

ESSENTIAL SURFACES OF NON-NEGATIVE EULER CHARACTERISTIC IN GENUS TWO HANDLEBODY EXTERIORS

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ABSTRACT. We provide a classification of the essential surfaces of non-negative Euler characteristic in the exteriors of genus two handlebodies embedded in the 3-sphere.

INTRODUCTION

As is well-known, the set of knots in the 3-sphere is classified into four classes, the *trivial knot*, *torus knots*, *satellite knots* and *hyperbolic knots*, depending on the types of the essential surfaces of non-negative Euler characteristic lying in their exteriors. The trivial knot is the only knot that contains an essential disk in its exterior, while the torus knot exteriors contain essential annuli but do not contain essential tori. The class of satellite knots consists of knots admitting essential tori. Classical studies on knots prove that the essential annuli in the exterior of torus knots or satellite knots are very limited, that is, each of them is either a cabling annulus or one which can be extended to decomposing spheres (cf. Lemma 1.2). The class of hyperbolic knots consists of knots whose exteriors are *simple*, that is, do not admit any essential surfaces of Euler characteristic at least zero. By Thurston's Hyperbolization Theorem [28, 35, 37, 38, 44], the complement of each hyperbolic knot admits a complete hyperbolic metric of finite volume. A great many studies on knots have been based on this classification.

A genus g handlebody V embedded in the 3-sphere S^3 , where g is a non-negative integer, is called a *genus g handlebody-knot* and denoted by (S^3, V) . When g equals one, the study of handlebody-knots coincides with classical knot theory. On the other hand, the study of handlebody-knots whose exteriors are also handlebodies is related to the theory of Heegaard splittings. By Thurston's Hyperbolization Theorem again, the exterior $E(V)$ of handlebody-knot V of genus at least two is simple if and only if $E(V)$ admits a hyperbolic structure with totally geodesic boundary. Otherwise, the configurations of essential surfaces of non-negative Euler characteristic in the exterior $E(V)$ are much more complicated in general compared to the case of knots. The aim of this paper is to classify these essential surfaces in

Received by the editors February 21, 2013 and, in revised form, May 29, 2013.

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 57M15.

Key words and phrases. Knot, handlebody, essential surface.

The first-named author was supported in part by Grant-in-Aid for Young Scientists (B) (No. 26800028), Japan Society for the Promotion of Science, and by JSPS Postdoctoral Fellowships for Research Abroad.

The second-named author was supported in part by Grant-in-Aid for Scientific Research (C) (No. 23540105), Japan Society for the Promotion of Science.

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the exteriors of genus two handlebody-knots. In fact, we classify, without overlap, the essential disks into three types (cf. Section 2), the essential annuli into four types (cf. Section 3), the essential Möbius bands into two types (cf. Section 4), and the essential tori into three types (cf. Section 5). This should be contrasted with the case of knots; the essential annuli, for example, in knot exteriors can be classified into two types, as was mentioned above. To obtain the above classification, we fully use the results on essential planar surfaces and punctured tori properly embedded in the exteriors of knots, which are strongly related to the study of Dehn surgeries on knots in the 3-sphere that produce reducible or toroidal 3-manifolds.

In [36], Motto gave an infinite family of genus two handlebody-knots, and using essential annuli lying in their exteriors, he showed that the handlebody-knots in the family are mutually distinct whereas they have homeomorphic exteriors. In [32], Lee and Lee provided other infinite families of genus two handlebody-knots such that the handlebody-knots in each of the families are mutually distinct whereas they have homeomorphic exteriors. A detailed description of essential annuli in the exteriors of the handlebody-knots again played an important role in their paper. Also, in [10], Eudave-Muñoz and the second-named author determined essential annuli that can be extended to 2-decomposing spheres in tunnel number one, genus two handlebody-knot exteriors, and they characterized their summands by 2-decomposing spheres. Each of the above families of essential annuli is entirely contained in one type of the essential surfaces studied in this paper.

On the other hand, the first-named author defined in [31] the *symmetry group* of a handlebody-knot. This is the group of isotopy classes of self-homeomorphisms of S^3 leaving the handlebody-knot invariant. When the exterior of a genus two handlebody-knot is boundary-reducible or simple, a finite presentation of its symmetry group can be obtained following [1, 6, 12, 31, 42]. However, apart from a few examples, the symmetry groups of the remaining handlebody-knots still remain unknown. The result in this paper would be a beginning step to developing the study of the symmetry groups.

In [26], Ishii, Kishimoto and the second-named author showed the unique decomposition theorem with respect to a special kind of 2-decomposing spheres for handlebody-knots of arbitrary genus whose exteriors are boundary-irreducible. In an appendix of the paper, we prove the same uniqueness theorem for arbitrary handlebody-knots.

Throughout this paper, we will work in the piecewise linear category.

Notation. Let X be a subset of a given polyhedral space Y . Throughout the paper, we will denote the interior of X by $\text{Int } X$ and the number of components of X by $\#X$. We will use $N(X; Y)$ to denote a closed regular neighborhood of X in Y . If the ambient space Y is clear from the context, we denote it briefly by $N(X)$. Let M be a 3-manifold. Let $L \subset M$ be a submanifold with or without boundary. When L is 1- or 2-dimensional, we write $E(L) = M \setminus \text{Int } N(L)$. When L is 3-dimensional, we write $E(L) = M \setminus \text{Int } L$. We shall often say surfaces, compression bodies, etc. in an ambient manifold to mean the isotopy classes of them.

1. PRELIMINARIES

Let M be a compact orientable 3-manifold. Let F be an orientable (possibly not connected) surface properly embedded in M . A disk D embedded in M is called a *compressing disk* for F if $D \cap F = \partial D$ and ∂D is an essential simple closed

curve on F . A disk D embedded in M is called a *boundary-compressing disk* for F if $D \cap F \subset \partial D$ is a single essential arc on F and $D \cap \partial M = \partial D \setminus \text{Int}(D \cap F)$. The surface F is said to be *incompressible* (*boundary-incompressible*, respectively) if there exists no compressing disk (boundary-compressing disk, respectively) for F . The surface F is said to be *essential* if F is incompressible, boundary-incompressible and not boundary parallel. A connected non-orientable surface F' properly embedded in M is said to be *essential* if the *frontier* of $N(F'; M)$, that is, the closure of $\partial N(F'; M) \setminus \partial M$, is essential.

We recall that a *handlebody* is a compact orientable 3-manifold containing pairwise disjoint essential disks such that the manifold obtained by cutting along the disks is a 3-ball. The *genus* of a handlebody is defined to be the genus of its boundary surface. The following well-known fact will be needed later. See e.g. [27].

Lemma 1.1. *Let F be an essential surface in a handlebody. Then F is a disk.*

The essential annuli in knot exteriors are classified as follows. See e.g. [5].

Lemma 1.2. *Let K be a knot in S^3 . If $E(K)$ contains an essential annulus A , then exactly one of the following holds:*

- (1) *K is a torus knot or a cable knot and A is its cabling annulus;*
- (2) *K is a composite knot and A can be extended to a decomposing sphere for K .*

We note that the above lemma can be generalized as a classification of the essential annuli in the exteriors of links in S^3 . In fact, if A is an essential annulus in the exterior of a link, then A is a cabling annulus, A can be extended to a decomposing sphere, or A connects two components of the link, where at least one of the boundary components of A has a meridional or integral boundary slope.

As a direct corollary of Lemma 1.2, we can also classify the essential Möbius bands in knot exteriors as follows:

Lemma 1.3. *Let K be a knot in S^3 . If $E(K)$ contains an essential Möbius band F , then K is either an $(n, 2)$ -torus knot or an $(n, 2)$ -cable knot for an odd integer n , and the frontier of $N(F)$ satisfies (1) in Lemma 1.2.*

In Sections 3 and 4, we obtain the same types of classifications as Lemmas 1.2 and 1.3, respectively, for genus two handlebody-knots.

Let M be a compact orientable 3-manifold. Let F be an orientable surface (possibly not connected) properly embedded in M . Let D be a compressing disk for F . Then we have a new proper surface F' by cutting F along ∂D and pasting two copies of D to it. We say that F' is *obtained by compressing F along D* .

Let M be a 3-manifold. We recall that M is said to be *reducible* if it contains a sphere that does not bound a 3-ball in M . Otherwise, M is said to be *irreducible*. Also, M is said to be *boundary-reducible* if it contains an essential disk. Otherwise, M is said to be *boundary-irreducible*.

Lemma 1.4. *Let M be a compact, orientable, irreducible, boundary-irreducible 3-manifold such that ∂M is a closed surface of genus at least two. Let A be an annulus properly embedded in M . If each component of ∂A is essential on ∂M and A is not parallel to the boundary of M , then A is essential in M .*

Proof. Assume that each component of ∂A is non-trivial on ∂M and that A is not parallel to ∂M . If A admits a compressing disk D_1 in M , then each of the disks obtained by compressing A along D_1 is an essential disk in M . This contradicts the assumption that M is boundary-irreducible. Thus it suffices to show that A is boundary-incompressible. Assume for contradiction that A admits a boundary-compressing disk D_2 in M . Let D be the disks obtained by boundary-compressing A along D_2 . We will show that D is an essential disk in M . Set $\gamma = \partial D_2 \cap \partial M$. We note that ∂D is the component of $\partial N(\partial A \cup \gamma; \partial M)$ that is not parallel to either component of ∂A . If the two simple closed curves ∂A are not parallel on ∂M , then ∂D is not trivial on ∂M . Hence D is an essential disk in M . Assume that ∂A consists of parallel simple closed curves on ∂M . Let A' be the sub-annulus of ∂M such that $\partial A' = \partial A$. If γ is not contained in A' , then ∂D is not trivial on ∂M . Hence D is an essential disk in M . If γ is contained in A' , then D_2 is an essential disk in a component N of S^3 cut off by the torus $A \cup A'$. This implies that N is a solid torus and D_2 is its meridian disk. Moreover, ∂D_2 intersects each component of ∂A once and transversely. Hence A is parallel to ∂M through N . This is a contradiction. \square

Let M be a compact orientable 3-manifold. Let F be an orientable surface (possibly not connected) properly embedded in M . An annulus A embedded in M is called a *peripherally compressing annulus* for F if $A \cap F$ is a single essential simple closed curve on F and $A \cap \partial M = \partial A \setminus (A \cap F)$ is a single essential simple closed curve on ∂M . We note that a peripherally compressing annulus is called an *accidental annulus* when it is considered in a knot exterior. See e.g. [24]. Let A be a peripherally compressing annulus for F . Then we have a new proper surface F' by cutting F along $F \cap A$ and pasting two copies of A to it. We say that F' is obtained by *peripherally compressing F along A* .

Lemma 1.5. *Let M be a compact, orientable, irreducible 3-manifold such that ∂M is a torus. Let T be an essential torus in M . Let A be a peripherally compressing annulus for T . Then the annulus obtained by peripherally compressing T along A is essential in M .*

Proof. Let T' be the annulus obtained by peripherally compressing T along A . Assume that there exists a compressing disk D_1 for T' . We can isotope D_1 so that $\partial D_1 \cap N(A) = \emptyset$. Then ∂D_1 is parallel to $A \cap T$, otherwise ∂A is not essential on the annulus T' . Since $A \cap T$ is essential on T , D_1 is a compressing disk for T . This is a contradiction.

Assume that there exists a boundary-compressing disk D_2 for T' . We note that the two components $\partial T'$ are parallel on the boundary of M . Let A' be the sub-annulus of ∂M such that $\partial A' = \partial A$ and $\partial D_2 \cap \partial M \subset A'$. Since M is irreducible, the component N of M cut off by T' which contains D_2 is a solid torus and D_2 is its meridian disk. Then T' and A' are parallel through N since each component of ∂A intersects ∂D_2 once and transversely. Now, if N contains A , T is compressible in $N \subset M$ (See the left-hand side in Figure 1). Otherwise, T is parallel to ∂M (See the right-hand side in Figure 1). Therefore both cases contradict the assumption that T is essential in M . This completes the proof. \square

Let (S^3, V) be a handlebody-knot. We say that (S^3, V) is *trivial* if $E(V)$ is also a handlebody. A 2-sphere S in S^3 is called an *n -decomposing sphere* for (S^3, V) if $S \cap V$ consists of n essential disks in V , and $S \cap E(V)$ is an essential surface

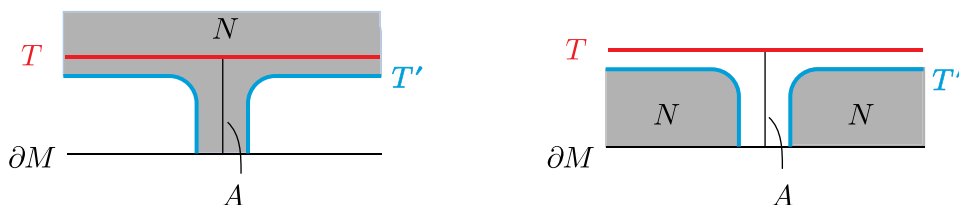


FIGURE 1

in $E(V)$. A handlebody-knot (S^3, V) is said to be n -decomposable if it admits an n -decomposing sphere. A 1-decomposable handlebody-knot is sometimes said to be *reducible*. Otherwise, it is said to be *irreducible*. We note that, by Lemma 1.1, trivial handlebody-knots are not n -decomposable for $n > 1$. It is proved in [3, 45] that a handlebody-knot (S^3, V) of genus two is 1-decomposable if and only if its exterior $E(V)$ is ∂ -reducible, i.e. $\partial E(V)$ is compressible in $E(V)$. See e.g. [25, 26] and the references given there for more details.

2. CLASSIFICATION OF THE ESSENTIAL DISKS IN GENUS TWO HANDLEBODY-KNOT EXTERIORS

We first review the notion of *characteristic compression body* introduced in [4]. Let M be an irreducible compact 3-manifold with boundary and let \mathcal{D} be the union of mutually disjoint compression disks for ∂M . Let W be the union of $N(\mathcal{D} \cup \partial M; M)$ and all the components of $M \setminus \text{Int}(N(\mathcal{D} \cup \partial M; M))$ that are 3-balls. Then we call W a *compression body* for ∂M . Also, $\partial_+ W = \partial M \subset \partial W$ is called the *exterior boundary* of W and $\partial_- W = \partial W \setminus \partial_+ W$ is called the *interior boundary* of W . A *characteristic compression body* W of M is a compression body for ∂M such that the closure of $M \setminus W$ is boundary-irreducible. Here, we remark that, if W is a characteristic compression body, every compressing disk D for ∂M can be isotoped so that $D \subset W$. We also remark that any closed incompressible surface in W is parallel to a sub-surface of $\partial_- W$ (see e.g. [4]).

Theorem 2.1 ([4]). *An irreducible compact 3-manifold with boundary has a unique characteristic compression body.*

Let (S^3, V) be a genus two handlebody-knot. Let W be the characteristic compression body for $E(V)$. We classify V into the following four types:

- (i): $\partial_- W$ is a closed orientable surface of genus two;
- (ii): $\partial_- W$ consists of two tori;
- (iii): $\partial_- W$ is a torus;
- (iv): $\partial_- W = \emptyset$.

Let (S^3, V) be a genus two handlebody-knot. As we mentioned in Section 1 V is of type (i) if and only if V is not 1-decomposable. We also note that V is of type (iv) if and only if V is trivial.

Let X be a handlebody of genus at least 1. A simple closed curve l on ∂X is said to be *primitive* with respect to X if there exists an essential disk E in X such that ∂E and l have a single transverse intersection on ∂X .

Let (S^3, V) be a genus two handlebody-knot. We introduce the following three types of essential disks in $E(V)$.

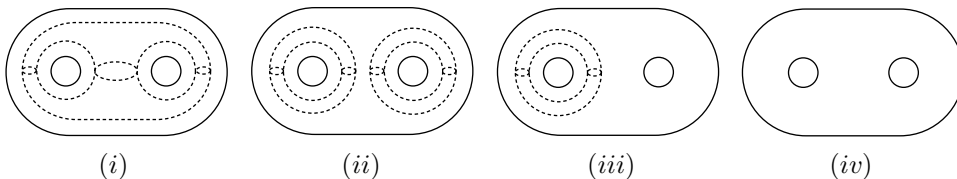


FIGURE 2. The four types of characteristic compression bodies $W \subset E(V)$.

Type 1 (1-decomposing sphere type): An essential disk D in $E(V)$ is called a *Type 1 disk* if ∂D bounds an essential disk D' in V . Here we remark that $D \cup D'$ becomes a 1-decomposing sphere for V ;

Type 2 (primitive disk type): An essential disk D in $E(V)$ is called a *Type 2 disk* if ∂D is primitive with respect to V ;

Type 3 (unknotting tunnel type): An essential disk D in $E(V)$ is called a *Type 3 disk* if there exists a tunnel number one 2-component link $l_1 \sqcup l_2$ and an unknotting tunnel τ of it such that

- l_1 is a trivial knot;
- there exists a re-embedding $h : E(l_1) \rightarrow S^3$ such that $V = h(E(l_1 \cup l_2 \cup \tau))$ and $D = h(D_*)$, where D_* is the co-core of the 1-handle $N(\tau; E(l_1 \cup l_2))$ attached to $N(l_1 \sqcup l_2)$.

Remark. For each essential disk D in $E(V)$ we define $c(D)$ to be the maximum number of mutually disjoint and mutually non-isotopic compression disks for ∂V in S^3 each of which is disjoint from and non-isotopic to D . It is easily verified that if D is of Type i ($i = 1, 2, 3$), then $c(D) = 4 - i$.

Example. Let $L = l_1 \sqcup l_2$ be the Whitehead link and τ be its unknotting tunnel as illustrated on the left-hand side in Figure 3. Let $h : E(l_1) \rightarrow S^3$ be the re-embedding such that $h(E(l_1))$ is a thickened trefoil. Then $V = h(E(l_1 \cup l_2 \cup \tau))$ is a genus two handlebody-knot and the image D of the co-core the 1-handle $N(\tau; E(l_1 \cup l_2))$ becomes an essential disk in $E(V)$ as shown on the right-hand side in Figure 3.

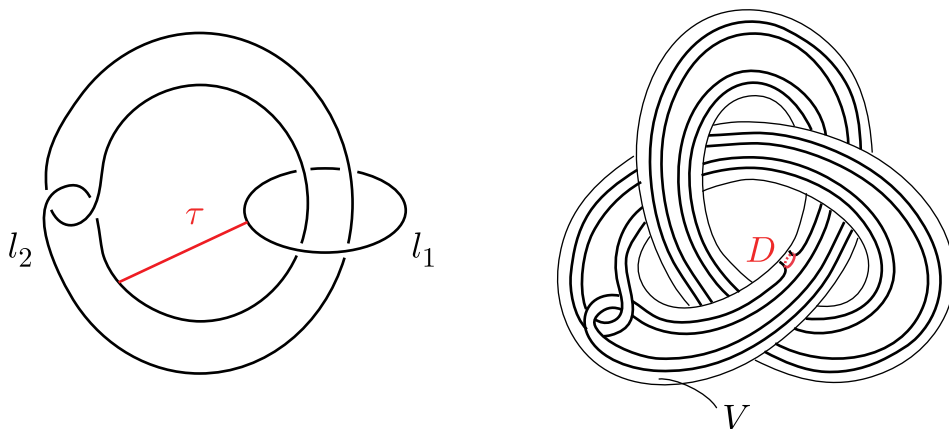


FIGURE 3. A Type 3 essential disk.

We remark that if D is an essential disk in the exterior W of the trivial genus two handlebody-knot V , then D is the dual disk of an unknotting tunnel of the tunnel number one knot or link which is the core of $W \setminus \text{Int } N(D; W)$.

Theorem 2.2. *Let (S^3, V) be a non-trivial genus two handlebody-knot. Then each essential disk D in the exterior of V belongs to exactly one of the above three types.*

Proof. Let D be an essential disk in $E(V)$. By definition, we may easily check that D cannot belong to more than one type.

Let W be the characteristic compression body of $E(V)$. We first consider the case where V is of type (ii). Set $\partial_- W = T_1 \sqcup T_2$, where each of T_1 and T_2 is a torus. It is clear that D is separating in $E(V)$. Since $T_1 \sqcup T_2$ is incompressible in $E(V \cup W)$ and compressible in S^3 , $T_1 \sqcup T_2$ is compressible in $V \cup W$. Let $D' \subset V \cup W$ be a compressing disk for T_1 and S' be a sphere obtained by compressing T_1 along D' . We note that S' is an essential sphere in $V \cup W$; otherwise $V \cup W$ is a solid torus, which is a contradiction. By Haken's lemma [19], there exists an essential sphere S'' in $V \cup W$ such that $S'' \cap V$ is a single disk. This implies that S'' is a 1-decomposing sphere for V . By Lemma 3.1 of [31], $S'' \cap W$ is a unique compressing disk of $\partial E(V)$ in $E(V)$, which implies $S'' \cap W$ is isotopic to D in $E(V)$. Therefore ∂D bounds a disk (parallel to $S'' \cap V$) in V , hence D is a Type 1 disk.

In the following we shall consider the case where V is of type (iii). In this case $V \cup W$ is a solid torus since $\partial_- W$ bounds a solid torus in S^3 while $E(V \cup W)$ is not a solid torus.

Suppose that D is non-separating in $E(V)$. Then there exists a simple arc γ properly embedded in W so that

- γ intersects D once and transversely; and
- $\gamma \cup \partial_- W$ is a spine of W .

See the left-hand side in Figure 4. Since $V \cap W$ is a solid torus and $V \cup W \setminus \text{Int } N(\gamma) \cong$

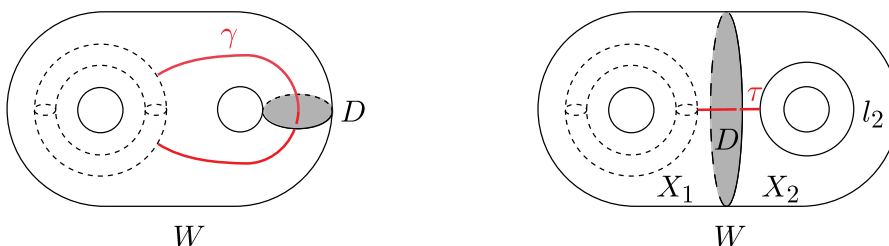


FIGURE 4

V is a genus two handlebody, it follows from [14] that γ is unknotted in $V \cup W$, that is, there exists a disk E in $V \cup W$ such that $E \cap \gamma = \partial E \cap \gamma = \gamma$ and $E \cap \partial_- W = \partial E \setminus \text{Int } \gamma$. This implies that D is a Type 2 disk.

Suppose that D is separating in $E(V)$. We set $W \setminus \text{Int } N(D; W) = X_1 \sqcup X_2$, where $X_1 \cong T^2 \times [0, 1]$ and X_2 is a solid torus. Let l_2 be the core of X_2 . Then there exists a simple arc τ in W such that

- τ connects $\partial_- W$ and l_2 ;
- $(\text{Int } \tau) \cap (\partial_- W \cup l_2) = \emptyset$;
- τ intersects D once and transversely; and
- $\Gamma = \tau \cup l_2 \cup \partial_- W$ is a spine of W .

See the right-hand side in Figure 4. We re-embed the solid torus $V \cup W$ into S^3 by a map $\iota : V \cup W \rightarrow S^3$ so that $E(\iota(V \cup W))$ is a solid torus. Let l_1 be the core of $E(\iota(V \cup W))$. Then $l_1 \cup \iota(l_2)$ is a tunnel number one link with an unknotting tunnel $\iota(\tau)$, hence D is a Type 3 disk. This completes the proof. \square

3. CLASSIFICATION OF THE ESSENTIAL ANNULI IN GENUS TWO HANDLEBODY-KNOT EXTERIORS

In this section, we provide a classification of the essential annuli in the exteriors of genus two handlebody-knots. Essential annuli in one of the four types in the classification are described using *Eudave-Muñoz knots*. We quickly review the definition and important properties of this class of knots.

In [8] Eudave-Muñoz provided an infinite family of hyperbolic knots $k(l, m, n, p)$ (where either n or p is equal to 0) that admit non-integral toroidal surgeries. The knots are now called *Eudave-Muñoz knots*. The construction of the knot $k(l, m, n, p)$ can be briefly explained as follows. Let (B, T) be the two-string tangle shown in Figure 5. In the figure, (B, T) lies outside of the small circle depicted in the middle. Then the double branched cover of the tangle (B, T) is the exterior of the Eudave-Muñoz knot $k(l, m, n, p)$. We note that the $(-2, 3, 7)$ -pretzel knot, which is

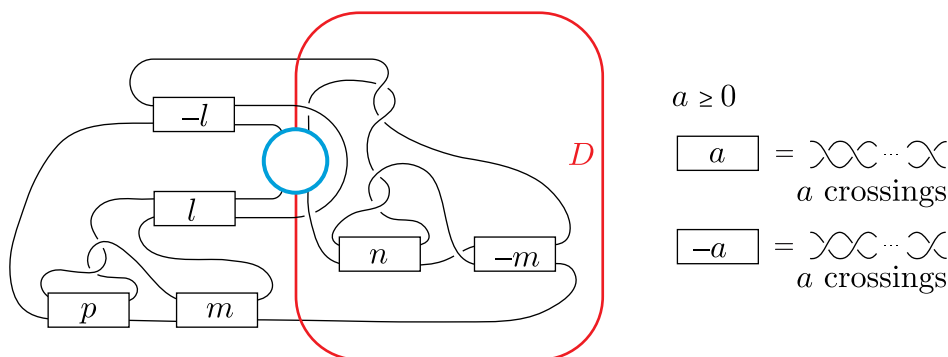


FIGURE 5

one of the most famous examples of knots that admits non-integral toroidal Dehn surgeries, is $k(3, 1, 1, 0)$. In [9], a non-integral toroidal slope r for $k(l, m, n, p)$ is described in terms of the parameters as

$$r = l(2m - 1)(1 - lm) + n(2lm - 1)^2 - 1/2$$

for $k(l, m, n, 0)$ and

$$r = l(2m - 1)(1 - lm) + p(2lm - l - 1)^2 - 1/2$$

for $k(l, m, 0, p)$. The slope r is obtained as a lift of the circle ∂D , where the disk D is depicted as in Figure 5. Gordon and Luecke [18] proved that these are the only hyperbolic knots which admit non-integral toroidal surgeries.

Theorem 3.1 ([18]). *Let K be a hyperbolic knot in S^3 that admits a non-integral toroidal surgery. Then K is one of the Eudave-Muñoz knots and the toroidal slope is r described above.*

Lemma 3.2. *Let K be an Eudave-Muñoz knot and let P be an incompressible twice-punctured torus properly embedded in $E(K)$ so that ∂P consists of the two parallel toroidal slopes of K . Then P cuts off $E(K)$ into two handlebodies of genus two.*

Proof. Let $K = k(l, m, n, p)$. Let (B, T) and D be the tangle and the disk, respectively, as shown in Figure 5. Let $p : E(K) \rightarrow B$ be the double branched covering of (B, T) . Then we have $P = p^{-1}(D)$. Since the disk D cuts off (B, T) into two trivial 3-string tangles (B_1, T_1) and (B_2, T_2) , P cuts off $E(K)$ into two genus two handlebodies $p^{-1}(B_1)$ and $p^{-1}(B_2)$. \square

Let (S^3, V) be a genus two handlebody-knot. We provide a list of annuli properly embedded in $E(V)$.

Type 1 (2-decomposing sphere type): Let $\Gamma \subset S^3$ be a spatial handcuff-graph.

Let S be a sphere in S^3 that intersects Γ in exactly one edge of Γ twice and transversely. Set $V = N(\Gamma)$. We call $A = S \setminus \text{Int } V$ a *Type 1 annulus* for the handlebody-knot (S^3, V) if A is not parallel to the boundary of V . See Figure 6.

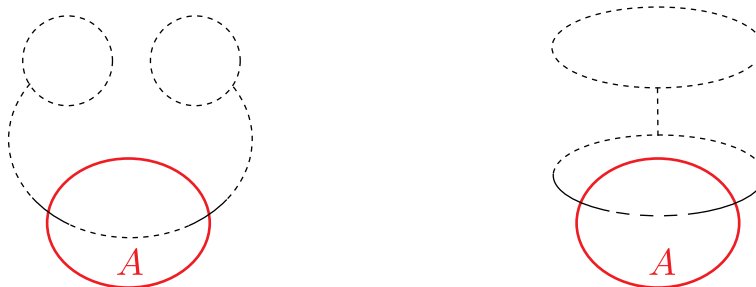


FIGURE 6. Type 1 annuli.

Type 2 (Hopf tangle type): Let $\Gamma \subset S^3$ be a spatial handcuff-graph. Assume that one of the two loops of Γ is a trivial knot bounding a disk D such that $\text{Int } D$ intersects Γ in an edge e once and transversely. Set $V = N(\Gamma)$ and $A = D \cap E(V)$. We call A a *Type 2 annulus* for the handlebody-knot (S^3, V) . See Figure 7.

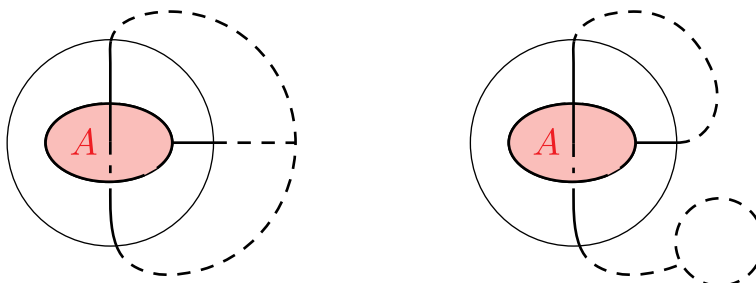


FIGURE 7. Type 2 annuli.

Type 3 (knot/link type): Let X be a solid torus embedded in S^3 . Let A be an annulus properly embedded in $E(X)$ so that $\partial A \cap \partial X$ consists of parallel non-trivial simple closed curves on ∂X .

- Let α be a properly embedded trivial simple arc in X such that $\partial\alpha \cap \partial A = \emptyset$. Set $V = X \setminus \text{Int } N(\alpha)$. Then we call A a *Type 3-1 annulus* for the handlebody-knot (S^3, V) provided that, if ∂A bounds an essential disk in X , then any meridian disk of X has non-empty intersection with α .
- Let ∂A not bound an essential disk in X . Let α be a properly embedded simple arc in $E(X)$ such that $\alpha \cap A = \emptyset$. Set $V = X \cup N(\alpha)$. Then we call A a *Type 3-2 annulus* for the handlebody-knot (S^3, V) if A is not parallel to the boundary of V .

Let X_1, X_2 be two disjoint solid tori embedded in S^3 . Assume that there exists an annulus A properly embedded in $E(X_1 \sqcup X_2)$ so that $A \cap \partial X_i$ is a non-trivial simple closed curve in ∂X_i for $i = 1, 2$. Further we require that no component of ∂A bounds a meridian disk in X_1 or X_2 . Let $e \subset E(X_1 \sqcup X_2) \setminus A$ be a proper arc connecting ∂X_1 and ∂X_2 . Set $V = X_1 \cup X_2 \cup N(e)$. Then we call A a *Type 3-3 annulus* for the handlebody-knot (S^3, V) . A proper annulus A in the exterior of a genus two handlebody-knot is said to be a *Type 3 annulus* if it is a Type 3-1, 3-2 or 3-3 annulus. Figure 8 shows schismatic pictures of Type 3 annuli.

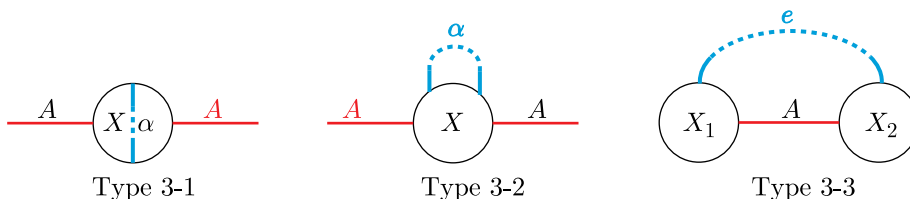


FIGURE 8. Type 3 annuli.

Type 4 (Eudave-Muñoz type): Let K be an Eudave-Muñoz knot and let P be an incompressible twice-punctured torus properly embedded in $E(K)$ so that ∂P consists of the two parallel toroidal slopes of K . By Lemma 3.2, P cuts off $E(K)$ into two handlebodies of genus two. Let V be one of them and set $A = \partial N(K) \setminus \text{Int } (\partial N(K) \cap \partial V)$.

- We call A a *Type 4-1 annulus* for the handlebody-knot (S^3, V) .
- Let $U \subset S^3$ be a knot or a two component link contained in $E(V \cup N(K))$ so that $E(V \cup N(K) \cup U)$ is a compression body for $E(V \cup N(K))$. Let $i : E(U) \rightarrow S^3$ be a re-embedding such that $E(i(E(U)))$ is not a solid torus or two solid tori. Then we call $i(A)$ a *Type 4-2 annulus* for the handlebody-knot.

A proper annulus A in the exterior of a genus two handlebody-knot is said to be a *Type 4 annulus* if it is a Type 4-1 or 4-2 annulus. Figure 9 depicts a schismatic picture of an essential annulus of Type 4.

Remark. The annuli listed above are not always essential. However, if $A \subset E(V)$ is an annulus of one of the above four types, at least we have the following by

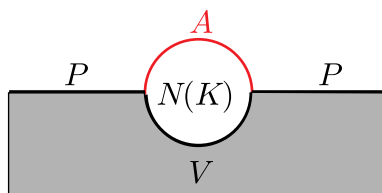


FIGURE 9. A Type 4 annulus.

definition:

- each component of ∂A is essential on ∂V .
- A is not parallel to the boundary of V .

In Corollary 3.18, we will prove that if (S^3, V) is irreducible, then the above annuli are actually essential.

As in the case of essential disks in $E(V)$, for each essential annulus A in $E(V)$, we define $c(A)$ to be the maximum number of mutually disjoint and mutually non-isotopic compression disks for ∂V in S^3 each of which is disjoint from A . Here if there exists no such compression disk, we define $c(A) = 0$. It is easily verified that if A is of Type i ($i = 1, 2, 3, 4$), then $c(A) = 4 - i$.

Example. Figure 10 shows several types of essential annuli in the exteriors of genus two handlebody-knots.

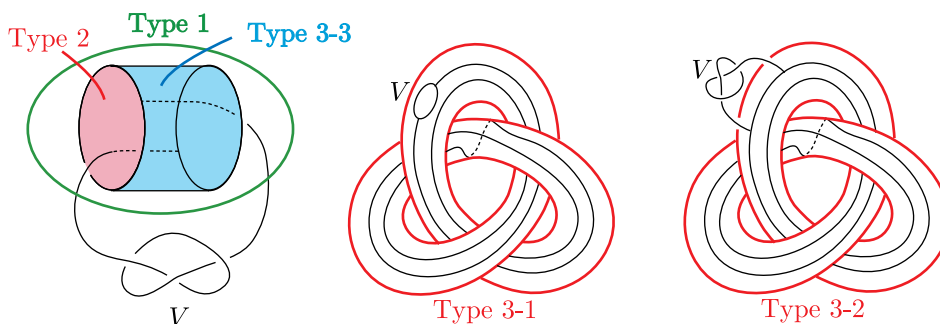


FIGURE 10. Essential annuli.

Now we are ready to state the classification theorem of the essential annuli in the exterior of genus two handlebody-knots. This should be contrasted with Lemma 1.2.

Theorem 3.3. *Let (S^3, V) be a genus two handlebody-knot. Then each essential annulus in the exterior of V belongs to exactly one of the four types listed above.*

Let (S^3, V) be a genus two handlebody-knot. Let A be an essential annulus A in the exterior $E(V)$. Set $\partial A = a_1 \sqcup a_2$. We classify the configurations of the boundary of A on ∂V into the following four cases:

Case 1: a_1 and a_2 are non-parallel, non-separating simple closed curves on ∂V .

Case 2: a_1 is non-separating and a_2 is separating on ∂V .

Case 3: a_1 and a_2 are parallel separating simple closed curves on ∂V .

Case 4: a_1 and a_2 are parallel non-separating simple closed curves on ∂V .

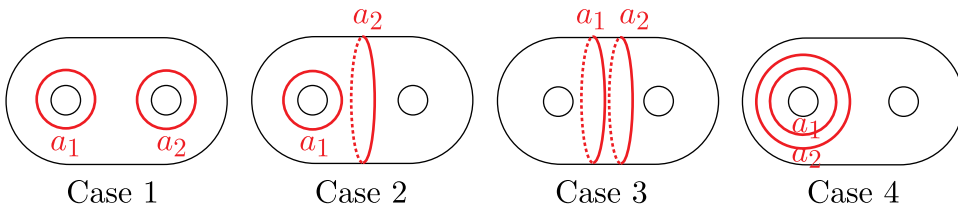


FIGURE 11

We note that, by Lemma 1.1, the trivial handlebody-knot does not contain essential annuli in its exterior.

Lemma 3.4. *Let (S^3, V) be a genus two handlebody-knot. Let $A \subset E(V)$ be an essential annulus.*

- (1) *If both a_1 and a_2 bound disks in V , then A is a Type 1 annulus.*
- (2) *If exactly one of a_1 and a_2 bounds a disk in V , then A is a Type 2 annulus.*

Proof. (1) is straightforward from the definition. Let exactly one of a_1 and a_2 , say a_1 , bound a disk E in V .

If E is non-separating in V , then we may assume that a_2 is an essential simple closed curve on the boundary of the solid torus $X = V \setminus \text{Int } N(E)$. Then the disk $A \cup E$ determines a Seifert surface of the core K of X . It follows that K is the trivial knot. Now, there is a handcuff-spine of V consisting of two loops e_1, e_2 and one cut edge e such that e_1 intersects D once and transversely, $e_2 = K$ and $e \cap E = \emptyset$. This implies that A is a Type 2 annulus.

If E is separating in V , then $V \setminus \text{Int } N(E)$ consists of two solid tori X_1 and X_2 , and a_2 is an essential simple closed curve on the boundary of one of them, say X_1 . Then, again, the disk $A \cup E$ determines a Seifert surface of the core K of X_1 . It follows that K_1 is the trivial knot. Fix meridian disks E_1 and E_2 of X_1 and X_2 , respectively. There is a handcuff-spine of V consisting of two loops e_1, e_2 and one cut edge e such that $e_1 = K$, e_2 is the core of X_2 , $e \cap (E_1 \cup E_2) = \emptyset$ and e intersects E once and transversely. This implies that A is also a Type 2 annulus. \square

Let P be a non-meridional, essential, planar surface properly embedded in the exterior of a knot K in S^3 . If P is a disk, it is clear that K is the trivial knot and P is its Seifert surface. If P is an annulus, then by Lemma 1.2, K is a torus knot or a satellite knot and P its cabling annulus. The next two lemmas, which play an important role throughout this section, show that P can be neither an n -punctured sphere for $n \geq 3$ odd nor a 4-punctured sphere.

Lemma 3.5. *Let P be a non-meridional planar surface with an odd number of boundary components properly embedded in the exterior $E(K)$ of a knot K . Then P is essential if and only if K is the trivial knot and P is a meridian disk of the solid torus $E(K)$.*

Proof. The sufficiency is clear. For necessity, let $F \subset E(K)$ be a non-meridional planar surface with an odd number of boundary components. Then by capping off the boundary components of P by meridian disks of the filling solid torus, we

obtain a non-separating sphere \hat{P} in the 3-manifold $S^3(K; p/q)$ obtained from S^3 by performing the Dehn surgery along K with the surgery slope p/q , where $p/q \neq 1/0$ is the boundary slope of P . Hence $S^3(K; p/q)$ can be presented as $(S^2 \times S^1) \# M$. It follows that $H_1(S^3(K; p/q)) \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z} \oplus H_1(M)$. This implies that $p = 0$ and $H_1(M) = 0$. By Corollary 8.3 of [11], the 3-manifold $S^3(K; 0)$ is prime and the genus of the knot is zero. Therefore K is the trivial knot and P is the meridian disk of $E(K)$. \square

Remark. It is proved in [16] that if there exists a non-trivial knot that contains an essential planar surface P of non-meridional boundary in its exterior, then the boundary-slope of P is integral.

Lemma 3.6. *The exterior of a knot in S^3 contains no properly embedded incompressible 4-punctured sphere with integral boundary slope.*

The proof of Lemma 3.6, is given in Appendix A by Cameron Gordon.

We remark that Lemmas 3.5 and 3.6 are strongly related to the famous Cabling Conjecture, which was proposed González-Acuña and Short.

Conjecture 3.7 (The Cabling Conjecture [13]). *A Dehn surgery on a knot K in S^3 can give a reducible manifold only when K is a cable knot and the surgery slope is that of the cabling annulus.*

The conjecture is known to hold for several classes of knots including satellite knots [41], strongly invertible knots [7], alternating knots [34], symmetric knots [20, 33] and the knots admitting bridge spheres with Hempel distance at least three [2, 22, 23]. However, the general case is still one of the most important open problems in knot theory. We note that if the exterior of every knot in S^3 contains no properly embedded essential planar surface of negative Euler characteristic with integral boundary slope, then the Cabling Conjecture is true.

Lemma 3.8 (Classification of Case 1). *Let $A \subset E(V)$ be an essential annulus of Case 1. Then A is a Type 2, 3-1 or 3-3 annulus.*

Proof. By Lemma 3.6, the 4-punctured sphere $P = \partial V \setminus \text{Int } N(a_1 \cup a_2)$ is compressible in $E(A)$. Let D be a compressing disk for P .

Assume first that D lies in V . Let D be separating in V . Then $V \setminus \text{Int } N(D)$ consists of two disjoint solid tori X_1 and X_2 such that $a_i \subset \partial X_i$ for $i = 1, 2$. If either a_1 or a_2 , say a_1 , is trivial on ∂X_1 , a_1 is parallel to ∂D on ∂V . This contradicts the assumption that a_1 is non-separating. Thus both a_1 and a_2 are non-trivial on ∂X_1 and ∂X_2 , respectively. Then A is a Type 3-3 annulus. Let D be non-separating in V . If either a_1 or a_2 bounds a disk in V , it follows from Lemma 3.4 that A is a Type 2 annulus since a_1 and a_2 are not parallel on ∂V . Otherwise, a_1 and a_2 are parallel essential simple closed curves on the boundary of $X = V \setminus \text{Int } N(D; V)$. Since $a_1 \cup a_2$ separates $\partial E(X)$, A is separating in $E(X)$. On the other hand, since a_1 and a_2 are not parallel on ∂V , each of the two annulus components of $\partial X \setminus \text{Int } N(\partial A; \partial X)$ meets $\partial N(D)$. It follows that $V \cap \text{Int } A \neq \emptyset$, whence a contradiction. See the left-hand side in Figure 12.

Next, assume that D lies in $E(V)$. Let D be separating in V . Since a_1 and a_2 are non-parallel and non-separating on ∂V , each of the two components of ∂V cut off by ∂D contains a_1 or a_2 . It follows that $D \cap A \neq \emptyset$, whence a contradiction. See the right-hand side in Figure 12. Let D be non-separating in V . Set $X = V \cup N(D)$.

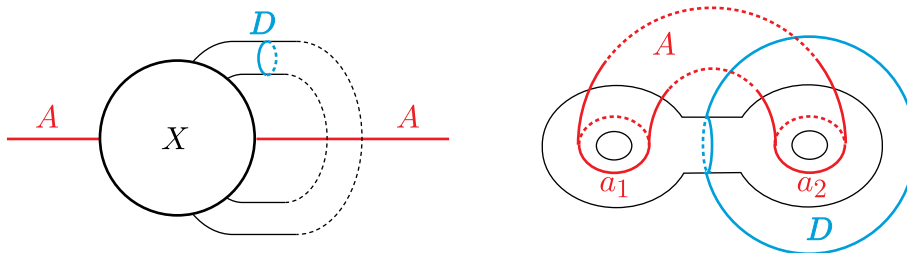


FIGURE 12

Since ∂X is a torus in S^3 , either X or $E(X)$ is a solid torus. If $E(X)$ is a solid torus, then V is the trivial genus two handlebody-knot. This contradicts Lemma 1.1. Hence X is a solid torus. Since A is essential in $E(V)$, neither a_1 nor a_2 is parallel to ∂D on ∂V . It follows that a_1 and a_2 are parallel essential simple closed curves on ∂X . Let $\alpha \subset X$ be the dual arc of D , that is, α is a simple arc properly embedded in X so that $N(D) = N(\alpha)$. By [14], α must be a trivial arc in X . This implies that A is a Type 3-1 annulus. This completes the proof. \square

Lemma 3.9. *Let A be an essential annulus in $E(V)$. Suppose that $E(V)$ is boundary-reducible. Then there exists an essential disk D in $E(V)$ such that $D \cap A = \emptyset$.*

Proof. Let D be an essential disk in $E(V)$. We minimize $\#(A \cap D)$ up to isotopy of D . If $A \cap D = \emptyset$, then we are done. Assume that $A \cap D \neq \emptyset$. Then a standard cut-and-paste argument allows us to retake an essential disk D in $E(V)$ so that $A \cap D$ consists of essential circles or essential arcs. However, the existence of an essential circle in $A \cap D$ implies that A is compressible, while the existence of an essential arc in $A \cap D$ implies that A is boundary-compressible. This is a contradiction. \square

Lemma 3.10. *If $E(V)$ contains an essential annulus of Case 2 or 3, then $E(V)$ is boundary-irreducible.*

Proof. Let $E(V)$ be boundary-reducible and assume that there exists an essential annulus $A \subset E(V)$ that is an essential annulus of Case 2 or 3. In what follows, we will prove that there exists an essential disk in $E(V)$ whose boundary is parallel to either a_1 or a_2 on ∂V . This implies that A is compressible, whence a contradiction.

By Lemma 3.9, there exists an essential disk D in $E(V)$ disjoint from A .

Assume that D is separating in $E(V)$. Since any mutually disjoint, separating, essential simple closed curves on a genus two closed surface are mutually parallel, ∂D is parallel to a_2 .

Assume that D is non-separating in $E(V)$. Suppose that A is of Case 2. Let P_1 and P_2 be the pair of pants component and the once-punctured component of ∂V cut off by ∂A . If ∂D is contained in P_1 , ∂D is parallel to a_1 on ∂V . If ∂D is contained in P_2 , then there exists a simple closed curve l on P_2 that intersects ∂D once and transversely. Then the closure D' of $\partial N(D \cup l; E(V)) \setminus \partial M$ is an essential separating disk in $E(V)$ disjoint from A . Then, by the above argument, $\partial D'$ is parallel to a_2 on ∂V . Suppose that A is of Case 3. Since D is non-separating, ∂D is contained in a once-punctured component of ∂V cut off by ∂A . Then we obtain an essential disk D' in $E(V)$ so that ∂D is parallel to a_2 on ∂V as above. \square

Lemma 3.11 (Classification of Case 2). *Let $A \subset E(V)$ be an essential annulus of Case 2. Then A is a Type 2 annulus.*

Proof. By Lemma 3.10, we may assume that $E(V)$ is boundary-irreducible. Let P be the component of $\partial V \setminus \text{Int } N(a_1 \cup a_2)$ that is homeomorphic to a pair of pants. Lemma 3.5 implies that P is compressible in $E(A)$. Since ∂V is incompressible in $E(V)$, P is compressible in $V \cap E(A)$. It follows that either a_1 or a_2 bounds a disk in V . By Lemma 3.4, A is a Type 1 or 2 annulus. Since a_1 and a_2 are not parallel by assumption, it follows that A is a Type 2 annulus. \square

Lemma 3.12 (Classification of Case 3). *Let $A \subset E(V)$ be an essential annulus of Case 3. Then A is a Type 1 annulus.*

Proof. By Lemma 3.10, we may assume that $E(V)$ is boundary-irreducible. Let $A' \subset \partial V$ be the annulus with $\partial A' = a_1 \sqcup a_2$. Then the torus $A \cup A'$ bounds a solid torus X in S^3 . Let P and Q be the once-punctured torus components of $\partial V \setminus \text{Int } A'$. Suppose first that $P \sqcup Q$ is contained in X . Then $P \sqcup Q$ is compressible in X since a solid torus does not contain incompressible once-punctured tori. Since ∂V is incompressible in $E(V)$, $P \sqcup Q$ is compressible in V . It follows that both a_1 and a_2 bound disks in V , which implies by Lemma 3.4 that A can be extended to a 2-decomposing sphere of V . Suppose next that $P \sqcup Q$ is contained in $E(X)$. Since both P and Q determine Seifert surfaces of the core of X , both ∂P and ∂Q are parallel to the preferred longitude of X . This implies that A and A' are parallel in X . However, this contradicts the assumption that A is essential. \square

We recall the following theorem by Hayashi and Shimokawa, which will be needed in the proof of Lemma 3.14.

Theorem 3.13 ([20]). *Let Y be a solid torus and $K \subset Y$ be a non-cabled knot. Assume that ∂Y is incompressible in $Y \setminus \text{Int } N(K)$. Let $Y(K; r)$ be the 3-manifold obtained from Y by performing the Dehn surgery along K with the surgery slope r . If $Y(K; r)$ contains a separating essential annulus \tilde{A} such that each component of $\partial \tilde{A}$ is primitive with respect to Y , then the slope r is integral.*

Lemma 3.14. *Let K be a knot in S^3 . If there exists an incompressible twice-punctured torus P in $E(K)$ with non-integral boundary slopes that cuts off $E(K)$ into two genus two handlebodies, then K is a hyperbolic knot.*

Proof. It is clear that K is neither the trivial knot nor a torus knot since it is well-known that these knots do not contain essential twice-punctured tori in their exteriors. Let K be a satellite knot. Then there exists an essential torus in $E(K)$. Each essential torus T cuts off S^3 into two components Y_1 and Y_2 , where Y_1 is a solid torus. We remark that $K \subset Y_1$, otherwise T is compressible in $E(K)$. Assume that $\#(P \cap T)$ is minimal up to isotopy of T . We note that $P \cap T \neq \emptyset$ since P cuts off $E(V)$ into two handlebodies V and V' . We also note that each component of $P \cap Y_2$ is essential since P is essential and $\#(P \cap T)$ is minimal. Let K_1 be the core of the solid torus Y_1 .

Claim 1. No component of $P \cap T$ is parallel to a component of ∂P on P .

Proof of Claim 1. Assume for contradiction that $P \cap T$ contains a simple closed curve l parallel to ∂P on P . Without loss of generality, we may assume that l cuts off an annulus P_0 from P so that $\text{Int } P_0 \cap T = \emptyset$. See Figure 13. Then P_0 is a peripherally

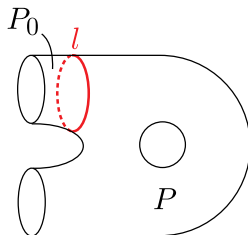


FIGURE 13

compressing annulus for T . By Lemma 1.5, we obtain by peripherally compressing T along P_0 an essential annulus T' in $E(K)$ with non-integral boundary-slope. This contradicts Lemma 1.2.

Claim 2. The number of mutually parallel loops of $P \cap T$ on P is at most two.

Proof of Claim 2. Assume for contradiction that $P \cap T$ contains mutually parallel $n \geq 3$ loops on P . Then there exist annulus components $P_1 \subset P \cap Y_1$ and $P_2 \subset P \cap Y_2$. Recall that P_2 is essential in Y_2 . By Lemma 1.2, P_2 is a cabling annulus for K_1 , or P_2 can be extended to a decomposing sphere for K_1 .

In the former case, the slopes $P \cap T$ are integral with respect to the meridian and preferred longitude of K_1 . Hence P_1 is parallel to ∂Y_1 from both sides. This implies that we can reduce the number of components of $P \cap T$, whence a contradiction.

In the latter case, the slopes $P \cap T$ bound meridian disks in Y_1 . By Claim 1, each component of P cut off by $P \cap T$ is either an annulus, a pair of pants, a 4-punctured sphere or a once-punctured torus. We see that $P \cap Y_2$ consists of only essential annuli as follows. Let Q be a component of $P \cap Y_2$. Since $P \cap \partial Y_2$ is meridional in Y_2 , $\# \partial Q$ is even; otherwise S^3 contains a non-separating sphere or torus, which is a contradiction. Thus Q is neither a pair of pants nor a once-punctured torus. On the other hand, by Claim 1, a 4-punctured sphere component of P cut off by $P \cap T$ (if any) lies in Y_1 . Thus Q is not a 4-punctured sphere. As a consequence, Q is an annulus. Among the essential annuli $P \cap Y_2$, take an outermost one P'_2 in Y_2 . By tubing P'_2 along a sub-annulus on T whose interior does not intersect P , we obtain an essential torus T' in $E(K)$ with $P \cap T' = \emptyset$. See Figure 14. This implies $T' \subset V$ or $T' \subset V'$. Then we have $T' \subset V$ or $T' \subset V'$. This contradicts Lemma 1.1.

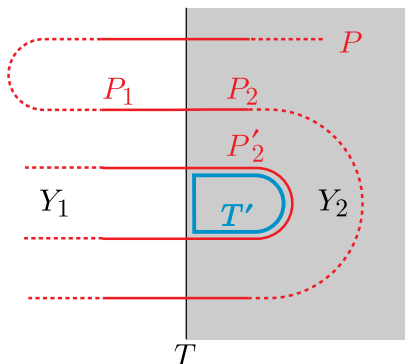


FIGURE 14

Claim 3. $P \cap T$ does not contain separating simple closed curves on P .

Proof of Claim 3. Assume for contradiction that $P \cap T$ contains a separating simple closed curve l on P . By Claim 1, l is parallel to no component of ∂P . If there exist

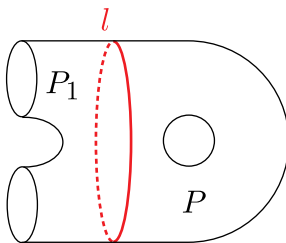


FIGURE 15

components of $P \cap T$ which are not parallel to l on P , they must be mutually parallel non-separating simple closed curves. Moreover, since T is separating in $E(K)$, the number of such components are exactly two by Claim 2. Let m be the number of components of $P \cap T$ parallel to l on P . Let n be the number of the non-separating components of $P \cap T$. Then (m, n) is $(1, 0)$, $(2, 0)$, $(1, 2)$ or $(2, 2)$. Let P_1 be the pair of pants component of P cut off by $P \cap T$ such that $\partial P \subset \partial P_1$. See the right-hand side in Figure 15. When $(m, n) = (1, 0)$, let P_2 be the once-punctured torus component of $P \setminus \text{Int } N(P \cap T; P)$. Then P_2 is a Seifert surface of the core K_1 of Y_1 . In particular, the slope l is the preferred longitude of Y_1 . Hence there is a re-embedding $h : Y_1 \rightarrow S^3$ such that $h(l)$ bounds a disk in $E(h(Y_1))$. Then by adding a disk, $h(P_1)$ can be extended to a proper annulus \hat{A} in $E(h(K))$ with non-integral boundary slope. It follows that \hat{A} is parallel to the boundary of $E(h(K))$. However, \hat{A} must be non-separating in $E(h(K))$ since \hat{A} intersects $E(h(Y_1))$ in a single meridional disk. This is a contradiction. When $(m, n) = (1, 2)$, $P \cap Y_2$ contains a component which is an essential pair of pants in $Y_2 = E(Y_1)$. This contradicts Lemma 3.5. When $(m, n) = (2, 0)$ or $(2, 2)$, by Lemma 1.2, the boundary-slope of $P \cap T$ is cabling or meridional for Y_1 . In the former case, we also have a contradiction by a similar argument of the case $(m, n) = (1, 0)$. In the latter case, there is a component of $P \cap Y_2$ that can be extended to a decomposing sphere for K_1 . Then by the same argument as the last part of the proof of Claim 2, there exists an essential torus in $E(K)$ which does not intersect P . This is a contradiction.

Claim 4. $T \cap P$ consists of two parallel non-separating simple closed curves on P .

Proof of Claim 4. By Claims 2 and 3, $P \cap T$ consists of two parallel non-separating simple closed curves on P (see the left-hand side in Figure 16), or four non-separating simple closed curves on P such that the two of them are parallel and the remaining two are also parallel (see the right-hand side in Figure 16). In the latter case, let P_1 be one of the two pairs of pants of P cut off by $P \cap T$. Since $P \cap T$ consists of mutually parallel integral slopes on T with respect to the knot K_1 , we can re-embed Y_1 by a map $h : Y_1 \rightarrow S^3$ so that each component of $h(P \cap T)$ bounds a disk in $E(h(Y_1))$. Then by adding disks to $h(P_1)$ along the boundary circles $h(\partial P_1 \setminus \partial N(K))$, we obtain a disk whose boundary is not integral with respect to $h(K)$. This is a contradiction.



FIGURE 16

Claim 5. K_1 is a torus knot and $Y_2 \cap T$ is the cabling annulus.

Proof of Claim 5. By Claim 4, both $V \cap T$ and $V' \cap T$ are separating incompressible annuli in the handlebodies V and V' , respectively. Then it follows from the classification of essential separating annuli in a genus two handlebody [30] that both $V \cap Y_2$ and $V' \cap Y_2$ are solid tori. This fact and Lemma 1.2 imply that K_1 is a torus knot and $P \cap Y_2$ is its cabling annulus.

Claim 6. There exist no essential tori in $Y_1 \setminus \text{Int}N(K)$.

Proof of Claim 6. Assume for contradiction that $Y_1 \setminus \text{Int}N(K)$ contains an essential torus T' . We also assume that $\#(P \cap T')$ is minimal up to isotopy in $Y_1 \setminus \text{Int}N(K)$. Clearly, T' is also essential in $E(K)$ and T' cuts S^3 into two components Y'_1 and Y'_2 , where Y'_1 is a solid torus. Then by Claim 5, Y'_2 is also a torus knot exterior. We note that Y_2 and Y'_2 are disjoint, otherwise T' is parallel to T in $E(K)$. Since $P \cap Y'_2$ is essential in Y'_2 , $P \cap Y'_2$ is a non-empty disjoint union of the cabling annuli in Y'_2 . Let γ be the core of the annulus $P \cap Y_2$ and let $\gamma_1, \gamma_2, \dots, \gamma_n$ be the cores of the annuli of $P \cap Y'_2$. Since Y_2 and Y'_2 are disjoint, we may assume (up to isotopy) that $\gamma \cap (\bigcup_{i=1}^n \gamma_i) = \emptyset$, $P \cap Y_2 = N(\gamma; P)$ and $P \cap Y'_2 = N(\bigcup_{i=1}^n \gamma_i; P)$. By the same argument as in the proof of Claim 3, none of $\gamma_1, \gamma_2, \dots, \gamma_n$ is separating in P . Assume that a component one of the circles $\gamma_1, \gamma_2, \dots, \gamma_n$, say γ_1 , is parallel to γ on P . Let $T_1 = T \cap V_1$ and let T'_1 be a component of $T' \cap V_1$ such that $\partial T'_1 = \partial N(\gamma_1; P)$. We remark that T_1 and T'_1 are separating incompressible annuli in V_1 and all components of ∂T_1 and $\partial T'_1$ are parallel on P . Then by [30], T_1 must be contained in the solid torus component of V_1 cut off by T'_1 . This is impossible since Y_2 and Y'_2 are disjoint. Therefore, P cut off by $T \cup T'$ contains a pair of pants component P_1 exactly one of whose boundary components lies on $\partial N(K)$. Now, as in the proof of Claim 4, we can re-embed $Y_1 \cap Y'_1$ by a map $h : Y_1 \cap Y'_1 \rightarrow S^3$ so that each component of $h(P \cap (T \cup T'))$ bounds a disk in $E(h(Y_1 \cap Y'_1))$. Then by adding disks to $h(P_1)$ along the boundary circles $h(\partial P_1 \setminus \partial N(K))$, we obtain a disk whose boundary is not integral with respect to $h(K)$. This is a contradiction.

We set $M = Y_1 \setminus \text{Int}N(K)$. By Claim 4, $P \cap M$ is a essential separating 4-punctured sphere in M . Then $P \cap M$ is naturally extend to a separating essential annulus \tilde{A} in the 3-manifold $Y_1(K; r)$ obtained from Y_1 by performing the Dehn surgery along K with the surgery slope r defined by the boundary slope of P on $\partial N(K)$. See Figure 17. Since $P \cap Y_2$ is the cabling annulus of the core K_1 of Y_1 , each component of $\partial \tilde{A} = P \cap \partial Y_1$ is primitive with respect to the solid torus Y_1 . By definition $\partial Y_1 = T$ is incompressible in M . Also, by Claim 6, M does not contain essential tori. Hence if K cannot be isotoped onto T , it follows from Theorem 3.13 that the slope r is integral. This is a contradiction. Otherwise, M is a Seifert fiber

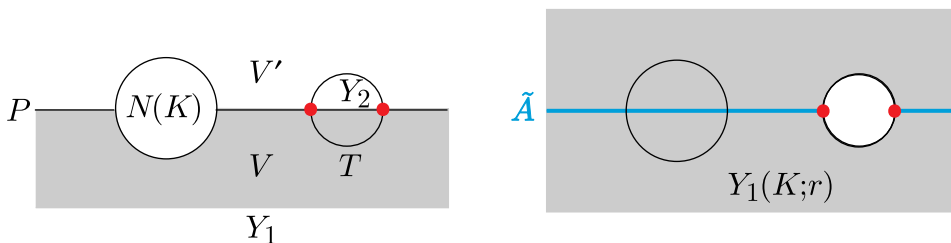


FIGURE 17

space, a so-called *cabling space*, and $P \cap M$ is an essential 4-punctured sphere in it. However, this is impossible by Lemma 3.1 in [15]. \square

The following theorem by Przytycki describes the incompressibility of surfaces before and after performing the Dehn filling.

Theorem 3.15 ([40]). *Let M be a compact 3-manifold whose boundary is a single torus. Let P be a compact orientable surface properly embedded in M so that*

- (1) P cuts off M into two handlebodies;
- (2) ∂P consists of two non-trivial simple closed curves on ∂M ; and
- (3) P is not parallel to ∂M .

Let \hat{M} be the 3-manifold obtained from M by performing the Dehn filling along the boundary slope of P . Let \hat{P} be the surface in \hat{M} naturally obtained by capping off the boundary of P . Then P is incompressible in M if and only if \hat{P} is incompressible in \hat{M} .

Theorems 3.1 and 3.15 together with Lemmas 3.2 and 3.14 provide the following corollary, which plays a key role for the classification of the essential annuli of Case 4.

Corollary 3.16. *Let K be a hyperbolic knot in S^3 . Let P be a compact twice-punctured torus properly embedded in $E(K)$ so that ∂P consists of parallel non-integral boundary slopes. Then P is an essential surface that cuts off $E(K)$ into two genus two handlebodies if and only if K is an Eudave-Muñoz knot and P is an incompressible twice-punctured torus properly embedded in $E(K)$ so that ∂P consists of the two parallel toroidal slopes of K .*

Lemma 3.17 (Classification of Case 4). *Let $A \subset E(V)$ be an essential annulus of Case 4. Then A is a Type 1, 3-1, 3-2 or 4 annulus.*

Proof. If both a_1 and a_2 bound a disk in V , A is a Type 1 annulus by Lemma 3.4. In the following, we assume that both a_1 and a_2 do not bound disks in V . Let $A' \subset \partial V$ be the annulus with $\partial A' = a_1 \sqcup a_2$. Then the torus $A \cup A'$ bounds a solid torus X in S^3 . Set $P = \partial V \setminus \text{Int } A'$.

Suppose first that P is contained in X . Then there is a compressing disk D for P .

Suppose that ∂D is non-separating on P and let $P' \subset X$ be the annulus obtained by compressing P along D .

Claim 7. $A' \cup P'$ bounds a solid torus in X .

Proof of Claim 7. When P' is incompressible in X , P' is parallel to either A or A' . In each case, it is clear that $A' \cup P'$ bounds a solid torus in X . Suppose that P is compressible in X . Then by compressing P' , we obtain two disks D_1 and D_2 bounded by a_1 and a_2 , respectively. The disks D_1 and D_2 cut off X into two 3-balls B_1 and B_2 such that $A \subset \partial B_1$ and $A' \subset \partial B_2$. Now, P' is obtained by tubing D_1 and D_2 along a simple arc γ connecting D_1 and D_2 . If $\gamma \subset B_1$, we are done (see the left-hand side in Figure 18). Assume that $\gamma \subset B_2$. Then $A' \cup P'$ is bounding

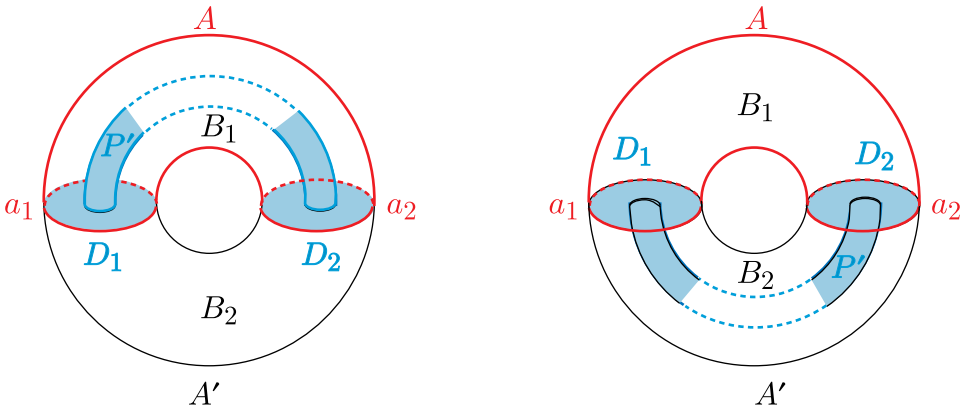


FIGURE 18

$V' = B_2 \setminus \text{Int } N(\gamma; B_2)$. The handlebody V is obtained from V' by attaching a 1-handle or drilling along a simple arc (see the right-hand side in Figure 18). The latter is impossible since A is incompressible in X . This implies that V' is also a solid torus, whence the claim.

By Claim 7, $A' \cup P'$ bounds a solid torus X' in X . Since $\partial A' = \partial P' = a_1 \cup a_2$, a_1 and a_2 are parallel non-trivial simple closed curves on $\partial X'$. If $D \subset E(V)$, then V is obtained from X' by drilling X' along a properly embedded simple arc α in X' such that $\partial\alpha \cap \partial A = \emptyset$. By [14], α is a trivial arc in X' . This implies that A is a Type 3-1 annulus. If $D \subset V$, then V is obtained from X' by adding a regular neighborhood of a properly embedded simple arc in $E(X \setminus \text{Int } X')$. This implies A is a Type 3-2 annulus.

Suppose that ∂D is separating on P . Since A is incompressible in $E(V)$, ∂D is parallel to neither a_1 nor a_2 on P . Hence ∂D is also separating on ∂V . By compressing P along D we obtain one annulus P'_1 and one torus P'_2 . In a similar argument as in Claim 7, we see that $A' \cup P'_1$ bounds a solid torus X'_1 in X . See Figure 19. Let $D \subset V$. Then P'_2 bounds a solid torus component X'_2 of $V \setminus \text{Int } N(D; V)$. There exists a properly embedded simple arc α in $X \setminus \text{Int } (X'_1 \cup X'_2)$ connecting $\partial X'_1$ and $\partial X'_2$ so that $N(\alpha, X \setminus \text{Int } (X'_1 \cup X'_2)) = N(D; V)$. Now using the solid torus X'_1 and $\alpha \cup X'_2$ it is easy to see that A is a Type 3-2 annulus. Let $D \subset E(V)$. Then P'_2 is contained in X'_1 . When P'_2 bounds a solid torus in X'_1 , then we can prove in the same way as above that A is a Type 3-1 annulus. Suppose P'_2 does not bound a

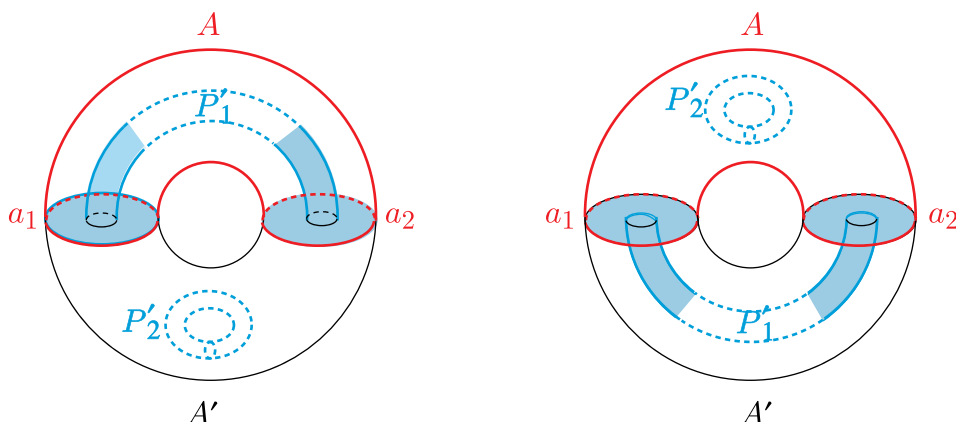


FIGURE 19

solid torus in X'_1 . Then P'_2 bounds in X'_1 a region Y homeomorphic to the exterior of a non-trivial knot in S^3 . Since ∂D is separating on ∂V , $D \subset E(V)$ is a Type 1 disk. If D is a Type 1 disk, both a_1 and a_2 bound disks in V ; this contradicts the assumption at the beginning of the proof. If D is a Type 3 disk, one of the components of $E(V) \setminus \text{Int } N(D) = E(X'_1) \sqcup Y$ must be a solid torus by definition. Since Y is not a solid torus, $E(X'_1)$ is a solid torus, i.e. X'_1 is a standard solid torus in S^3 . Then A is not essential in $E(X'_1)$, so is not in $E(V)$. This is a contradiction.

Next, suppose that P is contained in $E(X)$. Let K be the core of X . When K is the trivial knot, the above arguments immediately imply that A is a Type 3-1 or 3-2 annulus. Assume that K is not the trivial knot. Since A is essential, A and A' is not parallel in X . Hence the boundary-slope of P on ∂X is non-integral.

If P is compressible in $E(X)$, there exists a unique annulus component P' of the surface obtained by compressing P as in the above argument. Since $E(X)$ is boundary-irreducible, a_1 and a_2 do not bound disks in $E(X)$. Therefore P' is incompressible. Since A determines a cabling annulus of the core of A' , P' is parallel to A or A' . The former case is impossible since, if so, A is boundary-compressible in $E(V)$. In the latter case, applying a similar argument for the case of $P \subset X$, we can prove that A is a Type 3-1 or 3-2 annulus.

Suppose that P is incompressible in $E(X)$. By Lemma 3.9, $A' \cup P = \partial V$ is incompressible in $E(V)$, i.e. $E(V)$ is boundary-irreducible. Similarly, $A \cup P$ is incompressible in $X \cup V$. Therefore $A \cup P$ is compressible in $V' = E(X \cup V)$. It follows that the interior boundary $\partial_- W$ of the characteristic compression body W of V' is two tori, a single torus or the empty set.

Assume first that $E(V)$ does not admit an essential torus. Then it is clear from the definition that $\partial_- W = \emptyset$, i.e. V' is also a genus two handlebody. Then by Corollary 3.16 K is an Eudave-Muñoz knot. This implies that A is a Type 4-1 annulus.

Finally, assume that $E(V)$ contains essential tori. In this case, the interior boundary $\partial_- W$ is a single torus or two tori. Then, we can re-embed $X \cup V \cup W$ in S^3 so that $E(X \cup V)$ is a handlebody. This implies that A is a Type 4-2 annulus. This completes the proof. \square

Proof of Theorem 3.3. Let A be an essential annulus in $E(V)$. Then by Lemmas 3.8, 3.11, 3.12 and 3.17, A belongs to at least one of the four Types 1, 2, 3 and 4. Moreover, by Lemma 3.4 and the definition of the types of annuli in $E(V)$, we have the following characterization:

- (1) If both components of ∂A bound disks in V , then A is a Type 1 annulus and vice versa.
- (2) If exactly one of the components of ∂A bounds a disk in V , then A is a Type 2 annulus and vice versa.
- (3) If no component of ∂A bounds a disk in V , and there exists a compression disk of ∂V in S^3 disjoint from A , then A is a Type 3 annulus and vice versa.
- (4) If no component of ∂A bounds a disk in V , and there exist no compression disks of ∂V in S^3 disjoint from A , then A is a Type 4 annulus and vice versa.

The proof is then straightforward. \square

We recall that a handlebody knot (S^3, V) is said to be irreducible if it is not 1-decomposable. It is equivalent to saying that $E(V)$ is boundary-irreducible. For irreducible genus two handlebody-knots, we have a complete classification of the essential annuli in their exteriors as follows:

Corollary 3.18. *Let (S^3, V) be an irreducible genus two handlebody-knot. Let A be an annulus properly embedded in $E(V)$. Then A is essential in $E(V)$ if and only if A is a Type 1, 2, 3-2, 3-3 or 4 annulus.*

Proof. The “only if” part follows from Theorem 3.3 and the definition of the Type 3-1 annulus. In fact, if $E(V)$ contains an annulus of type 3-1, the dual disk of the drilling arc α in the definition of the Type 3-1 annulus gives an essential disk in $E(V)$.

Let A be a Type 1, 2, 3-2, 3-3 or 4 annulus. By definition, each component of A is essential on ∂V and A is not parallel to the boundary of $E(V)$. Hence by Lemma 1.4, A is essential. \square

4. CLASSIFICATION OF THE ESSENTIAL MÖBIUS BANDS IN GENUS TWO HANDLEBODY-KNOT EXTERIORS

The classification of the essential annuli in the exteriors of genus two handlebody-knots provided in the previous section directly provides a classification of the essential Möbius bands in them.

Let Y be a solid torus embedded in S^3 . Let K be an $(n, 2)$ -slope on ∂Y with respect to the core of Y , where n is an odd integer. Set $X = N(K; S^3)$ and $A = \partial Y \setminus \text{Int } X$. A Type 3-1 (3-2, respectively) essential annulus in the exterior of a genus two handlebody-knot is called a *Type 3-1* annulus* (a *Type 3-2* annulus*, respectively) if it is constructed using the above X and A in their definitions. An annulus F in the exterior of a genus two handlebody-knot is said to be a *Type 3** annulus if it is a Type 3-1* or 3-2* annulus.

Let (S^3, V) be a genus two handlebody-knot. An essential Möbius band F in $E(V)$ is called a *Type 1-1, 1-2 and 2 Möbius band*, respectively, if the frontier of its regular neighborhood is a Type 3-1*, 3-2* and 4 annulus, respectively. An essential Möbius band F in the exterior of a genus two handlebody-knot is said to be a *Type 1 Möbius band* if it is a Type 1-1 or 1-2 Möbius band.

Example. The left-hand side in Figure 20 shows a Type 1-1 essential Möbius band F_1 in the exterior of a genus two handlebody-knot V_1 . This example is provided in [36] to prove that handlebody-knots are not determined by their complements. The right-hand side in the same figure shows a Type 1-2 essential Möbius band F_2 in the exterior of a genus two handlebody-knot V_2 .

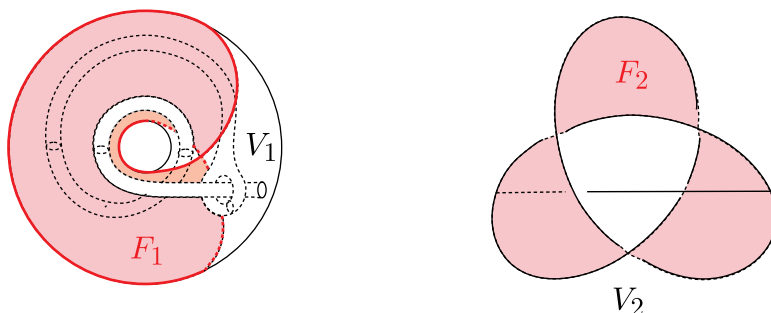


FIGURE 20. Type 1 essential Möbius bands.

Theorem 4.1. *Let (S^3, V) be a genus two handlebody-knot. Then each essential Möbius band F in the exterior of V belongs to exactly one of the above two types. Conversely, each of Types 3* and 4 essential annuli in $E(V)$ is the frontier of a regular neighborhood of an essential Möbius band of $E(V)$.*

Proof. Let A be the frontier of a regular neighborhood of an essential Möbius band F in $E(V)$. Then A satisfies the following:

- (1) ∂A cuts off an annulus A' from ∂V ;
- (2) $A \cup A'$ bounds a solid torus Y in $E(V)$;
- (3) ∂A is an $(n, 2)$ -slope with respect to the core of Y , where n is an odd integer.

By (1) and Lemmas 3.12 and 3.17, A is a Type 1, 3-1, 3-2 or 4 annulus. By (2), Type 1 is impossible. Let A be a Type 3-1 or Type 3-2 annulus. Then by definition there exists a compressing disk $D \subset E(Y)$ for $P = \partial V \setminus \text{Int } A'$ such that ∂D is non-separating on ∂V . Let P' be the surface obtained by compressing P along D . By (3), the boundary-slope of the annulus P' is an $(n, 2)$ -slope with respect to the core of Y . This implies that A is a Type 3* annulus. Since no essential annulus in $E(V)$ can be both Types 3 and 4 by Theorem 3.3, no essential Möbius band in $E(V)$ can be both Types 1 and 2.

For the other direction, let $A \subset E(V)$ be an essential annulus of Type 3* or Type 4. Then ∂A cuts off an annulus A' from ∂V and $A \cup A'$ bounds a solid torus Y in $E(V)$. If A is a Type 3* annulus, by definition, $\partial A \subset \partial Y$ consists of $(n, 2)$ -slopes with respect to the core of Y , where n is an odd integer. If A is a Type 4 annulus, by Theorem 3.1 and the definition of Type 4, $\partial A \subset \partial Y$ also consists of $(n, 2)$ -slopes with respect to the core of Y , where n is an odd integer. Hence in both cases, there exists a Möbius band F properly embedded in Y so that ∂F is the core of the annulus A' . Since the frontier of F is isotopic to A in $E(V)$, F is essential in $E(V)$. This completes the proof. \square

As a direct corollary of Theorem 4.1, we have the following:

Corollary 4.2. *Let (S^3, V) be a genus two handlebody-knot. Then there exists a one-to-one correspondence between the set of isotopy classes of essential Möbius bands in $E(V)$ and the set of isotopy classes of Type 3* or 4 essential annuli in $E(V)$.*

As for essential annuli, we have a complete classification of the essential Möbius bands in the exteriors of irreducible genus two handlebody-knots.

Corollary 4.3. *Let (S^3, V) be an irreducible genus two handlebody-knot. Let F be a Möbius band properly embedded in $E(V)$. Then F is essential in $E(V)$ if and only if F is a Type 1-2 or 2 Möbius band.*

Proof. This follows immediately from Corollary 3.18 and Theorem 4.1. □

5. CLASSIFICATION OF THE ESSENTIAL TORI IN GENUS TWO HANDLEBODY-KNOT EXTERIORS

Let (S^3, V) be a handlebody-knot and let T be a torus properly embedded in $E(V)$. A peripherally compressing annulus A for T in $E(V)$ is called, in particular, a *meridionally compressing annulus* if $A \cap \partial V$ bounds an essential disk in V . We say that T is *meridionally compressible* if it admits a meridionally compressing annulus. Otherwise, T is said to be *meridionally incompressible*.

Theorem 5.1. *Let (S^3, V) be a genus two handlebody-knot. Let T be an essential torus in $E(V)$. Then the following holds:*

- (1) *If T is meridionally compressible, then there exists a Type 1 essential annulus A in $E(V)$ such that ∂A cuts off from ∂V an annulus A' so that $A \cup A'$ is isotopic to T .*
- (2) *If T is not meridionally compressible but peripherally compressible, then there exists a Type 3-1 or 3-2 essential annulus A in $E(V)$ such that ∂A cuts off from ∂V an annulus A' so that $A \cup A'$ is isotopic to T .*
- (3) *If T is peripherally incompressible, then there exists a handlebody-knot (S^3, V') and a solid torus X in $E(V')$ such that $E(V' \cup X)$ does not contain an essential annulus A with $A \cap \partial V' \neq \emptyset$ and $A \cap \partial X \neq \emptyset$, and that there exists a re-embedding $h : E(X) \rightarrow S^3$ so that $h(V') = V$ and $h(\partial E(X)) = T$.*

Proof. Let $Y \subset S^3$ be the solid torus bounded by T . Since T is incompressible in $E(V)$, Y contains V .

Assume that there exists a peripherally compressing annulus \hat{A} for T . Let $A \subset E(V)$ be the annulus obtained by peripherally compressing T along \hat{A} . Since T is essential, it follows from a similar argument of Claim 3 in the proof of Lemma 3.14 that A is also essential. Let A' be the annulus component of ∂V cut off by ∂A . We note that T is ambient isotopic to $A \cup A'$. Then it is immediate from Lemma 3.4 that \hat{A} is a meridionally compressing annulus if and only if A is a Type 1 annulus. If \hat{A} is not a meridionally compressing annulus, by Lemmas 3.12 and 3.17, A is a Type 3-1, 3-2 or 4 annulus. However, Type 4 is impossible, since, if so, $A \cup A'$ bounds a solid torus in $E(V)$, which implies that T is compressible.

Next, assume that T is peripherally incompressible. Then $Y \setminus \text{Int } V$ does not contain an essential annulus A with $A \cap \partial V \neq \emptyset$ and $A \cap \partial Y \neq \emptyset$. We can re-embed Y into S^3 by a map i so that $X = E(i(Y))$ is a solid torus. The assertion is now easily seen by setting $V' = i(V)$. \square

APPENDIX A (BY CAMERON GORDON)

Proof of Lemma 3.6. Let K be a knot in S^3 , with exterior $E(K)$. The lemma is clearly true if K is trivial, so assume that K is non-trivial. Let P be a properly embedded incompressible 4-punctured sphere in $E(K)$ with integral boundary slope α .

We will assume familiarity with the terminology of labeled fat vertex intersection graphs, as described for example in [17].

Let $E(K)(\alpha) = E(K) \cup V_\alpha$ be the closed manifold obtained by α -Dehn filling on $E(K)$, where V_α is the filling solid torus. We may cap off the components of ∂P with meridian disks v_1, v_2, v_3, v_4 of V_α (numbered in order along V_α) to get a 2-sphere $\hat{P} \subset E(K)(\alpha)$. By [11], if we put K in thin position, then there is a level 2-sphere $\hat{Q} \subset S^3$, with corresponding meridional planar surface $Q = \hat{Q} \cap E(K) \subset E(K)$, such that the intersection graphs Γ_P and Γ_Q in \hat{P} and \hat{Q} respectively, defined in the usual way by the arc components of $P \cap Q$, have no monogon faces. Note that v_1, v_2, v_3, v_4 are the (fat) vertices of Γ_P .

Since $H_1(S^3) = 0$, Γ_P does not represent all types [39], and hence by [17] Γ_Q contains a Scharlemann cycle σ . Let f be the disk face of Γ_Q bounded by σ and let $k \geq 2$ be the number of edges in σ . Since P is incompressible we can assume by standard arguments that $(\text{Int } f) \cap P = \emptyset$. Without loss of generality σ is a (12)-Scharlemann cycle. The edges of σ give rise to k corresponding “dual” edges of Γ_P , joining vertices v_1 and v_2 . They thus divide \hat{P} into k segments.

Claim 8. Vertices v_3 and v_4 of Γ_P lie in the same segment.

Proof. Let q be the number of components ∂Q , which is equal to the valency of the vertices of Γ_P . Suppose v_3 and v_4 lie in different segments. Then of the q edges of Γ_P incident to v_3 , a_1 go to v_1 and a_2 to v_2 , where $a_1 + a_2 = q$. Similarly b_1 edges at v_4 go to v_1 and b_2 to v_2 , where $b_1 + b_2 = q$. It follows that either $a_1 + b_1$ or $a_2 + b_2$ is $\geq q$, a contradiction. \square

By Claim 8 there is a disk $D \subset \hat{P}$ containing fat vertices v_1 and v_2 and the k edges dual to σ , and disjoint from v_3 and v_4 .

Let H_{12} be that part of V_α that runs between fat vertices v_1 and v_2 of Γ_P . Let \hat{X} be a regular neighborhood of $D \cup H_{12} \cup f$, pushed slightly off \hat{P} . Then \hat{X} is a punctured lens space whose fundamental group has order k . The 2-sphere $\partial \hat{X}$ meets V_α in two meridian disks that are nearby parallel copies of v_1 and v_2 . Let A be the annulus $\partial \hat{X} \cap E(K)$. Then A separates $E(K)$ into X and Y , say, where $X \subset \hat{X}$. Note that \hat{X} is obtained by attaching the 2-handle H_{12} to X , and that $P \subset Y$.

Claim 9. A is essential in $E(K)$.

Proof. Clearly A is incompressible in $E(K)$.

Suppose A is boundary parallel in $E(K)$. Then either X or Y is homeomorphic to $A \times I$ (with A corresponding to $A \times \{0\}$). In the first case, \hat{X} is homeomorphic

to B^3 , a contradiction. In the second case, P compresses in Y , and therefore in $E(K)$, again a contradiction. \square

Claim 9 implies that K is a cable knot with cabling annulus A . Since P has the same boundary slope as A , it is easy to show that this is impossible. \square

APPENDIX B: DECOMPOSITION OF HANDLEBODY-KNOTS BY 2-DECOMPOSING SPHERES

In this appendix, we provide a unique decomposition theorem of handlebody-knots of arbitrary genus by decomposing spheres, which is a generalization of [26]. This is achieved by focusing only on a generalization of Type 1 annuli defined in Section 3 for higher genus case.

A 2-decomposing sphere S in S^3 is called a *knotted handle decomposing sphere* for a handlebody-knot (S^3, V) if $S \cap V$ consists of two parallel essential disks in V , and $S \cap E(V)$ is an essential annulus in $E(V)$.

Let (S^3, V) be a handlebody-knot and S be its knotted handle decomposing sphere. Then $S \cap \partial V$ cuts off an annulus A from ∂V . Let T be an essential torus in $E(V)$ obtained by tubing $S \cap E(V)$ along A . Let \hat{A} be a meridionally compressing annulus for T . Then by annulus-compressing T along \hat{A} , we get a new knotted handle decomposing sphere S' . We say that S' is obtained from S by an *annulus-move along A* .

A set S_1, \dots, S_n of knotted handle decomposing spheres for a handlebody-knot (S^3, V) is said to be *unnnested* if each sphere S_i bounds a 3-ball B_i in S^3 so that $B_i \cap V \cong B^3$ ($1 \leq i \leq n$) and $B_i \cap B_j = \emptyset$ ($1 \leq i < j \leq n$). We remark that a maximal unnnested set of knotted handle decomposing spheres always exists by the Kneser-Haken finiteness theorem [19, 29]. Moreover, the following is proved in [26].

Theorem B.2 ([26]). *Let (S^3, V) be a handlebody-knot such that $E(V)$ is boundary-irreducible. Then (S^3, V) admits a unique maximal unnnested set of knotted handle decomposing spheres up to isotopies and annulus-moves.*

In the following, we see that we can remove from the above theorem the assumption that $E(V)$ is boundary-irreducible.

Theorem B.3. *Every handlebody-knot (S^3, V) admits a unique maximal unnnested set of knotted handle decomposing spheres up to isotopies and annulus-moves.*

Lemma B.4. *Let $\{T_1, T_2, \dots, T_n\}$ be a maximal set of mutually disjoint, mutually non-parallel, essential, meridional-compressible tori in $E(V)$ satisfying the following:*

- for each $i = 1, 2, \dots, n$, let Y_i be the region in S^3 spanned by T_i such that $Y_i \cap V = \emptyset$. Then $Y_i \cap Y_j = \emptyset$ for $1 \leq i < j \leq n$; and
- the core K_i of $E(Y_i)$ is a prime knot.

Then any essential, meridional-compressible torus T in $E(V)$ can be isotoped so that $T \cap T_i = \emptyset$ for all i .

Proof. Assume for contradiction that $T \cap (\bigcup_{i=1}^n T_i) \neq \emptyset$ after minimizing the number of components of $T \cap (\bigcup_{i=1}^n T_i)$ by an isotopy. Let T_i intersect T . Let A be a

component of $T \cap Y_i$. Then by Lemma 1.2 and the assumption that K_i is a prime knot, A is a cabling annulus for K_i . Hence A intersects a meridionally compressing annulus A_i for T_i . It follows that $A_i \cap T$ consists of non-empty proper arcs with end points on $\partial A_i \setminus \partial V$.

Let $\delta \subset A_i$ be the disk cut off from A_i by an outermost arc α of $A_i \cap T$ in A_i . Let A' be the component of $T \cap E(Y_i)$ containing α . See Figure 21. By boundary-

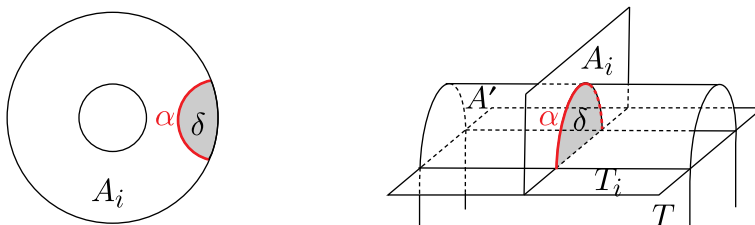


FIGURE 21

compressing A' along δ , we get a disk D whose boundary bounds a disk D' on T_i . Since a solid torus is irreducible, $D \cap D'$ bounds a 3-ball in $E(Y_i)$. This implies that A' can be isotoped to $E(Y_i)$ in $E(V) \cap E(Y_i)$. This contradicts the minimality of $\#(T \cap (\bigcup_{i=1}^n T_i))$. \square

Lemma B.5. *Let $\{T_1, T_2, \dots, T_n\}$ be a maximal set of mutually disjoint, mutually non-parallel, essential tori in $E(V)$ such that there exist peripherally compressing annuli A_i for T_i ($1 \leq i \leq n$) with $A_j \cap A_k = \emptyset$, $A_j \cap T_k = \emptyset$ for $1 \leq j < k \leq n$. For each $i = 1, 2, \dots, n$, let Y_i be the region in S^3 spanned by T_i such that $Y_i \cap V = \emptyset$. Then $Y_i \cap Y_j = \emptyset$ for $1 \leq i < j \leq n$.*

Proof. If $Y_i \cap Y_j \neq \emptyset$ for some i, j , we may assume without loss of generality that $Y_i \subset Y_j$ since every torus embedded in S^3 is separating. However, this is impossible since it is assumed that the compressing annulus A_j does not intersect T_i . \square

Proof of Theorem B.3. Let $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ and $\mathcal{S}' = \{S'_1, S'_2, \dots, S'_n\}$ be maximal unnested sets of knotted handle 2-decomposing spheres for a handlebody-knot (S^3, V) . Since they are unnested, each sphere S_i (S'_i , respectively) bounds a 3-ball B_i (B'_i) in S^3 such that $B_i \cap V \cong B^3$ ($B'_i \cap V \cong B^3$, respectively) ($1 \leq i \leq n$) and $B_i \cap B_j = \emptyset$ ($B'_i \cap B'_j = \emptyset$, respectively) ($1 \leq i < j \leq n$). Each sphere S_i (S'_i , respectively) separates an annulus A_i (A'_i , respectively) from ∂V . Let T_i (T'_i , respectively) be an essential torus in $E(V)$ obtained by tubing $S_i \cap E(V)$ ($S'_i \cap E(V)$) along A_i (A'_i , respectively). Let Y_i (Y'_i , respectively) be the region in S^3 spanned by T_i (T'_i , respectively) such that $Y_i \cap V = \emptyset$ ($Y'_i \cap V = \emptyset$, respectively). It is easy to check that the set $\mathcal{T} = \{T_1, T_2, \dots, T_n\}$ (resp. $\mathcal{T}' = \{T'_1, T'_2, \dots, T'_n\}$, respectively) satisfies the assumption of Lemma B.5. Therefore we have $Y_i \cap Y_j = \emptyset$ ($Y'_i \cap Y'_j = \emptyset$, respectively) for $1 \leq i < j \leq n$. Moreover, by Schubert's theorem [43], the core K_i (K'_i , respectively) of $E(Y_i)$ ($E(Y'_i)$, respectively) is prime for $1 \leq i \leq n$. Hence by Lemma B.4 that we have $\mathcal{T} = \mathcal{T}'$. This implies that \mathcal{S}' is obtained by at most n annulus-moves from \mathcal{S} .

ACKNOWLEDGMENTS

The authors would like to thank Mario Eudave-Muñoz for his valuable comments related to Lemma 3.2. This work was carried out while the first-named author was visiting Università di Pisa as a JSPS Postdoctoral Fellow for Research Abroad. A part of the work was done while the second-named author was visiting Università di Pisa. They are grateful to the university and its staff for their warm hospitality.

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