TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 4, April 2015, Pages 2527–2549
S 0002-9947(2014)05975-8
Article electronically published on August 8, 2014

TOPOLOGICAL BIRKHOFF

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ABSTRACT. One of the most fundamental mathematical contributions of Garrett Birkhoff is the HSP theorem, which implies that a finite algebra **B** satisfies all equations that hold in a finite algebra **A** of the same signature if and only if **B** is a homomorphic image of a subalgebra of a finite power of **A**. On the other hand, if **A** is infinite, then in general one needs to take an *infinite* power in order to obtain a representation of **B** in terms of **A**, even if **B** is finite.

We show that by considering the natural topology on the functions of **A** and **B** in addition to the equations that hold between them, one can do with finite powers even for many interesting infinite algebras **A**. More precisely, we prove that if **A** and **B** are at most countable algebras which are oligomorphic, then the mapping which sends each term function over **A** to the corresponding term function over **B** preserves equations and is *Cauchy-continuous* if and only if **B** is a homomorphic image of a subalgebra of a *finite* power of **A**.

Our result has the following consequences in model theory and in theoretical computer science: two ω-categorical structures are primitive positive bi-interpretable if and only if their topological polymorphism clones are isomorphic. In particular, the complexity of the constraint satisfaction problem of an ω-categorical structure only depends on its topological polymorphism clone.

1. Introduction

The algebraic result we present has a motivating application in model theory, which in turn has implications for the study of the computational complexity of constraint satisfaction problems in theoretical computer science. We start our introduction with this model-theoretic perspective on our result, and describe the central algebraic theorem of this article later in the introduction, in Section 1.2.

1.1. **The model-theoretic perspective.** A countable structure Γ is called *ω-categorical* iff all countable models of the first-order theory of Γ are isomorphic to Γ. A substantial amount of information about an ω-categorical structure Γ is already coded into the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ, viewed abstractly as a topological group whose topology is the topology of pointwise convergence. In particular, Ahlbrandt and Ziegler [AZ86] proved that two countable ω-categorical structures are *first-order bi-interpretable* if and only if their automorphism groups are isomorphic as topological groups. The concept of interpretation we use here is standard [Hod93] and will be recalled in Section 4.

Received by the editors March 25, 2012 and, in revised form, October 3, 2012.
2010 *Mathematics Subject Classification.* Primary 03C05, 03C40, 08A35, 08A30; Secondary 08A70.
The first author has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013 Grant Agreement no. 257039).
The second author is grateful for support through an APART-fellowship of the Austrian Academy of Sciences.

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Recently, the following variant of the theorem of Ahlbrandt and Ziegler has been shown, replacing the automorphism group by the endomorphism monoid (which, of course, contains more information about the original structure than the automorphism group) [BJ11]: two ω-categorical structures Γ and Δ without constant endomorphisms are *existential positive bi-interpretable* (i.e., bi-interpretable by means of existential positive first-order formulas) if and only if their endomorphism monoids $\mathrm{End}(\Gamma)$ and $\mathrm{End}(\Delta)$ are isomorphic as abstract topological monoids, i.e., iff there exists a bijective function $\xi\colon \mathrm{End}(\Gamma) \to \mathrm{End}(\Delta)$ which sends the identity function on Γ to the identity on Δ, which satisfies $\xi(f \circ g) = \xi(f) \circ \xi(g)$ for all $f, g \in \mathrm{End}(\Gamma)$, and such that both ξ and its inverse are continuous.

In the same paper, it is stated as an open problem whether this statement can be modified further to characterize *primitive positive bi-interpretability* if one replaces the endomorphism monoid by the *polymorphism clone*. A primitive positive interpretation is a first-order interpretation where all the involved formulas are *primitive positive*, i.e., of the form $\exists x_1, \ldots, x_n(\phi_1 \wedge \cdots \wedge \phi_m)$ where $\phi_1, \ldots, \phi_m$ are atomic formulas. A *polymorphism* of a structure Γ is a homomorphism from Γ^k to Γ for some finite $k \geq 1$; the polymorphism clone $\mathrm{Pol}(\Gamma)$ of Γ is the set of all polymorphisms of Γ and contains, in particular, at least the information of $\mathrm{End}(\Gamma)$, which is the unary fragment of $\mathrm{Pol}(\Gamma)$. In general, a *(concrete) clone* is a set of finitary functions on a fixed set which contains all projections and which is closed under composition; it is not hard to see that $\mathrm{Pol}(\Gamma)$ is a clone in this sense. Moreover, $\mathrm{Pol}(\Gamma)$ is a closed subset of the topological space $\mathscr{O}_\Gamma = \bigcup_{k\geq 1} \Gamma^{\Gamma^k}$ of all finitary functions on Γ, just like $\mathrm{Aut}(\Gamma)$ is a closed subset of the space of all permutations on Γ and $\mathrm{End}(\Gamma)$ is a closed subset of the space of unary functions on Γ. The topology of $\mathscr{O}_\Gamma$ is obtained by viewing this space as the sum space of the spaces Γ^{Γ^k}, and each Γ^{Γ^k} as a power of Γ, which itself is taken to be discrete. Similarly to automorphism groups and endomorphism monoids, where we distinguish between the concrete permutation groups and transformation monoids on the one hand and abstract topological groups and topological monoids with their laws of composition and topology on the other hand, polymorphism clones can be viewed abstractly as *topological clones* carrying an algebraic and a topological structure. The topology on $\mathrm{Pol}(\Gamma)$ is inherited from the space $\mathscr{O}_\Gamma$; note that for countably infinite Γ each Γ^{Γ^k}, and in fact also $\mathscr{O}_\Gamma$, is homeomorphic to the Baire space and that therefore the space $\mathrm{Pol}(\Gamma)$ is a closed subspace of the Baire space. The algebraic structure of $\mathrm{Pol}(\Gamma)$ is that of a multi-sorted algebra with operations that correspond to the composition of the elements of $\mathrm{Pol}(\Gamma)$ and constant symbols corresponding to the projections. We can avoid a formal description of this ghastly structure here (and refer the interested reader, for example, to [Tay93] or the survey paper [GP08]) since we only need the very natural notion of a *homomorphism between clones* $\mathscr{C}, \mathscr{D}$: these are functions $\xi\colon \mathscr{C} \to \mathscr{D}$ which send every projection in $\mathscr{C}$ to the corresponding projection in $\mathscr{D}$, and such that $\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$ for all n-ary $f \in \mathscr{C}$ and all m-ary $g_1, \ldots, g_n \in \mathscr{C}$. In particular, two polymorphism clones $\mathrm{Pol}(\Gamma), \mathrm{Pol}(\Delta)$ are isomorphic as topological clones iff there exists a bijection ξ from $\mathrm{Pol}(\Gamma)$ onto $\mathrm{Pol}(\Delta)$ such that both ξ and its inverse are continuous clone homomorphisms. The above-mentioned problem in [BJ11] asked whether for two ω-categorical structures Γ, Δ having isomorphic polymorphism clones and being bi-interpretable is one and the same thing.

Besides the theoretical interest they might have, primitive positive interpretations are additionally motivated by an application in theoretical computer science: every relational structure Γ with a finite signature defines a computational problem, called the *constraint satisfaction problem of* Γ and denoted by $\mathrm{CSP}(\Gamma)$, and it is known that when a relational structure Δ has a primitive positive interpretation in a relational structure Γ, then $\mathrm{CSP}(\Delta)$ has a polynomial-time reduction to $\mathrm{CSP}(\Gamma)$. Very general and deep complexity classification results rely on this fact; see for example the collection of survey articles in [CKV08]. More on this application can be found in Section 6.

In this paper, we give an affirmative answer to the question from [BJ11] about primitive positive interpretability. A *reduct* of a structure Δ' is a structure on the same domain obtained by forgetting some relations or functions of Δ'. We prove the following.

Theorem 1. *Let Γ and Δ be finite or countable ω-categorical structures. Then:*

- *Δ has a primitive positive interpretation in Γ if and only if Δ is a reduct of a finite or ω-categorical structure Δ' such that there exists a continuous homomorphism from $\mathrm{Pol}(\Gamma)$ into $\mathrm{Pol}(\Delta')$ whose image is dense in $\mathrm{Pol}(\Delta')$.*
- *Γ and Δ are primitive positive bi-interpretable if and only if their polymorphism clones are isomorphic as topological clones.*

It follows from this theorem and the remarks above that the computational complexity of the constraint satisfaction problem for a relational structure in a finite language only depends on its topological polymorphism clone.

Corollary 2. *Let Γ and Δ be finite or countable ω-categorical relational structures with finite signatures. If $\mathrm{Pol}(\Gamma)$ and $\mathrm{Pol}(\Delta)$ are isomorphic as topological clones, then $\mathrm{CSP}(\Gamma)$ and $\mathrm{CSP}(\Delta)$ are polynomial-time equivalent.*

1.2. **Topological Birkhoff.** To prove Theorem 1 we show an algebraic result which is of independent interest and which can be seen as a topological version of Birkhoff's HSP theorem.

An *algebra* is a structure with a purely functional signature. The *clone of an algebra* $\mathbf{A}$ with signature τ, denoted by $\mathrm{Clo}(\mathbf{A})$, is the set of all functions with finite arity on the domain A of $\mathbf{A}$ which can be written as τ-terms over $\mathbf{A}$. More precisely, every abstract τ-term t induces a function $t^{\mathbf{A}}$ on A, and $\mathrm{Clo}(\mathbf{A})$ consists precisely of the functions of this form.

Let $\mathbf{A}$, $\mathbf{B}$ be algebras of the same signature τ. The assignment ξ from $\mathrm{Clo}(\mathbf{A})$ to $\mathrm{Clo}(\mathbf{B})$ which sends every element $t^{\mathbf{A}}$ of $\mathrm{Clo}(\mathbf{A})$ to $t^{\mathbf{B}}$ is a well-defined function if and only if for all τ-terms s,t we have that $s^{\mathbf{B}} = t^{\mathbf{B}}$ whenever $s^{\mathbf{A}} = t^{\mathbf{A}}$. In that case, it is in fact a surjective homomorphism between clones; we then call ξ the *natural homomorphism* from $\mathrm{Clo}(\mathbf{A})$ onto $\mathrm{Clo}(\mathbf{B})$.

When $\mathcal{C}$ is a class of algebras with common signature τ, then $\mathrm{P}(\mathcal{C})$ denotes the class of all products of algebras from $\mathcal{C}$, $\mathrm{P}^{\mathrm{fin}}(\mathcal{C})$ denotes the class of all *finite* products of algebras from $\mathcal{C}$, $\mathrm{S}(\mathcal{C})$ denotes the class of all subalgebras of algebras from $\mathcal{C}$, and $\mathrm{H}(\mathcal{C})$ denotes the class of all homomorphic images of algebras from $\mathcal{C}$. A *pseudovariety* is a class $\mathcal{V}$ of algebras of the same signature such that $\mathcal{V} = \mathrm{H}(\mathcal{V}) = \mathrm{S}(\mathcal{V}) = \mathrm{P}^{\mathrm{fin}}(\mathcal{V})$, i.e., a class closed under homomorphic images, subalgebras, and finite products. The pseudovariety *generated* by a class of algebras $\mathcal{C}$ (or by a single algebra $\mathbf{A}$) is the smallest pseudovariety that contains $\mathcal{C}$ (contains $\mathbf{A}$, respectively).

For *finite* algebras, Birkhoff's HSP theorem takes the following form (see Exercise 11.5 in combination with the proof of Lemma 11.8 in [BS81]).

Theorem 3 (Birkhoff)**.** *Let* $\mathbf{A}, \mathbf{B}$ *be finite algebras with the same signature. Then the following three statements are equivalent:*

(1) *The natural homomorphism from* $\mathrm{Clo}(\mathbf{A})$ *onto* $\mathrm{Clo}(\mathbf{B})$ *exists.*
(2) $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$.
(3) $\mathbf{B}$ *is contained in the pseudovariety generated by* $\mathbf{A}$.

When $\mathbf{A}$ and $\mathbf{B}$ are of arbitrary cardinality, then the equivalence of (2) and (3) still holds. However, if one wants to maintain equivalence with item (1), then another version of Birkhoff's theorem states that one has to replace finite powers by arbitrary powers in the second item, that is, one has to replace $\mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$ by $\mathrm{HSP}(\mathbf{A})$. The third item has to be adapted using the notion of a *variety* of algebras, i.e., a class of algebras of common signature closed under the operators H, S and P.

Our topological variant of Birkhoff's theorem shows that one can keep finite powers for a large class of infinite algebras if one additionally requires that the natural homomorphism from $\mathrm{Clo}(\mathbf{A})$ onto $\mathrm{Clo}(\mathbf{B})$ is Cauchy-continuous (with respect to the metric of the Baire space) when we view $\mathrm{Clo}(\mathbf{A})$ and $\mathrm{Clo}(\mathbf{B})$ as topological clones as described above.

A permutation group $\mathscr{G}$ on a countable set A is called *oligomorphic* iff for each finite $n \geq 1$, the componentwise action of $\mathscr{G}$ on A^n has finitely many orbits. In our context it is worth noting that the theorem of Ahlbrandt and Ziegler implies that being oligomorphic is a property of the abstract topological group $\mathscr{G}$, i.e., for isomorphic permutation groups $\mathscr{G}$ and $\mathscr{H}$ one is oligomorphic iff the other one is. For a characterization by abstract properties see [Tsa]. An algebra $\mathbf{A}$ is called *oligomorphic* iff the unary invertible operations in $\mathrm{Clo}(\mathbf{A})$, that is, the unary bijective operations whose inverse is also in $\mathrm{Clo}(\mathbf{A})$, form an oligomorphic permutation group. We call it *locally oligomorphic* iff the topological closure $\overline{\mathrm{Clo}(\mathbf{A})}$ in the space $\mathscr{O}_{\mathbf{A}}$ of all finitary functions on the domain of $\mathbf{A}$ is oligomorphic. Clearly, oligomorphic algebras are also locally oligomorphic; the algebra on a countable set A which has all unary operations on A which are not permutations is an example which shows that the two notions are not equivalent.

One of the motivations for oligomorphic groups is the theorem of Engeler, Svenonius, and Ryll-Nardzewski (see e.g. the textbook [Hod93]): the automorphism group of a countable structure Γ is oligomorphic if and only if Γ is ω-categorical. This implies that any *polymorphism algebra* of Γ, i.e., any algebra on the domain on Γ whose functions are precisely the elements of $\mathrm{Pol}(\Gamma)$ indexed in some arbitrary way, is oligomorphic if and only if Γ is ω-categorical. Note that such polymorphism algebras are oligomorphic if and only if they are locally oligomorphic, since their clone $\mathrm{Pol}(\Gamma)$ is always a closed subset of $\mathscr{O}_\Gamma$. It is not hard to see that all algebras in the pseudovariety generated by an oligomorphic (locally oligomorphic) algebra are again oligomorphic (locally oligomorphic). In this paper, we will prove the equivalence of (1) and (2) in the following theorem, which is a topological characterization of pseudovarieties of oligomorphic algebras. As mentioned above, the equivalence of (2) and (3) holds for arbitrary algebras $\mathbf{A}, \mathbf{B}$ and is well known from Birkhoff's work.

Theorem 4. *Let* $\mathbf{A}, \mathbf{B}$ *be locally oligomorphic or finite algebras with the same signature. Then the following three statements are equivalent:*

(1) *The natural homomorphism from* $\mathrm{Clo}(\mathbf{A})$ *onto* $\mathrm{Clo}(\mathbf{B})$ *exists and is Cauchy-continuous.*
(2) $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$.
(3) $\mathbf{B}$ *is contained in the pseudovariety generated by* $\mathbf{A}$.

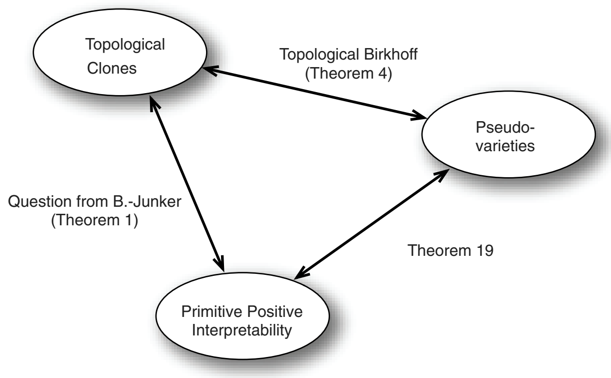

FIGURE 1. Topological clones, primitive positive interpretations, and pseudovarieties.

Note that Theorem 3 really is a special case of Theorem 4, since the topology of any clone on a finite set is discrete, and hence the natural homomorphism from the clone of a finite algebra to that of another algebra is always (Cauchy-) continuous.

We will see in Section 4 how to derive Theorem 1 from Theorem 4 and a certain correspondence between primitive positive interpretations and pseudovarieties; confer also Figure 1.

1.3. **Related work.** Pseudovarieties consisting of *finite* algebras have been studied by many researchers in many different contexts and are important in particular in formal language theory. There is also an equational characterization for pseudovarieties of finite algebras, the Eilenberg-Schützenberger theorem [ES76]. The topology used in subsequent publications [Ban83, Rei82] concerning pseudovarieties of finite algebras is different from the topology that we use here; also note that our results are about pseudovarieties that also contain infinite algebras.

[MP11], in connection with pioneering work on homomorphism-homogeneous structures [CN06], introduces the notion of *weakly oligomorphic* for relational structures via their endomorphism monoid. It could make sense to use their notion for algebras as another weakening of "oligomorphic" rather than "locally oligomorphic". In fact, every locally oligomorphic algebra would then be weakly oligomorphic, and yet it is quite possible that Theorem 4 holds and can be proven by our same methods even for the class of weakly oligomorphic algebras. However, we gave preference to "locally oligomorphic" with its more group theoretic flavor.

In our proof of Theorem 4 we will work with the closure $\overline{\mathrm{Clo}(\mathbf{A})}$ of $\mathrm{Clo}(\mathbf{A})$ in $\mathscr{O}_{\mathbf{A}}$ rather than with $\mathrm{Clo}(\mathbf{A})$ itself, allowing for a certain compactness argument. Even when the functions of $\mathbf{A}$ are assumed to form, say, a closed set, $\mathrm{Clo}(\mathbf{A})$ can be topologically complicated: [GPS13] gives an example of a (topologically) closed algebra $\mathbf{A}$ which has only unary operations and for which $\mathrm{Clo}(\mathbf{A})$ is not a Borel set.

1.4. **Outline of the paper.** This introduction is followed by Section 2, in which we will provide the proof of Theorem 4. We then give some examples in Section 3 which examine the differences between continuous and non-continuous clone homomorphisms in our context. Section 4 brings us back to the model-theoretic perspective in more detail and links Theorems 1 and 4. We will provide concrete instances of Theorem 1 in Section 5. In Section 6 we discuss applications to constraint satisfaction problems; the discussion will be followed by a concrete example in Section 7. We conclude the paper with an outlook and open problems in Section 8.

1.5. **Further conventions.** All ω-categorical structures in this paper are assumed to be countable.

If $\mathscr{F}$ is a set of finitary functions on a set and $k \geq 1$, then we write $\mathscr{F}^{(k)}$ for the k-ary functions in $\mathscr{F}$; this applies in particular to $\mathrm{Pol}(\Gamma)$ and $\mathrm{Clo}(\mathbf{A})$.

For an n-tuple a and $1 \leq i \leq n$, we write a_i for the i-th component of a. We do not always distinguish between the domain of a structure and the structure itself, so we write things like "$a \in \Gamma$" to refer to an element of Γ. In the case of algebras, however, we also write A for the domain of $\mathbf{A}$.

When we write fg for a composite of unary functions f, g, we mean that g is applied first.

2. Pseudovarieties and topological clones

2.1. **Continuity of the natural homomorphism.** The following lemma shows the easy direction of the equivalence of Theorem 4, namely that (2) implies (1).

Proposition 5. *Let $\mathbf{A}$ and $\mathbf{B}$ be countable algebras of the same signature τ. If $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$, then the natural homomorphism from $\mathrm{Clo}(\mathbf{A})$ onto $\mathrm{Clo}(\mathbf{B})$ exists and is Cauchy-continuous.*

Proof. We show the statement for the cases where $\mathbf{B}$ is a finite product of $\mathbf{A}$, or a subalgebra of $\mathbf{A}$, or a homomorphic image of $\mathbf{A}$; the full statement then follows by combining the three. It is well known that in all three cases, the natural homomorphism exists; this is because products, subalgebras, and homomorphic images of $\mathbf{A}$ satisfy at least the equations between τ-terms that hold in $\mathbf{A}$. It thus remains to show that the natural homomorphism ξ from $\mathrm{Clo}(\mathbf{A})$ onto $\mathrm{Clo}(\mathbf{B})$ is Cauchy-continuous.

Assume first that $\mathbf{B} = \mathbf{A}^n$ for some finite $n \geq 1$. Let U be an open set from the subbasis of the topology on $\mathrm{Clo}(\mathbf{A}^n)^{(k)}$, where $k \geq 1$; that is, there exist a k-tuple $a \in (A^n)^k$ and a value $v \in A^n$ such that U consists precisely of those k-ary terms of $\mathbf{A}^n$ which send a to v. Now viewing a as a matrix in $A^{k \times n}$, denote for all $1 \leq i \leq n$ by c_i the i-th column of a. Then $\xi^{-1}[U]$ consists of those k-ary terms of $\mathbf{A}$ which send c_i to v_i, for all $1 \leq i \leq n$.

Assume now that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Then the preimage of any subbasis set U of $\mathrm{Clo}(\mathbf{B})^{(k)}$ is equal to U, and hence also open in $\mathrm{Clo}(\mathbf{A})^{(k)}$.

Finally, let $\mathbf{B}$ be a homomorphic image of $\mathbf{A}$. Then $\mathbf{B}$ is isomorphic to $\mathbf{A}/{\sim}$ for a congruence relation $\sim$ of $\mathbf{A}$, and we may assume $\mathbf{B} = \mathbf{A}/{\sim}$. Let U be a subbasis set of the topology of $\mathrm{Clo}(\mathbf{B})^{(k)}$, so U consists of those functions in $\mathrm{Clo}(\mathbf{B})^{(k)}$ which send a certain tuple a of $(\mathbf{A}/{\sim})^k$ to some $v \in (\mathbf{A}/{\sim})$. Then a function $f \in \mathrm{Clo}(\mathbf{A})^{(k)}$ is an element of $\xi^{-1}[U]$ iff for any fixed k-tuple $c \in \mathbf{A}^k$ such that $c_i \in a_i$ for all $1 \leq i \leq n$, $f(c) \in v$. □

2.2. **The converse.** We will now show that (1) implies (2) in Theorem 4.

Let X, Y be countably infinite sets, and let G be a group acting on Y. We equip the set Y^X of all functions from X to Y with the topology of the Baire space, i.e., we consider Y as a discrete space and give Y^X the product topology. Now define an equivalence relation $\sim_G$ on Y^X which identifies two functions $f, g \in Y^X$ iff there exists $\alpha \in G$ such that $f = \alpha g$. We then consider the factor space $Y^X/{\sim_G}$ with the quotient topology, and also write Y^X/G for this space; therefore, a subset $O \subseteq Y^X/G$ is open iff $\bigcup O$ is open in Y^X.

Proposition 6. *Let X, Y be countably infinite sets, and let G be a group which acts on Y. Then Y^X/G is compact if and only if the action of G on Y is oligomorphic.*

Proof. We first prove that if the action of G is oligomorphic, then Y^X/G is compact. Say without loss of generality that $X = \omega$. Pick for every $n \geq 1$ and every orbit of the componentwise action of G on Y^n a representative tuple of this orbit in such a way that being a representative of an orbit is closed under taking initial segments; this can be done inductively. Write R for the set of representatives. When we partially order R by saying for $a, b \in R$ that a is smaller than or equal to b if and only if a is an initial segment of b, then R becomes a finitely branching tree, the branches of which are elements of Y^X. Consider the subspace B of Y^X of those functions which are branches of R; in other words, for $f \in Y^X$ we have $f \in B$ if and only if the restriction $f{\restriction_n}$ of f to $\{0, \ldots, n\}$ is in R, for all $n \geq 1$. Then B is compact by Tychonoff's theorem as it is homeomorphic with a closed subspace of $\prod_{n \in \omega} k(n)$, where $k(n)$ is the (finite) number of representatives of length n. Moreover, $G \cdot B := \{\alpha f \mid \alpha \in G \wedge f \in B\}$ is dense in Y^X, and so $(G \cdot B)/G$ is dense in Y^X/G. But no two elements f, g of B satisfy $f \sim_G g$, and so $(G \cdot B)/G$ is homeomorphic to B. Hence, Y^X/G has a dense compact subset, proving that Y^X/G is compact itself.

For the other direction, assume that the action of G is not oligomorphic. Pick an $n \geq 1$ such that the componentwise action of G on Y^n has infinitely many orbits, and enumerate these orbits by $(O_i)_{i \in \omega}$. Now for all $i \in \omega$, let U_i consist of all classes $[f]_{\sim_G}$ in Y^X/G with the property that $f{\restriction_n}$ belongs to O_i; this is well defined since for all $f, g \in Y^X$ with $f \sim_G g$ we have that $f{\restriction_n}$ belongs to O_i iff $g{\restriction_n}$ belongs to O_i. Then Y^X/G is the disjoint union of the U_i. But each U_i is open, and hence Y^X/G is not compact. □

We remark that the space Y^X/G is not Hausdorff, which explains that it can have a dense compact subset which is not equal to the whole space – some readers might have wondered about this.

Lemma 7. *Let X, Y be countable sets, and let G be a group with an oligomorphic action on Y. Let S be a closed subset of Y^X which is invariant under G, i.e., $G \cdot S \subseteq S$. Then S/G is compact.*

Proof. S/G is a closed subspace of the compact space Y^X/G. □

For a structure Δ, we write $\mathrm{Emb}(\Delta)$ for the set of self-embeddings of Δ.

Corollary 8. *Let Δ be an ω-categorical structure. Then the following spaces are compact:*

- $\operatorname{Emb}(\Delta)/\operatorname{Aut}(\Delta)$*;*
- $\operatorname{End}(\Delta)/\operatorname{Aut}(\Delta)$*;*
- $\operatorname{Pol}^{(k)}(\Delta)/\operatorname{Aut}(\Delta)$*, for all $k \geq 1$.*

Moreover, if $\mathbf{A}$ is a locally oligomorphic algebra and $\mathscr{G}$ is the group of all invertible unary bijections in $\overline{\operatorname{Clo}(\mathbf{A})}$, then $\overline{\operatorname{Clo}(\mathbf{A})}^{(k)}/\mathscr{G}$ is compact, for all $k \geq 1$.

Proof. $\operatorname{Emb}(\Delta)$, $\operatorname{End}(\Delta)$ and $\operatorname{Pol}^{(k)}(\Delta)$ are closed subsets of Δ^Δ and Δ^{Δ^k}, respectively, which are invariant under $\operatorname{Aut}(\Delta)$. Since Δ is ω-categorical, the action of $\operatorname{Aut}(\Delta)$ on Δ is oligomorphic by the theorem of Engeler, Svenonius, and Ryll-Nardzewski (see [Hod93]), and hence the first statement follows Lemma 7. The argument for the second statement is identical. $\square$

Note that $\operatorname{Pol}(\Delta)/\operatorname{Aut}(\Delta)$ is never compact since it is the disjoint union of the spaces $\operatorname{Pol}^{(k)}(\Delta)/\operatorname{Aut}(\Delta)$.

Notation 9. Let D be a set, and let f be a k-ary function on D for some $k \geq 1$. If $C \in D^{m\times k}$ for some $m \geq 1$, then we write $f(C)$ for the tuple of size m obtained by applying f to each row of the matrix C.

Lemma 10. *Let $\mathbf{A}, \mathbf{B}$ be algebras of the same signature, where $\mathbf{A}$ is locally oligomorphic. Assume that the natural homomorphism ξ from $\operatorname{Clo}(\mathbf{A})$ onto $\operatorname{Clo}(\mathbf{B})$ exists and is Cauchy-continuous. Then for all finite $F \subseteq B$ and all $k \geq 1$ there exist an $m \geq 1$ and $C \in A^{m\times k}$ such that for all $f, g \in \operatorname{Clo}^{(k)}(\mathbf{A})$ we have that $f(C) = g(C)$ implies $\xi(f)\restriction_F = \xi(g)\restriction_F$.*

Proof. We denote the unique continuous extension of ξ to $\overline{\operatorname{Clo}(\mathbf{A})}$ by $\bar\xi$. So $\bar\xi$ is a continuous mapping from $\overline{\operatorname{Clo}(\mathbf{A})}$ into $\overline{\operatorname{Clo}(\mathbf{B})}$. Moreover, it is a homomorphism: if $f \in \overline{\operatorname{Clo}(\mathbf{A})}^{(n)}$ and $g_1, \ldots, g_n \in \overline{\operatorname{Clo}(\mathbf{A})}^{(l)}$, where $n, l \geq 1$, then there exist sequences $(f^i)_{i\in\omega}$ and $(g_j^i)_{i\in\omega}$ of functions in $\operatorname{Clo}(\mathbf{A})^{(n)}$ and in $\operatorname{Clo}(\mathbf{A})^{(l)}$ which converge to f and g_j, respectively, and so

$$\begin{aligned}\bar\xi(f(g_1,\ldots,g_n)) &= \bar\xi(\lim_{i\to\infty}(f^i(g_1^i,\ldots,g_l^i))) = \lim_{i\to\infty}\xi(f^i(g_1^i,\ldots,g_n^i)) \\ &= \lim_{i\to\infty}\xi(f^i)(\xi(g_1^i),\ldots,\xi(g_n^i)) = \bar\xi(f)(\bar\xi(g_1),\ldots,\bar\xi(g_n)).\end{aligned}$$

We will prove the existence of $m \geq 1$ and $C \in A^{m\times k}$ such that for all $f, g \in \overline{\operatorname{Clo}(\mathbf{A})}^{(k)}$ we have that $f(C) = g(C)$ implies $\bar\xi(f)\restriction_F = \bar\xi(g)\restriction_F$; the lemma then clearly follows.

Recall that the basic open sets of $\overline{\operatorname{Clo}(\mathbf{A})}^{(k)}$ are precisely sets of the form

$$O_{D,a} := \{f \in \overline{\operatorname{Clo}(\mathbf{A})}^{(k)} \mid f(D) = a\},$$

for $l \geq 1$, a matrix $D \in A^{l\times k}$ and a vector $a \in A^l$; the basic open sets of $\overline{\operatorname{Clo}(\mathbf{B})}^{(k)}$ are defined similarly. Call a basic open set O of $\overline{\operatorname{Clo}(\mathbf{A})}^{(k)}$ an *island* iff $\bar\xi(f)\restriction_F = \bar\xi(g)\restriction_F$ for all $f, g \in O$. From the definition of the basic open sets it is clear that for $f \in \overline{\operatorname{Clo}(\mathbf{A})}^{(k)}$, the set of all $h \in \overline{\operatorname{Clo}(\mathbf{B})}^{(k)}$ which agree with $\bar\xi(f)$ on F is open in $\overline{\operatorname{Clo}(\mathbf{B})}$. Hence, the continuity of $\bar\xi$ implies that every $f \in \overline{\operatorname{Clo}(\mathbf{A})}^{(k)}$ is contained in a basic open island.

Write $\mathscr{G}$ for the group of unary invertible bijections in $\overline{\mathrm{Clo}(\mathbf{A})}$. Then $\mathscr{G}$ is oligomorphic as $\mathbf{A}$ is locally oligomorphic. Observe next that for any basic open island $O_{D,a}$ of $\overline{\mathrm{Clo}(\mathbf{A})}^{(k)}$, the set $\mathscr{G} \cdot O_{D,a} = \{\alpha f \mid \alpha \in \mathscr{G} \wedge f \in O_{D,a}\}$ is an open subset of $\overline{\mathrm{Clo}(\mathbf{A})}^{(k)}$ which is invariant under $\mathscr{G}$. Hence, it defines an open subset $V_{D,a}$ of $\overline{\mathrm{Clo}(\mathbf{A})}^{(k)}/\mathscr{G}$, namely the set of all $\sim_{\mathscr{G}}$-classes which have a representative in $O_{D,a}$. So every class $[f]_{\sim_{\mathscr{G}}}$ is contained in some set $V_{D,a}$ for a basic open island $O_{D,a}$. Since $\overline{\mathrm{Clo}(\mathbf{A})}^{(k)}/\mathscr{G}$ is compact by Proposition 6, there are finitely many basic open islands $O_{D_1,a_1}, \dots, O_{D_n,a_n}$ such that the corresponding sets V_{D_i,a_i} cover $\overline{\mathrm{Clo}(\mathbf{A})}^{(k)}/\mathscr{G}$. We then have that $\overline{\mathrm{Clo}(\mathbf{A})}^{(k)}$ is covered by the sets $\mathscr{G} \cdot O_{D_i,a_i}$. Set $m := l_1 + \dots + l_n$, where l_i denotes the number of rows of D_i, for $1 \leq i \leq n$. Let C be the matrix of dimension $m \times k$ which is obtained by superposing the D_i. To see that C satisfies the desired property, let $f, g \in \overline{\mathrm{Clo}(\mathbf{A})}^{(k)}$. Assume wlog that $f \in \mathscr{G} \cdot O_{D_1,a_1}$; then there exists $\alpha \in \mathscr{G}$ such that $f(D_1) = \alpha(a_1)$. Since $f(C) = g(C)$, we have $f(D_1) = g(D_1)$, and so also $g(D_1) = \alpha(a_1)$. Hence, $\alpha^{-1} f$ and $\alpha^{-1} g$ are in O_{D_1,a_1}, implying $\bar{\xi}(\alpha^{-1} f)\restriction_F = \bar{\xi}(\alpha^{-1} g)\restriction_F$ since O_{D_1,a_1} is an island. Thus, $\bar{\xi}(f)\restriction_F = \bar{\xi}(g)\restriction_F$ since $\bar{\xi}$ is a homomorphism. □

Definition 11. We say that an algebra $\mathbf{A}$ of signature τ is *finitely generated* iff there exists a finite subset F of the domain of $\mathbf{A}$ such that the only subalgebra of $\mathbf{A}$ containing F is $\mathbf{A}$ itself; in other words, every element a of $\mathbf{A}$ can be written as $t^{\mathbf{A}}(b_1, \dots, b_k)$ for some $k \geq 1$, a k-ary τ-term t, and $b_1, \dots, b_k \in F$.

Proposition 12. *Let* $\mathbf{A}, \mathbf{B}$ *be algebras of the same signature* τ*, where* $\mathbf{A}$ *is locally oligomorphic and* $\mathbf{B}$ *is finitely generated. If the natural homomorphism from* $\mathrm{Clo}(\mathbf{A})$ *onto* $\mathrm{Clo}(\mathbf{B})$ *exists and is Cauchy-continuous, then* $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$.

Proof. Let $F = \{b_1, \dots, b_k\}$ be a set of generators of $\mathbf{B}$, and let $m \geq 1$ and $C \in A^{m \times k}$ be given by Lemma 10. Let $\mathbf{S}$ be the subalgebra of $\mathbf{A}^m$ generated by the columns $c_1, \dots, c_k$ of C; so the elements of $\mathbf{S}$ are precisely those of the form $t^{\mathbf{A}^m}(c_1, \dots, c_k)$, for a k-ary τ-term t. Define a function $\mu \colon \mathbf{S} \to \mathbf{B}$ by setting $\mu(t^{\mathbf{A}^m}(c_1, \dots, c_k)) := t^{\mathbf{B}}(b_1, \dots, b_k)$. Then μ is well defined, for if $t^{\mathbf{A}^m}(c_1, \dots, c_k) = s^{\mathbf{A}^m}(c_1, \dots, c_k)$, then $t^{\mathbf{B}}(b_1, \dots, b_k) = s^{\mathbf{B}}(b_1, \dots, b_k)$ by the properties of C. Since $\mathbf{B}$ is generated by F, μ is onto. We claim that μ is moreover a homomorphism; it then follows that $\mathbf{B}$ is the homomorphic image of the subalgebra $\mathbf{S}$ of $\mathbf{A}^m$, and so $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$. To this end, let f be any function symbol of τ, let n be its arity, and let $s_1, \dots, s_n \in \mathbf{S}$. Write $s_i = t_i^{\mathbf{A}^m}(c_1, \dots, c_k) = t_i^{\mathbf{S}}(c_1, \dots, c_k)$ for all $1 \leq i \leq n$. Then

$$\begin{aligned}
\mu(f^{\mathbf{S}}(s_1, \dots, s_n)) &= \mu(f^{\mathbf{S}}(t_1^{\mathbf{S}}(c_1, \dots, c_k), \dots, t_n^{\mathbf{S}}(c_1, \dots, c_k))) \\
&= \mu(f^{\mathbf{S}}(t_1^{\mathbf{S}}, \dots, t_n^{\mathbf{S}})(c_1, \dots, c_k)) \\
&= \mu((f(t_1, \dots, t_n))^{\mathbf{S}}(c_1, \dots, c_k)) = (f(t_1, \dots, t_n))^{\mathbf{B}}(b_1, \dots, b_k) \\
&= f^{\mathbf{B}}(t_1^{\mathbf{B}}(b_1, \dots, b_k), \dots, t_n^{\mathbf{B}}(b_1, \dots, b_k)) \\
&= f^{\mathbf{B}}(\mu(s_1), \dots, \mu(s_n)).
\end{aligned}$$

□

Proposition 13. *Let* $\mathbf{B}$ *be an algebra which is locally oligomorphic. Then* $\mathbf{B}$ *is finitely generated.*

Proof. Let $\mathscr{G}$ be the permutation group of invertible unary bijections of $\overline{\mathrm{Clo}(\mathbf{B})}$. Since $\mathbf{B}$ is locally oligomorphic, the action of $\mathscr{G}$ on B has finitely many orbits. Picking a representative from each orbit one obtains a generating set for $\mathbf{B}$. □

Theorem 4 now follows from Propositions 5, 12, and 13. Note that in the theorem, it would have been sufficient to assume that $\mathbf{B}$ be finitely generated rather than locally oligomorphic, but since we are mainly interested in polymorphism clones of ω-categorical structures we have chosen to formulate the theorem as it is. The following is the stronger variant which follows from Propositions 5 and 12.

Theorem 14. *Let $\mathbf{A}, \mathbf{B}$ be countable algebras with the same signature, where $\mathbf{A}$ is locally oligomorphic and $\mathbf{B}$ is finitely generated. Then the following three statements are equivalent:*

(1) *The natural homomorphism from* $\mathrm{Clo}(\mathbf{A})$ *onto* $\mathrm{Clo}(\mathbf{B})$ *exists and is Cauchy-continuous.*
(2) $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$.
(3) $\mathbf{B}$ *is contained in the pseudovariety generated by* $\mathbf{A}$.

3. Pseudovariety examples

We now give two examples examining the continuity condition on the natural homomorphism in Theorem 4. The first example is due to Keith Kearnes [Kea07] and demonstrates that there are oligomorphic algebras $\mathbf{A}$ such that the variety generated by $\mathbf{A}$ contains finite members which the pseudovariety generated by $\mathbf{A}$ does not contain.

Proposition 15. *There are algebras $\mathbf{A}, \mathbf{B}$ with common signature such that*

- $\mathbf{A}$ *is locally oligomorphic;*
- $\mathbf{B}$ *is finite;*
- $\mathbf{B} \in \mathrm{HSP}(\mathbf{A})$*;*
- $\mathbf{B} \notin \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$.

Hence, the natural homomorphism from $\mathrm{Clo}(\mathbf{A})$ *onto* $\mathrm{Clo}(\mathbf{B})$ *exists but is not continuous.*

Proof. Let the signature τ consist of unary function symbols $(f_i)_{i\in\omega}$ and $(g_i)_{i\in\omega}$. Let $\mathbf{A}$ be any algebra on ω with signature τ such that the functions $f_i^{\mathbf{A}}$ form a locally oligomorphic permutation group, such that no $g_i^{\mathbf{A}}$ is injective, and such that $f_0^{\mathbf{A}}$ is contained in the topological closure of $\{g_i^{\mathbf{A}}\}_{i\in\omega}$. Let $\mathbf{B}$ be the τ-algebra on $\{0,1\}$ such that $f_i^{\mathbf{B}}$ is the identity function for all $i \in \omega$ and such that $g_i^{\mathbf{B}}$ is the constant function with value 0. It is easy to see that the natural homomorphism from $\mathrm{Clo}(\mathbf{A})$ onto $\mathrm{Clo}(\mathbf{B})$ exists. However, it is not continuous since $f_0^{\mathbf{A}}$ is contained in the topological closure of $\{g_i^{\mathbf{A}}\}_{i\in\omega}$, but $f_0^{\mathbf{B}}$ is not contained in the topological closure of $\{g_i^{\mathbf{B}}\}_{i\in\omega}$. □

We remark that one can easily modify the previous example to obtain algebras $\mathbf{A}, \mathbf{B}$ with finite signature and the same properties. On the other hand, by taking an uncountable signature, one can make $\mathbf{A}$ even oligomorphic.

The next example becomes relevant when one has (concrete) clones without a signature of a corresponding algebra; this is for example the case for polymorphism clones of structures, as in Theorem 1 and in the following sections. It shows that when we are given two such clones $\mathscr{C}, \mathscr{D}$, then it might happen that there exists a homomorphism from $\mathscr{C}$ onto $\mathscr{D}$ which is not continuous, as well as a continuous clone homomorphism onto $\mathscr{D}$. In other words, when we make algebras out of $\mathscr{C}$ and $\mathscr{D}$ by matching the functions in $\mathscr{C}$ and $\mathscr{D}$ with an appropriate functional signature τ, then we might do so in such a way that the natural homomorphism from $\mathscr{C}$ onto $\mathscr{D}$ exists and is continuous, and in another way such that the natural homomorphism from $\mathscr{C}$ onto $\mathscr{D}$ exists but is not continuous.

Proposition 16. *There are algebras* $\mathbf{A}, \mathbf{B}$ *such that*

- $\mathbf{A}$ *is oligomorphic;*
- $\mathbf{B}$ *is finite;*
- *there exists a non-continuous clone homomorphism from* $\mathrm{Clo}(\mathbf{A})$ *onto* $\mathrm{Clo}(\mathbf{B})$*;*
- *there exists a Cauchy-continuous clone homomorphism from* $\mathrm{Clo}(\mathbf{A})$ *onto* $\mathrm{Clo}(\mathbf{B})$.

Proof. Let $\mathbf{B}$ be as in Proposition 15. Let $\mathbf{A}$ be the algebra on ω which has the following three sets of functions:

$$\mathscr{F}_1 := \{f \in \omega^\omega \mid f(0) = f(1) = 1 \text{ and } (\forall n \geq 2\ f(n) \geq 2) \text{ and } f \text{ is not surjective}\},$$
$$\mathscr{F}_2 := \{f \in \omega^\omega \mid f(0) = f(1) = 1 \text{ and } f\restriction_{[2,\infty)} \text{ is a permutation on } [2,\infty]\},$$
$$\mathscr{F}_3 := \{f \in \omega^\omega \mid f(0) = 0 \text{ and } f(1) = 1 \text{ and } f \text{ is a permutation on } \omega\}.$$

Now observe that if $f \in \mathscr{F}_i$ and $g \in \mathscr{F}_j$, then $f \circ g \in \mathscr{F}_{\min(i,j)}$. The function which sends all elements of $\mathscr{F}_1 \cup \mathscr{F}_2$ to the constant function of $\mathbf{B}$ and all elements of $\mathscr{F}_3$ to the identity induces a continuous homomorphism from $\mathrm{Clo}(\mathbf{A})$ onto $\mathrm{Clo}(\mathbf{B})$. On the other hand, the function which sends all elements of $\mathscr{F}_1$ to the constant unary function of $\mathrm{Clo}(\mathbf{B})$ and all elements of $\mathscr{F}_2 \cup \mathscr{F}_3$ to the identity in $\mathrm{Clo}(\mathbf{B})$ induces a non-continuous homomorphism from $\mathrm{Clo}(\mathbf{A})$ onto $\mathrm{Clo}(\mathbf{B})$. ☐

4. Primitive positive interpretations

In this section we prove Theorem 1. Our definition of interpretations follows [Hod93] and is standard, and will be recalled in the following. Let τ be a signature, and let Γ be a τ-structure. If $\delta(x_1, \ldots, x_k)$ is a first-order τ-formula with k free variables $x_1, \ldots, x_k$, we write $\delta(\Gamma^k)$ for the k-ary relation that is defined by δ on Γ.

An atomic τ-formula is called *unnested* iff it is of the form $x_0 = x_1$, of the form $x_0 = f(x_1, \ldots, x_n)$, or of the form $R(x_1, \ldots, x_n)$, for some n-ary function symbol $f \in \tau$ or relation symbol $R \in \tau$, and variables $x_0, x_1, \ldots, x_n$. It is straightforward to see that every atomic τ-formula is equivalent to a primitive positive τ-formula whose atomic subformulas are unnested (see Theorem 2.6.1 in [Hod93]).

Definition 17. A σ-structure Δ has a *(first-order) interpretation* I in a τ-structure Γ iff there exists a natural number $d \geq 1$, called the *dimension* of I, and

- a τ-formula $\delta_I(x_1, \ldots, x_d)$ – called the *domain formula*,
- for each unnested atomic σ-formula $\phi(y_1, \ldots, y_k)$ a τ-formula $\phi_I(\overline{x}_1, \ldots, \overline{x}_k)$ where the $\overline{x}_i$ denote disjoint d-tuples of distinct variables – called the *defining formulas*,
- a surjective map $h\colon \delta_I(\Gamma^d) \to \Delta$ – called the *coordinate map*,

such that for every unnested atomic σ-formula ϕ and all tuples $\overline{a}_i \in \delta_I(\Gamma^d)$,

$$\Delta \models \phi(h(\overline{a}_1), \ldots, h(\overline{a}_k)) \ \Leftrightarrow\ \Gamma \models \phi_I(\overline{a}_1, \ldots, \overline{a}_k)\ .$$

If the formulas δ_I and ϕ_I are primitive positive (existential positive), we say that the interpretation I is *primitive positive* (*existential positive*). Note that the dimension d, the set $S := \delta_I(\Gamma^d)$, and the coordinate map h determine the defining formulas up to logical equivalence; hence, we sometimes denote an interpretation by $I = (d, S, h)$.

4.1. **Primitive positive interpretations and pseudovarieties.** For ω-categorical structures Γ, primitive positive interpretability in Γ can be characterized in terms of the pseudovariety generated by a polymorphism algebra of Γ. Via the results of the previous section, pseudovarieties also correspond to topological clones – so they provide the link between primitive positive interpretations and topological clones, which will be used to prove Theorem 1 in Section 4.2 (confer also Figure 1).

Definition 18. Let Γ be a structure and $\mathbf{A}$ an algebra. Then $\mathbf{A}$ is called a *polymorphism algebra* of Γ iff $\mathbf{A}$ and Γ have the same domain, and the set of operations of $\mathbf{A}$ is precisely the set of polymorphisms of Γ.

Clearly, every structure Γ has a polymorphism algebra, which can be obtained by assigning function names to the polymorphisms in some arbitrary way.

Theorem 19. *Let Γ be a finite or ω-categorical structure, and let Δ be an arbitrary structure. Then the following are equivalent:*

(1) *for every polymorphism algebra $\mathbf{C}$ of Γ there is an algebra $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{C})$ such that $\mathrm{Clo}(\mathbf{B}) \subseteq \mathrm{Pol}(\Delta)$;*
(2) *there is a polymorphism algebra $\mathbf{C}$ of Γ and an algebra $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{C})$ such that $\mathrm{Clo}(\mathbf{B}) \subseteq \mathrm{Pol}(\Delta)$;*
(3) *Δ has a primitive positive interpretation in Γ.*

The equivalence between (1) and (2) emphasizes the fact that for our purposes, it does not matter in what way we assign function names to the polymorphisms of Γ. Theorem 19 already appeared in the survey article [Bod08]; it was inspired by results obtained in the context of constraint satisfaction problems for finite structures [BKJ05]. Since we need Theorem 19 in a more detailed form (Proposition 21), we provide its full proof here.

Let Γ be a τ-structure with domain D, and $R \subseteq D^k$ a k-ary relation. We say that R is *primitive positive definable* in Γ iff there exists a primitive positive τ-formula $\phi(x_1, \ldots, x_k)$ such that for all $(c_1, \ldots, c_k) \in D^k$ it is true that $(c_1, \ldots, c_k) \in R$ if and only if Γ satisfies $\phi(c_1, \ldots, c_k)$. We say that a τ-formula ϕ with k free variables is *preserved* by a function $f \colon D^l \to D$ (over Γ) iff for all $t_1^1, \ldots, t_l^k \in D$, if $\Gamma \models \phi(t_i^1, \ldots, t_i^k)$ for all $i \leq l$, then $\Gamma \models \phi(f(t_1^1, \ldots, t_l^1), \ldots, f(t_1^k, \ldots, t_l^k))$. Note that f is a polymorphism of Γ if and only if f preserves all atomic unnested τ-formulas over Γ. We say that a relation $R \subseteq D^k$ (a function $g \colon D^k \to D$) is preserved by f iff f is a polymorphism of the structure $(D; R)$ (of $(D; g)$).

We need the following characterization of primitive positive definability in ω-categorical structures Γ; for finite structures Γ, this is due to [Gei68, BKKR69].

Theorem 20 (from [BN06])**.** *Let Γ be finite or ω-categorical. Then a relation R has a primitive positive definition in Γ if and only if R is preserved by all polymorphisms of Γ.*

For example, when D is the domain of an ω-categorical structure Γ and $\mathbf{C}$ is a polymorphism algebra of Γ, then an equivalence relation $R \subseteq D^2$ is a congruence of $\mathbf{C}$ if and only if R is primitive positive definable in Γ.

Proof of Theorem 19. The implication from (1) to (2) follows from the existence of a polymorphism algebra $\mathbf{C}$ of Γ.

(2) $\Rightarrow$ (3). Write τ for the signature of $\mathbf{C}$. There exists a finite number $d \geq 1$, a subalgebra $\mathbf{S}$ of $\mathbf{C}^d$ with domain S, and a surjective homomorphism h from $\mathbf{S}$ to $\mathbf{B}$. We claim that Δ has the primitive positive interpretation $I := (d, S, h)$ in Γ. All operations of $\mathbf{C}$ preserve S (viewed as a d-ary relation over Γ), since $\mathbf{S}$ is a subalgebra of $\mathbf{C}^d$. Theorem 20 implies that S has a primitive positive definition $\delta(x_1, \ldots, x_d)$ in Γ, which becomes the domain formula δ_I.

Let ψ be an unnested atomic formula over the signature of Δ and with k free variables $x_1, \ldots, x_k$. Let $R \subseteq C^{dk}$ be the relation defined by

$$(a_1^1, \ldots, a_1^d, \ldots, a_k^1, \ldots, a_k^d) \in R \;\Leftrightarrow\; \Delta \models \psi(h(a_1^1, \ldots, a_1^d), \ldots, h(a_k^1, \ldots, a_k^d)) \,,$$

and let $f \in \tau$ be arbitrary. By assumption, $f^{\mathbf{B}}$ preserves ψ. Since h is a homomorphism, it follows that $f^{\mathbf{C}}$ preserves R. We conclude that all polymorphisms of Γ preserve R. Since Γ is ω-categorical and by Theorem 20, the relation R has a primitive positive definition in Γ, which becomes the defining formula for $\psi(x_1, \ldots, x_k)$. So I is indeed a primitive positive interpretation of Δ in Γ.

To prove (3) $\Rightarrow$ (1), suppose that Δ has a primitive positive interpretation $I = (d, S, h)$ in Γ. Let $\mathbf{C}$ be a polymorphism algebra of Γ, and let τ be the signature of $\mathbf{C}$. We have to show that $\mathrm{HSP}^{\mathrm{fin}}(\mathbf{C})$ contains a τ-algebra $\mathbf{B}$ such that all operations in $\mathbf{B}$ are polymorphisms of Δ. The set S is preserved by all operations of $\mathrm{Clo}(\mathbf{C}) = \mathrm{Pol}(\Gamma)$, because it is primitive positive definable in Γ by the domain formula of I (Theorem 20). Therefore, S induces a subalgebra $\mathbf{S}$ of $\mathbf{C}^d$. Let K be the kernel of the coordinate map h of I. Then for all tuples $\overline{a}, \overline{b} \in S$, the $2d$-tuple $(\overline{a}, \overline{b})$ satisfies $=_I$ in Γ if and only if $(\overline{a}, \overline{b}) \in K$. Since $=_I$ is primitive positive definable in Γ, it is preserved by all operations of $\mathbf{C}$ by Theorem 20. It follows that K is a congruence of $\mathbf{S}$. As a consequence, h induces a τ-algebra $\mathbf{B}$ on its image, which equals the domain of Δ, in such a way that h is a homomorphism from $\mathbf{S}$ onto $\mathbf{B}$: let $f \in \tau$ be m-ary, and let $c_1, \ldots, c_m$ be arbitrary elements of Δ. Then pick $\overline{a}_1, \ldots, \overline{a}_m \in S$ such that $h(\overline{a}_i) = c_i$, and define $f^{\mathbf{B}}(c_1, \ldots, c_m) := h(f^{\mathbf{S}}(\overline{a}_1), \ldots, f^{\mathbf{S}}(\overline{a}_m))$. This is well defined since the kernel K of h is a congruence of $\mathbf{S}$, and by definition of $\mathbf{B}$, h is a homomorphism from $\mathbf{S}$ onto $\mathbf{B}$. It remains to verify that for all $f \in \tau$, $f^{\mathbf{B}}$ is a polymorphism of Δ, i.e., every unnested atomic formula ϕ over Δ is preserved by $f^{\mathbf{B}}$. From the definitions of ϕ_I and $f^{\mathbf{B}}$, one easily sees that $f^{\mathbf{B}}$ preserves ϕ over Δ if and only if $f^{\mathbf{C}}$ preserves ϕ_I over Γ. Since $f^{\mathbf{C}}$ is a polymorphism of Γ, and since ϕ_I is a primitive positive τ-formula over Γ, $f^{\mathbf{C}}$ indeed preserves ϕ_I, and hence $f^{\mathbf{B}}$ preserves ϕ. $\square$

The proof of Theorem 19 above gives more information about the link between polymorphism algebras and primitive positive interpretations, and we state it explicitly.

Proposition 21. *Let Γ be a finite or ω-categorical structure with domain D, and let Δ be an arbitrary structure with domain B. Then for all $d \geq 1$, $S \subseteq D^d$, and $h\colon S \to B$ the following are equivalent:*

(1) *For every polymorphism algebra $\mathbf{C}$ of Γ the set S induces a subalgebra $\mathbf{S}$ of $\mathbf{C}^d$, the kernel of h is a congruence of $\mathbf{S}$, and the homomorphic image $\mathbf{B}$ of $\mathbf{S}$ under h satisfies $\mathrm{Clo}(\mathbf{B}) \subseteq \mathrm{Pol}(\Delta)$;*

(2) *Δ has the primitive positive interpretation (d, S, h) in Γ.*

4.2. **Primitive positive interpretations and topological clones.** We can now show the first part of Theorem 1.

Proposition 22. *Let* Γ *be finite or* ω*-categorical, and* Δ *be arbitrary. Then* Δ *has a primitive positive interpretation in* Γ *if and only if* Δ *is the reduct of a finite or* ω*-categorical structure* Δ' *such that there exists a continuous clone homomorphism from* $\mathrm{Pol}(\Gamma)$ *to* $\mathrm{Pol}(\Delta')$ *whose image is dense in* $\mathrm{Pol}(\Delta')$*.*

Proof. Let $\mathbf{C}$ be a polymorphism algebra of Γ.

Suppose first that Δ has a primitive positive interpretation in Γ. By Theorem 19 there is an algebra $\mathbf{B}$ in the pseudovariety generated by $\mathbf{C}$ such that all operations of $\mathbf{B}$ are polymorphisms of Δ. Since Γ is finite or ω-categorical, $\mathbf{C}$ is finite or oligomorphic, and the algebra $\mathbf{B}$ is finite or oligomorphic as well. By Theorem 4 the natural homomorphism ξ from $\mathrm{Clo}(\mathbf{C})$ onto $\mathrm{Clo}(\mathbf{B})$ exists and is continuous. Let Δ' be the structure with the same domain as $\mathbf{B}$ that contains all relations and all functions preserved by all operations of $\mathbf{B}$. Since $\mathrm{Clo}(\mathbf{B}) \subseteq \mathrm{Pol}(\Delta')$, it follows that Δ' is finite or ω-categorical by the theorem of Engeler, Svenonius, and Ryll-Nardzewski. Moreover, it is easy to see and well known that $\mathrm{Pol}(\Delta') = \overline{\mathrm{Clo}(\mathbf{B})}$, so the image of ξ is dense in $\mathrm{Pol}(\Delta')$. Since all operations of $\mathbf{B}$ are polymorphisms of Δ, all relations and functions of Δ are relations and functions of Δ', and this shows that Δ is indeed a reduct of Δ'.

To prove the converse, let Δ' be a finite or ω-categorical structure such that Δ is a reduct of Δ', and such that there is a continuous homomorphism ξ from $\mathrm{Pol}(\Gamma)$ to $\mathrm{Pol}(\Delta')$ whose image is dense in $\mathrm{Pol}(\Delta')$. Let $\mathbf{B}$ be the algebra with the same domain as Δ, the same signature τ as $\mathbf{C}$, and where $f \in \tau$ denotes the operation $\xi(f^{\mathbf{C}})$ of $\mathrm{Pol}(\Delta')$. Then $\overline{\mathrm{Clo}(\mathbf{B})} = \mathrm{Pol}(\Delta')$ since the image of ξ is dense in $\mathrm{Pol}(\Delta')$. Hence, $\mathbf{B}$ is finite or locally oligomorphic since Δ' is finite or ω-categorical. We can therefore apply Theorem 4 to infer $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{C})$. By Theorem 19, Δ' has a primitive positive interpretation in Γ. It follows that in particular Δ has a primitive positive interpretation in Γ. $\square$

In Section 5 we will present an example showing that in Proposition 22 we cannot simply require the continuous clone homomorphism ξ to be surjective. In particular, the image of a closed oligomorphic clone under a continuous homomorphism need not be closed.

How do we recognize whether two structures Γ and Δ have isomorphic topological polymorphism clones?

Definition 23. Two structures Γ and Δ such that Γ has a primitive positive interpretation in Δ and Δ has a primitive positive interpretation in Γ are called *mutually primitive positive interpretable.*

We will see in Section 5 that there are ω-categorical structures Γ and Δ that are mutually primitive positive interpretable and have non-isomorphic topological polymorphism clones. To characterize the situation where Γ and Δ have isomorphic topological polymorphism clones, we need the following stronger notion.

Definition 24. Two structures Γ and Δ are called *primitive positive bi-interpretable*[1] iff there is an interpretation $I = (d_1, S_1, h_1)$ of Δ in Γ and an interpretation

[1] Here we follow the analogous definition for *first-order bi-interpretability* as introduced in [AZ86].

$J = (d_2, S_2, h_2)$ of Γ in Δ such that the $(1 + d_1 d_2)$-ary relation R_{IJ} defined by

$$x = h_1(h_2(y_{1,1}, \ldots, y_{1,d_2}), \ldots, h_2(y_{d_1,1}, \ldots, y_{d_1,d_2}))$$

is primitive positive definable in Δ and the $(1 + d_1 d_2)$-ary relation R_{JI} defined by

$$x = h_2(h_1(y_{1,1}, \ldots, y_{1,d_1}), \ldots, h_1(y_{d_2,1}, \ldots, y_{d_2,d_1}))$$

is primitive positive definable in Γ.

In the following, we write $h_1 \circ h_2$ for the function defined by

$$\begin{aligned}&(y_{1,1}, \ldots, y_{1,d_2}, \ldots, y_{d_1,1}, \ldots, y_{d_1,d_2}) \\ &\qquad \mapsto h_1(h_2(y_{1,1}, \ldots, y_{1,d_2}), \ldots, h_2(y_{d_1,1}, \ldots, y_{d_1,d_2}))\,.\end{aligned}$$

Proposition 25. *Let Γ and Δ be finite or ω-categorical. Then the following are equivalent:*

(1) $\mathrm{Pol}(\Gamma)$ *and* $\mathrm{Pol}(\Delta)$ *are isomorphic as topological clones.*
(2) Γ *has a polymorphism algebra* $\mathbf{A}$*, and* Δ *has a polymorphism algebra* $\mathbf{B}$ *such that* $\mathrm{HSP}^{\mathrm{fin}}(\mathbf{A}) = \mathrm{HSP}^{\mathrm{fin}}(\mathbf{B})$*.*
(3) Γ *and* Δ *are primitive positive bi-interpretable.*

Proof. We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Let $\mathbf{A}$ be a polymorphism algebra of Γ with signature τ, and suppose that $\mathrm{Pol}(\Gamma)$ and $\mathrm{Pol}(\Delta)$ are isomorphic via a homeomorphism ξ. Let $\mathbf{B}$ be the algebra with the same domain as Δ and signature τ such that $f^{\mathbf{B}} = \xi(f^{\mathbf{A}})$ for all $f \in \tau$. Then $\mathbf{B}$ is a polymorphism algebra of Δ, and it follows from Theorem 4 that $\mathrm{HSP}^{\mathrm{fin}}(\mathbf{A}) = \mathrm{HSP}^{\mathrm{fin}}(\mathbf{B})$. Thus (1) indeed implies (2).

(2) $\Rightarrow$ (3). Suppose that Γ has a polymorphism algebra $\mathbf{A}$ and Δ has a polymorphism algebra $\mathbf{B}$ such that $\mathrm{HSP}^{\mathrm{fin}}(\mathbf{A}) = \mathrm{HSP}^{\mathrm{fin}}(\mathbf{B})$. So there is a $d_1 \geq 1$, a subalgebra $\mathbf{S}_1$ of $\mathbf{A}^{d_1}$, and a surjective homomorphism h_1 from $\mathbf{S}_1$ to $\mathbf{B}$. Similarly, there is a $d_2 \geq 1$, a subalgebra $\mathbf{S}_2$ of $\mathbf{B}^{d_2}$, and a surjective homomorphisms h_2 from $\mathbf{S}_2$ to $\mathbf{A}$. By Proposition 21, $I := (d_1, S_1, h_1)$ is an interpretation of Δ in Γ and $J := (d_2, S_2, h_2)$ is an interpretation of Γ in Δ. Because the statement is symmetric it suffices to show that R_{IJ} is primitive positive definable in Δ. Theorem 20 asserts that this is equivalent to showing that $h_1 \circ h_2$ is preserved by all operations $f^{\mathbf{B}}$ of $\mathbf{B}$. So let k be the arity of $f^{\mathbf{B}}$, let D be the domain of Δ, and let $b^i = (b^i_{1,1}, \ldots, b^i_{d_1,d_2})$ be elements of $D^{d_1 d_2}$, for $1 \leq i \leq k$. Then

$$\begin{aligned}&f^{\mathbf{B}}((h_1 \circ h_2)(b^1), \ldots, (h_1 \circ h_2)(b^k)) \\ &\quad = h_1\big(f^{\mathbf{A}}(h_2(b^1_{1,1}, \ldots, b^1_{1,d_2}), \ldots, h_2(b^k_{1,1}, \ldots, b^k_{1,d_2})), \\ &\qquad\qquad \ldots, f^{\mathbf{A}}(h_2(b^1_{d_1,1}, \ldots, b^1_{d_1,d_2}), \ldots, h_2(b^k_{d_1,1}, \ldots, b^k_{d_1,d_2}))\big) \\ &\quad = (h_1 \circ h_2)(f^{\mathbf{B}}(b^1, \ldots, b^k))\,.\end{aligned}$$

(3) $\Rightarrow$ (1). Suppose that Γ and Δ are primitive positive bi-interpretable via an interpretation $I = (d_1, S_1, h_1)$ of Δ in Γ and an interpretation $J = (d_2, S_2, h_2)$ of Γ in Δ. Let $\mathbf{A}$ be a polymorphism algebra of Γ, and $\mathbf{B}$ be a polymorphism algebra of Δ. Then by Proposition 21 S_1 induces an algebra $\mathbf{S}_1$ in $\mathbf{A}^{d_1}$ and h_1 is a surjective homomorphism from $\mathbf{S}_1$ to an algebra $\mathbf{B}'$ satisfying $\mathrm{Clo}(\mathbf{B}') \subseteq \mathrm{Pol}(\Delta)$. Similarly, S_2 induces in $\mathbf{B}^{d_2}$ an algebra $\mathbf{S}_2$ and h_2 is a homomorphism from $\mathbf{S}_2$ onto an algebra $\mathbf{A}'$ such that $\mathrm{Clo}(\mathbf{A}') \subseteq \mathrm{Pol}(\Gamma)$. By Theorem 4 the natural homomorphisms ξ_1 from $\mathrm{Clo}(\mathbf{A})$ onto $\mathrm{Clo}(\mathbf{B}')$ and ξ_2 from $\mathrm{Clo}(\mathbf{B})$ onto $\mathrm{Clo}(\mathbf{A}')$ exist and are continuous.

We will verify that $\xi_2\xi_1$ is the identity on $\mathrm{Clo}(\mathbf{A})$; the proof that $\xi_1\xi_2$ on $\mathrm{Clo}(\mathbf{B})$ is the identity is analogous. It then follows that ξ_1 and ξ_2 are isomorphisms and homeomorphisms between $\mathrm{Clo}(\mathbf{A})$ and $\mathrm{Clo}(\mathbf{B})$.

Write τ for the signature of $\mathbf{A}$. Write $\mathbf{C}$ for the τ-algebra on the domain of $\mathbf{A}$ obtained by setting $f^{\mathbf{C}} := (\xi_2\xi_1)(f^{\mathbf{A}})$ for all $f \in \tau$. Let $f \in \tau$ be k-ary; we show $f^{\mathbf{C}} = f^{\mathbf{A}}$. Let $a_1, \dots, a_k \in \Gamma$ be arbitrary. Since $h_2 \circ h_1$ is surjective onto Γ, there are $b^i = (b^i_{1,1}, \dots, b^i_{d_1,d_2}) \in \Gamma^{d_1 d_2}$ such that $a_i = h_2 \circ h_1(b^i)$. Then

$$\begin{aligned} f^{\mathbf{C}}(a_1, \dots, a_k) &= f^{\mathbf{C}}(h_2 \circ h_1(b^1), \dots, h_2 \circ h_1(b^k)) \\ &= h_2\big(f^{\mathbf{B}'}(h_1(b^1_{1,1}, \dots, b^1_{d_1,1}), \dots, h_1(b^k_{1,1}, \dots, b^k_{d_1,1})), \\ &\qquad \dots, f^{\mathbf{B}'}(h_1(b^1_{1,d_2}, \dots, b^1_{d_1,d_2}), \dots, h_1(b^k_{1,d_2}, \dots, b^k_{d_1,d_2}))\big) \\ &= h_2 \circ h_1(f^{\mathbf{A}}(b^1, \dots, b^k)) \\ &= f^{\mathbf{A}}(h_2 \circ h_1(b^1), \dots, h_2 \circ h_1(b^k)) \\ &= f^{\mathbf{A}}(a_1, \dots, a_k), \end{aligned}$$

where the second and third equations hold since h_2 and h_1 are algebra homomorphisms, and the fourth equation holds because $f^{\mathbf{A}}$ preserves $h_2 \circ h_1$. This follows from Theorem 20 and the assumption that R_{JI} is primitive positive definable in Γ. Hence, $f^{\mathbf{A}} = f^{\mathbf{C}} = \xi_2\xi_1(f^{\mathbf{A}})$ for all $f \in \tau$, which is what we had to show. $\square$

The following fact has been proven recently for *finite* algebras, independently by Marković, Maroti, and McKenzie [MMM] and by Davey, Jackson, Pitkethly, and Szabó [DJPS]. An algebra $\mathbf{A}$ is called *finitely related* iff there exists a structure Γ with the same domain as $\mathbf{A}$ and with finite relational signature such that $\overline{\mathrm{Clo}(\mathbf{A})} = \mathrm{Pol}(\Gamma)$. We present a generalization to all locally oligomorphic algebras.

Corollary 26. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite or locally oligomorphic algebras such that* $\overline{\mathrm{Clo}(\mathbf{A})}$ *and* $\overline{\mathrm{Clo}(\mathbf{B})}$ *are isomorphic as topological clones. Then* $\mathbf{A}$ *is finitely related if and only if* $\mathbf{B}$ *is finitely related.*

Proof. Suppose that $\mathbf{A}$ is finitely related; that is, there exists a structure Γ with finite relational signature such that $\overline{\mathrm{Clo}(\mathbf{A})} = \mathrm{Pol}(\Gamma)$. Let Δ be the relational structure with the same domain as $\mathbf{B}$ that has all relations that are preserved by all operations of $\mathbf{B}$. Then $\mathrm{Pol}(\Delta) = \overline{\mathrm{Clo}(\mathbf{B})}$, and thus it suffices to show that Δ has a reduct Δ' with finite signature and the same polymorphisms as Δ.

Note that the automorphisms of Γ and Δ are exactly the unary invertible operations in $\overline{\mathrm{Clo}(\mathbf{A})}$ and $\overline{\mathrm{Clo}(\mathbf{B})}$, respectively. Since $\mathbf{A}$ and $\mathbf{B}$ are finite or locally oligomorphic, Γ and Δ are finite or ω-categorical. By Proposition 25, Γ and Δ are primitive positive bi-interpretable. Let I_1 and I_2 be the corresponding interpretations of Γ in Δ and Δ in Γ, respectively. Let σ be the signature of Δ, and let $\sigma' \subseteq \sigma$ be the set of all relation symbols that appear in all the formulas of I_1; since the signature τ of Γ is finite, σ' is finite as well. Let Δ' be the σ'-reduct of Δ. We will show that there is a primitive positive definition of Δ in Δ'; by Theorem 20, this implies that Δ and Δ' have the same polymorphisms.

Let ψ be an atomic σ-formula with k free variables $x_1, \dots, x_k$. We specify an equivalent primitive positive σ'-formula. Suppose that the interpretation I_1 of Γ in Δ is d_1-dimensional and that the interpretation I_2 of Δ in Γ is d_2-dimensional. Let $\phi(x, y_{1,1}, \dots, y_{d_1,d_2})$ be the primitive positive formula that defines $R_{I_2 I_1}$ in Δ. Note that the primitive positive τ-formula ψ_{I_2} has kd_2 free variables; we can assume without loss of generality that ψ_{I_2} only contains unnested atomic formulas as conjuncts. Let $(\psi_{I_2})_{I_1}$ be the primitive positive σ'-formula obtained from ψ_{I_2} by replacing each conjunct ψ' of ψ_{I_2} by $(\psi')_{I_1}$ and then pushing existential quantifiers to the front. Then the formula

$$\exists y^1_{1,1},\ldots,y^k_{d_1,d_2}\ \Big(\bigwedge_{i\leq k}\phi(x_i,y^i_{1,1},\ldots,y^i_{d_1,d_2})$$
$$\wedge\ (\psi_{I_2})_{I_1}(y^1_{1,1},\ldots,y^1_{d_1,d_2},\ldots,y^k_{1,1},\ldots,y^k_{d_1,d_2})\Big)$$

is a primitive positive σ'-formula that defines $\psi(x_1,\ldots,x_k)$ over Δ'. □

5. Primitive positive interpretation examples

Example 1. Let Γ be the structure with domain $\mathbb{N}^2$ and a single binary relation $M := \{((u_1,u_2),(v_1,v_2)) \mid u_2 = v_1 \text{ and } u_1,u_2,v_1,v_2 \in \mathbb{N}\}$. Then Γ and the structure $\Delta := (\mathbb{N};=)$ are primitive positive bi-interpretable. The interpretation I of Γ in Δ is 2-dimensional, the domain formula is *true*, and the coordinate map h is the identity. The interpretation J of Δ in Γ is 1-dimensional, the domain formula is *true*, and the coordinate map g sends (x,y) to x. Both interpretations are clearly primitive positive. Then $g(h(x,y)) = z$ is definable by the formula $x = z$, and $h(g(u),g(v)) = w$ is primitive positive definable by

$$M(w,v) \wedge \exists p\ (M(u,p) \wedge M(w,p))\ .$$

Example 2. An instructive example of two structures Γ and Δ that are *not* primitive positive bi-interpretable, even though they are mutually primitive positive interpretable, is

$$\Gamma := \big(\mathbb{N}^2; \{((u_1,u_2),(v_1,v_2)) \,|\, u_1 = v_1 \text{ and } u_1,u_2,v_1,v_2 \in \mathbb{N}\}\big)$$

and $\Delta := (\mathbb{N};=)$. The two structures are not even *first-order* bi-interpretable. To see this, observe that the binary relation of Γ is an equivalence relation and that $\mathrm{Aut}(\Gamma)$ has a proper closed normal subgroup that is distinct from the one-element group, namely the set of all permutations that setwise fix the equivalence classes of this equivalence relation. On the other hand, $\mathrm{Aut}(\Delta)$ is the symmetric permutation group of a countably infinite set, which has no proper closed normal subgroup that is distinct from the one-element group (it has exactly four proper normal subgroups [SS33], of which only the one-element subgroup is closed).

Example 3. The image of a continuous homomorphism ξ from $\mathrm{Pol}(\Gamma)$ to $\mathrm{Pol}(\Delta)$ might be dense in $\mathrm{Pol}(\Delta)$ without being surjective, for ω-categorical structures Γ and Δ. The basic idea of this example is due to Dugald Macpherson and can be found in [Hod93] (on page 354). Let Γ be the structure $(\mathbb{Q};<,P,P_4)$ where

- $<$ is the usual strict order of the rational numbers,
- $P \subseteq \mathbb{Q}$ is such that both P and $Q := \mathbb{Q}\setminus P$ are dense in $(\mathbb{Q};<)$, and
- P_4 is the relation $\{(x_1,x_2,x_3,x_4) \in \mathbb{Q}^4 \mid x_1 = x_2 \text{ or } x_3 = x_4\}$.

It is a well-known fact that all polymorphisms of Γ are essentially unary[2] since they have to preserve P_4 (see e.g. Lemma 5.3.2 in [Bod12]). The substructure Δ induced by P in Γ has the primitive positive interpretation $(1,P,\mathrm{id})$ in Γ. Indeed, since

[2]A function $f\colon D^l \to D$ is called *essentially unary* iff there exists an $i \leq l$ and a function $g\colon D \to D$ such that $f(x_1,\ldots,x_l) = g(x_i)$ for all $x_1,\ldots,x_l \in D$.

all functions of $\mathrm{Pol}(\Gamma)$ are essentially unary, the mapping which sends every unary function f of $\mathrm{Pol}(\Gamma)$ to $f\restriction_P$ induces a function ξ from $\mathrm{Pol}(\Gamma)$ to $\mathrm{Pol}(\Delta)$ which is a continuous homomorphism and whose image is dense in $\mathrm{Pol}(\Delta)$. We claim that ξ is not surjective.

A *Dedekind cut* (S,T) *of* P is a partition of P into subsets S,T with the property that for all $s \in S$, $t \in T$ we have $s < t$. Those cuts are obtained by choosing either an irrational number $r \in \mathbb{R} \setminus \mathbb{Q}$ or an element $r \in Q$, and setting $S := \{a \in P \mid a < r\}$ and $T := \{a \in P \mid a > r\}$. Let (S_1, T_1) be a Dedekind cut obtained from an element q in Q, and let (S_2, T_2) be a Dedekind cut obtained from an irrational number i. By a standard back-and-forth argument, there exists an $\alpha \in \mathrm{Aut}((P,<))$ that maps S_1 to S_2 and T_1 to T_2. Suppose for contradiction that there is $\beta \in \mathrm{Aut}(\Gamma)$ with $\beta\restriction_P = \alpha$. Then $s < \beta(q) < t$ for all $s \in S_2$, $t \in T_2$, contradicting the irrationality of i.

Example 4. We adapt the previous example to demonstrate that Δ might have a primitive positive interpretation in an ω-categorical structure Γ, but Δ is *not* the reduct of an ω-categorical structure Δ' such that there is a *surjective* homomorphism from $\mathrm{Pol}(\Gamma)$ to $\mathrm{Pol}(\Delta')$.

Let Γ be the structure $(\mathbb{Q}; <, P, E)$ where $<$ is the usual ordering of $\mathbb{Q}$, P is a dense subset of $\mathbb{Q}$ such that $\mathbb{Q} \setminus P$ is also dense in $\mathbb{Q}$, and E is a subset of P^2 such that (P,E) induces the random graph. Then the structure $\Delta := (P; <, E)$ has the primitive positive interpretation $(1, P, \mathrm{id})$ in Γ. Let Δ' be any ω-categorical structure such that Δ is the reduct of Δ' and such that there is a continuous clone homomorphism ξ from $\mathrm{Pol}(\Gamma)$ to $\mathrm{Pol}(\Delta)$. We claim that ξ cannot be surjective.

6. Constraint satisfaction problems

Primitive positive interpretations play an important role in the study of the computational complexity of constraint satisfaction problems. For a structure Γ with finite relational signature τ, the *constraint satisfaction problem for* Γ (denoted by $\mathrm{CSP}(\Gamma)$) is the computational problem of deciding whether a given primitive positive τ-sentence (that is, a primitive positive formula without free variables) is true in Γ. For example, when $\Gamma = (\{0,1,2\}; \neq)$, then $\mathrm{CSP}(\Gamma)$ is the 3-coloring problem. When $\Gamma = (\mathbb{Q}; <)$, then $\mathrm{CSP}(\Gamma)$ is the acyclicity problem for finite directed graphs. Many computational problems studied in qualitative reasoning in artificial intelligence, but also in many other areas of theoretical computer science, can be formulated as constraint satisfaction problems for ω-categorical structures.

The subclass of problems of the form $\mathrm{CSP}(\Gamma)$ for finite Γ has attracted considerable interest in recent years. Feder and Vardi [FV99] conjectured that such CSPs are either in P or are NP-complete. A very fruitful approach to this conjecture is the so-called *universal-algebraic approach*. One of the basic insights of this approach is that for finite Γ, the complexity of $\mathrm{CSP}(\Gamma)$ only depends on the pseudovariety generated by any of the polymorphism algebras of Γ. For ω-categorical Γ, the same statement follows from Theorem 19 and the following, which can be seen as a different formulation of results obtained in [BKJ05].

Theorem 27 (from [Bod08])**.** *Let* Γ *and* Δ *be structures with finite relational signatures. If there is a primitive positive interpretation of* Γ *in* Δ*, then there is a polynomial-time reduction from* $\mathrm{CSP}(\Gamma)$ *to* $\mathrm{CSP}(\Delta)$*.*

For finite structures Γ, this also shows that the complexity of $\mathrm{CSP}(\Gamma)$ is captured by the abstract polymorphism clone of Γ; see Theorem 3. In other words, if Γ and Δ are such that their abstract polymorphism clones are isomorphic, then $\mathrm{CSP}(\Gamma)$ and $\mathrm{CSP}(\Delta)$ are polynomial-time equivalent.

Corollary 2 gives a generalization of this fact for ω-categorical structures: the complexity of $\mathrm{CSP}(\Gamma)$ only depends on the topological polymorphism clone of Γ. In the following we explain that this is not only a fact of theoretical interest, but that Theorem 1 also provides a practical tool to prove the hardness of $\mathrm{CSP}(\Gamma)$. An example will be given in Section 7.

Note that all algebras with domain of size at least two and with the property that all their operations are projections have, up to isomorphism, the same abstract clone, which we denote by $\mathbf{1}$. For $1 \leq i \leq k$, we denote the element of $\mathbf{1}$ which correponds to the k-ary projection onto the i-th coordinate by π_i^k. So $\{\pi_i^k \mid i,k \in \mathbb{N}, i \leq k\}$ is the set of elements of the abstract clone $\mathbf{1}$. Note that the topology on $\mathbf{1}$ is the discrete topology since $\mathbf{1}$ has only finitely many elements for each arity.

An example of a structure whose polymorphism clone is isomorphic to $\mathbf{1}$ is the structure $(\{0,1\}; \mathrm{1IN3})$, where $\mathrm{1IN3} := \{(0,0,1),(0,1,0),(1,0,0)\}$. The CSP for this structure is the well-known positive 1-IN-3-3SAT problem, which can be found in [GJ78] and which is NP-complete.

Theorem 28. *Let Γ be an ω-categorical structure. Then the following are equivalent:*

(1) *All finite structures have a primitive positive interpretation in Γ.*
(2) *The structure $(\{0,1\}; \mathrm{1IN3})$ has a primitive positive interpretation in Γ.*
(3) *Γ has a polymorphism algebra $\mathbf{C}$ such that the pseudovariety generated by $\mathbf{C}$ contains a two-element algebra $\mathbf{A}$, all of whose operations are projections.*
(4) *There exists a continuous homomorphism from $\mathrm{Pol}(\Gamma)$ to $\mathbf{1}$.*

If one of those conditions applies and if Γ has a relational signature, then Γ has a finite signature reduct Γ' such that $\mathrm{CSP}(\Gamma')$ is NP-hard.

Proof. The equivalence of (1) and (2) with (4) follows from Theorem 1, and the equivalence of (3) with (4) from Theorem 4. We remark that the equivalence between (1), (2) and (3) can also be found in [Bod12].

To prove the statement about NP-hardness, let Γ' be the reduct of Γ that contains exactly those relations that appear in the formulas of the primitive positive interpretation of $(\{0,1\}; \mathrm{1IN3})$ in Γ. Note that Γ' has finite signature and still interprets $(\{0,1\}; \mathrm{1IN3})$ primitively positively. NP-hardness of $\mathrm{CSP}(\Gamma')$ now follows from the mentioned fact that $\mathrm{CSP}((\{0,1\}; \mathrm{1IN3}))$ is NP-hard and from Theorem 27. □

7. Constraint satisfaction example

Consider the structure $\Gamma = (\mathbb{Q}; \mathrm{Betw})$ where Betw is the ternary relation $\{(x,y,z) \in \mathbb{Q}^3 \mid x < y < z \vee z < y < x\}$. Then $\mathrm{CSP}(\Gamma)$ is a well-known NP-complete problem known as the *Betweenness problem* [Opa79, GJ78]. Applying the method presented in Section 6, we will show NP-hardness of this problem by exhibiting a continuous clone homomorphism ξ from $\mathrm{Pol}(\Gamma)$ to $\mathbf{1}$.

In the following, for $k \geq 1$ and $x, y \in \Gamma^k$, we write $\neq(x,y)$ iff $x_j \neq y_j$ for all $1 \leq j \leq k$.

Claim. Let $k \geq 1$ and let $f \in \mathrm{Pol}(\Gamma)$ be k-ary. Then one of the following holds:

(1) there is $1 \leq d \leq k$ such that $f(x) < f(y)$ for all $x, y \in \Gamma^k$ with $\neq(x, y)$ and $x_d < y_d$;

(2) there is $1 \leq d \leq k$ such that $f(x) > f(y)$ for all $x, y \in \Gamma^k$ with $\neq(x, y)$ and $x_d < y_d$.

Since d is clearly unique for each f, setting $\xi(f) := \pi^k_d$ defines a function ξ from $\mathrm{Pol}(\Gamma)$ onto $\mathbf{1}$. It is straightforward to check that ξ is a homomorphism. To see that ξ is continuous, observe that for $1 \leq d \leq k$ the preimage of any π^k_d under ξ equals the intersection of $\mathrm{Pol}(\Gamma)$ with the set of all k-ary functions on Γ which satisfy either (1) or (2). Since the set of functions satisfying (1) or (2) is closed, so is $\xi^{-1}[\{\pi^k_i\}]$.

So we are left with the proof of the claim above. Observe first that either $f(0, \ldots, 0) < f(1, \ldots, 1)$ or $f(0, \ldots, 0) > f(1, \ldots, 1)$ holds: for if the two values were equal, this would contradict $\mathrm{Betw}(f(0, \ldots, 0), f(1, \ldots, 1), f(2, \ldots, 2))$. We will now show that $f(0, \ldots, 0) < f(1, \ldots, 1)$ implies (1); then by symmetry of the statements, $f(0, \ldots, 0) > f(1, \ldots, 1)$ implies (2).

Observe the following: whenever $a, a', b, b' \in \Gamma^k$ are so that $\neq(a, a')$, $\neq(b, b')$, and $a_i < a'_i$ iff $b_i < b'_i$ for all $1 \leq i \leq k$, then $f(a) < f(a')$ iff $f(b) < f(b')$. To see this, suppose without loss of generality that $f(a) < f(a')$ and $f(b) \geq f(b')$. Pick for all $1 \leq i \leq k$ any $c_i < \min(a_i, a'_i, b_i, b'_i)$ if $a_i < a'_i$, and $c_i > \max(a_i, a'_i, b_i, b'_i)$ otherwise. Pick moreover $d_i < \min(a_i, a'_i, b_i, b'_i)$ if $a_i > a'_i$, and $d_i > \max(a_i, a'_i, b_i, b'_i)$ otherwise. Now $\mathrm{Betw}(c_i, a_i, a'_i)$ for all $1 \leq i \leq k$ and $f(a) < f(a')$ imply $f(c) < f(a)$; likewise, $\mathrm{Betw}(a_i, a'_i, d_i)$ for all $1 \leq i \leq k$ and $f(a) < f(a')$ imply $f(a') < f(d)$, and so $f(c) < f(d)$. However, the same argument with b and b' yields $f(d) < f(c)$, a contradiction.

Now suppose that (1) does not hold, and let $c^0 \in \Gamma^k$ be arbitrary. We will inductively define tuples $c^1, \ldots, c^k \in \Gamma^k$ such that $f(c^0) \geq f(c^1) \geq \cdots \geq f(c^k)$ and such that $c^0_i < c^k_i$ for all $1 \leq i \leq k$, which contradicts our observation since $f(0, \ldots, 0) < f(1, \ldots, 1)$. For $0 \leq j < k$, we define c^{j+1} from c^j as follows. Consider $x, y \in \Gamma^k$ witnessing the failure of (1) for $d = j$; that is, $\neq(x, y)$, $x_j < y_j$, and $f(x) \geq f(y)$ hold. Select $t \in \Gamma^k$ such that $\neq(c^j, t)$ and such that $c^j_i < t_i$ iff $x_i < y_i$ for all $1 \leq i \leq k$. Then $c^j_j < t_j$, and the observation shows that $f(c^j) \geq f(t)$. For $1 \leq i \leq k$, set $c^{j+1}_i := c^j_i + k$ if $t_i > c^j_i$, and $c^{j+1}_i := c^j_i - 1$ otherwise. By our observation, $f(c^j) \geq f(c^{j+1})$; since in the process every coordinate is increased by k at least once, and decreased by 1 at most $k - 1$ times, we have $c^0_i < c^k_i$ for all $1 \leq i \leq k$.

8. Discussion

Our results demonstrate that many properties of an ω-categorical structure Γ are already determined by the polymorphism clone of Γ viewed as a *topological clone*, i.e., viewed as an abstract algebraic structure additionally equipped with the topology of pointwise convergence. One might ask which properties of Γ are captured by the abstract algebraic structure of the polymorphism clone of Γ *without the topology.* Observe that for finite Γ, the two concepts coincide.

We would like to point out that there is considerable literature about ω-categorical structures where the topology on the automorphisms is uniquely determined by the abstract automorphism group; this is for instance the case if $\mathrm{Aut}(\Gamma)$

has the so-called *small index property*, that is, all subgroups of countable index are open. (This is equivalent to saying that all homomorphisms from $\mathrm{Aut}(\Gamma)$ to S_∞, the symmetric group on a countably infinite set, are continuous.) The small index property has for instance been shown

- for $\mathrm{Aut}(\mathbb{N};=)$ by Dixon, Neumann, and Thomas [DPT86];
- for $\mathrm{Aut}(\mathbb{Q};<)$ and for the automorphism group of the atomless Boolean algebra by Truss [Tru89];
- for the automorphism groups of the random graph [HHLS93];
- for all ω-categorical ω-stable structures [HHLS93];
- for the automorphism groups of the Henson graphs by Herwig [Her98].

An example of two ω-categorical structures (with infinite relational signature) whose automorphism groups are isomorphic as abstract groups but not as topological groups can be found in [EH90].

It is well known that every Baire measurable homomorphism between Polish groups is continuous (see e.g. [Kec95]). So let us remark that there exists a model of ZF+DC where every set is Baire measurable [She84]. For the structures Γ that we need to model computational problems as $\mathrm{CSP}(\Gamma)$, it therefore seems fair to assume that the abstract automorphism group of Γ always determines the topological automorphism group (thanks to Todor Tsankov for pointing this out to us; consistency of this statement with ZF has already been observed in [Las91]). However, this does not answer the question as to in which situations the abstract polymorphism clone determines the topological polymorphism clone.

In Theorem 3, if $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$, then $\mathbf{B}$ is in fact a homomorphic image of a subalgebra of $\mathbf{A}^n$, where $n = |\mathbf{A}|^{|\mathbf{B}|}$; that is, we have an explicit bound for the size of the power of $\mathbf{A}$ we have to take in order to represent $\mathbf{B}$. Peter Cameron has asked us whether a bound was known also in the locally oligomorphic case, i.e., in Theorem 4. By its nature of a compactness argument, our proof does not provide such a bound, and it would be interesting to find out whether a bound could also be given in our case.

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