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ASYMPTOTICS OF THE DENSITIES OF THE FIRST PASSAGE TIME DISTRIBUTIONS FOR BESSEL DIFFUSIONS

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ABSTRACT. This paper concerns the first passage times to a point $a > 0$, denoted by σ_a, of Bessel processes. We are interested in the case when the process starts at $x > a$ and we compute the densities of the distributions of σ_a to obtain the exact asymptotic forms of them as $t \to \infty$ that are valid uniformly in $x > a$ for every order of the Bessel process.

1. INTRODUCTION AND MAIN RESULTS

This paper concerns the first passage times to a point $a \geq 0$, denoted by σ_a, of Bessel processes of order $\nu \in \mathbf{R}$. We are interested in the case when the process starts at $x > a$ and we compute the densities of the distributions of σ_a to obtain the exact asymptotic forms of them as $t \to \infty$ that are valid uniformly in $x > a$ for each order ν. If $\nu = \pm 1/2$, we have well-known explicit expressions of them, which are often used in various circumstances; otherwise there has been quite restricted information on them until quite recently. In the case when $0 \leq x < a$ the distribution of σ_a solves a boundary value problem of the associated second order differential equation on the finite interval $(0, a)$ and the distribution of σ_a or its density is represented by means of eigenfunction expansion ([3], [8], [12], etc.), and thereby we can obtain accurate estimates of them. In the case $x > a$, however, the region for the differential equation is the infinite interval (a, ∞) and the corresponding representation is given by a Fourier-Bessel transform (cf. [16], Section 4.10), which it does not seem a simple matter to derive an asymptotic form of the density directly from. Also, there have been only a few partial results as given in [15], [18], [7] in which $\nu = 0$ or/and relative ranges of x are restricted at least for sharp estimates (in addition to the cases $\nu = \pm 1/2$). In the recent paper [2] Byczkowski, Malecki and Ryznar have computed an estimate of the density for σ_a for all values of ν by using a certain integral representation of it given in [1]: they obtain upper and lower bounds of the correct order of magnitude valid uniformly for all $t > 0, x > a$, which however does not give the exact asymptotic form as we shall obtain in this paper (although in some cases their results are very close to and even finer than ours; see (i) of Remark 1 of the present paper). Hamana and Matumoto [10] have derived a similar (but, in a significant point, quite different) integral representation of the density of σ_a for the case $x > a$ (as well as for the

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case $0 \le x < a$) and computed an exact asymptotic form of the density as $t \to \infty$ but with $x > a$ fixed.

The present investigation is originally motivated by a study of Wiener sausage of the Brownian bridge in $\mathbf{R}^d$ joining the origin to a point $\mathbf{x} \in \mathbf{R}^d$ over a time interval $[0, t]$ when $|\mathbf{x}|$ grows linearly with t ([20]). The evaluation of the expected volume of the sausage swept by a ball of radius a can be reduced to that of the density for σ_a with arbitrary starting point $> a$. Not that only the case of the order $\nu = (d-2)/2$ is concerned there. In fact, the results for all orders $\nu \ge 0$ turn out to take part in the evaluation.

Let X_t^ν be the Bessel process of order $\nu \in \mathbf{R}$, whose infinitesimal generator $\mathcal{L}^\nu$ is given by

$$\mathcal{L}^\nu = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x} \cdot \frac{d}{dx}.$$

If $2\nu+2$ is a positive integer, X_t^ν represents the radial part of the standard $(2\nu+2)$-dimensional Brownian motion. If $\nu \ge -1$ we write d for $2\nu+2$:

$$d = 2\nu + 2, \text{ or what is the same, } \nu = \frac{d}{2} - 1; \tag{1}$$

the process X_t^ν is sometimes called the d-dimensional Bessel process no matter whether d is an integer or not. Let P_x be the probability law of the process X_t^ν started at $x \ge 0$ and E_x be the expectation by P_x. Let σ_a denote the first passage time of X_t^ν to $a > 0$ and $q^\nu(t, x; a)$ the density of the distribution of σ_a:

$$q^\nu(x, t; a) = \frac{d}{dt} P_x[\sigma_a \le t].$$

We also write $q^{(d)}$ for q^ν, where $d = 2\nu + 2$ if $\nu \ge -1$.

In what follows we suppose $\nu \ge 0$ unless the contrary is stated explicitly. At the end of this introduction we shall observe that there is a simple relation between q^ν and $q^{-\nu}$ and the case $\nu < 0$ is reduced to the case $\nu > 0$ and vice versa. If $\nu \ge 0$, the origin is an entrance and non-exit boundary to the positive half line as is well known. We shall use the two indices d and ν interchangeably, understanding that they are related by (1). Put

$$p_t^\nu(x) = p_t^{(d)}(x) = (2\pi t)^{-d/2} e^{-x^2/2t}.$$

For the process X^ν started at the origin $p_t^{(d)}(x)$ is the density of the distribution of X_t^ν w.r.t. the invariant measure $c_d x^{d-1} dx$, where $c_d = 2\pi^{d/2}/\Gamma(\frac{1}{2}(d+1))$, the normalizing constant. We prefer $q^{(d)}$ and $p_t^{(d)}$ to q^ν and p_t^ν, and in order to avoid confusion we shall use the former notation throughout the paper except in a few occasions when the use of the latter one is definitely suitable.

In [18] the present author obtains the following result among others (see (38) of Section 2 for another one). Put $\kappa = 2e^{-2\gamma}$, where $\gamma = -\int_0^\infty e^{-u} \lg u \, du$ (Euler's constant).

Theorem 1 ([18]). *If $\nu = 0$, then uniformly for $x > a$, as $t \to \infty$,*

(2)

$$q^{(2)}(x, t; a) = \frac{\lg(\frac{1}{2}\kappa x^2/a^2)}{t(\lg(\kappa t/a^2))^2} e^{-\ x^2/2t} + \begin{cases} \dfrac{2\gamma \lg(t/x^2)}{t(\lg t)^3} + O\Big(\dfrac{1}{t(\lg t)^3}\Big) & \textit{for } x^2 < t, \\[2ex] O\Big(\dfrac{1 + [\lg(x^2/t)]^2}{x^2(\lg t)^3}\Big) & \textit{for } x^2 \ge t. \end{cases}$$

This theorem does not identify in any sense the asymptotic form of $q^{(2)}(x,t;a)$ for $x > \sqrt{4t \lg\lg t}$ (see Lemma 9 in Section 3). The objective of this paper is to complement this ((4) in Theorem 2 and Corollary 4), and at the same time to also obtain an asymptotic form of q^ν for $\nu > 0$ when the Bessel process is transient ((3) of Theorem 2 and Theorem 3).

Define a function $\Lambda_\nu(y), y \geq 0$ by

$$\Lambda_\nu(y) = \frac{(2\pi)^{\nu+1}}{2y^\nu K_\nu(y)}, \quad y > 0,$$

and $\Lambda_\nu(0) = \lim_{y\downarrow 0}\Lambda_\nu(y)$. Here K_ν is the modified Bessel function of the second kind of order ν; $\Lambda_\nu(0)$ is well defined (see (5) below) and

$$\Lambda_\nu(0) = \frac{2\pi^{\nu+1}}{\Gamma(\nu)} \quad \text{for} \quad \nu > 0; \text{ and}$$

$$\Lambda_0(y) \sim \frac{\pi}{-\lg y} \quad \text{as} \quad y \downarrow 0,$$

in particular $\Lambda_0(0) = 0$. (By definition the ratio of two sides of $\sim$ tends to 1 in the indicated process of taking limits.) The main result of this paper is then stated as follows.

Theorem 2. *Uniformly for $x > a$, as $t \to \infty$,*

$$q^{(d)}(x,t;a) = a^{2\nu}\Lambda_\nu\left(\frac{ax}{t}\right)p_t^{(d)}(x)\left[1 - \left(\frac{a}{x}\right)^{2\nu}\right]\Big(1 + o(1)\Big) \qquad \text{if} \quad \nu > 0 \tag{3}$$

and

$$q^{(2)}(x,t;a) = p_t^{(2)}(x) \times \begin{cases} \dfrac{4\pi \lg(x/a)}{(\lg t)^2}\Big(1 + o(1)\Big) & (x \leq \sqrt{t}), \\ \Lambda_0\left(\dfrac{ax}{t}\right)\Big(1 + o(1)\Big) & (x > \sqrt{t}). \end{cases} \tag{4}$$

If the right-hand sides are multiplied by $e^{-a^2/2t}$, both the formulae (3) *and* (4) *so modified hold true also as $x \to \infty$ uniformly for $t > 0$.*

From the estimates of the density $q^{(d)}$ we can easily compute those of the distribution $P_x[\sigma_a < t]$ and we shall carry out the computation that will be based on (3) and (4) in the last short section.

Remark 1. (i) In [2] Byczkowski, Malecki and Ryznar give estimates closely related to Theorem 2. The main result of [2], their Theorem 2 (in its section 3), is a weaker version of the estimates (3) and (4): the uniform upper and lower bounds of the correct asymptotic order of magnitude are obtained instead of the exact asymptotic form. For the case when $x/t \to \infty$, however, they derive a very precise estimate, finer than one given above, of which we present an explicit statement shortly (Lemma 5 below). For $\nu > 0$ their Proposition 5 identifies the asymptotic form of $q^{(d)}(x,t;a)$ in the case when x/t converges to a positive constant (as $t \to \infty$), the same result as included in Theorem 2 as a significant special case. It is also noted that for each $x > a$ fixed the formula (3) is given in [10] but with some coefficient not being explicit (see also [1]). The proofs in these papers rest on certain integral representations of $q^{(d)}(x,t)$ (given by [1] (in [10]) and by [9] (in [2])), and the methods adopted therein are quite different from ours.

(ii) For every ν, $\Lambda_\nu(y)$ is an increasing function of y. Indeed, we have

$$\Lambda_\nu(y) = \frac{2\pi^{\nu+1}}{\int_0^\infty \exp(-\frac{1}{4u}y^2)e^{-u}u^{\nu-1}du} \; (y>0), \tag{5}$$

as is readily deduced from the identity $K_\nu(z) = \frac{1}{2}(z/2)^\nu \int_0^\infty e^{-\frac{1}{4u}z^2-u}u^{-\nu-1}du$ ($|\arg z| < \frac{\pi}{4}$) ([13], p. 119), of which the right-hand side is invariant under the replacement of ν by $-\nu$. It is also noted that for each $\nu \geq 0$,

$$K_\nu(y) = \sqrt{\pi/2y}\, e^{-y}(1+O(1/y))\, (y\to\infty),$$

so that

$$\Lambda_\nu(y) = (2\pi)^{\nu+1/2}y^{-\nu+1/2}\, e^y(1+O(1/y))\; (y\to\infty). \tag{6}$$

The detailed estimation of $q^{(d)}(x,t;a)$ inside the parabolic regions $x^2 < Ct$ ($C > 1$) would be of fundamental importance. We next give an extension of Theorem 1 to $\nu > 0$ in this respect, which partially prepares us for the proof of Theorem 2 in an obvious way (see (iii) of Remark 2 below).

For $\nu > 0$ we have the Green function, $G^{(d)}(x,y)$ say; we need to bring in $G(x) = G^{(d)}(x); = G^{(d)}(x,0) = G^{(d)}(0,x)$, or explicitly

$$G(x) = \int_0^\infty p_t^{(d)}(x)dt = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}}\cdot\frac{1}{x^{d-2}}.$$

The next theorem (a reduced version of Propositions 6 and 7 in Section 2) gives a fairly fine estimate of $q^{(d)}$ for $\nu > 0$ in the case when $x \leq \sqrt{2(\nu\wedge 1)t\lg t}$ (as in Theorem 1 in a sense). Here as well as in what follows $c\vee b = \max\{c,b\}, c\wedge b = \min\{c,b\}$ for c,b real. It is noted that $a^{2\nu}\Lambda_\nu(0) = 1/G(a)$.

Theorem 3. *Let $\nu > 0$. If $\nu \neq 1$, then uniformly for $x > a$, as $t\to\infty$,*

$$q^{(d)}(x,t;a)$$
$$= \frac{(a^2/2)^\nu}{\Gamma(\nu)\, t^{\nu+1}}\left[e^{-x^2/2t} - \left(\frac{a}{x}\right)^{2\nu} e^{-a^2/2t}\right] + O\left(\frac{1\wedge(\sqrt{t}/x)^{\nu+(\frac{1}{2}\wedge\nu)}}{t^{\nu+1+(\nu\wedge 1)}}\right) \tag{7}$$
$$= \frac{1}{G(a)}\left[p_t^{(d)}(x) - \left(\frac{a}{x}\right)^{d-2} p_t^{(d)}(a)\right] + O\left(\frac{1\wedge(\sqrt{t}/x)^{[(d-1)/2]\wedge(d-2)}}{t^{d/2+(\nu\wedge 1)}}\right). \tag{8}$$

In the case $\nu = 1$ the same estimate holds true if the error term given by the O-symbol is replaced by

$$O\left(\frac{1+\lg(t/x^2)}{t^3}\right) \text{ for } a < x < \sqrt{t};\; O\left(\frac{\lg t}{t^3}\left(\frac{\sqrt{t}}{x}\right)^{3/2}\right) \text{ for } x > \sqrt{t}.$$

Remark 2. (i) For random walks on the d-dimensional square lattice $\mathbf{Z}^d$ we have analogues of Theorems 1 and 3 [17]. The form of the principal term in formula (8) is intrinsically the same as and in fact suggested by that corresponding to the walks. For $d = 2$ Theorem 3 (or Proposition 8) provides an improvement of Theorem 1.4 of [17] in view of Theorem 1.5 of it.

(ii) In the proof of Theorem 3 we give a more precise expression of the error term, which shows that its order of magnitude cannot be improved at least for $x < \sqrt{t}$.

(iii) If the range is restricted to $2a < x < \sqrt{t}$, (3) is an immediate consequence of Theorem 3; the case $a < x \leq 2a$ is contained in Proposition 6, which also implies (3). In view of these results for $x < \sqrt{t}$ the essential ingredient of (3) is now the estimate in the region $x > \sqrt{t}$. With certain additional results employed the same can be said for $\nu = 0$ (for details see the Remark given at the end of Section 2).

The two-dimensional case is particularly interesting and deserves to be described here in more detail. Restricting to the region $x > \sqrt{t}$ we may state formula (4) as follows: uniformly for $x > \sqrt{t}$, as $t \to \infty$,

$$q^{(2)}(x,t,a) = \frac{1}{2K_0(ax/t)} \cdot \frac{e^{-x^2/2t}}{t}\Big(1 + o(1)\Big). \tag{9}$$

Substitution from the formulae (6) and $K_0(u) = -\lg(\frac{1}{2}e^{\gamma}u) + O(u^2 \lg u)$ $(u \downarrow 0)$ makes the right-hand side above explicit if x/t goes to 0 or ∞. We shall actually compute errors in formula (9) (see Proposition 8 and Lemma 5). Taking into account these comments, the next result is essentially a corollary of Theorem 1 and the proof of Theorem 2.

Corollary 4. *Let $\nu = 0$. Uniformly in $x > a$, as $t \to \infty$,*

$$\begin{aligned} q^{(2)}(x,t,a) &= \frac{2\lg(x/a)}{(\lg t)^2} \cdot \frac{e^{-x^2/2t}}{t}\left[1 + O\left(\frac{1}{\lg t}\right)\right] && \text{if } a < x \le \sqrt{t} \\ &= \frac{1}{2\lg(t/x)} \cdot \frac{e^{-x^2/2t}}{t}\left[1 + O\left(\frac{1}{\lg(t/x)}\right)\right] && \text{if } x/t \to 0,\ x > \sqrt{t} \qquad (10)\\ &= \sqrt{\frac{ax}{2\pi t}} \cdot \frac{e^{-(x-a)^2/2t}}{t}\left[1 + O\left(\frac{t}{x}\right)\right] && \text{if } x/t \to \infty. \end{aligned}$$

The factor $e^{-(x-a)^2/2t}$ in the last formula of (10), asymptotically equivalent to $e^{-x^2/2t}$ when $x/t \to 0$, may be understood to be natural by comparing it with the Gaussian kernel $p_t^{(2)}(x-a)$ (see also (15) in Section 2).

The next result from [2] (Lemma 4) gives a fine estimate in the case when $x/t \to \infty$. It in particular shows that the dependence on d of the leading term in this case comes only from the factor $(a/x)^{(d-1)/2}$.

Lemma 5 ([2])**.** *For each $\nu \ge 0$ it holds that uniformly for all $t > 0$ and $x > a$,*

$$q^{(d)}(x,t;a) = \frac{x-a}{\sqrt{2\pi}\,t^{3/2}} e^{-(x-a)^2/2t}\left(\frac{a}{x}\right)^{(d-1)/2}\left[1 + \frac{\beta t}{ax}\left(1 + O\left(\sqrt{t} \wedge \frac{t}{x-a}\right)\right)\right], \tag{11}$$

where $\beta = (d-1)(3-d)/8 = (\frac{1}{4} - \nu^2)/2$.

From the scaling property of Bessel processes it follows that

$$q(x,t;a) = a^{-2}q(x/a, t/a^2; 1).$$

For the proofs of the foregoing theorems we shall mostly consider only the case $a = 1$ and write $q(x,t)$ for $q(x,t;1)$.

The estimation of $q(x,t)$ will be made in the following three cases

(i) $x < \sqrt{t}$;

(ii) $\sqrt{t} < x \le Mt$ (with M arbitrarily fixed);

(iii) $x/t \to \infty$,

of which the cases (i) and (ii) will be discussed in Sections 2 and 3, respectively. The methods employed in these cases are different from one another. Roughly speaking, for the case (i) the estimation is based on the well-known formula for the

Laplace transform of $q^{(d)}(x,\cdot)$, to which we apply the Laplace inversion formula; some computation using the Cauchy integral theorem then leads to somewhat finer estimates (Proposition 6) than those given in Theorem 3. For the case (ii) we exploit the fact that any Bessel process of order $\nu > -1$ can be decomposed as a sum of two independent Bessel processes and apply the result of the case (i). This gives some error estimate to the asserted asymptotic form of q in the case $x/t \to 0$ (Propositions 8 and 11). To include the case $x/t \to v > 0$ an additional argument is employed.

For (iii) Lemma 5 provides a better estimate than that required for Theorem 2. The proof of Lemma 5 rests on the integral representation obtained in [1] and the derivation from it is involved. A relatively easier proof for the relevant estimate in Theorem 2 can be provided by making use of the following probabilistic expression:

$$q^{(d)}(x,t;a) = \frac{x-a}{\sqrt{2\pi t^3}} e^{-(x-a)^2/2t}\left(\frac{a}{x}\right)^{(d-1)/2} E_x^{BM}\left[\exp\left\{\beta\int_0^t \frac{ds}{B_s^2}\right\}\middle|\sigma_a = t\right], \tag{12}$$

where the conditional expectation is taken w.r.t. the probability measure of the standard linear Brownian motion B_t. (This identity is readily derived from the well-known formula for $q^{(1)}$ by using the Cameron-Martin-Girsanov formula.) In the case $x/t \to \infty$, $t > 1$, estimate (11), but with the second term in the big square brackets replaced by a less exact $O(t/x)$, can be derived from (12) by some comparison argument based on the diffusion equation that is associated with the expectation in (12) via the Kac formula (cf. [20]), of which we will not go into further detail. By the way, it is noted that the expression of $q^{(d)}$ above verifies, on expanding the exponential, the estimate (11) of the case $t \downarrow 0$ with the same replacement for the second term as above.

If x/t is large enough, one can evaluate the conditional expectation in (12) for large t directly as mentioned above, of which the dependence on ν comes only from β. Otherwise, however, a direct evaluation of it seems hard. Our results on $q^{(d)}$ rather give a precise estimate of it valid uniformly in x, which turns out to be useful: in [20] we exploit the estimate to derive an asymptotic form of the space-time distribution of the hitting of a ball by d-dimensional Brownian motion.

Throughout the paper C, C', C'', etc. will be used to denote constants whose precise values are not important for the present purpose; the same letter may designate different constants depending on the occasions where it occurs.

We conclude the present section by mentioning some simple facts for the case $\nu < 0$. The probability that the Bessel process $X_t^{|\nu|}$ of order $|\nu|$ started at $x > a$ hits a in a finite time is given by $h(x) = (a/x)^{2|\nu|}$, which is a harmonic function for the process restricted on $[a,\infty)$ with killing at a, and the conditional process conditioned on this event is an h-transform of it. On identifying the generator this conditional process is a Bessel process of order $\nu\,(<0)$. Hence

$$q^{\nu}(x,t;a) = q^{|\nu|}(x,t;a)\frac{x^{2|\nu|}}{a^{2|\nu|}}.$$

(This also follows from (14) below.) Every Bessel process of negative order visits the origin in a finite time with probability one and we have the explicit formula

$$q^{\nu}(x,t;0) = \Lambda_{|\nu|}(0)x^{2|\nu|}p_t^{|\nu|}(x) \tag{13}$$

(for the derivation let $a \downarrow 0$ in (14) and use (16), both given in Section 2). By a comparison argument we have the inequality

$$\int_0^t q^\nu(x-a,s;0)ds < \int_0^t q^\nu(x,s;a)ds \ (x>a, t>0) \text{ if } -2^{-1} < \nu < 0,$$

and the same one but in the opposite direction if $\nu < -2^{-1}$.

2. Proof of Theorem 3

For any $\nu \in \mathbf{R}$,

$$E_x[\exp\{-\lambda\sigma_a\}] = \frac{K_\nu(x\sqrt{2\lambda})x^{-\nu}}{K_\nu(a\sqrt{2\lambda})a^{-\nu}} \qquad (\lambda > 0, x > a > 0), \tag{14}$$

as is well known and may be derived by solving the problem: $\mathcal{L}^\nu U = \lambda U$ $(x > a)$ with the lateral conditions $U(a+0) = 1$ and U being positive and decreasing, of which the solution is unique. (See (17) below.)

For $\nu = 1/2$ (i.e., $d = 3$) we have a particularly simple expression of $q^{(3)}$: for $x > a$

$$q^{(d=3)}(x,t;a) = \frac{ae^{-(x-a)^2/2t}}{t\sqrt{2\pi t}}\left(1 - \frac{a}{x}\right), \tag{15}$$

which trivializes this special case of Theorems 3 and 2 and is helpful for making a guess at the asymptotic form of $q^{(d)}$ in general cases.

In what follows we let $\nu \geq 0$ and, when there is no risk of confusion, we suppress the superscript $^{(d)}$ from $q^{(d)}(x,t;a)$ and $p_t^{(d)}(x)$ except for the statement of propositions or lemmas.

Put $G_\lambda(x) = \int_0^\infty p_t(x)e^{-\lambda t}dt$. We know

$$G_\lambda(x) = \frac{2}{(2\pi)^{d/2}}\left(\frac{\sqrt{2\lambda}}{x}\right)^\nu K_\nu(x\sqrt{2\lambda}) \tag{16}$$

([5], p. 146). It is convenient (and natural) to write the representation (14) in the form

$$E_x[\exp\{-\lambda\sigma_a\}] = \frac{G_\lambda(x)}{G_\lambda(a)}. \tag{17}$$

For $\nu > 0$ let $G(x) = \lim_{\lambda\downarrow 0} G_\lambda(x)$, so that

$$G(x) = \int_0^\infty p_t(x)dt = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}x^{d-2}}.$$

If ν is not an integer, $K_\nu(z) = \Big(\pi/2\sin(\pi\nu)\Big)\Big[I_{-\nu}(z) - I_\nu(z)\Big]$, where I_ν is the modified Bessel function of the first kind of order ν and given by

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(z/2)^{2k}}{\Gamma(\nu+k+1)\Gamma(k+1)} \tag{18}$$

for $|\arg z| < \pi$ ([13], p. 108).

Proposition 6. *Let $\nu > 0$ and $M > 0$. For $a < x < M\sqrt{t}$,*

$$q^{(d)}(x,t;a) = \frac{1}{G(a)}\left[p_t^{(d)}(x) - \left(\frac{a}{x}\right)^{d-2} p_t^{(d)}(a)\right] + a^{-2}\eta(x/a, t/a^2), \tag{19}$$

with the function $\eta(x,t)$, $x>1, t>2$, admitting the estimate

$$\eta(x,t) = O\Big(\frac{1-x^{-1}}{t^{\nu+2}}\Big) \quad \textit{if } \nu>1; \; = O\Big(\frac{1-x^{-1}}{t^{2\nu+1}}\Big) \quad \textit{if } 0<\nu<1$$
$$\textit{and} \;\; \eta(x,t) = O\Big(\frac{1-x^{-1}}{t^{\nu+2}}\lg t\Big) \quad \textit{if } \nu=1.$$

(See (25), (36) *and* (37) *for more exact forms of $\eta(x,t)$.)*

Remark 3. One might suspect that the function

$$q^*(x,t) := [G(a)]^{-1}p_t^{(d)}(x-a)[1-(a/x)^{d-2}],$$

an analogue to the explicit form of $q^{(3)}$, can take the place of the leading term in (19). Since the difference of them is at most the magnitude of $O(x^{(2-2\nu)\vee 1}/t^{\nu+2})$, this is true if $\nu<1$; in the case $\nu>1$, however, the difference becomes much larger than $\eta(t,x)$ as x gets large, so that the replacement causes a larger error term.

Proof of Proposition 6. The Laplace inversion of (17) gives

$$q(x,t;a) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{G_z(x)}{G_z(a)}e^{tz}dz. \tag{20}$$

Here and in what follows the functions $z^{\nu/2}$ and $\lg z$ involved in the integrand (see (18), (34)) take their principal values. For the evaluation of the integral above we follow the argument made in [17] for the random walk of dimensions $d\geq 3$. Motivated by it we decompose

$$\begin{aligned}\frac{G_\lambda(x)}{G_\lambda(a)} &= \frac{G_\lambda(x)-(a/x)^{2\nu}G_\lambda(a)}{G_\lambda(a)} + \frac{a^{2\nu}}{x^{2\nu}}\\ &= \frac{G_\lambda(x)-(a/x)^{2\nu}G_\lambda(a)}{G(a)} + R(\lambda;x),\end{aligned} \tag{21}$$

where

$$R(\lambda;x) = \Big[\frac{1}{G_\lambda(a)}-\frac{1}{G(a)}\Big]\Big[G_\lambda(x)-(a/x)^{2\nu}G_\lambda(a)\Big] + \frac{a^{2\nu}}{x^{2\nu}}.$$

By the definition of $G_\lambda(x)$ the contribution to (20) of the first term on the right-hand side of (21) equals

$$\frac{1}{G(a)}\Big[p_t(x)-(a/x)^{2\nu}p_t(a)\Big].$$

The error term in Proposition 6 is then written as

$$\eta(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}R(z,x)e^{tz}dz. \tag{22}$$

As $z\to\infty$,

$$K_\nu(z) = (\pi/2z)^{1/2}e^{-z}(1+O(1/z)) \qquad (|\arg z|\leq\pi). \tag{23}$$

Hence $G_z(x)/G_z(a) = O(e^{-(x-a)\sqrt{2z}})$ as $z\to\infty$ (with x fixed), and we see that the function $R(z,x)$ rapidly approaches zero as $z\to\infty$ in the sector $|\arg z|>\pi-\delta$ for any $\delta>0$. This together with the fact that $K_\nu(z)$ has no zeros on the right half plane $\Re z \geq 0$ (cf. [21], p. 511) (hence $G_z(x)$ has no zeros on $-\pi \leq \arg z \leq \pi$) permits us to apply Cauchy's integral formula to transform (22) into

$$\eta(x,t) = \frac{1}{2\pi i}\int_0^\infty \Big[-R(-u+i0,x) + R(-u-i0,x)\Big] e^{-tu}du, \tag{24}$$

where $R(-u+i0,x) = \lim_{y\downarrow 0} R(-u+iy,x)$ and analogously for $R(-u-i0,x)$. It is noted that the integral in (24) is not affected by subtraction of any entire function from R.

Let ν be not an integer. We show that uniformly for $a < x < M\sqrt{t}$,

$$\eta(x,t) = \frac{-\nu-1}{2^{\nu+1}(\nu-1)\Gamma(\nu)}\left(1-\frac{1}{x^{2\nu}}+\frac{x^2-1}{x^{2\nu}}\right)\frac{1}{t^{\nu+2}} - B_0^2\frac{1-x^{-2\nu}}{\Gamma(-2\nu)t^{2\nu+1}} + r(t,x) \tag{25}$$

with

$$\begin{aligned} |r(t,x)| &\leq \frac{C}{t^{2\nu+1}}\left[\left(\frac{x^2}{t}\right)\vee\frac{1}{t^\nu}\right]\left(1-\frac{1}{x}\right) \qquad (0<\nu<1) \\ &\leq \frac{C}{t^{\nu+2}}\cdot\frac{x^2}{t}\left(1-\frac{1}{x}\right) \qquad (\nu>1), \end{aligned}$$

where the constant B_0 is given in (27) below. The estimation of η is simple apart from the uniformity in x, which we must take care of in dealing with the dependence on x. Let $a = 1$ for simplicity. Recall the definitions of $G_\lambda(1)$ and $G(1)$. From the power series expansion of I_ν given in (18) we then deduce straightforwardly

$$\begin{aligned} \frac{G_\lambda(1)}{G(1)} &= \frac{(\sqrt{2\lambda})^\nu[I_{-\nu}(\sqrt{2\lambda}) - I_\nu(\sqrt{2\lambda})]}{\lim_{\lambda\downarrow 0}(\sqrt{2\lambda})^\nu I_{-\nu}(\sqrt{2\lambda})} \\ &= \Gamma(1-\nu)\left(\sum_{k=0}^\infty \frac{(\lambda/2)^k}{\Gamma(-\nu+k+1)k!} + \left(\frac{\sqrt{2\lambda}}{2}\right)^{2\nu}\sum_{k=0}^\infty\frac{(\lambda/2)^k}{\Gamma(\nu+k+1)k!}\right) \\ &= \Big[1 + A_1\lambda + A_2\lambda^2 + \cdots\Big] - \lambda^\nu\Big[B_0 + B_1\lambda + \cdots\Big] \end{aligned} \tag{26}$$

with

$$A_1 = [2(1-\nu)]^{-1},\ B_0 = 2^{-\nu}\Gamma(1-\nu)/\Gamma(\nu+1). \tag{27}$$

Let $f(\lambda) = G_\lambda(1)/G(1) - 1$. Then

$$\frac{1}{G_\lambda(1)} - \frac{1}{G(1)} = \frac{1}{G(1)}\left[\frac{1}{1+f(\lambda)} - 1\right] = -\frac{f(\lambda)}{G(1)} + \frac{[f(\lambda)]^2}{G_\lambda(1)}. \tag{28}$$

Also, noting

$$x^{2\nu}G_\lambda(x) = G_{x^2\lambda}(1),$$

we obtain

$$\begin{aligned} x^{2\nu}\times\frac{G_\lambda(x) - (1/x)^{2\nu}G_\lambda(1)}{G(1)} &= \frac{G_{x^2\lambda}(1) - G_\lambda(1)}{G(1)} \\ &= A_1\lambda(x^2-1) - B_0\lambda^\nu(x^{2\nu}-1) + H_\lambda(x), \end{aligned} \tag{29}$$

where $H_\lambda(x) = \Big[A_2\lambda^2(x^4-1)+\cdots\Big] + \Big[B_1\lambda^{\nu+1}(x^{2\nu+2}-1)+\cdots\Big]$, the remainder term. Since $f(z) = O(|z|^{\nu/2-1/4})$ as $z\to\infty$ along the negative real line, the equality (26) entails

$$f(\lambda) = A_1\lambda - B_0\lambda^\nu + O(|\lambda|^{(\nu\wedge 1)+1})\ (|\lambda|<1) \text{ and } |f(-u\pm i0)| \leq C|u|^{\nu/2}\ (u>0).$$

From the power series expansions of $I_{\pm\nu}(z)$ it follows that

$$A_k = O(1/\Gamma(-\nu+k+1)k!), \quad B_k = O(1/\Gamma(\nu+k+1)k!).$$

With these preliminary discussions we now compute the integral in (24). Using (28) and (29) we make the decomposition

$$x^{2\nu}R(\lambda;x) = -f(\lambda)\Big(A_1\lambda(x^2-1) - B_0\lambda^\nu(x^{2\nu}-1)\Big) + T_1(\lambda) + f(\lambda)H_\lambda(x) + 1, \tag{30}$$

where $T_1(\lambda) = [f(\lambda)]^2[G_{x^2\lambda}(1) - G_\lambda(1)]/G_\lambda(1)$; the principal part would be involved in the first term in view of (21), (27) and (29).

First consider the contribution of T_1 and observe that for $u>0$,

$$\begin{aligned}|T_1(-u+i0) - T_1(-u-i0)| \le C_0 u^{2(\nu\wedge 1)}&\Big[u^\nu(x^{2\nu}-1) + C_1u^{\nu+1}(x^{2\nu+2}-1)+\cdots\Big]\\ &+ C_0' u^{2\nu}\Big[u(x^2-1) + C_1'u^2(x^4-1)+\cdots\Big]\end{aligned}$$

with certain constants C_k, C_k' that are dominated by a constant multiple of $2^k k^{3\nu}/(k!)^2$. Here we have exploited the fact that the terms of integral powers $c_n\lambda^n$ involved in $T_1(\lambda)$ cancel out in the difference on the left-hand side. Employing the simple inequality $x^s - 1 \le (1\vee s)(1-x^{-1})x^s$ valid for all $x>1, s>0$, we infer that for $1<x<M\sqrt{t}$,

$$\begin{aligned}&\int_0^\infty \Big|T_1(-u+i0) - T_1(-u-i0)\Big| e^{-tu}du\\ &\le C\Big(1-\frac{1}{x}\Big)\Bigg(\frac{1}{t^{2(\nu\wedge1)+1}}\sum_{k=0}^\infty \frac{k^{3\nu+1}4^k x^{2(\nu+k)}}{t^{\nu+k}k!} + \frac{1}{t^{2\nu+1}}\sum_{k=1}^\infty \frac{k^{3\nu+1}4^k x^{2k}}{t^k k!}\Bigg)\\ &\le C'\Big(1-\frac{1}{x}\Big)\Bigg[\frac{1}{t^{2(\nu\wedge1)+1}}\Big(\frac{x^2}{t}\Big)^\nu + \frac{1}{t^{2\nu+1}}\Big(\frac{x^2}{t}\Big)\Bigg], \end{aligned}\tag{31}$$

where C' depends on M.

Secondly, in the same way we see that if $T_2(\lambda) = f(\lambda)H_\lambda(x)$,

$$\int_0^\infty |T_2(-u+i0) - T_2(-u-i0)| e^{-tu}du \le C\Big(1-\frac{1}{x}\Big)\Big(\frac{x^2}{t}\Big)^{\nu+1}\frac{1}{t^{\nu\wedge1+1}}. \tag{32}$$

Thirdly, putting

$$F(\lambda) = -f(\lambda)\Big(A_1\lambda(x^2-1) - B_0\lambda^\nu(x^{2\nu}-1)\Big),$$

we have

$$F(\lambda) = A_1B_0\lambda^{\nu+1}\Big(x^{2\nu}-1+x^2-1\Big)(1+C_1\lambda+\cdots) - B_0^2\lambda^{2\nu}(x^{2\nu}-1)(1+C_1'\lambda+\cdots),$$

apart from the difference by an entire function, and for $s>-1$,

$$\frac{1}{2\pi i}\int_0^\infty \Big[-(-u+i0)^s + (-u-i0)^s\Big]e^{-tu}du = \frac{1}{\Gamma(-s)t^{s+1}}.$$

(Here the identity $\Gamma(1+s)\sin\pi s = -\pi/\Gamma(-s)$ is used; remember that $1/\Gamma(-n) = 0$ if n is non-negative integer.) Hence

$$\begin{aligned}(33)\quad &\frac{1}{2\pi i}\int_0^\infty [-F(-u+i0)+F(-u-i0)]e^{-tu}du \\ &= \left(A_1B_0\frac{x^{2\nu}-1+x^2-1}{\Gamma(-\nu-1)t^{\nu+2}} - B_0^2\frac{x^{2\nu}-1}{\Gamma(-2\nu)t^{2\nu+1}}\right)\Big[1+O(1/t)\Big].\end{aligned}$$

Finally, collecting the bounds (24) and (26) through (32) (of which we divide each formula by $x^{2\nu}$ since we have multiplied it in (29) and (30)), noting $B_0/\Gamma(-\nu-1) = (\nu+1)/2^\nu\Gamma(\nu)$ and making an elementary comparison of terms that appear on the right-hand sides of them, we find (25) to be true.

Let ν be a positive integer. The arguments are similar to the above. In place of (26) we have

(34)
$$\begin{aligned}\frac{G_\lambda(1)}{G(1)} &= \frac{2}{(\nu-1)!}\left[\sum_{k=0}^{\nu-1}\frac{(\nu-k-1)!}{2(k!)}\left(\frac{-\lambda}{2}\right)^k \right.\\ &\qquad\left. -\left(\frac{-\lambda}{2}\right)^\nu\left(\gamma+\lg\sqrt{\frac{\lambda}{2}}\right)\sum_{k=0}^\infty\frac{(\lambda/2)^k}{k!(\nu+k)!}\right] + \left(\frac{-\lambda}{2}\right)^\nu g(\lambda) \\ &= \Big[1+A_1\lambda+A_2\lambda^2+\cdots\Big] - \lambda^\nu(\lg\lambda)\Big[B_0+B_1\lambda+\cdots\Big],\end{aligned}$$

where $g(\lambda) = a_0 + a_1\lambda + \cdots$ is a certain entire function (with $a_0 = 1$ for $\nu = 1$), and

$$(35)\qquad A_1 = \begin{cases} -2^{-1}[1+\lg(2e^{-2\gamma})] & (\nu=1), \\ -[2(\nu-1)]^{-1} & (\nu\ge 2)\end{cases} \quad \text{and} \quad B_0 = \frac{(-1)^\nu}{2^\nu\nu!(\nu-1)!}.$$

Noting

$$(2\pi)^{-1}\Im\Big(-[\lg(-u+i0)]^k + [\lg(-u-i0)]^k\Big) = -k(\lg u)^{k-1}\ (k=1,2,u>0)$$

we compute the integral in (24) to see that if ν is an integer greater than 1, then

$$(36)\quad \eta(x,t) = -(-1)^{\nu+1}(\nu+1)!A_1B_0\left(1-\frac{1}{x^{2\nu}}+\frac{x^2-1}{x^{2\nu}}\right)\frac{1}{t^{\nu+2}} + O\left(\frac{1-x^{-1}}{t^{\nu+2}}\cdot\frac{x^2}{t}\right).$$

The coefficient of the leading term coincides with one appearing in (25) [since $1/\Gamma(-n) = 0$ if n is a non-negative integer], so that (25) is valid for $\nu = 2,3,4,\ldots$. If $\nu = 1$ and f is defined as before, the leading term of $-f(\lambda)\Big[G_{x^2\lambda}(1) - G_\lambda(1)\Big]/G(1)$ being

$$(B_0^2\lambda^2\lg\lambda)(-x^2\lg(x^2\lambda)+\lg\lambda),$$

we see

$$(37)\qquad \eta(x,t) = -4B_0^2\frac{(1-x^{-2})\lg t - \lg x}{t^{\nu+2}} + O\left(\frac{1-x^{-1}}{t^{\nu+2}}\right).$$

The foregoing two formulae obviously imply the desired bounds for $\eta(x,t)$. The proof of Proposition 6 is complete. $\square$

We still need to obtain error estimates for $|x| > \sqrt{t}$.

Proposition 7. *Let $\nu > 0$. Then the function $\eta(t,x)$ defined via* (19) *admits the estimates*

$$\eta(t,x) = O\left(\frac{1}{t^{\nu+2}}\left(\frac{\sqrt{t}}{x}\right)^{\nu+\frac{1}{2}}\right) \text{ if } \nu > 1; \ = O\left(\frac{1}{t^{2\nu+1}}\left(\frac{\sqrt{t}}{x}\right)^{\nu+(\frac{1}{2}\wedge\nu)}\right) \text{ if } 0 < \nu < 1$$

and $$\eta(t,x) = O\left(\frac{\lg t}{t^{\nu+2}}\left(\frac{\sqrt{t}}{x}\right)^{\nu+\frac{1}{2}}\right) \text{ if } \nu = 1$$

that are valid uniformly for $x > \sqrt{t} > 2$.

Proof. We can proceed as in the preceding proof except that in place of (29) we make the decomposition

$$x^{2\nu}\Big(G_\lambda(x) - (1/x)^{2\nu}G_\lambda(1)\Big) = G_{x^2\lambda}(1) - G(1) + [G(1) - G_\lambda(1)]$$

and estimate the contributions of the three terms on the right-hand side separately. Let $\nu \neq 1$. It follows from the preceding proof that the contributions of the last two terms to $\eta(x,t)$ are bounded by a constant multiple of $x^{-2\nu}t^{-\nu-1} = t^{-2\nu-1}(\sqrt{t}\,/x)^{2\nu}$. As for the first term, on the one hand, we recall that $G_z(1) = C_\nu(2z)^{\nu/2}K_\nu(\sqrt{2z}) = C'_\nu z^{(2\nu-1)/4}e^{-\sqrt{2z}}(1+o(1))$ as $z \to \infty$ to see that

$$x^{-2\nu}|f(-u \pm i0)G_{-x^2u\pm i0}(1)| \le Cu^{1\wedge\nu}x^{-2\nu}(x^2u)^{(2\nu-1)/4} = Cu^{1\wedge\nu+\nu}(x^2u)^{-\nu/2-1/4}$$

for $u > 1/x^2$ (f is the same as in (28)) and the integration over $u > 1/x^2$ of e^{-tu} times the last member yields a magnitude of the order $O\Big(t^{-(1\wedge\nu)+\nu+1}(\sqrt{t}/x)^{\nu+1/2}\Big)$. On the other hand we have $x^{-2\nu}|f(-u\pm i0)G_{-x^2u\pm i0}(1)| \le Cu^{1\wedge\nu}x^{-2\nu}$ for $0 < u < 1/x^2$ and the corresponding integral does not exceed the foregoing magnitude. By (24) we find the asserted bound for $\nu \neq 1$. The case $\nu = 1$ is omitted, it being similarly dealt with. □

Remark about the case $\nu = 0$ and proof of Corollary 4. Theorem 1 (as well as any other results of [18]) does not give a precise asymptotic form for x near a, while the first formula of Corollary 4 does. Here we indicate a manner by which such an estimate can be obtained by following the method employed in the proof of Proposition 6, and thereby prove the first formula of Corollary 4. By the way, this entails the case $x/\sqrt{t} \to 0$ of the formula (4).

For $\nu = 0$ we must replace $G_z(x)/G_z(a)$ by $K_0(x\sqrt{2z})/K_0(a\sqrt{2z})$ in the inversion formula (20) that represents $p(x,t;a)$. Put $g(\lambda) = -\lg(a\sqrt{\kappa\lambda})$ ($\kappa = 2e^{-2\gamma}$) as in [18]; $g(\lambda)$ is the principal part of $K_0(a\sqrt{2\lambda})$ as $\lambda \downarrow 0$. The analogue of the decomposition (21) should be

$$\frac{K_0(x\sqrt{2\lambda})}{K_0(a\sqrt{2\lambda})} = \frac{K_0(x\sqrt{2\lambda}) - K_0(a\sqrt{2\lambda})}{g(\lambda)} + R(\lambda;x),$$

where

$$R(\lambda;x) = \left[\frac{1}{K_0(a\sqrt{2\lambda})} - \frac{1}{g(\lambda)}\right]\left[K_0(x\sqrt{2\lambda}) - K_0(a\sqrt{2\lambda})\right] + 1.$$

With this R define η by (24) for which we have (25). Then we can proceed as in the proof of Theorem 1 of [18] with the same computation but taking care of the effect of the subtraction of $K_0(a\sqrt{2\lambda})$ in the expressions above, which results in the additional factor $(x-a)\wedge 1$ in front of the error term and hence allows us to replace

$1 + \lg_+ x$ by $\lg(x/a)$ in the error estimate of Theorem 1 of [18] so that uniformly for $x > a$, as $t \to \infty$,

$$q^{(2)}(x,t;a) = 2[\lg(x/a)]\frac{\kappa}{a^2}W\left(\frac{\kappa}{a^2}t\right) + O\left(\frac{\lg(x/a)}{t(\lg t)^2}\cdot\frac{x^2 \wedge t}{t}\right), \tag{38}$$

where $W(\lambda) = \int_0^\infty \frac{e^{-\lambda u}du}{[\lg u]^2+\pi^2} = \frac{1}{\lambda(\lg\lambda)^2} - \frac{2\gamma}{(\lg\lambda)^3} + \cdots$.

Now let $a = 1$ and substitute $\frac{1}{2}\kappa = e^{-2\gamma}$ into the numerator of the leading term in the formula of Theorem 1, and we find that for $1 < x < \sqrt{t}$,

$$q^{(2)}(x,t) = \frac{2\lg x}{t[\lg(\kappa t)]^2}e^{-x^2/2t} + \frac{2\gamma(1-e^{-x^2/2t})}{t[\lg(\kappa t)]^2} + \frac{-4\gamma\lg x}{t[\lg(\kappa t)]^3} + O\left(\frac{1}{t[\lg t]^3}\right).$$

Using this estimate for $\sqrt{t}/\lg t < x < \sqrt{t}$ and (38) for $1 < x \le \sqrt{t}/\lg t$ we obtain

$$q^{(2)}(x,t) = \frac{2\lg x}{t[\lg(\kappa t)]^2}e^{-x^2/2t} + O\left(\frac{\lg x}{t[\lg t]^3}\right) \qquad (a < x < \sqrt{t})$$

as $t \to \infty$, which is the same as the first formula of Corollary 4.

3. Proof of Theorem 2

For the proof of Theorem 2 it suffices to verify the formula of it in the case $\sqrt{t} < x < Mt$ for each $M > 1$ in view of Theorem 2 and the Remark given at the end of Section 2. It is convenient to treat the cases $\nu = 0$ and $\nu > 0$ separately. In both cases one may suppose x/t to tend to a constant $v \ge 0$, and the subcases $v = 0$ and $v > 0$ are also separately treated since different arguments are employed for them, although the framework is the same. In the case $v = 0$ we shall provide estimates of the error terms that are not given in Theorem 2.

3.1. The case $\nu = 0$.

In the course of proof of Theorem 2 given below we shall derive the following proposition, which entails the formula of Theorem 2 in the case $x/t \to 0$.

Proposition 8. *Let $\nu = 0$. It holds that uniformly for $\sqrt{t} < x < t/2$, as $t \to \infty$,*

$$q^{(2)}(x,t,a) = \frac{\pi}{K_0(ax/t)}\,p_t^{(2)}(x)\left[1 + O\left(\frac{1}{\lg(t/x)}\right)\right].$$

For the proof of Proposition 8 we shall use Theorem 1, which it is convenient to reduce to the following slightly weaker form.

Lemma 9. *Let $\nu = 0$. Uniformly for $x > a$, as $t \to \infty$,*

$$\begin{aligned} q^{(2)}(x,t;a) &= 2\pi p_t^{(2)}(x)\left[\frac{2\lg(x/a)}{[\lg(t/a^2)]^2} + O\left(\frac{1}{(\lg t)^2}\right)\right] \quad \textit{if } x^2 < 2t\lg(\lg t) \\ &= 2\pi p_t^{(2)}(x)\left[\frac{2\lg(x/a)}{[\lg(t/a^2)]^2} + o\left(\frac{1}{\lg t}\right)\right] \quad \textit{if } 2t\lg(\lg t) \le x^2 \le 4t\lg(\lg t). \end{aligned}$$

In what follows we let $a = 1$. We need the following lemma from [19] in which $\nu = 0$.

Lemma 10. *For any ν, there is a constant $c = c_\nu > 0$ such that for all $x > 1$ and $t > 1$,*

$$q^{(d)}(x,t) \le cp_{t+1}^{(d)}(x). \tag{39}$$

Proof. The proof follows from the parabolic Harnack inequality (cf. eg., [6], [22]) as in the case $\nu = 0$. $\square$

We use the fact that the Bessel process of order $\nu = 0$ is the radial process of the standard two-dimensional Brownian motion $B_t^{(2)}$. Let $\mathbf{x}$ denote a generic point of $\mathbf{R}^2$ and $P_{\mathbf{x}}^{BM(2)}$ the probability of $B^{(2)}$ started at $\mathbf{x}$. We can suppose that the initial point $B_0^{(2)} = \mathbf{x}$ is on the upper vertical axis so that $\mathbf{x} = (0, x)$. Write ξ_t and Y_t for the horizontal and vertical components of $B_t^{(2)}$, respectively, and let T_K be the first hitting time of the vertical level K by $B_t^{(2)}$: $T_K = \inf\{t > 0 : Y_t = K\}$. Then the space-time distribution of (T_K, ξ_{T_K}) is given by

$$\frac{P_{(0,x)}^{BM(2)}[T_K \in dt, \xi_{T_K} \in d\xi]}{dtd\xi} = \frac{x-K}{t}\, p_t^{(1)}(x-K)p_t^{(1)}(\xi) \tag{40}$$

(cf. [11], page 25), which yields the representation

$$q(x,t) = \int_0^t ds \int_{-\infty}^{\infty} \frac{x-K}{t-s}\, p_{t-s}^{(2)}\Big(\sqrt{\xi^2 + (x-K)^2}\Big) q\Big(\sqrt{\xi^2 + K^2}, s\Big) d\xi. \tag{41}$$

K may be any number between 1 and $x = |\mathbf{x}|$, but we suppose $4 \le K < x/2$. With a fixed K we are to compute the repeated integral on the right-hand side by using the formula of Lemma 9. It is remarked that we make no use of Lemma 9 in the case when $x/t \to v > 0$.

We break the rest of the proof into four parts. For the case $x/t \to 0$ certain elaborate computations directly yield the desired formula of Proposition 8 (Parts 1 and 2). In the case $x/t \to v \ne 0$, on the other hand, we first show the existence of limit of the ratio $q(x,t)/p_t(x)$ (Part 3). While it is difficult to identify the limit along the same line as in the case $x/t \to 0$, with its existence at hand another way determines the limit (Part 4).

Throughout the proof we suppose that for some $M > 0$,

$$\sqrt{t} < x < Mt.$$

The constant $K \ge 4$ may be fixed arbitrarily prior to Part 3, in which we need to take K large enough, so we do not assign K a specific value. We write p_t, q for $p_t^{(2)}$, $q^{(2)}$ to be consistent to our convention that the superscript ${}^{(d)}$ is dropped, while we continue to write $p_t^{(\kappa)}$ if $\kappa \ne 2$. We put, for $0 \le c < \tau \le t$,

$$I_{c,\tau} = I_{c,\tau}(x,t) := \int_c^\tau ds \int_{-\infty}^{\infty} \frac{x-K}{t-s} p_{t-s}\Big(\sqrt{\xi^2 + (x-K)^2}\Big) q\Big(\sqrt{\xi^2 + K^2}\,, s\Big) d\xi,$$

the contribution to the integral in (41) from the interval $c < s < \tau$.

Part 1: *Estimation of* $I_{c,t}$. Here c is a constant not less than 4. In the identity $p\alpha^2 + q\beta^2 = pq(\alpha - \beta)^2 + (p\alpha + q\beta)^2$, where $p, q \in \mathbf{R}$ with $p + q = 1$ and α, β may be vectors in any Euclidian space, take $p = (t-s)/t$ and divide both sides of it by

$$T = pqt = \frac{s(t-s)}{t}$$

to obtain

$$\frac{1}{s}\alpha^2 + \frac{1}{t-s}\beta^2 = \frac{1}{t}(\beta - \alpha)^2 + \frac{1}{T}\left(\alpha + \frac{s}{t}(\beta - \alpha)\right)^2. \tag{42}$$

Then substituting the two-dimensional vectors $\alpha = (\xi, K), \beta = (\xi, K - x)$ leads to

$$
\begin{aligned}
&p_{t-s}\Big(\sqrt{\xi^2 + (x-K)^2}\Big)p_s\Big(\sqrt{\xi^2+K^2}\Big) \\
&\qquad = \frac{1}{2\pi T}\, p_t(x) e^{-\xi^2/2T} \exp\left[-\frac{1}{2T}\Big(K - \frac{s}{t}x\Big)^2\right].
\end{aligned} \tag{43}
$$

In the repeated integral of $I_{c,t}$ we split the range of integration w.r.t. ξ at $\xi = \pm\sqrt{4s \lg\lg s}$. We claim that

$$
\begin{aligned}
I_{c,t} \;=\; & p_t(x) \int_c^t \frac{x-K}{(t-s)\sqrt{T}} \exp\left\{-\frac{1}{2T}\Big(K - \frac{s}{t}x\Big)^2\right\} \\
& \times \Bigg[\int_{\sqrt{\xi^2+K^2} < \sqrt{4s\lg\lg s}} \frac{\lg(\xi^2+K^2)}{(\lg s)^2\sqrt{T}} e^{-\xi^2/2T} d\xi \\
& \qquad + R(s,t) + O\Big(\frac{1}{(\lg s)^2}\Big)\Bigg] ds,
\end{aligned} \tag{44}
$$

where $R(s,t)$, the term that comes from the remainder term in Lemma 9, is $o(1/\lg s)$. For the part $|\xi| \geq \sqrt{4s \lg s}$ we have only to substitute the expression of q given in Lemma 9 and apply (43) (note that the bound in Lemma 9 actually holds uniformly for $t \geq 4$ simply because $q(x,t)$ is bounded there). For the integral on $|\xi| \geq \sqrt{4s\lg s}$ we need to take p_{s+1} in place of p_s in (43) so that the corresponding contribution to $I_{c,t}$ becomes

$$
\begin{aligned}
& p_{t+1}(x) \int_c^t \frac{x-K}{(t-s)\sqrt{T'}} \exp\left\{-\frac{1}{2T'}\Big(K - \frac{s+1}{t+1}x\Big)^2\right\} \\
& \quad \times \int_{\sqrt{\xi^2+K^2} \geq \sqrt{4s\lg\lg s}} \frac{e^{-\xi^2/2T'}}{\sqrt{T'}} d\xi ds,
\end{aligned} \tag{45}
$$

where $T' = (s+1)(t-s)/(t+1)$; since the inner integral is $O(1/(\lg s)^2)$ uniformly in t, a simple change of variable gives the error term in (44). Thus we have verified the claim.

Scaling the variable ξ by $\sqrt{s}$ and dominating $e^{-\xi^2/2T}$ by $e^{-\xi^2/2s}$ if necessary, we see that the quantity in the big square brackets is evaluated to be

$$
\frac{\sqrt{2\pi}}{\lg s} + O\bigg(\frac{1}{(\lg s)^2\sqrt{(t-s)/t}}\bigg) + R(s,t). \tag{46}
$$

We must compute the integral

$$
J := \int_c^t \frac{x-K}{(t-s)\sqrt{T}} \exp\left\{-\frac{1}{2T}\Big(K - \frac{s}{t}x\Big)^2\right\} \frac{\sqrt{2\pi}}{\lg s} ds. \tag{47}
$$

Now we suppose $x/t < 1/2$ so that $\lg t/x > \lg 2$ and take $c = 4$. By a simple change of the variables of integration, e.g., according to $u = (x/t)\sqrt{s}$, which transforms s/t to $u^2 t/x^2$ one can easily find that $J \sim p_t(x)\pi/\lg(t/x)$ as $x/t \to 0$, $t/x^2 \to 0$ (which is enough for Theorem 2 restricted to the case $x/t \to 0$). But this way does not give the error estimate asserted in Proposition 8. To improve the evaluation of the integral we transform the variable of integration by

$$\rho = \frac{s}{t-s} \quad \left(\text{i.e.,} \quad s = \frac{t\rho}{1+\rho}\right), \tag{48}$$

entailing the relation $ds = (t-s)^2 d\rho/t = (t-s)\sqrt{T/t\rho}\, d\rho$.

Noting the inequalities

$$0 < \frac{1}{T} - \frac{1}{t\rho} \le \frac{2}{t-s} \quad \text{and} \quad \frac{1}{T} \cdot \frac{sx}{t} = \frac{x}{t-s} < \frac{2x}{t} \vee \left(\frac{2}{x} \cdot \frac{x^2\rho}{t}\right)$$

we may write the exponent appearing in the integral of (47) in the form

$$-\frac{1}{2T}\left(K - \frac{s}{t}x\right)^2 = -\frac{K^2}{2t\rho} - (1-\delta)\frac{x^2\rho}{2t} + O\left(\frac{x}{t}\right),$$

where $\delta = \delta(t, x, s, K)$ is a function of t, x, s, K that satisfies $0 < \delta < 4K/x$ (provided that $K < x$). Further, transform the variable ρ to $u = x\sqrt{\rho/t}$. Then

$$\frac{x ds}{(t-s)\sqrt{T}} = \frac{x d\rho}{\sqrt{t\rho}} = 2du,$$

and we obtain

$$J = \int_{(x/t)\sqrt{c/(1-c/t)}}^{\infty} \frac{2\sqrt{2\pi}}{\lg\left[(t/x)^2 m(u)\right]} \exp\left(-\frac{K^2x^2}{2t^2u^2} - (1-\delta)\frac{u^2}{2}\right) du \left(1 + O\left(\frac{x}{t}\right)\right), \tag{49}$$

where $m(u) = u^2/(1 + u^2t/x^2)$. We apply the inequality

$$|1 - 1/(1+r)| \le |r| + r^2/(1+r) \qquad (r > -1)$$

with $r = [\lg m(u)]/\lg(t^2/x^2)$ for which $(1+r)^{-1} \le (\lg c)^{-1} \lg(t^2/x^2)$ to see that

$$\frac{2\sqrt{2\pi}}{\lg[(t/x)^2 m(u)]} = \frac{\sqrt{2\pi}}{\lg(t/x)} + O\left(\frac{1 + [\lg m(u)]^2}{[\lg(t/x)]^2}\right)$$

uniformly valid if u is confined to the range of integration. Owing to the identity

$$\int_0^\infty e^{-b/2u^2 - \lambda u^2/2} du = \sqrt{\frac{\pi}{2\lambda}}\, e^{-\sqrt{2b\lambda}} \qquad (\lambda > 0, b \ge 0) \tag{50}$$

and the bound $\int_0^\infty |\lg m(u)|^2 e^{-u^2/2} du \le C$, formula (49) reduces to

$$J = [\pi/\lg(t/x)][1 + O(1/\lg(t/x))]. \tag{51}$$

Taking the computation carried out just above into account one also observes that the contribution of the error term in (46) is $O\Big(p_t(x)/[\lg(t/x)]^2\Big)$ and concludes that uniformly for $\sqrt{t} < x < t/2$, as $t \to \infty$,

$$I_{c,t} = p_t(x)\left[\frac{\pi}{\lg(t/x)}\left(1 + O\left(\frac{1}{\lg(t/x)}\right) + \tilde{R}\right)\right], \tag{52}$$

where $\tilde{R}$, the term corresponding to $R(s,t)$, is $o(1)$.

Part 2: *Estimation of* $I_{0,c}$. The integral $I_{0,c}(t,x)$ is dominated by

$$\begin{aligned}\int_{-\infty}^{\infty} & P_{\sqrt{\xi^2+K^2}}[\sigma_1 < c] \sup_{0<s<c} \frac{x-K}{t-s} p_{t-s}\Big(\sqrt{\xi^2+(x-K)^2}\Big) d\xi \\ & \le C\sqrt{c}\, e^{-K^2/2c} \frac{x}{tK} p_t(x-K) \le \Big[C' e^{-K^2/2c} e^{Kx/t}\Big] p_t(x) \frac{x}{t},\end{aligned}$$

where for the first inequality we have used the bound

$$P_y[\sigma_1 < c] \le C\sqrt{c} e^{-(y-1)^2/2c}/(y-1)$$

for $y > 1$, a bound obtained from the one-dimensional result (cf. Lemma 3.2 of [19]). Combined with (52) this shows that $q(x,t)/p_t(x) = O(1/\lg t)$ at least for $\sqrt{2t \lg\lg t} < x \le \sqrt{4t\lg\lg t}$. Using this bound instead of the second one of Lemma 9 we obtain

$$R(s,t) = O(1/(\lg s)^2)$$

so that $\tilde{R} = O(1/\lg(t/x))$. The proof of Proposition 8 is complete.

Part 3: *Proof of convergence.* Here we suppose $x/t \to v > 0$ and prove that there exists $\lim q(x,t)/p_t(x)$, the limit value depends only on v and the convergence is locally uniform in v. Here we use Lemma 10 but not Lemma 9. With the help of (43) it gives

$$I_{K^2,t/2}(x,t) \le C p_t(x) \int_{K^2}^{t/2} \frac{x}{t} \exp\left[-\frac{1}{2(1-s'/t')}\left(\frac{K}{\sqrt{s'}} - \frac{x}{t'}\sqrt{s'}\right)^2\right] \frac{ds}{\sqrt{s}}, \tag{53}$$

where $s' = s+1$, $t' = t+1$ and (43) is applied with $p_{s+1}\Big(\sqrt{\xi^2+K^2}\Big)$ in place of $p_s\Big(\sqrt{\xi^2+K^2}\Big)$ (see (45)). One observes that the integral above is at most $O(e^{-vK/4})$ (use e.g. (50)). Also a quite crude estimation shows $I_{t/2,t}(x,t) \le C p_t(x) e^{-(2K-x)^2/8t}$. Combined with the result of Part 2 these show that for any $\varepsilon > 0$ one can choose K large so that

$$\limsup_{t\to\infty, x/t\to v} \left|\frac{q(x,t) - I_{c,K^2}(x,t)}{p_t(x)}\right| < \varepsilon. \tag{54}$$

Define $h_K(\xi,s)$ by

$$h_K(\xi,s) = q\Big(\sqrt{\xi^2+K^2}\,, s\Big)\Big/ p_s\Big(\sqrt{\xi^2+K^2}\Big).$$

By Lemma 10

$$h_K(\xi,s) \le C\exp\left[\frac{\xi^2+K^2}{2s(s+1)}\right],$$

and keeping this bound in mind we see that

$$
\begin{aligned}
&\frac{I_{c,K^2}(x,t)}{p_t(x)} \\
&= \frac{1}{p_t(x)} \int_c^{K^2} ds \int_{-\infty}^{\infty} \frac{x-K}{t-s}\, p_{t-s}\Big(\sqrt{\xi^2+(x-K)^2}\Big) p_s\Big(\sqrt{\xi^2+K^2}\Big) h_K(\xi,s)d\xi \\
&= \int_c^{K^2} \frac{ds}{2\pi T} \int_{-\infty}^{\infty} \frac{x-K}{t-s} \exp\bigg[-\frac{1}{2T}\Big(\Big(K-\frac{x}{t}s\Big)^2+\xi^2\Big)\bigg] h_K(\xi,s)d\xi \\
&\longrightarrow\ v\int_c^{K^2} \frac{ds}{2\pi s} \int_{-\infty}^{\infty} \exp\bigg[-\frac{(K-vs)^2}{2s}-\frac{\xi^2}{2s}\bigg] h_K(\xi,s)d\xi
\end{aligned}
$$

as $x/t \to v$. This together with (54) shows that $q(x,t)/p_t(x)$ is convergent and the limit value does not depend on the manner of x/t approaching to v. The required uniformity of the convergence is easily ascertained from the arguments made above.

Part 4: *Identification of the limit.* The proof rests on the identity

$$
p_t(x) = \int_0^t q(x,t-s)p_s(1)ds, \tag{55}
$$

which follows from the Markov property of the Bessel process and also from the identity (17). We may suppose that $x = tv$, $v \neq 0$. By Part 3

$$
q(tv,t-s) = \lambda p_{t-s}(tv)(1+o(1)) \quad \text{as} \quad s/t \to 0, t \to \infty \tag{56}
$$

for some constant $\lambda = \lambda(v) \geq 0$. Since

$$
\frac{p_{t-s}(tv)p_s(1)}{p_t(tv)} = \frac{1}{2\pi s(1-s/t)} \exp\bigg(-\frac{v^2 s}{2(1-s/t)} - \frac{1}{2s}\bigg),
$$

substitution of (56) into (55) yields

$$
\begin{aligned}
\frac{1}{\lambda} &= \lim \frac{1}{p_t(tv)} \int_0^t p_{t-s}(tv)p_s(1)ds \\
&= \frac{1}{2\pi} \int_0^{\infty} \exp\bigg(-\frac{v^2 s}{2} - \frac{1}{2s}\bigg)\frac{ds}{s} \\
&= K_0(v)/\pi
\end{aligned} \tag{57}
$$

(see [5], Eq. (29) on page 146 for the last equality). Hence $\lambda = \pi/K_0(v)$, as desired. This completes the proof of Theorem 2 in the case when x/t is bounded. □

Remark 4. In the case $x/t \to v > 0$, it seems hard to compute the value $\lim q(x,t)/p_t(x)$ along the same line as in Part 1 since our knowledge of the behavior of $q((\xi,K),s)$ is poor for small values of s that significantly contribute to the integral of (41). This point would be well understood from the argument made in Part 3 above. One notes that from Part 2 we know only that $I_{0,c}(x,t)$ becomes small if K/c is large enough, while the error term $O(1/(\lg T)^2)$ in the estimate (44) depends on c.

3.2. **The case $\nu > 0$.**

Proposition 11. *Let $\nu > 0$. It holds that uniformly for $\sqrt{t} < x < t/2$, as $t \to \infty$,*

$$\begin{aligned}\frac{q^{(d)}(x,t,a)}{p_t^{(d)}(x)} - \frac{1}{G(a)} &= O\left(\left(\frac{x}{t}\right)^{d-2}\sqrt{\lg\frac{t}{x}}\right) \qquad (0 < \nu < 1/2)\\ &= O(x/t) \qquad (\nu > 1/2).\end{aligned}$$

We use the fact that the square of the Bessel process of dimension $d > 2$ is the sum of those of two independent Bessel processes of dimensions 1 and $d' = d - 1$ ([14]). Let (Y_t) be the one-dimensional standard Brownian motion started at x and (ξ_t) the Bessel process of the dimension d' started at 0 and independent of (Y_t). Then the law of the process (X_t^2) is the same as the law of $(\xi_t^2 + Y_t^2)$. Let $T_K = \inf\{t > 0 : Y_t = K\}$. Then in place of (40) we have

$$\frac{P[T_K \in dt, \xi_{T_K} \in d\xi]}{dtd\xi} = \frac{x-K}{t} p_t^{(1)}(x-K)p_t^{(d')}(\xi)c_{d'}\xi^{d-2}$$

so that

$$q(x,t) = \int_0^t ds \int_{-\infty}^{\infty} \frac{x-K}{t-s} p_{t-s}^{(1)}(x-K)p_{t-s}^{(d')}(\xi) q\Big(\sqrt{\xi^2+K^2}, s\Big)c_{d'}\xi^{d-2}d\xi. \tag{58}$$

The proof of Proposition 11 given below is analogous to the one given for $\nu = 0$, and we proceed parallel to the lines of the preceding proof.

Part 1: *Estimation of $I_{c,t}$.* We write $I_{b,c}(x,t)$ as before for the integral in (58) restricted on the interval $[b,c]$. The product $p_{t-s}^{(1)}(x-K)p_{t-s}^{(d')}(\xi)$ appearing in the integrand may be written as

$$p_{t-s}\Big(\sqrt{\xi^2+(x-K)^2}\Big) = p_{t-s}(x-K)e^{-\xi^2/2(t-s)},$$

which we further rewrite in the form

$$\left(\frac{t}{t-s}\right)^{d/2} p_t(x-K)\exp\left\{-\frac{(x-K)^2s}{2t(t-s)}\right\}e^{-\xi^2/2(t-s)}. \tag{59}$$

We split the range of ξ-integration at $\xi = \pm\sqrt{4s\lg s}$ in the repeated integral of $I_{c,t}$. The integral on $|\xi| \geq \sqrt{4s\lg s}$ is disposed of by employing Lemma 10 as before (see (45)). As for the integral on the other part we first evaluate the contribution of the term $(\xi^2+K^2)^{-\nu}p_s(1)/G(1)$, which, using (59), is dominated by a constant multiple of

$$R_1 := p_t(x)\int_c^t ds \frac{xt^{d/2}p_s(1)}{(t-s)^{d/2+1}}\exp\left\{-\frac{(x-K)^2s}{2t(t-s)}\right\}\int_{|\xi|<\sqrt{4s\lg s}} e^{-\xi^2/2(t-s)}d\xi.$$

It is convenient to split the outer integral at $s = t/2$ and let $R_1^>$ and $R_1^<$ be the parts corresponding to $s > t/2$ and $s \leq t/2$, respectively. By performing the ξ-integration and changing the variable by $u = t - s$ we deduce

$$R_1^> \leq Cp_t(x)\int_0^{t/2} \frac{x}{u^{(d+1)/2}}e^{-x^2/4u}du \leq C''p_t(x)x^{2-d}.$$

For the evaluation of $R_1^<$ we replace the integrand by unity in the inner integral and have the bound

$$R_1^< \leq \frac{Cp_t(x)x}{t}\int_c^{t/2} \frac{\sqrt{\lg s}}{s^{(d-1)/2}}\exp\left\{-\frac{x^2s}{2t^2}\right\}ds.$$

Since the integral on the right-hand side is evaluated to be $O\Big((x/t)^{d-3}\sqrt{\lg t/x}\Big)$ or $O(1)$ according to whether $\nu < 1/2$ or $\nu > 1/2$, by taking into account the estimate for $R_1^>$ obtained above we deduce that

$$R_1 \le C'p_t(x)\Big(\frac{x}{t}\Big)^{d-2}\sqrt{\lg\frac{t}{x}} \text{ if } \nu < \tfrac{1}{2} \text{ and } R_1 \le C'p_t(x)\frac{x}{t} \text{ if } \nu > \tfrac{1}{2}. \tag{60}$$

Let $0 < \nu < 1/2$. Then, in a similar way to the above, we evaluate the contribution of the error term in (8), denoted by R_2, and make decomposition $R_2 = R_2^> + R_2^<$. For $R_2^>$ we note that $\int_{\mathbf{R}} e^{-\xi^2/2(t-s)}\xi^{d-2}d\xi = O((t-s)^{(d-1)/2})$ and deduce that

$$|R_2^>| \le Cp_t(x)xt^{-\nu}\int_0^{t/2} u^{-3/2}e^{-x^2/4u}du \le C'p_t(x)t^{-\nu};$$

also

$$\begin{aligned} |R_2^<| &\le \frac{Cp_t(x)x}{t}\int_c^{t/2}\frac{e^{-(x^2/2t^2)s}ds}{s^{d/2+\nu}}\int_0^{\sqrt{4s\lg s}}\xi^{d-2}d\xi \\ &= \frac{Cp_t(x)x}{t}\int_c^t\frac{(\lg s)^{(d-1)/2}}{s^{(d-1)/2}}e^{-(x^2/2t^2)s}ds \le C'p_t(x)\Big(\frac{x}{t}\Big)^{d-2}\sqrt{\lg\frac{t}{x}}, \end{aligned} \tag{61}$$

so that

$$|R_2| \le C'''p_t(x)\Big(\frac{x}{t}\Big)^{d-2}\sqrt{\lg\frac{t}{x}}. \tag{62}$$

Putting $T = s(t-s)/t$ we have in place of (43)

$$\begin{aligned} &p_{t-s}\Big(\sqrt{\xi^2+(x-K)^2}\Big)p_s\Big(\sqrt{\xi^2+K^2}\Big) \\ &= p_t(x)p_T^{(d')}(\xi)\frac{1}{\sqrt{2\pi T}}\exp\bigg[-\frac{1}{2T}\Big(K-\frac{s}{t}x\Big)^2\bigg]. \end{aligned} \tag{63}$$

Applying this together with (60) and (62) and making use of Lemma 10 in the same manner as before we find that

$$\begin{aligned} I_{c,t} &= \frac{p_t(x)}{G(1)}\int_c^t\frac{x-K}{(t-s)\sqrt{2\pi T}}\exp\bigg\{-\frac{1}{2T}\Big(K-\frac{s}{t}x\Big)^2\bigg\} \\ &\quad\times\bigg[\int_{\sqrt{\xi^2+K^2}<\sqrt{4s\lg s}}p_T^{(d')}(\xi)c_{d'}\xi^{d-2}d\xi + O\Big(\frac{1}{s}\Big)\bigg]ds \\ &\quad + O\bigg(p_t(x)\Big(\frac{x}{t}\Big)^{d-2}\sqrt{\lg\frac{t}{x}}\bigg). \end{aligned}$$

The quantity in the big square brackets may be evaluated to be

$$1 + O\Big(1/s[(t-s)/t]^{(d-1)/2}\Big).$$

In order to evaluate the whole integral we employ the transformation (48) and follow the succeeding arguments up to (51). We then conclude that

$$I_{c,t} = \frac{p_t(x)}{G(1)}\bigg[1 + O\bigg(\Big(\frac{x}{t}\Big)^{d-2}\sqrt{\lg\frac{t}{x}}\bigg)\bigg].$$

Let $\nu > 1/2$. Then, the integral of the third member in (61) becomes bounded, so that we have $|R_2^<| \le C' p_t(x)x/t$ in place of the bound given therein. The other computations may be carried out in a similar way and we obtain $I_{c,t} = [p_t(x)/G(1)](1 + O(x/t))$.

Part 2: *Estimation of* $I_{0,c}$. The same computation as before gives the same bound of $I_{0,c}$ (but here $p_t = p_t^{(d)}$), which is sufficient for the present purpose. Thus Proposition 11 has been proved.

Part 3: *Proof of convergence.* The proof is quite similar to the one given for $\nu = 0$. The bound (53) and hence the relation (54) holds true without any alteration except that here q and p_t are defined with $d > 2$. Define $h_K(\xi, s)$ as before. Then

$$\frac{I_{c,K^2}(x,t)}{p_t(x)} \longrightarrow v \int_a^{K^2} \frac{ds}{(2\pi s)^{d/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(K - vs)^2}{2s} - \frac{\xi^2}{2s} \right] h_K(\xi, s) c_{d'} \xi^{d-2} d\xi$$

as $x/t \to v$, and as before we conclude the desired convergence.

Part 4. Let $\lambda_d(v)$ be the limit of $q(x,t)/p_t(x)$ as $x/t \to v > 0$. The functional equation (55) holds true for all $\nu > 0$ in view of the corresponding identity for the Laplace transforms. In place of (57) we then have that if $x/t \to v$, then

$$\begin{aligned} \frac{1}{\lambda_d(v)} = \lim \int_0^t \frac{p_{t-s}(tv)p_s(1)}{p_t(tv)} ds &= \frac{1}{(2\pi)^{d/2}} \int_0^\infty \exp\left(-\frac{v^2 s}{2} - \frac{1}{2s} \right) \frac{ds}{s^{d/2}} \\ &= 2v^{d/2-1} K_{d/2-1}(v)/(2\pi)^{d/2}, \end{aligned}$$

so that $\lambda_d(v) = \Lambda_d(v)$. Thus we conclude the formula of Theorem 2. □

4. Asymptotics of the distribution of σ_a

We derive estimates of the distribution $P_x[\sigma_a < t]$ from those of the density. Here we compute only the principal parts of $P_x[\sigma_a < t]$ or $P_x[t < \sigma_a < \infty]$ (according to whether $t < x^2$ or $t \ge x^2$). With a little more labor one can obtain the error term by employing Propositions 6, 7, 8, 11 or Lemma 5. Let $\gamma(y, \nu)$ be the incomplete gamma function and put

$$\gamma_\nu(y) = \frac{\gamma(y,\nu)}{\Gamma(\nu)} = \frac{1}{\Gamma(\nu)} \int_0^y e^{-u} u^{\nu-1} du$$

and

$$\Gamma_\nu(y) = 1 - \gamma_\nu(y) = \frac{1}{\Gamma(\nu)} \int_y^\infty e^{-u} u^{\nu-1} du.$$

Theorem 12. *Let $\nu > 0$. Uniformly for $x > a$, as $t \to \infty$,*

$$\frac{P_x[t < \sigma_a < \infty]}{P_x[\sigma_a < \infty]} = \left[1 - \left(\frac{a}{x} \right)^{2\nu} \right] \gamma_\nu\left(\frac{x^2}{2t} \right)(1 + o(1)) \tag{64}$$

and

$$\frac{P_x[\sigma_a < t]}{P_x[\sigma_a < \infty]} = \frac{1}{\Lambda_\nu(0)} \Lambda_\nu\left(\frac{ax}{t} \right) \Gamma_\nu\left(\frac{x^2}{2t} \right)(1 + o(1)). \tag{65}$$

If $x^2/t \to \infty$, the first formula (64) is poor (at least in comparison with the second one), since then $\gamma_\nu(x^2/2t)$ tends to 1 and it says simply that the conditional probability of escaping from a after t tends to 1 and nothing more; such a result may be verified more directly (e.g., the crude bound given in Lemma 10 may be used to derive a fairly nice estimate of the speed of convergence). Similarly, in the case $x^2/t \to 0$, (65) asserts that the conditional probability of arriving at a before t tends to 1, which also readily follows from (13) as well as from (64).

Taking the limit along $x^2/2t = 1/y$, either of (64) or (65) shows that the scaled variable $2\sigma_a/x^2$ conditioned on the event $\sigma_a < \infty$ converges in law to a variable whose distribution function is $1-\gamma_\nu(1/y)$. This, however, follows immediately from (14) by knowing the formulae $K_\nu(t) \sim 2^{\nu-1}\Gamma(\nu)t^{-\nu}$ $(t \downarrow 0)$ ([13], page 136) and $-\int_0^\infty e^{-\lambda y} d\gamma_\nu(1/y) = 2K_\nu(2\sqrt{\lambda})\lambda^{\nu/2}/\Gamma(\nu)$ ([5], (29) on page 146).

The proofs of two formulae of Theorem 12 are similar. Since (64) is easier, we prove only (65). By what is remarked above we have only to prove it for $x > \sqrt{t/\lg t}$. With this restriction we can include the case $\nu = 0$. We remind the readers that

$$P_x[\sigma_a < \infty] = \left(\frac{a}{x}\right)^{2\nu}.$$

Theorem 13. *Let $\nu \geq 0$. Uniformly for $x > \sqrt{t/\lg t}$, as $t \to \infty$,*

$$P_x[\sigma_a < t] = \Lambda_\nu\left(\frac{ax}{t}\right)\left(\frac{a}{x}\right)^{2\nu} \frac{2^\nu}{(2\pi)^{\nu+1}} \int_{x^2/2t}^\infty e^{-y} y^{\nu-1} dy(1+o(1)).$$

Proof. Employing Lemma 5 (if necessary) as well as Theorem 2 and recalling $\Lambda_\nu(y) = Cy^{-\nu+1/2}e^y(1+o(1))$ for $y > 1$, one observes first that $P_x[\sigma_a < \sqrt{t}]$ is negligible and then that

$$P_x[\sigma_a < t] = a^{2\nu} \int_0^t \Lambda_\nu\left(\frac{ax}{s}\right) p_s^{(d)}(x) ds(1+o(1)).$$

By a simple change of the variable the right-hand side is transformed into

$$\left(\frac{a}{x}\right)^{2\nu} \frac{1}{(2\pi)^{d/2}} \int_{x^2/t}^\infty \Lambda_\nu\left(\frac{ay}{x}\right) e^{-y/2} y^{d/2-2} dy(1+o(1)).$$

If $\nu > 0$, the proof is easy from this expression and the following argument is made to include the case $\nu = 0$. If $\sqrt{t/\lg t} < x < \sqrt{t \lg t}$ (for instance), then $x^2/t > 1/\lg t$ and $(\lg t)^2 x/t \to 0$, and hence the range of integration may be restricted to the interval $[x^2/t, (\lg t)^2 x^2/t]$ in which $\Lambda_\nu(ay/x) = \Lambda_\nu(ax^2/t)(1+o(1))$ so that one may replace $\Lambda_\nu(ay/x)$ by $\Lambda_\nu(ax^2/t)$, yielding the desired formula after a simple change of the variable of integration. The other case may be dealt with in a similar manner. If $\sqrt{t \lg t} \leq x < t$, then x^2/t goes to infinity so that the upper limit of the integral may be replaced by $(1+\delta)x^2/t$ for any $\delta > 0$ and the required relation is reduced to $\Lambda((1+\delta)ax/t)/\Lambda_\nu(ax/t) \to 1$ as $\delta \downarrow 0$ uniformly in this region, which is plainly true. As for the case $x \geq t$ one has only to replace δ by K/x with large K and argue analogously. The proof of Theorem 13 is complete. □

In the case when $\nu = 0$ and $x < \sqrt{2t \lg \lg t}$ a precise asymptotic form is obtained in [18]. Combined with it Theorem 13 shows

Corollary 14. *Let $\nu = 0$. Uniformly for $x > a$, as $t \to \infty$,*

$$P_x[\sigma_a < t] = 1 - \frac{2\lg(x/a)}{\lg t}\left[1 - \frac{\lg(2e^{-\gamma})}{\lg t} + O\left(\frac{\frac{1}{\lg t} V \frac{1}{t} x^2}{\lg t}\right)\right] \quad \textit{for} \ \ x < \sqrt{t} \tag{66}$$

$$= \frac{1}{2K_0(ax/t)} \int_{x^2/2t}^\infty \frac{e^{-y}}{y} dy\,(1+o(1)) \quad \textit{for} \ \ x > \sqrt{t/\lg t}. \tag{67}$$

From Corollary 14 it follows that if $\nu = 0$ and $x = \mu t^\alpha$ with $\mu > 0$ fixed, then as $t \to \infty$,

$$P_x[\sigma_1 < t] \longrightarrow (1 - 2\alpha) \quad \text{if} \quad 0 \le \alpha < 1/2,$$

$$P_x[\sigma_1 < t] \sim \int_{\frac{1}{2}\mu^2 t^{2\alpha-1}}^\infty \frac{e^{-y}}{2y} dy \times \begin{cases} \dfrac{1}{(1-\alpha)\lg t} & \text{if} \quad 1/2 \le \alpha < 1, \\ \dfrac{1}{K_0(\mu)} & \text{if} \quad \alpha = 1, \\ \left(\pi^{-1} 2\mu t^{\alpha-1}\right)^{1/2} e^{\mu t^{\alpha-1}} & \text{if} \quad \alpha > 1. \end{cases}$$

In view of the identity $P_x[\sigma_a < t] = P_{x/a}[\sigma_1 < t/a^2]$, the case $\alpha = 1/2$ of this formula is the same as (1.6) of Spitzer [15], where it is used to derive his well-known test for a parabolic thinness at infinity of space-time boundaries. By the same token an equivalent form in the case $0 < \alpha < 1/2$ is $\lim_{a \downarrow 0} P_1[\lg \sigma_a \le \gamma \lg a^{-1}] = \gamma(2+\gamma)^{-1}$ $(\gamma = \alpha^{-1} - 2)$, which is found in [11], Problem 4.6.4, but in terms of the Laplace transform.

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