} \\ &\leq |g_1^{-1} u^{-1} g_1|_{w^{-1}} \cdot |(w\gamma u g_1)^{-1} v_L|^{-1} \\ &= |g_1^{-1} u^{-1} g_1|_{w^{-1}} h_1(w\gamma u g_1). \end{aligned}$$

Assume $u \in \widehat{U}_{\mathcal{D}}$, and we shall estimate $|g_1^{-1}u^{-1}g_1|_{w^{-1}}$, which is bounded by

$$\begin{aligned} & |(a_g\sigma(\mathbf{q}_1))^{-1}u_g^{-1}(a_g\sigma(\mathbf{q}_1))|_{w^{-1}}|(a_g\sigma(\mathbf{q}_1))^{-1}u^{-1}(a_g\sigma(\mathbf{q}_1))|_{w^{-1}} \\ & \times |(a_g\sigma(\mathbf{q}_1))^{-1}u_g(a_g\sigma(\mathbf{q}_1))|_{w^{-1}}. \end{aligned}$$

Let us estimate $|(a_g\sigma(\mathbf{q}_1))^{-1}u^{-1}(a_g\sigma(\mathbf{q}_1))|_{w^{-1}}$. Other factors can be treated similarly. Since $u^{-1} \in \widehat{U}_{\mathcal{D}}$, u^{-1} is a product of the elements $\tilde{x}_{\alpha}(u_{\alpha})$ where either $\alpha \in \Phi_+$, $u_{\alpha} \in -\mathcal{D}\langle t \rangle_+$ or $\alpha \in \Phi_-$, $u_{\alpha} \in -t\mathcal{D}\langle t \rangle_+$, and the elements $\tilde{h}_{\alpha_i}(u_i)$, $u_i \in (1+t\mathcal{D}\langle t \rangle_+)^{-1}$, $i = 1, \dots, n$.

By our choice of \mathcal{D} , there exists $M_{\mathcal{D}} > 0$ such that for any $x = (x_v)_v \in \mathcal{D}$ we have $|x_v| \leq M_{\mathcal{D}}$ for each $v|\infty$. Then for any $\eta_{\mathcal{D}} > 2M_{\mathcal{D}}$ there exists $M > 0$ such that if $x = 1 + \sum_{j=1}^{\infty} x_j t^j \in (1+t\mathcal{D}\langle t \rangle_+)^{-1}$, then $|x_{jv}| \leq M\eta_{\mathcal{D}}^j$, $j = 1, 2, \dots$, for each $v|\infty$. In fact $x \in (1+t\mathcal{D}\langle t \rangle)^{-1}$ implies that x_v^{-1} defines a non-vanishing series absolutely convergent in the range $|t| < (2M_{\mathcal{D}})^{-1}$, hence so is x_v itself. This implies

$$\frac{1}{\limsup \sqrt[j]{|x_{jv}|}} \geq \frac{1}{2M_{\mathcal{D}}} > \frac{1}{\eta_{\mathcal{D}}},$$

whence the assertion follows.

Consider

$$(a_g\sigma(\mathbf{q}_1))^{-1}\tilde{h}_{\alpha_i}(u_i)(a_g\sigma(\mathbf{q}_1)) = \tilde{h}_{\alpha_i}(\sigma(\mathbf{q}_1^{-1}) \cdot u_i).$$

Let $C > 1$ be any constant such that

$$(5.14) \quad C > \eta_{\mathcal{D}} > 2M_{\mathcal{D}}.$$

Applying Lemma 5.9 with M chosen as above and $\tau = C^{-1}\eta_{\mathcal{D}}$, we get a constant $\kappa_{C,\mathcal{D}} := 2\tilde{\kappa}_{M,\tau}$ such that

$$|\tilde{h}_{\alpha_i}(\sigma(\mathbf{q}_1^{-1}) \cdot u_i)_v|_{w^{-1}} \leq \exp(\kappa_{C,\mathcal{D}}l(w))$$

for each $v|\infty$. On the other hand it is clear that $\tilde{h}_{\alpha_i}(\sigma(\mathbf{q}_1^{-1}) \cdot u_i)_v \in \widehat{K}_v$ for $v < \infty$. Therefore

$$(5.15) \quad |\tilde{h}_{\alpha_i}(\sigma(\mathbf{q}_1^{-1}) \cdot u_i)|_{w^{-1}} \leq \exp(|S_{\infty}|\kappa_{C,\mathcal{D}}l(w)),$$

where S_{∞} is the set of infinite places of F .

Now consider

$$(a_g\sigma(\mathbf{q}_1))^{-1}\tilde{x}_{\alpha}(u_{\alpha})(a_g\sigma(\mathbf{q}_1)) = \tilde{x}_{\alpha}(a_g^{\alpha}\sigma(\mathbf{q}_1^{-1}) \cdot u_{\alpha}).$$

Since Ω is a compact subset of $T(\mathbb{A})$, we may assume that

$$\Omega \subset \prod_{v \in S_{\Omega}} T(F_v) \times \prod_{v \notin S_{\Omega}} T(\mathcal{O}_v),$$

where $S_{\Omega} \supset S_{\infty}$ is a finite set of places, and we can find $M_{\Omega} > 0$ such that $|a_v^{\alpha}| < M_{\Omega}$ for any $a \in \Omega$, $v \in S_{\Omega}$ and $\alpha \in \Phi$. Applying Lemma 5.8 with $M' = M_{\Omega}M_{\mathcal{D}}$ and $\tau' = C^{-1}$, we get a constant $\kappa_{C,\mathcal{D},\Omega} := 2\kappa_{M',\tau'}$ such that

$$|\tilde{x}_{\alpha}(a_g^{\alpha}\sigma(\mathbf{q}_1^{-1}) \cdot u_{\alpha})_v|_{w^{-1}} \leq \exp(\kappa_{C,\mathcal{D},\Omega}l(w))$$

for $v \in S_{\infty}$. Applying Lemma 5.10 we get

$$|\tilde{x}_{\alpha}(a_g^{\alpha}\sigma(\mathbf{q}_1^{-1}) \cdot u_{\alpha})_v|_{w^{-1}} \leq \exp(\kappa_{M_{\Omega}}l(w))$$

for $v \in S_{\Omega} \setminus S_{\infty}$. Therefore

$$(5.16) \quad |\tilde{x}_{\alpha}(a_g^{\alpha}\sigma(\mathbf{q}_1^{-1}) \cdot u_{\alpha})|_{w^{-1}} \leq \exp((|S_{\infty}|\kappa_{C,\mathcal{D},\Omega} + |S_{\Omega} \setminus S_{\infty}|\kappa_{M_{\Omega}})l(w)).$$

Combining (5.15) and (5.16), there exists a constant $\tilde{\kappa}_{C,\mathcal{D},\Omega}$ such that

$$|g_1^{-1}ug_1|_{w^{-1}} \leq \exp(\tilde{\kappa}_{C,\mathcal{D},\Omega}l(w))$$

for any $u \in \widehat{U}_{\mathcal{D}}$, and hence for any $u \in \widehat{U}(F) \backslash \widehat{U}(\mathbb{A})$. From (5.12) and (5.13), it follows that

$$(5.17) \quad h_1(w\gamma g) \leq \exp(\tilde{\kappa}_{C,\mathcal{D},\Omega}l(w)) \sigma(\mathfrak{q}_2^{-1})^{w^{-1}L} h_1(w\gamma ug_1).$$

Now we are ready to finish the proof of the theorem. We may assume that $s \in H_{\varepsilon}$ is a real number. Taking the s -th power of both sides of (5.17) and integrating over $\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})$, we obtain

$$\begin{aligned} E(s, h, g) &\leq \sum_{w \in W(\Delta, \emptyset)} \exp(s\tilde{\kappa}_{C,\mathcal{D},\Omega}l(w)) \sigma(\mathfrak{q}_2^{-1})^{sw^{-1}L} \int_{\widehat{U}(F) \backslash \widehat{U}(\mathbb{A})} \sum_{\gamma \in U_w(F)} h_s(w\gamma ug_1) du \\ &= \sum_{w \in W(\Delta, \emptyset)} \exp(s\tilde{\kappa}_{C,\mathcal{D},\Omega}l(w)) \sigma(\mathfrak{q}_2^{-1})^{sw^{-1}L} (a_g \sigma(\mathfrak{q}_1))^{\tilde{\rho} + w^{-1}(sL - \tilde{\rho})} c_w(s) \\ &= \sum_{w \in W(\Delta, \emptyset)} \exp(s\tilde{\kappa}_{C,\mathcal{D},\Omega}l(w)) \sigma(\mathfrak{q}_2)^{\tilde{\rho} - w^{-1}\tilde{\rho}} (a_g \sigma(\mathfrak{q}))^{\tilde{\rho} + w^{-1}(sL - \tilde{\rho})} c_w(s). \end{aligned}$$

Let us keep track of the proof of Lemma 4.5. Let

$$c_{\Omega} = \max_{a \in \Omega} c_a = \max_{a \in \Omega, \alpha \in \Phi} |a^{\alpha}| \leq M_{\Omega}^{|S_{\Omega}|}.$$

Then (4.24) reads

$$a_g^{\tilde{\rho} - w^{-1}\tilde{\rho} + w^{-1}sL} \leq c_{\Omega}^{l(w) + c_1 s \|\lambda\|}.$$

Plugging in (4.23), (4.25) and $c_w(s) \leq c_{\varepsilon}^{l(w)}$, we see that

$$\begin{aligned} E(s, h, g) &\leq |W| \exp(s\tilde{\kappa}_{C,\mathcal{D},\Omega}|\Phi_+|) (c_{\varepsilon} c_{\Omega})^{|\Phi_+|} \\ &\quad \times \sum_{\lambda \in Q^{\vee}} (c_{\varepsilon}^{c_2} c_{\Omega}^{c_1 s + c_2})^{\|\lambda\|} |\exp(s\tilde{\kappa}_{C,\mathcal{D},\Omega} c_2 \|\lambda\|)| |\mathfrak{q}_2|^{c_1 h \|\lambda\| + \frac{h^{\vee}}{2} \|\lambda\|^2} \\ &\quad \times |\mathfrak{q}|^{c_1 h \|\lambda\| - \frac{s - h^{\vee}}{2} \|\lambda\|^2}. \end{aligned}$$

The last summation converges if and only if

$$|\mathfrak{q}_2|^{\frac{h^{\vee}}{2}} |\mathfrak{q}|^{-\frac{s - h^{\vee}}{2}} < 1,$$

i.e.

$$(5.18) \quad s > c_{\mathfrak{q}} := h^{\vee} \left(1 + \frac{\log |\mathfrak{q}_2|}{\log |\mathfrak{q}|} \right).$$

It is clear that convergence is uniform for all s satisfying (5.18). \square

When $|\mathfrak{q}|$ is large enough, we can replace $c_{\mathfrak{q}}$ by a constant which does not depend on \mathfrak{q} . We need the following lemma [30, p.143].

Lemma 5.14. *There exists a constant $c_F > 1$ depending on the number field F such that, for all $\mathfrak{q} \in \mathbb{I}$, $|\mathfrak{q}| \geq c_F$, there exists $x \in F^{\times}$ such that $1 \leq |x\mathfrak{q}_v|_v \leq |\mathfrak{q}|$ for each place v .*

Since the Eisenstein series are left $\sigma(F^\times)$ -invariant, and $\sigma(F^\times)$ normalizes $\widehat{U}(\mathbb{A})T(\mathbb{A})$, we may replace \mathfrak{q} in Theorem 5.13 by $x\mathfrak{q}$ for any $x \in F^\times$. Therefore we may assume that \mathfrak{q} satisfies the conclusion of Lemma 5.14 whenever $|\mathfrak{q}| \geq c_F$. In this case \mathfrak{q}_2 can be chosen such that $|\mathfrak{q}_{2v}| = 1$ for $v < \infty$, and $|\mathfrak{q}_{2v}| \leq C$ for each $v|\infty$. Then we have

$$(5.19) \quad |\mathfrak{q}_2| \leq C^{2|S_\infty|}, \quad c_{\mathfrak{q}} \leq c'_F := h^\vee \left(1 + 2|S_\infty| \frac{\log C}{\log c_F} \right).$$

In summary we obtain the following result.

Theorem 5.15. *For any $\varepsilon > 0$ and compact subset Ω of $T(\mathbb{A})$, $E(s, f, g)$ and $E(s, h, g)$ converge absolutely and uniformly for*

$$s \in \{z \in \mathbb{C} | \Re z \geq \max(h + h^\vee + \varepsilon, c'_F)\}$$

and $g \in \widehat{U}(\mathbb{A})\Omega\sigma_{c_F-1}\widehat{K}$, where c_F and c'_F are given by Lemma 5.14 and (5.19) respectively.

The constants in the theorem can be made explicit for the case $F = \mathbb{Q}$. In fact we get the same range of convergence as that of the constant term of the Eisenstein series. Namely Theorem 4.4 also holds for the Eisenstein series itself.

Corollary 5.16. *Let $F = \mathbb{Q}$. Then for any $\varepsilon, \eta > 0$, $E(s, f, g)$ and $E(s, h, g)$ converge absolutely and uniformly for $s \in H_\varepsilon$ and $g \in \widehat{U}(\mathbb{A})\sigma_\eta\widehat{K}$.*

Proof. It is clear that $c_{\mathbb{Q}}$ in Lemma 5.14 can be chosen to be an arbitrary constant greater than 1. Fix $c_{\mathbb{Q}} = 1 + \eta$ with $\eta > 0$. We may choose

$$\mathcal{D} = [-\frac{1}{2}, \frac{1}{2}] \times \prod_p \mathbb{Z}_p$$

and $M_{\mathcal{D}} = \frac{1}{2}$. It follows from (5.14) and (5.19) that we can choose C to be close enough to 1 such that $c'_{\mathbb{Q}} \leq h^\vee + \varepsilon$. \square

As mentioned in the introduction, we conjecture that Corollary 5.16 holds for an arbitrary number field or in general for a global field F . For the geometric case that F is the function field of a smooth projective curve X over a finite field \mathbb{F}_q , Lemma 5.14 boils down to the Riemann-Roch theorem. Namely, let $D_{\mathfrak{q}}$ be the divisor corresponding to $\mathfrak{q} \in \mathbb{I}$; in order that $H^0(-D_{\mathfrak{q}}) \neq 0$ it is sufficient that $-\deg(D_{\mathfrak{q}}) + 1 - g > 0$, i.e.

$$|\mathfrak{q}| = q^{-\deg D_{\mathfrak{q}}} \geq q^g,$$

where g is the genus of X . Hence we set $c_F = q^g + \eta$ and $c'_F = h^\vee$, since $S_\infty = \emptyset$. In particular, the condition for s reduces to $s \in H_\varepsilon$, and Corollary 5.16 is true for $X = \mathbb{P}_{\mathbb{F}_q}^1$.

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