THE STABILITY OF SELF-SHRINKERS OF MEAN CURVATURE FLOW IN HIGHER CO-DIMENSION

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ABSTRACT. We generalize Colding and Minicozzi's work (2012) on the stability of hypersurface self-shrinkers to higher co-dimension. The first and second variation formulae of the *F*-functional are derived and an equivalent condition to the stability in general co-dimension is found. We also prove that \mathbb{R}^n is the only stable product self-shrinker and show that the closed embedded Lagrangian self-shrinkers constructed by Anciaux are unstable.

1. INTRODUCTION

Let $X : \Sigma \to \mathbb{R}^m$ be an isometric immersion of an *n*-dimensional manifold Σ in the Euclidean space \mathbb{R}^m . Mean curvature flow of X is a family of immersions $X_t : \Sigma \to \mathbb{R}^m$ that satisfies

$$\begin{cases} \left(\frac{\partial}{\partial t}X_t(x)\right)^{\perp} &= H(x,t),\\ X_0 &= X, \end{cases}$$

where H(x,t) is the mean curvature vector of $X_t(\Sigma)$ at $X_t(x)$ and v^{\perp} denotes the projection of v into the normal space of $X_t(\Sigma)$. Mean curvature flow of a submanifold in a Riemannian manifold can be defined similarly. Because the mean curvature vector points in the direction in which the area decreases most rapidly, mean curvature flow is a canonical way to construct minimal submanifolds. It also improves the geometric properties of an object along the flow (e.g., see [7]).

A submanifold Σ in \mathbb{R}^m is called a *self-shrinker* if its position vector $X : \Sigma \to \mathbb{R}^m$ satisfies

$$H = -\frac{1}{2}X^{\perp}.$$

The terminology comes from the fact that $\sqrt{1-t}X(\Sigma)$ is a solution of mean curvature flow, i.e., a self-shrinker evolves homothetically along mean curvature flow in a shrinking way. Moreover, self-shrinkers describe all possible central blow-up limits of a finite-time singularity of mean curvature flow. This follows from Huisken's monotonicity formula [8], and its generalization to type II singularity by Ilmanen [10] and White. Singularities occur along mean curvature flow in general and are obstacles to continue the flow. It is therefore an important issue to understand singularities and the candidates of their blow-up limits, self-shrinkers.

Standard sphere $\mathbb{S}^n(\sqrt{2n})$ and cylinder $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ are simple examples of self-shrinkers in \mathbb{R}^m . Abresch and Langer [1] found all immersed closed selfshrinkers in the plane. In the case of high dimensional complete hypersurface, Huisken [9] classified all embedded self-shrinkers with nonnegative mean curvature,

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polynomial volume growth, and the second fundamental form bounded. The condition on second fundamental form is later removed by Colding and Minicozzi in [6]. However, there are still many other different hypersurface self-shrinkers (e.g., see [3]), and a classification of all self-shrinkers is not expected. For the higher co-dimensional case, our understanding is even more limited. One result is due to Smoczyk in [13] who obtained a classification of self-shrinkers with parallel principal normal $\nu \equiv H/|H|$ and bounded geometry. The parallel principal normal condition mainly reduces the problem to a similar situation as the co-dimensional one case. Several different families of Lagrangian self-shrinkers are constructed by Anciaux in [2], and Joyce-Lee-Tsui in [11] which generalizes examples constructed by Lee-Wang in [12]. Lagrangians are submanifolds of middle dimension. See §5 for the definition of Lagrangian.

Adapted from the back heat kernel introduced by Huisken in [8], Colding and Minicozzi [6] defined a functional F by

(1)
$$F(\Sigma, x, t) = \frac{1}{\sqrt{4\pi t^n}} \int_{\Sigma} e^{\frac{-|X-x|^2}{4t}} d\mu,$$

for any submanifold $X : \Sigma^n \to \mathbb{R}^{n+1}$, $x \in \mathbb{R}^{n+1}$ and t > 0. One of the main properties of this functional is that (Σ, x_0, t_0) is a critical point of F iff Σ satisfies $H = -\frac{(X-x_0)^{\perp}}{2t_0}$. Specifically, it is a self-shrinker when $x_0 = 0$ and $t_0 = 1$. They proved that if an *n*-dimensional complete smooth embedded self-shrinker Σ^n in \mathbb{R}^{n+1} has polynomial volume growth and is F-stable with respect to compactly supported variations, then it must be the round sphere or a hyperplane. Here being F-stable means that for every compactly supported smooth variation Σ_s with $\Sigma_0 = \Sigma$, there exist variations x_s of 0 and t_s of 1 such that $\frac{\partial^2}{\partial s^2} F(\Sigma_s, x_s, t_s) \geq 0$ at s = 0. Relating to functional F, a notion of entropy and entropy stable can be defined, and F-unstable implies entropy unstable if the self-shrinker does not split off a line. Moreover, entropy decreases along mean curvature flow. The importance and goal of Colding-Minicozzi's work is to conclude that the blow-up near the first time singularity of mean curvature flow for generic initial data gives stable selfshrinkers (see [6] for the exact statement).

In this paper, we intend to generalize Colding and Minicozzi's work [6] to higher co-dimensional cases. The domain of the functional F is now (Σ, x, t) for $\Sigma^n \subset \mathbb{R}^m$, $x \in \mathbb{R}^m$ and t > 0. The critical points satisfy the same equation as in co-dimension one. Colding and Minicozzi's classification on F-stable hypersurface self-shrinkers is first to conclude that the mean curvature function h is the first eigenfunction of an elliptic operator; it then implies $h \ge 0$. Huisken's classification on embedded selfshrinkers with $h \ge 0$ and the generalization by Colding-Minicozzi will lead to the conclusion. Although the counterpart of Huisken's result in higher co-dimension is not available, we can still relate the stability of self-shrinkers in higher co-dimension to the mean curvature vector being the first vector-valued eigenfunction for an elliptic system. More precisely, the equivalent condition of stabilities is as in the following theorems.

Theorem 4. Suppose $\Sigma \subset \mathbb{R}^m$ is an n-dimensional smooth closed self-shrinker satisfying $H = -\frac{X^{\perp}}{2}$. The following statements are equivalent: (i) Σ is F-stable. (ii) $\int_{\Sigma} \langle V, -L^{\perp}V \rangle e^{-\frac{|X|^2}{4}} d\mu \geq 0$ for any admissible vector field V, namely, a smooth normal vector field V which satisfies

$$\int_{\Sigma} \langle V, H \rangle e^{-\frac{|X|^2}{4}} d\mu = 0 \quad and \quad \int_{\Sigma} \langle V, y^{\perp} \rangle e^{-\frac{|X|^2}{4}} d\mu = 0$$

for any constant vector $y \in \mathbb{R}^m$, where

$$L^{\perp}V = \Delta^{\perp}V + \langle A_{ij}, V \rangle g^{ki}g^{jl}A_{kl} + \frac{V}{2} - \frac{1}{2}\nabla^{\perp}_{X^{\top}}V$$

is a second order elliptic operator and A_{ij} is the second fundamental form as defined in (2), and ∇^{\perp} is the normal connection on the normal bundle of Σ .

Note that in high co-dimension F-unstable also implies entropy unstable if the self-shrinker does not split off a line. We will only discuss F-stability in this paper, and stable is always referred to as F-stable. From the standard spectrum theory for unbounded domain, it is natural to consider the set $\mathcal{H}_0^1(\Sigma)$, which is the closure of compact support smooth normal vector fields with respect to the norm $|| \cdot ||_{1,e}$. See the definition of $|| \cdot ||_{1,e}$ in (3). We also obtain the following equivalent condition for the stability of F in the complete noncompact case.

Theorem 5. Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional smooth complete noncompact selfshrinker satisfying $H = -\frac{X^{\perp}}{2}$. Suppose that the second fundamental form A of Σ is of polynomial growth and Σ has polynomial volume growth. The following statements are equivalent:

(i) Σ is *F*-stable.

(ii) $\int_{\Sigma} \langle V, -L^{\perp}V \rangle e^{-\frac{|X|^2}{4}} d\mu \ge 0$ for any admissible vector field V in $\mathcal{H}^1_0(\Sigma)$.

Recall that the notion of admissible vector field is defined in Theorem 4.

In the case of hypersurfaces, $\mathbb{S}^n(\sqrt{2n})$ and \mathbb{R}^n are the only complete smooth F-stable self-shrinkers with polynomial volume growth [6]. When considered as self-shrinkers in \mathbb{R}^m for m > n + 1, they are still F-stable. But the stability for all other higher co-dimensional self-shrinkers is not clear. We will employ the above equivalent condition to investigate the F-stability of product self-shrinkers and Anciaux's Lagrangian self-shrinkers [2].

For smooth self-shrinkers $\Sigma_1^{n_1} \subset \mathbb{R}^{m_1}$ and $\Sigma_2^{n_2} \subset \mathbb{R}^{m_2}$, it is easy to see that $\Sigma = \Sigma_1 \times \Sigma_2$ is also a self-shrinker in $\mathbb{R}^{m_1+m_2}$. Conversely, considering a self-shrinker $\Sigma \subset \mathbb{R}^{m_1+m_2}$, if Σ can be expressed as $\Sigma_1^{n_1} \times \Sigma_2^{n_2}$ for smooth $\Sigma_1^{n_1} \subset \mathbb{R}^{m_1}$ and $\Sigma_2^{n_2} \subset \mathbb{R}^{m_2}$, then both $\Sigma_1^{n_1}$ and $\Sigma_2^{n_2}$ are self-shrinkers. Such Σ is called a product self-shrinker in this paper. In §4, we prove

Theorem 6. The n-plane is the only complete smooth F-stable product self-shrinker in \mathbb{R}^m whose volume and second fundamental form are of polynomial growth.

Now we introduce Anciaux's examples in [2]. They are *n*-dimensional selfshrinkers in \mathbb{C}^n , $n \geq 2$, and are expressed as $\gamma(s)\psi(\sigma)$, where $\psi: M^{n-1} \to \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ is a minimal Legendrian immersion and γ is a complex-valued function that satisfies the system of ordinary differential equations (25). Because the *F*-value is infinite on the complete noncompact Lagrangian examples constructed by Anciaux, we will only discuss the closed case. That is, the corresponding curves γ are closed and the immersions $\psi: M \to \mathbb{S}^{2n-1}$ are closed. We prove that

Theorem 7. Anciaux's closed embedded examples as described in Lemma 1 are *F*-unstable.

Since Anciaux's examples are Lagrangian in \mathbb{C}^n , it is natural to ask whether these examples are still *F*-unstable under the restricted Lagrangian variations. We have the following

Theorem 8. Anciaux's closed embedded examples are *F*-unstable under Lagrangian variations for the following cases:

(i) $n = 2 \text{ or } n \ge 7$,

(ii) 2 < n < 7, and $E \in \left[\sqrt{\frac{7-n}{8}}E_{max}, E_{max}\right]$,

where E and E_{max} are described in (26).

Theorems 7 and 8 also work for the case with transversal intersections. It will be interesting to understand whether Joyce-Lee-Tsui's Lagrangian self-shrinkers are F-stable or not. The answer to this question is still not clear to us. By a suggestion of Mu-Tao Wang, we recently also studied Hamiltonian stability of F-functional for Lagrangian self-shrinkers. We can prove that Clifford torus (the product of circles) are Hamiltonian F-stable and find that our variation in the proof of Theorem 8 is in fact a Hamiltonian variation for $n \geq 3$. It thus shows that the cases are Hamiltonian F-unstable. These and related issues will be investigated in a forthcoming paper.

We learned after this paper was finished that Andrews-Li-Wei also obtained part of the results in this paper independently [5]. However, they focused on the classification of self-shrinkers with parallel principal normal instead in the second part of their paper. We remark that part of our results was first presented by the second author at the annual meeting of the Taiwan Mathematical Society in December of 2010.

2. The 1st and 2nd variation formulae of F

2.1. Notation and preliminaries. Let $X : \Sigma^n \to \mathbb{R}^m$ be a smooth isometric immersion and continue to denote the image as Σ which has co-dimension m - n. Suppose $\{e_i\}$ and $\{e_\alpha\}$ are orthonormal frames for the tangent bundle $T\Sigma$ and the normal bundle $N\Sigma$, respectively. The coefficients of the second fundamental form and the mean curvature vector are defined to be

(2)
$$A_{ij} = A^{\alpha}_{ij} e_{\alpha} \equiv \langle \overline{\nabla}_{e_i} e_j, e_{\alpha} \rangle e_{\alpha}$$

and $H = H^{\alpha} e_{\alpha} \equiv A_{ii},$

where by convention we are summing over repeated indices and $\overline{\nabla}$ is the standard connection of the ambient Euclidean space. For a submanifold B in an ambient manifold C, we use $A^{B,C}$ and $H^{B,C}$ to denote the associated second fundamental form and mean curvature vector, respectively. When the ambient space is \mathbb{C}^n , we denote them as A^B (or A) and H^B (or H) for simplicity. For a normal vector field $V, \langle A, V \rangle$ is a (2,0)-tensor and $|\langle A, V \rangle|^2$ is defined as $\sum_{i,j=1}^n \langle A_{ij}, V \rangle^2$. When Σ is a hypersurface, the mean curvature vector H and the second fundamental form reduce to the function $h = -\langle H, \mathbf{n} \rangle$ and the 2-tensor $h_{ij} = -\langle A_{ij}, \mathbf{n} \rangle$, respectively. Here \mathbf{n} is the unit outer normal vector of Σ .

Definition 1. Let Σ be a submanifold in \mathbb{R}^m and $B_r(0)$ be the geodesic ball in \mathbb{R}^m with radius r. Σ is said to have *polynomial volume growth* if there are constants C_1, C_2 and $k \in \mathbb{N}$ so that for all $r \geq 0$

$$Vol(B_r(0) \cap \Sigma) \le C_1 r^k + C_2.$$

Definition 2. A normal vector field V (or the second fundamental form A) of Σ is of *polynomial growth* if there are constants C_1 , C_2 and $k \in \mathbb{N}$ so that for all $r \ge 0$

$$|V| \le C_1 r^k + C_2$$
 (or $|A| \le C_1 r^k + C_2$) on $B_r(0) \cap \Sigma$.

For any smooth normal vector fields V and W in the space of sections $\Gamma(N\Sigma)$, their weighted L^2 inner product is defined to be $\int_{\Sigma} \langle V, W \rangle e^{-\frac{|X|^2}{4}} d\mu$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^m . Denote this weighted L^2 inner product by $\langle V, W \rangle_e$ and call $(\Gamma(N\Sigma), \langle \cdot, \cdot \rangle_e)$ the weighted L^2 inner product space. For $V \in \Gamma(N\Sigma)$, we define norms $||V||_e = \langle V, V \rangle_e^{\frac{1}{2}}$ and

(3)
$$||V||_{1,e} = \left(\int_{\Sigma} |V|^2 e^{-\frac{|X|^2}{4}} d\mu\right)^{1/2} + \left(\int_{\Sigma} |\nabla^{\perp} V|^2 e^{-\frac{|X|^2}{4}} d\mu\right)^{1/2}.$$

Let $N_c(\Sigma)$ be the collection of all smooth normal vector fields in $\Gamma(N\Sigma)$ with compact support and denote the space $\mathcal{H}_0^1(\Sigma)$ as the closure of $N_c(\Sigma)$ with respect to the norm $|| \cdot ||_{1,e}$.

2.2. The first variation formula of F. Colding and Minicozzi derived the first and second variation formulae of the F-functional of a hypersurface in [6]. These can be generalized to higher co-dimensional cases by similar calculation. We derive the first variation formula of F in the following theorem.

Theorem 1. Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional complete manifold with polynomial volume growth. Suppose that $\Sigma_s \subset \mathbb{R}^m$ is a normal variation of Σ , x_s , t_s are variations of x_0 and t_0 , and

$$\frac{\partial \Sigma_s}{\partial s} = V, \quad \frac{dx_s}{ds} = y, \quad and \quad \frac{dt_s}{ds} = \tau,$$

where V has compact support. Then

$$\begin{aligned} \frac{\partial}{\partial s}F(\Sigma_s, x_s, t_s) &= \frac{1}{\sqrt{4\pi t_s}^n} \int_{\Sigma_s} \left(-\langle V, H_s + \frac{X_s - x_s}{2t_s} \rangle + \tau \left(\frac{|X_s - x_s|^2}{4t_s^2} - \frac{n}{2t_s} \right) \right. \\ (4) &\qquad \qquad + \frac{\langle X_s - x_s, y \rangle}{2t_s} \right) e^{\frac{-|X_s - x_s|^2}{4t_s}} d\mu, \end{aligned}$$

where X_s is the position vector of Σ_s and H_s is its mean curvature vector.

Proof. From the first variation formula for area, we have

(5)
$$\frac{\partial}{\partial s}(d\mu) = -\langle H_s, V \rangle d\mu.$$

The variation of the weight $\frac{1}{\sqrt{4\pi t_s}}e^{-|X_s-x_s|^2/4t_s}$ contains terms coming from the variation of X_s , the variation of x_s and the variation of t_s , respectively. Using the following equations:

$$\frac{\partial}{\partial t_s} \log\left((4\pi t_s)^{-n/2} e^{-\frac{|X_s - x_s|^2}{4t_s}}\right) = \frac{-n}{2t_s} + \frac{|X_s - x_s|^2}{4t_s^2},$$
$$\frac{\partial}{\partial x_s} \log\left((4\pi t_s)^{-n/2} e^{-\frac{|X_s - x_s|^2}{4t_s}}\right) = \frac{X_s - x_s}{2t_s}$$
and
$$\frac{\partial}{\partial X_s} \log\left((4\pi t_s)^{-n/2} e^{-\frac{|X_s - x_s|^2}{4t_s}}\right) = -\frac{X_s - x_s}{2t_s},$$

we obtain

$$\frac{\partial}{\partial s} \log\left((4\pi t_s)^{-n/2} e^{-\frac{|X_s - x_s|^2}{4t_s}}\right)$$
$$= -\frac{\langle X_s - x_s, V \rangle}{2t_s} + \tau\left(\frac{|X_s - x_s|^2}{4t_s^2} - \frac{n}{2t_s}\right) + \frac{1}{2t_s} \langle X_s - x_s, y \rangle.$$

Combining this with (5) gives (4).

Definition 3. We call (Σ, x_0, t_0) a *critical point* of F if it is critical with respect to all normal variations which have compact support in Σ and all variations in x and t.

From the definition of F in (1), we have $F(\Sigma, x, t) = F(\frac{\Sigma-x}{\sqrt{t}}, 0, 1)$ and it is easy to see the following property:

$$(\Sigma, x_0, t_0)$$
 is a critical point of F if and only if $(\frac{\Sigma-x_0}{\sqrt{t_0}}, 0, 1)$

(6) is a critical point of F.

Therefore, we only consider the case $x_0 = 0$, $t_0 = 1$. In the case of hypersurfaces, Colding and Minicozzi proved that $(\Sigma, 0, 1)$ is a critical point of F if Σ satisfies that $h = \frac{\langle X, \mathbf{n} \rangle}{2}$. Their result, when written in the vector form $H = -\frac{X^{\perp}}{2}$, also holds for higher co-dimensional cases. The proof needs the following propositions.

Proposition 1. If $\Sigma \subset \mathbb{R}^m$ is an n-dimensional complete submanifold with $H = -\frac{X^{\perp}}{2}$, then

(7)
$$\mathcal{L}X_i = -\frac{1}{2}X_i \quad and$$
$$\mathcal{L}|X|^2 = 2n - |X|^2.$$

Here X_i is the *i*-th component of the position vector X, *i.e.*, $X_i = \langle X, \partial_i \rangle$ and the linear operator $\mathcal{L}f = \Delta f - \frac{1}{2} \langle X, \nabla f \rangle = e^{\frac{|X|^2}{4}} div(e^{\frac{-|X|^2}{4}} \nabla f).$

Proposition 2. If $\Sigma \subset \mathbb{R}^m$ is an n-dimensional complete submanifold with polynomial volume growth, and $H = -\frac{X^{\perp}}{2}$, then

(8)
$$\int_{\Sigma} X e^{\frac{-|X|^2}{4}} d\mu = \overrightarrow{0} = \int_{\Sigma} X |X|^2 e^{\frac{-|X|^2}{4}} d\mu \quad and$$
$$\int_{\Sigma} (|X|^2 - 2n) e^{\frac{-|X|^2}{4}} d\mu = 0.$$

Moreover, if $W \in \mathbb{R}^m$ is a constant vector, then

(9)
$$\int_{\Sigma} \langle X, W \rangle^2 e^{-\frac{|X|^2}{4}} d\mu = 2 \int_{\Sigma} |W^{\top}|^2 e^{-\frac{|X|^2}{4}} d\mu$$

The proofs for these propositions are similar to the hypersurface case proved by Colding and Minicozzi (see Lemma 3.20 and Lemma 3.25 in [6]), and will thus be omitted. Combining (4), (6) and (8), we get

Proposition 3. For any $x_0 \in \mathbb{R}^m$, $t_0 \in \mathbb{R}^+$, (Σ, x_0, t_0) is a critical point of F if and only if $H = -\frac{(X-x_0)^{\perp}}{2t_0}$.

2.3. The general second variation formula of F.

Theorem 2. Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional complete submanifold with polynomial volume growth. Suppose that Σ_s is a normal variation of Σ , x_s , t_s are variations of x_0 and t_0 , and

$$\frac{\partial \Sigma_s}{\partial s} = V, \quad \frac{dx_s}{ds} = y, \quad \frac{dt_s}{ds} = \tau, \quad \frac{d^2x_s}{ds^2} = y', \quad and \quad \frac{d^2t_s}{ds^2} = \tau',$$

where V has compact support. Then

$$\begin{aligned} \frac{\partial^2 F}{\partial s^2}(\Sigma, x_0, t_0) \\ &= \frac{1}{\sqrt{4\pi t_0}^n} \int_{\Sigma} e^{-\frac{|X-x_0|^2}{4t_0}} \Big\{ -\langle V, L_{x_0, t_0}^{\perp} V \rangle + \frac{\langle X-x_0, V \rangle}{t_0^2} \tau + \frac{\langle V, y \rangle}{t_0} \\ &- \frac{(|X-x_0|^2 - nt_0)\tau^2}{2t_0^3} - \frac{|y|^2}{2t_0} - \frac{\tau \langle X-x_0, y \rangle}{t_0^2} \\ &+ \left(-\langle V, H + \frac{X-x_0}{2t_0} \rangle + \tau (\frac{|X-x_0|^2}{4t_0^2} - \frac{n}{2t_0}) + \langle \frac{X-x_0}{2t_0}, y \rangle \right)^2 \\ (10) &- \langle \overline{\nabla}_V^{\perp} V, H + \frac{X-x_0}{2t_0} \rangle + \tau' (\frac{|X-x_0|^2}{4t_0} - \frac{n}{2t_0}) + \frac{\langle X-x_0, y' \rangle}{2t_0} \Big\} d\mu, \end{aligned}$$

where

$$L_{x_0,t_0}^{\perp} V = \Delta^{\perp} V + \langle A_{ij}, V \rangle g^{ki} g^{jl} A_{kl} + \frac{V}{2t_0} - \frac{1}{2t_0} \nabla_{(X-x_0)^{\top}}^{\perp} V$$

and A_{ij} is the second fundamental form as defined in (2).

Proof. Apply one more derivative on equation (4). It gives

$$\begin{aligned} \frac{\partial^2 F}{\partial s^2}(\Sigma, x_0, t_0) \\ = & \frac{1}{\sqrt{4\pi t_0}^n} \int_{\Sigma} e^{-\frac{|X-x_0|}{4t_0}} \left\{ -\langle V, \frac{\partial}{\partial s} (H_s + \frac{X_s - x_s}{2t_s}) \Big|_{s=0} \rangle \right. \\ & + \tau \frac{\partial}{\partial s} (\frac{|X_s - x_s|^2}{4t_s^2} - \frac{n}{2t_s}) \Big|_{s=0} + \left\langle \frac{\partial}{\partial s} (\frac{X_s - x_s}{2t_s}) \Big|_{s=0}, y \right\rangle \\ & + \left(-\langle V, H + \frac{X - x_0}{2t_0} \rangle + \tau (\frac{|X - x_0|^2}{4t_0^2} - \frac{n}{2t_0}) + \left\langle (\frac{X - x_0}{2t_0}), y \right\rangle \right)^2 \\ (11) & - \langle V', (H + \frac{X - x_0}{2t_0}) \rangle + \tau' (\frac{|X - x_0|^2}{4t_0^2} - \frac{n}{2t_0}) + \left\langle (\frac{X - x_0}{2t_0}), y' \right\rangle \right\} d\mu. \end{aligned}$$

Similar to the derivation of the second variation formula for the area, we have

(12)
$$\langle (\frac{\partial H_s}{\partial s}), V \rangle = \langle \Delta^{\perp} V + \langle A_{ij}, V \rangle g^{ki} g^{jl} A_{kl}, V \rangle.$$

On the other hand, since the Lie bracket $[V, (\frac{X-x_0}{2t_0})^{\top}]$ is tangent to Σ_s , it follows that

(13)
$$\langle \overline{\nabla}_V^\top V, \frac{X - x_0}{2t_0} \rangle = -\langle V, \overline{\nabla}_V (\frac{X - x_0}{2t_0})^\top \rangle = -\langle V, \overline{\nabla}_{(\frac{X - x_0}{2t_0})^\top} V \rangle.$$

$$\begin{split} \text{Using } & \frac{\partial X_s}{\partial s} = V, \ \frac{dt_s^{-1}}{ds} = -\tau t_s^{-2} \text{ and } \frac{dx_s}{ds} = y, \text{ we simplify} \\ & - \langle V, \frac{\partial}{\partial s} (H_s + \frac{X_s - x_s}{2t_s}) \Big|_{s=0} \rangle - \langle V', H + \frac{X - x_0}{2t_0} \rangle \\ & = - \langle V, \frac{\partial H_s}{\partial s} \Big|_{s=0} \rangle - \langle V, \frac{\partial}{\partial s} (\frac{X_s - x_s}{2t_s}) \Big|_{s=0} \rangle - \langle \overline{\nabla}_V^{\perp} V, H + \frac{X - x_0}{2t_0} \rangle - \langle \overline{\nabla}_V^{\top} V, \frac{X - x_0}{2t_0} \rangle \\ & = - \langle V, L_{x_0, t_0}^{\perp} V \rangle - \langle \overline{\nabla}_V^{\perp} V, H + \frac{X - x_0}{2t_0} \rangle + \langle V, \frac{y}{2t_0} \rangle + \frac{\tau}{2t_0^2} \langle V, X - x_0 \rangle, \end{split}$$

where the second equality is from (12), (13), and the definition of L_{x_0,t_0}^{\perp} . The second term in (11) is given by

$$\begin{aligned} \frac{\partial}{\partial s} \left(\frac{|X_s - x_s|^2}{4t_s^2} - \frac{n}{2t_s} \right) \Big|_{s=0} &= \frac{\langle X - x_0, V - y \rangle}{2t_0^2} - \frac{\tau |X - x_0|^2}{2t_0^3} + \frac{n\tau}{2t_0^2} \\ &= \frac{\langle X - x_0, V \rangle}{2t_0^2} - \frac{|X - x_0|^2 - nt_0}{2t_0^3} \tau - \frac{\langle X - x_0, y \rangle}{2t_0^2}. \end{aligned}$$

For the third term in (11), observe that

$$\left\langle \frac{\partial}{\partial s} \left(\frac{X_s - x_s}{2t_s} \right) \right|_{s=0}, y \rangle = \left\langle \frac{V}{2t_0}, y \right\rangle - \frac{|y|^2}{2t_0} - \frac{\tau}{2t_0^2} \langle X - x_0, y \rangle.$$

Combining these gives the theorem.

2.4. The second variation formula at a critical point. From now on we denote $\frac{\partial^2 F}{\partial s^2}(\Sigma, 0, 1)$ in (10) as $D^2_{(V,y,\tau)}F$ for clarity. When $(\Sigma, 0, 1)$ is a critical point of F, we have $H = -\frac{X^{\perp}}{2}$ and the second variation formula of F at $(\Sigma, 0, 1)$ can be simplified as the following:

Theorem 3. Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional complete submanifold with polynomial volume growth. Suppose that Σ_s is a normal variation of Σ , x_s , t_s are variations of $x_0 = 0$ and $t_0 = 1$, and

$$\frac{\partial \Sigma_s}{\partial s}\Big|_{s=0} = V, \quad \frac{dx_s}{ds}\Big|_{s=0} = y, \quad \frac{dt_s}{ds}\Big|_{s=0} = \tau,$$

where V has compact support. If $(\Sigma, 0, 1)$ is a critical point of F, then

$$D^{2}_{(V,y,\tau)}F$$
(14)
$$=\frac{1}{\sqrt{4\pi^{n}}}\int_{\Sigma} \left(-\langle V, L^{\perp}V\rangle - 2\tau\langle H, V\rangle - \tau^{2}|H|^{2} + \langle V, y\rangle - \frac{1}{2}|y^{\perp}|^{2}\right)e^{-\frac{|X|^{2}}{4}}d\mu.$$

Here the operator $L^{\perp} = L_{0,1}^{\perp}$, and

(15)
$$L^{\perp}V = \Delta^{\perp}V + \langle A_{ij}, V \rangle g^{ki}g^{jl}A_{kl} + \frac{V}{2} - \frac{1}{2}\nabla^{\perp}_{X^{\top}}V.$$

Proof. Since $(\Sigma, 0, 1)$ is a critical point of F, by (4) we have that

(16)
$$H = -\frac{X^{\perp}}{2}$$

It follows from (8) that (17)

$$\int_{\Sigma} X e^{\frac{-|X|^2}{4}} d\mu = \overrightarrow{0} = \int_{\Sigma} X |X|^2 e^{\frac{-|X|^2}{4}} d\mu \quad \text{and} \quad \int_{\Sigma} (|X|^2 - 2n) e^{\frac{-|X|^2}{4}} d\mu = 0.$$

Theorem 2 (with $x_0 = 0$ and $t_0 = 1$) gives

$$D_{(V,y,\tau)}^{2}F = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left(-\langle V, L^{\perp}V \rangle + \tau \langle X, V \rangle + \langle V, y \rangle - \frac{(|X|^{2} - n)\tau^{2}}{2} - \frac{|y|^{2}}{2} - \tau \langle X, y \rangle + \left\{ \tau (\frac{|X|^{2}}{4} - \frac{n}{2}) + \langle \frac{X}{2}, y \rangle \right\}^{2} \right) e^{-\frac{|X|^{2}}{4}} d\mu,$$

where we use (16) and (17) to conclude the vanishing of a few terms in (10). Note that y is a constant vector and τ is a constant. Squaring out the last term of $D^2_{(V,y,\tau)}F$ and using (16) and (17) again leads to

$$D_{(V,y,\tau)}^{2}F = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left(-\langle V, L^{\perp}V \rangle - 2\tau \langle H, V \rangle + \langle V, y \rangle - \frac{|y|^{2}}{2} + \tau^{2} (\frac{|X|^{2}}{4} - \frac{n}{2})^{2} + \frac{1}{4} \langle X, y \rangle^{2} - \frac{(|X|^{2} - n)\tau^{2}}{2} \right) e^{-\frac{|X|^{2}}{4}} d\mu.$$

Using the equality (7) and Stokes' theorem, we have that

$$\begin{split} \int_{\Sigma} \tau^2 (\frac{|X|^2}{4} - \frac{n}{2})^2 e^{-\frac{|X|^2}{4}} d\mu &= -\int_{\Sigma} \tau^2 (\frac{|X|^2}{4} - \frac{n}{2}) \mathcal{L} \frac{|X|^2}{4} e^{-\frac{|X|^2}{4}} d\mu \\ &= \int_{\Sigma} \tau^2 \nabla \frac{|X|^2}{4} \cdot \nabla \frac{|X|^2}{4} e^{-\frac{|X|^2}{4}} d\mu \\ &= \int_{\Sigma} \tau^2 \frac{|X|^2 - |X|^2}{4} e^{-\frac{|X|^2}{4}} d\mu \\ &= \int_{\Sigma} \tau^2 \frac{|X|^2 - |X^{\perp}|^2}{4} e^{-\frac{|X|^2}{4}} d\mu. \end{split}$$

Combining (8) and (9), the second variation $D^2_{(V,y,\tau)}F$ can be further simplified as

$$\frac{1}{\sqrt{4\pi^n}} \int_{\Sigma} \left(-\langle V, L^{\perp}V \rangle - 2\tau \langle H, V \rangle - \tau^2 |H|^2 + \langle V, y \rangle - \frac{1}{2} |y^{\perp}|^2 \right) e^{-\frac{|X|^2}{4}} d\mu.$$

In [6], Colding and Minicozzi defined the following concept.

Definition 4. A critical point $(\Sigma, 0, 1)$ of F is called F-stable if for every compactly supported smooth variation Σ_s with $\Sigma_0 = \Sigma$ and $\frac{\partial \Sigma_s}{\partial s}\Big|_{s=0} = V$, there exist variations x_s of 0 and t_s of 1 such that $D^2_{(V,y,\tau)}F \ge 0$, where $y = \frac{dx_s}{ds}\Big|_{s=0}$ and $\tau = \frac{dt_s}{ds}\Big|_{s=0}$.

Remark 1. When Σ is fixed, i.e. V = 0, we see from (14) that the second variation formula of F is nonpositive for any variations of x_s and t_s .

3. An equivalent condition for F-stability

3.1. Vector-valued eigenfunctions and eigenvalues of L^{\perp} . Let $X : \Sigma \to \mathbb{R}^m$ be a closed self-shrinker. Recall that the second order operator L^{\perp} is defined by

$$L^{\perp}V = \Delta^{\perp}V - \frac{1}{2}\nabla^{\perp}_{X^{\top}}V + \langle A_{ij}, V \rangle g^{ki}g^{jl}A_{kl} + \frac{V}{2}$$

for $V \in \Gamma(N\Sigma)$. Therefore, we have

$$\int_{\Sigma} \langle -L^{\perp}V, W \rangle e^{-\frac{|X|^2}{4}} d\mu$$

=
$$\int_{\Sigma} \left(\langle \nabla^{\perp}V, \nabla^{\perp}W \rangle - \langle A_{ij}, V \rangle \langle A_{kl}, W \rangle g^{ik} g^{jl} - \frac{1}{2} \langle V, W \rangle \right) e^{-\frac{|X|^2}{4}} d\mu$$

for $V, W \in \Gamma(N\Sigma)$. It is easy to see that the operator L^{\perp} is self-adjoint in the weighted L^2 inner product space and the standard spectral theory also works. Direct computation leads to the following proposition.

Proposition 4. Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional smooth complete self-shrinker satisfying $H = -\frac{X^{\perp}}{2}$. Then the mean curvature vector H and the normal projection y^{\perp} of a constant vector field y satisfy

(18)
$$L^{\perp}H = H \text{ and } L^{\perp}y^{\perp} = \frac{1}{2}y^{\perp},$$

respectively. That is, they are vector-valued eigenfunctions of L^{\perp} with eigenvalue 1 and $\frac{1}{2}$, respectively.

Proof. Fix $p \in \Sigma$ and choose an orthonormal frame $\{e_i\}$ in a neighborhood of p such that $\nabla_{e_i} e_j(p) = 0$, $g_{ij}(p) = \delta_{ij}$. Using $H = -\frac{1}{2}X^{\perp}$, we have

(19)
$$\nabla_{e_i}^{\perp} H = \nabla_{e_i}^{\perp} (-\frac{1}{2} X^{\perp}) = \frac{1}{2} \nabla_{e_i}^{\perp} (\langle X, e_j \rangle e_j - X) = \frac{1}{2} \langle X, e_j \rangle A_{ij}.$$

In the second equality of (19), we use $X^{\top} = \langle X, e_j \rangle e_j$. Taking another covariant derivative at p, it gives

(20)
$$\nabla_{e_k}^{\perp} \nabla_{e_i}^{\perp} H = \frac{1}{2} (\nabla_{e_k} \langle X, e_j \rangle) A_{ij} + \frac{1}{2} \langle X, e_j \rangle \nabla_{e_k}^{\perp} A_{ij}$$
$$= \frac{1}{2} A_{ik} + \frac{1}{2} \langle X, A_{kj} \rangle A_{ij} + \frac{1}{2} \langle X, e_j \rangle \nabla_{e_j}^{\perp} A_{ik}$$

where $\nabla_{e_k} e_j(p) = 0$; the Codazzi equation and (19) are used in the last equality. Taking the trace of (20) and using $H = -\frac{1}{2}X^{\perp}$, we conclude that

$$\Delta^{\perp} H = \frac{1}{2} H - \langle H, A_{ij} \rangle A_{ij} + \frac{1}{2} \nabla_{X^{\top}}^{\perp} H$$

Therefore,

$$L^{\perp}H = \Delta^{\perp}H - \frac{1}{2}\nabla_{X^{\top}}^{\perp}H + \langle A_{ij}, H \rangle A_{ij} + \frac{1}{2}H = H.$$

For a constant vector y in \mathbb{R}^m , the covariant derivative of y^{\perp} is

(21)
$$\nabla_{e_i}^{\perp} y^{\perp} = \nabla_{e_i}^{\perp} (y - \langle y, e_j \rangle e_j) = -\langle y, e_j \rangle A_{ij}.$$

Taking another covariant derivative at p, it gives

(22)
$$\nabla_{e_k}^{\perp} \nabla_{e_i}^{\perp} y^{\perp} = -(\nabla_{e_k} \langle y, e_j \rangle) A_{ij} - \langle y, e_j \rangle \nabla_{e_k}^{\perp} A_{ij}$$
$$= -\langle y, A_{kj} \rangle A_{ij} - \langle y, e_j \rangle \nabla_{e_j}^{\perp} A_{ki}$$

by applying $\nabla_{e_k} e_j(p) = 0$ and the Codazzi equation. Taking the trace of (22) and using (19), (21), we conclude that

$$\begin{split} \Delta^{\perp}y^{\perp} &= -\langle y, A_{ij} \rangle A_{ij} - \langle y, e_j \rangle \nabla_{e_j}^{\perp} H \\ &= -\langle y^{\perp}, A_{ij} \rangle A_{ij} - \frac{1}{2} \langle y, e_j \rangle \langle X, e_i \rangle A_{ij} \\ &= -\langle y^{\perp}, A_{ij} \rangle A_{ij} + \frac{1}{2} \langle X, e_i \rangle \nabla_{e_i}^{\perp} y^{\perp} \\ &= -\langle y^{\perp}, A_{ij} \rangle A_{ij} + \frac{1}{2} \nabla_{X^{\top}}^{\perp} y^{\perp}. \end{split}$$

Therefore,

$$L^{\perp}y^{\perp} = \Delta^{\perp}y^{\perp} - \frac{1}{2}\nabla^{\perp}_{X^{\top}}y^{\perp} + \langle A_{ij}, y^{\perp} \rangle A_{ij} + \frac{1}{2}y^{\perp} = \frac{1}{2}y^{\perp}.$$

For the hypersurface case, we have the following immediate corollary.

Corollary 1 (Theorem 5.2 in [6]). Let $\Sigma \subset \mathbb{R}^{n+1}$ be an n-dimensional smooth complete self-shrinker satisfying $h = \frac{\langle X, \mathbf{n} \rangle}{2}$, where $h = -\langle H, \mathbf{n} \rangle$ is the mean curvature function. Then h and the functions $\langle y, \mathbf{n} \rangle$ for constant vector fields y are eigenfunctions of L with Lh = h and $L\langle y, \mathbf{n} \rangle = \frac{1}{2} \langle y, \mathbf{n} \rangle$, where

$$Lf = \Delta f - \frac{1}{2} \langle X, \nabla f \rangle + |\langle A_{ij}, n \rangle|^2 f + \frac{1}{2} f.$$

3.2. An equivalent condition. In this subsection, we give an equivalent condition for the stability of a critical point $(\Sigma, 0, 1)$ of F. Namely, it is F-stable if and only if H and y^{\perp} are the only vector-valued eigenfunctions of L^{\perp} with positive eigenvalue, where y in \mathbb{R}^m is a constant vector field. The formulation is inspired by the proof of Lemma 4.23 of Colding and Minicozzi in [6].

Theorem 4. Suppose $\Sigma \subset \mathbb{R}^m$ is an n-dimensional smooth closed self-shrinker satisfying $H = -\frac{X^{\perp}}{2}$. The following statements are equivalent: (i) Σ is F-stable.

(ii) $\int_{\Sigma} \langle V, -L^{\perp}V \rangle e^{-\frac{|X|^2}{4}} d\mu \geq 0$ for any admissible vector field V, namely, a smooth normal vector field V which satisfies

(23)
$$\int_{\Sigma} \langle V, H \rangle e^{-\frac{|X|^2}{4}} d\mu = 0 \quad and \quad \int_{\Sigma} \langle V, y^{\perp} \rangle e^{-\frac{|X|^2}{4}} d\mu = 0$$

for any constant vector $y \in \mathbb{R}^m$.

Proof. (i) \Rightarrow (ii) Assume to the contrary that there is an admissible vector field V satisfying $\int_{\Sigma} \langle V, -L^{\perp}V \rangle e^{\frac{-|X|^2}{4}} d\mu < 0$. For any real value τ and constant vector yin \mathbb{R}^m , using the second variation formula (14) of F, we have

$$\begin{split} &D_{(V,y,\tau)}^2 F\\ =& \frac{1}{\sqrt{4\pi}^n} \int_{\Sigma} \left(-\langle V, L^{\perp}V \rangle - 2\tau \langle H, V \rangle - \tau^2 |H|^2 + \langle V, y \rangle - \frac{1}{2} |y^{\perp}|^2 \right) e^{-\frac{|X|^2}{4}} d\mu \\ =& \frac{1}{\sqrt{4\pi}^n} \int_{\Sigma} \left(-\langle V, L^{\perp}V \rangle - \tau^2 |H|^2 - \frac{1}{2} |y^{\perp}|^2 \right) e^{-\frac{|X|^2}{4}} d\mu \\ <& 0. \end{split}$$

This contradicts the stability of F.

(ii) \Rightarrow (i) From the standard spectral theory, a smooth normal vector field V can be decomposed as $aH + z^{\perp} + V_0$, where aH and z^{\perp} are the projections of V with respect to weighted L^2 inner product into H and $\{y^{\perp}|y \in \mathbb{R}^m\}$, respectively. Note that V_0 is an admissible vector field. For any real value τ and constant vector $y \in \mathbb{R}^m$, by plugging the decomposition of V into (14), we have

$$\begin{split} &D_{(V,y,\tau)}^{2}F\\ =&\frac{1}{\sqrt{4\pi^{n}}}\int_{\Sigma}\left(-\langle V,L^{\perp}V\rangle-2\tau\langle H,V\rangle-\tau^{2}|H|^{2}+\langle V,y\rangle-\frac{1}{2}|y^{\perp}|^{2}\right)e^{-\frac{|X|^{2}}{4}}d\mu\\ =&\frac{1}{\sqrt{4\pi^{n}}}\int_{\Sigma}\left(-a^{2}|H|^{2}-\frac{1}{2}|z^{\perp}|^{2}-\langle V_{0},L^{\perp}V_{0}\rangle-2\tau a|H|^{2}-\tau^{2}|H|^{2}\\ &+\langle z^{\perp},y^{\perp}\rangle-\frac{1}{2}|y^{\perp}|^{2}\right)e^{-\frac{|X|^{2}}{4}}d\mu\\ \geq&\frac{1}{\sqrt{4\pi^{n}}}\int_{\Sigma}\left(-|H|^{2}(a+\tau)^{2}-\frac{1}{2}|z^{\perp}-y^{\perp}|^{2}\right)e^{-\frac{|X|^{2}}{4}}d\mu, \end{split}$$

where condition (ii) is used in the last inequality. Choosing $\tau = -a$ and y = z gives $D^2_{(V,z,-a)}F \ge 0$. That is, Σ is *F*-stable.

Recall that $\mathcal{H}_0^1(\Sigma)$ is the closure of $N_c(\Sigma)$ with respect to the norm $||\cdot||_{1,e}$, where $N_c(\Sigma)$ is the set of smooth normal vector fields with compact support. When Σ is a smooth self-shrinker with polynomial volume growth and its second fundamental form A is of polynomial growth, it is easy to see that $|H|_e$ and $|y^{\perp}|_e$ are finite. Hence H and y^{\perp} all belong to $\mathcal{H}_0^1(\Sigma)$. For any $V \in \mathcal{H}_0^1(\Sigma)$, the integral $\int_{\Sigma} \left(|\nabla^{\perp} V|^2 - |\langle A, V \rangle|^2 - \frac{1}{2} |V|^2 \right) e^{-\frac{|X|^2}{4}} d\mu$ is finite and

(24)
$$\langle V, -L^{\perp}V \rangle_{e} = \int_{\Sigma} \left(|\nabla^{\perp}V|^{2} - |\langle A, V \rangle|^{2} - \frac{1}{2}|V|^{2} \right) e^{-\frac{|X|^{2}}{4}} d\mu,$$

if Σ has no boundary. We also obtain the following equivalent condition for F-stability in the complete noncompact case.

Theorem 5. Let $\Sigma \subset \mathbb{R}^m$ be an n-dimensional smooth complete noncompact selfshrinker satisfying $H = -\frac{X^{\perp}}{2}$. Suppose that the second fundamental form A of Σ is of polynomial growth and Σ has polynomial volume growth. The following statements are equivalent:

(i)
$$\Sigma$$
 is *F*-stable.

(ii) $\int_{\Sigma} \langle V, -L^{\perp}V \rangle e^{-\frac{|X|^2}{4}} d\mu \ge 0$ for any admissible vector field V in $\mathcal{H}_0^1(\Sigma)$.

Remark 2. The conditions can be weakened to that the integrals $\int_{\Sigma} e^{-\frac{|X|^2}{4}} d\mu$ and $\int_{\Sigma} |A|^2 e^{-\frac{|X|^2}{4}} d\mu$ are finite. Admissible vector fields are characterized by (23).

Proof. (i) \Rightarrow (ii) Assume to the contrary that there is an admissible vector field V in $\mathcal{H}_0^1(\Sigma)$ satisfying $\langle V, -L^{\perp}V \rangle_e < 0$. Here V may not be compact supported. For $j \in \mathbb{N}$, consider smooth functions $\phi_j : \mathbb{R}^+ \bigcup \{0\} \to \mathbb{R}$ that satisfy $0 \le \phi_j \le 1$, $\phi_j \equiv 1$ on $[0, j), \phi_j \equiv 0$ outside [0, j + 2) and $|\phi'_j| \le 1$. Define cutoff functions $\psi_j(X) = \phi_j(\rho(X))$, where $X \in \Sigma$ and $\rho(X)$ is the distance function from a fixed point $p \in \Sigma$ to X with respect to the metric g_{ij} . Let $V_j(X) = \psi_j(X)V(X)$; then we have

$$\begin{split} |\nabla^{\perp} V_j|^2 &= \sum_{i=1}^n |(\nabla_{e_i} \psi_j) V + \psi_j \nabla_{e_i}^{\perp} V|^2 \\ &\leq 2 |\nabla \psi_j|^2 |V|^2 + 2 |\psi_j|^2 |\nabla^{\perp} V|^2 \\ &\leq 2 |V|^2 + 2 |\nabla^{\perp} V|^2. \end{split}$$

Here $\{e_i\}$ is an orthonormal basis for $T_X\Sigma$. Since the second fundamental form A of Σ is of polynomial growth and $V \in \mathcal{H}^1_0(\Sigma)$, the weighted L^2 inner product $\langle V, -L^{\perp}V \rangle_e$ is finite. Using the dominant convergence theorem and the admissible condition, it follows that

$$\lim_{j \to \infty} \langle V_j, -L^{\perp} V_j \rangle_e = \langle V, -L^{\perp} V \rangle_e \text{ and } \lim_{j \to \infty} \langle V_j, H \rangle_e = \lim_{j \to \infty} \langle V_j, y^{\perp} \rangle_e = 0.$$

For any small positive ϵ , choose a sufficiently large j such that

$$\langle V_j, -L^{\perp} V_j \rangle_e < \frac{1}{2} \langle V, -L^{\perp} V \rangle_e < 0, |\langle V_j, H \rangle_e| < \epsilon |H|_e, \text{ and } \max_{|y^{\perp}|_e = 1} |\langle V_j, y^{\perp} \rangle_e| < \epsilon.$$

For any real value τ and constant vector y in \mathbb{R}^m , we get

$$D^{2}F_{(V_{j},y,\tau)} = \frac{1}{\sqrt{4\pi^{n}}} \int_{\Sigma} \left(-\langle V_{j}, L^{\perp}V_{j} \rangle - 2\tau \langle H, V_{j} \rangle - \tau^{2} |H|^{2} + \langle V_{j}, y^{\perp} \rangle - \frac{1}{2} |y^{\perp}|^{2} \right) e^{-\frac{|X|^{2}}{4}} d\mu$$

$$< \frac{1}{\sqrt{4\pi^{n}}} \left(-\frac{1}{2} \langle V, L^{\perp}V \rangle_{e} + 2\tau\epsilon |H|_{e} - \tau^{2} |H|_{e}^{2} + \epsilon |y^{\perp}|_{e} - \frac{1}{2} |y^{\perp}|_{e}^{2} \right)$$

$$= \frac{1}{\sqrt{4\pi^{n}}} \left(-\frac{1}{2} \langle V, L^{\perp}V \rangle_{e} + \epsilon^{2} - (\tau |H|_{e} - \epsilon)^{2} + \frac{1}{2} \epsilon^{2} - \frac{1}{2} (|y^{\perp}|_{e} - \epsilon)^{2} \right).$$

Choosing $\epsilon^2 < \frac{1}{10} \langle V, L^{\perp}V \rangle_e$, we get $D^2 F_{(V_j, y, \tau)} < 0$ for every τ and y. This contradicts the stability of F.

(ii) \Rightarrow (i) From the standard spectral theory, a smooth normal vector field $V \in \mathcal{H}_0^1(\Sigma)$ can be decomposed as $aH + z^{\perp} + V_0$, where V_0 is an admissible vector field. Here we use the fact that V, H, and z^{\perp} belong to $\mathcal{H}_0^1(\Sigma)$ to conclude that V_0 belongs to $\mathcal{H}_0^1(\Sigma)$, too. The remaining part of the proof is essentially the same as the proof of (ii) \Rightarrow (i) in Theorem 4.

4. Classification of stable product self-shrinkers

In this section, we discuss the F-stability of product self-shrinkers.

Theorem 6. The n-plane is the only complete smooth F-stable product self-shrinker in \mathbb{R}^m whose volume and second fundamental form are of polynomial growth.

Proof. The mean curvature H of $\Sigma_1 \times \Sigma_2$ is equal to $(H_1, H_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ and $\Sigma_1 \times \Sigma_2$ is a self-shrinker because $H_1 = -\frac{X_1^{\perp}}{2}$ and $H_2 = -\frac{X_2^{\perp}}{2}$. Since a smooth minimal self-shrinker must be a plane through 0 (Corollary 2.8 in [6]), there are only two cases, namely $\Sigma_1 \times \Sigma_2$ with $H_1 \neq 0, H_2 \neq 0$ or $\Sigma_1 \times \mathbb{R}^{n_2}$ with $H_1 \neq 0$, needing to be excluded. By Theorem 4 and Theorem 5, it suffices to construct an admissible vector field V with $\int_{\Sigma_1 \times \Sigma_2} \langle V, -L^{\perp}V \rangle e^{-\frac{|X|^2}{4}} d\mu < 0$.

For the first case, let $V = (aH_1, bH_2)$, where a and b would be chosen later. Note that V is not a zero vector field since $H_i \neq 0$ for i = 1, 2. The first integral in the admissible condition is

$$\begin{split} & \int_{\Sigma_1 \times \Sigma_2} \langle V, H \rangle e^{-\frac{|X|^2}{4}} d\mu \\ &= \int_{\Sigma_1} \int_{\Sigma_2} (a|H_1|^2 + b|H_2|^2) e^{-\frac{|X_1|^2}{4}} e^{-\frac{|X_2|^2}{4}} d\mu_2 d\mu_1 \\ &= a \int_{\Sigma_1} |H_1|^2 e^{-\frac{|X_1|^2}{4}} d\mu_1 \int_{\Sigma_2} e^{-\frac{|X_2|^2}{4}} d\mu_2 + b \int_{\Sigma_1} e^{-\frac{|X_1|^2}{4}} d\mu_1 \int_{\Sigma_2} |H_2|^2 e^{-\frac{|X_2|^2}{4}} d\mu_2. \end{split}$$

Now choose nonzero constants a and b such that $\int_{\Sigma_1 \times \Sigma_2} \langle V, H \rangle e^{-\frac{|X^2|}{4}} d\mu = 0$. Recall that H_i and y_i^{\perp} are vector-valued eigenfunctions of L_i^{\perp} for $y_i \in \mathbb{R}^{m_i}$, where L_i^{\perp} is the corresponding operator on Σ_i , i = 1, 2. Hence the second integral in the admissible condition for $y = (y_1, y_2) \in \mathbb{R}^{m_1 + m_2}$ is

$$\int_{\Sigma_1 \times \Sigma_2} \langle V, y^{\perp} \rangle e^{-\frac{|X_1|^2}{4}} d\mu$$

= $a \int_{\Sigma_1} \langle H_1, y_1 \rangle e^{-\frac{|X_1|^2}{4}} d\mu_1 \int_{\Sigma_2} e^{-\frac{|X_2|^2}{4}} d\mu_2 + b \int_{\Sigma_1} e^{-\frac{|X_1|^2}{4}} d\mu_1 \int_{\Sigma_2} \langle H_2, y_2 \rangle e^{-\frac{|X_2|^2}{4}} d\mu_2$
=0.

Hence V is an admissible vector field and the weighted L^2 inner product $\langle V,-L^\perp V\rangle_e$ can be computed as

$$\begin{split} &\int_{\Sigma_1 \times \Sigma_2} \langle V, -L^{\perp} V \rangle e^{-\frac{|X|^2}{4}} d\mu \\ &= \int_{\Sigma_1 \times \Sigma_2} \langle (aH_1, bH_2), -(aH_1, bH_2) \rangle e^{-\frac{|X|^2}{4}} d\mu \\ &= -a^2 \int_{\Sigma_1} |H_1|^2 e^{-\frac{|X_1|^2}{4}} d\mu_1 \int_{\Sigma_2} e^{-\frac{|X_2|^2}{4}} d\mu_2 - b^2 \int_{\Sigma_1} e^{-\frac{|X_1|^2}{4}} d\mu_1 \int_{\Sigma_2} |H_2|^2 e^{-\frac{|X_2|^2}{4}} d\mu_2 \\ &< 0. \end{split}$$

It shows that $\Sigma_1 \times \Sigma_2$ is *F*-unstable.

For the second case, choose $V = s(H_1, 0)$, where s is the first component coordinate function in \mathbb{R}^{n_2} . Using the fact that $\int_{\mathbb{R}^{n_2}} se^{-\frac{|X_2|^2}{4}} d\mu_2 = 0$, the first integral in the admissible condition is

$$\int_{\Sigma_1 \times \mathbb{R}^{n_2}} \langle V, H \rangle e^{-\frac{|X_1|^2}{4}} d\mu = \int_{\mathbb{R}^{n_2}} s e^{-\frac{|X_2|^2}{4}} d\mu_2 \int_{\Sigma_1} |H_1|^2 e^{-\frac{|X_1|^2}{4}} d\mu_1 = 0$$

and the second integral in the admissible condition is

$$\int_{\Sigma_1 \times \mathbb{R}^{n_2}} \langle V, y^{\perp} \rangle e^{-\frac{|X|^2}{4}} d\mu = \int_{\mathbb{R}^{n_2}} s e^{-\frac{|X_2|^2}{4}} d\mu_2 \int_{\Sigma_1} \langle H_1, y_1^{\perp} \rangle e^{-\frac{|X_1|^2}{4}} d\mu_1 = 0$$

for $y = (y_1, y_2) \in \mathbb{R}^{m_1+m_2}$. Therefore, the smooth normal vector field $V = s(H_1, 0)$ is an admissible vector field and belongs to $\mathcal{H}_0^1(\Sigma)$. Using the fact $L^{\perp}H = H$, direct computation shows that V is a vector-valued eigenfunction of L^{\perp} with eigenvalue $\frac{1}{2}$ and the weighted L^2 inner product

$$\langle V, -L^{\perp}V\rangle_e = \int_{\Sigma_1 \times \mathbb{R}^{n_2}} -\frac{1}{2}|V|^2 e^{-\frac{|X|^2}{4}}d\mu < 0.$$

Hence $\Sigma_1 \times \mathbb{R}^{n_2}$ is *F*-unstable.

5. The unstability of Anciaux's examples

5.1. Anciaux's examples. Let $\langle \langle \cdot, \cdot \rangle \rangle = \sum_{i=1}^{n} dz_i \otimes d\overline{z}_i$ be the standard Hermitian inner product on \mathbb{C}^n , where $z_i = x_i + \sqrt{-1}y_i$, $i = 1, \ldots, n$ are the standard complex coordinates. It gives the standard Riemannian metric $\langle \cdot, \cdot \rangle = \operatorname{Re}\langle \langle \cdot, \cdot \rangle \rangle = \sum_{i=1}^{n} (dx_i^2 + dy_i^2)$ and standard symplectic form $\omega(\cdot, \cdot) = -\operatorname{Im}\langle \langle \cdot, \cdot \rangle \rangle = \sum_{i=1}^{n} dx_i \wedge dy_i$ on \mathbb{C}^n . We have $\omega(\cdot, \cdot) = \langle J \cdot, \cdot \rangle$, where $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$ and $J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$.

Recall that an immersion ψ from an (n-1)-dimensional manifold M into \mathbb{S}^{2n-1} is said to be Legendrian if $\alpha|_{\psi(M)} = 0$, where $\alpha(\cdot) = \omega(X, \cdot)$ is the contact 1-form on \mathbb{S}^{2n-1} induced from the standard symplectic form ω on \mathbb{C}^n and X is the position vector for points in \mathbb{S}^{2n-1} . Moreover, $d\alpha = 2\omega$ and on a Legendrian $\psi(M)$ we have $\langle Jy, z \rangle = \omega(y, z) = \frac{1}{2} d\alpha(y, z) = 0$, $\langle JX^M, y \rangle = \omega(X^M, y) = \alpha(y) = 0$ for all y, $z \in T\psi(M)$. For $y \in T\psi(M) \subset T(\mathbb{S}^{2n-1})$, we always have $\langle X^M, y \rangle = 0$, therefore it follows that y, Jz, X^M , and JX^M are mutually orthogonal for any $y, z \in T\psi(M)$. It is easy to see that the complex scalar product $\gamma\psi$ of a smooth regular curve $\gamma : I \to \mathbb{C}^*$ and ψ is a Lagrangian submanifold in \mathbb{C}^n , i.e., $\omega|_{\gamma\psi} \equiv 0$. Anciaux proved the following result in [2].

Lemma 1 (Anciaux [2]). Let $\psi : M \to \mathbb{S}^{2n-1}$ be a minimal Legendrian immersion for $n \geq 2$ and $\gamma : I \to \mathbb{C}^*$ be a smooth regular curve parameterized by its arclength s. Then the immersion

$$\begin{array}{ll} \gamma \ast \psi : I \times M & \to \mathbb{C}^n \\ (s, \sigma) & \to \gamma(s) \psi(\sigma) \end{array}$$

is a Lagrangian. Moreover, $\gamma * \psi$ satisfies the self-shrinker equation

$$H + \frac{1}{2}(\gamma * \psi)^{\perp} = 0$$

if and only if γ satisfies the following system of ordinary differential equations:

(25)
$$\begin{cases} r'(s) = \cos(\theta - \phi), \\ \theta'(s) - \phi'(s) = (\frac{r}{2} - \frac{n}{r})\sin(\theta - \phi). \end{cases}$$

where the curve γ is denoted as $r(s)e^{i\phi(s)}$ and θ is the angle between the tangent vector of γ and the positive x-axis. From (25), we can derive a conservation law

(26)
$$r^n e^{-\frac{r^2}{4}} \sin(\theta - \phi) = E,$$

where $0 < E \leq E_{max} = (\frac{2n}{e})^{n/2}$ is a constant determined by the initial data $(r(s_0), \theta(s_0) - \phi(s_0)).$

5.2. The unstability for general variations. Because Anciaux's complete noncompact examples are contained in a ball of finite radius, their F-functional values will be infinite. Hence we will only discuss the closed cases.

Theorem 7. Anciaux's closed embedded examples as described in Lemma 1 are *F*-unstable.

To prove the result, we first set up the notation and derive a few lemmas. Any point $p \in \Sigma = \gamma * \psi(I \times M)$ can be represented as $\gamma(s_0)q$ for some $s_0 \in I$ and $q \in \psi(M)$. Choose a local normal coordinate system x^1, \ldots, x^{n-1} at q. Denote $u_s = \frac{\partial X}{\partial s} = \gamma' X^M$, $e_i = \frac{\partial X^M}{\partial x^i}$, and $u_i = \frac{\partial X}{\partial x^i} = \gamma e_i$ for $i = 1, \ldots, n-1$, where X^M is the position vector of $\psi(M)$ and $X = \gamma X^M$. The induced metric on Σ in the $u_1, \ldots, u_{n-1}, u_s$ basis has

(27)
$$g_{ss} = 1, \quad g_{js} = g_{sj} = 0, \quad g_{jk} = r^2 h_{jk}, \quad \text{and} \quad h_{jk}(q) = \delta_{jk}$$

for j, k = 1, ..., n-1. The Levi-Civita connections on Σ and $\psi(M)$ are denoted by ∇ and ∇^M , respectively. Define

$$N_0 = \{ V | V = J(\gamma w), \ w \in \Gamma(T\psi(M)) \}.$$

For $V \in N_0$, the operator $\langle V, -L^{\perp}V \rangle_e$ can be simplified as below.

Lemma 2. Assume that Σ is a closed Lagrangian self-shrinker as in Lemma 1 and $V \in N_0$ is represented by $J(\gamma w)$. The second fundamental forms of Σ in \mathbb{C}^n and $\psi(M)$ in \mathbb{S}^{2n-1} are denoted by A^{Σ} and $A^{M,\mathbb{S}}$, respectively. Then we have

(28) (i)
$$|\langle A^{\Sigma}, V \rangle|^2 = |\langle A^{M, \mathbb{S}}, Jw \rangle|^2 + 2\sin^2(\theta - \phi)|w|^2$$
,

(29) (ii)
$$|\nabla^{\perp}V|^{2} = |\nabla^{M}w|^{2} + 2\cos^{2}(\theta - \phi)|w|^{2}$$
,
(iii) $\langle V, -L^{\perp}V \rangle_{e} = -\int_{\gamma} \left(\frac{1}{2}r^{2} - 2 + 4\sin^{2}(\theta - \phi)\right)e^{\frac{-r^{2}}{4}}r^{n-1}ds\int_{M}|w|^{2}d\mu_{M}$
(30) $+\int_{\gamma}e^{\frac{-r^{2}}{4}}r^{n-1}ds\int_{M} \left(|\nabla^{M}w|^{2} - |\langle A^{M,\mathbb{S}}, Jw\rangle|^{2}\right)d\mu_{M}.$

Proof. (i) For $V \in N_0$, it can be represented by $J(\gamma w)$ for some vector field $w \in \Gamma(T\psi(M))$. Using $\gamma \overline{\gamma} = r^2$ and $\gamma' \overline{\gamma} = re^{i(\theta - \phi)}$, we conclude that

$$\langle A_{kl}^{\Sigma}, V \rangle = \operatorname{Re}\langle\langle \gamma \frac{\partial^2 X^M}{\partial x^k \partial x^l}, J(\gamma w) \rangle\rangle = r^2 \operatorname{Re}\langle\langle A_{kl}^M, Jw \rangle\rangle = r^2 \langle A_{kl}^{M,S}, Jw \rangle,$$

$$\langle A_{ks}^{\Sigma}, V \rangle = \operatorname{Re}\langle\langle \gamma' \frac{\partial X^M}{\partial x^k}, J(\gamma w) \rangle\rangle = r \sin(\theta - \phi) \langle e_k, w \rangle,$$

$$\langle A_{ss}^{\Sigma}, V \rangle = \operatorname{Re}\langle\langle \gamma'' X^M, J(\gamma w) \rangle\rangle = \operatorname{Re}(\gamma'' \overline{\gamma} \langle\langle X^M, Jw \rangle\rangle) = 0$$

for k, l = 1, ..., n - 1. Here the second equalities of the second and third equations of (31) follow from the fact that e_k , Jw, X^M , and JX^M are mutually orthogonal. Combining (27) and (31), it gives

$$\begin{split} |\langle A^{\Sigma}, V \rangle|^2 &= \sum_{k,l=1}^{n-1} \langle A_{kl}^{\Sigma}, V \rangle^2 \frac{1}{r^4} + 2 \sum_{k=1}^{n-1} \langle A_{ks}^{\Sigma}, V \rangle^2 \frac{1}{r^2} + \langle A_{ss}^{\Sigma}, V \rangle^2 \\ &= |\langle A^{M,\mathbb{S}}, Jw \rangle|^2 + 2 \sin^2(\theta - \phi)|w|^2 \quad \text{at } p. \end{split}$$

(ii) Since Σ is a Lagrangian, $\{Ju_{\alpha}\}_{\alpha=1,\dots,n-1,s}$ is an orthogonal basis at p for the normal bundle. We now calculate the projections of $(\nabla_{u_{\alpha}}^{\perp}J(\gamma w))_{\alpha=1,\dots,n-1,s}$

on Ju_j and Ju_s . Using the properties that w, Je_k , X^M , and JX^M are mutually orthogonal, $\gamma \overline{\gamma} = r^2$ and $\gamma' \overline{\gamma} = re^{i(\theta - \phi)}$, we can show that

$$(32) \qquad \begin{array}{l} \langle \nabla_{u_{k}}^{\perp} J(\gamma w), Ju_{j} \rangle = \operatorname{Re}\langle \langle i\gamma \frac{\partial}{\partial x^{k}} w, i\gamma e_{j} \rangle \rangle = r^{2} \langle \nabla_{e_{k}}^{M} w, e_{j} \rangle, \\ \langle \nabla_{u_{k}}^{\perp} J(\gamma w), Ju_{s} \rangle = -\operatorname{Re}\langle \langle i\gamma w, \frac{\partial}{\partial x^{k}} i\gamma' X^{M} \rangle \rangle = -r \cos(\theta - \phi) \langle w, e_{k} \rangle, \\ \langle \nabla_{u_{s}}^{\perp} J(\gamma w), Ju_{j} \rangle = \operatorname{Re}\langle \langle i\gamma' w, i\gamma e_{j} \rangle \rangle = r \cos(\theta - \phi) \langle w, e_{j} \rangle, \\ \langle \nabla_{u_{s}}^{\perp} J(\gamma w), Ju_{s} \rangle = \operatorname{Re}\langle \langle i\gamma' w, i\gamma' X^{M} \rangle \rangle = 0. \end{array}$$

From (32), it follows that

$$\begin{split} |\nabla^{\perp}V|^{2} &= \langle \nabla_{u_{\alpha}}^{\perp}J(\gamma w), \nabla_{u_{\beta}}^{\perp}J(\gamma w) \rangle g^{\alpha\beta} \\ &= \sum_{k=1}^{n-1} \langle \nabla_{u_{k}}^{\perp}J(\gamma w), \nabla_{u_{k}}^{\perp}J(\gamma w) \rangle \frac{1}{r^{2}} + \langle \nabla_{u_{s}}^{\perp}J(\gamma w), \nabla_{u_{s}}^{\perp}J(\gamma w) \rangle \\ &= \left(\sum_{j,k=1}^{n-1} \langle \nabla_{u_{k}}^{\perp}J(\gamma w), \frac{Ju_{j}}{r} \rangle^{2} + \sum_{k=1}^{n-1} \langle \nabla_{u_{k}}^{\perp}J(\gamma w), Ju_{s} \rangle^{2} \right) \frac{1}{r^{2}} + \sum_{j=1}^{n-1} \langle \nabla_{u_{s}}^{\perp}J(\gamma w), \frac{Ju_{j}}{r} \rangle^{2} \\ &= \sum_{j,k=1}^{n-1} \langle \nabla_{e_{k}}^{M}w, e_{j} \rangle^{2} + \sum_{j=1}^{n-1} 2\cos^{2}(\theta - \phi) \langle w, e_{j} \rangle^{2} \\ &= |\nabla^{M}w|^{2} + 2\cos^{2}(\theta - \phi)|w|^{2}. \end{split}$$

(iii) Plugging (28) and (29) into (24), and using $e^{\frac{-|X|^2}{4}}d\mu_{\Sigma} = e^{-\frac{r^2}{4}}r^{n-1}ds\,d\mu_M$, we get

$$\begin{split} \langle V, -L^{\perp}V \rangle_{e} \\ &= \int_{\Sigma} \left(|\nabla^{\perp}V|^{2} - |\langle A^{\Sigma}, V \rangle|^{2} - \frac{1}{2}|V|^{2} \right) e^{-\frac{|X|^{2}}{4}} d\mu_{\Sigma} \\ &= \int_{\gamma} \int_{M} \left(|\nabla^{M}w|^{2} + 2\cos^{2}(\theta - \phi)|w|^{2} - \left(|\langle A^{M,\mathbb{S}}, Jw \rangle|^{2} + 2\sin^{2}(\theta - \phi)|w|^{2} \right) \\ &- \frac{1}{2}r^{2}|w|^{2} \right) e^{-\frac{r^{2}}{4}}r^{n-1}d\mu_{M} \, ds \\ &= -\int_{\gamma} \left(\frac{1}{2}r^{2} - 2 + 4\sin^{2}(\theta - \phi) \right) e^{\frac{-r^{2}}{4}}r^{n-1}ds \int_{M} |w|^{2}d\mu_{M} \\ &+ \int_{\gamma} e^{\frac{-r^{2}}{4}}r^{n-1}ds \int_{M} \left(|\nabla^{M}w|^{2} - |\langle A^{M,\mathbb{S}}, Jw \rangle|^{2} \right) d\mu_{M}. \end{split}$$

Thus (iii) is proved.

To further simplify $\langle V, -L^{\perp}V \rangle_e$, we need to derive some integral equalities on the curve γ .

Lemma 3. Let $\gamma : I \to \mathbb{C}^*$ be a closed smooth regular curve parameterized by the arclength s and satisfying (25). That is, $\gamma * \psi$ in Lemma 1 defines a closed self-shrinker. Then one has

(33)
$$\int_{\gamma} (\frac{1}{2}r^2 - n)r^{n-1}e^{-\frac{r^2}{4}}ds = 0$$

and

(34)
$$\int_{\gamma} (\frac{1}{2r^2} - \frac{n}{r^4}) r^{n-1} e^{-\frac{r^2}{4}} ds = -\int_{\gamma} \frac{4\cos^2(\theta - \phi)}{r^4} r^{n-1} e^{-\frac{r^2}{4}} ds.$$

Remark 3. The equality (33) will be used to simplify (30) while the equality (34) will be used to simplify (48) for Lagrangian variations.

Proof. Equality (33) follows from a simplification of (8),

$$0 = \int_{\gamma} \int_{M} (r^2 - 2n) e^{-\frac{r^2}{4}} r^{n-1} d\mu_M ds = \int_{\gamma} (r^2 - 2n) e^{-\frac{r^2}{4}} r^{n-1} ds \int_{M} d\mu_M,$$

and $\int_M d\mu_M \neq 0$. Recall from Proposition 1 that we have

$$\mathcal{L}f = \Delta f - \frac{1}{2} \langle X, \nabla f \rangle = e^{\frac{|X|^2}{4}} div (e^{-\frac{|X|^2}{4}} \nabla f).$$

Hence

(35)
$$\int_{\Sigma} \mathcal{L}(\frac{1}{|X|^2}) e^{-\frac{|X|^2}{4}} d\mu_{\Sigma} = \int_{\Sigma} div (e^{-\frac{|X|^2}{4}} \nabla \frac{1}{|X|^2}) d\mu_{\Sigma} = 0,$$

since $\partial \Sigma = \emptyset$. On the other hand,

(36)
$$\mathcal{L}(\frac{1}{|X|^2}) = \frac{-\mathcal{L}|X|^2}{|X|^4} + \frac{2|\nabla|X|^2|^2}{|X|^6} = \frac{-2n + |X|^2}{|X|^4} + \frac{8|X^\top|^2}{|X|^6}$$

by equation (7) and $\nabla |X|^2 = 2X^{\top}$.

Combining (35), (36), and $|X^{\top}| = \operatorname{Re}\left(re^{i(\phi-\theta)}\right) = r\cos(\theta-\phi)$, it gives

$$0 = \int_{\gamma} \int_{M} \left(\frac{-2n+r^2}{r^4} + \frac{8r^2\cos^2(\theta-\phi)}{r^6}\right) e^{-\frac{r^2}{4}} r^{n-1} d\mu_M ds$$
$$= \int_{\gamma} \left(\frac{-2n+r^2}{r^4} + \frac{8r^2\cos^2(\theta-\phi)}{r^6}\right) e^{-\frac{r^2}{4}} r^{n-1} ds \int_{M} d\mu_M.$$

Then the equality (34) follows since $\int_M d\mu_M \neq 0$.

Next, we want to find a vector field w_0 in $\Gamma(T\psi(M))$ with nice special properties that will be needed in proving Theorem 7 and Theorem 8.

Lemma 4. Let $\psi : M^{n-1} \to \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ be a minimal Legendrian immersion. Then there exists a nonzero vector field w_0 in $\Gamma(T\psi(M))$ satisfying

(37)
$$\frac{\int_{M} (|\nabla^{M} w_{0}|^{2} - |\langle A^{M, \mathbb{S}}, Jw_{0} \rangle|^{2}) d\mu}{\int_{M} |w_{0}|^{2} d\mu} \leq 1 \quad and \quad \langle \nabla^{M}_{x} w_{0}, y \rangle = \langle \nabla^{M}_{y} w_{0}, x \rangle$$

for any $x, y \in T\psi(M)$.

Remark 4. The condition $\langle \nabla_x^M w_0, y \rangle = \langle \nabla_y^M w_0, x \rangle$ implies that $\frac{1}{r^2} J(\gamma w_0)$ induces a Lagrangian variation.

Proof. Define

$$f(y) = \int_M (|\nabla^M y|^2 - |\langle A^{M,\mathbb{S}}, Jy \rangle|^2) d\mu$$

for $y \in \Gamma(T\psi(M))$. Let E_1, \ldots, E_{2n} be the standard basis for \mathbb{C}^n with $E_{\alpha+n} = JE_{\alpha}$ for $\alpha = 1, \ldots, n$. We claim that there exists a β_0 in $\{1, \ldots, 2n\}$ such that $w_0 = E_{\beta_0}^{\top}$ is a nonzero vector field satisfying $f(w_0) \leq \int_M |w_0|^2 d\mu$, where $E_{\beta_0}^{\top}$ is the projection

of E_{β_0} into the tangent space of $\psi(M)$. For fixed $q \in \psi(M)$, choose a local normal coordinate system x^1, \ldots, x^{n-1} at q. Denote $\partial_j = \frac{\partial}{\partial x^j}$. We have

(38)
$$\langle \nabla^{M}_{\partial_{k}}(E^{\top}_{\beta}), \partial_{j} \rangle = \langle \frac{\partial}{\partial x^{k}}(E_{\beta} - E^{\perp}_{\beta}), \partial_{j} \rangle = -\langle \frac{\partial}{\partial x^{k}}E^{\perp}_{\beta}, \partial_{j} \rangle = \langle E_{\beta}, A^{M}_{jk} \rangle$$

Since the map ψ is a Legendrian immersion into \mathbb{S}^{2n-1} , the space spanned by $\{\partial_1, \ldots, \partial_{n-1}, X^M\}$ is a Lagrangian plane in \mathbb{C}^n . It gives

(39)
$$A_{kj}^M = A_{kj}^{M,\mathbb{S}} + \langle A_{kj}^M, X^M \rangle X^M = A_{kj}^{M,\mathbb{S}} - \delta_{kj} X^M \quad \text{at } q$$

and the second fundamental form $A_{jk}^{M,\mathbb{S}}$ of the submanifold $\psi(M)$ in \mathbb{S}^{2n-1} is orthogonal to JX^M because

$$\langle A_{kj}^{M,\mathbb{S}}, JX^M \rangle = \langle \frac{\partial}{\partial x^k} (\partial_j), JX^M \rangle = -\langle \partial_j, J\partial_k \rangle = 0.$$

Recall that ψ is a minimal immersion in \mathbb{S}^{2n-1} and hence $H^{M,\mathbb{S}} = 0$. Combining the equations (38) and (39), the first term of $f(E_{\beta}^{\top})$ can be simplified as

$$\begin{aligned} |\nabla^{M}(E_{\beta}^{\top})|^{2} &= \sum_{j,k=1}^{n-1} |\langle E_{\beta}, A_{kj}^{M,\mathbb{S}} \rangle - \langle E_{\beta}, \delta_{kj} X^{M} \rangle|^{2} \\ &= |\langle E_{\beta}, A^{M,\mathbb{S}} \rangle|^{2} - 2 \langle E_{\beta}, H^{M,\mathbb{S}} \rangle \langle E_{\beta}, X^{M} \rangle + (n-1) \langle E_{\beta}, X^{M} \rangle^{2} \\ (40) &= |\langle E_{\beta}, A^{M,\mathbb{S}} \rangle|^{2} + (n-1) \langle E_{\beta}, X^{M} \rangle^{2} \quad \text{at} \quad q. \end{aligned}$$

Since ∂_l and X^M are orthogonal, we have $(JA^{M,\mathbb{S}})^{\top} = JA^{M,\mathbb{S}}$ and the second term of $f(E_{\beta}^{\top})$ can be simplified as

(41)
$$\langle A^{M,\mathbb{S}}, J(E_{\beta}^{\top}) \rangle = -\langle JA^{M,\mathbb{S}}, E_{\beta}^{\top} \rangle = -\langle JA^{M,\mathbb{S}}, E_{\beta} \rangle = \langle A^{M,\mathbb{S}}, JE_{\beta} \rangle.$$

Combining (40) and (41) gives

(42)
$$f(E_{\alpha}^{\top}) = \int_{M} \left(|\langle E_{\alpha}, A^{M, \mathbb{S}} \rangle|^{2} + (n-1) \langle E_{\alpha}, X^{M} \rangle^{2} - |\langle E_{\alpha+n}, A^{M, \mathbb{S}} \rangle|^{2} \right) d\mu,$$

(43)
$$f(E_{\alpha+n}^{\top}) = \int_{M} \left(|\langle E_{\alpha+n}, A^{M, \mathbb{S}} \rangle|^2 + (n-1) \langle E_{\alpha+n}, X^M \rangle^2 - |\langle E_{\alpha}, A^{M, \mathbb{S}} \rangle|^2 \right) d\mu$$

for $\alpha = 1, \ldots, n$. Summing (42) and (43) over $\alpha = 1, \ldots, n$ gives

(44)
$$\sum_{\alpha=1}^{n} \left(f(E_{\alpha}^{\top}) + f(E_{\alpha+n}^{\top}) \right) = (n-1) \sum_{\beta=1}^{2n} \int_{M} \langle E_{\beta}, X^{M} \rangle^{2} d\mu = (n-1) \int_{M} d\mu$$

since $|X^M| = 1$.

On the other hand, we have $\sum_{\beta=1}^{2n} |E_{\beta}^{\top}|^2 = \sum_{\beta=1}^{2n} \sum_{j=1}^{n-1} \langle E_{\beta}, \partial_j \rangle^2 = \sum_{j=1}^{n-1} |\partial_j|^2 = n-1$ at q because $\partial_1, \ldots, \partial_{n-1}$ is an orthonormal basis for $T_q \psi(M)$. Plugging it into (44), we get

$$\sum_{\beta=1}^{2n} \int_M \left(|\nabla^M(E_\beta^\top)|^2 - |\langle A^{M,\mathbb{S}}, J(E_\beta^\top)\rangle|^2 \right) d\mu = \sum_{\beta=1}^{2n} \int_M |E_\beta^\top|^2 d\mu.$$

Therefore, there exists a β_0 in $\{1, \ldots, 2n\}$ such that $E_{\beta_0}^{\top}$ is a nonzero vector field and

$$\int_M \left(|\nabla^M(E_{\beta_0}^\top)|^2 - |\langle A^{M,\mathbb{S}}, J(E_{\beta_0}^\top) \rangle|^2 \right) d\mu \le \int_M |E_{\beta_0}^\top|^2 d\mu$$

which is exactly the first inequality in (37). Using (38) and the fact that $\langle E_{\beta_0}, A_{jk}^M \rangle$ is symmetric for j, k, it follows that the vector field $w_0 = E_{\beta_0}^{\top}$ also satisfies the second condition in (37).

Now we are ready to prove Theorem 7:

Proof of Theorem 7. By Theorem 4, it suffices to construct an admissible vector field V satisfying $\int_{\Sigma} \langle V, -L^{\perp}V \rangle e^{-\frac{|X|^2}{4}} d\mu < 0$. Assume $V = J(\gamma w)$, where $w \in \Gamma(T\psi(M))$ would be chosen later. Because H is parallel to Ju_s (see [2], p. 40), the first integral $\int_{\Sigma} \langle V, H \rangle e^{-\frac{|X|^2}{4}} d\mu$ in the admissible condition is equal to zero. The second integral in the admissible condition is

$$\int_{\Sigma} V e^{-\frac{|X|^2}{4}} d\mu = i \int_{\gamma} \gamma e^{-\frac{r^2}{4}} r^{n-1} ds \int_M w d\mu_M$$

Recall that the construction of γ in [2] is made by m > 1 pieces $\Gamma_1, \ldots, \Gamma_m$ which each corresponds to one period of curvature function. (When γ is the circle $\mathbb{S}^1(\sqrt{2n})$, we may take m = 2.) Every piece Γ_i is the same as Γ_1 up to a rotation. Suppose the rotation index of γ is l. Then we have

$$\int_{\gamma} \gamma e^{-\frac{r^2}{4}} r^{n-1} ds = \sum_{j=1}^{m} \int_{\Gamma_j} e^{-\frac{r^2}{4}} r^n e^{i\phi} ds$$
$$= \int_{\Gamma_1} e^{-\frac{r^2}{4}} r^n e^{i\phi} (1 + e^{i\frac{2l\pi}{m}} + \dots + e^{i\frac{(m-1)l}{m} \cdot 2\pi}) ds = 0,$$

since $1 + e^{i\frac{2l\pi}{m}} + \cdots + e^{i\frac{(m-1)l}{m}\cdot 2\pi} = 0$ for m > 1. Therefore, the second integral in the admissible condition is equal to zero.

For the case $n \ge 3$, we choose $w = w_0$ satisfying (37) and $V_0 = J(\gamma w_0)$. Plugging the first inequality of (37) into (30) and using (33), the weighted L^2 inner product $\langle V_0, -L^{\perp}V_0 \rangle_e$ becomes

$$\begin{split} &\int_{\Sigma} \langle V_0, -L^{\perp} V_0 \rangle e^{-\frac{|X|^2}{4}} d\mu \\ &\leq -\int_{\gamma} \left(\frac{1}{2} r^2 - 3 + 4 \sin^2(\theta - \phi) \right) e^{\frac{-r^2}{4}} r^{n-1} ds \int_M |w_0|^2 d\mu_M \\ &= -\int_{\gamma} \left(\left(n - 3 + 4 \sin^2(\theta - \phi) \right) \right) e^{-\frac{r^2}{4}} r^{n-1} ds \int_M |w_0|^2 d\mu_M \\ &< 0. \end{split}$$

For the case n = 2, the only minimal Legendrian curves in \mathbb{S}^3 are great circles. They are totally geodesic in \mathbb{S}^3 . Therefore, the weighted L^2 inner product $\langle V, -L^{\perp}V \rangle_e$ can be simplified as

$$\begin{split} &\int_{\Sigma} \langle V, -L^{\perp}V \rangle e^{-\frac{|X|^2}{4}} d\mu \\ &= \int_{\gamma} e^{-\frac{r^2}{4}} r \Big(\int_{\mathbb{S}^1} |\nabla^{\mathbb{S}^1} w|^2 - \left(\frac{1}{2}r^2 - 2 + 4\sin^2(\theta - \phi)\right) |w|^2 d\mu_{\mathbb{S}^1} \Big) ds \\ &= \int_{\gamma} e^{-\frac{r^2}{4}} r \Big(\int_{\mathbb{S}^1} |\nabla^{\mathbb{S}^1} w|^2 - 4\sin^2(\theta - \phi) |w|^2 d\mu_{\mathbb{S}^1} \Big) ds. \end{split}$$

Here we use (33) again to get the last equality. Finally, by choosing w to be the tangent vector of the great circle, which is a parallel vector field, we can make the weighted L^2 inner product negative. This completes the proof.

5.3. The unstability for Lagrangian variations. Since Anciaux's examples are Lagrangian, it is natural to investigate whether these examples are still unstable under the more restricted Lagrangian variations, that is, for variations from the deformation of Lagrangian submanifolds. A simple calculation shows that a vector field V induces a Lagrangian variation if and only if the associated one form $\alpha_V = \omega(V, \cdot)$ is closed, i.e.

(45)
$$\langle \nabla_X^{\perp} V, JY \rangle = \langle \nabla_Y^{\perp} V, JX \rangle,$$

where ∇^{\perp} is the normal connection on $N\Sigma$ and $X, Y \in T\Sigma$. For this question, we can prove the following results.

Theorem 8. Let Σ be an n-dimensional closed embedded Lagrangian self-shrinker as in Lemma 1. Then Σ is F-unstable under Lagrangian variations for the following cases:

(i) $n = 2 \text{ or } n \ge 7$, (ii) 2 < n < 7, and $E \in [\sqrt{\frac{7-n}{8}}E_{max}, E_{max}]$,

where E and E_{max} are described in (26).

Because $\langle \nabla_{u_s}^{\perp} V, Ju_j \rangle \neq \langle \nabla_{u_j}^{\perp} V, Ju_s \rangle$ for $V \in N_0$, it does not induce a Lagrangian variation. Thus to prove the theorem, we need to consider variations different from those in §5.2. We now define a new set N_1 by

$$N_1 = \{ V | V = \frac{1}{r^2} J(\gamma w), \text{ where } w \in \Gamma(T\psi(M)) \text{ satisfies} \\ \langle \nabla_x^M w, y \rangle = \langle \nabla_y^M w, x \rangle, \text{ for all } x, y \in T\psi(M) \}.$$

For $V \in N_1$, written as $V = \frac{1}{r^2} J(\gamma w)$, we claim that V satisfies the equation (45) and hence indeed induces a Lagrangian variation. Noting that $\gamma' = e^{i\theta}$, $\langle V, Ju_s \rangle = 0$, and r' satisfying (25), we therefore have

$$\begin{split} \langle \nabla_{u_s}^{\perp} V, Ju_j \rangle &= -\frac{2r'}{r^3} \langle J(\gamma w), J(\gamma e_j) \rangle + \frac{1}{r^2} \langle J(\gamma' w), J(\gamma e_j) \rangle \\ &= -\frac{\cos(\theta - \phi)}{r} \langle w, e_j \rangle, \\ \langle \nabla_{u_j}^{\perp} V, Ju_s \rangle &= -\langle V, \nabla_{u_j}^{\perp} Ju_s \rangle = -\frac{1}{r^2} \langle J(\gamma w), J(\gamma' e_j) \rangle = -\frac{\cos(\theta - \phi)}{r} \langle w, e_j \rangle \\ \langle \nabla_{u_k}^{\perp} V, Ju_j \rangle &= \frac{1}{r^2} \langle \frac{\partial}{\partial x_k} J(\gamma w), J(\gamma e_j) \rangle = \langle \nabla_{e_k}^M w, e_j \rangle \\ &= \langle \nabla_{e_j}^M w, e_k \rangle = \langle \nabla_{u_j}^{\perp} V, Ju_k \rangle. \end{split}$$

Thus (45) is satisfied.

For $V \in N_1$, the operator $\langle V, -L^{\perp}V \rangle_e$ can be simplified as in the following lemma.

Lemma 5. Assume that Σ is a closed Lagrangian self-shrinker as in Lemma 1 and $V \in N_1$ is represented by $\frac{1}{r^2}J(\gamma w)$. The second fundamental forms of Σ in \mathbb{C}^n and of $\psi(M)$ in \mathbb{S}^{2n-1} are denoted by A^{Σ} and $A^{M,\mathbb{S}}$, respectively. Then we have

(46) (i)
$$|\langle A^{\Sigma}, V \rangle|^2 = \frac{1}{r^4} |\langle A^{M,\mathbb{S}}, Jw \rangle|^2 + \frac{2}{r^4} \sin^2(\theta - \phi)|w|^2$$
,

$$(47) \quad (ii) \ |\nabla^{\perp}V|^{2} = \frac{1}{r^{4}} |\nabla^{M}w|^{2} + \frac{2\cos^{2}(\theta-\phi)}{r^{4}} |w|^{2},$$

$$(iii) \ \langle V, -L^{\perp}V \rangle_{e} = -\int_{\gamma} \left(\frac{1}{2}r^{2} - 2 + 4\sin^{2}(\theta-\phi)\right) e^{\frac{-r^{2}}{4}} r^{n-5} ds \int_{M} |w|^{2} d\mu_{M}$$

$$(48) \qquad +\int_{\gamma} e^{\frac{-r^{2}}{4}} r^{n-5} ds \int_{M} \left(|\nabla^{M}w|^{2} - |\langle A^{M,\mathbb{S}}, Jw\rangle|^{2}\right) d\mu_{M}.$$

Proof. (i) For $V \in N_1$, denoting $V_0 = J(\gamma w) \in N_0$, we then have $V = \frac{1}{r^2}V_0$. Using equation (28) gives

$$|\langle A^{\Sigma}, V \rangle|^{2} = \frac{1}{r^{4}} |\langle A^{\Sigma}, V_{0} \rangle|^{2} = \frac{1}{r^{4}} |\langle A^{M, \mathbb{S}}, Jw \rangle|^{2} + \frac{2}{r^{4}} \sin^{2}(\theta - \phi)|w|^{2}.$$

(ii) From equation (32), we conclude that

$$\langle \nabla_{u_k}^{\perp} \frac{1}{r^2} J(\gamma w), Ju_j \rangle = \frac{1}{r^2} \langle \nabla_{u_k}^{\perp} J(\gamma w), Ju_j \rangle = \langle \nabla_{e_k}^M w, e_j \rangle,$$

$$\langle \nabla_{u_k}^{\perp} \frac{1}{r^2} J(\gamma w), Ju_s \rangle = \frac{1}{r^2} \langle \nabla_{u_k}^{\perp} J(\gamma w), Ju_s \rangle = -\frac{1}{r} \cos(\theta - \phi) \langle w, e_j \rangle,$$

$$\langle \nabla_{u_s}^{\perp} \frac{1}{r^2} J(\gamma w), Ju_s \rangle = \frac{-2r'}{r^3} \langle J(\gamma w), Ju_s \rangle + \frac{1}{r^2} \langle \nabla_{u_s}^{\perp} J(\gamma w), Ju_s \rangle = 0.$$

Using (49) and (45), a computation at p leads to

$$\begin{split} |\nabla^{\perp}V|^{2} &= \langle \nabla_{u_{\alpha}}^{\perp} \frac{1}{r^{2}} J(\gamma w), \nabla_{u_{\beta}}^{\perp} \frac{1}{r^{2}} J(\gamma w) \rangle g^{\alpha\beta} \\ &= \sum_{k=1}^{n-1} \langle \nabla_{u_{k}}^{\perp} \frac{1}{r^{2}} J(\gamma w), \nabla_{u_{k}}^{\perp} \frac{1}{r^{2}} J(\gamma w) \rangle \frac{1}{r^{2}} + \langle \nabla_{u_{s}}^{\perp} \frac{1}{r^{2}} J(\gamma w), \nabla_{u_{s}}^{\perp} \frac{1}{r^{2}} J(\gamma w) \rangle \\ &= \left(\sum_{j,k=1}^{n-1} \langle \nabla_{u_{k}}^{\perp} \frac{1}{r^{2}} J(\gamma w), \frac{J u_{j}}{r} \rangle^{2} + \sum_{k=1}^{n-1} \langle \nabla_{u_{k}}^{\perp} \frac{1}{r^{2}} J(\gamma w), J u_{s} \rangle^{2} \right) \frac{1}{r^{2}} \\ &+ \sum_{j=1}^{n-1} \langle \nabla_{u_{s}}^{\perp} \frac{1}{r^{2}} J(\gamma w), \frac{J u_{j}}{r} \rangle^{2} \\ &= \frac{1}{r^{4}} \sum_{j,k=1}^{n-1} \langle \nabla_{e_{k}}^{M} w, e_{j} \rangle^{2} + \frac{2}{r^{4}} \sum_{j=1}^{n-1} \cos^{2}(\theta - \phi) \langle w, e_{j} \rangle^{2} \\ &= \frac{1}{r^{4}} |\nabla^{M} w|^{2} + \frac{2 \cos^{2}(\theta - \phi)}{r^{4}} |w|^{2}. \end{split}$$

(iii) Plugging (46) and (47) into (24), and using $e^{\frac{-|X|^2}{4}}d\mu_{\Sigma} = e^{-\frac{r^2}{4}}r^{n-1}dsd\mu_M$, we get

$$\begin{split} \langle V, -L^{\perp}V \rangle_{e} \\ &= \int_{\Sigma} \left(|\nabla^{\perp}V|^{2} - |\langle A^{\Sigma}, V \rangle|^{2} - \frac{1}{2}|V|^{2} \right) e^{-\frac{|X|^{2}}{4}} d\mu_{\Sigma} \\ &= \int_{\gamma} \int_{M} \left(|\nabla^{M}w|^{2} + 2\cos^{2}(\theta - \phi)|w|^{2} - \left(|\langle A^{M,\mathbb{S}}, Jw \rangle|^{2} + 2\sin^{2}(\theta - \phi)|w|^{2} \right) \\ &- \frac{1}{2}r^{2}|w|^{2} \right) e^{-\frac{r^{2}}{4}}r^{n-5}d\mu_{M}ds \\ &= -\int_{\gamma} \left(\frac{1}{2}r^{2} - 2 + 4\sin^{2}(\theta - \phi) \right) e^{\frac{-r^{2}}{4}}r^{n-5}ds \int_{M} |w|^{2}d\mu_{M} \\ &+ \int_{\gamma} e^{\frac{-r^{2}}{4}}r^{n-5}ds \int_{M} \left(|\nabla^{M}w|^{2} - |\langle A^{M,\mathbb{S}}, Jw \rangle|^{2} \right) d\mu_{M}. \end{split}$$

Thus (iii) is proved.

Proof of Theorem 8. By Theorem 4, it suffices to construct an admissible Lagrangian variation V satisfying $\int_{\Sigma} \langle V, -L^{\perp}V \rangle e^{-\frac{|X|^2}{4}} d\mu < 0$. Assume $V = \frac{1}{r^2} J(\gamma w) \in N_1$, where $w \in \Gamma(T\psi(M))$ will be chosen later. Similar to the proof of Theorem 7, V is an admissible Lagrangian variation.

We now further specify V, so that $\int_{\Sigma} \langle V, -L^{\perp}V \rangle e^{-\frac{|X|^2}{4}} d\mu < 0$. When $n \geq 3$, we choose $w = w_0$ satisfying (37). Then $V_1 = \frac{1}{r^2} J(\gamma w_0)$ is in N_1 . From (37) and (48), the weighted L^2 inner product $\langle V_1, -L^{\perp}V_1 \rangle_e$ becomes

$$\begin{split} &\int_{\Sigma} \langle V_1, -L^{\perp} V_1 \rangle e^{-\frac{|X|^2}{4}} d\mu \\ &\leq -\int_{\gamma} \left(\frac{1}{2} r^2 - 3 + 4 \sin^2(\theta - \phi) \right) e^{\frac{-r^2}{4}} r^{n-5} ds \int_M |w_0|^2 d\mu_M \\ &= -\int_{\gamma} \Big(\left(n - 3 + 4 \sin^2(\theta - \phi) - 4 \cos^2(\theta - \phi) \right) \Big) e^{-\frac{r^2}{4}} r^{n-5} ds \int_M |w_0|^2 d\mu_M \\ &= -\int_{\gamma} \Big(\left(n - 7 + 8 \sin^2(\theta - \phi) \right) \Big) e^{-\frac{r^2}{4}} r^{n-5} ds \int_M |w_0|^2 d\mu_M, \end{split}$$

where (34) is used to conclude the first equality. Thus to prove Lagrangian unstability, it suffices to show that $f(s) = n - 7 + 8 \sin^2(\theta - \phi)$ is nonnegative and positive at some point. For $n \ge 7$, this is clearly true. Because we have $\sin(\theta - \phi) \ge \frac{E}{E_{max}}$ from (26), f(s) is nonnegative and positive somewhere when 2 < n < 7 and $E \in [\sqrt{\frac{7-n}{8}}E_{max}, E_{max}].$

In the case n = 2, the only minimal Legendrian curves in \mathbb{S}^3 are great circles which are totally geodesic. Choosing w_1 to be the tangent vector of the great circle, we have $|\nabla^{\mathbb{S}^1} w_1| = 0$ and $|w_1| = 1$. The vector field $V_1 = \frac{1}{r^2} J(\gamma w_1)$ gives a Lagrangian variation and the weighted L^2 inner product $\langle V_1, -L^{\perp} V_1 \rangle_e$ in (48) can

be simplified as

$$\int_{\Sigma} \langle V_1, -L^{\perp} V_1 \rangle e^{-\frac{|X|^2}{4}} d\mu$$

= $-\int_{\gamma} \left(\frac{1}{2} r^2 - 2 + 4 \sin^2(\theta - \phi) \right) e^{-\frac{r^2}{4}} r^{-3} ds \int_{\mathbb{S}^1} |w|^2 d\mu_{\mathbb{S}^1}$
= $-2\pi \int_{\gamma} \left(\frac{1}{2} r^2 + 2 \left(\sin^2(\theta - \phi) - \cos^2(\theta - \phi) \right) \right) e^{-\frac{r^2}{4}} r^{-3} ds$

Using (34), it follows that

$$\int_{\gamma} \frac{1}{2} r^2 e^{-\frac{r^2}{4}} r^{-3} ds = \int_{\gamma} 2 \left(\sin^2(\theta - \phi) - \cos^2(\theta - \phi) \right) e^{-\frac{r^2}{4}} r^{-3} ds.$$

Therefore, $\langle V_1, -L^{\perp}V_1 \rangle_e = -2\pi \int_{\gamma} r^2 e^{-\frac{r^2}{4}} r^{-3} ds < 0$, and concludes the Lagrangian unstability in Theorem 8.

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